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THEORY OF REACTIONS WITH PRODUCTION OF THREE PARTICLES NEAR THRESHOLD

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INTRODUCTION

REACTIONS in which several particles are produced have been recently under intensive experimental investigation. These reactions are for the time being the only source of information regarding the scattering amplitudes of unstable particles. In particular, their investigation has led to observation of many resonances. Yet the methods of theoretically interpreting these experimental data are very limited. To observe resonances, one usually analyzes whether the spectrum of the produced particles agrees with the Breit-Wigner or Watson–Migdal formulas.[1,2] In the nonresonance situation, attempts are sometimes made to use the Chew and Low method of analytic continuation in the momentum transfer, which necessitates, however, very detailed experimental data. The latter method makes it possible in principle to determine the scattering amplitudes of unstable particles at arbitrary energies.

We shall consider in this paper another approach which allows us to find the scattering amplitudes of unstable particles at zero energy. This approach is based on an investigation of processes connected with production of several particles near the threshold. It turns out that in this case it is possible to develop a consistent theory that describes reactions with creation of low-energy particles in terms of a certain number of independent parameters and in terms of the scattering amplitudes of the pairs of produced particles. For the case when two particles are produced, this theory is the well-known theory of the deuteron effective radius.

We shall be interested essentially in reactions in which three particles are produced, although a similar analysis can be carried out also for a larger number of particles. The existence of a consistent theory of reactions in which low-energy particles are produced is of interest also regardless of the possibilities that are connected with the determination of scattering amplitudes of particle pairs.

The first steps in the creation of the theory described below were made essentially by G. V. Skornyakov and K. A. Ter-Martirosyan[3], who considered the quantum-mechanical problem of three resonantly interacting particles, and by V. N. Gribov, who investigated the angular correlations in $K \to 3\pi$ decay[4]. Gribov's latest paper served as the basis for further development of the theory.

Questions connected with production of three low-energy particles were considered further by several workers[5-19]. Since the understanding of the situation has deepened from investigation to investigation, it has by now become quite difficult to study the theory from the original articles. This is the reason for writing the present review.

In certain recently published papers, problems related to those discussed here were investigated in a manner which is not quite correct. Some authors have very recently[20-22] noted errors in earlier papers. Within the framework of the nonrelativistic threshold approach of interest to us, the results contained in these papers overlap for a greater part with the results of[3-16]. We have therefore taken the liberty of not discussing these papers.

Many of the results presented below were initially obtained by a quantum-mechanical analysis. It will turn out more convenient in the future to use a method based on the study of the analytic properties of the amplitudes. From this point of view we shall consider consecutively the entire theory of reactions in which particles are produced near a threshold.

In the first section of the paper we give some kinematic relations which will be useful in what follows. In the second section, we consider general problems connected with the selection of the essential diagrams. The third section is devoted to an illustration of our method, using as an example the scattering of two particles at low energy. In the fourth section, we give the expansion of the amplitude for creation of three particles in powers of the states with different total angular momenta. Since further calculations are essentially connected with the use of the unitarity condition, we show in the fifth section how to use the unitarity condition to calculate the discontinuities in the amplitudes near the singularities of interest to us. In the sixth and seventh section we obtain an expansion of the amplitude with zero total angular momentum in terms of the momenta of the produced particles, accurate to third-order terms. In the eighth section we consider the amplitude for the production of three particles in a state with unity total angular momentum. The ninth section is devoted to a study of reactions in which two of the produced particles interact resonantly. In the tenth section the developed theory is applied to certain concrete reactions ($\pi + N \to N + \pi + \pi$, $\gamma + N \to N + \pi + \pi$, and $K \to 3\pi$ decay). In this review we barely touch upon the question of reactions in which three resonantly interacting particles are produced (for example, $N+D \rightarrow N+N+N$). These reactions are treated in [17-19], in which more detailed references can be found.

1. KINEMATICS

In this section we introduce the principal symbols employed and write out several useful kinematic relations. The amplitude for the transformation of two particles into three (Fig. 1) depends on five invariant variables* which can be chosen to be two momentum transfers and three relative particle energies of particles in the final state. Let p_1 and p_2 be the 4-momenta of the colliding particles in the center-of-mass system, and let k_1, k_2 and k_3 be the 4-momenta of the produced particles in the same system. Then the relative energies in the three particles in the final state, $\sqrt{s_{12}}$, $\sqrt{s_{13}}$, and $\sqrt{s_{23}}$ and the momentum transfers $\sqrt{-t_1}$ and $\sqrt{-t_2}$ can be written in the form

$$\left.\begin{aligned} s_{12} &= (k_{10}+k_{20})^2-(\mathbf{k}_1+\mathbf{k}_2)^2, & t_1 &= (p_{10}-k_{10})^2-(\mathbf{p}_1-\mathbf{k}_1)^2,\\ s_{13} &= (k_{10}+k_{30})^2-(\mathbf{k}_1+\mathbf{k}_3)^2, & t_2 &= (p_{10}-k_{20})^2-(\mathbf{p}_1-\mathbf{k}_2)^2,\\ s_{23} &= (k_{20}+k_{30})^2-(\mathbf{k}_2+\mathbf{k}_3)^2, \end{aligned}\right\} \quad (1)$$

where k_{i0}, p_{i0} and $\mathbf{k}_i$, $\mathbf{p}_i$ are the time-dependent and space-dependent components of k_i and p_i.

All other invariant variables characterizing the amplitude under consideration can be expressed in terms of the five invariants which have been written out.

In the nonrelativistic approximation, near the particle production threshold, the quantities s_{il} are expanded in series in the relative momenta of the produced particles k_{il}:

$$\sqrt{s_{il}} \cong m_i + m_l + \frac{k_{il}^2}{2\mu_{il}}, \qquad \mu_{il}^{-1} = m_i^{-1} + m_l^{-1}, \tag{2}$$

where m_i are the masses of the particles in the final state.

The kinetic energy E which is released in the reaction is connected with the total energy

$$\sqrt{s} = [(p_{10}+p_{20})^2-(\mathbf{p}_1+\mathbf{p}_2)^2]^{1/2}$$

by the relation

$$\sqrt{s} = m_1+m_2+m_3+E. \tag{3}$$

From the equality

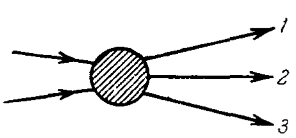


FIG. 1.

$$s = s_{12}+s_{13}+s_{23}-m_1^2-m_2^2-m_3^2$$

it follows that

$$E = \frac{m_1+m_2}{m_1+m_2+m_3}\frac{k_{12}^2}{2\mu_{12}} + \frac{m_1+m_3}{m_1+m_2+m_3}\frac{k_{13}^2}{2\mu_{13}} + \frac{m_2+m_3}{m_1+m_2+m_3}\frac{k_{23}^2}{2\mu_{23}}. \tag{4}$$

This quantity can also be represented in a different form

$$E = \frac{k_{12}^2}{2\mu_{12}} + \frac{k_3^2}{2\mu_3}, \qquad \mu_3^{-1} = m_3^{-1} + (m_1+m_2)^{-1}. \tag{5}$$

The change over from the momenta $\mathbf{k}_{12}$ and $\mathbf{k}_3$, for example, to the momenta $\mathbf{k}_{13}$ and $\mathbf{k}_2$ is by means of the formulas

$$\mathbf{k}_2 = -\frac{m_2}{m_1+m_2}\mathbf{k}_3 - \mathbf{k}_{12},$$

$$\mathbf{k}_{13} = -\frac{m_1(m_1+m_2+m_3)}{(m_1+m_2)(m_1+m_3)}\mathbf{k}_3 + \frac{m_3}{m_1+m_3}\mathbf{k}_{12}. \tag{6}$$

Other analogous equations are obtained by cyclic permutation of the indices.

Near the creation threshold, expressions (1) for t_1 and t_2 can be rewritten, accurate to the linear terms in the momenta of these particles, in the form

$$\begin{aligned} t_1 &= t_1^{(0)} + 2p_1^{(0)}k_1z_1,\\ t_2 &= t_2^{(0)} + 2p_1^{(0)}k_2z_2. \end{aligned} \tag{7}$$

Here $t_1^{(0)}$ and $t_2^{(0)}$ are the threshold values of the invariants t_1 and t_2, $p_1^{(0)}$ is the absolute value of the momentum $\mathbf{p}_1$ at the threshold energy, and z_1 and z_2 are the cosines of the angles between the vectors $\mathbf{p}_1$, $\mathbf{k}_1$ and $\mathbf{p}_1$, $\mathbf{k}_2$.

In what follows it will be frequently convenient to use in place of the variables k_{il} the variables

$$\mathbf{x}_{il} = \frac{\mathbf{k}_{il}}{\sqrt{2\mu_{il}E}}. \tag{8}$$

We introduce a special symbol for the frequently encountered mass combination

$$\beta_1 = \frac{m_1(m_1+m_2+m_3)}{(m_1+m_2)(m_1+m_3)} \tag{9}$$

(β_2 and β_3 are determined in similar fashion). In this notation, we present a relation, which will be useful in what follows, between the quantities x_{13}, x_{12}, and z — the cosine of the angle between the momentum $\mathbf{k}_{12}$ (or $\mathbf{x}_{12}$) and the momentum of the third particle in the c.m.s. of particles 1 and 2:

$$x_{13}^2 = (1-\beta_1)x_{12}^2 + \beta_1(1-x_{12}^2) + 2z\sqrt{\beta_1(1-\beta_1)x_{12}^2(1-x_{12}^2)}. \tag{10}$$

2. FUNDAMENTAL PRINCIPLES OF DIAGRAM SELECTION

We are interested in the amplitude of the conversion of two particles into three near the threshold of the reaction, that is, in the condition when the total released kinetic energy is much smaller than the mass of any of the particles. In this case we can attempt to expand the reaction amplitude in a series in powers

*This statement pertains, strictly speaking, to the case of neutral spinless particles. We shall verify later that the presence of the spin and the isotopic spin does not complicate the situation in principle.

of $s_{ik} - s_{ik}^{(0)}$ and $t_i - t_i^{(0)}$, where $\sqrt{s_{ik}}$ and $\sqrt{-t_i}$ are the invariant variables (the relative energies and momentum transfers) on which the amplitude depends, while $\sqrt{s_{ik}^{(0)}}$ and $\sqrt{-t_i^{(0)}}$ are the threshold values. It is obvious that such an expansion is hindered by the amplitude singularities near the threshold values of the invariants. However, after separating these singularities we can expand the amplitude in powers of $s_{ik} - s_{ik}^{(0)}$ and $t_i - t_i^{(0)}$. By near singularities of the amplitudes we mean either singularities which lie exactly at the threshold values of the invariants, or singularities which are separated from the threshold by a distance which is much smaller than the square of the mass of any of the particles. Here, of course, we must take into consideration the singularities which are located both on the physical and on the unphysical sheets. The remaining singularities ("far singularities") are located at distances of the order of the particle mass squared ($\sim m^2$), so that after separating the near singularities the expansion is essentially carried out in powers of $(s_{ik} - s_{ik}^{(0)})/m^2$ and $(t_i - t_i^{(0)})/m^2$. In other words, after separating the near singularities we deal with expansion in powers of $(kr_0)^2$, where k is the momentum of any of the produced particles and r_0 is the interaction radius.

Let us examine first the amplitude singularities that correspond exactly to the threshold values of the invariants. In this simple case the location and character of the singularities can be obtained directly from the unitarity condition in the s-channel ($\sqrt{s}$ is the total energy). The situation here is simpler than in the derivation of the Landau curves for the unitarity condition, where, as shown by Mandelstam, continuation in the momentum transfer is also necessary.*

The reaction amplitude has no singularities in the momentum transfer at the threshold values $t_i = t_i^{(0)}$, and the appearance of singularities in s and s_{il} is connected with the appearance of new terms in the unitarity condition. The singularities arise at energy-squared values s_{il} and s equal to the squares of the sums of the masses of all possible intermediate states. If we are interested in singularities lying at the threshold values of the energies, we must consider those unitarity-condition terms which are connected with the intermediate states with the same three particles as in the final states. In the unitarity condition these terms correspond to the Feynman diagrams shown symbolically in Fig. 2.

All other diagrams which do not contain three-particle fission, have no singularities at the threshold of the interaction in question (provided, of course, that the sum of any two other particles does not accidentally coincide with $m_1 + m_2 + m_3$; this case will not be considered). The amplitude for the transformation of three particles into three, which enters in the diagram of Fig. 2, includes cases when one of the particles does not interact with the two others (Fig. 3).

*A reader who is not familiar with the Landau rules [26] for determining the singularities of Feynman diagrams can read first the appendix to this review, where we present a simple derivation of the Landau rules and write out the concrete formulas used subsequently for the case of a triangular diagram.

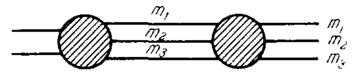


FIG. 2.

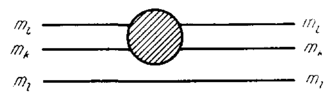


FIG. 3.

If the interaction within the six-point diagram actually takes place between all three particles, then the corresponding Feynman diagram has a three-particle singularity in the square of the total energy s when $s = (m_1 + m_2 + m_3)^2$ ($E = 0$). In the case of the degenerate six-point diagram shown in Fig. 3, a two-particle singularity in s_{ik} occurs at $s_{ik} = (m_i + m_k)^2$ ($k_{il} = 0$). It is clear that many diagrams can have simultaneously either type of singularity. Such diagrams are shown, for example, in Figs. 4,b,d and f. Let us now imagine that we have separated from the diagram of Fig. 2 all the possible scatterings of particle pairs in the final state and the transformations of three particles into three.

We then arrive at the diagrams shown in Fig. 4, where the shaded irreducible blocks no longer contain transformations of two particles into two or three particles into three. The singular terms of the amplitudes are obtained from these diagrams in those cases when all the lines are real ($q_i^2 = m_i^2$). This, for example, follows from the Landau rules (see the Appendix). In calculating the singular terms in this case, the irreducible blocks of Fig. 3 go on the mass shell, and since they no longer contain transformations of two particles into two or three particles into three, they have no singularities at the threshold values of their invariants.

We shall therefore expand from now on these blocks in powers of the deviations of the invariants from the threshold values. The constant term in the block for the transformation of two particles into two is obviously the scattering length in the S-state, and the succeeding terms are connected both with the interaction radius in S-scattering and with the higher partial waves. As will be clear from what follows, if we are interested in not too high terms in the expansion of the entire amplitude in powers of $s_{ik} - s_{ik}^{(0)}$ and $t_i - t_i^{(0)}$, then we can confine ourselves to the expansion of the irreducible blocks in their own invariants to the first few terms.

So far we have investigated only singularities located exactly at the threshold values of the invariants. We are really interested also in other near singulari-

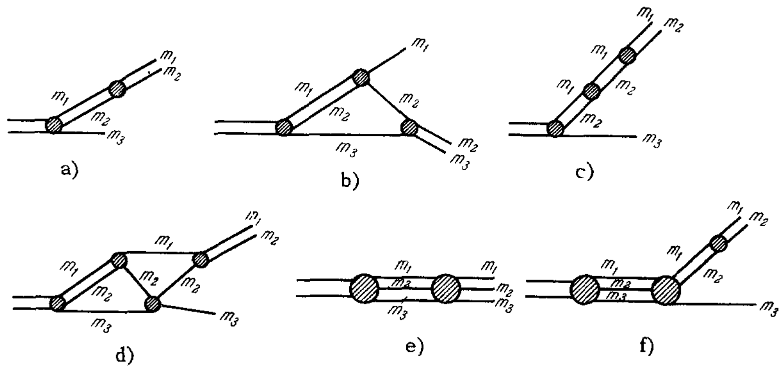


FIG. 4.

ties, which are situated at much shorter distances from the threshold than the square of the mass of any of the particles. Such singularities can result, first, from the existence of "compound particles" with low coupling energy, such as the deuteron. In particular, the amplitude of the reaction $\pi + D \rightarrow N + N + \pi$ has, for example, near-threshold singularities in the momentum transfer, not connected with diagrams of the type shown in Fig. 4. Such singularities arise in the diagrams shown in Fig. 5. In situations of this type it is necessary to take into account, besides, the diagrams of the type in Fig. 4, also diagrams in the vertices of which there occur "almost real" transformations of a deuteron into two nucleons. It is clear that the positions of the singularities connected with these latter diagrams will be very close to the diagrams of the physical region, since the coupling energy of the deuteron is low.

Second, such singularities can be situated on other sheets, where they are not determined directly by the unitarity condition. In order for these singularities to be located not far from the physical region they should, of course, be under cuts that go either from the point $s_{il} = (m_i + m_l)^2$ in the s_{il} plane, or from the point $s = (m_1 + m_2 + m_3)^2$ in the s plane. Since such cuts are contained only in the diagrams shown in Fig. 4, the singularities of interest to us can also arise only in diagrams of the type of Fig. 4. The singularities on other unphysical sheets defined by other cuts (connected with remote singularities) will of course not influence the expansion of the amplitudes near threshold.

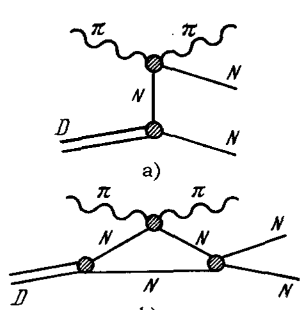


FIG. 5.

We thus arrive at the following final conclusion. In order to separate correctly the singularities in the expansion of the amplitude of transformation of two particles into three near threshold, we must investigate the diagrams connected with the scattering of particles and with the transformation of three particles into three, of the type shown in Fig. 4. The irreducible blocks must in this case be replaced by their series expansion near the threshold values of the invariants. In the presence of weakly bound particles of the deuteron type, it is also necessary to take into account diagrams with almost real decay of such particles. The foregoing diagrams contain all the singularities which are close to threshold, both on the physical and on the unphysical sheets.

In conclusion we wish to emphasize the following important circumstance. Since all our expansions converge in the regions defined by the position of the "remote" singularities, the totality of the diagrams under consideration determines the amplitude of the reaction in the form of some series in powers of $(kr_0)^2$, where r_0 is the interaction radius and k is the momentum of any of the produced particles. Here, of course, it is understood that the analytic terms (the powers of the quantities $s_{ik} - s_{ik}^{(0)}$ and $t_i - t_i^{(0)}$) which result both from the diagrams under consideration and from all other diagrams, must be added to the contribution of the separated diagrams. As will be shown later, the summation of the selected diagrams will cause the amplitude to acquire singularities whose positions are connected not with the effective interaction radius r_0, but with the particle-pair scattering amplitudes. If these amplitudes coincide with r_0 in order of magnitude, $a \sim r_0$ (nonresonant situation), the reaction amplitude must be expanded further in powers of (ka). It becomes possible in fact to confine oneself in this case only to the simplest diagrams shown in Fig. 4. This is precisely the procedure which we shall use in Secs. 4–6 to calculate terms of different order of magnitude in the threshold momenta. On the other hand, if $a \gg r_0$ (resonant case), as is the situation, for example, in the scattering of low-energy nucleons, then all the diagrams of Fig. 4 must be taken into account. A simplest example of this kind (scattering of two particles) is presented for illustration in the next section. In the case of three particles we arrive, generally speaking, to the Skornyakov-Ter-Martirosyan equations[3], which were investigated in detail by Danilov[17]. A detailed examination of these equations is beyond the scope of the present review.

3. SCATTERING OF PARTICLES NEAR THRESHOLD

The consideration advanced in the preceding section can be extended, of course, without modification to the case of an arbitrary number of particles in the final state. We shall now consider by way of illustration the simplest case of scattering of nucleons by nucleons at low energies and we shall show that the rules for selecting diagrams, formulated in the preceding section, lead us immediately to the well-known Bethe-Peierls effective-radius theory.

The expression for the amplitude of the low-energy nucleon-nucleon scattering is best obtained in the following fashion. The partial amplitude for the scattering $f_l(k) = k^{-1}\exp(i\delta_l)\sin\delta_l$ with angular momentum l satisfies the unitarity condition

$$\operatorname{Im} f_l = k\,|f_l|^2 \quad (k^2>0), \qquad \operatorname{Im} f_l = 0 \quad (k^2<0).$$

Putting $h_l = -1/f_l$, we have

$$\operatorname{Im} h_l = k \quad (k^2>0), \qquad \operatorname{Im} h_l = 0 \quad (k^2<0),$$

that is,

$$h_l = ik + \alpha_l(k^2). \tag{11}$$

The function $\alpha_l(k^2)$ (the real part of h_l) should be an analytic function of k^2 at $k^2 = 0$, for otherwise $\alpha_l(k^2)$ could not be a real function simultaneously when $k^2 > 0$ and when $k^2 < 0$. It follows from (11) that

$$f_l = -\frac{1}{ik+\alpha_l(k)^2} = \frac{a_l(k^2)}{1-ika_l(k^2)}, \qquad a_l(k^2) \equiv -\frac{1}{\alpha_l(k^2)}. \tag{12}$$

It is obvious that in S-scattering at low energies the function $a_0(k^2)$ can be replaced by a constant $a_0(0) = a_0$, which is the amplitude of nucleon scattering at zero energy; then

$$f_0 = \frac{a_0}{1-ika_0}. \tag{13}$$

Formula (13) yields the well-known expression for the low-energy nucleon-nucleon scattering amplitude. If we retain the next higher term in the expansion of the function $a_0(k^2)$ in powers of k^2, we immediately obtain an expression for f_0 in the effective-radius approximation.

We shall now show that the diagrams shown in Fig. 6 with nucleons in the intermediate state also lead to formula (13). These diagrams have a singularity at low nucleon energy and therefore, in accordance with the statements made in the preceding section, they should be considered for the separation of the singularity in the amplitude. When separating the singularity, the blocks in the vertices of the diagrams can be expanded in powers of the invariants, and in the lowest approximation they should be replaced by a_0, which is the scattering amplitude at zero energy.

The latter condition makes the vertices of the diagrams of Fig. 6 pointlike, after which the calculation becomes exceedingly simple. Let us calculate the contribution of the simplest diagram shown in Fig. 6. Its imaginary part is obviously equal to ka_0^2, and consequently, the entire diagram can be written in the form of the dispersion integral

$$a_0^2\frac{k^2}{\pi}\int_0^\infty \frac{k'\,dk'^2}{k'^2(k'^2-k^2-i\varepsilon)} = a_0^2 ik. \tag{14}$$

In formula (14) subtraction was effected at the point $k^2 = 0$ and the computational constant has not been written out, since it pertains to the nonsingular part of the amplitude. It is furthermore easy to note that the integration over the individual loops in the highest-order diagrams shown in Fig. 6 is carried out independently. Therefore, a diagram consisting of n loops contains the term $a_0^{n+1}(ik)^n$ and the terms connected with the interference between the terms of type ika_0 and the analytic terms. The entire amplitude of nucleon-nucleon scattering (S wave) can be written in the form

$$f_0 = (ika_0)\,a_0 + (ika_0)^2\,a_0 + (ika_0)^3\,a_0 + \ldots \tag{15}$$

The terms written out constitute the contribution of the diagrams of Fig. 6, while the remaining terms are due both from the diagrams of Fig. 6 and from all other diagrams. According to the statements made in the preceding section, when the powers $(kr_0)^2$ are neglected the entire contribution of the terms which were not written out can be replaced by a constant C. Its value is $C = a_0$, so that by definition $f_0 = a_0$ when $k^2 = 0$. Summing the geometric progression (15) we

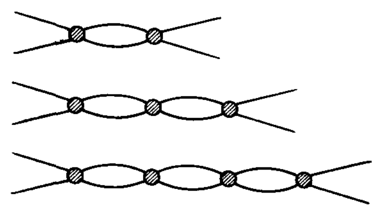


FIG. 6.

arrive directly at (13). In accord with the statements made at the end of the preceding section, we obtain a singularity at $k = 1/ia_0$. The pole obtained can lie on the physical sheet when $a_0 < 0$ in which case it corresponds to the case of a deuteron; it can also be on the physical sheet not at $a_0 > 0$, and then it constitutes a singlet virtual level in the nucleon-nucleon system. If the quantity a_0 were of the order of r_0, it would be necessary to rewrite (13) in the form

$$f_0 = a_0(1+ika_0).$$

As indicated in the preceding section, this expression would reduce to a contribution (containing a singularity) from the simplest diagram (Fig. 6). Finally, we note that allowance for the analytic term $\approx k^2$ would bring us to the effective-radius approximation.

4. EXPANSION OF THE AMPLITUDE IN STATES WITH DIFFERENT TOTAL ANGULAR MOMENTA

In the second section we indicated that reactions with production of three particles are divided into two types, the study of which calls for several different approaches. One type includes reactions in which "compound particles" such as the deuteron are present in the initial state. In this case the singularities in the square of the momentum transfer $-t_i$ are located not far from the physical region of the solution, near the threshold. An example is the reaction $\pi + D \to N + N + \pi$. The second type of reaction, which is dealt with in this article, is one in which the singularities in t_i lie far from the physical region (at distances $\sim m^2$). In this case, the amplitude for the production of three particles can be expanded near threshold in powers of $t_i - t_i^0$ or else in powers of $k_i z_i$ (see formula (7)):

$$A(k_{12}^2, k_{13}^2, k_{23}^2, t_1, t_2) = \sum_{p, s=0}^{\infty} T_{ps}(k_{12}^2, k_{13}^2, k_{23}^2)(k_1 z_1)^p (k_2 z_2)^s. \quad (16)$$

The terms of this expansion with large p and large s are small, so that the expansion is essentially in powers of k_i/m.

It will be more convenient to rewrite (16) in the form of a series in the states with different total angular momenta. Since the c.m.s. momenta of the produced particles satisfy the relation $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$, they all lie in one plane. We introduce a coordinate system in which the Z axis is perpendicular to this plane and the X axis is parallel to $\mathbf{k}_1$. The direction of the momentum of the incident particles p_i is characterized in this coordinate system by two angles ϑ and φ. z_1 and z_2 are then expressed in terms of ϑ and φ in the following manner:

$$z_1 = \sin\vartheta\cos\varphi, \quad z_2 = \sin\vartheta\cos(\varphi - \gamma), \quad (17)$$

where γ is the angle between the vectors $\mathbf{k}_1$ and $\mathbf{k}_2$, expressed in terms of energy of the particles in the final state.

Substituting (17) in (16), we obtain a function of the angles ϑ and φ, which can be expanded in a series in the spherical functions $Y_{LM}(\vartheta, \varphi)$:

$$A(k_{12}^2, k_{13}^2, k_{23}^2, t_1, t_2) = \sum_{L=0}^{\infty} \sum_{M=-L}^{L} A_{LM}(k_{12}^2, k_{13}^2, k_{23}^2) Y_{LM}(\vartheta, \varphi). \quad (18)$$

The coefficients A_{LM} are the amplitudes of the transition with total momentum L and projection M on the Z axis.

It is obvious that inasmuch as the expansion of a polynomial of degree $p+s$ (of trigonometric functions of the angles ϑ and φ) contains only spherical functions with $L \le p+s$, the coefficients A_{LM} are expressed in terms of the quantities $T_{ps}k_1^p k_2^s$ with $p+s \ge L$. This means that the amplitudes A_{LM} are of the order $(k_i/m)^L$, that is, they are small when L is large. By reflection in the XOY plane and total reflection of the coordinate frame, we can easily verify that the parity of the numbers L and M is the same, and that the A_{LM} are expressed in terms of those T_{ps} for which the parity of $p+s$ is the same as the parity of L.

The amplitudes $A_{LM}(k_{12}^2, k_{13}^2, k_{23}^2)$ constitute, essentially, the amplitudes of decay of a particle with spin L and with a mass equal to the total energy of the incoming particles $\sqrt{s}$. Accordingly, they can be represented graphically by the diagram shown in Fig. 7. In the following sections we shall consider in detail the amplitude $A_{00}(k_{12}^2, k_{13}^2, k_{23}^2)$ which we shall denote henceforth by $A(k_{12}^2, k_{13}^2, k_{23}^2)$.

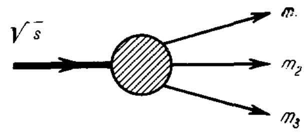


FIG. 7.

5. UNITARITY CONDITION AND CALCULATION OF THE DISCONTINUITIES

In calculating the amplitude of the reaction near threshold we shall make extensive use of the unitarity condition. In the simplest form, the unitarity condition for the partial wave was already used in Sec. 3 to determine the amplitudes of nucleon-nucleon scattering.

We shall also find useful in what follows the three-particle unitarity condition and expressions for the amplitude discontinuities connected with one definite singularity. We present now some of the necessary relations.

If, as usual, we introduce in lieu of the S matrix the T matrix, $S = 1 + iT$, then the unitarity condition $SS^+ = 1$ for the S matrix is rewritten for the T matrix in the form $-i(T - T^+) = TT^+$.

It is convenient to use the invariant amplitudes M connected with the matrix elements of the T matrix by the relation

$$T_{fi} = (2\pi)^4 \delta^4\left(\sum p_i - \sum p_f\right) \frac{M_{fi}}{\prod_n \sqrt{2\omega_n}}. \quad (19)$$

Here M_{fi} is the amplitude of the transition from the state $|i\rangle$ into the state $|f\rangle$, $\delta^4(\Sigma p_i - \Sigma p_f)$ is a four-dimensional δ-function which expresses the energy-conservation law, and ω_n are the energies corresponding to all the initial and final particles. It is easy to see that the unitarity condition for the amplitude M_{fi} then takes the form

$$\operatorname{Im} M_{fi} = \sum_N \frac{1}{2(2\pi)^{3N-4}} \int d^4q_1 \ldots d^4q_N \delta^4\left(\sum p_i - \sum q_n\right) \delta(q_1^2 - m_1^2) \ldots \delta(q_N^2 - m_N^2) M_{fn} M_{in}^*, \quad (20)$$

where N is the number of particles, q_n are their 4-momenta in the intermediate state, and M_{fn} and M_{in} are the amplitudes of the transition from the intermediate state into the final state and into the initial state, respectively.

We shall need in what follows the two-particle and the three-particle unitarity condition $(N=2)$ and $(N=3)$. In the first of these cases we can readily transform (20) to the usual form

$$\operatorname{Im} M_{fi} = \frac{q}{8\pi\sqrt{s}} \int \frac{d\Omega}{4\pi} M_{fn} M_{in}^*. \quad (21a)$$

Here $\sqrt{s}$ is the total energy of the two particles, q the c.m.s. momentum of the particle in the intermediate state, and $d\Omega$ the solid-angle element of the momentum **q**.

In the case when only elastic scattering is possible, it is frequently convenient to use states with definite total momentum l in place of states with definite momentum. We then get in place of (21a)

$$\operatorname{Im} M_l = \frac{q}{8\pi\sqrt{s}} |M_l|^2. \tag{21b}$$

The M_l coincide, apart from a factor of course, with the partial amplitudes $f_l = e^{i\delta_l} \sin\delta_l/q$, where δ_l is the scattering phase shift. Since $\operatorname{Im} f_l = q|f_l|^2$, we have

$$f_l = 8\pi\sqrt{s}\, M_l. \tag{22}$$

At zero energy $(q=0)$ the quantity f_0 goes over into the particle scattering length a_{12}. (The indices 1 and 2 denote the numbers of the scattered particles.) If we denote the value of M_0 at zero energy by g_{12}, we obtain from (22) the connection between g_{12} and a_{12}:

$$g_{12} = 8\pi(m_1 + m_2)\, a_{12}, \tag{23}$$

where m_1 and m_2 are the masses of the scattered particles. The quantity g_{12} plays the role of a "charge" in the Feynman diagrams which will be used in what follows. This means that the factor $i(2\pi)^4 g_{12}$ corresponds to the vertices of the diagrams in which particles 1 and 2 are scattered.

As an example, we present an expression for the diagram shown in Fig. 8, which we shall need later. The quantity M_{fi} corresponding to this diagram is

$$-g_{12}g_{13}(q_1'^2 - m_1^2)^{-1} = -64\pi^2(m_1+m_2)(m_1+m_3)(q_1'^2 - m_1^2)^{-1}, \tag{24}$$

where q_1' is the particle momentum in the intermediate state.

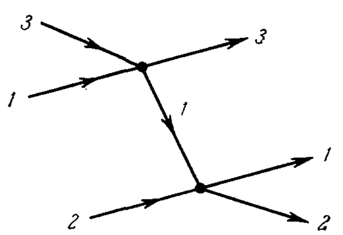


FIG. 8.

Let us imagine that the unitarity condition (20) contains the terms corresponding to two- and three-particle intermediate states. The amplitude M_{fi} has then two threshold singularities in the square of the total energy s at the points $s = (M_1+M_2)^2$ and $s = (m_1+m_2+m_3)^2$ (Fig. 9), where M_1 and M_2 are the masses of the particles of the two-particle intermediate state, and m_1, m_2, m_3 are the masses in the three-particle state. Assume that $(m_1+m_2+m_3)^2 > (M_1+M_2)^2$. The expression (20) for the imaginary part of the amplitude M_{fi} for $s > (m_1+m_2+m_3)^2$ determines simultaneously the discontinuity of this amplitude on going around both branch points ($\operatorname{Im} M_{fi}$ is equal to the difference between the values of the amplitude M_{fi} at the points a and a′ in Fig. 9, divided by 2i). To separate the singular part of the amplitude, connected, for example, with the three-particle threshold, we shall need to know the value of the discontinuity on the cut that begins with the three-particle singularity, that is, the difference between the values of the amplitude at points b and b′ (Fig. 9,a). We are interested here in the case when the amplitude is taken on the upper (physical) edge of the two-particle cut. In order to illustrate the resultant situation, we have shifted in Fig. 9,a a part of the two-particle cut into the lower half-plane. Thus discontinuity can be conveniently found in the following manner. We continue analytically the amplitude M_{fi} in terms of the masses M_1 and M_2 so that the two-particle threshold is to the right of the three-particle threshold $(M_1+M_2)^2 > (m_1+m_2+m_3)^2$, as shown in Fig. 9,b. Then the discontinuity of interest to us at the points b and b′ can be calculated from the unitarity condition in which there remains one three-particle division. The amplitudes M^*, which appear in (20), represent in this case the values of the amplitudes on the lower edge of the three-particle cut, at the point b′. The unitarity condition (20) can therefore be rewritten, for $(m_1+m_2+m_3)^2 < s < (M_1+M_2)^2$ in the form

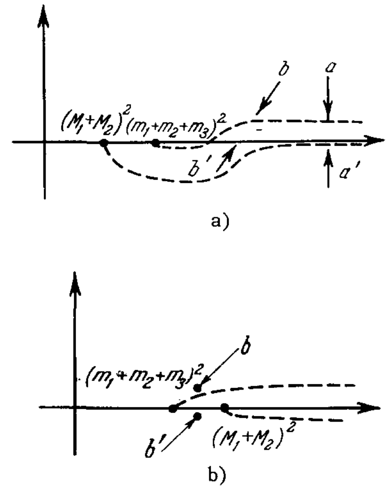


FIG. 9.

$$\frac{1}{2i}[M_{fi}(b) - M_{fi}(b')] = \frac{1}{2(2\pi)^5}\int d^4q_1\, d^4q_2\, d^4q_3 \times \delta^4\Big(\sum p_i - \sum q_n\Big)\,\delta(q_1^2 - m_1^2)\,\delta(q_2^2 - m_2^2) \times \delta(q_3^2 - m_3^2)\, M_{fn}(b)\, M_{in}(b'). \tag{25}$$

Expression (25) has been written in analytic form (it does not contain the complex-conjugation sign) and can therefore be continued directly into the region $(M_1+M_2)^2 < (m_1+m_2+m_3)^2$. We see then that the discontinuity of interest to us, on the three-particle singularity, is determined by an integral in a form customary for the unitarity condition with respect to the three-particle phase volume, of the product of the amplitude M_{fn}, taken on the upper edge of both cuts (at the point b in Fig. 9,a), and of the amplitude M_{in}, taken on the upper edge of the two-particle cut and on the lower edge of the three-particle cut (at the point b′ in Fig. 9,a). The value of M_{in} at the point b′ differs from the quantity M^*_{in} in (20).

We have considered by way of an example a definite case when two- and three-particle singularities are present. The generalization of the obtained rules to the calculation of the discontinuity on an arbitrary singularity is perfectly obvious.

In the following sections we shall investigate in detail, with the aid of the unitarity condition, diagrams connected with the scattering of particles in the final state. The parameter used in all the expressions will be the constant λ —the constant for the production of three particles at zero energy. The quantity λ can be complex, but it is easy to see that its real and imaginary parts are connected by the unitarity condition. To verify this it is sufficient to write out the unitarity condition at an energy corresponding directly to the threshold reaction

$$\operatorname{Im}\lambda = e^{i\delta}\sin\delta\cdot\lambda^*. \tag{26}$$

Here δ is the scattering phase of the incident (initial) particles at threshold energy in a state with zero total angular momentum. The three-particle term in the unitarity condition vanishes at the threshold because of the vanishing of the phase value. We see from (26) that

$$\lambda = \varrho e^{i\delta}, \tag{27}$$

where ρ is a real number (positive or negative). Formula (27) illustrates essentially the Fermi rule for the determination of the phases of matrix elements.

6. LINEAR AND QUADRATIC TERMS OF THE EXPANSION OF THE AMPLITUDES WITH L = 0 IN TERMS OF THE THRESHOLD MOMENTA

In this section we consider the amplitude $A_{00}(k^2_{12}, k^2_{13}, k^2_{23}) \equiv A(k^2_{12}, k^2_{13}, k^2_{23})$ corresponding to zero total angular momentum, and calculate it with accuracy to terms that are quadratic in k_{12}, k_{13} and k_{23}. The scattering lengths of the pairs of produced particles a_{12}, a_{13} and a_{23} will be assumed to be not too large: $a_{ik} \lesssim r_0$ (r_0 is the interaction radius). In other words, we assume that there is no resonance situation in the scattering of the produced particles at the zero energy.

Let us consider first the simplest diagrams of particle scattering in the final state, which, as explained in Sec. 2, have singularities at threshold values of the invariants. We investigate now in detail the analytic properties of these diagrams, calculate them, and then show that it is these simplest diagrams which contribute to the terms that are linear and quadratic in the threshold momenta or, more accurately speaking, to the linear and quadratic terms which are nonanalytic near the threshold.

The simplest diagrams are connected with single scattering of particles in the final state (Fig. 10). In

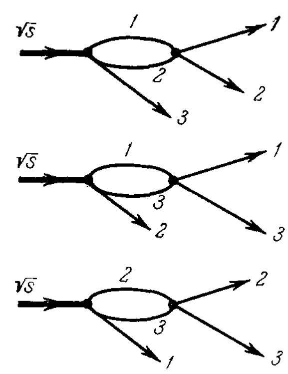


FIG. 10.

calculating those singular parts of these diagrams which are largest in the threshold momenta, we should replace their vertices by the scattering amplitudes of the particle pairs at zero energy a_{12}, a_{13}, and a_{23} and replace the quantity λ by the amplitude for the transformation of the initial particles into three particles at the threshold value of the energy. This procedure is perfectly analogous to that used in Sec. 3. The singular part of the diagrams in Fig. 10 can then be calculated in the same manner as the contribution of the diagrams in Fig. 6. Using the result (14) of the third section, we obtain directly the singular part of the diagrams in Fig. 10:

$$i\lambda a_{12}k_{12} + i\lambda a_{13}k_{13} + i\lambda a_{23}k_{23}. \tag{28}$$

We now proceed to consider diagrams of the type shown in Fig. 11. There is obviously a total of six diagrams of this type, differing in the permutation of the final and intermediate states. The vertices of the diagrams in question are again paired scattering amplitudes of the particles at zero energy and the amplitudes for the production of three particles with zero kinetic energy. The diagram of Fig. 11 depends on two variables, k^2_{12} and $E = \sqrt{s} - m_1 - m_2 - m_3$. To calculate the contribution which contains a singularity

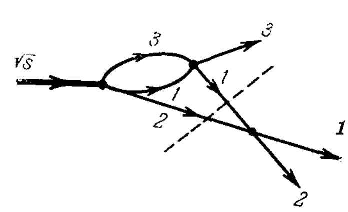


FIG. 11.

in k_{12}^2 at small k_{12}^2, made by the diagram of Fig. 11, it is convenient to use the dispersion relation in terms of k_{12}^2. The diagram of Fig. 11 is essentially a "three-point diagram" with decay masses $\sqrt{s}$. In this situation it is difficult, generally speaking, to write out directly the dispersion relation in terms of k_{12}^2. It can be obtained, however, by starting from a simple dispersion relation in k_{12}^2 at low values of s, and then carrying out analytic continuation in this variable.

For small s the dispersion relation in k_{12}^2 is

$$I(k_{12}^2, E) = I(0, E) + \frac{k_{12}^2}{\pi} \int_0^\infty \frac{I_1(k_{12}'^2, E)}{k_{12}'^2 (k_{12}'^2 - k_{12}^2 - i\varepsilon)} dk_{12}'^2, \quad (29)$$

where the subtractional constant $I(0, E)$ does not depend, of course, on k_{12}^2 and therefore does not pertain to the terms that are singular in k_{12}^2.

One might think that the diagram of the type in Fig. 11 could have, besides trivial threshold singularities, also singularities of the Landau type, corresponding to the vanishing of all the Feynman denominators. We stipulate immediately that there are in fact no Landau singularities in the case in question. To understand the reason for the absence of these singularities, we must recall the usual situation in triangular diagrams. At normal values of the two external mass M_1 and M_2, triangular diagrams usually have two singularities of the Landau type in the Landau mass M_3, located on the unphysical sheet. These singularities are simultaneously singularities of the absorption part in M_3. If one of the external masses (for example, M_1) increases, then one of the indicated singularities of the absorption part in the third mass links with the integration contour, as a result of which the dispersion relation acquires a nontrivial form, and the singularity of the amplitude in M_3 goes itself onto the physical sheet. In the case of the diagram shown in Fig. 11, the positions of the two indicated singularities in M_3 coincide. In fact, this causes these singularities to cancel each other. This will follow rigorously from our subsequent calculation of the magnitude of the diagram in Fig. 11, but for the time being, we confine ourselves to the following remarks. The position of the singularities referred to above is determined by the Mandelstam relation (see the Appendix) $z_1^2 + z_2^2 + z_3^2 - 2z_1z_2z_3 - 1 = 0$ between the three cosines z_1, z_2 and z_3, which are connected with the masses at the vertices of the triangular diagram. (Each cosine is equal to $(\mu_1^2 + \mu_2^2 - \mu_3^2)(2\mu_1\mu_2)^{-1}$ where μ_1 and μ_2 are the masses of the internal lines, while μ_3 is the mass of the external line.) For the diagram shown in Fig. 11, the cosine connected with the vertex at which scattering of particles 1 and 3 takes place is equal to unity, as a result of which the position of the point suspected by us (corresponding to the usual presence of the Landau singularity) is determined by the relation

$$s_{12} = (m_1 + m_2)^2 + \frac{m_1}{m_1 + m_3} [s - (m_1 + m_2 + m_3)^2]. \quad (30)$$

We see that regardless whether this point is indeed a singular point of the absorption part or not, it uniformly moves from left to right with increasing s and does not deform the integration contour in the dispersion integral (29). Thus, the dispersion relation (29) retains at any rate its form, a fact which we shall make use of in what follows.

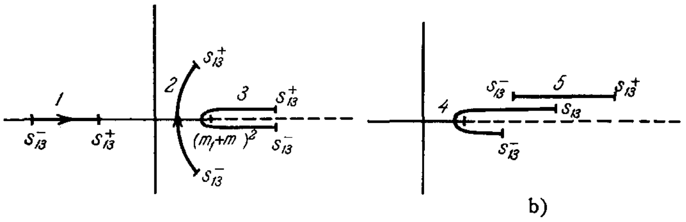
We now proceed to calculate the absorption part $I_1(k_{12}^2, E)$. We shall show presently that it actually has no singularities at the point defined by (30), but it turns out here that $I_1(k_{12}^2, E)$ has a non-Landau singularity in k_{12}^2 (or s_{12}) near threshold, that is, a singularity which does not correspond to the vanishing of the denominators of the Feynman diagram[10-20]. At small values of s we can obtain $I_1(k_{12}^2, E)$ with the aid of the unitarity condition in the channel where s_{12} is the energy:

$$I_1(k_{12}^2, E) = \frac{k_{12}}{\sqrt{s_{12}}} a_{12}(m_1 + m_2) \int_{-1}^{1} \frac{dz}{2} B(s_{13}'). \quad (31)$$

The normalization of the unitarity condition which we have written out follows from (21) and (23); $B(s_{13}')$ is the amplitude for the transformation of the initial particle into three particles, shown in Fig. 11 to the left of the dashed line. It depends, of course, only on s_{13}' —the square of the total energy of relative motion of particle 3 in the final state and of particle 1 in the intermediate state. z is the cosine of the angle between the momentum of the third particle in the final state in the c.m.s. of particles 1 and 2 and the relative momentum of particles 1 and 2 in the intermediate state. It is easy to write out the connection between the invariant s_{13}' and the variable z:

$$s_{13}' = m_1^2 + m_3^2 - \frac{(s_{12} + m_3^2 - s)(s_{12} + m_1^2 - m_2^2)}{2s_{12}} + \frac{z}{2s_{12}} \sqrt{[s_{12} - (m_1 + m_2)^2][s_{12} - (m_1 - m_2)^2][s - (\sqrt{s_{12}} + m_3)^2][s - (\sqrt{s_{12}} - m_3)^2]}. \quad (32)$$

We go over in (31) with the aid of (32) to integration with respect to s_{13}'. The integrand $B(s_{13}')$ has obviously a single singularity in the s_{13}' plane at $s_{13}' = (m_1 + m_3)^2$ (Fig. 12). Integration with respect to s_{13}' in (31) for small values of s ($s < (\sqrt{s_{12}} - m_3)^2$) is along the segment of the negative axis between the values s_{13}^- and s_{13}^+ (contour 1 in Fig. 12,a). To obtain the correct expression for large values of s, it is necessary to carry out analytic continuation in s, assigning to s (the square of the external mass) a positive imaginary increment: $s \rightarrow s + i\epsilon$. When $s = (\sqrt{s_{12}} - m_3)^2$, the quantities s_{13}^- and s_{13}^+ go over into the complex plane, and when $s > (\sqrt{s_{12}} - m_3)^2$ the integration is carried out between the complex-conjugate points (contour 2 in Fig. 12,a). We note that the point $s = (\sqrt{s_{12}} - m_3)^2$ is not singular, since the continuation



a) b)

FIG. 12

from $s \to s + i\epsilon$ gives here the same result as from $s \to s + i\epsilon$. This is connected with the fact that the changes in the direction of integration, resulting from these two continuations, are canceled by the change in the sign of the root in the factor that relates dz with ds'_{13}. When $s = (\sqrt{s_{12}} + m_3)^2$ the limits of integration s^+_{13} and s^-_{13} fall on a cut of the function $B(s'_{13})$ going from the point $(m_1+m_3)^2$ along the positive real axis. Then s^+_{13} falls on the upper edge of the cut, s^-_{13} on the lower one, and the integration contour encompasses the cut, as shown by contour 3 in Fig. 12,a. As can be readily seen from (31), the point where s^+_{13} and s^-_{13} coincide in the s'_{13} plane always lies to the right of $(m_1+m_3)^2$, provided, as is assumed, $s_{12} > (m_1+m_3)^2$. The point $s = (\sqrt{s_{12}} + m_3)^2$ is singular, for in the case of the continuation from $s \to s + i\epsilon$ the upper limit of integration s^+_{13} moves along the real axis to the right with further increase of s, while s^-_{13} moves to the left (Fig. 12,b, contour 4), whereas after circuiting the point $s = (\sqrt{s_{12}} + m_3)^2$ with negative imaginary increment $(s \to s + i\epsilon)$, s^+_{13} moves to the left and s^-_{13} to the right. In addition, these continuations correspond to different signs of the factor relating dz with ds'_{13} (different signs of the root $\sqrt{s - (\sqrt{s_{12}} + m_3)^2}$). In the absence of a cut in the function $B(s'_{13})$ the change in the sign of the root would offset the permutation of the integration limits, as was the case at the point $s = (\sqrt{s_{12}} - m_3)^2$. However, owing to the presence of a cut in $B(s'_{13})$, circuiting of the point $s = (\sqrt{s_{12}} + m_3)^2$ with positive and negative imaginary additions leads to different results, and therefore the point $s = (\sqrt{s_{12}} + m_3)^2$ is singular. With further increase in s, the point s^-_{13} circuits the start of the cut $(m_1+m_3)^2$ and falls on the upper edge, after which integration contour assumes the form 5 on Fig. 12,b. The point s^-_{13} coincides with the start of the cut precisely when s is given by relation (30). The value of s corresponding to (30) could, generally speaking, be a singular point of the absorption part (coincidence of the end of the integration contour with the singular point of the integrand). As already noted, this point is in fact not singular, as will be seen directly from the explicit expression for the absorption part which we shall derive below.

We now proceed to calculate $I_1(k^2_{12}, E)$ directly. The function of $B(s'_{13})$ under the integral sign in (31) contains a certain constant part, a part linear in k'_{13}, and terms of higher order in k'_{13}. The constant term in $B(s'_{13})$ leads again to expression (28), so that an account of this term reduces essentially to a renormalization of the constant λ. So long as this constant is taken to be the already observed value of the amplitude for the production of three particles at zero energy, the constant term in $B(s'_{13})$ should be discarded. We now consider the term of $B(s'_{13})$ which is linear in k'_{13}. It is easy to see from the subsequent calculations that terms of higher order in k'_{13} of $B(s'_{13})$ lead to terms of higher order in the higher threshold momenta in the final expression for the amplitude. With the accuracy indicated, $B(s'_{13})$ must be replaced by the quantity (see formula (28))

$$B(s'_{13}) = i\lambda a_{13} k_{13}. \tag{33}$$

Going over to the nonrelativistic approximation with the aid of the formulas of Sec. 1 in expression (32), we obtain the connection between k'^2_{13} and z. This connection is expressed by (10) of Sec. 1, where $x_{13} = k_{13}/\sqrt{2\mu_{13}E}$ must be replaced by $x'_{13} = k'_{13}/\sqrt{2\mu_{13}E}$. Substituting (33) in (31), we get

$$I_1(x^2_{12}, E) = \int_{x^-_{13}}^{x^+_{13}} dx'_{13} x'^2_{13} i\lambda a_{12} a_{13} \sqrt{\frac{\mu_{12}\mu_{13}}{\beta_1(1-\beta_1)}} \frac{E}{\sqrt{1-x^2_{12}}}$$

$$= i\lambda a_{12} a_{13} E \frac{2\sqrt{\beta_1 \mu_{12}\mu_{13}}\, x_{12}}{\sqrt{1-x^2_{12}}} \left(1 + \frac{1-4\beta_1}{3\beta_1} x^2_{12}\right),$$

$$x^\pm_{13} = \sqrt{\beta_1(1-x^2_{12})} \pm \sqrt{(1-\beta_1) x^2_{12}}. \tag{34}$$

The limits of integration $x^\pm_{13} = \sqrt{\beta_1(1 - x^2_{12})} \pm \sqrt{(1-\beta_1)x^2_{12}}$ in the integral of (34) are obtained in the following manner. The integration with respect to z reduces directly to integration with respect to x'^2_{13} between the limits $(\sqrt{\beta_1(1-x^2_{12})} \pm \sqrt{(1-\beta_1)x^2_{12}})^2$. On going over to integration with respect to x'_{13}, the question arises of the choice of the sign of the root when the new limits of integration are determined. This choice should be made in accordance with the above-described continuation of the unitarity condition in s (see Fig. 12). We see from this continuation, in particular, that at large values of s (large E, small x_{12}) both limits of integration with respect to x'^2_{13} lie on the upper edge of the cut, and accordingly both limits of integration in x'^2_{13} should be positive. This condition is indeed satisfied in the integral of (34).

The expression (34) for the absorption part $I_1(x^2_{12}, E)$ has a singularity at $x^2_{12} = 1$, i.e., at $k^2_{12} = 2\mu_{12}E$. This singularity, as already mentioned, has a non-Landau character and means that the integration limits for the absorption part fall on the cut (the position of this singularity was given above in the form $s = (\sqrt{s_{12}} + m_3)^2$). The obtained singularity is located on the boundary of the physical region at the point where the momentum of the third particle vanishes. On the other hand, as stated above, (34) does not contain a characteristic Landau-type singularity corresponding to the vanishing of all denominators of the Feynman diagram. Such a singularity, were it to exist, should be located at a value of s_{12} given by formula (30), that is, at $x_{12}^2 = \beta_2$.

To calculate the special contribution to the amplitude made by the diagram of Fig. 11, we must now substitute (34) in the dispersion integral (29). This raises the question of the correct circuiting around the singularity $k_{12}'^2 = 2\mu_{12}E(x_{12}' = 1)$ when integrating with respect to $k_{12}'^2$. The expression (34) is obviously valid both for $E > 0$ and $E < 0$ when the singularity $k_{12}'^2 = 2\mu_{12}E$ lies outside the integration contour. A correct analytic continuation from the region $E < 0$ into the region $E > 0$ will be one in which E receives a positive imaginary increment (since E plays the role of the external mass). Therefore the singularity $k_{12}'^2 = 2\mu_{12}E$ is located above the integration contour with respect to $k_{12}'^2$.

If we substitute (34) in (29) we readily see that the resultant integral diverges logarithmically. This divergence is a result of the expansion of the absorption part in powers of $k_{12}'^2$, whereas the exact expression would cut off the integral at a value of k_{12}^2 of the order of the particle masses. It is meaningless to make more precise the character of this cutoff by starting from the form of the concrete diagram, since the cutoff can also result, for example, from the decrease in the exact amplitudes, which we have replaced by constants at the vertices of the diagram. On the other hand, two results obtained at different cutoffs, differ from each other by an amount Ck_{12}^2, where C is a certain constant. Terms of this type are analytic, they are contained in a large number of diagrams and, in accordance with the developed approach, they cannot be calculated but should be added to the amplitude with arbitrary coefficients. By virtue of the foregoing, we shall cut off the integral (29) at $k_{12}' \sim m$, where m is a quantity of the order of the masses of the particles encountered in the reactions.

Taking the foregoing remarks into account, we can carry out the integration of (29) in a rather simple manner, and obtain the following expression for the terms that are singular in k_{12}^2 and are connected with the diagram in Fig. 11:

$$I(x_{12}^2, E) - I(0, E) = -2\lambda a_{12}a_{13}E\sqrt{\frac{m_1m_2m_3}{m_1+m_2+m_3}}\left[\frac{2x_{12}\arccos x_{12}}{\pi\sqrt{1-x_{12}^2}}\right.$$
$$\left.\times\left(\beta_1+\frac{1-4\beta_1}{3}x_{12}^2\right)-\frac{1-4\beta_1}{3\pi}x_{12}^2\left(\ln\frac{m}{E}+i\pi\right)\right]. \qquad (35)$$

We have thus separated the terms containing the singularity in k_{12}^2 from the total contribution to the amplitude of the diagram of Fig. 11. The first terms in (35) have, as can be seen, a characteristic root-type threshold singularity at $k_{12}^2 = 0$ and a curious singularity in k_{12}^2 at $k_{12}^2 = 2\mu_{12}E$, which, however, is located on the unphysical sheet connected with the cut drawn from the point $k_{12}^2 = 0$. In fact, when $x_{12} > 0$ (on the upper edge of the cut) arccos x_{12} behaves as $x_{12}^2 \to 1$, like the root $\sqrt{2(1-x_{12}^2)}$, which cancels out the root in the denominator. On the lower edge of the cut, however, where $x_{12} < 0$, we get $x_{12}^2 \to 1$ when arccos $x_{12} \to \pi$, and we have a singularity of the type $(1-x_{12}^2)^{-1/2}$. The last term in (35) does not contain singularities in k_{12}^2, although it does have a singularity of the type $k_{12}^2 \ln E$ at $E = 0$. Inclusion of this term in (35) is somewhat arbitrary and is justified by the fact that our next task is precisely the separation of terms containing the singularity at the total energy E in the diagram of Fig. 11. We shall now show that, in addition to the last term in (35), such a contribution is contained also in the subtraction constant $I(0, E)$. This contribution behaves like $E \ln E$ and reflects the presence in the total energy of an ordinary logarithmic singularity connected with the three-particle intermediate state.

In calculating $I(0, E)$ (or, more accurately, in calculating the singular contribution), it is convenient to use the three-particle unitarity condition in the channel where E is the energy (Fig. 13). We are calculat-

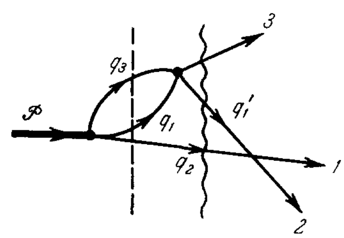


FIG. 13.

ing here in this case, of course, the discontinuity only on the three-particle cut (we recall that λ has "its own complexity" connected with scattering of particles in the initial state (see formula (27)). This discontinuity is determined, in accordance with (25), by the expression

$$I_2(0, E) = -\frac{1}{\pi^3}\lambda a_{12}a_{13}(m_1+m_2)(m_1+m_3)\int\frac{d^4q_1\,d^4q_2\,d^4q_3}{q_1'^2-m_1^2}$$
$$\times\,\delta^4(q_1+q_2+q_3-P)\,\delta(q_1^2-m_1^2)\,\delta(q_2^2-m_2^2)\,\delta(q_3^2-m_3^2). \qquad (36)$$

The notation in this formula is quite obvious. The meaning of the 4-momenta q_1, q_2, q_3, q_1', and P is explained in Fig. 13. The momentum P has only a time-dependent component, equal to $p_0 = \sqrt{s} \approx m_1 + m_2 + m_3 + E$. In formula (36) we take into account only the contribution from the cut shown in Fig. 13 by the dashed line. The contribution to the three-particle jump, connected with the division shown by the wavy line, is missing, since the relative momentum of particles 1 and 2 is equal to zero. The normalization factor in (36) is obtained from (25) and (24).

Calculation of the interval (36), although somewhat cumbersome, is quite elementary. It is convenient, for example, to proceed in the following manner. After integrating with respect to d^4q_3 with the aid of a four-dimensional δ-function, we can carry out the following integrations with respect to d^4q_1 in the c.m.s. of particles 1 and 3. These integrations then yield the two-particle phase volume of particles 1 and 3. It is expedient to introduce next additional integration with respect to the mass (total energy) of particles 1 and 3: $\delta((q_1+q_3)^2 - m_{13}^2)dm_{13}$, and carry out integration over d^4q_2 in the common center-of-mass system. The integration $d^4q_2\delta(q_2^2 - m_2^2) - \delta((P-q_2)^2 - m_{13}^2)$ is in this case again integration over the two-particle phase volume of the particles with masses m_2 and m_{13}. After the last integration with respect to dm_{13}^2, we arrive at the following result, which is obtained, of course, by going over to the nonrelativistic approximation:

$$I_2(0, E) = \lambda a_{12}a_{13}\sqrt{\frac{m_1m_2m_3}{m_1+m_2+m_3}}\frac{1}{3}(1+2\beta_1)E. \quad (37)$$

Expression (37) makes it possible to obtain the singular part of the function I(0, E) in terms of E, with the aid of the dispersion relation

$$I(0, E) - I(0, 0) = \frac{E}{\pi}\int_0^\infty dE' \frac{I_2(0, E')}{E'(E'-E-i\varepsilon)}. \quad (38)$$

The subtraction constant I(0, 0) contributes only to the amplitude of the process at zero energy. This constant term should be included in λ, the observed value of the amplitude for the production of three particles at zero energy. Taking this into account, we shall leave out I(0, 0) and present only an expression for I(0, E), which contains a singularity in E. Substitution of (37) in (38) yields a logarithmically diverging integral, which we cut off at a value of the order of the particle mass m, using the same considerations as in the derivation of formula (35) from the dispersion integral (29). We thus have

$$I(0, E) = \lambda a_{12}a_{13}\sqrt{\frac{m_1m_2m_3}{m_1+m_2+m_3}}\frac{1}{3}(1+2\beta_1)\frac{E}{\pi}\left(\ln\frac{m}{E}+i\pi\right). \quad (39)$$

The last term in (39) does not contain any singularities in E and is therefore included in (39) somewhat arbitrarily. We shall see, however, that similar imaginary analytic terms, unlike the real terms in I(0, E), occur in our problem only as a result of diagrams connected with multiple scattering of particles, so that they have a unique meaning.

We have calculated the nonanalytic terms which are linear (formula (28)) and quadratic (formulas (35) and (39)) in the threshold momenta and are connected with the diagrams shown in Figs. 10 and 11. By quadratic terms we mean here also terms of the order of E ln E (or k_{12}^2 ln E). We shall now explain why the nonanalytic and quadratic terms which we have obtained enter in the amplitude only as a result of diagrams of the type considered (the three diagrams of Fig. 10 and the six diagrams similar to that shown in Fig. 11). We recall first that nonanalytic terms can arise only in diagrams which are connected with the scattering of the produced particles. If we consider diagrams connected with double scattering of any pair of particles (Fig. 14,a), and we separate the linear terms from both

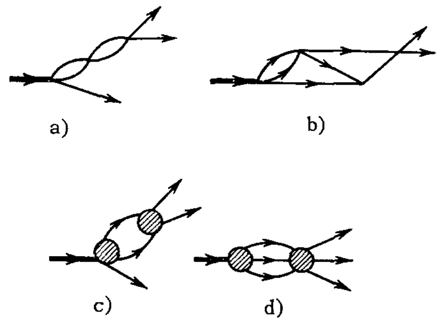


FIG. 14.

loops, then we can readily see that their contribution is equal to $-\lambda k_{12}^2 a_{12}$, which is an analytic expression in k_{12}^2. Therefore the contribution of such diagrams cannot be separated from other analytic terms. Let us consider further diagrams connected with a large number of scatterings (for example, the diagram of Fig. 14,b, where the produced particles are rescattered three times). From the calculations given in the present section we see, in fact, that each additional scattering leads to an extra power of the threshold momentum in the expression for the amplitude. Thus, the diagram of Fig. 10 gives linear terms and the diagram shown in Fig. 11 leads to quadratic terms. In the next section we shall verify that diagrams with triple scattering (for example, Fig. 14,b) give cubic terms. The reason for this fact is perfectly clear. Each additional scattering gives an extra power of the momentum in the absorption part of the diagram. This momentum represents, in essence, the phase volume of the scattered particles near threshold. The region which is always important in reconstructing the singular terms in the amplitude from the absorption part of the dispersion integral is that of small momenta (or energies), since large momenta correspond only to analytic terms. In other words, in any dispersion integral we can make a sufficient number of integrations such that small integration momenta become important in the case of small external momenta. On the other hand, the resultant subtraction polynomial gives an analytic dependence, and terms of this type are anyway included separately. Thus, we arrive at the conclusion that the diagrams with a large number of scattering events (for example, of the type shown in Fig. 14,b) make no contribution to the nonanalytic linear and quadratic terms.

Singular terms can arise in diagrams describing the scattering of particles, in which the blocks corresponding to scattering or transformation into three particles are not replaced by a constant. One such diagram is shown symbolically in Fig. 14,c. Let us expand the blocks contained in the diagram in powers of the deviations of the corresponding invariant from the threshold values (we recall that after separating all the scatterings these blocks depend near threshold on the invariants in analytic fashion). The constant terms in these expressions lead us precisely to one of the diagrams which we have taken into account, shown in Fig. 10. The next terms introduced into the absorption part are at least cubic in the threshold momenta. As already explained, the corresponding nonanalytic terms in the amplitude itself turn out to be in this case also at least cubic.

Finally, let us turn to consider the diagram shown in Fig. 14,d, which includes a block for the transformation of three particles into three. We recall that diagrams of this type contain terms which are nonanalytic near threshold, regardless of whether they contain particle-pair scattering. If the block for transformation of three particles into three is a constant, then the absorption part of such a diagram is of the order of E^2 (the phase volume of three particles near threshold), and the amplitude itself contains a nonanalytic term of the order of $E^2 \ln E$. In other words, in this case the diagram contributes only to terms of fourth order of smallness in the threshold momenta. The block of transformation of three particles into three can, however, become infinite at small particle momenta. This occurs, for example, if the amplitude for the transformation of three particles into three has a pole-like character (Fig. 15,a). In this case we arrive at the already-investigated diagram of Fig. 11. Inasmuch as the magnitude of the pole "six-point diagram" is of the order of E^{-1}, the magnitude of the absorption part turns out to be of the order of E,

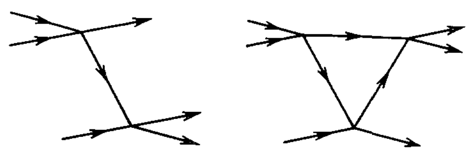


FIG. 15.

as is also seen from formulas (34) and (37). The entire amplitude likewise turns out to be of the order of E (or E ln E). The amplitude of the transformation of three particles into three becomes infinite also in a slower manner. As can be seen from the following section, the "six-point diagram," as seen in Fig. 15,b, will be of the order of $E^{-1/2}$ near threshold. Accordingly, the contribution to the amplitude from the diagram of Fig. 14,b which includes such a "six-point diagram," is, as already mentioned, of the order of $E^{3/2}$. Since the amplitude for the transformation of three particles into three cannot become infinite faster than E^{-1}, to calculate the amplitude with quadratic accuracy it is sufficient to consider diagrams of Fig. 11.

We can now present a complete expression for the amplitude, accurate to terms that are quadratic in the threshold momenta. From the statements made above we see that this expression is obtained from a contribution of three diagrams shown in Fig. 10 (linear terms), six diagrams of the type shown in Fig. 11 (quadratic nonanalytic terms), and analytic terms of the following type: λ (amplitude of the reaction threshold energy) and $\alpha_3 k_{12}^2$, $\alpha_2 k_{13}^2$, $\alpha_1 k_{23}^2$. The latter are analytic terms which should be added to the amplitude with certain unknown coefficients. Since these terms characterize the contribution of the remote singularities, the order of magnitude of the coefficients is λ/m^2, where m is a quantity of the order of the masses of the particles encountered in the reaction. Accordingly, we put $\alpha_i = \lambda C_i$, where $C_i \sim 1/m^2$. Thus we have

$$A(k_{12}^2, k_{13}^2, k_{23}^2) = \lambda [1 + ik_{12}a_{12} + ik_{13}a_{13} + ik_{23}a_{23} + a_{12}a_{13}(I_1(x_{12},E) + I_1(x_{13}, E)) + a_{12}a_{23}(I_2(x_{12},E) + I_2(x_{23},E)) + a_{13}a_{23}(I_3(x_{23},E) + I_3(x_{13},E)) + C_1k_{23}^2 + C_2k_{13}^2 + C_3k_{12}^2], \tag{40}$$

and

$$I_\alpha(x,E) = -2E\sqrt{\frac{m_1m_2m_3}{m_1+m_2+m_3}}\left\{\frac{2x\arccos x}{\pi\sqrt{1-x^2}}\left[\beta_\alpha + x^2\frac{1-4\beta_\alpha}{3}\right] - \frac{1}{\pi}\left(\ln\frac{m}{E} + i\pi\right)\left[\frac{1}{6}(1+2\beta_\alpha) + x^2\frac{1-4\beta_\alpha}{3}\right]\right\}. \tag{41}$$

We recall that $x_{il} = k_{il}/\sqrt{2\mu_{il}E}$ and, for example, $\beta_1 = m_1(m_1+m_2+m_3)(m_1+m_2)^{-1}(m_1+m_3)^{-1}$. The expression for $I_1(x_{12}, E)$ was obtained in formulas (35) and (39), in which $I_1(x_{12})$ was denoted as $I(x_{12}^2, E)$.

Formula (40) contains three undetermined constants C_1, C_2, and C_3; in addition, each of the $I_\alpha(x, E)$ contains a term in the form E ln m, so that ln m is also a certain undetermined constant. Since, however, E is the sum of the squares of k_{12}^2, k_{13}^2, and k_{23}^2 (formula (4)), the expression (40) contains essentially only three undetermined constants.

From (40) follows directly an expression for the differential cross section of the reaction:

$$\frac{d\sigma}{d\Gamma} = |\lambda|^2\{1 + 2a_{12}a_{13}[k_{12}k_{13} + I_1(x_{12},E) + I_1(x_{13},E)] + 2a_{12}a_{23}[k_{12}k_{23} + I_2(x_{12},E) + I_2(x_{23},E)] + 2a_{13}a_{23}[k_{13}k_{23} + I_3(x_{13},E) + I_3(x_{23},E)] + 2C_1k_{23}^2 + 2C_2k_{13}^2 + 2C_3k_{12}^2\},$$

$$d\Gamma = \delta\left(\frac{k_{12}^2}{2\mu_{12}} + \frac{k_3^2}{2\mu_3} - E\right) d^3k_{12}\, d^3k_3. \tag{42}$$

The phase volume element dΓ can, of course, be represented in a different form, for example, $d\Gamma \sim dk_{12}^2 dk_{13}^2$ or $d\Gamma \sim k_{12}^2 dk_{12} d\Omega_{k3}$ ($d\Omega_{k3}$—element of solid angle of the momentum of the third particle in the common center of mass). Expression (42) for the differential cross section has one curious property.

At the end points of the spectrum, for example, as $k_{12} \to 0$, the cross section does not contain a term linear in k_{12}. This follows from the fact that in the physical region of the reaction we have $x_{13}^2 \to \beta_1$ as $k_{12} \to 0$ (see formula (10)), that is, $k_{13} \to \sqrt{2\beta_1\mu_{13}E}$. On the other hand, the function $I_1(x_{12}, E)$ contains at small values of k_{12} a term of the form $-k_{12}\sqrt{2\beta_1\mu_{13}E}$ which exactly cancels out the term linear in k_{12}. This behavior of the cross section at the edge of the spectrum is quite general. The point is that for any total energy E (even if it is not small compared with the particle

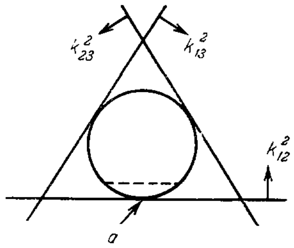


FIG. 16.

masses) the amplitude of the reaction at the edge of the spectrum, for example as $k_{12} \to 0$, takes the form

$$A(k_{12}^2, k_{13}^2, k_{23}^2) = A(0, k_{13}^2, k_{23}^2)(1 + ik_{12}a_{12}). \tag{43}$$

It must be borne in mind here that this property is satisfied only in the physical region of the process. In Fig. 16 is shown the Dalitz diagram for the reaction in question. The physical region is a circle inscribed in a triangle. In the physical region $k_{12}^2 = 0$ corresponds to one point a, shown by the arrow in the figure. Expression (43) is applicable precisely in the vicinity of this point, and therefore the variables k_{13}^2 and k_{23}^2 in (43) are expressed in terms of the total energy E. Formula (43) is easiest to derive in the following manner. For arbitrary total energy E but small k_{12}^2, the terms that are singular in k_{12}^2 must be sought in diagrams of the type shown in Fig. 4,a. It is convenient to seek again the singular term with aid of a dispersion relation in k_{12}^2. In calculating the singular part of the amplitude for small k_{12}^2, a major role will again be played by small momenta of integration in the dispersion integral with respect to k_{12}^2. For small k_{12}^2 it is necessary to replace the particle scattering block in the absorption part by the particle scattering amplitude at zero energy, after which there enters in the unitarity condition the amplitude for production of three particles integrated over the angle of one of the particles in the intermediate state. For small k_{12}^2, the region of integration with respect to the angle corresponds to the very small region of variation of the invariants on which the amplitude depends. For example, for the case of transformation of one particle into three, the integration over the angle corresponds to integration along the dashed line in Fig. 16, when the invariants k_{13}^2 and k_{23}^2 vary within narrow limits. Therefore the amplitude for particle production can be replaced by its value at the point a, and taken outside of the integral sign. This leads us directly to (43). We see from (43) that the cross section (the square of the modulus of (43)) no longer contains a term linear in k_{12}. This property changes radically if charge exchange of particles 1 and 2 is possible. This circumstance can play an essential role, for example, in interpretation of the experimental data for the reactions $\pi + N \to N + \pi + \pi$ or $\gamma + N \to N + \pi + \pi$. We shall return to a discussion of this question later. From (42) we easily obtain an expression for the total cross section of the reactions:

$$\sigma = \text{const} \cdot E^2 \left[1 + AE \ln \frac{m}{E} + BE\right],$$

$$A = \frac{8}{3\pi} \sqrt{\frac{m_1 m_2 m_3}{m_1 + m_2 + m_3}} [a_{12}a_{13}(1 - \beta_1) + a_{12}a_{23}(1 - \beta_2) + a_{13}a_{23}(1 - \beta_3)]. \tag{44}$$

The constant B is not calculated, since it is connected with the constants C_1 which enter in (42). The near-threshold dependence of the cross section (44) on the energy may not take place in the presence of charge exchange between the produced particles. This case will also be discussed later in an examination of the reactions $\pi + N \to N + \pi + \pi$ and $\gamma + N \to N + \pi + \pi$.

7. CUBIC TERMS IN THE EXPANSION IN THRESHOLD MOMENTA AND GENERAL STRUCTURE OF THE EXPANSION OF THE AMPLITUDE WITH L = 0

We have derived an expression for the amplitude of the reaction with production of three low-energy particles with zero total momentum, accurate to terms which are quadratic in the threshold momenta. The general character of the foregoing analysis allows us to describe qualitatively the complete structure of the expansion of the amplitude with L = 0 near the threshold. It is obvious that the amplitude can be represented by the sum

$$A(k_{12}^2, k_{13}^2, k_{23}^2) = \sum_{n=0}^{\infty} A^{(n)}(k_{12}^2, k_{13}^2, k_{23}^2), \tag{45}$$

where each of the quantities $A^{(n)}$ is of the order of $(E/m)^{n/2}$, E is the kinetic energy of the particles, and m is a quantity of the order of the particle mass. When n = 0 we have $A_0 = \lambda$, which is the amplitude for the production of three particles at zero energy. When n = 1, the value of $A^{(1)}(k_{12}^2, k_{13}^2, k_{23}^2)$ is given by the expression (28) of the preceding section. A characteristic feature of these terms (which are linear in the threshold momenta) is that they all contain as parameters only the scattering amplitudes of the particle-pairs.

We recall that the reason for this is that the linear terms are, by definition, nonanalytic if regarded as functions of the complex variables k_{12}^2, k_{13}^2, and k_{23}^2. The quadratic terms in the expansion of the amplitude (n = 2 in formula (45)) already contain new unknown real parameters C_1, C_2, and C_3. The physical meaning of these parameters is obvious: they characterize the interaction of the particles at short distances. The introduction of the constants C_i in the expression for the amplitude is equivalent to introduction of the interaction radius in the problem of low-energy particle scattering. When considering the quadratic terms, we encounter for the first time the complicated functions of dimensional relations of the type $x_{12}^2 = k_{12}^2/2\mu_{12}E$, the existence of which is due primarily to the existence of several different variables of one order of magnitude. The appearance of such functions is closely related with the existence of simplest threshold singularities for the following reason. Each of such functions should, obviously, have singularities in its own variable (x_{il}^2) for certain numerical values of x_{il}^2 (for example, when $x_{il}^2 = 1$, as is the case with the quadratic terms). In the opposite case, these functions would be simply polynomials or would have an essential singularity at infinity. Any singularity of the type $x_{il}^2 = 1$ is simultaneously a singularity in the k_{il}^2 plane when $k_{il}^2 = 2\mu_{il}E$ and a singularity in the E plane when $E = k_{il}^2/2\mu_{il}$. Such singularities cannot be present on the physical sheath and should be "hidden" under cuts connected with the threshold singularities.

What are the parameters that enter in the terms of higher order in (45), and foremost in the cubic terms? As explained in the preceding section, the cubic terms must be sought in diagrams that are connected with not more than triple scattering of particles in the final state. The absorption terms of the corresponding diagrams will, obviously, always be expressed in terms of the paired scattering amplitudes and the amplitude for production of three particles, calculated accurate to quadratic terms. The latter contains, besides the paired amplitudes, the constants C_1, C_2, and C_3, which will thus enter in the final expression for the cubic terms. However, no new unknown constants will appear, and this constitutes the manifestation of the nonanalyticity of all the cubic terms.

The fourth-order terms contain a large number of new unknown parameters. Their number indeed becomes so large in this case that the calculation of the fourth-order terms is no longer meaningful. First of all, we get the terms of the type $B_1k_{23}^4$, $B_2k_{13}^4, \ldots,$ $D_1k_{12}^2k_{13}^2$, $D_2k_{12}^2k_{23}^2$, etc., which cannot be calculated. In addition, it is necessary, as already explained in the preceding section, to take into account in this order the diagram shown in Fig. 14,d. This diagram gives a term of the form $E^2 \ln E$, which is nonanalytic in the total energy and has a coefficient proportional to the constant part of the amplitude for the transformation of three particles into three—a quantity not encountered in the lower-order terms.

The remainder of the structure of the expansion (45) is clear. The fifth-order terms, for example, do not contain any new unknown parameters compared with the terms in the first to fourth orders. In general, however, with increasing order there appears an ever increasing number of constants that cannot be evaluated, and their presence deprives the series (45) of any practical meaning.

We now proceed to a detailed calculation of the cubic terms in the three-particle production amplitude. However, we shall not calculate all the third-order terms, but only the real cubic terms.

The point is that only these terms interfere in the expression for the cross section, and add to it a contribution of third-order of smallness in the threshold momenta. The imaginary cubic terms, on the other hand, interfere only with the linear (imaginary) terms and make a fourth-order contribution to the cross section. We stipulate first that such a situation takes place only in the absence of charge exchange of particles in the final state, and in a few other cases of practical interest, which will be analyzed later ($K \to 3\pi$-decay, pion production in collisions between a positive pion and a proton). In some other cases ($\pi^- + p \to N + \pi + \pi$, $\gamma + p \to N + \pi + \pi$) the third-order terms appear in the cross section of the reaction also as a result of imaginary cubic terms in the reaction amplitude. It is essential, however, that for reactions of the first type (the neutral case, etc.) the expression for the cross section does not contain linear terms, so that the cubic terms are here the first correction to the fundamental, quadratic terms. In the second case, however, the quadratic terms themselves already constitute a correction to the linear terms present in the cross section. Therefore the calculation of the cubic terms is of lesser interest here.

The amplitude of the process can contain complex terms connected with the scattering of particles in the final state, and a complexity results from interaction of the particles in the initial state. Thus, in particular, in the absence of possible intermediate states other than the initial and final states, the amplitude for transformation of two particles into three contains a factor $e^{i\delta(E)}$, where $\delta(E)$ is the phase of the particle scattering in the initial state.

We shall now be interested only in the interaction in the final state, and will separate the real and imaginary terms in the expression for the quantity $e^{-i\delta(E)} \times A(k_{12}^2, k_{13}^2, k_{23}^2)$, and not in the amplitude itself. Later on, the results will be applied to the case of $K \to 3\pi$-decay, where the complexity connected with the interaction in the initial state does not appear at all, and to the case of pion production in collisions between a positive pion and a proton, where it can also be shown that the amplitude contains no essential cubic terms other than those calculated.

Like all other nonanalytic terms, the real cubic terms must be sought in diagrams connected with particle scattering. It is seen, for example, that we deal with the diagram shown in Fig. 17, where for concreteness the particles 1 and 2 are scattered. We assume that the blocks entering in the diagram can also include different particle scatterings. To obtain the cubic terms in the expression for the reaction amplitude it is necessary to take into account the linear and quad-

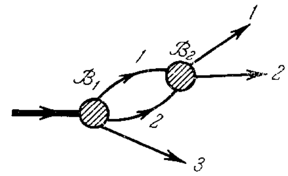


FIG. 17.

ratic terms in the blocks B_1 and B_2. It is easy to see that if account is taken of the linear terms in k_{12} in both blocks B_1 and B_2, we shall arrive at pure imaginary cubic terms for the entire amplitude.

In fact, in this case we immediately obtain for the absorption part a real expression proportional to k_{12}^3, and a cubic term in the amplitude, which itself is proportional to ik_{12}^3. If we choose from the block B_2 a term linear in k_{12}, and from the block B_1 a term linear, say, in k_{13}, then we arrive at the diagram shown in Fig. 18,a. We shall investigate this diagram in detail later. If the block B_1 is replaced by its threshold value λ and we separate from B_2 the quadratic terms proportional to k_{12}^2, we again arrive at a pure imaginary contribution to the amplitude. This follows from the fact that the quadratic terms in the amplitude of the scattering of two particles are always real (see formula (12)), so that the absorption part turns out to be real here, too, and the amplitude is pure imaginary. Let us attempt now to take into account the quadratic terms in B_1, by replacing the block B_2 with the amplitude of the scattering of particles at zero energy. We must bear in mind first that the block B_1 contains analytic and real terms of the type $C_3k_{12}^2$, $C_2k_{13}^2$, and $C_1k_{23}^2$, which are not connected with any definite class of diagrams. It is easily seen, however, that these terms give a pure imaginary contribution to the amplitude of the process. This statement is obvious for the term $C_3k_{12}^2$.

On the other hand, if we replace the block B_1, say, by $C_2k_{13}^2$ then the calculation of the diagram of Fig. 17 is analogous to the calculation carried out above for the diagram of Fig. 11 (the only difference is that we had there for the block B_1 the expression $ik_{13}\lambda a_{13}$). The absorption part in k_{12} is found to be real, and the contribution to the amplitude is pure imaginary.

Finally, we can retain in the block B_1 the nonanalytic quadratic terms or the pure imaginary terms that are proportional to iE and are connected with the diagrams of double scattering of the particles. We then arrive at the two sets of diagrams shown in Fig. 18, a and b. The first is readily seen to coincide with the diagram obtained by separating the linear terms in B_1 and B_2.

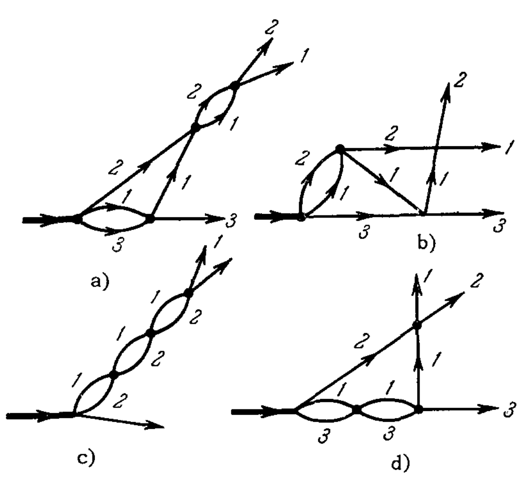


FIG. 18.

It is also useful to note that the diagrams shown in Fig. 18, c and d, make a pure imaginary contribution to the amplitude. The diagram of Fig. 18,c was in fact already discarded when we ascertained that retention of the terms linear in k_{12} in the two blocks B_1 and B_2 leads to an imaginary contribution to the amplitude, while the diagram of Fig. 18,d can be considered in perfect analogy with the contribution of the already discussed term $C_2k_{13}^2$.

So far we have always started out from diagrams connected with the scattering of a particle pair (see Fig. 17). We see readily that if we start the analysis with an arbitrary diagram that contains the transformation of three particles into three (Fig. 19), we again arrive at the diagrams of Fig. 18,a and b. In fact, inasmuch as the three-particle phase volume is equal in order of magnitude to E^2, the only possibility for obtaining a cubic term is the replacement of B_1 in Fig. 19 by a constant (λ) and the separation from the block B_2 of the contribution which behaves like $E^{-1/2}$ at small values of E. We can verify by direct calculation that this property is possessed by the amplitudes of transformation of three particles into three, shown in Fig. 20,a and b. There are no other "six-point diagrams" possessing the required property. Substitution of the diagrams shown in Fig. 20,a and b for the block B_2 in the diagram of Fig. 19 brings us to the diagrams of Fig. 18,a and b.

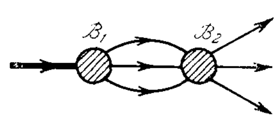


FIG. 19.

Thus we have shown that the real cubic terms of interest to us can be contained only in the diagrams of Fig. 18,a and b. We now calculate these cubic terms. Analysis of the diagram of Fig. 18,a, is perfectly elementary, since integration over the loop consisting of particles 1 and 2 is independent of the other integrations. This means that to obtain cubic terms we need here only multiply the second-order terms that depend on k_{12}^2 (and E) by the quantity $ik_{12}a_{12}$. For the contribution of the diagram 18,a we then obtain the following expression (see formula (41)):

$$-k_{12}a_{12}^2a_{13}2E\sqrt{\frac{m_1m_2m_3}{m_1+m_2+m_3}}\left[\frac{1}{6}(1+2\beta_1)+\frac{1}{3}(1-4\beta_1)x_{12}^2\right]. \quad (46)$$

The analysis of the diagram 18,b is somewhat more complicated. It is convenient to use for its calculation the dispersion relations in k_{12}^2 and E, in a manner similar to that used in the calculation of the quadratic terms.

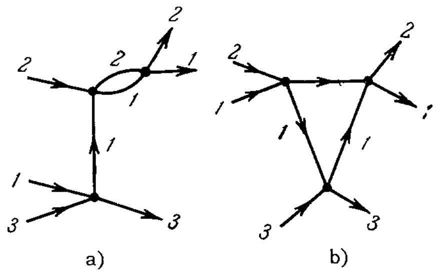


FIG. 20.

The absorption part in k_{12}^2 is determined by the cross section of diagram 18,b. Part of the diagram, situated to the left of this line, is the quantity $I_1(x_{13}, E)$ calculated in the preceding section. We can show that if we take the real parts of $I_1(x_{13}, E)$ then the corresponding contribution to the absorption part is real when $k_{12}^2 < 2\mu_{12}E$ and imaginary when $k_{12}^2 > 2\mu_{12}E$. On the other hand, the contribution to the amplitude itself turns out to be pure imaginary in this case. A check on the foregoing statement calls for rather cumbersome calculations; we shall not present even the final expression, since the explicit form of the obtained imaginary cubic terms is of no interest to us at all. On the other hand, if we separate from $I_1(x_{13}E)$ the pure imaginary terms, the final contribution to the amplitude turns out to be real. The calculation of the absorption part and of the dispersion integral is very similar here to the corresponding calculation for the quadratic terms. The final expression for the contribution of diagram 18,b takes the form

$$k_{12}a_{12}^2 a_{13} 2E \sqrt{\frac{m_1 m_2 m_3}{m_1+m_2+m_3}} \Big[-\frac{1}{6}(1+4\beta_1-8\beta_1^2) - \frac{1}{3}(1-4\beta_1)(1-2\beta_1)x_{12}^2 \Big]. \tag{47}$$

To obtain (47) we can either use the dispersion relation in k_{12}^2 with two subtractions, or make no subtractions at all but cut off the dispersion integral in k_{12}^2 at a certain value Λ^2. The diverging parts of the integral, which depend on Λ^2, turn out here to be pure imaginary and of no interest to us. The latter remark allows us to avoid calculations of subtraction constants that depend on the total energy in the dispersion relation in k_{12}^2. Direct calculation of these constants, similar to that made in the preceding section, also shows that they are pure imaginary.

The complete expression for the cubic terms in the amplitude is obtained by summing the contributions from six diagrams that differ in the permutation of particles of the type shown in Fig. 18,a, and twelve diagrams of the type shown in 18,b.

8. PRODUCTION OF THREE PARTICLES IN A STATE WITH UNITY TOTAL MOMENTUM

So far we have investigated the amplitude for the production of three particles with total angular momentum $L = 0$. In this section we consider the amplitude with $L = 1$, accurate to terms quadratic in the momenta.

The expansion of the amplitude of production of three particles in powers of amplitudes with different total angular momenta was carried out in Sec. 3. In the same section we introduced the notation for many of the quantities used in the present section, which we shall now employ without further elaboration.

The amplitude for the production of three particles with total angular momentum $L = 1$ is completely determined, accurate to terms quadratic in the momenta, by two terms of the series (16):

$$T_{10}(k_{12}^2, k_{13}^2, k_{23}^2)k_1 z_1 + T_{01}(k_{12}^2, k_{23}^2, k_{13}^2)k_2 z_2. \tag{48}$$

This follows from the fact that the amplitude with $L = 1$ is connected only with those T_{mn} for which $m+n$ is an odd number. On the other hand, the next terms of the series (16) with odd $m+n$, although contributing to the amplitude with $L = 1$, are obviously terms of higher order.

The terms linear in the momenta are obtained from (48) by replacing T_{10} and T_{01} by their threshold values. The quadratic terms result from the fact that the amplitudes T_{10} and T_{01} contain nonanalytic terms that are linear in k_{il}. These terms, naturally, result from the diagrams shown in Fig. 21. The purpose of the present section is in fact to calculate these linear corrections to T_{10} and T_{01}. We note first, however, that since $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$, the following relation holds

$$k_1 z_1 + k_2 z_2 + k_3 z_3 = 0. \tag{49}$$

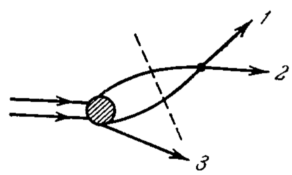


FIG. 21.

This means that expression (48) can be rewritten also in the form of a linear combination of $k_1 z_1$ and $k_3 z_3$ or of $k_2 z_2$ and $k_3 z_3$. For concreteness, we shall use the angles z_1 and z_2 and we shall show at the end of the section how to write down the answer in symmetrical form.

We denote the threshold values of the amplitudes T_{10} and T_{01} by α_1 and α_2, respectively. The terms linear, say, in k_{12} are then obtained by considering the diagram shown in Fig. 21, in which the block to the left of the dashed line should be replaced by $\alpha_1 k_1 z_1 + \alpha_2 k_2 z_2$.

The diagram of Fig. 21 can be calculated with the aid of the methods developed by us in the preceding section. The linear corrections to this diagram, which are of interest to us, are determined by the dispersion integral in k_{12}^2:

$$\frac{k_{12}^2}{\pi} \int_0^\infty dk_{12}'^2 \frac{B(k_{12}')}{k_{12}'^2 (k_{12}'^2 - k_{12}^2 - i\varepsilon)}, \tag{50a}$$

where the absorption part $B(k'_{12})$ is equal to

$$B(k'_{12}) = k'_{12}a_{12}\int_{-1}^{1}\frac{dz}{2}(\alpha_1 k'_1 z'_1 + \alpha_2 k'_2 z'_2). \tag{50b}$$

Here $\mathbf{k}'$ and $\mathbf{k}'_2$ are the momenta of the intermediate particles 1 and 2 in the c.m.s., z is the cosine of the angle between the relative momentum of particles 1 and 2 in the intermediate state ($\mathbf{k}'_{12}$) and the momentum of one of the incident particles ($\mathbf{P}$), z'_1 or z'_2 are the cosines of the angle between $\mathbf{P}$ and $\mathbf{k}'_1$ or $\mathbf{k}'_2$, respectively. Multiplying in scalar fashion $\mathbf{P}$ by the relation (6) between $\mathbf{k}'_1$, $\mathbf{k}'_2$ and $\mathbf{k}'_3$, $\mathbf{k}'_{12}$, we obtain directly

$$k'_1z'_1 = -\frac{m_1}{m_1+m_2}k_3z_3 + k'_{12}z, \qquad k'_2z'_2 = -\frac{m_2}{m_1+m_2}k_3z_3 - k'_{12}z, \tag{51}$$

which allows us to integrate directly with respect to z in (50b); we get

$$B(k'_{12}) = k_3z_3a_{12}k'_{12}\left(-\frac{m_1}{m_1+m_2}\alpha_1 - \frac{m_2}{m_1+m_2}\alpha_2\right). \tag{52}$$

From this it follows immediately that the integral in (50a) is equal to $iB(k_{12})$. By considering in similar fashion the corrections connected with the scattering of other pairs of particles, we obtain the following expression for the amplitude with L = 1, accurate to terms quadratic in the momenta:

$$\sum_M A_{1M}(k_{12}^2, k_{13}^2, k_{23}^2)Y_{1M}(\vartheta, \varphi) = k_1z_1\left[\alpha_1 + ik_{12}a_{12}\frac{m_1\alpha_1+m_2\alpha_2}{m_1+m_2} + ik_{23}a_{23}\left(\alpha_1 - \frac{m_2}{m_2+m_3}\alpha_2\right)\right] + k_2z_2\left[\alpha_2 + ik_{12}a_{12}\frac{m_1\alpha_1+m_2\alpha_2}{m_1+m_2} + ik_{13}a_{13}\left(\alpha_2 - \frac{m_1}{m_1+m_3}\alpha_1\right)\right]. \tag{53}$$

The structure of the amplitude for the production of three particles with L = 1 is thus similar to the structure of the amplitude with L = 0. The first terms of this amplitude are also determined by the unknown constants α_i (the amplitudes at threshold energy), while the correction terms are determined in terms of the same constants and in terms of the scattering lengths of the produced particles. The imaginary and real parts of the complex constant (just as the imaginary and real parts of the amplitude λ) are connected by the unitarity condition.

Expression (53) is asymmetrical in the indices 1, 2, and 3. This is connected with the special choice of the amplitude in the form (48). If we use from the very outset a symmetrical notation for the amplitude in the form

$$T_1k_1z_1 + T_2k_2z_2 + T_3k_3z_3, \tag{54}$$

then we readily obtain in place of (53) the following answer, which is symmetrical in the indices:

$$\sum_M A_{1M}(k_{12}^2, k_{13}^2, k_{23}^2)Y_{1M}(\vartheta\varphi) = k_1z_1\left[\beta_1 + ik_{23}a_{23}\left(\beta_1 - \frac{m_2}{m_2+m_3}\beta_2 - \frac{m_3}{m_2+m_3}\beta_3\right)\right] + k_2z_2\left[\beta_2 + ik_{13}a_{13}\left(\beta_2 - \frac{m_1}{m_1+m_3}\beta_1 - \frac{m_3}{m_1+m_3}\beta_3\right)\right] + k_3z_3\left[\beta_3 + ik_{12}a_{12}\left(\beta_3 - \frac{m_1}{m_1+m_2}\beta_1 - \frac{m_2}{m_1+m_2}\beta_2\right)\right], \tag{55}$$

where β_1, β_2 and β_3 are the threshold values of T_1, T_2 and T_3. Owing to the presence of the connection (49) between k_1z_1, k_2z_2 and k_3z_3 the functions T_i in (54) (and also β_i) are not uniquely determined. In fact, one of the quantities T_i is arbitrary; (53) corresponds to the choice $T_3 = 0$.

9. RESONANT INTERACTION OF PRODUCED PARTICLES

The entire analysis in the preceding sections pertains to the case of nonresonant interaction of produced particles, that is, to the case when the scattering lengths of the produced particles are of the order of the interaction radius $a_{il} \sim r_0$. If the amplitudes a_{il} are large ($a_{il} \gg r_0$), then there exists an energy region in which the condition $k_{il}r_0 \ll 1$ is satisfied, but nevertheless $k_{il}a_{il} \sim 1$. Let us consider the simplest case when only one of the paired scattering amplitudes is large, say a_{12}. From an examination of the preceding section it is clear that it is necessary in this case to take into account diagrams that contain an arbitrarily large number of scatterings of particles 1 and 2. Whereas earlier, in the zeroth approximation, the amplitude was determined by the single diagram of Fig. 22, to obtain the amplitudes in the zeroth approximation we must now sum all the diagrams shown in Fig. 22. This summation obviously is perfectly analogous to that carried out in Sec. 3 when considering the amplitude of two-particle scattering. As a result we obtain for the amplitude in the zeroth approximation the well-known Migdal-Watson formula

$$A = \lambda(1 - ik_{12}a_{12})^{-1}, \tag{56}$$

where λ is the amplitude of the process at zero energy.

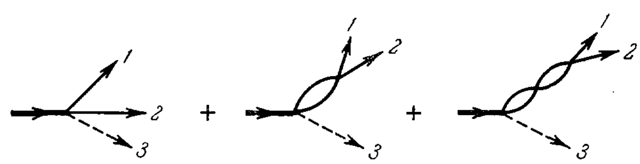


FIG. 22.

The corrections linear in the momenta to this expression can be of two types: corrections of the order of $k_{12}r_0$ connected with the effective radius of interaction of particles 1 and 2, and corrections of the order $k_{il}a_{13}$ and $k_{il}a_{23}$ connected with single scattering of the nonresonantly interacting particles. The corrections of the first type are accounted for in a manner similar to that used in the case of two-particle scattering. It can be shown that it is sufficient to make in (56) the substitutions

$$a_{12} \to a_{12}\left(1 - \frac{1}{2}k_{12}^2r_0a_{12}\right) \text{ and } \lambda \to \lambda\left(1 - \frac{1}{2}k_{12}^2r_0a_{12}\right).$$

A rigorous proof of the foregoing substitution can be obtained, for example, with the aid of a quantum-mechanical analysis [17]. The reason for the substitution $\lambda \to \lambda(1 - \frac{1}{2}k_{12}^2a_{12}r_0)$ lies in the fact that the three-particle wave functions pertaining to different values of k_{12} turn out to be proportional to one another in the region of action of the forces, with accuracy linear in r_0, provided only $a_{12} \gg r_0$. We can present also the following explanation. Diagrams which do not contain pair scatterings were replaced in the zeroth approximation by the constant λ. We are now interested essentially in the expansion of these diagrams in k_{12}^2. Let us consider, for example, the process $N + N \to N + N + \pi$. The other diagrams which do not contain pair scatterings of the produced particles include, for example, the diagram shown in Fig. 23. The block shown in the figure

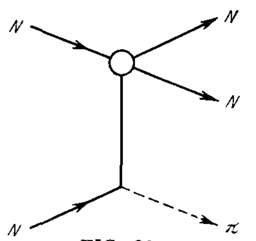


FIG. 23.

can be replaced, with accuracy linear in kr_0, by the quantity $a_{12}(1 - \frac{1}{2}k_{12}^2a_{12}r_0)$. From this we see that in order to take into account diagrams of the type shown in Fig. 23 the quantity λ must be replaced by $\lambda(1 - \frac{1}{2}k_{12}^2a_{12}r_0)$. On the other hand, the contribution of these diagrams to λ turns out be larger than the contribution of the other diagrams if the quantity a_{12} is large ($a_{12} \gg r_0$). Thus, we obtain the following correction to the Migdal-Watson formula:

$$A = \frac{\lambda\left(1 - \frac{1}{2}k_{12}^2 r_0 a_{12}\right)}{1 - ik_{12}a_{12}\left(1 - \frac{1}{2}k_{12}^2 r_0 a_{12}\right)}. \tag{57}$$

As already mentioned, other corrections linear in the momenta result from single scattering of the non-resonantly interacting particles. The situation reduces here to a calculation of a sum of the diagrams shown in Fig. 24, a and b. The diagrams of Fig. 24, b must also be multiplied by the quantity $(1 - ik_{12}a_{12})^{-1}$, since additional multiple scattering of particles 1 and 2 is possible in these diagrams. (We do not show the corresponding diagrams to save space.) The calculation of diagrams 24, a and b was carried out in [8,14]. The results obtained in these papers are quite complicated in form, and are therefore not presented here. We note, however, the following curious circumstance. When dealing with a reaction in which particles 1 and 2 are much heavier than particle 3, for example, the reaction $N + N \to N + N + \pi$, all the corrections connected with diagrams 24, a and b contain an additional small factor $\sqrt{\mu/M}$ (μ and M are the masses of the pion and the nucleon). Therefore, for reactions of this type, the corrections connected with the interaction radius turn out to be more significant and the simple formula (57) can be used in the analysis of these reactions.

We shall now deal briefly with the question of resonant interaction of all three particles, where all three pair amplitudes are much larger than r_0. In this case it is necessary to sum all the diagrams describing the scattering of the produced particles even in the zeroth approximation. For the sum of these diagrams we can obtain a certain integral equation [15,16], previously derived by Skornyakov and Ter-Martirosyan from a quantum-mechanical analysis of the problem [3]. We can also obtain in similar fashion equations in an approximation linear in kr_0 [17,18]. A detailed study of the question of resonant interaction of three particles is beyond the scope of the present review.

We should perhaps also note that the formulas derived in the preceding sections (where a nonresonant interaction of the produced particles was assumed) remain in force if $a_{il} \gg r_0$ but the total kinetic energy is so small that $k_{il}a_{il} \ll 1$. The only circumstances which must be borne in mind in this case is that when $a_{il} \gg r_0$ all the undetermined constants connected with the analytic terms are also expressed in terms of the pair scattering amplitudes. The reason for this is as follows. The analytic quadratic terms in (40) were essentially the result of cutting off the integrals with respect to the momenta at $k_{il}^2 \sim 1/r_0^2 \sim m^2$. In the resonance situation this cutoff occurs much lower, at $k_{il}^2 \sim 1/a_{il}^2$, since the blocks of the pair resonances contain factors of the type $(1 - ik_{22}a_{12})^{-1}$.

10. THE REACTIONS $\pi + N \to N + \pi + \pi$ AND $\gamma + N \to N + \pi + \pi$, AND $K \to 3\pi$-DECAY

We consider now certain real processes involving participation of three particles at low energies in the final state, and discuss the possibility of their experimental analysis from the point of view of the theory developed above. One of the first questions arising in this connection is to separate experimentally the states with definite total angular momenta. Although such a separation does not raise any special difficulty in principle, we wish to note one simple circumstance. The expression obtained for the cross section of the reaction with total angular momentum L = 0, accurate to quadratic terms (formula (42)), can be used directly for an analysis of the experimental data if the latter are averaged over the angles that determine the relative orientation of the plane of particle production and the direction of the incident beam. The complete expression for the cross section of the reaction contains both terms with L = 0 and terms corresponding to the angular momentum L = 1 (formula (53)). If we average the square of the modulus of the sum of the terms (40) and (53) over the angles ϑ and φ, then the interference terms, of course, drop out and the entire contribution of the terms with L = 1 to the average cross section turns out to be

$$\int |k_1z_1\alpha + k_2z_2\alpha_2|^2 \sin\vartheta \, d\varphi \, d\vartheta.$$

If we use the expressions (17) for z_1 and z_2, then we readily see that the foregoing integral contains terms of three types: k_1^2, k_2^2 and $\mathbf{k}_1 \cdot \mathbf{k}_2$. By virtue of the equality $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$ all these terms can be written in terms of a linear combination of k_{12}^2, k_{13}^2, and k_{23}^2. We see therefore that in the averaged cross section the terms with L = 1 lead only to a redetermination of the constants C_1, C_2 and C_3 in expression (40).

The theory developed in the preceding sections for neutral and spinless particles can be generalized to include the case of creation of real particles. It is sufficient then to bear in mind only that the vertices of the diagrams in question can correspond not only to elastic scattering of particles, but also to their charge exchange. Thus, for example, to calculate the linear terms in the reaction $\pi^- + p \to$

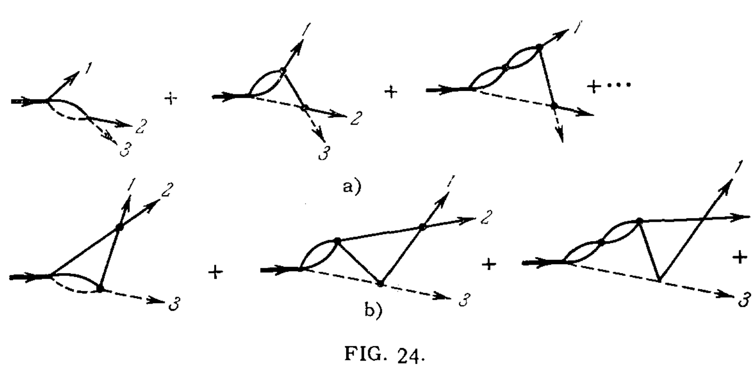


FIG. 24.

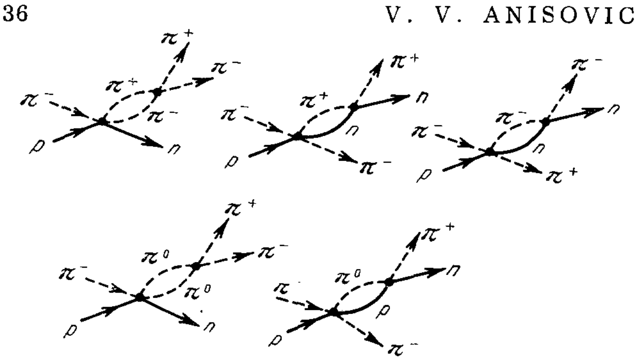


FIG. 25.

$n + \pi^+ + \pi^-$ it is necessary to consider all the diagrams shown in Fig. 25.

Whereas in the first three diagrams the vertices corresponding to the transformation of two particles into three contain the amplitude of the process $\pi^- + p \to n + \pi^+ + \pi^-$ at zero energy, the two other diagrams contain the amplitudes of the processes $\pi^- + p \to n + \pi^0 + \pi^0$ and $\pi^- + p \to p + \pi^- + \pi^0$. We now proceed to consider the concrete reactions.

a) The reaction $\pi + N \to N + \pi + \pi$. We consider first the reaction for the production of two pions when a pion collides with a proton:

$$\pi^- + p \to \pi^+ + \pi^- + n, \qquad (58a)$$
$$\pi^- + p \to \pi^0 + \pi^0 + n, \qquad (58b)$$
$$\pi^- + p \to \pi^- + \pi^0 + p. \qquad (58c)$$

We denote the amplitudes of these reactions at zero energy respectively by

$$\lambda_1 = \varrho_1 e^{i\varphi_1}, \quad \lambda_2 = \varrho_2 e^{i\varphi_2}, \quad \lambda_3 = \varrho_3 e^{i\varphi_3}$$

and put

$$\alpha_{ik} = \varrho_{ik} \sin \varphi_{ik}, \quad \beta_{ik} = \varrho_{ik} \cos \varphi_{ik}, \quad \varrho_{ik} = \varrho_k / \varrho_i, \quad \varphi_{ik} = \varphi_i - \varphi_k. \qquad (59)$$

If we introduce the amplitudes of scattering of pions at zero energy in states with total isotopic spin T = 0 – a_0 and T = 2 – a_2, normalized as the limit of the quantity $k^{-1}e^{i\delta} \sin \delta$, as $k \to 0$, then the different amplitudes for the scattering and charge exchange of the pions at zero energy are expressed in terms of a_0 and a_2 by means of the formula

$$a_{++}^{++} = a_{--}^{--} = 2a_2, \quad a_{+0}^{+0} = a_{-0}^{-0} = a_2, \quad a_{+-}^{+-} = \frac{2}{3} a_0 + \frac{1}{3} a_2 \equiv a_s,$$
$$a_{00}^{00} = \frac{2}{3} a_0 + \frac{4}{3} a_2 \equiv a_s^0, \quad a_{+-}^{00} = a_{00}^{+-} = \frac{2}{3}(a_2 - a_0) \equiv a_l. \qquad (60)$$

Here, for example, the quantity a_{++}^{++} denotes the amplitude for the scattering $\pi^+ + \pi^+ \to \pi^+ + \pi^+$, while a_{+-}^{00} denotes the amplitude of charge exchange $\pi^+ + \pi^- \to \pi^0 + \pi^0$, etc.

The normalization of the amplitudes is chosen such that the total cross section is $\sigma = 4\pi \mid a \mid^2$ for non-identical produced pions and $\sigma = 2\pi \mid a \mid^2$ for identical pions. Thus, for example,

$$\sigma_{+-}^{00} = 2\pi \mid a_l \mid^2, \qquad \sigma_{00}^{+-} = 4\pi \mid a_l \mid^2,$$
$$\sigma_{++}^{++} = 2\pi \mid 2a_2 \mid^2 = 8\pi \mid a_2 \mid^2, \qquad \sigma_{+0}^{+0} = 4\pi \mid a_2 \mid^2. \qquad (61)$$

The amplitudes for the scattering and charge exchange of a pion with a nucleon can also be readily expressed in terms of the isotopic amplitudes $b_{1/2}$ and $b_{3/2}$:

$$b_{p+}^{p+} = b_{n-}^{n-} = b_{3/2}, \quad b_{p-}^{p-} = b_{n+}^{n+} = \frac{1}{3} b_{3/2} + \frac{2}{3} b_{1/2} \equiv b_s,$$
$$b_{p0}^{p0} = b_{n0}^{n0} = \frac{1}{3} b_{1/2} + \frac{2}{3} b_{3/2} \equiv b_s^0, \quad b_{p-}^{n0} = b_{n+}^{p0} = \frac{1}{3}\sqrt{2}\,(b_{3/2} - b_{1/2}) \equiv b_l. \qquad (62)$$

Finally, we introduce numbering of the particles in the sequence in which they have been written out in the reactions (58). Then, for example, the quantity k_{12} for the first reaction will denote henceforth momentum of relative motion of the first and second particles, that is, of the π^+ and π^- mesons, k_{23} for the third reaction is the momentum of relative motion of π^0 and p, etc.

If we disregard the dependence of the cross sections of the reactions on the energy of the incident beam, that is, on the total energy of the produced particles, then the terms of the type $k_{il}^2 \times \ln(\mu/E)$ and $E \ln(\mu/E)$ are included in the terms of the type Ck_{il}^2. We then have for the squares of the matrix elements of the reactions (58), accurate to quadratic terms,

$$\left.\begin{aligned}
\frac{d\sigma(\pi^+\pi^-n)}{d\Gamma} &= \varrho_1^2 \{1 + k_{12}\alpha_{12}a_l + 2k_{13}\alpha_{13}b_l + \beta_1 [k_{12}k_{13} + I_1(x_{12}) \\
&\quad + I_1(x_{13})] + \beta_2 [k_{12}k_{23} + I_1(x_{12}) + I_1(x_{23})] + \beta_3 [k_{13}k_{23} + I_3(x_{13}) \\
&\quad + I_3(x_{23})] + \beta_4 [I_1(x_{12}) + I_1(x_{13})] + \beta_5 I_3(x_{13}) + C_1 k_{12}^2 + C_2 k_{13}^2\}, \\
\beta_1 &= 2\left(a_s + \frac{1}{2}\beta_{12}a_l\right)(b_s + \beta_{13}b_l) + \alpha_{12}\alpha_{13}a_l b_l, \\
\beta_2 &= 2\left(a_s + \frac{1}{2}\beta_{12}a_l\right)b_{3/2}, \quad \beta_3 = 2(b_s + \beta_{13}b_l)\,b_{3/2}, \\
\beta_4 &= 2a_l b_l \beta_{13} - a_l b_l (\beta_{12}\beta_{13} + \alpha_{12}\alpha_{13}), \quad \beta_5 = 2(b_l)^2(\beta_{12} - \sqrt{2}\beta_{13});
\end{aligned}\right\} \qquad (63a)$$

$$\left.\begin{aligned}
\frac{d\sigma(\pi^-\pi^0 n)}{d\Gamma} &= \varrho_2^2 \{1 + 2k_{12}\alpha_{21}a_l + 2(k_{13} + k_{23})\alpha_{23}b_l \\
&\quad + \gamma_1 [k_{12}(k_{13} + k_{23}) + 2I_1(x_{12}) + I_1(x_{13}) + I_1(x_{23})] \\
&\quad + \gamma_2 [k_{13}k_{23} + I_3(x_{13}) + I_3(x_{23})] + \gamma_3 [2I_1(x_{12}) + I_1(x_{13}) + I_1(x_{23})] \\
&\quad + \gamma_4 [I_3(x_{13}) + I_3(x_{23})] + D_1 k_{12}^2\}, \\
\gamma_1 &= 2\alpha_{21}\alpha_{23}a_l b_l + 2\left(\frac{1}{2}a_s^0 + \beta_{21}a_l\right)(b_s^0 + b_l\beta_{23}), \\
\gamma_2 &= 2(b_s^0 + \beta_{23}b_l)^2 + 2\alpha_{23}^2 b_l^2, \quad \gamma_4 = 2(b_l)^2(\beta_{21} - \beta_{23}^2) - 2(b_l)^2\alpha_{23}^2, \\
\gamma_3 &= -2(\alpha_{21}\alpha_{23} + \beta_{21}\beta_{23})a_l b_l + \beta_{23}a_l b_l;
\end{aligned}\right\} \qquad (63b)$$

$$\left.\begin{aligned}
\frac{d\sigma(\pi^-\pi^0 p)}{d\Gamma} &= \varrho_3^2 \{1 + 2k_{13}\alpha_{32}b_l + 2k_{23}\alpha_{31}b_l + \delta_1 [k_{12}k_{13} \\
&\quad + I_1(x_{12}) + I_1(x_{13})] + \delta_2 [k_{12}k_{23} + I_1(x_{12}) + I_1(x_{23})] \\
&\quad + \delta_3 [k_{13}k_{23} + I_3(x_{13}) + I_3(x_{23})] + \delta_4 [I_1(x_{13}) - I_1(x_{23})] \\
&\quad + \delta_5 I_3(x_{13}) + \delta_6 I_3(x_{23}) + F_1 k_{12}^2 + F_2 k_{13}^2\}, \\
\delta_1 &= 2a_2(b_s + \beta_{32}b_l), \quad \delta_2 = 2a_2(b_s^0 + \beta_{31}b_l), \\
\delta_3 &= 2(b_s + \beta_{32}b_l)(b_s^0 + \beta_{31}b_l) + 2\alpha_{31}\alpha_{32}(b_l)^2, \\
\delta_4 &= 2a_l b_l\left(\beta_{31} - \frac{1}{2}\beta_{32}\right), \quad \delta_5 = 2(b_l)^2[1 - \alpha_{31}\alpha_{32} - \beta_{31}\beta_{32}], \\
\delta_6 &= -2(b_l)^2[\alpha_{31}\alpha_{32} + \beta_{31}\beta_{32}] + 2\beta_{31}(b_l)^2\sqrt{2}.
\end{aligned}\right\} \qquad (63c)$$

The functions I_1 and I_3 are defined as follows:

$$I_1(x) = -2E\sqrt{\frac{M}{M+2}}\frac{x}{\sqrt{1-x^2}}\left[\frac{M+2}{2(M+1)} - x^2\frac{M+3}{3(M+1)}\right]\frac{2}{\pi}\arccos x,$$

$$I_3(x) = -2E\sqrt{\frac{M}{M+2}}\frac{x}{\sqrt{1-x^2}}\left[\frac{M(M+2)}{(M+1)^2} - x^2\frac{3M^2+6M-1}{3(M+1)^2}\right]\frac{2}{\pi}\arccos x,$$

$$x_{12} = k_{12}/\sqrt{E}, \quad x_{13} = k_{13}\sqrt{(M+1)/2ME}, \quad x_{23} = k_{23}\sqrt{(M+1)/2ME}. \qquad (64)$$

Here E is the total kinetic energy of the three particles and M is the nucleon mass; the pion mass is set equal to zero.

Formula (63) includes terms of the type const · k_{il}^2. Here, however, there is no longer any need to include Ck_{23}^2 in (63a) and (63c), since k_{23}^2 is expressed in terms of k_{12}^2, k_{13}^2 and E, and the dependence of the total energy will not be written out in these formulas. In expression (63b) there is no need to write out both squares k_{23}^2 and k_{13}^2, since it contains only their sum which is expressed in terms of k_{12}^2 and E.

Let us indicate immediately how to modify (63) in order to separate also the dependence on the total energy. In all three formulas it is necessary to add the term const · E and to make the substitutions

$$I_1(x) \to I_1(x) + K_1(x), \qquad I_3(x) \to I_3(x) + K_3(x),$$

where

$$K_1(x) = -\frac{2E}{\pi}\ln\left(\frac{\mu}{E}\right)\sqrt{\frac{M}{M+2}}\left[\frac{-2M-3}{6(M+1)} + \frac{M+3}{3(M+1)}x^2\right],$$
$$K_3(x) = -\frac{2E}{\pi}\ln\left(\frac{\mu}{E}\right)\sqrt{\frac{M}{M+2}}\left[\frac{-(3M^2+6M+1)}{6(M+1)^2}\right.$$
$$\left.+\frac{3M^2+6M-1}{3(M+1)^2}x^2\right]. \qquad (65)$$

Expressions of the type $k_{12}k_{13} + I_1(x_{12}) + I_1(x_{13})$ which have been separated out in formulas (63), behave at small values of k_{12} and k_{13} like k_{12}^2 and k_{13}^2. It is more difficult to distinguish them experimentally from the terms $C_1k_{12}^2$ and $C_1k_{13}^2$ than the terms that follow them. Therefore, for example, the determination of the coefficients β_1, β_2, and β_3 is more complicated than that of β_4 and β_5. The latter, however, are always proportional to the product of charge-exchange amplitudes, and therefore (like the linear terms) they yield information not regarding the two pion scattering lengths but only regarding the combination $a_2 - a_0$.

The quantities α_{ik} and β_{ik} in (63) can be expressed, owing to the unitarity condition, in terms of the πN scattering phase shifts δ_{11} and δ_{31} in the states $p_{1/2}$ with isotopic spin 1/2 and 3/2 at an energy corresponding to the threshold of production of two pions.

If we write out the isotopically invariant matrix elements for production of two pions in states with T = 0 (total isotopic spin 1/2) and with T = 2 (total spin 3/2) in the form

$$\left\langle \frac{1}{2}\,0\,|\,S\,|\,\frac{1}{2}\right\rangle = F_{11}e^{i\delta_{11}}, \qquad \left\langle \frac{3}{2}\,2\,|\,S\,|\,\frac{3}{2}\right\rangle = F_{31}e^{i\delta_{31}}, \qquad (66)$$

then it can be readily shown that

$$\left.\begin{aligned} \lambda_1 &= -\frac{\sqrt{2}}{3}F_{11}e^{i\delta_{11}} + \frac{1}{3\sqrt{5}}F_{31}e^{i\delta_{31}},\\ \lambda_2 &= \frac{\sqrt{2}}{3}F_{11}e^{i\delta_{11}} + \frac{2}{3\sqrt{5}}F_{31}e^{i\delta_{31}},\\ \lambda_3 &= -\frac{1}{\sqrt{10}}F_{31}e^{i\delta_{31}}. \end{aligned}\right\} \qquad (67)$$

From this we readily find, for example, that

$$\alpha_{12} = \frac{3\sin(\delta_{31}-\delta_{11})}{x\sqrt{10} + 1/x\sqrt{10} - 2\cos(\delta_{31}-\delta_{11})},$$

$$\beta_{12} = \frac{(2/x\sqrt{10}) - x\sqrt{10} - \cos(\delta_{31}-\delta_{11})}{x\sqrt{10} + 1/x\sqrt{10} - 2\cos(\delta_{31}-\delta_{11})}, \qquad (68)$$

$x = F_{11}/F_{31}$, and also the connection between α_{12} and α_{13} or β_{12} and β_{13}:

$$\alpha_{12} = -\sqrt{2}\,\alpha_{13}, \qquad 1 + \beta_{12} = -\sqrt{2}\,\beta_{13}. \qquad (69)$$

If the kinetic energy of the produced particles is particularly low, we confine ourselves in formula (63) only to the terms that are linear in the momenta. Then the analysis of the experimental data becomes much simpler. If we represent the experimental dependence of the cross section, say, of the reaction $\pi^- + p \to \pi^+ + \pi^- + n$ on k_{12} and k_{13} in the form

$$\frac{d\sigma}{dk_{12}^2dk_{13}^2} = \text{const}\cdot[1 + Ak_{12} + Bk_{13}], \qquad (70)$$

then, as seen from (63a) and (69), the coefficient ratio, A/B is determined only by the amplitudes of the charge exchange of the pions with one another and of the pions with the nucleon:

$$\frac{A}{B} = -\frac{a_l}{\sqrt{2}\,b_l} = -\frac{a_2 - a_0}{b_{3/2} - b_{1/2}}. \qquad (71)$$

Inasmuch as $b_{1/2}$ and $b_{3/2}$ are known, an experimental analysis of the reaction $\pi^- + p \to n + \pi^+ + \pi^-$ makes it possible in this manner to determine the amplitude of the charge exchange of pions at zero energy. The corresponding experiment is presently under way in Dubna. The preliminary value of the amplitude for the charge exchange of pions turned out to be [23]

$$a_2 - a_0 = (-0.25 \pm 0.05)\frac{\hbar}{\mu c}.$$

We proceed to consider the production of a pion in collisions between a π^+ meson and a proton. In this case two reactions are possible:

$$\pi^+ + p \to \pi^+ + \pi^+ + n, \qquad (72a)$$

$$\pi^+ + p \to \pi^+ + \pi^0 + p. \qquad (72b)$$

Both amplitudes (72) can be readily expressed at zero energy in terms of the previously introduced matrix elements for the production of a pion in a state with total isotopic spin 3/2 (the pion spin is 2). We have

$$\lambda_1 = \frac{2}{\sqrt{5}}e^{i\delta_{31}}F_{31}, \qquad \lambda_2 = -\frac{1}{\sqrt{10}}e^{i\delta_{31}}F_{31}. \qquad (73)$$

Since these amplitudes do not have a relative phase shift, no terms linear in k_{il} will enter in the expression for the cross section. On the other hand, the expression for the cross section depends only on the amplitude of $\pi\pi$ and πN scattering at zero energy. We write out the result, expressed in terms of the isotopic amplitude a_2 and $b_{1/2}$, $b_{3/2}$:

$$\left.\begin{aligned} \frac{d\sigma(\pi^+\pi^+n)}{d\Gamma} &= \frac{4}{5}|F_{31}|^2\{1 + \beta_1[k_{12}(k_{13}+k_{23}) + 2I_1(x_{12}) + I_1(x_{13}) + I_1(x_{23})]\\ &\quad + \beta_2[k_{13}k_{23} + I_3(x_{13}) + I_3(x_{23})] + \beta_3[I_3(x_{13}) + I_3(x_{23})] + C_1k_{12}^2\},\\ \beta_1 &= 2a_2\left(\frac{1}{6}b_{3/2} + \frac{5}{6}b_{1/2}\right), \qquad \beta_2 = 2\left(\frac{1}{6}b_{3/2} + \frac{5}{6}b_{1/2}\right)^2,\\ \beta_3 &= -\frac{5}{18}(b_{3/2} - b_{1/2})^2; \end{aligned}\right\} \qquad (74a)$$

$$\left.\begin{aligned} \frac{d\sigma(\pi^+\pi^0p)}{d\Gamma} &= \frac{1}{10}|F_{31}|^2\{1 + \gamma_1[k_{12}k_{13} + I_1(x_{12}) + I_1(x_{13})]\\ &\quad + \gamma_2[k_{12}k_{23} + I_1(x_{12}) + I_1(x_{23})] + \gamma_3[k_{13}k_{23} + I_3(x_{13})\\ &\quad + I_3(x_{23})] + \gamma_4I_3(x_{23}) + D_1k_{12}^2 + D_2k_{13}^2\},\\ \gamma_1 &= 2a_2b_{3/2}, \qquad \gamma_2 = 2a_2\left(\frac{5}{3}b_{1/2} - \frac{2}{3}b_{3/2}\right),\\ \gamma_3 &= 2b_{3/2}\left(\frac{5}{3}b_{1/2} - \frac{2}{3}b_{3/2}\right), \qquad \gamma_4 = \frac{20}{9}(b_{3/2} - b_{1/2})^2. \end{aligned}\right\} \qquad (74b)$$

The functions I_1 and I_3 are defined as before by means of formulas (64), while D_1 and D_2 are unknown constants. If we are interested in the energy dependence of the cross sections, then we must again add to the right side of (74) a term proportional to E, with an undetermined coefficient, and make the substitution (65). Of the two $\pi\pi$ scattering amplitudes, only a_2, of course, enters in (74).

Finally, let us write out the expressions for the total cross sections of the reaction (58), retaining terms of order $E^{1/2}$ and $E\times\ln(\mu/E)$ compared with unity. The terms proportional to E include already the undetermined constants and are not expressed in terms of the scattering amplitudes. For the reactions occurring in π^- meson–proton collisions we have

$$\left.\begin{aligned}
&\sigma(\pi^+\pi^-n)=\varrho_1^2E^2(1+A_1\sqrt{E}+B_1E\ln(\mu/E)),\\
&\sigma(\pi^0\pi^0n)=\frac{1}{2}\varrho_2^2E^2(1+A_2\sqrt{E}+B_2E\ln(\mu/E)),\\
&\sigma(\pi^-\pi^0p)=\varrho_3^2E^2(1+A_3\sqrt{3}+B_3E\ln(\mu/E));\\
&\quad A_1=\frac{32}{15\pi}\left(\alpha_{12}a_l+2\sqrt{\frac{2M}{M+1}}\alpha_{13}b_l\right),\\
&\quad A_2=\frac{32}{15\pi}\left(2\alpha_{21}a_l+4\sqrt{\frac{2M}{M+1}}\alpha_{23}b_l\right),\\
&\quad A_3=\frac{32}{15\pi}\left(2\alpha_{32}b_l+2\sqrt{\frac{2M}{M+1}}\alpha_{31}b_l\right),\\
&B_1=\frac{4}{3\pi}\sqrt{\frac{M}{M+2}}\frac{M}{M+1}[2a_sb_s^0+\beta_{12}a_lb_s^0+\beta_{13}a_2b_l]\\
&\qquad+\frac{8}{3\pi}\sqrt{\frac{M}{M+2}}\frac{1}{(M+1)^2}\left[b_sb_{3/2}+\frac{1}{2}\beta_{12}(b_l)^2+\beta_{13}b_s^0b_l\right],\\
&B_2=\frac{4}{3\pi}\sqrt{\frac{M}{M+2}}\frac{M}{M+1}[a_s^0b_s^0+2\beta_{21}a_lb_s^0+2\beta_{23}a_2b_l]\\
&\qquad+\frac{8}{3\pi}\sqrt{\frac{M}{M+2}}\frac{1}{(M+1)^2}[(b_s^0)^2+\beta_{21}(b_l)^2+2\beta_{23}b_s^0b_l],\\
&B_3=\frac{4}{3\pi}\sqrt{\frac{M}{M+2}}\frac{M}{M+1}[a_2(b_s^0+b_s)+\beta_{31}a_2b_l+\beta_{32}a_2b_l]\\
&\qquad+\frac{8}{3\pi}\sqrt{\frac{M}{M+2}}\frac{1}{(M+1)^2}\left[b_sb_s^0+\frac{1}{2}(b_l)^2+\beta_{31}b_s^0b_l+\beta_{32}b_s^0b_l\right].
\end{aligned}\right\} \tag{75}$$

In collisions between a π^+ meson and a proton the total cross sections take the form

$$\left.\begin{aligned}
&\sigma(\pi^+\pi^+n)=\frac{2}{5}|F_{31}|^2E^2\left(1+BE\ln\frac{\mu}{E}\right),\\
&\sigma(\pi^+\pi^0p)=\frac{1}{10}|F_{31}|^2E^2\left(1+B'E\ln\frac{\mu}{E}\right),\\
&\quad B=\frac{4}{3\pi}\sqrt{\frac{M}{M+2}}\frac{M}{M+1}\left[2a_2\left(\frac{1}{6}b_{3/2}+\frac{5}{6}b_{1/2}\right)\right]\\
&\qquad+\frac{8}{3\pi}\sqrt{\frac{M}{M+2}}\frac{1}{(M+1)^2}\left[\frac{1}{9}(5b_{1/2}^2+5b_{1/2}b_{3/2}-b_{3/2}^2)\right],\\
&\quad B'=B.
\end{aligned}\right\} \tag{76}$$

It is easy to show that the equality of the coefficients B and B' is a simple consequence of isotopic invariance.

Inasmuch as for the reactions (72) the expressions for the cross section do not contain linear terms, it is meaningful to calculate for them the terms cubic in the momenta, which serve as the first correction to the quadratic term. The amplitudes of the reactions (72a) and (72b) have no relative phase shift, and therefore the situation with the cubic terms turns out to be very similar to the neutral case considered in Sec. 7. An explicit expression for the cubic terms in the cross sections of reactions (72a) and (72b) was obtained in [16] and will not be presented here, to save space.

b) The reactions $\gamma + p \to N + \pi + \pi$. We now consider the reaction for the photoproduction of two pions:

$$\gamma+p\to\pi^-+\pi^++p, \tag{77a}$$

$$\gamma+p\to\pi^0+\pi^0+p, \tag{77b}$$

$$\gamma+p\to\pi^++\pi^0+n. \tag{77c}$$

It is easy to note that since the charge states of the produced particles are obtained by substituting all the projections of the isotopic spin of the final states in reactions (58), formulas (63) and (65) will be applicable also in this case if we adhere to the numbering of the particles in the same sequence as they are written out in reaction (77). Now, for example, the momentum k_{13} for the reaction (77a) is the momentum of relative motion of the pion and the proton, the momentum k_{23} in the third reaction pertains to the motion of the pion relative to the neutron, etc.

The amplitudes for photoproduction at zero energy λ_i have, of course, nothing in common with the amplitudes of the reaction $\pi + N \to N + \pi + \pi$. They can be expressed in terms of the matrix elements for photoproduction in states with total isotopic spin 1/2 and 3/2, namely, $G_{11}e^{i\alpha_{11}}$ and $G_{31}e^{i\alpha_{31}}$, for a total angular momentum 1/2:

$$\lambda_1=\frac{1}{\sqrt{3}}G_{11}e^{i\alpha_{11}}-\frac{1}{\sqrt{15}}G_{31}e^{i\alpha_{31}},$$

$$\lambda_2=-\frac{1}{\sqrt{3}}G_{11}e^{i\alpha_{11}}-\frac{2}{\sqrt{15}}G_{31}e^{i\alpha_{31}},\qquad \lambda_3=\sqrt{\frac{3}{10}}G_{31}e^{i\alpha_{31}},$$

from which we get in this case

$$\alpha_{12}=\frac{3\sin(\alpha_{31}-\alpha_{11})}{y\sqrt{5}+1/y\sqrt{5}-2\cos(\alpha_{31}-\alpha_{11})},$$
$$\beta_{12}=\frac{2/y\sqrt{5}-y\sqrt{5}-\cos(\alpha_{31}-\alpha_{11})}{y\sqrt{5}+1/y\sqrt{5}-\cos(\alpha_{31}-\alpha_{11})}. \tag{78}$$

The phases α_{11} and α_{31} can be related with the aid of the unitarity condition to other processes. It is easy to note that relation (69) remains in force between the quantities α_{12}, α_{13} and β_{12}, β_{13}.

c) The decay $K^+ \to 3\pi$. The expression for the probabilities of the decay $K^+ \to 3\pi$ was obtained, accurate to terms quadratic in the momenta, by Gribov [9]. Since the rule $\Delta T = 1/2$, is satisfied in K^+ decay, the produced pions can only be in states with unity total isotopic spin. Taking this into account, we can readily rewrite the formulas of [11] for the probabilties of the decays $K^+ \to 2\pi^+ + \pi^-$ and $K^+ \to 2\pi^0 + \pi^+$ in the following form:

$$\begin{aligned}
\frac{dW}{d\Gamma}=4\lambda^2\Big\{&1+\beta_1[k_{12}(k_{13}+k_{23})+2I(x_{12})+I(x_{13})+I(x_{23})]\\
&+\beta_2[k_{13}k_{23}+I(x_{13})+I(x_{23})]+\beta_3[I(x_{13})+I(x_{23})]\\
&+2\alpha k_{12}^2+\left[\alpha+\delta-\frac{25}{36}(a_0-a_2)^2\right](k_{13}^2+k_{23}^2)\Big\},
\end{aligned} \tag{79a}$$

$$\begin{aligned}
\frac{dW'}{d\Gamma}=\lambda^2\{&1+\gamma_1[k_{12}(k_{13}+k_{23})+2I(x_{12})+I(x_{13})+I(x_{23})]\\
&+\gamma_2[k_{13}k_{23}+I(x_{13})+I(x_{23})]+2\gamma_3I(x_{12})+2\delta k_{12}^2+2\alpha(k_{13}^2+k_{23}^2)\}.
\end{aligned} \tag{79b}$$

Here k_{il} are the momenta of relative motion of the i-th and l-th pions. The indices 1 and 2 pertain to identical particles

$$I(x)=-\frac{\sqrt{3}}{\pi}\varkappa^2\frac{x\arccos x}{(1-x^2)^{1/2}}\left(1-\frac{8}{9}x^2\right),$$
$$x=k/\varkappa,\qquad \varkappa^2=M_K-3\approx 0.56; \tag{80}$$

M_K is the mass of the K meson, the pion mass is equal to unity, and λ, α, and δ are real numbers determined from experiment; β and γ_i are numbers expressed in terms of the pion scattering lengths a_0 and a_2:

$$\left.\begin{aligned}
&\beta_1=\frac{5}{3}a_0a_2+\frac{1}{3}a_2^2, && \gamma_1=\frac{10}{3}a_0a_2-\frac{4}{3}a_2^2,\\
&\beta_2=\frac{1}{18}(5a_0+a_2)^2, && \gamma_2=2a_2^2,\\
&\beta_3=-\frac{5}{18}(a_0-a_2)^2, && \gamma_3=\frac{10}{9}(a_0-a_2)^2.
\end{aligned}\right\} \tag{81}$$

Using formula (79a) and (79b) we can obtain expressions for the total probabilities of the processes $K^+ \to 2\pi^+ + \pi^-$ and $K^+ \to 2\pi^0 + \pi^+$:

$$W = 4\lambda^2\Omega\{1 + \varkappa^2[2\alpha + \delta - 0.71(a_0 - a_2)^2 + 0.04\beta_1 + 0.02\beta_2 - 0.42\beta_3]\}, \quad (82a)$$

$$W' = \lambda^2\Omega'\{1 + \varkappa^2[2\alpha + \delta + 0.04\gamma_1 + 0.02\gamma_2 - 0.42\gamma_3]\}; \quad (82b)$$

Ω and Ω' are the phase volumes of the produced pions. Since $2\mu_{\pi^+} - 2\mu_{\pi^0} \approx 9.2$ MeV, Ω' is somewhat larger than Ω.

The ratio of the decay probabilities is

$$W'/W = 0.315[1 + 0.05(a_0 - a_2)^2]. \quad (83)$$

In this formula, terms of the order of unity take into account the difference in the phase volumes of the produced pions [24].

From formulas (79a) and (79b) we can also obtain the energy distributions of the produced pions:

$$W_{\pi^-} = 1 + \varkappa^2[\beta_1 F_1(x) + \beta_2 F_2(x) + \beta_3 F_3(x) + \frac{25}{36}(a_0 - a_2)^2 x^2 + (\alpha - \delta)x^2], \quad (84a)$$

$$W_{\pi^+} = 1 + \varkappa^2\Big[\beta_1\left(\frac{1}{2}F_1(x) + F_2(x)\right) + \frac{1}{2}\beta_2 F_1(x) + \beta_3\left(I(x) + \frac{1}{2}F_3(x)\right) - \frac{25}{72}(a_0 - a_2)^2 x^2 - \frac{1}{2}(\alpha - \delta)x^2\Big], \quad (84b)$$

$$W'_{\pi^+} = 1 + \varkappa^2[\gamma_1 F_1(x) + \gamma_2 F_2(x) + 2\gamma_3 I(x) - 2(\alpha - \delta)x^2]; \quad (84c)$$

the quantity x^2 is connected with the energy E of the meson under consideration in the following manner: $\kappa^2 = \kappa^2 x^2 + 3E/2$.

In formulas (84) we can leave out all the constant terms of order κ^2, since after including them in the normalizing factor they affect only terms of order κ^4. The functions of $F_1(x)$, $F_2(x)$, and $F_3(x)$ are relatively easy to calculate, and it turns out that they can be well approximated in the region $0 < x < 1$ by the following polynomials of x^2:

$$F_1(x) = -0.33 + 0.74x^2, \qquad F_2(x) = 0.42 - 0.74x^2,$$
$$F_3(x) = -0.33 - 0.16x^2. \quad (85)$$

If we introduce $\epsilon = E/E_{max}$ then the formulas (84) can be rewritten in the form

$$W_{\pi^-} = 1 + (\varepsilon - 1/2)[0.2a_0^2 + 0.4a_0a_2 - 0.5a_2^2 - 0.5(\alpha - \delta)]; \quad (86a)$$

$$W_{\pi^+} = 1 + (\varepsilon - 1/2)[-0.1a_0^2 - 0.2a_0a_2 + 0.2a_2^2 + 0.25(\alpha - \delta)] - 0.2(a_0 - a_2)^2[I(\sqrt{1-\varepsilon}) + 0.1]. \quad (86b)$$

$$W'_{\pi^+} = 1 + (\varepsilon - 1/2)[-1.4a_2a_0 + 1.4a_2^2 + 1.2(\alpha - \delta)] + 1.3(a_0 - a_2)^2[I(\sqrt{1-\varepsilon}) + 0.1]. \quad (86c)$$

The energy spectra in the reactions under consideration were measured by several workers [25]. Experiment yields

$$W_{\pi^-} = 1 + (\varepsilon - 1/2)(0.53 \pm 0.07), \qquad W_{\pi^+} = 1 - (\varepsilon - 1/2)(0.26 \pm 0.09),$$
$$W'_{\pi^+} = 1 - (\varepsilon - 1/2)(1.0 \pm 0.4). \quad (87)$$

Formulas (86) contain one constant $\alpha - \delta$ which is not known beforehand, and therefore an investigation of the energy spectrum of one of the pions cannot yield any information concerning a_0 and a_2. Since $|I(\sqrt{(1-\epsilon)}) + 0.1| \leq 0.1$ when $0 < \epsilon < 1$, the last term in (86b) is of the order of 10^{-2}. We therefore have approximately $(W_{\pi^-} - 1)/(W_{\pi^+} - 1) \approx -2$, which agrees with experiment. To obtain information concerning a_0 and a_2 from the energy distributions, it is necessary to study both the reaction $K^+ \to 2\pi^+ + \pi^-$ and the reaction $K^+ \to 2\pi^0 + \pi^+$.

Experimental data on the energy distribution of π^+ mesons in the reaction $K^+ \to 2\pi^0 + \pi^+$ are quite crude. They merely allow us to estimate the combination of the quantities a_0 and a_2. The last term in (86c) is of the order of 0.1. Neglecting this term, we can verify that comparison of (86) and (87) yields $a_0^2 - a_0a_2 + 0.5a_2^2 \approx 0.7 \pm 1$.

Using formulas (79), we can also obtain the distribution with respect to z (difference in energy between the identical mesons, divided by its maximum value $z = \sqrt{3}\kappa^{-2}(E_1 - E_2)$). For the reaction $K^+ \to 2\pi^+ + \pi^-$ this distribution is of the form

$$W(z) = 1 + z^2(0.04a_0^2 + 0.4a_0a_2). \quad (88)$$

Experiment yields [25]

$$W(z) = 1 + z^2(0.0 \pm 0.1). \quad (89)$$

It follows therefore that $|a_0^2 + 10a_0a_2| \lesssim 2.5$. If a_0 is not much larger than a_2, then $|a_0a_2| \lesssim 0.25$.

Thus, we see from the foregoing analysis that the experimental data on $K^+ \to 3\pi$ decay do not contradict the formulas derived under the assumption that a_0 and a_2 are small. However, the experimental accuracy does not allow us yet to determine a_0 and a_2 from an analysis of $K^+ \to 3\pi$ decay.

The terms that are cubic in the momenta of the produced particles were calculated in [11]. They turn out to be very small, and it is presently meaningless to compare them with experimental data.

APPENDIX

In this appendix we present a simple derivation of the Landau rules [26] for finding the singularities of Feynman diagrams. For concreteness we shall consider the triangular diagram shown in Fig. 26, although the analysis that follows has a general character.

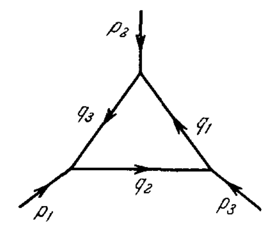


FIG. 26.

Let the momenta of the external particles be p_1, p_2 and p_3 and let their masses be m_1, m_2 and m_3, and by definition $p_i^2 = m_i^2$ (i = 1, 2, 3). From the momentum conservation law we have $p_1 + p_2 + p_3 = 0$. The momenta of the internal (virtual) particles will be denoted by q_1, q_2, q_3, and their masses by μ_1, μ_2, μ_3. Generally speaking $q_i^2 \neq \mu_i^2$.

The Feynman integral for the diagram in Fig. 26 is, apart from a factor,

$$\int \frac{d^4q_1}{(q_1^2 - \mu_1^2)(q_2^2 - \mu_2^2)(q_3^2 - \mu_3^2)}, \qquad q_2 = q_1 - p_3, \; q_3 = q_1 + p_2. \quad (A.1)$$

We consider first integration with respect to dq_{10}. The integrand has in the complex q_{10} plane poles at

$$\left.\begin{aligned} q_{10} &= \pm\sqrt{\mathbf{q}_1^2 + \mu_1^2} \mp i\delta, \\ q_{20} &= \pm\sqrt{\mathbf{q}_2^2 + \mu_2^2} \mp i\delta, \quad \text{i.e.} \quad q_{10} = p_{30} \pm \sqrt{(\mathbf{q}_1 - \mathbf{p}_3)^2 + \mu_2^2} \mp i\delta, \\ q_{30} &= \pm\sqrt{\mathbf{q}_3^2 + \mu_3^2} \mp i\delta, \quad \text{i.e.} \quad q_{10} = -p_{20} \pm \sqrt{(\mathbf{q}_1 + \mathbf{p}_2)^2 + \mu_3^2} \mp i\delta. \end{aligned}\right\} \quad (A.2)$$

One of the possible arrangements of these poles is shown in Fig. 27. The dots, crosses, and circles denote respectively poles written out in the first, second, and third lines of (A.2). It is obvious that the integral in (A.1) can have a singularity only when the poles in the q_{10} plane of the integrand expression compress the integration contour, for example, when the position of the cross located above the real axis coincides (apart from the sign of an infinitesimally small imaginary part) with the position of the circle

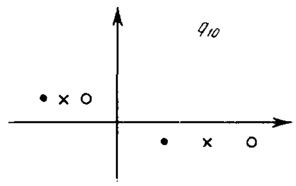


FIG. 27.

located below the real axis. Thus, the integral (A.1) has a singularity only when at least two or all three denominators in (A.1) vanish simultaneously:

$$q_i^2=\mu_i^2 \qquad (i=1,\ 2,\quad \text{or}\quad i=1,\ 3,\quad \text{or}\quad i=2,\ 3) \tag{A.3}$$

or else

$$q_i^2=\mu_i^2 \qquad (i=1,\ 2,\ 3). \tag{A.4}$$

The conditions (A.3) constitute essentially the conditions for determining the singularities of the simpler diagrams, in which one of the internal lines contracts to a point (as, for example, in Fig. 28). This can be immediately verified by writing out the expression for such a simplified diagram. We shall now stop to discuss only the singularities connected with the condition (A.4). Using a reasoning which is perfectly analogous to that which follows, we can easily show that the conditions (A.3) lead to simple threshold singularities $p_1^2 = (\mu_3 + \mu_2)^2$, $p_2^2 = (\mu_1 + \mu_3)^2$, $p_3^2 = (\mu_2 + \mu_1)^2$.

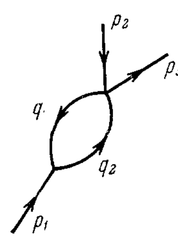


FIG. 28.

Conditions (A.4) are certainly not sufficient for appearance of a singularity, in fact, it is clear from the foregoing that it is also necessary that when the singularities of the integrand coincide, some be located above and some below the real axis. Since the sign of the infinitesimally small imaginary increment is opposite to the sign of q_{i_0} (see (A.2)), it is necessary for this purpose that one of the three quantities q_{10}, q_{20}, or q_{30} have a sign opposite to the other two. In the more general case, when the large number $q_i^2 - \mu_i^2$ vanishes, it is required that some of the quantities q_{i_0} have opposite signs. We note that this requirement has an invariant character, since the sign of the time-dependent component of a time-like vector does not depend on the choice of the reference frame.

We now consider three positive numbers $\alpha_1, \alpha_2, \alpha_3$, which we shall determine later. We write out a three-dimensional vector $\alpha_1\mathbf{q}_1 + \alpha_2\mathbf{q}_2 + \alpha_3\mathbf{q}_3$ and choose a coordinate frame such that the momentum vanishes in it.* Thus, by definition

$$\alpha_1\mathbf{q}_1+\alpha_2\mathbf{q}_2+\alpha_3\mathbf{q}_3=0. \tag{A.5}$$

*In this case the 4-vector $\alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3$ is assumed at first to be time-like. Later, by choosing suitable values of α_1, it becomes identically equal to zero. It is obvious that such a zero vector can always be obtained as a particular (limiting) case of a time-like vector.

We now consider the quantity $\alpha_1 q_{10} + \alpha_2 q_{20} + \alpha_3 q_{30}$. Since one of the quantities q_{i_0} has a sign opposite to that of the other two, we can always choose α_1, α_2 and α_3 in such a way that the combination $\alpha_1 q_{10} + \alpha_2 q_{20} + \alpha_3 q_{30}$ vanishes:

$$\alpha_1 q_{10}+\alpha_2 q_{20}+\alpha_3 q_{30}=0. \tag{A.6}$$

The conditions (A.5) and (A.6) can now be formulated in the following invariant form. For those values of q_i which correspond to the appearance of a singularity in the integral (A.1) we can always find such positive numbers α_i that the 4-vector $\alpha_1 q_1 + \alpha_2 q_2 + \alpha_2 q_3$ vanishes:

$$\sum_{i=1}^{3}\alpha_i q_i=0. \tag{A.7}$$

The conditions (A.4) (or (A.3)) and (A.7) have a perfectly general character and are called the Landau rules. It can be shown that when considering real singularities, these conditions are sufficient for the appearance of a singularity. The statements that follow pertain to the concrete case of the diagrams in Fig. 26. Taking the scalar products of (A.7) with q_1, q_2 and q_3, we obtain the system

$$\left.\begin{aligned}\alpha_1 q_1^2+\alpha_2(q_1q_2)+\alpha_3(q_1q_3)&=0,\\ \alpha_1(q_1q_2)+\alpha_2 q_2^2+\alpha_3(q_2q_3)&=0,\\ \alpha_1(q_1q_3)+\alpha_2(q_2q_3)+\alpha_3 q_3^2&=0.\end{aligned}\right\} \tag{A.8}$$

(A.8) defines α_i which differ from zero only under the additional condition that the following determinant vanishes:

$$\begin{vmatrix} q_1^2, & (q_1q_2), & (q_1q_3)\\ (q_1q_2), & q_2^2, & (q_2q_3)\\ (q_1q_3), & (q_2q_3), & q_3^2\end{vmatrix}=0. \tag{A.9}$$

By virtue of (A.4) we have $q_i^2 = \mu_i^2$ and, for example, $(q_1q_2) = -\frac{1}{2}\left[(q_1 - q_2)^2 - q_1^2 - q_2^2\right] = -\frac{1}{2}\left[m_3^2 - \mu_1^2 - \mu_2^2\right] \equiv -\mu_1\mu_2 z_3$. With the aid of the variables z introduced in this manner, we can rewrite (A.9) in the form (see page 135):

$$z_1^2+z_2^2+z_3^2-2z_1z_2z_3-1=0. \tag{A.10}$$

If we solve (A.10) with respect to one of the invariants m_1^2, m_2^2, m_3^2 or, which is the same, with respect to one of the quantities z_1, z_2, z_3, we obtain two solutions. We can show that only one leads to positive α_i. The corresponding singularity turns out to be the only one on the physical sheets specified by the exact definition of arrangement of the singularities in the integral (A.1), that is, by formulas (A.2).

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