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Methodological Notes

# A NATURAL SYSTEM OF UNITS IN CLASSICAL ELECTRODYNAMICS AND ELECTRONICS

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THE purpose of this note is to indicate a system of units which can be called natural for classical electrodynamics. The basic point of such a system is to introduce scales in numerical estimates of various electronic phenomena and to simplify the writing down of formulas.

Natural units were first introduced by Hartree in the solution of the problems of atomic theory, starting with nonrelativistic quantum mechanics (the Schrödinger equation).[1] The atomic system is based on Planck's constant $\hbar$, the charge e, or more accurately $e^2$, and the mass of the electron m. The unit of length is the Bohr radius of the atom

$$r_B=\frac{\hbar^2}{me^2}=0.53\cdot 10^{-8}\text{ cm},$$

and the energy unit is the rydberg—the atomic unit of energy:

$$\frac{me^4}{\hbar^2}=27\text{ eV}.$$

Since the system is nonrelativistic, the speed of light c does not enter into this system, and the unit of speed is the atomic unit $e^2/\hbar$, smaller than c by a factor of 137.

Subsequently, in relativistic quantum electrodynamics preference was given to a slightly different system based on $\hbar$, c, and m. The square of the charge—the interaction constant—$e^2 = 1/137$ was then found to have a value relating $\hbar$, e, and c by means of the fine-structure constant:

$$\alpha=\frac{e^2}{\hbar c}=\frac{1}{137}$$

(more accurately, 1/137.0390).

There exists, however, another possibility for a natural system based on $e^2$, c, and m. The mutual correspondence of these units is clear from the table.

Thus in the proposed system the unit of length is the classical radius of the electron $r_0$, and the unit of energy—the rest energy of the electron $mc^2$. The unit of mass in all three systems is the mass of the electron m. In the proposed system of units Planck's constant is equal to 137. In essence, this emphasizes the fact that classical radiation theory is no longer applicable at distances smaller than $137r_0$ (see [2], Sec. 37).

From the way in which the units are generated it can be seen that within the framework of the indicated theories one cannot generate more units, since only three dimensional quantities enter the expression for $\alpha$.

| Theory | Constants | | | | | |
|---|---|---|---|---|---|---|
| | Action quantum $\hbar=1.05\times10^{-27}$ erg-sec | Charge squared $e^2$ | Speed of light $=3\times10^{10}$ cm/sec | Electron mass $m=9.1\times10^{-28}$ g | Unit of length, cm | Energy unit, eV |
| Quantum mechanics of an atom | 1 | 1 | 137 | 1 | Bohr radius of atom $a_0=\frac{\hbar^2}{me^2}=137^2\ r_0=0.53\cdot10^{-8}$ cm | Atomic unit—rydberg $\frac{mc^2}{137^2}=\frac{me^4}{\hbar^2}=27$ eV |
| Quantum electrodynamics | 1 | $\frac{1}{137}$ | 1 | 1 | Compton length $=137\ r_0=\frac{\hbar}{mc}=3.8\cdot10^{-11}$ cm | Rest energy of electron $mc^2=511$ keV |
| Classical electrodynamics and electronics | 137 | 1 | 1 | 1 | Classical radius of electron $r_0=\frac{e^2}{mc^2}=2.8\cdot10^{-13}$ cm | Rest energy of electron $mc^2=511$ keV |

The new system is above all very convenient for solving problems in classical electrodynamics and electronics. To demonstrate this, it is most useful to generate the units for voltage, current, and power.

For the voltage unit it is natural to set

$$V_0 = \frac{mc^2}{e} = 511\,000 \text{ V},$$

corresponding to the rest energy of the electron. For the unit of current one then obtains the expression

$$I_0 = \frac{mc^3}{e} = 17\,000 \text{ A},$$

known as the "running-electron" current. This quantity can be readily visualized as the current of a small circuit of electrons moving at the speed of light in steps of $r_0$. Clearly the ratio $V_0/I_0 = 1/c = R_0 = 30$ ohm corresponds to the wave impedance, the reciprocal of the speed of light. Finally, for the power one obtains

$$P_0 = \frac{m^2c^5}{e^2} = 511\,000 \text{ V} \times 17\,000 \text{ A} = 8700 \text{ MW}.$$

This value of the power is characteristic for relativistic electron devices and appears in the theory of electron accelerators. For instance, in the calculation of a microtron $P_0$ is the scale of the reactive power in the resonator of this accelerator.[3] Estimates of the limiting current of a microtron are also expressed in terms of the cited current units.[4] The current $I_0$ also determines the scale of the relativistic self-contracting beam considered by Budker.[5]

An elementary calculation shows that when a power $P_0$ crosses an area

$$S = \bar{\lambda}^2 = \frac{\lambda^2}{4\pi^2}$$

the electromagnetic field will impart to the electrons an energy $\sim mc^2$. In the propagation of this power in a wave-guide the quadratic forces due to the electromagnetic field, proportional to grad $E^2$, will be of the same order as the forces $eE$ acting on a particle which is moving synchronously with the wave, and will be proportional to $P^{1/2}$.[6] We emphasize that the value of $P_0$ and the effects produced by it do not depend on the frequency of the field so long as phenomena connected with the quantum nature of the interaction need not be taken into account.

Accelerators operate in the centimeter wavelength region and in their case this is certainly correct. It is also correct for optical frequencies. Present-day Q-switched lasers operating in the giant-pulse mode develop a power of the order of $P_0$. Therefore one expects in the focus of such generators of coherent light or in a system of standing waves with such an energy density asynchronous electron acceleration to large velocities. For such an acceleration mechanism the dependence of the electron energy U will be proportional to the laser power P:

$$\frac{U}{mc^2} \sim \frac{P}{P_0},$$

unlike in synchronous interaction where $U \sim P^{1/2}$. Thus in optics $P_0$ indicates a region of phenomena which could be called relativistic optics.

For electron devices such as accelerators, klystrons, and magnetrons, $P_0$ is the physical scale of high-power relativistic electronics. However, in the corresponding calculation of the parameters of a device, for example the traveling-wave tube, it is by no means necessary to consider the case of large voltages and currents, since the indicated quantities also appear in expressions describing nonrelativistic devices.

As an example let us apply these units to the plane-diode formula

$$i = \frac{1}{9\pi} \sqrt{\frac{2e}{m}} \frac{V^{3/2}}{l^2},$$

relating the current density i with the voltage V and the interelectrode distance $l$. Using the expressions for $V_0$ and $I_0$, we obtain

$$i = \frac{\sqrt{2}}{9\pi} \frac{I_0}{l^2} \left(\frac{V}{V_0}\right)^{3/2} = \frac{I_0}{36\pi l^2} \beta^3,$$

where

$$\beta = \frac{v}{c} \approx \sqrt{\frac{2V}{V_0}}$$

and v is the velocity of the electron. The formulas for a plasma are also simpler. For example, for the dielectric permittivity we obtain

$$\varepsilon = 1 - \frac{4\pi e^2 N}{m\omega^2} = 1 - \frac{N\lambda^2 r_0}{\pi},$$

and the plasma wavelength is

$$\lambda_{\text{pl}} = \sqrt{\frac{\pi}{N r_0}},$$

where N is the number of electrons per unit volume.

It is instructive to transform the expressions for the intensity of a radiating charge, expressing the power in terms of $P_0$. The expression for the power of the radiation of a charge e moving on a circle of radius R with a velocity v takes on the form

$$P = \frac{2e^2v^4}{3c^3R^2\left(1 - \frac{V^2}{c^2}\right)^2} = \frac{2\beta^4 r_0^2}{3(1-\beta^2)^2 R^2} P_0.$$

Analogously we obtain the expression for the intensity of the Cerenkov radiation in the spectral interval $d\omega$ or $d\lambda$:

$$P = \frac{e^2}{c}\left(1 - \frac{c^2}{n^2v^2}\right)\omega\, d\omega = \frac{r_0^2}{\lambda^2}\left(1 - \frac{1}{\beta^2 n^2}\right)\frac{d\lambda}{\lambda} P_0.$$

We see thus that the use of the proposed system results in an appreciable simplification of the formulas and the cumbersome products of e, m, and c disappear from them; a series of relations whose number can be easily multiplied takes on a clearer form.

Naturally, new physical results are not obtained. However, the resulting simplification is convenient both in estimates of various phenomena, and also in explaining and teaching electrodynamics.

In essence, the indicated procedure for transforming the units is similar to the idea of rationalizing units in electrodynamics. In the so-called rationalization the geometric factor $4\pi$ is transferred from one part of the formulas to another where its presence is more convenient. The number $\alpha$, being an internal characteristic of quantum theory and electrodynamics, enters the calculations in the same way, and we can, depending on the type of problem, freely decide its position in such a way that it should disturb us as little as possible. It is curious to note that $\alpha \sim 4\pi^{-2}$.

Thus for problems of classical electrodynamics, radiation theory, and electronics, in which quantum phenomena are not considered, the proposed system has indeed some advantages. It may nevertheless be assumed that the system of units of quantum electrodynamics is of a more fundamental nature. It includes all the quantities h, e and c, and $\alpha$ is the small parameter of which solutions are constructed. This assumption would be better founded, were the development of the theory to show us a method of calculating $\alpha$, and making thus clear what exactly constitutes the violation of electrodynamics, giving rise to the appearance of quantum phenomena.

Anyone who has taught or studied electromagnetic theory is well aware of the arguments and difficulties which arise because of the conviction that one system of units has advantages over another. The only thing which should not be lost sight of is that a system should correspond to the requirements of all of physics, and consequently also of contemporary technology as a whole, and should satisfy the most fundamental laws of nature known to us. For this reason one must fully agree with M. A. Leontovich's criticism of the international system of units; he showed that its units of induction and field do not conform with the principle of relativistic invariance, and a single quantity, the magnetic field in vacuum, can be expressed in terms of different units.[7] One should, on the other hand, also emphasize that one must not expect the introduction of a single system, in view of the fact that the above discussion has indicated that even among the natural systems there is no desirable uniqueness and the choice of a system is determined by the range of problems considered.

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