SOVIET PHYSICS USPEKHI VOLUME 9, NUMBER 2 SEPTEMBER-OCTOBER 1966

532.5

MAGNETOHYDRODYNAMIC WAVES

R. V. POLOVIN and K. P. CHERKASOVA

Physico-technical Institute, Academy of Sciences, Ukrainian S.S.R., Khar'kov

Usp. Fiz. Nauk 88, 593-617 (April, 1966)

INTRODUCTION

ALFVEN'S idea that magnetic force lines are "glued" to particles[1] has made it possible to explain the dynamics of sun spots. This approximation turned out to be too crude under terrestrial conditions, but it served as the basis for engineering applications of magnetohydrodynamics (electromagnetic measuring instruments and pumps, magnetohydrodynamic generators, and plasma accelerators).

A the focus of magnetohydrodynamic research has shifted towards an exact account of all the concomitant phenomena: viscosity, thermal conductivity, final electric conductivity, Hall current, inhomogeneity of the medium, nonequilibrium phenomena, etc. The solutions obtained in this case for the magnetohydrodynamic equations are cumbersome and difficult to visualize.

On the other hand, some new and unexpected results were obtained in recent years in magnetohydrodynamics. The present review is aimed at a description of these results, but still within the framework of an ideal medium, and is devoted to magnetohydrodynamic waves in a homogeneous medium. Even under such a limitation, in order to keep the review to a reasonable size, we had to omit almost all proofs. Cumbersome formulas have been purposely left out of the text, and we present principally qualitative results. The lack of proofs is compensated in part by references to the original articles.

Investigations of the stability of electromagnetic flow call for the use of the evolutionality conditions. These conditions have made it possible not only to discard the unrealizable solutions of the magnetohydrodynamic equations, but also to explain from a unified point of view several subtle questions in ordinary hydrodynamics (flow in nozzles, transonic flow around bodies, oblique and conical shock waves).

1. LINEAR WAVES

We begin our investigation of magnetohydrodynamic waves with small-amplitude waves in a stationary homogeneous medium. Linearizing the magnetohydrodynamic equations and assuming that the perturbed quantities depend on the coordinates and on the time like $\exp[i(kx-\omega t)]$, we find that the perturbations break up into seven waves[2,3]:

1. Two fast magnetic-sound waves[4], in which the non-vanishing perturbations are δv_x, δv_y, $\delta\rho$, δp, δH_y (v—velocity, ρ—density, p—pressure, **H**—magnetic field; the coordinate frame is chosen such as to make the unperturbed magnetic field H_z equal to zero). The phase velocity ω/k of such waves is equal to $\pm V_+$, where

$$V_+=\sqrt{\frac{U^2+c^2+\sqrt{(U^2+c^2)^2-4U^2c^2\cos^2\theta}}{2}} \qquad (1.1)$$

(c—speed of sound, $U=H\sqrt{\mu_0/\rho}$—Alfven velocity, θ—angle between the x axis and the vector **H**).

2. Two slow magnetic sound waves, in which the nonvanishing quantities are also δv_x, δv_y, $\delta\rho$, δp, and δH_y, and the phase velocity is $\pm V_-$, where

$$V_-=\sqrt{\frac{U^2+c^2-\sqrt{(U^2+c^2)^2-4U^2c^2\cos^2\theta}}{2}}. \qquad (1.2)$$

The magnetic-sound waves are plane-polarized; this means that there exists a reference frame in which $\delta v_z=\delta H_z=0$.

3. Two Alfven waves[1], in which the nonvanishing perturbations are δv_y, δv_z, δH_y, and δH_z, and the phase velocity is $\pm U_x$, where

$$U_x=|H_x|\sqrt{\frac{\mu_0}{\rho}}=H\sqrt{\frac{\mu_0}{\rho}}\,|\cos\theta|. \qquad (1.3)$$

The Alfven wave has circular polarization; this means that the quantity $H_y^2+H_z^2$ remains constant.

4. Entropy wave, in which only the perturbation of $\delta\rho$ differs from zero, and the perturbations of all the other quantities (δp, δv, δH) vanish. The phase velocity of the entropy wave is zero.

The phase velocity of propagation of magnetohydrodynamic waves depends on the angle θ between the direction of the magnetic field and the direction of propagation of the wave. A plot of this dependence, called a phase polar[5-7], is shown for the fast and slow magnetic-sound waves in Fig. 1. In this figure

$$OA=\min(U,\ c),\quad OB=\max(U,\ c),$$
$$OC=\sqrt{U^2+c^2}.$$

The internal curves (OAO and the curves symmetrical to it) are called the slow phase polar and the external curve BC, the fast phase polar.

An arbitrary perturbation of magnetohydrodynamic quantities can be represented in the form of a superposition of linear waves of the type $\exp[i(kr\cos\theta-\omega t)]$. The propagation velocity of the magnetic-sound

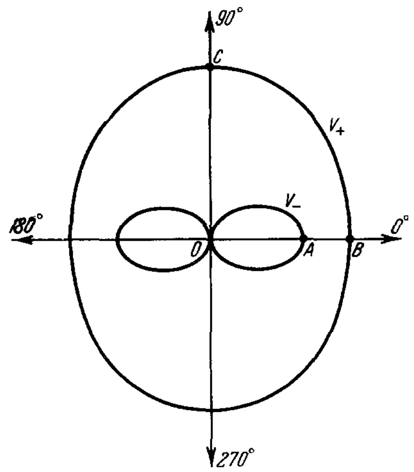

FIG. 1. Phase polars. V+—fast polar; V—slow polar. The magnetic field is directed along the polar axis.

perturbation does not coincide, however, with the phase velocity of the wave $\omega/k = V_\pm$, but is equal to the group velocity [8]

$$\mathbf{U}_\pm = \frac{\partial\omega}{\partial\mathbf{k}}. \tag{1.4}$$

In the one dimensional case, when the quantity θ is the same for all waves ($r\cos\theta = x$), the phase and group velocities coincide (i.e., there is no dispersion).

In the two- and three-dimensional cases the phase velocity does not coincide with the group velocity. In fact, it follows from (1.4) that

$$U_\pm \cos\theta = \frac{\partial\omega}{\partial k_x}, \quad U_\pm \sin\theta = \frac{\partial\omega}{\partial k_y}. \tag{1.5}$$

Substituting in (1.5) $\omega = V_\pm(\theta)k$, $k = \sqrt{k_x^2 + k_y^2}$, we obtain the parametric equation of the group polars [6,7,9-10]

$$x \equiv U_\pm(\theta)\cos\theta = V_\pm(\theta)\cos\theta - V'_\pm(\theta)\sin\theta,$$
$$y \equiv U_\pm(\theta)\sin\theta = V_\pm(\theta)\sin\theta + V'_\pm(\theta)\cos\theta \tag{1.6}$$

(see Fig. 2; the plus sign corresponds to the fast group polar and the minus sign to the slow polar). In this figure

$$OB = \max(U,\ c), \tag{1.7}$$

$$OA = \min(U,\ c), \tag{1.8}$$

$$OC = \sqrt{U^2 + c^2}, \quad OD = \frac{Uc}{\sqrt{U^2+c^2}}. \tag{1.9}$$

The perturbations emitted from a point source at the instant $t = 0$ will be situated at the instant $t = 1$ in the region contained inside the fast group polar. We note that in the two- and three-dimensional cases no splitting of the initial perturbation into fast and slow magnetic-sound waves occurs. All the perturbations propagate with the fast group velocity $U_+(\theta)$. On the other hand, as shown by calculations [12,13], the perturbations inside the slow group polars are equal to zero.

At an arbitrary instant of time t, the perturbations will be different from zero in the region enclosed between the lines $r = U_+(\theta)t$ and $r = U_-(\theta)t$, which we shall also call the fast and slow group polars corresponding to the instant t.

We note also that in magnetohydrodynamics, unlike ordinary hydrodynamics, the Huygens principle is not satisfied for three-dimensional waves. According to this principle the hydrodynamic perturbations emitted at an instant $t = 0$ from a point $x = y = z = 0$ will be different from zero at any succeeding instant $t > 0$ only on the surface of the sphere (in ordinary hydrodynamics)

$$x^2 + y^2 + z^2 = c^2t^2 \tag{1.10}$$

and will be identically equal to zero inside this sphere. To the contrary, in ordinary hydrodynamics the Huygens principle is not satisfied for two-dimensional waves; the perturbations will differ from zero at all points inside the circle

$$x^2 + y^2 \leqslant c^2t^2. \tag{1.11}$$

In magnetohydrodynamics the Huygens principle is not satisfied for either three dimensional or two dimensional waves [14]. The perturbations are different from zero at any point contained between the fast and slow group polars.

2. CHARACTERISTICS

We confine ourselves to two-dimensional stationary flow. The characteristics are defined as lines on which infinitesimally small discontinuities of magnetohydrodynamic quantities are possible. These lines can be regarded as shock waves of infinitesimally small intensity.

In ordinary hydrodynamics the characteristics exist only if the velocity of the medium is supersonic. They form with the velocity vector, an angle α defined by the relation $\sin\alpha = c/v$ (α is the Mach angle).

In magnetohydrodynamics the situation is much more complicated.

First, there are two types of two-dimensional waves —fast and slow magnetic-sound waves. Therefore there exist two types of characteristics—fast and slow. In addition, in many cases the perturbations propagate not only downstream, as in ordinary hydrodynamics, but also upstream [15].

In the case of stationary flow, the characteristic is the front of the wave emitted from a pointlike source moving with the same velocity v as the medium. Therefore the characteristic is the envelope of a family of group polars $U_\pm(\theta)t$, whose centers are located at the points vt (the parameter t runs through the values from zero to infinity).

Figure 2 shows the group polar corresponding to the instant of time $t = 1$. In the course of time the dimensions of the group polars increase, and the symmetry centers are displaced with the same velocity v as the liquid. If the liquid moves parallel to the mag-

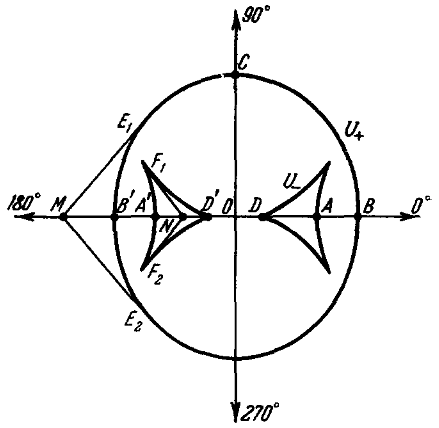

FIG. 2. Group polars. U_+—fast polar; U_- – slow polar. The magnetic field is directed along the polar axis.

netic field, then at the instant of time t = 1 the center of symmetry is shifted to the right along the x axis by a distance which is numerically equal to the velocity v of the liquid (the liquid flows from left to right). In other words, the point at which the perturbation was situated at the instant of time t = 0 constitutes at the instant t = 1 the end of the vector (- v) drawn from the symmetry center (in Fig. 2 these are the points M and N for two different values of the velocity). Therefore the characteristic coincides with the tangent to the group polar $r = U_\pm(\theta)t$ drawn from the point $(-\mathbf{v})$ [7,12]. In Fig. 2 the characteristics are the straight lines NF_1, NF_2, and ME_1, ME_2. The characteristics NF_1 and NF_2 are drawn to the left, i.e., upstream, and the characteristics ME_1 and ME_2 to the right, i.e., downstream.

If one cannot draw a tangent to the group polar from the point $(-\mathbf{v})$, then there are no characteristics. Such a flow is called elliptic. If the characteristics exist, then the flow is called hyperbolic.

Let us determine the conditions for hyperbolicity in the case when the velocity of the medium is parallel to the magnetic field[15-19]. We see from Fig. 2 that when v and H are parallel, the vector $(-\mathbf{v})$ is directed along the ray OB′. Since the fast group polar is convex, the fast characteristic exists when v > OB′ (see the point M on Fig. 2). From (1.7) follows the condition for the existence of fast characteristics

$$\max(U,\ c) < v. \tag{2.1}$$

Since the slow group polar is concave, the slow characteristic exists if OD′ < v < OA′ (see point N on Fig. 2). From (1.8) and (1.9) follows the condition for the existence of slow characteristics

$$\frac{Uc}{\sqrt{U^2+c^2}} < v < \min(U,\ c). \tag{2.2}$$

Conditions (2.1) and (2.2) coincide with the conditions for Cerenkov generation of magnetic-sound waves[20].

We note that if the velocity of the medium is parallel to the magnetic field, simultaneous existence of fast and slow characteristics is impossible. In addition, fast characteristics are directed downstream, and slow ones upstream. If on the other hand the velocity is not parallel to the magnetic field, then this rule is not satisfied—existence of fast and slow characteristics is possible; in addition, fast characteristics can be directed upstream and slow ones downstream.

In contrast to stationary flows, for which characteristics may not exist, in the case of nonstationary flows the characteristics always exist. Thus, for example, for one-dimensional flows in ordinary hydrodynamics the characteristics in the (x, t) plane are the lines defined by the differential equations

$$\frac{dx}{dt} = v_x, \tag{2.3}$$

$$\frac{dx}{dt} = v_x + c, \tag{2.4}$$

$$\frac{dx}{dt} = v_x - c. \tag{2.5}$$

Perturbations of the entropy and of the curl of the velocity propagate along the characteristic (2.3), a sound wave moving downstream propagates along the characteristic (2.4), and a sound wave moving upstream along the characteristic (2.5).

3. TRANSONIC FLOWS

In ordinary hydrodynamics the term "transonic" designates flow in which the velocity of the medium goes through the speed of sound. In magnetohydrodynamics the role of the speed of sound is played by the velocities of propagation of small perturbations.

The speed of sound is exceeded in nozzles and in flow around various bodies. We start with flow in a nozzle. Averaging all the quantities over the transverse cross section of the nozzle, we arrive at a one-dimensional problem, in which all the quantities depend only on the distance x along the nozzle axis.

In ordinary hydrodynamics infinitesimally small perturbations of the velocity, pressure, and entropy propagate along the characteristics (2.3)—(2.5). The characteristics (2.3) and (2.4) are always directed downstream (dx/dt > 0). The characteristic (2.5) is directed downstream for a supersonic medium ($v_x > c$) and upstream for subsonic velocity ($v_x < c$).

If the transition through the speed of sound in one-dimensional flow is with acceleration (see Fig. 3a; the index 1 pertains to the entrance to the nozzle, the index 2 to the exit), then the perturbations that arise on the sound line (v = c) travel into the upstream ($v_1 < c_1$) and downstream ($v_2 > c_2$) regions. Therefore the flow ahead and behind the sound line (v = c) can be "tuned" to the perturbation[21]. This explains why a Laval nozzle is stable. Flow having the number of waves necessary to move the perturbations from any point are called evolutional. Otherwise we speak of non-evolutional flow.

On the other hand, continuous passage through the

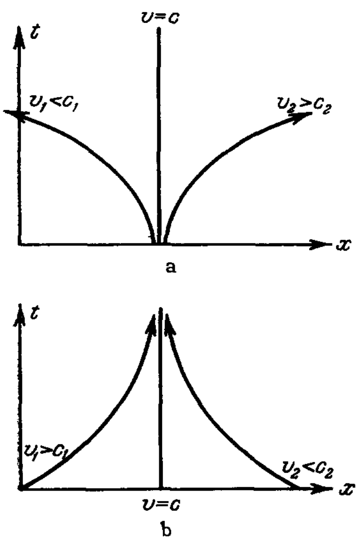

FIG. 3. Characteristics of one-dimensional flow with passage through the speed of sound: a) with acceleration, b) with deceleration. The index 1 pertains to the entrance to the nozzle and the index 2 to the exit from the nozzle, v = c is the sound line.

speed of sound with deceleration (supersonic diffusor) is impossible. This is explained by the fact that different perturbations which occur at the entrance and exit of the nozzle are transported to the sonic line and "stick" on it (see Fig. 3b). Since these perturbations are independent, a discontinuity occurs on the sound line[22], i.e., a shock wave is formed. (This idea was first advanced by Reynolds (see [23]). The formation of discontinuities on the sound line was demonstrated in [24-27] by an investigation of the evolution of small perturbations.)

Similar arguments can be advanced in the case of magnetohydrodynamics[28]. The magnetic field and the velocity of the medium will in this case have two components: H_x and v_x along the nozzle axis, and H_y and v_y in the transverse direction (the coordinate system is chosen such that $H_z \equiv 0$ and $v_z \equiv 0$). Such a flow cannot be realized in an ordinary nozzle, for the transverse velocity component v_y will make the particles strike the wall. A flow of this type can be realized in an annular nozzle[29-31]. The x axis is in this case directed along the nozzle axis, the y axis in the azimuthal direction, and the z axis radially. We note two cases in which magnetohydrodynamic flow in an ordinary (non-annular) nozzle is possible:

1) the velocity is parallel to the magnetic field[32],

2) the velocity is perpendicular to the magnetic field.

In the second case the magnetohydrodynamic equations reduce to the equations of ordinary hydrodynamics if the pressure p and the internal energy per unit mass ϵ are replaced by the quantities

$$p^* = p + \frac{1}{2}\mu_0 H^2, \quad \varepsilon^* = \varepsilon + \frac{1}{2\varrho}\mu_0 H^2, \tag{3.1}$$

and the speed of sound is replaced by

$$c^* = \sqrt{c^2 + U^2}, \tag{3.2}$$

where U is the Alfven velocity.

In the case of magnetohydrodynamic flow in nozzles, two limiting cases are possible, depending on the value of the electric conductivity σ.[33,34]

1. If the magnetic Reynolds number $R_m \equiv lv/\nu_m$ (l — characteristic dimension, $\nu_m = 1/\mu_0\sigma$ —magnetic viscosity) is large[34-36], then there exist three phase velocities of propagation of infinitesimally small disturbances: the Alfven velocity U_x and two velocities of propagation of magnetic-sound waves $V_\pm$ (if the magnetic field is perpendicular to the velocity of medium, then $U_x = V_- = 0$ and $V_+ = \sqrt{c^2 + U^2}$).

2. If the magnetic Reynolds number is small[34,35,37-42], then the magnetic field induced by the motion of the plasma can be neglected. The change in the magnetic field constitutes then an external action, and infinitesimally small perturbations will propagate with the usual speed of sound.

In the case of large magnetic Reynolds numbers, the following types of flow are possible without going through the characteristic velocity:

1) "slow" flow:

$$v_x < V_-, \tag{3.3}$$

2) "pre-Alfven" flow:

$$V_- < v_x < U_x, \tag{3.4}$$

3) "super-Alfven" flow:

$$U_x < v_x < V_+, \tag{3.5}$$

4) "fast" flow:

$$V_+ < v_x. \tag{3.6}$$

When $R_m \ll 1$, as in ordinary hydrodynamics, two types of flow are possible without going through the characteristic velocity:

1) subsonic flow: $v_x < c$,

2) supersonic flow: $v_x > c$.

Continuous passage through any characteristic velocity is possible with acceleration and is impossible with deceleration (shock waves are always produced).

We now proceed to transonic flow around a body. We start with ordinary hydrodynamics.

Let a stationary bounded body be situated in a subsonic gas stream. Since the gas velocity on the surface of the body exceeds the velocity at infinity, at some critical value of the Mach number at infinity the velocity of the gas reaches the speed of sound near some point on the surface of the body. Starting with this value of the Mach number, shock waves can appear. This raises the question: what will occur if the velocity of the incoming gas is increased—will shock waves appear or is continuous flow around the body possible such that the velocity of the gas at infinity is subsonic, but limited regions of supersonic velocity exist at the surface of the body?

Such continuous flows exist formally[43], but in practice shock waves are always produced on going over from supersonic to subsonic velocity[44-46].

The impossibility of continuous transonic flow around a body can be readily explained by noting that near the body the flow can be regarded as one-dimensional. In this region the conclusion made above with respect to flow in nozzles, that a continuous transition from supersonic to subsonic flow is impossible, also holds true.

The same conclusion was reached by Kuo[47] in an investigation of the evolution of small perturbations.

The impossibility of continuous transonic flow around the body in the case of an infinitesimally change in the contour of the body was demonstrated in several papers[43,48-55]. Bers raised the following objection to this proof[56]: the influence of the boundary layer makes it necessary to consider in the theory of an ideal medium not the true profile, but some effective profile formed by the boundary layer. Therefore not every deformation of the profile is admissible.

Bers' objection does not pertain to the work of Morawetz[57], in which it is shown that an infinitesimally small change in the Mach number makes a continuous transonic flow impossible.

Sometimes the impossibility of transonic flow around a body is attributed to changes in the boundary layer, i.e., to viscosity. If this explanation were true, then transonic flow could be realizable by pumping out the air[50].

Some authors have related the impossibility of transonic flow to formation of limit lines[54,58-60]— envelopes of characteristics. Further investigation has shown, however, that transonic flows without limit lines are also possible.[54,61,62].

We now proceed to transonic flow in magnetohydrodynamics.

As noted earlier, if the flow velocity is perpendicular to the magnetic field, then the flow coincides qualitatively with the flow in ordinary hydrodynamics. Therefore, passage of the velocity of the medium through the phase velocity of the magnetic-sound wave (3.2) with deceleration is unstable[26]. For arbitrary orientation of the magnetic field, any decelerated transition of the velocity of the medium through the phase velocity of the magnetic-sound wave or the Alfven velocity is unstable[28].

Transonic magnetohydrodynamic flow was investigated in [32,63-67] without consideration of stability, for a medium with velocity parallel to the magnetic field.

We note that the transitions which occur in this case, from the region of ellipticity into the region of hyperbolicity of the stationary flows and vice versa, have nothing in common with the possibility of realizing such flows. The realizability of such flows is determined by the evolutionality conditions, which are obtained from consideration of nonstationary perturbations.

Let us proceed, finally, to two-dimensional transonic flow. In ordinary hydrodynamics arbitrary two-dimensional perturbations can be represented in the form of the superposition of the perturbation of the velocity curl, the perturbations of the entropy, and of the potential isentropic sound perturbation. The latter is described by the equation

$$\Phi_{tt}+2(\Phi_x\Phi_{xt}+\Phi_y\Phi_{yt})+(\Phi_x^2-c^2)\Phi_{xx}+(\Phi_y^2-c^2)\Phi_{yy}+2\Phi_x\Phi_y\Phi_{xy}=0, \tag{3.7}$$

where Φ is the velocity potential.

Let us see how the perturbations behave near the sonic line $v = c$. Directing the x axis parallel to the velocity, we obtain

$$\Phi_x=c, \quad \Phi_y=0. \tag{3.8}$$

Assuming that on going from the supersonic velocity to the subsonic the characteristics that fall on the sound line cannot leave this line, let us differentiate (3.8) with respect to t:

$$\Phi_{xt}=0, \quad \Phi_{yt}=0. \tag{3.9}$$

Substituting (3.8) and (3.9) in (3.7), we obtain the equation

$$\Phi_{tt}-c^2\Phi_{yy}=0,$$

the solution of which is

$$\Phi=f_1(y-ct)+f_2(y+ct). \tag{3.10}$$

The function (3.10) is a superposition of two waves that move apart from the plane $y = 0$ in the positive and negative y directions. Thus, two converging waves correspond to two diverging waves, i.e., the transition through the speed of sound is evolutional.

This reasoning no longer holds if the sound line bears against a solid wall at a point at which the velocity vector has a definite direction. Then only one wave can move out of the point $x = 0$, $y = 0$, and the flow is non-evolutional.

If on the other hand the sound line bears, say, against a conical point, then the transition through the speed of sound with deceleration is evolutional. Such flows were observed experimentally[68].

4. SELF-SIMILAR PLANE WAVES

The equations of magnetohydrodynamics constitute a system of nonlinear partial differential equations. An investigation of such a system entails great mathematical difficulties even in the one-dimensional case, when all the magnetohydrodynamic quantities depend only on a single spatial coordinate x and the time t.

The investigation becomes much simpler if the initial data do not contain a parameter with dimension of length. The magnetohydrodynamic equations are then invariant against the change of variables $x \to Cx$ and $t \to Ct$. Therefore a change of variables $x/t = \xi$ transforms these equations into a system of ordinary nonlinear equations which do not contain implicitly the independent variable ξ. Solutions of such a system of equations are called self-similar waves[69].

Self-similar waves arise during the decay of an arbitrary initial discontinuity of magnetohydrodynamic quantities. The piston problem and the problem of the collision of shock waves reduce to this problem.

Fast and slow magnetic-sound linear waves correspond to fast and slow self-similar waves. In these waves the variables are v_x, v_y, ρ, p, and H_y (the wave moves in the x direction).

The velocities of propagation of the fast and slow self-similar waves are determined by formulas (1.1) and (1.2). These formulas determine the velocity of propagation of the wave in a medium at rest. In a moving medium, the velocity of propagation of the wave is $v_x \pm V_\pm$. Unlike linear waves, in which v_x is constant, in self-similar waves v_x will be different at different points.

The main difference between self-similar waves and linear waves is that in linear waves all the magnetohydrodynamic quantities—v_x, v_y, ρ, p, and H_y—executes small oscillations about equilibrium values, whereas in self-similar waves these quantities vary monotonically.

Let us see now how the different magnetohydrodynamic quantities vary in a self-similar wave. First of all, self-similar waves are rarefaction waves[70,71]: $\Delta\rho < 0$, $\Delta p < 0$. The variation of the quantities H_y, v_x, and v_y in self-similar waves is shown in the table:

Variation of H_y, v_x, and v_y in waves of different types. D^+ and D^-—detonations (fast and slow); I^+, I^-—ionization (fast and slow); C_f, C_s, C_p, C_{s1}—combustion (fast, super-Alfven, pre-Alfven, and slow); S^+ and R^+—fast waves (shock and rarefaction); A—180° Alfven discontinuity; S^- and R^-—slow waves (shock and rarefaction). It is assumed that the wave moves to the right and that $H_x > 0$ and $H_y > 0$. In the opposite case ΔH_y, Δv_x, and Δv_y must be replaced by sign $H_y \cdot \Delta H_y$, ϵv_x, and ϵ sign$(H_x H_y) \cdot \Delta v_y$, where $\epsilon = +1$ if the wave moves to the right and $\epsilon = -1$ if the wave moves to the left.

			ΔH_y	Δv_x	Δv_y			
D^+	I^+	C_f	+	+	—	S^+		
		C_s	—	—	+	R^+		
			—	0	+		A	
D^-	I^-	C_p	—	+	+			S^-
		C_{s1}	+	—	—			R^-

A fast self-similar wave corresponds to the second line of the table (see R^+—"rarefaction" in the right side of the table). The slow self-similar wave corresponds to the lowest line (see R^- in the right side of the table). For concreteness, the coordinate system is chosen such that the wave moves in the positive x direction (relative to the medium) and the conditions $H_x > 0$ and $H_y > 0$ are satisfied.

Self-similar waves are plane-polarized: if $H_z = 0$ and $v_z = 0$ in front of the wave, then this relation will be satisfied at all points of the self-similar wave.

A particular case of a self-similar wave is a perpendicular wave in which the projection of the magnetic field on the direction of propagation of the wave is equal to zero: $H_x = 0$.

We note that Alfven and entropy waves cannot be self-similar.

5. DISCONTINUITIES

In the preceding sections we have neglected dissipative effects—viscosity, heat conduction, and electric resistance, assuming the corresponding coefficients to be small. This is incorrect for those regions in space where the gradients of the magnetohydrodynamic quantities are large. Within the framework of the theory of an ideal medium, such regions must be regarded as surfaces on which ρ, p, **v**, and **H** experience discontinuities.

The values of the magnetohydrodynamic quantities on both sides of the discontinuity surface are connected by boundary conditions obtained not from the equations of magnetohydrodynamics, but directly from the laws of conservation of the mass, momentum, energy, and from the continuity of the tangential component of the electric field and the normal component of the magnetic field[3,72].

It follows from the boundary conditions that there are two different types of discontinuities[2,3,5,72]:

1. Shock waves, on which the quantities v_x, v_y, ρ, p, and H_y are discontinuous (the coordinate system is chosen such as to make the x axis directed along the normal to the surface of the discontinuity, $v_z = 0$, and $H_z = 0$).

Shock waves are either fast or slow, depending on whether or not they exceed the Alfven velocity. However, the propagation velocity of shock waves is determined not only by the parameters of the medium in front of the wave, but depends also on the intensity of the shock wave.

We note that if the velocity and the magnetic field are parallel to each other in front of the shock wave, they will also be parallel behind the shock wave.

Two particular cases of shock waves are of interest:

a) Parallel wave, in which the transverse component of the velocity and of the magnetic field are equal to zero on both sides of the discontinuity surface:

$$v_y = v_z = 0, \quad H_y = H_z = 0.$$

From the boundary conditions it follows that the discontinuities of the quantities v_x, ρ, and p on a parallel shock wave will be the same as in the absence of a magnetic field.

b) Singular wave, on one side of which the magnetic field and the velocity are directed normal to the discontinuity surface.

2. Alfven (or rotational) discontinuity, on which the quantities v_x, ρ, and p are continuous, while the magnetic-field vector experiences a rotation through a certain angle about the x axis. The propagation velocity

of an Alfven discontinuity in a stationary medium is equal to the Alfven velocity $U = |H|\sqrt{\mu_0/\rho}$.

An Alfven discontinuity, generally speaking, is not plane-polarized: if $H_{1z} = 0$ and $v_{1z} = 0$ ahead of the discontinuity, then $H_{2z} \neq 0$ and $v_{2z} \neq 0$ behind the discontinuity. However, if the magnetic field is rotated through 180° the Alfven discontinuity is plane-polarized. The character of variation of the magnetic field and velocity on such discontinuity is shown in the table.

3. Contact discontinuity, in which the quantities p, **v**, and **H** are continuous, and only ρ experiences a discontinuity. The propagation velocity of a contact discontinuity in a stationary medium is equal to zero.

In the case when the normal component of the magnetic field vanishes, the propagation velocities of the Alfven discontinuity and of the slow shock wave also become equal to zero. Therefore these discontinuities merge with the contact discontinuity, which is also at rest relative to the medium. Such a combined discontinuity is called tangential. Unlike shock waves, on which it is possible to specify arbitrarily the discontinuity of one of the magnetohydrodynamic quantities, on a tangential discontinuity it is possible to specify arbitrarily the discontinuities of five quantities.

6. CONDITIONS FOR THE EVOLUTIONALITY OF DISCONTINUITIES

Specification of the boundary conditions at the discontinuity is not enough to define in unique fashion the discontinuous solution. This difficulty is encountered also in ordinary hydrodynamics. Thus, for example, when a piston moves out of a tube there are two possible formal solutions: 1) self-similar rarefaction wave, 2) shock rarefaction wave. The second solution is discarded in ordinary hydrodynamics, since the entropy decreases in a shock rarefaction wave.

In magnetohydrodynamics rarefaction shock waves are impossible, since the entropy decreases on them, too[73,74]. However, in magnetohydrodynamics there exist too many compression shock waves and the problem of the motion of a medium with specified initial and boundary conditions frequently has several solutions (see, for example, [75] and [76]). Thus, the increasing entropy criterion, by which it is possible to exclude "excessive" discontinuities in ordinary hydrodynamics, is much too weak in magnetohydrodynamics.

Actually, however, not all shock waves on which the boundary conditions are satisfied and the entropy increases are realizable.

For a solution to be realizable it is necessary that it be stable. Investigation of the stability is usually carried out in the following manner. Infinitesimally small perturbations $\delta u_1, \delta u_2, \ldots$ are superimposed on the unperturbed values of the density, velocity, magnetic field, etc. Linearization yields a system of differential equations with constant coefficients, a solution of which is a superposition of plane waves $\exp[r(kx - \omega t)]$. The system of differential equations reduces in this case to a homogeneous system of linear algebraic equations:

$$\left.\begin{array}{l} A_{11}\delta u_1 + \ldots + A_{1n}\delta u_n = 0, \\ \ldots\ldots\ldots\ldots\ldots \\ A_{n1}\delta u_1 + \ldots + A_{nn}\delta u_n = 0, \end{array}\right\} \qquad (6.1)$$

where A_{ik} are functions of ω and k. The system (6.1) has a nontrivial solution if its determinant is equal to zero, i.e., if ω and k are connected by a relation called the dispersion equation:

$$F(\omega, k) = 0.$$

By specifying a real value of k (i.e., by specifying the wavelength $\lambda = 2\pi/k$ of the perturbation), it is possible to obtain from the dispersion equation the corresponding value of ω. Real ω means that the solution is stable against perturbations with a given wavelength λ. Complex ω (with positive imaginary part) is evidence of an exponential growth of the perturbation with time, i.e., of instability of the initial solution.

In many cases, however, the scheme described above for the investigation of stability is not applicable, since it may happen that the number of equations in the system (6.1) is not equal to the number of unknowns.* Then the solutions either do not exist, or their number is infinitely large.

On the other hand, in ordinary hydrodynamics and magnetohydrodynamics the Cauchy problem (the problem of finding the values of the magnetohydrodynamic quantities for $t > 0$, if the values for $t = 0$ are known) always has a unique solution. The absence or nonuniqueness of the solution constitutes a violation of the causality principle.

Since the only assumption made in the derivation of the system (6.1) was that linearization is possible, it follows therefore that perturbations which are infinitesimally small at $t = 0$ cannot become large immediately.

For example, if the initial solution is the shock wave shown in Fig. 4a, then infinitesimally small perturbation splits it into two shock waves. The perturbation

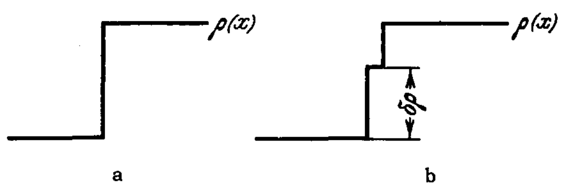

FIG. 4. Splitting of a shock wave. a) Initial wave; b) splitting wave; $\delta\rho$ – density perturbation. The shock wave moves from right to left.

*The number of equations is equal to the number of independent boundary conditions on the discontinuity surface, and the number of unknowns is equal to the number of waves of infinitesimally small amplitude, which diverge on both sides of the discontinuity surface.

$\delta\rho$ then immediately becomes large, although for small values of p this perturbation is localized in only a small region (Fig. 4b).

Such solutions, in which infinitesimally small perturbations cause a finite change in the solution, will be called, following I. L. Gel'fand[77], non-evolutional* (in other words, unstable against the appearance of new discontinuities).

An investigation of evolutionality is much simpler than an investigation of ordinary stability, since it reduces simply to counting the number of outgoing waves. At the same time, the evolutionality conditions make it possible to explain from a unified point of view the non-realizability of many solutions of the equations of ordinary hydrodynamics and to predict the non-realizability of many solutions of the equations of magnetohydrodynamics. Without the use of the evolutionality it is impossible to solve the magnetohydrodynamic problem in the presence of shock waves.

In magnetohydrodynamics there exist two regions of evolutionality of shock waves[76,89]:

1) Fast shock waves, for which

$$V_{1+} \leqslant v_{1x}, \quad U_{2x} < v_{2x} \leqslant V_{2+} \tag{6.2}$$

(V_+, U_x —phase velocities of the fast magnetic-sound and Alfven waves in the direction of the normal to the surface of the discontinuity, v_x —normal component of the velocity of the medium in a reference frame in which the discontinuity is at rest; the subscript 1 pertains to the region ahead of the discontinuity and the subscript 2 to the region behind the discontinuity).

2) Slow shock waves, for which

$$V_{1-} \leqslant v_{1x} < U_{1x}, \quad v_{2x} \leqslant V_{2-} \tag{6.3}$$

(V_- is the phase velocity of the slow magnetic-sound wave in the normal direction).

A certain discussion arose in connection with the conditions for evolutionality of singular shock waves[9]. Behind a fast singular shock wave the relation $v_{2x} = U_{2x}$ is satisfied, and in front of a slow singular wave $v_{1x} = U_{1x}$. Therefore, according to the evolutionality conditions (6.2) and (6.3), singular shock waves are non-evolutional. On the other hand, an infinitesimally small transverse magnetic field H_y is sufficient to realize an evolutional shock wave that is close to singular. Since the magnetic field cannot be specified with absolute accuracy, the question of evolutionality of singular shock waves has no physical meaning—such waves must be regarded as the limit of evolutional shock waves when the transverse magnetic field tends to zero. Therefore singular shock waves must be classified as evolutional[91].

From the evolutionality condition it follows[92] that shock waves are always compression waves ($\Delta\rho > 0$, $\Delta p > 0$). Using this result, we can determine the character of variation of the remaining magnetohydrodynamic quantities in shock waves (see the first and fourth lines of the table). We note that in slow shock waves the transverse magnetic field H_y decreases but does not reverse sign[2] (reversal of the sign of H_y takes place in unrealizable non-evolutional shock waves).

*The idea of evolutionality was first advanced in connection with a study of discontinuities in ordinary hydrodynamics by L. D. Landau and E. M. Lifshitz ([78] p. 405), and also by Curant and Friedrichs ([79] p. 215); the origin of the idea dates back to Hugoniot[80] (see[81], p. 99), and Jouguet[82] (see [83]). The evolutionality conditions for the general case, which includes also the equations of magnetohydrodynamics, were formulated in an article by Lax[84]. It is surprising that following Lax's article, the possible existence of non-evolutional shock waves was admitted in several papers[85-88] on the basis of the premise that the entropy increases in such waves. The impossibility of the existence of non-evolutional shock waves in magnetohydrodynamics was demonstrated in[89].

7. EXOTHERMAL AND ENDOTHERMAL DISCONTINUITIES

Let us consider discontinuities on which energy is released (exothermal discontinuity) or absorbed (endothermal discontinuity) as a result of chemical reactions, phase transitions, radiation or absorption of photons, dissociation, ionization, or recombination. The boundary conditions on such discontinuities are obtained by including the reaction energy in the energy conservation law.

In exothermal discontinuities the temperature increases rapidly in a narrow layer. The medium is heated by thermal conduction or else in a shock wave. In the former case we speak of conduction waves (which include condensation discontinuities[78,93], photoionization discontinuities,[94,95] and recombination discontinuities[96]) and in the latter case, of detonation waves. The propagation velocity of a combustion wave is determined by the characteristics of the medium. To the contrary, the propagation velocity of a detonation wave depends not only on the characteristics of the medium but also on the intensity of the shock wave.

The only endothermal discontinuities known are shock waves accompanied by dissociation, ionization, and radiation.

It follows from the evolutionality conditions that in ordinary hydrodynamics the combustion waves can be of two types;

1) subsonic combustion

$$v_1 < c_1, \quad v_2 < c_2, \tag{7.1}$$

2) supersonic combustion[97]

$$v_1 > c_1, \quad v_2 > c_2. \tag{7.2}$$

For shock detonation waves and for shock waves accompanied by ionization, the evolutionality conditions in ordinary hydrodynamics are

$$v_1 > c_1, \ v_2 \leqslant c_2. \tag{7.3}$$

Let us proceed to magnetohydrodynamics. The regions of evolutionality of detonation waves and of shock waves accompanied by ionization[98,99] coincide with the regions (6.2) and (6.3) of evolutionality of shock waves.

A special role is played by detonation in the Chapman–Jouguet regime, for which the velocity of the reaction products relative to the discontinuity is equal to the phase velocity of propagation of small perturbations:

$$v_{2x} = V_{2\pm}. \quad (7.4)$$

In the opposite ($v_{2x} < V_{2\pm}$) the detonation is called overcompressed.

In the case of detonation in the Chapman–Jouguet regime, the medium is heated at the expense of the released reaction energy. The amplitude of the detonation wave does not depend in this case on the velocity of the piston bounding the tube in which the detonation takes place. This amplitude is determined by the properties of the medium.

In the case of an overcompressed detonation, the medium is heated both by the released reaction energy and by the kinetic energy of the piston. The amplitude of the detonation wave will then be larger than in the Chapman–Jouguet regime, and depends on the velocity of the piston.

Detonation in the Chapman-Jouguet regime is stable in the sense that small changes of the medium parameters do not cause it to become overcompressed. A change in the parameters of the medium causes a change in the amplitudes of the magnetohydrodynamic waves accompanying the detonation in the Chapman–Jouguet regime (see Sec. 8).

As is well known, in ordinary hydrodynamics the Chapman-Jouguet regime is represented on the detonation adiabat by a point at which a straight line drawn from the point of the initial state is tangent to the detonation adiabat. Let us see how the Chapman-Jouguet detonation is plotted in magnetohydrodynamics. Figure 5 shows the magnetohydrodynamic detonation adiabat[100] (the connection between the total pressure $p^* = p + \mu_0 H^2/2$ and the specific volume V behind the discontinuity). The initial state (p_1^*, V_1) is represented by the point 4. The vertical and horizontal straight lines drawn through the point 4 intercept on the detonation adiabat the segment 18–0, which cannot be realized, since it corresponds to an imaginarity mass flux density $\rho_1 v_{1x}$. The straight lines 4–19, 4–10, 4–15, and 4–16 are tangent to the detonation adiabat at the points 19, 10, 15, and 16, at which the velocity of the discontinuity relative to the reaction products is equal to the velocity of propagation of the small perturbations. The points 19 and 15 correspond to detonation in the Chapman–Jouguet regime for the fast and slow waves. The sections 19–20 and 15–14 correspond to fast and slow overcompressed detonation. The line 4–14 has a slope $-\rho_1^2 U_{1x}^2$ relative to the abscissa axis. At the point 14 the velocity of propagation of the discontinuity in the stationary medium is equal to the Alfven velocity.

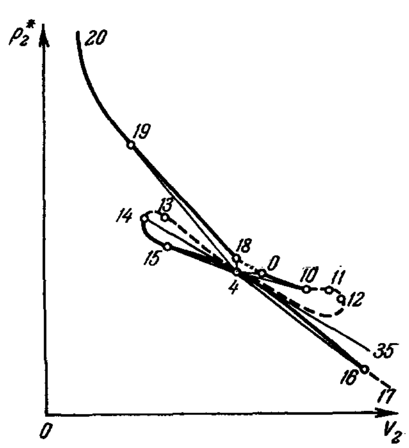

FIG. 5. Detonation adiabat in magnetohydrodynamics.

Let us proceed to magnetohydrodynamic combustion. From the conditions of evolutionality it follows that four regimes of magnetohydrodynamic combustion are possible[101,102]:

1) slow combustion:

$$v_{1x} < V_{1-}, \quad v_{2x} < V_{2-}, \quad (7.5)$$

2) pre-Alfven combustion:

$$V_{1-} < v_{1x} < U_{1x}, \quad V_{2-} < v_{2x} < U_{2x}, \quad (7.6)$$

3) super-Alfven combustion:

$$U_{1x} < v_{1x} < V_{1+}, \quad U_{2x} < v_{2x} < V_{2+}, \quad (7.7)$$

4) fast combustion:

$$V_{1+} < v_{1x}, \quad V_{2+} < v_{2x}. \quad (7.8)$$

These regimes correspond to segments 0–10, 15–4, 4–16, and 19–18 on Fig. 5.*

The character of variation of the magnetic field and of the velocity in magnetohydrodynamic detonation and combustion waves is shown in the table.

We have assumed that the magnetic field makes an acute angle with the direction of wave propagation. In the case of transverse wave propagation ($H_x = 0$) the magnetohydrodynamic detonation is qualitatively similar to ordinary detonation[39,102–104].

The results presented above are applicable, strictly speaking, only to a thermonuclear detonation, since the medium ahead of the detonation wave is assumed to be ideally conducting. In the case of chemical detonation the temperature of the medium ahead of the wave should be smaller than the ignition point; this contradicts the premise that thermal ionization causes the conductivity of the medium to be large.

It is of interest to develop a detonation theory for the case when the conductivity of the medium ahead of

*We note that when the reaction energy is sufficiently large the slow detonation, ionization on a slow shock wave, and also pre-Alfven and super-Alfven combustion are impossible[99].

the discontinuity is equal to zero, and is equal to infinity behind the discontinuity[105]. In this case the hydrodynamic waves do not interact with the electromagnetic waves in the region ahead of the discontinuity. Consequently, the discontinuity surface can radiate forward an electromagnetic wave whose amplitude is not determined by the conservation laws and can be obtained only from an analysis of the structure of the discontinuity[106,107] (see also [108,109]).

8. SEQUENCE OF WAVES

From the evolutionality conditions it follows that the number of waves that can move simultaneously in one direction is limited.

In ordinary hydrodynamics, if two shock waves, two self-similar waves, or a shock and self-similar wave move in one direction, then they must ultimately come together. Therefore if these waves were produced simultaneously, not more than one wave can move in one direction.

No waves can move ahead of the wave of supersonic combustion and behind the wave of subsonic combustion. At the same time, a shock or self-similar wave can move behind a supersonic combustion wave. In exactly the same way, a shock or self-similar wave can move ahead of a subsonic combustion wave.

In magnetohydrodynamics one of two shock waves of the same type (both fast or both slow) also overtakes the other. As regards the waves of different types, the Alfven discontinuity overtakes the slow wave, and the fast shock wave overtakes all types of discontinuities. In similar fashion, the shock wave overtakes the self-similar wave[69] if the latter belongs to the same type as the shock wave, or if it belongs to the slower type. A self-similar wave overtakes a shock wave of the same type and a shock wave of a slower type.

Thus, if all the waves were produced simultaneously, then not more than three waves can move in one direction: a fast wave (shock or self-similar) in front, followed by an Alfven wave, and finally a slow wave (shock or self-similar).

Let us determine now the magnetohydrodynamic waves that can accompany detonation and ionization waves.

Since the evolutionality conditions of the ionization and overcompressed-detonation waves coincide with the evolutionality conditions of the shock waves, the fast ionization wave or the overcompressed detonation wave cannot be accompanied by a fast shock wave or a fast self-similar wave (the same holds for the slow wave). Thus, not more than three waves can move in either direction.

If the overcompressed detonation or the ionization occurs on a fast wave, then the sequence of the waves will be as follows: first the overcompressed detonation (or ionization) wave, followed by the Alfven wave, and finally the slow wave (shock or self-similar).

In the case of a Chapman-Jouguet detonation, the velocity of the wave relative to the products of the reaction is equal to the velocity of propagation of small perturbations, therefore a fast self-similar wave can follow behind the fast wave in the Chapman-Jouguet regime (the same holds for a slow wave).

If the Chapman-Jouguet detonation occurs on the fast wave, then the sequence of the waves will be as follows: first the fast detonation wave in the Chapman-Jouguet regime, followed by the fast rarefaction (self-similar) wave, and then the Alfven wave, followed finally by a slow wave (shock or self-similar).

Some of the foregoing waves may not be produced.

The character of the waves accompanying the detonation depends on the boundary conditions. If the detonation occurs in a tube closed on one end by an ideally conducting piston, the character of these waves depends on the normal and tangential components of the piston velocity u_y and u_x. This relation is shown in Fig. 6.

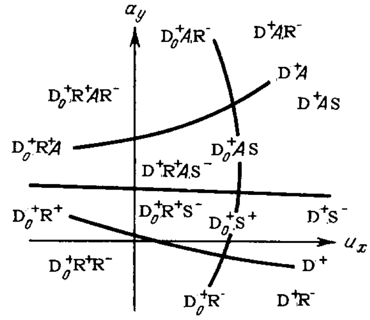

FIG. 6. Possible detonation regimes on the fast wave. The abscissas represent the longitudinal components of the piston velocity u_x, and the ordinates of the transverse components u_y. The letters D^+, D_0^+, S^-, R^+, R^- and A denote respectively the presence of a fast overcompressed detonation wave, fast detonation wave in the Chapman-Jouguet regime, slow shock wave, fast and slow rarefaction waves and the Alfven discontinuity.

As seen from Fig. 6, if the piston is at rest, then the detonation occurs in the Chapman-Jouguet regime[99,110,111]. We note that a one-dimensional detonation with the piston at rest is qualitatively similar to a spherical detonation (since the velocity of the medium at the center vanishes, from symmetry considerations). It is therefore to be expected that a spherical magnetohydrodynamic detonation also occurs in a Chapman-Jouguet regime.

Let us proceed to determine the possible types of magnetohydrodynamic waves accompanying combustion waves.

It follows from the evolutionality condition that the fast-combustion wave moves ahead of the magnetohydrodynamic waves of all types, the super-Alfven combustion wave lies between the fast (shock or self-similar) and Alfven waves, the pre-Alfven combustion wave lies between the Alfven and the slow waves, and finally, the slow-combustion wave moves behind all these magnetohydrodynamic waves. For example, in the case of slow combustion, the fast wave (shock or self-similar) moves in front, followed by the Alfven wave, then by the slow wave (shock or self-similar), and finally the slow-combustion wave.

9. BREAKUP OF DISCONTINUITY

When two shock waves collide, a discontinuity is produced, on which the boundary conditions are not satisfied. Therefore such a discontinuity breaks up into a series of discontinuous and self-similar waves. The number of produced waves is seven: three waves moving to the right (fast, Alfven, and slow), three similar waves moving to the left, and between them a contact discontinuity which is at rest relative to the medium. The fast and slow waves can be either shock or self-similar.

The breakup of the discontinuity of low intensity was investigated in [84,112]. Breakup of non-evolutional shock waves in several particular cases was considered in [113-115]. A qualitative investigation of the general problem of breakup of arbitrary magnetohydrodynamic discontinuity and of a non-evolutional shock wave of arbitrary intensity was carried out in [116].

A particular variant of the breakup of the discontinuity of the initial conditions is the piston problem. If a piston begins to move at the instant $t = 0$ with constant velocity, then a discontinuity occurs between the piston velocity and the velocity of the medium. The problem of the piston was solved for different cases in [75,86,117-121].

Problems involving the collision of shock waves and the reflection of shock waves from a wall and from the region of a strong magnetic field likewise reduce to the problem of breakup of a discontinuity [122,123]

10. SELF-SIMILAR STATIONARY WAVES

In supersonic two-dimensional flow the plane of flow can be divided into a constant-flow region and a region of self-similar waves. If the angle of flow differs little from 180°, then the self-similar wave will have low intensity, and the region occupied by the self-similar wave contracts to the characteristic of the stationary flow. In stationary self-similar waves, all the quantities depend only on the angle φ (in a polar coordinate system r, φ). Such waves arise in the case of flow around a corner of angle less than 180° (the angle is measured in the region occupied by the corner).

We confine ourselves to the two dimensional case when $v_z = 0$. The boundary conditions on the surface of the body in the stream depend on its electric conductivity. The magnetohydrodynamic effects appear most strongly when the body in the stream has infinite conductivity. The tangential component of the electric field on the surface of the body is then equal to zero:

$$E_t = 0. \qquad (10.1)$$

If the fluid flowing around body also has infinite conductivity, then the electric field $\mathbf{E}'$ is equal to zero in a reference frame in which the fluid is at rest. Using the Lorentz transformation, we can express $\mathbf{E}'$ in terms of an electric field E in the laboratory frame:

$$\mathbf{E}' = \mathbf{E} + \mu_0 [\mathbf{vH}] = 0. \qquad (10.2)^*$$

From (10.1) and (10.2) follows the boundary condition on the surface of the conductor in the stream

$$[\mathbf{vH}]_t = 0. \qquad (10.3)$$

Since the body is in a stream of an ideal fluid, we have $v_t \neq 0$. Therefore it follows from (10.3) that the normal component of the magnetic field is equal to zero:

$$H_n = 0. \qquad (10.4)$$

The boundary condition (10.4) is satisfied in two cases:

1) The velocity lies in the (x, y) plane and the magnetic field is directed along the z axis. This case reduces to ordinary gasdynamics.

2) The velocity and the magnetic field lie in the (x, y) plane are parallel to each other.

According to the results of Sec. 2, in this case two types of self-similar waves are possible, bounded by the characteristics from the region of constant flow:

a) Slow waves, if $cU(c^2 + U^2)^{-1/2} < v < \min(U, c)$. The slow waves move upstream.

b) Fast waves, if $\max(U, c) < v$. The fast waves move downstream. Stationary self-similar waves are rarefaction waves [5,124].

11. OBLIQUE SHOCK WAVES

In the case of flow around a corner larger than 180°, or the flow around a wedge, the turning of the velocity of the medium is realized in many cases in an oblique shock wave which is attached to the vertex of the corner.

Let us consider first oblique shock waves in ordinary hydrodynamics.

Knowing the velocity of the medium ahead of the shock wave ($v_{1x} = v_1$, $v_y = 0$), we can determine from the mass, momentum, and energy conservation laws on the shock wave the possible values of v_{2x} and v_{2y} behind the shock waves. The corresponding points are located on a curve called the shock polar [125] (Fig. 7).

To determine the state (v_{2x}, v_{2y}) behind the shock wave if the half-apex angle θ of the wedge is specified, it is necessary to draw from the origin O a straight line making an angle θ with the abscissa axis. This line crosses the shock polar at two points, S and W, corresponding to two possible oblique shock waves. The point S which is closest to the origin determines the shock wave of the strong family, and the other point

*$[\mathbf{vH}] \equiv \mathbf{v} \times \mathbf{H}$.

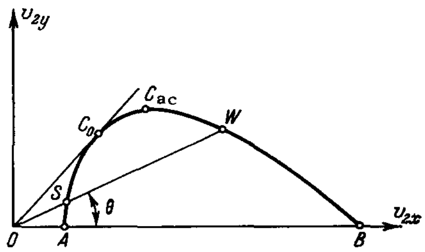

FIG. 7. Shock polar. A—Normal wave; B—oblique shock wave of infinitesimally small intensity; θ—half the wedge apex angle; S and W—oblique shock waves of the strong and weak families; C_0—point corresponding to maximum possible θ; C_{ac}—point at which the velocity of the medium behind the shock wave is equal to the velocity of sound.

W the shock wave of the weak family. If a tangent OC_0 is drawn to the shock polar, then the point of tangency separates the waves of the strong family (corresponding to the points on the arc AC_0) from the waves of the weak family (arc C_0B). If the half-apex angle of the wedge is larger than θ_0 (corresponding to the point C_0), then no oblique shock wave can exist.

Near the point C_0 is located the point C_{ac} (corresponding to a half-apex angle of the wedge θ_{ac}), such that the velocity of flow behind the shock wave is equal to the velocity of sound[126]. On the section AC_{ac} we have $v_2 < c_2$, and on the section $C_{ac}B$ we have $v_2 > c_2$. (The difference between θ_0 and θ_{ac} is less than half a degree for air; see [62], p. 398, and [127], p. 432.)

In spite of the fact that the shock waves of the strong family are formally possible, they do not occur in practice.

The impossibility of existence of shock waves of the strong family follows from the evolutionality conditions. This idea was first advanced in vague form by Prandtl[46]: if $v_2 < c_2$, then the perturbations are carried to the corner, and this leads to a breakup of the shock wave; on the other hand if $v_2 > c_2$, then the perturbations are carried out of the corner.

The proof of Prandtl's statement consists in the following[128]. Since the flow ahead of the shock wave is supersonic, the perturbations cannot penetrate upstream. Therefore the state of the medium ahead of the shock wave cannot change. If the state of the medium ahead of the shock wave is specified and also its apex half-angle θ, then according to Fig. 7 the state behind the shock wave is defined. Consequently, a perturbation produced at the instant t = 0 at the apex of the corner should be "swept out" by any instant t > 0. In the two-dimensional case the perturbations in a moving medium at the instant t fill the circle (Figs. 8a and b)

$$(x - v_x t)^2 + (y - v_y t)^2 \leqslant c^2 t^2. \tag{11.1}$$

The condition for the perturbation to vanish at the point x = y = 0 is $v_2 > c_2$. When this condition is not satisfied, detachment of the shock wave takes place. Experiments on the collision of metallic plates[129] indicate that the oblique shock wave becomes detached when $\theta < \theta_0$.

We now proceed to magnetohydrodynamic oblique shock waves.

Like the characteristics, the magnetohydrodynamic oblique shock waves can be directed not only downstream but also upstream[130] (see also [64,65,131-135]).

Let us consider in greater detail the case when the velocity of the medium is parallel to the magnetic field[128]. Allowance for the evolutionality conditions (6.2) and (6.3) greatly narrows down the number of possible solutions[136,137].

We determine first the angle at which an oblique magnetohydrodynamic shock wave can be inclined. From the boundary conditions it follows that when the medium passes through a shock wave the normal component of the magnetic field does not change. We see from the table that the tangential component of the magnetic field increases in fast shock waves and decreases in slow shock waves (without reversing sign). Therefore in fast shock waves the angle between the vector of the magnetic field and the normal to the discontinuity surface increases, while in slow ones it decreases. Since the velocity vector is directed parallel to the magnetic field, it is refracted by the shock wave in the same manner. Therefore in flow around a corner larger than 180° the discontinuity line can be located only in the sector shown in Fig. 9. It is clear from Fig. 9 that the fast shock wave is directed downstream and the slow one upstream.

The limits of the regions F and S (corresponding to the angles of inclination 90° and 90° + θ in Fig. 9) are reached in the fast and slow singular shock waves.

It can be shown that when the velocity vector is parallel to the magnetic field not more than one wave can be attached to the apex of the corner (this statement is not valid if the velocity and magnetic-field vectors are not parallel, as can be seen from the fact that in this

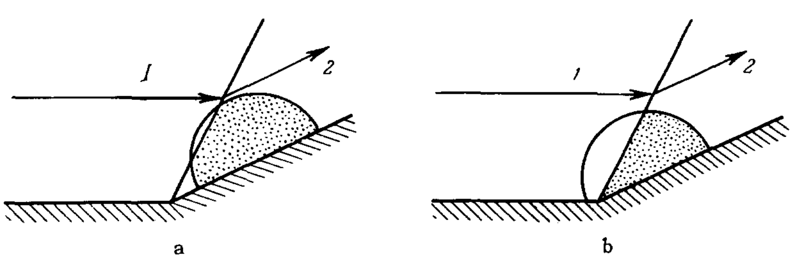

FIG. 8. Sweep-out conditions. The perturbed region is shown dotted. a) $v_2 > c_2$—perturbation is swept out from the apex; b) $v_2 < c_2$—perturbation is not swept out.

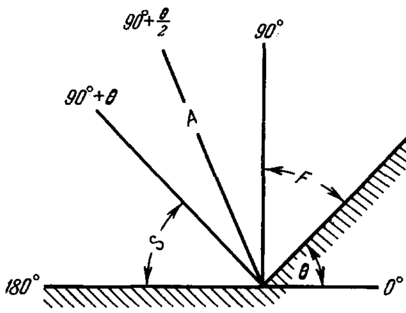

FIG. 9. Sectors in which a discontinuity line is possible. F–Sector in which a fast shock wave is possible; S–sector in which a slow shock wave is possible; A–direction of the Alfven discontinuity. The medium moves in the unshaded region from left to right, the velocity vector is parallel to the magnetic field.

case several characteristics can emerge from a single point, which can be regarded as shock waves of infinitesimally low intensity).

We now proceed to the sweep-out conditions.

From the evolutionality conditions it follows that[128] in front of the shock wave the condition satisfied is

$$v_1 > (\max U_1,\ c_1). \tag{11.2}$$

This means that the perturbation cannot penetrate upstream. Therefore they should be swept out from the region near the vertex of the angle behind the shock wave. Consequently the velocity vector should lie either outside the fast group polar or inside the slow group polar, i.e., one of the following conditions should be satisfied: either

$$v_2 > \max(U_2,\ c_2) \tag{11.3}$$

or

$$\frac{U_2c_2}{\sqrt{U_2^2+c_2^2}} < v_2 < \min(U_2,\ c_2). \tag{11.4}$$

It can be shown that condition (11.4) contradicts the conservation laws for the fast shock wave.

For the slow shock wave, condition (11.2) is impossible. If the condition

$$\frac{U_1c_1}{\sqrt{U_1^2+c_1^2}} < v_1 < \min(U_1,\ c_1) \tag{11.5}$$

is satisfied ahead of the wave, then no perturbations can occur upstream. Therefore the perturbations should be swept out from the corner behind the shock wave, i.e., the condition (11.4) should be satisfied. If condition (11.5) is not satisfied, then condition (11.4) should likewise not be satisfied.

12. CONICAL WAVES

In the case of supersonic flow around a cone, conical shock waves are produced. Conical shock waves in ordinary hydrodynamics and magnetohydrodynamics differ noticeably because the Huygens principle holds in ordinary hydrodynamics but not in magnetohydrodynamics.

Let us start with ordinary hydrodynamics. Conical flow is three dimensional, therefore a perturbation produced at the instant $t = 0$ at a point $x = y = z = 0$ will change the flow at an arbitrary instant $t > 0$, by virtue of the Huygens principle, only on the surface of the sphere

$$(x-v_xt)^2+(y-v_yt)^2+(z-v_zt)^2=c^2t^2, \tag{12.1}$$

but not inside the sphere. Consequently, the perturbation is always swept out from the apex. Thus, the flow behind the conical shock wave can be subsonic, as was indeed observed experimentally[138]. As regards conical shock waves of the strong family, they are impossible[139,140]. This can be proved in the following manner[141]. If the velocity behind the conical shock wave is subsonic, then in the case of flow around a cone the equation of the discontinuity surface in a spherical coordinate system (r, φ, θ) is of the form

$$\theta=\theta_w+Ar^m+\ldots, \tag{12.2}$$

where θ_W, A, and m are constants. As shown by a numerical calculation[141], for shock waves of the strong family $m < 0$, and for shock waves of the weak family $m > 0$. From this it follows directly that the conical shock waves of the strong family are impossible, since the discontinuity surface (12.1) does not pass for them through the point $r = 0$.

An investigation of the discontinuity of hydrodynamic quantities on conical shock waves is reported in [142]. Conical shock waves are always accompanied by continuous conical flows. Their investigation is the subject of [142-145].

The motion of a cone at an angle of attack is investigated in [146-149].

We now proceed to conical shock waves in magnetohydrodynamics (such waves have been considered in [150,151] in the case when the velocity of the medium at infinity is parallel to the magnetic field).

In magnetohydrodynamics oblique and conical shock waves are qualitatively similar, since the Huygens principle is satisfied for neither two-dimensional nor three-dimensional waves. Therefore for conical shock waves there should be satisfied the same sweep-out conditions as for oblique ones, namely, conditions (11.2) and (11.3) should be satisfied for fast conical shock waves (which are directed downstream); for slow shock waves (which are directed upstream) there should be satisfied either both conditions (11.4) and (11.5), or none at all.

[1] H. Alfven, Cosmical Electrodynamics, Oxford, 1950.
[2] S. I. Syrovat-skiĭ, UFN **62**, 247 (1957).
[3] L. D. Landau and E. M. Lifshitz, Elektrodinamika sploshnykh sred (Electrodynamics of Continuous Media) Gostekhizdat, 1957.
[4] G. Herlofson, Nature **165**, 1020 (1950).

[5] A. G. Kulikovskiĭ and G. A. Lyubimov, Magnitnaya gidrodinamika (Magnetohydrodynamics), Fizmatgiz, 1962.
[6] H. Grad, in: Magnetodynamics of Conducting Fluids (Ed. D. Bershader), Stanford Univ. Press, 1959, p. 37.
[7] W. R. Sears, Rev. Mod. Phys. **32**, 701 (1960).
[8] M. J. Lighthill, Phil. Trans. Roy. Soc. (London), **A252** (No. 1014), 397 (1960).
[9] J. Bazer and O. Fleischman, Phys. Fluids **2**, 366 (1959).
[10] Y. M. Lynn, Phys. Fluids **5**, 626 (1962).
[11] J. E. McCune and E. L. Resler, J. Aero and Space Sci. **27**, 493 (1960).
[12] H. Grad, Rev. Mod. Phys. **32**, 831 (1960).
[13] H. Weitzner, Bull. Amer. Phys. Soc. **5**, 321 (1960).
[14] R. Courant, Partial Differential Equations (Russ. transl.), Mir, 1964.
[15] M. N. Kogan, PMM **23**, 70 (1959).
[16] S. I. Syrovat-skiĭ, Trudy, FIAN (Phys. Inst. Acad. Sci.) **8**, 13 (1956).
[17] T. Taniuti, Progr. Theor. Phys. **19**, 749 (1958).
[18] J. Jmai, Rev. Mod. Phys. **32**, 992 (1960).
[19] P. Germain, Rech. aĕrhonaut, Nr. **7**, 13 (1960).
[20] A. I. Morozov, in: Fizika plazmy i problema upravlyaemykh termoyadernykh reaktsiĭ (Plasma Physics and the Problem of Controlled Thermonuclear Reactions), v. 4, AN SSSR, 1958, p. 331.
[21] L. Crocco, in: Osnovy gazovoĭ dinamiki (Principles of Gas Dynamics), IL, 1963, p. 64.
[22] R. V. Polovin, JETP **41**, 394 (1961), Soviet Phys. JETP **14**, 284 (1962).
[23] J. I. Taylor, in: Gazovya dinamika (Gas Dynamics), GONTI, 1939, pp. 29-48.
[24] A. Kantrowitz, Phys. Rev. **71**, 465 (1947).
[25] R. E. Meyer, Proc. Symp. Appl. Math., vol. 4, McGraw-Hill Co., N.Y.-Toronto-London, 1953, p. 41.
[26] J. Naze, Compt. rend. **246**, 3315 (1958); **248**, 362, 525 (1959).
[27] A. Kantrowitz, in: Osnovy gazovoĭ dinamiki (Fundamentals of Gas Dynamics) IL, 1963, p. 325.
[28] R. V. Polovin, PMTF No. 6, 3 (1961).
[29] C. L. Longmire, The Use of Plasma for Propulsion of Interplanetary Rockets, Plasma Acceleration, Stanford Univ. Press, 1960.
[30] K. P. Cherkasova, PMTF, No. 5, 144 (1962).
[31] A. I. Morozov and L. S. Solov'ev, ZhTF **34**, 1154 (1964), Soviet Phys. Tech. Phys. **9**, 898 (1965).
[32] C. K. Chu, Phys. Fluids **5**, 550 (1962).
[33] L. A. Vulis and P. L. Gusika, ZhTF **31**, 806 (1961), Soviet Phys. Tech. Phys. **6**, 582 (1962).
[34] J. A. Shercliff, Rev. Mod. Phys. **32**, 980 (1960).
[35] K. A. Lur'e, PMTF, No. 5, 3 (1961).
[36] K. B. Pavlov, JETP **39**, 304 (1960), Soviet Phys. JETP **12**, 216 (1961).
[37] G. S. Golitsyn and K. P. Stanyukovich, JETP **33**, 1417 (1957), Soviet Phys. JETP **6**, 1090 (1958).
[38] I. B. Chekmarev, PMM **24**, 382 (1960).
[39] R. A. Gross, J. Aero and Space Sci. **25**, 788 (1958).
[40] J. E. McCune and W. R. Sears, J. Aero and Space Sci. **27**, 139 (1960).
[41] E. L. Resler and W. R. Sears, ZAMP **96**, 509 (1958).
[42] W. R. Sears, ARS Journ. **29**, 397 (1959).
[43] K. G. Guderley, Theory of Transonic Flow (Russ. transl.), IL, 1960.
[44] T. von Karman, Supersonic Aerodynamics (Russ. transl.) IL, 1948, p. 61.
[45] G. V. Lipman and A. E. Puckett, Introduction to Aerodynamics of Compressible Liquids (Russ. transl.), IL, 1949.
[46] L. Prandtl, In collection "Gas Dynamics" GONTI, 1939, p. 5.
[47] Y. H. Kuo, J. Aero and Space Sci. **18**, 1 (1951).
[48] F. I. Frankl', UMN **15**, 163 (1960).
[49] F. I. Frankl', PMM **11**, 199 (1947) and **20**, 196 (1956).
[50] W. R. Sears, J. Appl. Phys. **21**, 771 (1950).
[51] C. S. Morawetz, Comm. Pure Appl. Math. **7**, 697 (1954); **9**, 45 (1956); **10**, 107 (1957).
[52] A. Buseman, J. Aero and Space Sci. **16**, 337 (1949); Proc. Symp. on Appl. Math., vol. 4, McGraw-Hill McGraw-Hill, New York - Toronto - London, 1953, p. 29.
[53] A. R. Manwell, Quart. J. Mech. Appl. Math. **7**, 40 (1954).
[54] L. Bers, Comm. Pure and Appl. Math. **7**, 40 (1954).
[55] J. Serrin, Mathematical Principles of Classical Mechanics of Liquids (Russ. transl.), IL, 1963.
[56] L. Bers, Mathematical Problems of Transonic and Subsonic Gas Dynamics (Russ. transl.), IL, 1961.
[57] C. S. Morawetz, Comm. Pure Appl. Math. **11**, 129 (1958).
[58] O. O. Ryzhov, PMM **27**, 309 (1963).
[59] V. S. Tatarenchik, ibid **8**, 401 (1944).
[60] J. W. Craggs, Proc. Cambr. Phil. Soc. **44**, 360 (1948).
[61] I. Colodner and C. S. Morawetz, Comm. Pure Appl. Math. **6**, 97 (1953).
[62] R. Mises, Mathematical Theory of Compressible Fluid Flow, Academic Press, New York, 1958.
[63] K. Tamada, Phys. Fluids **5**, 871 (1962).
[64] N. Geffen, Phys. Fluids **6**, 566 (1963).
[65] M. N. Kogan, PMM **25**, 132 (1961).
[66] M. D. Cowley, ARS Journ. 30, 271 (1960).
[67] R. Peyret, Compt. rend. **250**, 1971 (1960).
[68] G. J. Taylor and J. W. Maccoll, Proc. Roy. Soc. **A139**, 278 (1933).
[69] A. I. Akhiezer, G. Ya. Lyubarskiĭ, and R. V. Polovin, ZhTF **29**, 933 (1959), Soviet Phys. Tech. Phys. **4**, 849 (1960). Ukr. fiz. zh. **3**, 433 (1958).
[70] A. G. Kulikovskiĭ, DAN SSSR **121**, 987 (1958), Soviet Phys. Doklady **3**, 743 (1959).
[71] G. Ya. Lyubarskiĭ and R. V. Polovin, JETP **35**, 509 (1958), Soviet Phys. JETP **8**, 351 (1959). Ukr. fiz. zh. **3**, 567 (1958).
[72] F. Hoffmann and E. Teller, Phys. Rev. **80**, 692 (1950).

[73] S. V. Iordanskiĭ, DAN SSSR **121**, 610 (1958), Soviet Phys. Doklady **3**, 736 (1959).
[74] R. V. Polovin and G. Ya. Lyubarskiĭ, JETP **35**, 510 (1958), Soviet Phys. JETP **8**, 351 (1959); Ukr. fiz. zh. 3, 571 (1958).
[75] R. V. Polovin, JETP **39**, 1005 (1960), Soviet Phys. JETP **12**, 699 (1961).
[76] R. V. Polovin, UFN **72**, 33 (1960), Soviet Phys. Uspekhi **3**, 677 (1961).
[77] I. M. Gel'fand, UMN **14**, 87 (1959).
[78] L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Mechanics of Continuous Media), Gostekhizdat, 1954.
[79] R. Courant and K. Friedrichs, Supersonic Flow and Shock Wave, Interscience, 1948.
[80] H. Hugoniot, Journ. Ecole polytechn. **58**, 1 (1889).
[81] Ya. B. Zel'dovich, Teoriya udarnykh voln i vvednenie v gazodinamiku (Theory of Shock Waves and Introduction to Gas Dynamics), AN SSSR, 1946.
[82] E. Jouguet, Mechanique des explosifs, Paris, 1917.
[83] Ya. B. Zel'dovich, JETP **10**, 542 (1940).
[84] P. Lax, Comm. Pure Appl. Math. **10**, 537 (1957).
[85] R. Thibaut, Compt. rend. **255**, 834 (1962).
[86] J. Bazer, Astroph. J. **128**, 686 (1958).
[87] J. Bazer and W. B. Ericson, Astrophys. J. **129**, 758 (1959).
[88] W. B. Ericson and J. Bazer, Phys. Fluids **3**, 63 (1960).
[89] A. I. Akhiezer, G. Ya. Lyubarskiĭ, and R. V. Polovin, JETP **35**, 731 (1958), Soviet Phys. JETP **8**, 507
[90] N. H. Kemp, P. Germain, and H. Grad, Revs. Mod. Phys. **32**, 958 (1960).
[91] R. V. Polovin, In collection Fizika plazmy i problema upravlyaemogo termoyadernogo sinteza (Plasma Physics and the Problem of Controlled Thermonuclear Fusion), No. 3, Kiev, 1963, p. 169.
[92] T. Taniuti, Prog. Theor. Phys. **28**, 756 (1962).
[93] S. Z. Belen'kiĭ, DAN SSSR **48**, 172 (1945).
[94] F. A. Goldsworthy, Phil. Trans. Roy. Soc. (London) **A253** (No. 1028), 277 (1961); Rev. Mod. Phys. **30**, 1062 (1958).
[95] W. T. Axford, Phil. Trans. Roy. Soc. (London), **A253** (No. 1029), 301 (1961).
[96] S. Feldman, Phys. Fluids **1**, 546 (1958).
[97] R. Dunlap, R. L. Brehm, and J. A. Nicolls, Jet Propulsion **28**, 451 (1958).
[98] V. P. Demutskiĭ and R. V. Polovin, ZhTF **31**, 419 (1961), Soviet Phys. Tech. Phys. **6**, 302 (1961).
[99] A. A. Barmin, DAN SSSR **138**, 77 (1961), Soviet Phys. Doklady **6**, 374 (1961). PMM **26**, 801 (1962).
[100] V. P. Demutskiĭ, ZhTF **31**, 1392 (1961), Soviet Phys. Tech. Phys. **6**, 1014 (1962).
[101] R. V. Polovin and V. P. Demutskiĭ, JETP **40**, 1746 (1961), Soviet Phys. JETP **13**, 1229 (1961).
[102] E. Larish and I. Shekhtman, JETP **35**, 203 (1958), Soviet Phys. JETP **8**, 139 (1959).
[103] R. A. Gross, W. Chinitz, and T. J. Rivlin, J. Aero and Space Sci. **27**, 283 (1960).
[104] G. A. Lyubimov, DAN SSSR **126**, 513 (1959).
[105] J. B. Helliwell and D. C. Pack, Phys. Fluids **5**, 738 (1962).
[106] A. G. Kulikovskiĭ and G. A. Lyubimov, DAN SSSR **129**, 52 (1959), Soviet Phys. Doklady **4**, 1185 (1960); Izv. AN SSSR OTN, Mechanics—Machine Building, No. 4, 130 (1959).
[107] G. A. Lyubimov, DAN SSSR **126**, 532 (1959), Soviet Phys. Doklady **4**, 526 (1959); Izv. AN SSSR OTN, Mechanics and Machine Building, No. 5, 9 (1959).
[108] C. K. Chu, Phys. Fluids **7**, 1349 (1964).
[109] Yu. L. Zhilin, PMM **24**, 543 (1960).
[110] V. P. Demutskiĭ, ZhTF **32**, 445 (1961), Soviet Phys. JETP 7, 324 (1961).
[111] R. V. Polovin, Dissertation, Khar'kov State University, 1963.
[112] G. Ya. Lyubarskiĭ and R. V. Polovin, JETP **35**, 1291 (1958), Soviet Phys. JETP **8**, 901 (1959).
[113] G. Ya. Lyubarskiĭ and R. V. Polovin, JETP **36**, 1272 (1959), Soviet Phys. JETP **9**, 902 (1959).
[114] R. V. Polovin and K. P. Cherkasova, JETP **41**, 263 (1961), Soviet Phys. JETP **14**, 190 (1962).
[115] K. P. Cherkasova, PMTF, No. 6, 169 (1961).
[116] V. V. Gogosov, PMM **25**, 108 (1961) and **26**, 88 (1962); DAN SSSR **142**, 292 (1962), Soviet Phys. Doklady **7**, 10 (1962).
[117] G. Ya. Lyubarskiĭ and R. V. Polovin, DAN SSSR **128**, 684 (1959), Soviet Phys. Doklady **4**, 977 (1960).
[118] A. I. Akhiezer and R. V. Polovin, JETP **38**, 529 (1960), Soviet Phys. JETP **11**, 383 (1961).
[119] R. V. Polovin, JETP **38**, 1544 (1960), Soviet Phys. JETP **11**, 1113 (1960).
[120] A. A. Barmin and V. V. Gogosov, DAN SSSR **134**, 1041 (1960), Soviet Phys. Doklady **5**, 961 (1961).
[121] V. V. Gogosov, DAN SSSR **135**, 30 (1960), Soviet Phys. Doklady, **5**, 1160 (1961).
[122] V. V. Gogosov, PMM **25**, 456 (1961); DAN SSSR **141**, 55 (1961), Soviet Phys. Doklady **6**, 971 (1962).
[123] S. E. Grebenshchikov, M. D. Raĭzer, A. A. Rukhadze, and A. G. Frank, ZhTF **31**, 529 (1961), Soviet Phys. Tech. Phys. **6**, 381 (1961).
[124] G. A. Lyubimov, DAN SSSR **126**, 733 (1959), Soviet Phys. Doklady **4**, 529 (1959).
[125] F. Schubert, Gasdynamics, (Russ. transl.) IL, 1950, p. 92.
[126] S. Z. Belen'kiĭ, Trudy FIAN **10**, 5 (1958).
[127] A. Ferri, Elements of Aerodynamics of Supersonic Flow, Macmillan, 1949.
[128] R. V. Polovin, ZhTF **34**, 1798 (1964), Soviet Phys. Tech. Phys. **9**, 1390 (1965).
[129] W. A. Allen and J. W. Rogers, Phys. Fluids **4**, 493 (1961).
[130] M. N. Kogan, PMM **23**, 70, 557 (1959); **24**, 370, 530 (1960).
[131] Y. Kato and T. Taniuti, Progr. Theor. Phys. **21**, 606 (1959).
[132] H. Cabannes, Compt. rend. **245**, 1379 (1957); **250**, 1968 (1960); Rev. Mod. Phys. **32**, 973 (1960).

[133] S. Morioka, J. Phys. Soc. Japan **16**, 2346 (1961).
[134] M. I. Mimura, Raketn. tekhnika i kosmonavtika (Rocket Technology and Astronautics), No. 10, 40 (1963).
[135] M. I. Kiselev and N. I. Kolosnitsyn, DAN SSSR **131**, 773 (1960), Soviet Phys. Doklady **5**, 246 (1960).
[136] M. N. Kogan, in: Voprosy magnitnoĭ gidrodinamiki i dinamiki plazmy (Problems of Magnetohydrodynamics and Plasma Dynamics) No. 2, Riga, AN LatvSSR, 1962, p. 35.
[137] H. Cabannes, Compt. rend. **257**, 375 (1963).
[138] G. T. Taylor and J. W. Maccoll, Proc. Roy. Soc. **A139**, 298 (1933).
[139] H. Richter, ZAMM **28**, 341 (1948).
[140] T. von Karman, op. cit. [46], p. 45.
[141] H. Cabannes and C. Stael, J. Fluid Mech. **10**, 289 (1961); Rech. aeron., Nr. 27, 7 (1952); Compt. rend. **251**, 1960 (1960).
[142] A. Busemann, op. cit. [125], p. 197.
[143] B. M. Bulakh, PMM **18**, 451 (1954); **19**, 393 (1955).
[144] G. W. Zumwalt and G. G. Tang, op. cit. [134], No. 10, 190 (1963).
[145] R. Sednyi and N. Gerber, ibid. No. 11, 22 (1964).
[146] A. Stone, op. cit. [125], p. 239.
[147] R. Sauer, Anfangswertproblem bi partiellen Differentialgleichungen, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1952, Sec. 38.
[148] Ya. G. Sapunkov, PMM **26**, 190, 930 (1963).
[149] B. M. Bulakh, PMM **26**, 300, 973 (1962).
[150] M. Bausset, Compt. rend. **257** (1963).
[151] J. P. Barthel and P. S. Lykoudis, Phys. Fluids **4**, 1566 (1961).

Translated by J. G. Adashko