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DIFFRACTION RADIATION

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1. INTRODUCTION

THE purpose of this survey is to familiarize the reader with a class of radiation effects that can be combined under the title of "diffraction radiation." This type of radiation has various specific features that distinguish it from the well known radiation effects such as radiation from a source in nonuniform motion, Vavilov-Cerenkov radiation and transition radiation.

At the start of the survey we give a brief description of the well known radiation effects and a definition of diffraction radiation. This is followed by a discussion of the character of the field of moving sources of different types. Then there is a brief presentation of the mathematical techniques for getting exact solutions of problems of diffraction radiation. The major part of the survey is devoted to considering various cases of diffraction radiation.

The radiation from a moving charged particle can be divided into several types, depending on the physical conditions under which the radiation occurs.

By diffraction radiation we shall mean the broad class of phenomena associated with the radiation from field sources moving uniformly in the neighborhood of some optical homogeneity, provided this radiation does not reduce to Vavilov-Cerenkov radiation or to transition radiation.

To explain the physical nature of diffraction radiation we consider a simple example. Suppose a point charge moves uniformly in a straight line in free space past some ideally conducting body, which here plays the role of an optical inhomogeneity. The field of a uniformly moving charged particle can be represented as a superposition of plane waves of different frequencies. All of the waves are damped exponentially as we move away from the trajectory of the particle. This corresponds to the familiar fact that there is no radiation during uniform motion of a particle. The presence of the optical inhomogeneity leads to scattering of the partial waves. Now the total field is given by the sum of the field of the particle alone and the scattered electromagnetic field. Even though the particle's own field is expanded in damped components, the scattered field may contain undamped waves, and this corresponds to radiation.

We can give another explanation of the nature of diffraction radiation without resorting to an expansion of the field in plane waves. During the motion of a charge past an ideally conducting body, varying currents are induced on the surface of the body. These currents are *the source of the diffraction radiation.*

From our remarks it follows that the solution of problems of diffraction radiation involve the same difficulties as the solution of problems of diffraction of electromagnetic waves by fixed optical inhomogeneities. The two problems are closely related. Nevertheless, the use for diffraction radiation of methods that have been developed and applied successfully in the mathematical theory of diffraction meets with difficulties. In diffraction theory the use of approximate methods usually means that severe restrictions are imposed on the frequency of the scattered field, or, more precisely, on the ratio of the characteristic dimension of the obstacle to the wavelength. For diffraction radiation, on the other hand, an essential feature is that the sources excite a continuous spectrum of frequencies, so that the width of the spectral region is such that the criterion for validity of any approximation is not satisfied over the whole region. For this reason we shall devote most of our attention to exact solutions of problems of diffraction radiation. Rigorous solutions of problems of this type were first obtained in [25]. Mathematically the solution of these problems is very similar to that of classical problems of diffraction, with the field of the moving source taking the place of the incident electromagnetic wave. We shall consider the character of the field of moving sources in Sec. 2. For what follows it is important to know the main features of the well-known radiation effects, such as the radiation from a charge in nonuniform motion, Vavilov-Cerenkov radiation and transition radiation.

a) Radiation in nonuniform motion. This is the electromagnetic radiation that accompanies a change in velocity of the particle. There is no electromagnetic radiation from uniform motion of a charge in free space. But if the motion of the particle is not uniform, radiation occurs whose character can be judged from the expression for the vector potential A_ω of the field of the particle at large distances from its line of motion.[1] Suppose that the law of motion of the particle has the form $\mathbf{r} = \mathbf{r}(t)$. Then

$$\mathbf{A}_\omega = q\frac{e^{ikR_0}}{2\pi c R_0}\int_{-\infty}^{\infty}\mathbf{u}(t)\,e^{i[\omega t-\mathbf{kr}(t)]}\,dt, \tag{1.1}$$

where q is the charge on the particle, $\mathbf{u}(t) = d\mathbf{r}(t)/dt$ is its velocity, R_0 is the distance from the point of observation to the trajectory. If we have a line instead of a point source, A_ω is given by

$$\mathbf{A}_\omega = \frac{q}{c}\sqrt{\frac{i}{2\pi kr}}\,e^{i\mathbf{kr}}\int\mathbf{u}(t)\,e^{i[\omega t-\mathbf{kr}(t)]}\,dt, \tag{1.1a}$$

where $\mathbf{u}$ and $\mathbf{r}$ are two-dimensional vectors in the plane perpendicular to the line of the source. The integral in (1.1) has the dimensions of length. We call it the length of formation of the radiation or simply the shaping length. This quantity characterizes the length of path over which the particle is in phase with the radiation. For a uniformly moving particle, for example, the shaping length is

$$\mathbf{l} = \int_{-\infty}^{\infty} \mathbf{u}(t)\, e^{i[\omega t - \mathbf{kr}(t)]}\, dt = 2\pi \mathbf{u} \delta(\omega - \mathbf{ku}) = 2\pi \frac{\mathbf{u}}{\omega} \delta\left(1 - \frac{u}{c}\cos\theta\right), \quad (1.2)$$

where u is the velocity of the particle, and θ is the angle between the direction of radiation and the velocity. Since u/c is always less than unity, the shaping length is always zero for a particle in uniform motion.

Let us consider the expression for the shaping length in the case of instantaneous stopping of a particle. Suppose that for t < 0 the velocity of the particle was u, while at the instant t = 0 the particle stops, so that for t > 0 the velocity is zero; then

$$\mathbf{l} = \int_{-\infty}^{\infty} \mathbf{u}(t)\, e^{i[\omega t - \mathbf{kr}(t)]}\, dt = \mathbf{u} \int_{-\infty}^{0} e^{it(\omega - \mathbf{ku})} dt$$
$$= 2\pi \mathbf{u} \delta_{+}(\omega - \mathbf{ku}) = \pi \mathbf{u} \delta(\omega - \mathbf{ku}) + \frac{1}{i} \frac{\mathbf{u}}{\omega - \mathbf{ku}}. \quad (1.3)$$

The term containing the δ function is zero by the same argument as was used for the case of uniform motion. Thus, except for an unimportant phase factor i,

$$l = -\frac{u}{\omega\left(1 - \frac{u}{c}\cos\theta\right)}. \quad (1.4)$$

This same result can be obtained from simple qualitative arguments. Let us define the shaping length for radiation of frequency ω, propagating at an angle θ to the velocity of the particle, as the length over which the phase of the radiated wave changes by π. This length is easily calculated using Fig. 1. Suppose that a particle with velocity u radiates, at the beginning and end of the segment l, a wave $e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}$ of frequency ω, at an angle θ to the direction of its motion.

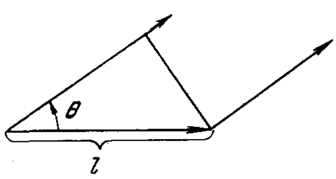

FIG. 1.

The phase difference between the waves radiated at the beginning and end of the segment is

$$\Delta\varphi = \omega\Delta t - \mathbf{k}\Delta\mathbf{r} = \omega \frac{l}{u} - kl\cos\theta = \frac{\omega l}{u}\left(1 - \frac{u}{c}\cos\theta\right). \quad (1.5)$$

Equating it to π, we find for l the value

$$l = \frac{\pi u}{\omega\left(1 - \frac{u}{c}\cos\theta\right)}, \quad (1.6)$$

which is close to (1.4). This simple qualitative derivation is due to I. M. Frank.

We also give the expression for the shaping length when the velocity of the charged particle changes abruptly. Let the velocity of the charged particle be u_1 for t < 0, and u_2 for t > 0. Then, again dropping the factor i, we get

$$l = \frac{u_1}{\omega - \mathbf{ku}_1} - \frac{u_2}{\omega - \mathbf{ku}_2} = \frac{u_1}{\omega\left(1 - \frac{u_1}{c}\cos\theta\right)} - \frac{u_2}{\omega\left(1 - \frac{u_2}{c}\cos\theta\right)}. \quad (1.7)$$

In this case the shaping length is equal to the difference between the shaping lengths corresponding to instantaneous stopping of the particle. We mention that when we speak of instantaneous stopping of the particle we mean that the velocity is changed during a finite time interval Δt, and that we are considering radiation frequencies small compared to $1/\Delta t$. Using the concept of shaping length, we can rewrite (1.1) for the field of a particle moving according to an arbitrary law in the form

$$\mathbf{A}_{\omega} = q \frac{e^{ikR_0}}{2\pi c R_0} \mathbf{l}. \quad (1.8)$$

The energy radiated by the moving charge into the angle range $d\theta\, d\varphi$ in the frequency range $d\omega$ is given by the formula

$$dW_{\omega} = \frac{q^2\omega^2}{4\pi^2 c^2} \,|\, l(\omega, \theta)\,|^2 \sin^3\theta\, d\theta\, d\varphi\, d\omega. \quad (1.9)$$

b) Vavilov-Cerenkov radiation. The source of this radiation is a charged particle in uniform motion. We know that in free space a uniformly moving charge does not radiate (the shaping length is zero in this case). But, as we see from (1.2), when

$$\cos\theta = \frac{c}{u}, \quad (1.10)$$

the shaping length becomes infinite. This means that the particle is in phase with its radiation over its entire path. Equation (1.10) cannot be satisfied in free space, since u < c. But if the particle moves in a refracting medium, the velocity of light in vacuum should be replaced by the phase velocity in the medium, c/n; then, if

$$\cos\theta = \frac{c}{nu}, \quad (1.11)$$

the shaping length becomes infinite. This means that the particle radiates uniformly over its whole path. The radiation has a characteristic directionality, determined by the optical properties of the medium and the velocity of the particle (cf. (1.11)). A necessary condition for radiation is that the velocity of the particle exceed the phase velocity of light in the medium.

If the dielectric constant of the medium is $\epsilon(\omega)$, while the velocity of the particle in the medium is u, the loss of energy in Vavilov-Cerenkov radiation is given by the Tamm-Frank formula [2]

$$\frac{dW}{dz} = \frac{q^2}{c^2} \int \left(1 - \frac{c^2}{\varepsilon(\omega) u^2}\right) \omega\, d\omega, \quad (1.12)$$

where the region of integration includes frequencies for which the radiation condition

$$\frac{c}{\sqrt{\varepsilon(\omega)}\, u} < 1 \quad (1.12)$$

is satisfied.

c) Transition radiation. As for Vavilov-Cerenkov radiation, the source of transition radiation is a uniformly moving charged particle. Here the velocity of motion may be smaller than the phase velocity of light in the medium through which the particle moves. The important thing for transition radiation is the change in the optical properties of the medium along the particle trajectory. The simplest and at the same time the most important example for the principle of transition radiation was first treated by V. L. Ginzburg and I. M. Frank.[3] They considered a uniformly moving charge, crossing along the normal between two dielectric media with dielectric constants $\epsilon_1(\omega)$ and $\epsilon_2(\omega)$. They determined the field and the energy loss in radiation in the backward direction (into the medium in which the charge was moving initially). Later G. M. Garibyan [4] calculated the energy loss in transition radiation in the forward direction, and showed that it increases linearly with energy. We shall give the formula for the energy of the transition radiation at angle θ to the particle velocity and for frequency ω. We shall restrict ourselves to the case of forward radiation. Suppose that a particle moving uniformly emerges from a medium with dielectric constant $\epsilon(\omega)$ into vacuum. Then the energy of the transition radiation in the vacuum at angle θ to the particle velocity has the form

$$W=\frac{2q^2\beta^2}{\pi c}\frac{\sin^3\theta\cos^2\theta\,d\theta}{(1-\beta^2\cos^2\theta)^2}\int\limits_0^\infty\frac{(\varepsilon-1)(1-\beta^2-\beta\sqrt{\varepsilon-\sin^2\theta})}{(\varepsilon\cos\theta+\sqrt{\varepsilon-\sin^2\theta})(1-\beta\sqrt{\varepsilon-\sin^2\theta})}\,d\omega, \tag{1.14}$$

where β denotes the ratio u/c. This expression not only describes the true transition radiation but also the Vavilov-Cerenkov radiation (if it is present) emitted by a particle in a medium and emerging into vacuum after refraction at the boundary.

The polarization of the transition radiation (in the case of normal incidence of the charge on the boundary of separation) is similar to the polarization of the Vavilov-Cerenkov radiation. In both cases the electric vector lies in the plane containing the trajectory of the particle.

The qualitative features of the transition radiation can be explained pictorially, following I. M. Frank. To do this we write a different form of (1.7) for the shaping length in the case of a jump in velocity:

$$l=\frac{c}{\omega}\left(\frac{\frac{u_1}{c}}{1-\frac{u_1}{c}\cos\theta}-\frac{\frac{u_2}{c}}{1-\frac{u_2}{c}\cos\theta}\right), \tag{1.15}$$

from which we see that the quantity l, and consequently the intensity of the radiation, is determined by the change in the parameter u/c during the motion of the particle. Formula (1.15) is valid for the motion of a particle in empty space. If the particle moves in a material medium with dielectric constant $\epsilon(\omega)$, we should consider the parameter $u\sqrt{\epsilon}/c$ instead of u/c. The quantity $u\sqrt{\epsilon}/c$ can change, even though the particle velocity does not, if the optical properties of the medium change along the path of the particle. Thus the parameter $u\sqrt{\epsilon}/c$, which determines the radiation, can change both because of changes in the particle velocity u and changes in the dielectric constant ϵ of the medium. This crude treatment shows that radiation arising from nonuniform motion of a charged particle in an optically homogeneous medium is analogous to the radiation that appears for uniform motion of the particle, but in an optically inhomogeneous medium. In the case of transition radiation that was considered above, where the particle emerges from the medium into vacuum, the parameter changes from its value $u\sqrt{\epsilon}/c$ in the medium to the value u/c in the vacuum. In accordance with our remarks, the transition radiation here is analogous to the radiation from a particle moving in empty space, whose velocity changes from $u\sqrt{\epsilon}$ to u. The shaping length in this case is

$$l=\frac{u}{i\omega}\left(\frac{\sqrt{\varepsilon}}{1-\frac{u}{c}\sqrt{\varepsilon}\cos\theta}-\frac{1}{1-\frac{u}{c}\cos\theta}\right). \tag{1.16}$$

Substituting this value of l in (1.9), we find for the energy of the radiation

$$dW=\frac{q^2u^2}{4\pi^2c^2}\left[\frac{\sqrt{\varepsilon}-1}{\left(1-\frac{u}{c}\sqrt{\varepsilon}\cos\theta\right)\left(1-\frac{u}{c}\cos\theta\right)}\right]^2\sin^3\theta\,d\theta\,d\varphi\,d\omega. \tag{1.17}$$

This expression is in good agreement with that obtained from the exact formula (1.14) in the limiting case of large particle energy $u \sim c$, high frequencies and small angles of the radiation. Under the conditions enumerated, one can disregard the reflection and refraction of the radiated light at the boundary of separation, which simplifies the treatment. For relativistic velocities of the particle, the transition radiation forward may contain extremely high frequencies. To determine the upper limit of the spectrum of the transition radiation, we use the asymptotic expression for the dielectric constant ϵ at high frequencies

$$\varepsilon(\omega)=1-\frac{\omega_0^2}{\omega^2},\qquad \omega_0^2=\frac{4\pi ne^2}{m}. \tag{1.18}$$

Substituting this expression for $\epsilon(\omega)$ in (1.17), we find that the spectrum of the transition radiation forward extends out to frequencies

$$\omega_{\lim}\approx\frac{\omega_0}{\sqrt{1-\beta^2}}, \tag{1.19}$$

and then drops off rapidly. Thus the total loss in transition radiation in the relativistic case is proportional to the limiting frequency $\omega_{\lim}$:[4]

$$W=\frac{e^2}{3c}\frac{\omega_0}{\sqrt{1-\beta^2}}. \tag{1.20}$$

From (1.20) we also see that the ultrarelativistic case the energy loss in radiation is proportional to the particle energy.

We note an interesting feature of the transition radiation. From formula (1.19) we see that the limiting frequency in the spectrum of the transition radiation increases proportionally to the particle energy, and for sufficiently high energies may lie in the region of hard x- or γ-radiation. It might seem that in this case the classical treatment becomes invalid. But, as we see from (1.16), the shaping length also increases with the particle energy, and is proportional to the square of the energy. Thus as the energy of the particle increases, we arrive at the paradoxical situation where the radiation has a very small wave length (less than interatomic separations), but is formed over very large macroscopic segments of the path. This indicates that the classical approach is valid.

2. THE ELECTROMAGNETIC FIELD OF UNIFORMLY MOVING SOURCES.

We shall need expressions for the field of a source moving uniformly in free space or in a homogeneous refracting medium. We restrict our treatment to sources of the simplest kind—line sources (charged wire and current-carrying wire) and point charges.

The electromagnetic field in free space satisfies the Maxwell equations

$$\left.\begin{aligned}&\operatorname{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t},\\&\operatorname{rot}\mathbf{H}=\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}+\frac{4\pi}{c}\mathbf{j},\\&\operatorname{div}\mathbf{E}=4\pi\rho,\\&\operatorname{div}\mathbf{H}=0.\end{aligned}\right\} \tag{2.1*}$$

Here E and H are the electric and magnetic field intensities, and ρ and j are the charge and current densities produced by the source.

Usually one goes over from the system (2.1) of Maxwell equations to equations for the scalar potential φ and vector potential **A**, in terms of which the fields are given by the formulas

$$\begin{aligned}&\mathbf{E}=-\operatorname{grad}\varphi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t},\\&\mathbf{H}=\operatorname{rot}\mathbf{A}.\end{aligned} \tag{2.2}$$

The equations for the potentials have the form

$$\begin{aligned}&\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}=-\frac{4\pi}{c}\mathbf{j},\\&\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi=-4\pi\rho,\end{aligned} \tag{2.3}$$

*rot ≡ curl.

where the potentials A and φ are related by the Lorentz condition

$$\operatorname{div} \mathbf{A} + \frac{1}{c}\frac{\partial \varphi}{\partial t} = 0. \tag{2.4}$$

To describe the electromagnetic field of uncharged current-carrying sources (for example, a neutral wire with current), where one can set $\varphi = 0$, it is convenient to use the vector potential A. In other cases, where one must know both the vector and scalar potentials for a description of the complete field, and where both potentials must satisfy suitable boundary conditions, it is more convenient to determine the field by means of the Hertz vector $\mathbf{\Pi}$.

We shall describe the field sources by a function $\mathcal{P}$, in terms of which the charge density ρ and current density j are given as follows:

$$\varrho = -\operatorname{div} \mathcal{P}, \qquad \mathbf{j} = \frac{1}{c}\frac{\partial \mathcal{P}}{\partial t}. \tag{2.5}$$

(It is obvious that this definition does not violate charge conservation.) We also introduce the Hertz vector $\mathbf{\Pi}$, which is related to A and φ by

$$\mathbf{A} = \frac{1}{c}\frac{\partial \mathbf{\Pi}}{\partial t}, \qquad \varphi = -\operatorname{div} \mathbf{\Pi}. \tag{2.6}$$

As a result we arrive at the following equation for the Hertz vector:

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{\Pi} = -4\pi \mathcal{P}. \tag{2.7}$$

From (2.2) and (2.6) it follows that the electromagnetic field vectors are expressed in terms of the vector $\mathbf{\Pi}$ by the relations

$$\mathbf{E} = \left(\operatorname{grad}\operatorname{div} - \frac{1}{c^2}\cdot\frac{\partial^2}{\partial t^2}\right)\mathbf{\Pi}, \qquad \mathbf{H} = \frac{1}{c}\operatorname{rot}\frac{\partial \mathbf{\Pi}}{\partial t}. \tag{2.8}$$

We shall obtain explicit expressions for the Hertz vector for electromagnetic field sources of the simplest type: a charged wire, a line current and a point charge, moving uniformly in free space. We note that these elementary solutions are the Green's functions respectively for the d'Alambert equation in two dimensions (line sources) and three dimensions (point charge). The solution for an arbitrary distribution of charges and currents can be expressed in terms of these functions.

a) The field of sources moving uniformly in vacuum. We consider a wire parallel to the x axis (Fig. 2), carrying a charge with linear density q, and moving with constant velocity $\mathbf{u} = \{u_x, u_y\}$ in the y, z plane. We denote the distance from the origin to the trajectory by a. The charge density ρ and the corresponding current density j are given by the expressions

$$\varrho = q\delta(\mathbf{r} - \mathbf{a} - \mathbf{u}t), \qquad \mathbf{j} = q\mathbf{u}\delta(\mathbf{r} - \mathbf{a} - \mathbf{u}t). \tag{2.9}$$

The source density vector $\mathcal{P}$, defined by (2.5) is equal to

$$\mathcal{P} = \frac{qui}{(2\pi)^2}\int e^{i\mathbf{k}(\mathbf{r}-\mathbf{a}-\mathbf{u}t)}\frac{dk_x\,dk_y}{\mathbf{k}\mathbf{u}}, \tag{2.10}$$

where we have used the well known expansion of the δ function in (2.9) in plane waves

$$\delta(\mathbf{r} - \mathbf{a} - \mathbf{u}t) = \frac{1}{(2\pi)^2}\int e^{i\mathbf{k}(\mathbf{r}-\mathbf{a}-\mathbf{u}t)}\,dk_x\,dk_y.$$

The solution of the inhomogeneous d'Alambert equation with right side of the form (2.10) is given by the formula

$$\Pi^0 = \frac{qui}{\pi}\int \frac{e^{i\mathbf{k}(\mathbf{r}-\mathbf{a}-\mathbf{u}t)}}{|\mathbf{k}|^2 - \frac{(\mathbf{k}\mathbf{u})^2}{c^2}}\,\frac{dk_x\,dk_y}{\mathbf{k}\mathbf{u}}. \tag{2.11}$$

We change to new integration variables ω and κ, using the relations

$$\omega = \mathbf{k}\mathbf{u}, \qquad \varkappa = \frac{\mathbf{k}\mathbf{a}}{a}. \tag{2.12}$$

With this substitution we take as new variables the components of the wave vector k along two mutually perpendicular directions: along the trajectory of the wire (the vector u) and along the normal to the wire (the vector a):

$$\mathbf{k} = \omega\frac{\mathbf{u}}{u^2} + \varkappa\frac{\mathbf{a}}{a}. \tag{2.13}$$

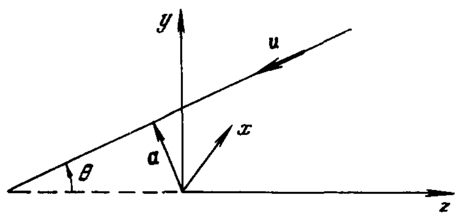

FIG. 2.

The quantity $\omega = \mathbf{k}\cdot\mathbf{u}$ determines the frequency of the plane waves appearing in the expansion (2.11). The Hertz vector is written in terms of the new integration variables as follows:*

$$\Pi^0 = \frac{iq\mathbf{u}}{\pi u}\int_{-\infty}^{\infty}\!\!\int \frac{e^{i\frac{\omega}{u}\left(\frac{\mathbf{u}\mathbf{r}}{u} - ut\right) + i\varkappa\left(\frac{\mathbf{a}\mathbf{r}}{a} - a\right)}}{\varkappa^2 + \frac{\omega^2}{u^2}(1-\beta^2)}\,\frac{d\varkappa\,d\omega}{\omega}. \tag{2.14}$$

Performing the κ integration by using residues, we get

$$\Pi^0 = \frac{iq\mathbf{u}}{\gamma u}\int e^{i\frac{\omega}{u}\left(\frac{\mathbf{u}\mathbf{r}}{u} - ut\right) - k\gamma\left|\frac{\mathbf{a}\mathbf{r}}{a} - a\right|}\,\frac{d\omega}{k\omega}, \tag{2.15}$$

where we use the notation

$$k = \frac{\omega}{c}, \qquad \beta = \frac{u}{c}, \qquad \gamma = \frac{\sqrt{1-\beta^2}}{\beta}. \tag{2.16}$$

We have obtained the frequency resolution of the Hertz vector. For elementary sources one could also carry out the ω integration and get the explicit dependence of the Hertz vector on the coordinates and time. We shall not do this, since later we shall define the characteristics of the diffraction radiation in terms of just these spectral resolutions. As we see from (2.15) the field of a charged wire is a superposition of waves of the form

$$e^{i\frac{\omega}{u}\frac{\mathbf{u}\mathbf{r}}{u} - k\gamma\left|\frac{\mathbf{a}\mathbf{r}}{a} - a\right|}.$$

These are "inhomogeneous" plane waves of frequency ω, propagating in the direction of motion of the source and damping exponentially with damping coefficient $k\gamma$ as we move away from the trajectory of the source.

To describe the field of a line current moving uniformly in free space, one needs only the vector potential A, which is determined by the first equation in (2.3). We introduce a spectral resolution for the vector potential, analogous to (2.16):

$$\mathbf{A}^0 = \frac{\mathbf{j}_0}{u\gamma}\int_{-\infty}^{\infty} e^{i\frac{\omega}{u}\left(\frac{\mathbf{u}\mathbf{r}}{u} - ut\right) - k\gamma\left|\frac{\mathbf{a}\mathbf{r}}{a} - a\right|}\,\frac{d\omega}{\omega}, \tag{2.17}$$

where j_0 is the current along the wire.

We now proceed to determine the field of a uniformly moving

*This expansion for the Hertz vector Π^0 diverges at low frequencies. But this divergence has no physical significance and disappears when we go over directly to the electromagnetic field vectors.

point particle with charge q. Let the velocity of the particle be $\mathbf{u}$ and the distance from the origin to the trajectory of the particle be $\mathbf{a}$. After calculations similar to those described above, we get the following expansion of the Hertz vector in plane waves:

$$\Pi^0 = \frac{iq\mathbf{u}}{2\pi^2 u} \int \frac{e^{i\frac{\omega}{u}\left(\frac{\mathbf{ur}}{u}-ut\right)+i\varkappa(\mathbf{r}-\mathbf{a})}}{\varkappa^2+\frac{\omega^2}{u^2}(1-\beta^2)} \frac{d\omega}{\omega} d\varkappa, \tag{2.18}$$

where κ is the component of the wave vector perpendicular to the particle.

Expression (2.18) can be integrated over κ, which gives the spectral resolution of the Hertz vector describing the field of a point charge. It has the form

$$\Pi^0 = \frac{iq\mathbf{u}}{\pi u} \int e^{i\frac{\omega}{u}\left(\frac{\mathbf{ur}}{u}-ut\right)} K_0(k\gamma\,|\mathbf{r}-\mathbf{a}|_\perp) \frac{d\omega}{\omega}, \tag{2.19}$$

where $(\mathbf{r}-\mathbf{a})_\perp$ is the projection of the vector $\mathbf{r}-\mathbf{a}$ on the plane normal to the particle velocity, and K is the MacDonald function. Formula (2.19) gives the expansion of the field of a point charge in cylindrical waves which damp exponentially as we move away from the path of the particle. The distance R from the particle's path at which the wave of frequency ω damps by a factor e is given by the formula

$$R \sim \frac{1}{k\gamma} = \frac{u}{\omega\sqrt{1-\beta^2}}. \tag{2.20}$$

We note that this same estimate is valid for the plane waves in terms of which the field of a line source is expanded (formula (2.15) for the Hertz vector of a charged wire and (2.17) for the field of a line current). As the velocity of the particle increases, R increases, and when the velocity of the particle approaches the velocity of light, the waves in terms of which the field is expanded become undamped. The properties of the field of a particle in the ultrarelativistic case approach those of a transverse electromagnetic wave. This enables an approximate determination of the field of diffraction radiation by using the methods of geometrical optics.

In many cases it is necessary that one not carry out the integration over κ in (2.18) completely, but rather represent the Hertz vector as a superposition of plane waves for which one is given not only the frequencies but also the projections of the wave vector on a selected direction. Usually the need to do this is related to the symmetry character of the optical inhomogeneity. We give such an expansion for the two most interesting specific cases.

Suppose that a point particle moves in the y, z plane (cf. Fig. 2). the geometry of the problem requires that the field of the particle be written as a superposition of waves of given frequency with a given projection of the wave vector on the x axis. Integrating (2.18) over the components of the vector κ, perpendicular to the x axis, we get

$$\Pi^0 = \frac{iq\mathbf{u}}{2\pi u} \int e^{i\frac{\omega}{u}\left(\frac{\mathbf{ur}}{u}-ut\right)+i\varkappa_x x} \frac{e^{-\sqrt{\varkappa_x^2+k^2\gamma^2}\left|\frac{\mathbf{ra}}{a}-a\right|}}{\sqrt{\varkappa_x^2+k^2\gamma^2}} \frac{d\omega}{\omega} d\varkappa_x. \tag{2.21}$$

This expression is convenient to use if the scattering obstacle has cylindrical symmetry with its symmetry axis parallel to the x axis (in other words, if the optical inhomogeneity is a cylindrical surface all of whose generators are parallel to the x axis). Formula (2.21) can be regarded as the representation of the field of a point particle in terms of the fields of infinitely long line sources whose charge density is modulated according to a harmonic law.

If the scattering obstacle has axial symmetry, and the particle moves parallel to the symmetry axis, it is conveneient to expand the field of the particle in harmonics of a given frequency and with a given value of the azimuthal component of the wave vector. Let the optical inhomogeneity have a symmetry axis coinciding with the z axis, and let the particle move parallel to the z axis at a distance a from it. We introduce a cylindrical coordinate system r, φ, z, setting the z axis along the symmetry axis of the obstacle (optical inhomogeneity). Writing (2.19) in the new coordinate system and using the addition theorem for Bessel functions, we get

$$\Pi_z^0 = \frac{iq}{\pi} \sum_m e^{im\varphi} \int e^{i\frac{\omega}{u}(z-ut)} \left\{ \begin{matrix} J_m(k\gamma a)\,K_m(k\gamma r) \\ J_m(k\gamma r)\,K_m(k\gamma a) \end{matrix} \right\} \frac{d\omega}{\omega} \tag{2.22}$$

Here the upper line in the curly brackets applies for $r > a$, and the lower for $r < a$.

b) Field of a source moving in a refracting medium. The Maxwell equations are written for a refracting medium in the form

$$\left.\begin{aligned} \operatorname{rot}\mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \\ \operatorname{rot}\mathbf{H} &= \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{j}, \\ \operatorname{div}\mathbf{D} &= 4\pi\varrho, \\ \operatorname{div}\mathbf{B} &= 0. \end{aligned}\right\} \tag{2.23}$$

E and H are the electric and magnetic field intensities, D and B are the corresponding electric and magnetic inductions. The system (2.2) must be supplemented by material equations relating the fields and the inductions. We take the material equations in the following form:

$$\mathbf{D} = \hat{\varepsilon}\mathbf{E}, \qquad \mathbf{B} = \hat{\mu}\mathbf{H}, \tag{2.24}$$

where $\hat{\epsilon}$ and $\hat{\mu}$ are linear operators. For monochromatic fields, in which all quantities are proportional to $e^{-i\omega t}$, the act on of the operators $\hat{\epsilon}$ and $\hat{\mu}$ reduces to multiplication of the spectral components of the fields by certain functions $\epsilon(\omega)$ and $\mu(\omega)$:

$$\mathbf{D}(\omega) = \varepsilon(\omega)\,\mathbf{E}(\omega), \qquad \mathbf{B}(\omega) = \mu(\omega)\,\mathbf{H}(\omega). \tag{2.25}$$

If we assume that ϵ and μ depend on ω but not on k, we automatically eliminate treating spatial dispersion.

The potentials A and φ in the refracting medium are given by the relations

$$\begin{aligned} \mathbf{H} &= \frac{1}{\mu}\operatorname{rot}\mathbf{A}, \\ \mathbf{E} &= -\operatorname{grad}\varphi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} \end{aligned} \tag{2.26}$$

and satisfy the equations

$$\begin{aligned} \left(\Delta - \frac{\varepsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} &= -\frac{4\pi\mu}{c}\mathbf{j}, \\ \left(\Delta - \frac{\varepsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi &= -\frac{4\pi}{\varepsilon}\varrho, \end{aligned} \tag{2.27}$$

which are valid if the potentials satisfy the supplementary condition

$$\operatorname{div}\mathbf{A} + \frac{\varepsilon\mu}{c}\frac{\partial\varphi}{\partial t} = 0. \tag{2.28}$$

The equation for the Hertz vector has the form

$$\left(\Delta - \frac{\varepsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}\right)\Pi = -4\pi \quad ,$$

where the vector $\mathcal{P}$ is defined by (2.5).

The potentials A and φ are expressed in terms of the Hertz vector by the formulas

$$\mathbf{A} = \frac{\mu}{c}\frac{\partial\Pi}{\partial t}, \qquad \varphi = -\frac{1}{\varepsilon}\operatorname{div}\Pi, \tag{2.29}$$

and the fields by the formulas

$$\begin{aligned} \mathbf{H} &= \frac{1}{c}\frac{\partial}{\partial t}\operatorname{rot}\Pi, \\ \mathbf{E} &= \frac{1}{\varepsilon}\operatorname{grad}\operatorname{div}\Pi - \frac{\mu}{c^2}\frac{\partial^2\Pi}{\partial t^2}. \end{aligned} \tag{2.30}$$

For simplicity we shall consider nonmagnetic refracting media and set $\mu = 1$.

The field of a line charge in a refracting medium is described by the following formula:

$$\Pi^0 = \frac{iq\mathbf{u}}{\pi u} \int \frac{e^{i\frac{\omega}{u}\left(\frac{\mathbf{ur}}{u} - ut\right) + i\varkappa\left(\frac{\mathbf{ar}}{a} - a\right)}}{\varkappa^2 + \frac{\omega^2}{u^2}(1 - \varepsilon\beta^2)} \frac{d\varkappa\, d\omega}{\omega}. \tag{2.31}$$

This formula is very similar to the expression for the Hertz vector of the field of a line charge in vacuum (2.14) and differs from it only in the replacement of $1 - \beta^2$ by $1 - \epsilon\beta^2$ in the denominator.

Integrating over κ, we get the spectral resolution of the Hertz vector

$$\Pi^0 = iq\mathbf{u} \int e^{i\frac{\omega}{u}\left(\frac{\mathbf{ur}}{u} - ut\right) - \frac{|\omega|}{u}\sqrt{1-\varepsilon\beta^2}\left|\frac{\mathbf{ar}}{a} - a\right|} \frac{d\omega}{\omega^2\sqrt{1 - \varepsilon\beta^2}}. \tag{2.32}$$

From formula (2.31) we see the essential difference between the field of a source moving in a refracting medium and the field of a source moving in vacuum. If $\epsilon\beta^2 < 1$, then, as for motion in vacuum, the field of the moving source consists only of damped waves. But if the condition $\epsilon\beta^2 > 1$ is satisfied, i.e., if the source moves with a velocity exceeding the phase velocity of propagation of electromagnetic waves in the medium, undamped harmonic waves appear. This is the so-called Vavilov-Cerenkov radiation, which is produced at the expense of the energy of the source.

By using these formulas one can easily obtain the Hertz vector for a line current or point charge moving uniformly in a refracting medium. In all cases when the condition $\epsilon\beta^2 > 1$ is satisfied we get Vavilov-Cerenkov radiation, which goes out to infinity (if the medium is not absorbing), while the direction of propagation of the Vavilov-Cerenkov radiation makes an angle θ, satisfying (1.11), with the direction of the velocity of the charge.

When $\epsilon\beta^2 > 1$ the sign of the quantity $\sqrt{1 - \epsilon\beta^2}$ should be taken so that the resolution of the Hertz vector contains only outgoing waves.

3. MATHEMATICAL METHODS FOR OBTAINING EXACT SOLUTIONS OF THE PROBLEM OF DIFFRACTION RADIATION.

To get exact solutions of problems of diffraction radiation we shall use the same method as was developed in the mathematical theory of diffraction for solving problems of scattering of an electromagnetic wave by obstacles of a special type. This method, which is called the Wiener-Hopf-Fock method, has been used to get rigorous solutions of many problems of diffraction theory.[5-7]

The problem of diffraction radiation differs from the classical problems of diffraction theory in having an incident wave which is a superposition of damped waves. We shall see what changes this difference causes in the results.

We shall illustrate the use of the Wiener-Hopf-Fock method on the example of the diffraction of a plane electromagnetic wave incident on an ideally conducting halfplane.

Suppose that a plane electromagnetic wave is incident on the ideally conducting halfplane $y = 0$, $z > 0$. For simplicity we shall assume that the wave vector of the incident wave lies in the y, z plane.

We shall consider the case where the magnetic field of the incident wave is parallel to the edge of the screen, i.e., to the x axis. Following L. A. Vaĭnshteĭn, [6] we shall call such a polarization of the incident wave (one can see that the scattered wave will have the same polarization) magnetic polarization.

The incident wave can be described by a Hertz vector Π^0 with the one nonzero component

$$\Pi_y^0 = e^{i(\mathbf{kr} - \omega t)}, \tag{3.1}$$

where **k** and **r** are two-dimensional vectors in the y, z plane.

We shall describe the scattered field by a Hertz vector

$$\Pi^1 = \Pi^1(y, z).$$

Thus, the total field, including both incident and scattered waves, is described by the Hertz vector

$$\Pi = \Pi^0 + \Pi^1. \tag{3.2}$$

The total field must satisfy the following requirements:

1) be a solution of the wave equation (2.7) with zero on the right hand side;

2) satisfy the boundary conditions on the ideally conducting halfplane:

$$E_z = 0 \quad \text{for} \quad y = 0, \quad z > 0; \tag{3.3}$$

3) satisfy the radiation condition at infinity: the energy flux in the scattered field must be directed away from the edge;

4) finally, satisfy the so-called screen edge condition: the energy density near the edge of the screen must be integrable over space.

We now proceed to solve this problem, which was first solved by A. Sommerfeld by a different method.[8]

The incident wave induces on the halfplane variable currents $\mathrm{j} = \mathrm{j}_z(z)$, which are the source of the scattered field.

The Hertz vector of the scattered field can be expressed in terms of the induced currents as follows:

$$\Pi^1 = \frac{i}{\omega} \int \frac{e^{ikR}}{R} \mathrm{j}(z)\, ds. \tag{3.4}$$

The integration is over all of the halfplane, R is the distance from the point of integration (x', $y' = 0$, z') to the point of observation (x, y, z):

$$R = \sqrt{(x - x')^2 + y^2 + (z - z')^2}. \tag{3.5}$$

Carrying out the x' integration by using the formula

$$H_0^{(1)}(k\,|D|) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{e^{ik\sqrt{D^2 + \xi^2}}}{\sqrt{D^2 + \xi^2}}\, d\xi, \tag{3.6}$$

we get

$$\Pi^1 = -\frac{\pi}{\omega} \int_0^{\infty} H_0^{(1)}\left(k\sqrt{y^2 + (z - \xi)^2}\right) \mathrm{j}(\xi)\, d\xi. \tag{3.7}$$

We represent the unknown current distribution j(z) as an expansion in a Fourier integral in z:

$$j(z) = \int f(w)\, e^{iwz}\, dw. \tag{3.8}$$

Substitution of this expression in (3.7) and integration over ξ gives

$$\Pi^1 = -\frac{2\pi}{\omega} \int_{-\infty}^{\infty} \frac{e^{iv|y|}}{v} f(w)\, e^{iwz}\, dw, \tag{3.9}$$

where

$$v = \sqrt{k^2 - w^2} \qquad (\operatorname{Im} v \geqslant 0). \tag{3.10}$$

We note that this expression shows an important property of the scattered radiation. The Hertz vector of the scattered field is an even function of y. It is easily seen that this is the cause of the symmetry of the angular distribution of the scattered radiation relative to the plane of the screen. This statement is valid for diffraction by any plane screen. We shall therefore from now on look at the scattered field on only one side of the screen.

It is easy to see that (3.9) is a solution of the homogeneous wave equation (3.7). To determine the explicit form of the function f(w) we use the boundary condition (3.3) on the halfplane. As a result we get the following inhomogeneous integral equation:

$$\int_{-\infty}^{\infty} v f(w) e^{iwz} dw = -\frac{\omega k_y k_z}{2\pi} e^{ik_z z} \quad \text{for} \quad z > 0, \tag{3.11}$$

where

$$k_z = -k \cos \theta.$$

Here π-θ is the angle between the direction of propagation of the plane wave and the positive direction of the axis (cf. Fig. 2).

To solve the problem we must still add to this equation a condition determining the behavior of f(z) on the extension of the scattering halfplane. For this condition we choose the obvious equation

$$j(z) = \int_{-\infty}^{\infty} f(w) e^{iwz} dw = 0 \quad \text{for} \quad z < 0, \tag{3.12}$$

which states that there are no currents on the extension of the halfplane. Thus the function f(w) is determined as the solution of the pair of integral equations (3.11) and (3.12), of which the first is valid for positive z, and the second for negative z. The solution of such equations is accomplished by the Wiener-Hopf-Fock method, [5] and we shall use the technique for solution developed in the papers of L. A. Vaĭnshteĭn.[6] We write the kernel of (3.11) as a product of two factors:

$$v = L_+(w) L_-(w) \frac{1}{w^2 - k_z^2}, \tag{3.13}$$

where the function $L_+(w)$ is holomorphic and has no zeros in the upper half of the complex w plane (Im w > 0), while the function $L_-(w)$ has similar properties in the lower half of the w plane (Im w < 0). For the function v defined by (3.10), this splitting is done in elementary fashion:

$$L_+(w) = \sqrt{k+w}\,(w+k_z), \qquad L_-(w) = \sqrt{k-w}\,(w-k_z), \tag{3.14}$$

where it is understood that the wave number k has a small positive imaginary part.

From (3.12) it follows that the required function f(w) must be holomorphic in the lower w halfplane. Thus we must look for a solution of the form

$$f(w) = \frac{C}{L_-(w)}, \tag{3.15}$$

where C is a constant independent of w.

With such a choice of the solution Eq. (3.12) is satisfied identically. To determine the constant C we substitute (3.15) in Eq. (3.11). Calculating the integral on the left side by residues, we get

$$C = \frac{i\omega k_y k_z^2}{2\pi^2 L_+(k_z)} = \frac{i\omega k_y k_z}{4\pi^2 \sqrt{k+k_z}}. \tag{3.16}$$

Formulas (3.8), (3.15), and (3.16) completely determine the scattered field arising from incidence of the plane wave (3.1) on the ideally conducting halfplane. As shown in [6], the integral representation of the solution found here is identical with the well-known representation of the solution of this problem found by Sommerfeld. We mention that the problem of the diffraction of a wave packet consisting of homogeneous plane waves, which was first treated in [28], can be solved in similar fashion.

As one sees from the method described for solving the pair of integral equations, the decisive point in finding the solution is to split the kernel v (w) into factors. This splitting is not unique. Multiplying $L_+(w)$ by an entire function and dividing $L_-(w)$ by the same function, we get a new decomposition of the kernel. Expression (3.15) for f (w) with the new function $L_-(w)$ is again a solution of the system of equations (3.11)-(3.12). A unique choice of solution is achieved by using the "edge conditions." According to these conditions, near the sharp edge of an obstacle the component of the current density normal to the edge is proportional to $r^{1/2}$, and the tangential component is proportional to $r^{-1/2}$, where r is the distance from the edge. Using well known formulas relating the asymptotic form of a function to the asymptotic form of its Fourier transform, we obtain the requirements that are imposed on the behavior of the Fourier components of the current density for $w \to \infty$. For the current component normal to the edge we have the condition

$$f(w) \sim \frac{1}{w^{3/2}} \quad \text{for} \quad w \to \infty. \tag{3.17}$$

In the example we are considering only the current component normal to the edge is excited. Obviously the solution (3.15) satisfies the "edge condition" (3.17).

If the component of the current density tangent to the edge is different from zero, the "edge conditions" require that the behavior of its Fourier components at infinity be

$$f(w) \sim \frac{1}{w^{1/2}} \quad \text{for} \quad w \to \infty. \tag{3.18}$$

We also give without proof the solution of the more complicated pair of integral equations:

$$\left.\begin{aligned} &\int_{-\infty}^{\infty} vF(w) e^{iwz} dw = g_1(z) \quad &\text{for} \quad z > 0, \\ &\int_{-\infty}^{\infty} F(w) e^{iwz} dw = g_2(z) \quad &\text{for} \quad z < 0. \end{aligned}\right\} \tag{3.19}$$

The method for solving this system of equations is described in the monograph [7] and leads to the following final form of the solution:

$$F(w) = \frac{1}{2\pi^{3/2}\sqrt{k-w}} \left\{ e^{-i\frac{\pi}{4}} \int_0^\infty e^{-iwz} dz \int_0^\infty g_1(z+\xi) e^{ik\xi} \frac{d\xi}{\sqrt{\xi}} + e^{i\frac{\pi}{4}} \int_{-\infty}^0 dz e^{-i(w-k)z} \frac{d}{dz} \int_0^\infty g_2(z-\xi) e^{-ik(z-\xi)} \frac{d\xi}{\sqrt{\xi}} \right\} \tag{3.20}$$

It is easy to see that the solution of the system (3.11)-(3.12) is obtained from (3.20) if we get $g_2 = 0$, $g_1 = -\frac{\omega k_y k_z}{2\pi} e^{ik_z z}$.

4. TWO-DIMENSIONAL PROBLEMS OF DIFFRACTION RADIATION IN FLIGHT OF A SOURCE PAST A SEMI-INFINITE CONDUCTING SCREEN

It is convenient to begin the treatment of diffraction radiation with two-dimensional problems. Such problems are obviously simpler mathematically than three-dimensional ones, and this simplifies the explanation of the physical picture of the phenomenon. In addition, two-dimensional problems may have a definite practical value. One problem in this class is that of the excitation of radiation by a plane modulated electron flow.

a) Radiation from a charged wire or current-carrying wire moving at constant velocity past the edge of an ideally conducting half-plane.[9] Suppose that an infinitely thin, ideally conducting halfplane is located at y = 0, z > 0 (cf. Fig. 2). A uniformly charged wire, parallel to the x axis (the edge of the halfplane), moves past the edge at constant velocity $\mathbf{u} = (u_z, u_y)$. We first consider the case where the wire during its motion does not intersect the halfplane. As in Sec. 1, where we calculated the field of a charged wire moving in vacuum, we denote the charge density per unit length of the wire by q, the distance from the edge to the trajectory of the wire by **a**, and the angle between the velocity vector **u** of the wire and the positive z axis by $\pi-\theta$. We assume that θ varies between 0 and π. The charged wire moving past the halfplane induces in it variable currents that are sources of radiation. We represent the total field as a superposition of the field of the wire in free space and the field of the induced currents. Correspondingly we set

$$\Pi = \Pi^0 + \Pi^1, \tag{4.1}$$

where Π^0 is the Hertz vector describing the field of the charged wire in free space, and is given by (2.15). In the case we are considering it is clear that the induced currents have only the one nonzero component j_z. It is therefore convenient to describe the radiation field by the one-component Hertz vector Π_z^1, of which the integral representation is given by formula (3.9)

$$\Pi_z^1 = -2\pi \int_{-\infty}^{\infty} \frac{e^{iv|y|}}{v} F(w) e^{iwz-i\omega t} \frac{dw\, d\omega}{\omega}. \tag{4.2}$$

We remind the reader that $F(w, \omega)$ is the Fourier component of the current induced on the screen, and has to be determined from the conditions of the problem.

For the boundary conditions we take (3.3), requiring the vanishing of the tangential component of the total electric field on the halfplane, and (3.12), which expresses the obvious requirement that there be no currents on the extension of the halfplane. As a result we get the following pair of integral equations for the quantity $F(w, \omega)$:

$$\left.\begin{aligned} &\int_{-\infty}^{\infty} F(w)\, v e^{iwz}\, dw \\ &= \frac{qk}{2\pi}\left[\frac{1}{\beta}\sin\theta + i\gamma\cos\theta\right] e^{-k\gamma a - iz\left(\frac{\omega}{u}\cos\theta - ik\gamma\sin\theta\right)} \quad \text{for } z>0, \\ &\int_{-\infty}^{\infty} F(w) e^{iwz}\, dw = 0 \qquad \text{for } z<0. \end{aligned}\right\} \tag{4.3}$$

From now on we shall deal with quantities at a fixed frequency ω, and abbreviate $F(w, \omega)$ to $F(w)$. The system (4.3) is similar to the pair of integral equations (3.11)—(3.12) in the Sommerfeld problem. The difference is that the right hand side of the equation valid for $z > 0$ now contains an inhomogeneous plane wave that damps exponentially with increasing absolute value of z. Physically this difference corresponds to the previously observed fact that the field of a moving source is a superposition of damped waves. The solution of (4.3) is carried out in the same way as in Sec. 3, and gives

$$F(w) = \frac{qk}{4\pi^2 i} \times \frac{\left(\frac{1}{\beta}\sin\theta + i\gamma\cos\theta\right) e^{-k\gamma a}}{\sqrt{k - \frac{\omega}{u}\cos\theta + ik\gamma\sin\theta}\,\sqrt{k-w}\left(w + \frac{\omega}{u}\cos\theta - ik\gamma\sin\theta\right)}. \tag{4.4}$$

In this formula we take the branch of the expression $\sqrt{k-w}$ that is positive for $w \to -\infty$. Formulas (4.2) and (4.4) completely determine the scattered field arising from uniform motion of the charged wire past the conducting halfplane. To get the total field we must add to the scattered field the field of the wire moving in vacuum.

As we see from (2.8), the nonzero components of the scattered field are H'_x, E'_y, E'_z. We give as an example the expression for H'_x:

$$H_x^1 = -ik\frac{\partial \Pi^1}{\partial y} = -\frac{2\pi}{c}\operatorname{sign} y \int e^{iv|y|} F(w) e^{iwz}\, dw. \tag{4.5}$$

The expressions for E_y and E_z have a similar structure, so we shall consider only H'_x in more detail. First we note the following point. If we introduce the angle θ_0 defined by the equations

$$\cos\theta_0 = \frac{1}{\beta} = \frac{c}{u}, \quad \sin\theta_0 = i\gamma = i\frac{\sqrt{1-\beta^2}}{\beta}, \tag{4.6}$$

the expression (4.4) for F(w) can be rewritten (for $\omega > 0$) in the form

$$F(w) = \frac{q\sqrt{2k}}{4\pi^2 i} e^{-k\gamma a} \frac{\cos\frac{\theta+\theta_0}{2}}{\sqrt{k-w}\,[w + k\cos(\theta+\theta_0)]}. \tag{4.7}$$

In this form F(w) coincides, except for a factor, with the solution (3.15)—(3.16) of the problem of diffraction of a plane wave incident on a halfplane at angle $\theta + \theta_0$. From (4.6) we see that this similarity is purely formal, since the angle θ_0 is imaginary. But the physical significance of such a coincidence can be understood if we consider the case where the velocity of the wire exceeds the speed of light, as occurs for the Vavilov-Cerenkov radiation in a medium with index of refraction $n > 1$. Then the quantity $\beta = u/c$ becomes $\beta' = (u/c)n$, and the angle θ_0 becomes real; it is the angle between the direction of propagation of the Vavilov-Cerenkov radiation and the direction of the velocity of the wire. This problem will be discussed in more detail below.

Formula (4.7) acquires a physical meaning in still another case. When $\beta = 1$, as we see from (4.6), the angle θ_0 goes to zero and formula (4.7) coincides to within a factor with the solution (3.5)—(3.6) of the Sommerfeld problem. This also follows from the fact that when $\beta = 1$ ($\gamma = 0$) the field of the charged source is expanded in plane undamped electromagnetic waves whose propagation direction coincides with the direction of the velocity of the source (cf., for example, the expressions (2.15) and (2.17), which determine the field of a charged wire and of a wire carrying current). The problem of the diffraction of such waves by a halfplane is just the Sommerfeld problem.

By using formula (4.7) the integral representation (4.5) of the field H_x^1 can be rewritten in the form

$$H_x^1 = -\frac{1}{c}\,\mathrm{sign}\,y\,\frac{q\sqrt{2k}}{2\pi i}\,e^{-k\gamma a}\cos\frac{\theta+\theta_0}{2} \times \int_{-\infty}^{\infty}\frac{e^{iv|y|+iwz}}{[w+k\cos(\theta+\theta_0)]\sqrt{k+w}}\,dw. \tag{4.8}$$

This expression is valid only for positive ω; when $\omega < 0$ one should take the complex conjugate expression. The expression (4.8) for the magnetic field can be transformed and expressed in terms of a Fresnel integral with complex argument, as is done in the theory of diffraction of a plane electromagnetic wave with "electric" polarization.[6]

We give the expression for $H^1_{x\omega}$ that is valid at large distances from the edge of the halfplane. We write $z = r\cos\varphi$, $y = r\sin\varphi$, $0 \le \varphi \le \pi$ and, using the method of steepest descent for the integration over w, we get

$$H^1_{x\omega} = \frac{2q}{c}\,\frac{e^{i\left(kr+\frac{\pi}{4}\right)}}{\sqrt{2\pi kr}}\,e^{-k\gamma a}\,\frac{\cos\frac{\varphi}{2}\cdot\cos\frac{\theta+\theta_0}{2}}{\cos\varphi+\cos(\theta+\theta_0)}. \tag{4.9}$$

Let us discuss the limits of validity of this formula. They are determined by the limits of validity of the saddle-point method by which the integral expression (4.8) was evaluated. The integrand in (4.8) has a pole at $w = -k\cos(\theta+\theta_0)$. The saddle point $w = k\cos\varphi$ must be sufficiently far from this pole. If this is not the case formula (4.9) gives an infinite value for $H'_{x\omega}$ at arbitrarily large distances r, which is physically meaningless. This condition can be written as an inequality requiring that the denominator in (4.9) be large compared to unity:

$$\left|\sqrt{2\pi kr}\,[\cos\varphi+\cos(\theta+\theta_0)]\right| \gg 1. \tag{4.10}$$

By using the definition (4.6), this inequality can be transformed to

$$\sqrt{2\pi kr}\,\sqrt{\frac{1+\beta\cos(\varphi+\theta)}{2\beta}}\,\sqrt{\frac{1+\beta\cos(\varphi-\theta)}{2\beta}} \gg 1 \tag{4.11}$$

or

$$r \gg \frac{1}{2\pi k}\,\frac{2\beta}{1+\beta\cos(\varphi+\theta)}\,\frac{2\beta}{1+\beta\cos(\varphi-\theta)}. \tag{4.12}$$

This inequality determines the distances at which the asymptotic expression (4.9) for the magnetic field is valid. For small velocities of the wire, $\beta \ll 1$, the expression (4.9), which was gotten by the saddle point method, is valid starting after a distance from the screen edge of the order of the wave length of the radiation. Now suppose that the velocity of the source moving past the edge of the conducting halfplane is close to the velocity of light. Then the denominator of one of the factors in (4.12) may be small. The angles φ near which this occurs are given by the equations

$$\begin{aligned} &1+\beta\cos(\varphi+\theta)=0,\\ &1+\beta\cos(\varphi-\theta)=0 \end{aligned} \qquad (\beta\approx 1). \tag{4.13}$$

from which we get

$$\varphi = \pi \pm \theta. \tag{4.14}$$

Here the plus sign corresponds to satisfying the second equation (4.13), and the minus sign to the first equation. Near these angle values one of the factors on the right of (4.12) is large, while the other is of order unity. Thus the inequality (4.12) can be rewritten as two inequalities:

$$r \gg \frac{u}{\pi\omega\,[1+\beta\cos(\varphi\pm\theta)]}. \tag{4.15}$$

Denoting by $\alpha = \pi + \theta$ the angle between the positive direction of the z axis and the direction of motion of the wire (Fig. 3), we get

$$r \gg \frac{u}{\pi\omega\,[1-\beta\cos(\varphi\pm\alpha)]}. \tag{4.16}$$

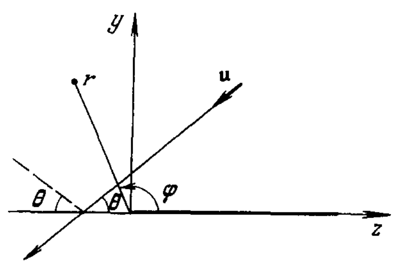

FIG. 3.

The expression on the right of (4.16) is, except for a factor, just the shaping length of the radiation, as defined by formula (1.6). Thus the expression (4.9) for the magnetic field of a charged wire moving past a conducting halfplane is valid at distances from the edge of the screen that exceed the shaping length of the radiation. We note that in the ultrarelativistic case ($\beta \to 1$) the distance from the screen edge given by (4.16) must be large in two directions: in the direction of the forward radiation, $\varphi = \alpha$, and in the direction of the mirror reflection of the radiation, $\varphi = -\alpha$. The fundamental point is that for all values of β less than unity there should exist such large distances from the screen edge that at these or larger distances the expression (4.9) for H_x^1 (and also the asymptotic expressions for all the other nonzero components of the field) are valid for all values of the angle φ. We note that in the problem of the diffraction of a plane electromagnetic wave incident on a semi-infinite conducting screen there exist angles at which the asymptotic expression for the scattered field is not valid, no matter how large the distance from the point of observation to the edge of the screen. These are the directions of the incident and the specularly reflected rays (the region of transition from light to shadow).

We can arrive at the inequality (4.16), which determines the conditions for applicability of the method of steepest descent, by a different method, by requiring that the distance from the saddle point to the pole exceed the characteristic size of the region of steepest descent, which in this case is not greater than $\sqrt{k/r}\,\sin^2\varphi$ in order of magnitude. This requirement gives

$$|k\cos\varphi + k\cos(\theta+\theta_0)| \gg \sqrt{\frac{k}{r}}\sin\varphi,$$

i.e., we arrive at an inequality that coincides with (4.10).

From the integral representation (4.8) for the magnetic field component H_x^1, we see that the magnetic field has a different character in different regions of space. This follows from the behavior of the integrand in (4.8) in the plane of the complex variable w. For the region $y > 0$, $z > 0$ (the space above the halfplane) the integral reduces to the residue at the pole $w = -k\cos(\theta+\theta_0)$ and to an integral along a cut in the upper w halfplane. Cuts arise because the function $v = \sqrt{k^2 - w^2}$ has branch points at $w = \pm k$. Accordingly the function v is determined uniquely on the plane with cuts extending out from the branch points, as shown in Fig. 4. The residue at $w = -k\cos(\theta+\theta_0)$ gives the so-called "image field", i.e., the field of a source of opposite sign, whose trajectory is obtained by reflecting the trajectory of the original source in the plane $y = 0$. The integral over the cut determines the radiation field of the charged wire. For $y < 0$, $z > 0$ (the region below the halfplane), the residue at the pole $w = -k\cos(\theta+\theta_0)$ compensates the primary field given by the Hertz vector $\mathbf{\Pi}^0$. This region corresponds to the region of geometrical shadow in the Sommerfeld problem, where only the radiation field of the charge exists. Finally, when $z < 0$ (the space to the left of the halfplane) the integral (4.8) reduces to an integral along the edges of the cut in the lower halfplane. In this case the total field is made up of the field of the source in empty space and the field of the radiation. It is obvious that in calculating the integral (4.8) by the saddle point method we do not get the image field, since it is damped exponentially as one moves away from the image trajectory. Thus the expression for $H^1_{x\omega}$ (4.9), obtained by steepest descent from (4.8) describes only the field of the radiation.

The magnetic field H_x^1 at large distances from the edge of the screen consists of diverging cylindrical waves. As we see from (4.9), the amplitude of these

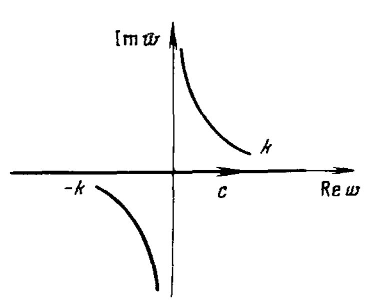

FIG. 4.

waves depends on a number of parameters: the velocity of the charge, the angle of observation, the impact parameter a, etc.

Let us calculate the intensity of the radiation at frequency ω in the angular interval $d\varphi$. The flux of the Poynting vector through the surface element $r\,d\varphi$ at large distances from the edge is

$$I(\varphi)\,d\varphi = rc\,|H^1_{x\omega}|^2\,d\varphi$$

$$= \frac{q^2}{\pi\omega}e^{-2k\gamma a}\frac{1+\beta\cos\theta}{\beta}\frac{\cos^2\frac{\varphi}{2}\,d\varphi}{\left(\cos\varphi+\frac{1}{\beta}\cos\theta\right)^2+\gamma^2\sin^2\theta}. \tag{4.17}$$

As we see from (4.17), the angular distribution is essentially different at small and at large velocities of the source. For small velocities of the charged wire ($\beta \ll 1$, $\gamma \gg 1$) the radiation is proportional to $\cos^2\varphi/2$. At large velocities ($\beta \to 1$, $\gamma \to 0$) the radiation has maxima along the directions $\varphi = \pm(\pi+\theta)$. These angles correspond to radiation in the directions of motion of the wire (radiation forward) and radiation in the mirror image direction relative to the conducting halfplane. The width of the maxima of the angular distribution is of order γ in the ultrarelativistic case.

Let us consider the dependence of the radiation intensity on frequency. As we see from (4.17), the spectral dependence is essentially given by the factor

$$e^{-2k\gamma a} = e^{-2a\frac{\omega}{u}\sqrt{1-\beta^2}}. \tag{4.18}$$

The spectrum of the radiation is bounded above, where the limiting frequency of the radiation is of order

$$\omega_{\max} \approx \frac{u}{a\sqrt{1-\beta^2}}. \tag{4.19}$$

Accordingly we may speak of the duration of the burst of radiation

$$\Delta t \approx \frac{1}{\omega_{\max}} = \frac{a\sqrt{1-\beta^2}}{u}. \tag{4.20}$$

We can arrive at the estimate (4.20) for the duration of the burst of diffraction radiation in a different way. To do this we determine the time dependence of the field H:

$$H(t) = 2\mathrm{Re}\int_0^\infty H_\omega e^{-i\omega t}\,d\omega.$$

Using the approximate expression (4.9) for H_ω, which is valid at large distances from the edge of the screen, we find that the time dependence of the field H is given by the factor

$$H(r, t) \sim \frac{1}{[(r-ct)^2+\gamma^2a^2]^{1/4}}.$$

From this we see that the spatial extension of the wave packet of diffraction radiation is of order γa and the duration of the burst of radiation is of order $\gamma a/c$. In empty space the wave packet of diffraction radiation propagates without spreading.

The presence of the factor $e^{-2k\gamma a}$ (4.18) can be explained as follows. As we see from (2.15), the field of a charged wire falls off exponentially as we move away from its path, with the decrement $k\gamma$. Thus the amplitude of the currents induced on the screen by the wire is proportional to $e^{-k\gamma a}$. The same exponential dependence also appears in the expression for the radiation field determined by the currents in the screen (cf. (4.4)). The intensity of the radiation is proportional to the square of this factor.

As we see from (4.19), the limiting frequency of the radiation increases proportionally to the energy of the moving source. So, for ultrarelativistic velocities of the source the spectral region will extend out to hard γ radiation. But for such high frequencies the screen can no longer be assumed to be ideally reflecting. This limits the region of applicability of the above treatment.

Integration of (4.17) over angles gives the expression for the spectral density of energy loss per unit length of a linear source,

$$W_\omega = 2\int_0^\pi I(\varphi)\,d\varphi = \frac{q^2}{\omega\gamma}e^{-2k\gamma a}. \tag{4.21}$$

The result is independent of the angle of incidence of the wire. This can be understood pictorially by using geometrical optics. If the source velocity is sufficiently high, the field has properties similar to those of the field of plane free electromagnetic waves. The diffraction radiation from the moving source can then be described approximately as the specular reflection of these waves by the halfplane. Thus, for example, the radiation in the upper halfspace reduces to the reflection of the part of the field of the wire that impinges on the screen. The energy of the field incident on the screen is independent of the angle of incidence of the wire and is determined only by the impact parameter a. A calculation of the energy of the radiation in the approximation of geometrical optics leads to a result that differs from (4.21) only by a factor $(c/u)^2$.

The fact that the total energy of the radiation is independent of the angle of incidence θ of the wire was gotten on the basis of asymptotic formulas, which are not always applicable. Thus, in the special case when $\theta = \pi$ and the velocity of the wire is equal to the velocity of light ($\beta \to 1$), the exact expression for the radiation field goes to zero. Physically this is explained by the fact that in this case the electric vector of the incident field is normal to the halfplane, so that there are no currents induced in the screen.

To find the total loss to radiation from a moving source, we must integrate (4.21) over the whole spectral range. It is obvious that the integral of W_ω (4.21) diverges logarithmically at low frequencies. This is explained by the character of the field of a moving wire at large distances from its path: the field of the wire falls off inversely with distance from it. Consequently the energy in the field of the moving wire diverges logarithmically. The energy in the radiation field has the same kind of divergence.

We mention that taking account of the finite conductivity and the finite thickness of the screen removes the divergence in the radiated energy at low frequencies. Our treatment is then valid only for those frequencies for which the skin depth is smaller than the thickness of the screen.

If the source of the diffraction radiation is a wire carrying current, the determination of the field, as in the case of a charged wire, reduces to solving a pair of integral equations. Physically this case differs from the previous one only in the polarization of the field of the source and, consequently, of the radiation field. In the case of the charged wire the electric field was perpendicular to the wire, and the magnetic field parallel to the wire and the screen edge. In the case of the current-carrying wire the electric field is parallel to the wire (the screen edge), while the magnetic field is perpendicular to the wire. Thus the problem of finding the field of a current-carrying wire moving near the edge of a conducting screen is analogous to the Sommerfeld problem for the case where the magnetic vector of the incident wave is parallel to the edge of the screen. We give the result of the calculations of the spectral and angular distributions of radiation energy from a wire carrying current:

$$I(\varphi)\,d\varphi = \frac{j_0^2}{\pi\omega\gamma^2u^2}e^{-2k\gamma a}\frac{1-\beta\cos\theta}{\beta}\,\frac{\sin^2\frac{\varphi}{2}\,d\varphi}{\left(\cos\varphi+\frac{1}{\beta}\cos\theta\right)^2+\gamma^2\sin^2\theta}, \tag{4.22}$$

where j_0 is the current in the wire and all the other notations have been used earlier.

The main characteristic features of the angular distribution in this case are similar to those already discussed for the charged wire.

Integrating over the angle of observation, taking account of the symmetry relative to the plane of the screen, we find for the total loss in radiation at frequency ω the expression

$$W_\omega = 2\int_0^\pi I(\varphi)\,d\varphi = \frac{j_0^2}{\omega u^2\gamma^3}e^{-2k\gamma a}. \tag{4.23}$$

In the case of the current-carrying wire the radiation losses depend more strongly on source velocity than for the charged wire. The reason is the stronger dependence of the incident field on u, as can be seen by comparing (2.15) and (2.17).

The angular distribution of the radiation and the energy loss from the wire can be estimated approximately by using a pictorial method. We know that the field of a source moving above a plane screen is equivalent to the field of the source and its image, both moving in free space. If the screen has an edge, then when the source gets to the edge, the image appears or disappears, depending on the direction of the velocity of the source. For certain angles of observation the source also may be visible from behind the screen or be covered by it. We know that vanishing or appearance of a uniformly moving source is accompanied by a burst of radiation whose energy can be estimated from (1.1a). For high velocities we get results that differ from the exact results obtained here because they lack the factor $\exp(-2k\gamma a)$. This factor takes account of the influence of the diffraction on the process of appearance or vanishing of the source and its image.

b) Radiation from a charged wire which penetrates a semi-infinite screen during its motion. Suppose that during its motion the source of the electromagnetic field passes through an optical inhomogeneity. We then get a field with a complex diffraction structure with contributions from both transition and diffraction radiation. We shall consider the simplest case of this kind: the passage through a semi-infinite metal screen of a line source of field. We shall assume that the charged wire intersects the semi-infinite screen at a distance d from its edge (Fig. 5).

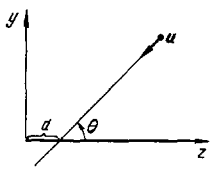

FIG. 5.

In this case the problem again reduces to a pair of integral equations for the unknown current distribution F(w) on the screen:

$$\left.\begin{aligned}&\int F(w)\,ve^{iwz}\,dw=\frac{qk}{2\pi}\left[\frac{\operatorname{sign}(z-d)}{\beta}\sin\theta+i\gamma\cos\theta\right]\\&\qquad\times e^{-i\frac{\omega}{u}z\cos\theta-k\gamma\sin\theta|z-d|}\qquad\text{for}\quad z>0,\\&\int F(w)\,e^{iwz}\,dw=0\qquad\qquad\text{for}\quad z<0.\end{aligned}\right\}\quad(4.24)$$

Unlike the problems treated previously, in which the source did not penetrate the screen, the right side of the first equation in (4.24) has different analytic expressions depending on the sign of the difference $(z-d)$ (the electric field tangent to the screen falls off exponentially toward both sides from the point of intersection of the halfplane by the source). Thus there are difficulties in using the method applied earlier for finding a solution. It is more convenient to use the results of solving (3.19), which were given at the end of Sec. 3. Setting $g_2(z) = 0$ in (3.20) and equating $g_1(z)$ to the right side of the first of Eqs. (4.24), we get the required expression for F(w):

$$F(w)=\frac{qke^{-i\frac{\pi}{4}}}{i4\pi^{5/2}\sqrt{k-w}}\times\left\{\left[\frac{i\gamma\cos\theta+\frac{1}{\beta}\sin\theta}{w+\frac{\omega}{u}\cos\theta-ik\gamma\sin\theta}-\frac{i\gamma\cos\theta-\frac{1}{\beta}\sin\theta}{w+\frac{\omega}{u}\cos\theta+ik\gamma\sin\theta}\right]\right.$$
$$\times e^{i\frac{\omega}{u}\cos\theta\cdot d-iwd}\int_0^d e^{i(k+w)\xi}\frac{d\xi}{\sqrt{\xi}}+\frac{i\gamma\cos\theta+\frac{1}{\beta}\sin\theta}{w+\frac{\omega}{u}\cos\theta-ik\gamma\sin\theta}$$
$$\times e^{k\gamma d\sin\theta}\int_d^\infty e^{i\left[k-\frac{\omega}{u}\cos\theta+ik\gamma\sin\theta\right]\xi}\frac{d\xi}{\sqrt{\xi}}+\frac{i\gamma\cos\theta-\frac{1}{\beta}\sin\theta}{w+\frac{\omega}{u}\cos\theta+ik\gamma\sin\theta}$$
$$\left.\times e^{-k\gamma d\sin\theta}\int_0^d e^{i\left[k-\frac{\omega}{u}\cos\theta-ik\gamma\sin\theta\right]\xi}\frac{d\xi}{\sqrt{\xi}}\right\}.\qquad(4.25)$$

Before analyzing this formula we rewrite it in a more convenient form, using the definition of the complex angle of incidence θ_0 from formula (4.6):

$$F(w)=\frac{qke^{-i\frac{\pi}{4}}}{4i\pi^{5/2}\sqrt{k-w}}\left\{\left[\frac{\sin(\theta+\theta_0)}{w+k\cos(\theta+\theta_0)}+\frac{\sin(\theta-\theta_0)}{w+k\cos(\theta-\theta_0)}\right]\right.$$
$$\times e^{i\left(\frac{\omega}{u}\cos\theta-w\right)d}\int_0^d e^{i(k+w)\xi}\frac{d\xi}{\sqrt{\xi}}$$
$$+\frac{\sin(\theta+\theta_0)}{w+k\cos(\theta+\theta_0)}e^{k\gamma d\sin\theta}\int_d^\infty e^{2ik\sin^2\frac{\theta+\theta_0}{2}\xi}$$
$$\left.\times\frac{d\xi}{\sqrt{\xi}}+\frac{\sin(\theta-\theta_0)}{w+k\cos(\theta-\theta_0)}e^{-k\gamma d\sin\theta}\int_0^d e^{2ik\sin^2\frac{\theta-\theta_0}{2}\xi}\frac{d\xi}{\sqrt{\xi}}\right\}.\qquad(4.26)$$

Formulas (4.26), (3.9) and (2.30) completely determine the field of radiation of a linear source intersecting a screen. Let us compare (4.26) with the simpler formula (4.7) which gives the radiation field for the problem where the wire moves without intersecting the screen. In contrast to (4.7), where the angles θ and θ_0 appear only in the combination $\theta + \theta_0$, the currents given by (4.26) also depend on $\theta - \theta_0$. The reason is that when the source penetrates the screen both of the "Cerenkov" waves accompanying the source suffer diffraction. One of them is incident at angle $\theta - \theta_0$, the other at angle $\theta + \theta_0$. But, if the source does not intersect the screen, only the one which is incident at angle $\theta + \theta_0$ is diffracted. We state once more that it is largely a matter of convention to call the incident waves Cerenkov waves, since for uniform motion of the source in vacuum the angle θ_0 is imaginary.

Let us consider the field in the wave zone. As in the preceding problem, we restrict our consideration to calculating the components of the magnetic field H_x at large distances from the screen. Using the method of steepest descent, we get

$$H_x=\frac{q\cos\frac{\varphi}{2}e^{ikr}}{\pi c\sqrt{r}}\left\{\left[\frac{\sin(\theta+\theta_0)}{\cos\varphi+\cos(\theta+\theta_0)}+\frac{\sin(\theta-\theta_0)}{\cos\varphi+\cos(\theta-\theta_0)}\right]\right.$$
$$\times e^{i\left(\frac{\omega}{u}\cos\theta-k\cos\varphi\right)d}\int_0^d e^{2ik\cos^2\frac{\varphi}{2}\xi}\frac{d\xi}{\sqrt{\xi}}$$
$$+\frac{\sin(\theta+\theta_0)}{\cos\varphi+\cos(\theta+\theta_0)}e^{k\gamma d\sin\theta}\int_d^\infty e^{2ik\sin^2\frac{\theta+\theta_0}{2}\xi}\frac{d\xi}{\sqrt{\xi}}$$
$$\left.+\frac{\sin(\theta-\theta_0)}{\cos\varphi+\cos(\theta-\theta_0)}e^{-k\gamma d\sin\theta}\int_0^d e^{2ik\sin^2\frac{\theta-\theta_0}{2}\xi}\frac{d\xi}{\sqrt{\xi}}\right\}.\qquad(4.27)$$

The field of the radiation from a charged wire, given by (4.27), has a complex diffraction structure. In the two limiting cases of $d \to 0$ and $d \to \infty$, the expression for the field simplifies considerably and the result can be understood pictorially. If the point where the source penetrates the screen approaches the edge of the screen ($d \to 0$), formula (4.27) goes over into (4.9), which gives the radiation field from a source passing near the edge of the halfplane but not intersecting it; for comparison with (4.9) we should set $a = 0$. Thus the solution depends continuously on the impact parameter.

We also consider the behavior of the field when the point of intersection $z = d$ is far from the edge of the screen, $d \to \infty$. Then (4.27) becomes

$$H_x = \frac{q}{c\sqrt{2\pi kr}} e^{ikr + i\frac{\pi}{4} + i\left(\frac{\omega}{u}\cos\theta - k\cos\varphi\right)d} \times \left[\frac{\sin(\theta+\theta_0)}{\cos\varphi + \cos(\theta+\theta_0)} + \frac{\sin(\theta-\theta_0)}{\cos\varphi+\cos(\theta-\theta_0)}\right]. \tag{4.28}$$

The expression found for the field coincides with the field of the transition radiation that appears when a linear source is incident on an ideally conducting screen. Formula (4.27) enables us to determine the effect of the distant edge of the semi-infinite screen on the transition radiation, i.e., to compute the correction terms in (4.28) for large values of d. From the asymptotic form of the Fresnel integral it follows that these corrections have relative order $1/\sqrt{kd}$. This result is not obvious beforehand. Since the field of a moving charge falls off exponentially with distance from its trajectory, one might expect that as the point of intersection moves away from the edge of the screen the field scattered by the edge would also fall off exponentially. This is just the dependence of scattered field on impact parameter that holds when the source does not intersect the screen during its motion. But if the trajectory of the particle intersects the screen, as shown above, the amplitude of the field scattered by the edge of the screen is damped much more slowly, according to an algebraic law. The reason for this difference in the field scattered by the screen can be given as a pictorial argument. The point of intersection of the screen by the charged wire is a source of transition radiation. This radiation goes outward in the form of cylindrical waves whose amplitude falls off inversely as $\sqrt{kr}$. At the edge of the screen the amplitude will be of relative order $1/\sqrt{kd}$. The corrections to the transition radiation caused by the scattering of waves at the edge of the screen will be of the same order of magnitude.

The diffraction radiation from a current-carrying wire has similar features in the case where the trajectory of the source intersects the screen. We shall not discuss that case separately.

c) Diffraction of Vavilov-Cerenkov radiation in the motion of a line source near a semi-infinite screen.[23] To conclude Sec. 4 we consider the special features of the diffraction radiation when the source moves in a refracting medium. We shall assume that the velocity of the source exceeds the phase velocity of propagation of electromagnetic waves in the medium. Then uniform motion of a source is accompanied by radiation of electromagnetic waves, whose propagation direction makes an angle with the direction of motion of the source that is given by (1.11). The field of a uniformly moving source in an unbounded refracting medium is determined from the Hertz vector (2.31) through formulas (2.30). We shall restrict our treatment to the case where the source does not intersect the semi-infinite plane screen during the course of its motion. Suppose that the screen is located within an unbounded refracting medium with dielectric constant ϵ. For the case of a charged wire (or a modulated plane electron beam) we shall describe the radiation by means of the Hertz vector $\mathbf{\Pi}^1$, which has components only along the z axis:

$$\Pi^1 = -\frac{2\pi}{\omega}\int_{-\infty}^{\infty}\frac{e^{iv'|y|}}{v'} f(w)\, e^{iwz}\, dw, \tag{4.29}$$

where

$$v' = \sqrt{\frac{\omega^2}{c^2}\varepsilon - w^2} = \sqrt{k'^2 - w^2}, \qquad \operatorname{Im} v' > 0. \tag{4.30}$$

The quantity v' defined here differs from the v introduced previously (3.10) in having $k = \omega/c$ replaced by $k' = (\omega/c)\sqrt{\epsilon}$, i.e., by taking account of the refracting properties of the medium. Without writing down the pair of integral equations for the function f(w), to which the problem reduces, we give the expression for the Fourier components of the currents induced in the halfplane:

$$f(w) = \frac{q\sqrt{2k'}}{4\pi^2 i}\,\frac{\cos\frac{\theta+\theta_0}{2}\, e^{i\frac{\omega}{u}\sqrt{\varepsilon\beta^2-1}\,a}}{\sqrt{k'-w}\,[w + k'\cos(\theta+\theta_0)]}. \tag{4.30'}$$

Right from the start we assume that the velocity of motion of the source exceeds the phase velocity of propagation of electromagnetic waves in the medium i.e., $\epsilon\beta^2 > 1$. Then the angle θ_0, given by the equation

$$\cos\theta_0 = \frac{1}{\beta\sqrt{\varepsilon}}, \tag{4.31}$$

is the real angle between the direction of propagation of the Vavilov-Cerenkov radiation and the velocity of the source.

From formulas (4.29) and (4.30) we get the following expression for the magnetic field of the radiation:

$$H'_x = -\frac{q\sqrt{2k'}}{2\pi c i}\cos\frac{\theta+\theta_0}{2}\, e^{i\frac{\omega}{u}a\sqrt{\varepsilon\beta^2-1}} \times \operatorname{sign} y \int \frac{e^{iv'|y| + iwz}}{\sqrt{k'-w}\,[w+k'\cos(\theta+\theta_0)]}\, dw. \tag{4.32}$$

Let us compare this expression with (4.8) for the magnetic field of the radiation from a source moving in vacuum ($\epsilon = 1$). The presence of the refracting medium results in the appearance of various new physical features of the problem of diffraction radiation. We have already pointed out one of them: the angle θ_0 given by (4.31) becomes real for superlight velocities. The second difference is that for superluminal velocities of the source the exponential falloff of the radiated field with increasing impact parameter is not present. This is related to the fact that the field of a superlight source has wavelike character, i.e., is not damped as one moves away from its path. Thus the intensity of the diffraction radiation, which is proportional to $|H^1|^2$, is independent of the impact parameter a.

The scattered field H^1 (4.32) can be expressed in terms of the Fresnel integrals, well known from the theory of diffraction:

$$H_x^1 = -\frac{q}{\sqrt{\pi i c}} e^{i\frac{\omega}{u} a \sqrt{\varepsilon\beta^2-1}} \times \left\{ e^{-ik'r\cos(\varphi-\theta-\theta_0)} \int_\infty^{-\sqrt{2k'r}\cos\frac{\varphi-\theta-\theta_0}{2}} e^{it^2}dt - e^{-ik'r\cos(\varphi+\theta+\theta_0)} \int_\infty^{\sqrt{2k'r}\cos\frac{\varphi+\theta+\theta_0}{2}} e^{it^2}dt \right\}. \tag{4.33}$$

We mention that if we take as the limit in the first integral in the curly brackets $-\infty$ instead of ∞, the formula will describe the total field, which is the sum of the field of the source and the scattered field. Expression (4.33), aside from the factor in front of the curly brackets, coincides with the solution discussed above for the Sommerfeld problem of diffraction of a plane electromagnetic wave whose magnetic vector is parallel to the edge of the screen and which is incident on the screen at an angle $\theta + \theta_0$. Thus, in the case of superluminal motion of the source the scattered field is the result of diffraction of the Vavilov-Cerenkov radiation that accompanies the motion of the charge.

We also note that the losses of energy from a line source to radiation in the presence of the screen remain the same as in an unbounded refracting medium, because the source runs ahead of the scattered Vavilov-Cerenkov field and does not interact with it.

A completely analogous solution exists for the diffraction by a halfplane of Vavilov-Cerenkov radiation for the case of a uniformly moving line source of current. We give the expression for the scattered electric field for this case:

$$E_x^1 = -\frac{j_0 e^{i\frac{\omega}{u} a\sqrt{\varepsilon\beta^2-1}}}{\sqrt{\pi i}\, c^2\sqrt{\varepsilon\beta^2-1}} \left\{ e^{-ik'r\cos(\varphi-\theta-\theta_0)} \int_\infty^{-\sqrt{k'r}\cos\frac{\varphi-\theta-\theta_0}{2}} e^{it^2}dt + e^{-ik'r\cos(\varphi+\theta+\theta_0)} \int_\infty^{\sqrt{k'r}\cos\frac{\varphi+\theta+\theta_0}{2}} e^{it^2}dt \right\}. \tag{4.34}$$

Formulas (4.33) for the scattered field H^1 of a charged two-dimensional source and (4.34) for the scattered field E^1 of a current source are exact. As we see from these formulas, the scattered fields depend on the argument $\sqrt{k'r}\cos\{1/2[\varphi \pm (\theta+\theta_0)]\}$. For large values of these arguments one can use asymptotic expressions for the Fresnel integral

$$\int_\infty^s e^{it^2}dt \approx \frac{e^{is^2}}{2is}, \tag{4.35}$$

which gives the approximate dependence of E and H in the far zone.

5. RADIATION OF A LINE SOURCE MOVING NEAR THE OPEN END OF A PLANE WAVEGUIDE

It is of interest to consider the properties of the diffraction radiation when the scattering bodies form a resonant system in which normal modes can be excited. Under these conditions the passing source excites within the scattering system a discrete set of its norm modes, while outside we get a radiation that is characterized by a continuous spectrum of frequencies. We shall consider the excitation of the simplest scattering structure of this type—a plane waveguide with an open end. The treatment is of additional interest because we are here dealing with one of the few exactly solvable problems of excitation of electromagnetic oscillations in an open resonator. The geometry of the problem is clear from Fig. 6.

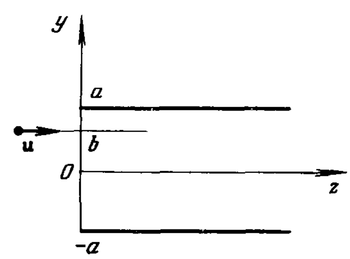

FIG. 6.

The plane waveguide is formed by two semi-infinite thin plates ($y = \pm a$, $z > 0$). We first consider the case when the wire, parallel to the x axis and carrying a current with linear density $j_x = j_0$, moves into the wave guide with constant velocity $u_z = u$. The distance of the trajectory of the current-carrying wire from the axis of the waveguide is b (b < a). The one-component vector potential $\mathbf{A} = \{A_x, 0, 0\}$ describing the total electromagnetic field is conveniently written in this case as a sum

$$A_\omega(y,z) = A_\omega^0(y,z) + A_\omega^1(y,z), \tag{5.1}$$

where

$$A_\omega^0(y,z) = \frac{j_0}{u\gamma\omega} e^{i\frac{\omega}{u}z} \left\{ e^{-|y-b|k\gamma} - e^{-k\gamma a} \times \left[\frac{\operatorname{ch}(k\gamma y)\operatorname{ch}(k\gamma b)}{\operatorname{ch}(k\gamma a)} + \frac{\operatorname{sh}(k\gamma y)\operatorname{sh}(k\gamma b)}{\operatorname{sh}(k\gamma a)} \right] \right\} \tag{5.2*}$$

*ch ≡ cosh, sh ≡ sinh.

is the vector potential describing the field of the current-carrying wire moving in an unbounded plane waveguide. This choice of A^0_ω is not necessary. We could choose for A^0_ω any other solution of the inhomogeneous problem, for example, the vector potential (2.17) of a current-carrying wire moving in free space. The vector potential A^1_ω describes the required free field which must be added to A^0_ω in order to satisfy the boundary conditions on the walls of the semi-infinite waveguide. The vector potential of the free field A^1_ω can be expressed in terms of currents $j^1_\omega(z)$ flowing in the walls of the waveguide and caused by the presence of the open end:

$$A^1_\omega(y, z) = \frac{1}{c}\int \frac{e^{ikR}}{R} j^1_\omega(z)\, ds. \tag{5.3}$$

Formula (5.3) enables us to reduce the determination of the vector potential A^1 to finding the currents induced on the plates of the wave guide by the moving source.

The currents $j^1(z)$ drop with increasing z and can be expanded in Fourier integrals. The currents on the top plate ($y = +a$), $j^1_a(z)$, and those on the bottom plate ($y = -a$), $j^1_{-a}(z)$, can be written in the form

$$j^1_a(z) = j_+(z) + j_-(z), \qquad j^1_{-a}(z) = j_+(z) - j_-(z),$$

where

$$j_\pm(z) = \int F_\pm(w) e^{iwz}\, dw. \tag{5.4}$$

The subscripts + and − denote, respectively, the even and odd parts of the currents and their Fourier transforms.

The condition for vanishing of the tangential electric field E_z on the walls of the waveguide and the requirement that there be no total current on the extension of the walls leads to the following independent system of equations for determining $F_+(w)$ and $F_-(w)$:

$$\left.\begin{aligned} &\int_{-\infty}^{\infty} F_\pm(w) L_\pm(w) e^{iwz}\, dw = 0 \qquad \text{for} \quad z > 0, \\ &\int_{-\infty}^{\infty} F_\pm(w) e^{iwz}\, dw = B_\pm e^{i\frac{\omega}{u}z} \qquad \text{for} \quad z < 0, \end{aligned}\right\} \tag{5.5}$$

where

$$L_\pm(w) = \frac{1 \pm e^{i2av}}{v}, \qquad B_+ = \frac{j_0}{4\pi u}\frac{\text{ch}\,(k\gamma b)}{\text{ch}\,(k\gamma a)}, \qquad B_- = \frac{j_0}{4\pi u}\frac{\text{sh}\,(k\gamma b)}{\text{sh}\,(k\gamma a)},$$
$$v = \sqrt{k^2 - w^2} \qquad (\text{Im}\, v > 0). \tag{5.6}$$

The solution of the system of integral equations (5.5) is found in precisely the same way as the solution of the system (4.3). The result is

$$F_+(w) = -\frac{j_0}{8\pi^2 ui}\frac{\psi_2\left(\frac{\omega}{u}\right)}{\psi_2(w)}\frac{\sqrt{k-w}}{\sqrt{k-\frac{\omega}{u}}}\frac{\text{ch}\,(k\gamma b)}{\text{ch}\,(k\gamma a)}\frac{1}{w-\frac{\omega}{u}}, \tag{5.7}$$

$$F_-(w) = -\frac{j_0}{8\pi^2 ui}\frac{\varphi_2\left(\frac{\omega}{u}\right)}{\varphi_2(w)}\frac{\sqrt{k-w}}{\sqrt{k-\frac{\omega}{u}}}\frac{\text{sh}\,(k\gamma b)}{\text{sh}\,(k\gamma a)}\frac{1}{w-\frac{\omega}{u}}. \tag{5.8}$$

The functions φ and ψ appearing in the solution are determined by splitting the kernel $L_\pm(w)$ of the system (5.5) into factors analytic in the upper (φ_1 and ψ_1) and the lower (φ_2 and ψ_2) halfplanes of the complex variable w:

$$L_+(w) = \frac{1 + e^{i2av}}{v} = \frac{\psi_1(w)\psi_2(w)}{v},$$
$$L_-(w) = \frac{1 - e^{i2av}}{v} = \frac{\varphi_1(w)\varphi_2(w)}{v}. \tag{5.9}$$

The explicit expressions for the functions φ and ψ are quite involved, but since the properties of these functions determine the character of the diffraction radiation, we give them:

$$\left.\begin{aligned} &\varphi_1(w) = \sqrt{-2i\frac{\sin ka}{k}}\sqrt{k+w}\, e^{-\frac{wa}{2} + i\frac{ka}{\pi}M_1(\tau)} \times \prod_{m=1}^{\infty}\left(1 + \frac{w}{k\sin\sigma_m}\right), \\ &M_1(\tau) = \left(\frac{\pi}{2} - \tau\right)\cos\tau + \sin\tau, \\ &\tau = \arcsin\left(\frac{w}{k}\right), \qquad \sigma_m = \arccos\left(\frac{\pi m}{ka}\right); \\ &\varphi_2(w) = \varphi_1(-w), \end{aligned}\right\} \tag{5.10}$$

$$\left.\begin{aligned} &\psi_1(w) = \sqrt{2\cos(ka)}\, e^{-\frac{wa}{2} + i\frac{ka}{\pi}M_1(\tau)}\prod_{m=1}^{\infty}\left(1 + \frac{w}{k\sin\tau_m}\right), \\ &\tau_m = \arccos\left[\frac{\pi}{ka}\left(m - \frac{1}{2}\right)\right] \qquad (m = 1, 2, \ldots), \\ &\psi_2(w) = \psi_1(-w). \end{aligned}\right\} \tag{5.11}$$

Formulas (5.7) and (5.8) give the expressions for the Fourier components of the even and odd parts of the currents induced on the plates of the waveguide. In going from the Fourier components $F_+(w)$ and $F_-(w)$ to the currents $j_\pm(z)$ the contour of integration in (5.4) should be chosen so that it circles above the pole at $w = \omega/u$. If we choose A^0 in the form of (2.17) instead of (5.2), the expressions for $F_\pm(w)$ would remain the same, but the contour of integration in (5.4) would then have to go around below the pole at $w = \omega/u$.

The vector potential $A^1_\omega(y, z)$ of the radiation field is expressed in terms of $F_+(w)$ and $F_-(w)$ as follows:

$$A^1_\omega(y, z) = \frac{2\pi i}{c}\int e^{iwz}[F_+(w)(e^{iv|y-a|} + e^{iv|y+a|}) + F_-(w)(e^{iv|y-a|} - e^{iv|y+a|})]\frac{dw}{v}. \tag{5.12}$$

The nonvanishing components of the radiation field are expressed in terms of A^1_ω by the formulas

$$E^1_x = ikA^1_\omega, \qquad H^1_y = \frac{\partial A^1_\omega}{\partial z} = \frac{1}{ik}\frac{\partial E^1_x}{\partial z},$$
$$H^1_z = -\frac{\partial A^1_\omega}{\partial y} = -\frac{1}{ik}\frac{\partial E^1_x}{\partial y}. \tag{5.13}$$

As we see from (5.13), all the components of the radiation field for the problem considered here ("electric polarization") can be expressed in terms of E^1_x. We therefore limit our investigation to this one electric field component. We start by studying the field inside the waveguide, $|y| < a$, $z > 0$. Then formulas (5.12) and (5.13) give

$$E_x^1 = -\frac{4\pi k}{c}\int e^{iwz+iva}\left[F_+(w)\cos(vy) - iF_-(w)\sin(vy)\right]\frac{dw}{v}. \tag{5.14}$$

This expression is also valid outside the wave guide ($z < 0$), so long as $|y| < a$.

The field inside the waveguide is determined by the analytic properties of the integrand in the upper half-plane of the complex variable w. One finds that the only singularities of the integrand of (5.14) in the upper halfplane are poles. The function $F_+(w)$ has poles at the points $w_m = k \sin \tau_m$ (the zeros of the function $\psi_2(w)$), while $F_-(w)$ has poles at the points $w_m = k \sin \sigma_m$ (the zeros of the function $\varphi_2(w)$). The quantities σ_m and τ_m are determined by formulas (5.10) and (5.11) and represent the angles made by the wave vectors of the asymmetric and symmetric waveguide modes with the y axis. The expression (5.14) for the field thus reduces to a sum of residues at these poles. Physically this corresponds to the excitation of a discrete spectrum of norm waveguide modes when the source passes in through the open end of the waveguide. By calculating the integral (5.14) using residues, we get the radiation field inside the waveguide in the form

$$E_x^1(y, z) = \sum_m \left[R_{+m}e^{ik\sin(\tau_m z)}\cos(k\cos\tau_m y) + R_{-m}e^{ik\sin(\sigma_m z)}\sin(k\cos\sigma_m y)\right], \tag{5.15}$$

where the respective coefficients of excitation for the symmetric magnetic waves, R_{+m}, and the asymmetric magnetic waves, R_{-m}, are

$$R_{+m} = \frac{j_0}{uc}\frac{(-1)^{m-1}\psi_1(k\sin\tau_m)\psi_2\left(\frac{\omega}{u}\right)}{2a\sin\tau_m} \times \frac{\sqrt{k-k\sin\tau_m}}{\sqrt{k-\frac{\omega}{u}}\left(k\sin\tau_m-\frac{\omega}{u}\right)}\frac{\mathrm{ch}(k\gamma b)}{\mathrm{ch}(k\gamma a)},$$

$$R_{-m} = \frac{j_0}{uc}\frac{(-1)^{m-1}\varphi_1(k\sin\sigma_m)\varphi_2\left(\frac{\omega}{u}\right)}{2a\sin\sigma_m} \times \frac{\sqrt{k-k\sin\sigma_m}}{\sqrt{k-\frac{\omega}{u}}\left(k\sin\sigma_m-\frac{\omega}{u}\right)}\frac{\mathrm{sh}(k\gamma b)}{\mathrm{sh}(k\gamma a)}. \tag{5.16}$$

From formula (5.15) we see that the diffraction field inside the guide is a superposition of normal mode waves coming from the open end into the body of the waveguide. The total flux of energy radiated into the waveguide can be determined by integrating the longitudinal component of the Umov-Poynting vector over the cross section of the waveguide. The result is

$$W_\omega = c\int_{-a}^{a} E_{\omega x}H_{-\omega y}\,dy = ac\sum_m\{|R_{+m}|^2\sin\tau_m + |R_{-m}|^2\sin\sigma_m\}. \tag{5.17}$$

This last expression is valid only if we are at a distance from the open end of the waveguide that exceeds the shaping length of the radiation for the lowest harmonics (i.e., the waveguide modes with the smallest values of m). With increasing harmonic number the size of the region of shaping decreases. Thus, if the distance from the open end of the waveguide exceeds the size of the shaping region for the lowest harmonics, this condition is also satisfied for the higher harmonics. The characteristic linear dimension of the shaping region, which is given by (1.4), is of order

$$l_{\mathrm{sh}} \sim \frac{u}{\omega(1-\beta\sin\tau_m)} \tag{5.18}$$

for the symmetric waveguide modes; for the asymmetric modes, τ_m should be replaced by σ_m. The summation in (5.17) should be taken only over those values of m for which $\sin\tau_m$ or $\sin\sigma_m$ is real at the particular frequency. Physically this means that contributions to the loss in radiation inside the waveguide come only from waves propagating without damping.

Formula (5.17) determines the losses of energy of such a source in exciting a plane waveguide, as a function of the sign and absolute value of the source velocity. We examine the dependence of the energy loss on source velocity for the example of excitation of the symmetric harmonic with index m (the H_m wave). Losses to excitation of this harmonic are described in expression (5.17) by the term proportional to $|R_m|^2$. Dropping the factor independent of source velocity, we get

$$W_m(\beta) \approx \frac{|\beta|}{(1-\beta)(1-\beta\sin\tau_m)^2}\left|\psi_2\left(\frac{\omega}{u}\right)\right|^2\frac{\mathrm{ch}^2(k\gamma b)}{\mathrm{ch}^2(k\gamma a)}. \tag{5.19}$$

As $u \to 0$ the function $\psi_2(\omega/u)$ tends to unity[6] and we get

$$W_m(\beta) \approx |\beta|e^{-\frac{2k}{|\beta|}(a-b)} \qquad (\beta\to 0), \tag{5.20}$$

i.e., at low velocities the losses in radiation of the m-th symmetric harmonic are exponentially small. The dependence on velocity is the same as for the passage of a current-carrying wire at a distance $(a - b)$ from an ideally conducting halfplane. As we see from (5.20), the energy radiated in this case is independent of whether the source moves into or out of the waveguide. With increasing velocity of the source a dependence on the sign of u appears. When $|\beta| \to 1$ the energy radiated into the waveguide depends essentially on the sign of u. If the velocity of the wire is close to light velocity and the wire moves out of the waveguide,

$$W_m \approx |\psi_2(-k)|^2\frac{\mathrm{ch}^2(k\gamma b)}{\mathrm{ch}^2(k\gamma a)}. \tag{5.21}$$

If $\beta \approx 1$ and the wire moves into the waveguide

$$W_m \approx \frac{|\psi_2(k)|^2}{(1-\beta)(1-\beta\sin\tau_m)^2}\frac{\mathrm{ch}^2(k\gamma b)}{\mathrm{ch}^2(k\gamma a)}. \tag{5.22}$$

We see that the excitation of the waveguide is greater when the source moves in than when it moves out.

The functions $|\psi_1(w)|^2$ and $|\psi_2(w)|^2$ can be expressed in closed form, without infinite products. For those frequencies at which only the first harmonics are excited in the waveguide, the expressions for $|R_{+1}|^2$ and $|R_{-1}|^2$ become simple. For example, the square modulus of the excitation coefficient R_{+1} for the fundamental symmetric harmonic is

$$|R_{+1}|^2 = \frac{4j_0^2|\beta|a}{\pi c} \times \frac{(1-\sin\tau_1)}{(1-\beta)(1-\beta^2\sin^2\tau_1)} \frac{\mathrm{ch}^2(k\gamma b)}{\mathrm{ch}^2(k\gamma a)} e^{\frac{-\omega}{u}a(1-\beta\sin\tau_1)-k\gamma a(1-\mathrm{sign}\,u)}. \quad (5.22')$$

Thus, the passage of a uniformly moving source through the end of the waveguide is accompanied by the excitation of normal waveguide modes, which move from the open end into the body of the guide. At low source velocities the amplitudes of the excited waves are exponentially small and independent of the sign of the velocity of the source. At relativistic velocities a marked asymmetry in the dependence on the sign of the velocity develops. Excitation of the guide on entry of the source is greater than on its emergence. In all cases the intensity of the radiation drops off exponentially at high frequencies according to the law

$$W_\omega \sim e^{-2k\gamma(a-b)} \qquad (k\gamma(a-b) \gg 1). \quad (5.23)$$

The asymmetry of the radiation intensity as a function of the sign of the source velocity is analogous to the corresponding asymmetry for the transition radiation.

We now consider the field excited by the motion of a current-carrying wire outside the waveguide. From formula (5.12), with $y > a$, we find the field in the space above the waveguide:

$$E_x = -\frac{4\pi k}{c}\int e^{iwz+ivy}[F_+(w)\cos(va) - iF_-(w)\sin(va)]\frac{dw}{v}. \quad (5.24)$$

The field in the space below the waveguide $y < -a$ is given by a formula that is obtained from (5.24) by changing the signs in front of y and $F_-(w)$. We calculate the radiation field at large distances from the open end of the waveguide. To do this we change to polar coordinates r, φ according to the formulas $z = r\cos\varphi$, $y = r\sin\varphi$. Evaluating the integral (5.24) by the saddle-point method, we get the following expression for the radiation field in the far zone:

$$E_x = H_\varphi = -\frac{4\pi k}{c}\sqrt{\frac{2\pi}{kr}}\,e^{i\left(kr-\frac{\pi}{4}\right)}[F_+(k\cos\varphi)\cos(ka\sin\varphi) - iF_-(k\cos\varphi)\sin(ka\sin\varphi)]. \quad (5.25)$$

This formula is valid over the whole space external to the waveguide, at distances from the open end exceeding the shaping length of the radiation (1.4).

The energy radiated into the angular range from φ to $\varphi + d\varphi$ is written in the form

$$W(\varphi)\,d\varphi = c|E_x|^2 r\,d\varphi, \quad (5.26)$$

where E_x is given by (5.25).

In the special case where the current-carrying wire enters or emerges along the axis of the waveguide ($b = 0$) and the excited field is symmetric with respect to the plane $y = 0$, we have

$$W(\varphi) = \frac{j_0^2\beta\sin^2\frac{\varphi}{2}\left|\psi_2\left(\frac{\omega}{u}\right)\right|^2|\psi_1(k\cos\varphi)|^2}{4c^2\pi\omega(1-\beta)(1-\beta\cos\varphi)^2\mathrm{ch}^2(k\gamma a)}. \quad (5.27)$$

At large frequencies $\omega \to \infty$ the function ψ_2 goes to unity and (5.27) coincides with (4.22), which gives the radiation loss for the case of a semi-infinite screen. At low frequencies, where the wave length of the radiation is comparable to the height of the waveguide, the angular distribution of the radiation is determined by the type of wave excited in the waveguide. For example, when $\pi/2 < ka < 3\pi/2$, which corresponds to the condition for undamped propagation only of the fundamental symmetric mode H_{01}, we have

$$|\psi_1(k\cos\varphi)|^2 = 2e^{-ka\cos\varphi}\frac{\cos\varphi+\sin\tau_1}{\cos\varphi-\sin\tau_1}\cos(ka\sin\varphi). \quad (5.28)$$

Substituting this value in (5.27), we get the angular distribution of the radiation in this frequency range.

As we see from (5.27), the radiation from a current-carrying wire into the external region depends on the sign of u, i.e., on the direction of motion of the source.

Diffraction radiation arises not only when the source passes through the open end of the guide, but also when the source moves in free space past the open end of the waveguide.[12,13] Suppose, for example, that a current-carrying wire is moving along the straight line characterized by the impact parameter **b** and the angle θ (Fig. 7). We assume that the trajectory of the wire does not intersect the walls of the waveguide. The currents induced by the wire on the guide walls are found by the same method as in the preceding problem of entry or emergence of the wire. We shall give the expressions for the Fourier components of the even and odd components of the induced current, $F_+(w)$ and $F_-(w)$:

$$F_+(w) = \frac{j_0 v_0}{8\pi^2\beta\omega\gamma}e^{-k\gamma b+iav_0}\frac{\sqrt{k-w}}{\sqrt{k-w_0}}\frac{\psi_2(w_0)}{\psi_2(w)}\frac{1}{w-w_0},$$

$$F_-(w) = \frac{j_0 v_0}{8\pi^2\beta\omega\gamma}e^{-k\gamma b+iav_0}\frac{\sqrt{k-w}}{\sqrt{k-w_0}}\frac{\varphi_2(w_0)}{\varphi_2(w)}\frac{1}{w-w_0}, \quad (5.29)$$

where we use $w_0 = -k/\beta\cos\theta + ik\gamma\sin\theta$, $v_0 = v(w_0) = -k/\beta\sin\theta - ik\gamma\cos\theta$. The quantities w_0 and v_0 are the components of the wave vector (along the z and y axes respectively) associated with the source of the

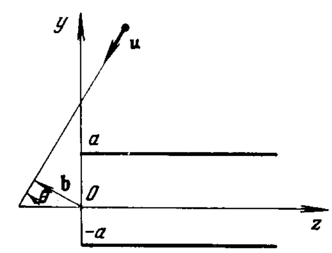

FIG. 7.

inhomogeneous plane wave incident on the plates of the waveguide. The vector potential of the scattered field is again given by (5.12). By now we must assume that the contour of integration circles the pole at $w = w_0$ from below. The reason is that we have chosen for A^0_ω in (5.1) the field (2.17) of the moving wire in free space. Analysis of the expressions for the radiation field in the far zone, and of the coefficients of excitation of normal modes of the waveguide in the region between the plates gives the same qualitative conclusions as for the problem considered above of motion through the open end of the waveguide.

If the separation of the plates of the waveguide goes to zero, the odd component $F_-(w)$ of the current density on the plates goes to zero, while $F_+(w)$ gives the Fourier component of the current density induced in a semi-infinite screen.

The case where the source of the field is a charged wire (or a plane modulated wave of charge density) differs from the problem considered here in the fact that the excited waves have a different polarization (the waves excited are not "magnetic", but "electric" waves, for which the magnetic vector is parallel to the edges of the plates forming the guide). Since the method of solution and the physical features of the resulting radiation are similar to that considered earlier, we shall not present the results, but rather refer the interested reader to the papers.[12,13]

We note, in conclusion, that the diffraction radiation arising when a source passes by the open end of a waveguide can be used to detect beams of charged particles.

6. RADIATION FROM A SOURCE MOVING UNIFORMLY IN THE NEIGHBORHOOD OF A DIFFRACTING LATTICE FORMED BY A SYSTEM OF EQUALLY SPACED IDEALLY CONDUCTING HALFPLANES

If the optical inhomogeneities are arranged periodically in space, the diffraction radiation is characterized by definite resonance properties. We explain these properties for the example of a diffracting lattice formed by equally spaced ideally conducting parallel halfplanes. In this case, in contrast to most of the examples previously considered of the radiation from uniformly moving sources in linear periodic media,[14-18] the problem admits of an exact solution.[19,20] The treatment of this problem is also of interest from the point of view of possibly generating electromagnetic radiation by beams of charged particles.

Consider the system of parallel ideally conducting halfplanes, described by the equations $z = na$, $y > 0$ ($n = 0, \pm 1, \pm 2, \ldots$) (Fig. 8). Suppose that a uniformly charged wire with linear charge density q moves past this system with constant velocity u. We denote the distance from the trajectory of the wire to the system by b, so the equation of the trajectory of the wire is

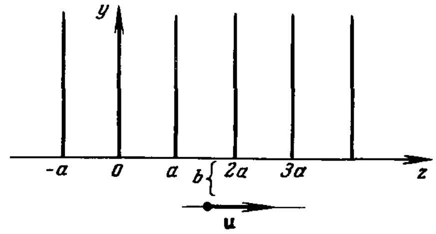

FIG. 8.

$y = -b$, $z = ut$. The problem is to determine the fields **E** and **H** excited by this source and satisfying the boundary conditions on the plates of the system (which is sometimes called a "comb").

The field for this problem is conveniently described by the Hertz vector $\mathbf{\Pi}_\omega$. We write $\mathbf{\Pi}_\omega$ as a sum

$$\mathbf{\Pi}_\omega = \mathbf{\Pi}^0_\omega + \mathbf{\Pi}^1_\omega, \tag{6.1}$$

where, in accordance with (2.15), the field of the wire in vacuum is described by the Hertz vector

$$\Pi^0_\omega = \Pi^0_{\omega z} = -\frac{q}{i\omega k\gamma} e^{-k\gamma |y+b| + i\frac{\omega}{u}z}. \tag{6.2}$$

The vector $\mathbf{\Pi}^1_\omega$ describes the free field that should be added to $\mathbf{\Pi}^0_\omega$ in order to satisfy the boundary conditions on the metal plates. The vector Π^1_ω can be expressed in terms of the currents induced by the source on the plates. From the geometry of the problem we see that the induced currents will have only a y component; thus

$$\Pi^1_{\omega y}(y, z) = \frac{i}{\omega} \sum_{m=-\infty}^{\infty} \int_0^{\infty} d\eta \int_{-\infty}^{\infty} d\xi \frac{e^{ikR_m}}{R_m} j_m(\eta), \tag{6.2'}$$

where R_m is the distance from the observation point (x, y, z) to the point (ξ, η) on the surface of the plate with index m:

$$R_m = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-am)^2}, \tag{6.3}$$

while $j_m(\eta)$ is the spectral component of the current in that plate. These formulas differ from (3.4) for the problem of the halfplane, because the field is determined by the currents induced on all of the planes.

Since the currents in the plates are induced by the source, whose velocity is u, we have the relation

$$j_m(t) = j_0\left(t - \frac{ma}{u}\right). \tag{6.4}$$

This relation enables us to express the currents induced in any plate in terms of one of them, say, the zeroth. In Fourier components (6.4) has the form

$$j_{m\omega}(y) = e^{i\frac{\omega}{u}am} j_{0\omega}(y). \tag{6.5}$$

Using this relation and representing $j_{0\omega}(y)$ by its Fourier integral expansion in y (cf., for example, formula (3.8)), we can bring the expression (6.2) for the Hertz vector to the form

$$\Pi^1_{\omega y} = -\frac{2\pi}{\omega} \int_{-\infty}^{\infty} F(w) \frac{e^{iwy}}{v} \sum_{m=-\infty}^{\infty} e^{iv|z-am| + i\frac{\omega}{u}am} dw. \tag{6.6}$$

The summation over m in the integrand can be performed if we restrict our investigation of the field to some one fixed period. Let us choose the (n + 1)-st space period, i.e., let us consider the range of values of z, limited by the inequalities

$$na \leqslant z \leqslant (n+1)\,a. \tag{6.7}$$

Then the summation over m gives

$$\Pi^{1}_{\omega y}(y,\ z) = -\frac{2\pi i}{\omega}\int_{-\infty}^{\infty} F(w)$$

$$\times \frac{\sin\{v\,[z-a(n+1)]\} - e^{i\frac{\omega}{u}a}\sin[v\,(z-an)]}{\cos(av)-\cos\left(a\,\frac{\omega}{u}\right)}\,e^{i\frac{\omega}{u}an+iwy}\,\frac{dw}{v}. \tag{6.8}$$

By imposing the requirements of vanishing of the tangential component of the electric field on the plates (for $z = an$, $y > 0$) and the absence of current on the extensions of the plates ($y < 0$), we arrive at a system of two integral equations for the Fourier amplitude F(w) of the current induced on the zeroth plate

$$\left.\begin{aligned} &\int_{-\infty}^{\infty} F(w)\,e^{iwy}\,dw = 0 && \text{for } y<0,\\ &\int_{-\infty}^{\infty} F(w)\,L(w)\,e^{iwy}\,dw = \frac{q\omega}{2\pi i u}\,e^{-k\gamma(y+b)} && \text{for } y>0, \end{aligned}\right\} \tag{6.9}$$

where

$$L(w) = \frac{v\sin va}{\cos av - \cos a\,\frac{\omega}{u}}$$

$$= \frac{2}{a}\,v^{2}\,\frac{\sin va}{va}\,\frac{\frac{a}{2}\left(v-\frac{\omega}{u}\right)}{\sin\left[\frac{a}{2}\left(v-\frac{\omega}{u}\right)\right]}\,\frac{\frac{a}{2}\left(v+\frac{\omega}{u}\right)}{\sin\frac{a}{2}\left[\left(v+\frac{\omega}{u}\right)\right]}\,\frac{1}{w^{2}+k^{2}\gamma^{2}}. \tag{6.10}$$

The solution of the system (6.9) can be obtained by the same method as in the earlier examples. We write the kernel L(w) in (6.9) in the form

$$L(w) = \frac{2}{a}\,v^{2}\,\frac{L_1(w)\,L_2(w)}{w^{2}+k^{2}\gamma^{2}}, \tag{6.11}$$

where $L_1(w)$, as usual, is holomorphic in the upper halfplane of the complex variable w and has no zeros there, while $L_2(w)$ has these same properties in the lower halfplane. Then the solution of the system (6.9) can be represented in the form

$$F(w) = \frac{aq\omega\gamma}{4\pi^{2} i u\,(1+i\gamma)\,L_1(ik\gamma)}\,e^{-k\gamma b}\,\frac{1}{(k-w)\,L_2(w)}. \tag{6.12}$$

We note that, in contrast to the problems considered earlier, the kernel (6.10) has poles at $w = \pm ik\gamma$, while the expression (6.12) for the Fourier component F(w) of the induced current is regular at these points. We recall that in the earlier problems the poles of the function F(w) corresponded to the image of the moving source.

The functions $L_1(w)$ and $L_2(w)$ appearing in the decomposition (6.11) can be written as infinite products:

$$L_{1,2}(w) = \prod_{n=1}^{\infty}\left(\frac{wa}{n\pi} \pm \sqrt{\left(\frac{ka}{n\pi}\right)^{2}-1}\right)e^{\pm i\frac{wa}{n\pi}}$$

$$\times\left\{\left[\frac{wa}{2n\pi} \pm \sqrt{\left(\frac{ka}{2n\pi}\right)^{2}-\left(1+\frac{a\omega}{2n\pi u}\right)^{2}}\right]e^{\pm i\frac{a}{2n\pi}\left(w-\frac{\omega}{u}\right)}\right.$$

$$\left.\times\left[\frac{wa}{2n\pi} \pm \sqrt{\left(\frac{ka}{2n\pi}\right)^{2}-\left(1-\frac{a\omega}{2n\pi u}\right)^{2}}\right]e^{\pm i\frac{a}{2n\pi}\left(w+\frac{\omega}{u}\right)}\right\}^{-1}, \tag{6.13}$$

where the upper signs should be taken for L_1 and the lower signs for L_2. In the representation (6.13) it is assumed that the values of the radicals are chosen so that the conditions

$$\left.\begin{aligned} &\operatorname{Im}\sqrt{\left(\frac{ka}{n\pi}\right)^{2}-1}>0,\\ &\operatorname{Im}\sqrt{\left(\frac{ka}{2n\pi}\right)^{2}-\left(1\pm\frac{a\omega}{2n\pi u}\right)^{2}}>0 \end{aligned}\right. \qquad (n=1,\ 2,\ \ldots). \tag{6.14}$$

are satisfied. Relations (6.12) and (6.13) determine completely the required function F(w) and, consequently, the radiation field of the charged wire. The components of the radiation field are expressed in terms of $\Pi^{1}_{y\omega}(y, z)$ by the formulas

$$H^{1}_{\omega x} = ik\,\frac{\partial \Pi^{1}_{y}}{\partial z},\qquad E^{1}_{\omega z} = \frac{\partial^{2}\Pi^{1}_{y}}{\partial z\,\partial y},\qquad E^{1}_{\omega y} = \left(k^{2}+\frac{\partial^{2}}{\partial y^{2}}\right)\Pi^{1}_{y}. \tag{6.15}$$

From here on we calculate only the one nonvanishing component of the magnetic field H^1_x. For the regions $an \le z \le a(n + 1)$ we get, from (6.8), (6.12) and (6.15), the expression

$$H^{1}_{x}(y,\ z) = -\frac{qk\gamma e^{-k\gamma b+i\frac{\omega}{u}an}}{\pi i u\,(1+i\gamma)\,L_1(ik\gamma)}\int_{-\infty}^{\infty}\{\cos\{v\,[z-a(n+1)]\}$$

$$-e^{i\frac{\omega}{u}a}\cos[v\,(z-an)]\}\,\sqrt{\frac{k+w}{k-w}}\,\frac{L_1(w)}{\sin(va)\cdot(w^{2}+k^{2}\gamma^{2})}\,e^{iwy}\,dw. \tag{6.16}$$

The integral representation (6.16) for the radiation field can be changed to a representation in the form of a series of residues at the singularities of the integrand. The different sets of poles in the upper and lower halfplanes of the complex variable w correspond to a different character of the field in the region between the metallic plates ($y > 0$) and in the free halfspace ($y < 0$).

When $y > 0$ the quantity H^1_x is determined by the first-order poles of the integrand in (6.16) in the upper w halfplane:

$$w = ik\gamma$$

and

$$w_m = \sqrt{k^{2}-\left(\frac{m\pi}{a}\right)^{2}} \qquad (m=0,\ 1,\ 2,\ \ldots). \tag{6.17}$$

The residue at the pole $w = ik\gamma$ cancels the contribution H^0_x in the expression for the total field, which thus is a superposition of symmetric electric waveguide modes, coming from the open end of the cell:

$$H_{x\omega}(y,\ z) = \sum_{m=0}^{\infty} R_m \cos\frac{m\pi}{a}\,z\,e^{iw_m y}. \tag{6.18}$$

The coefficients for excitation of waveguide modes are

$$\left.\begin{aligned} R_m &= (-1)^{mn}\alpha_m \frac{2qk\gamma e^{-k\gamma b+i\frac{\omega}{u}an}(k+w_m)L_1(w_m)\left[1-(-1)^m e^{i\frac{\omega}{u}a}\right]}{ua(1+i\gamma)w_m(w_m^2+k^2\gamma^2)L_1(ik\gamma)}, \\ \alpha_m &= \begin{cases} -\frac{1}{2} & \text{for } m=0, \\ 1 & \text{for } m\neq 0. \end{cases} \end{aligned}\right\} \tag{6.19}$$

We note that among the waveguide modes excited is the fundamental TEM mode, which propagates between the plates at the velocity of light. The field of this wave (corresponding to the term with m = 0 in the sum (6.18)) is independent of z. A wave of the TEM type is excited and propagates without damping into each waveguide cell no matter how small the frequency. We note that the excitation coefficient R_m contains the factor $1 - (-1)^m e^{i(\omega/u)a}$, which can vary from zero to a maximum value of two, depending on the quantity $\omega a/u$. This last quantity has a simple physical meaning. The factor a/u is the time during which the source passes over one period of the structure. Thus the quantity $a\omega/u$ is proportional to the ratio of the time for passing through one period of the comb to the period of the radiated wave. Obviously this ratio determines the work done on the source of the field by the excited wave in a period of the structure. This work is a maximum when

$$\frac{\omega}{u}a + m\pi = (2k+1)\pi, \tag{6.20}$$

where k is an arbitrary integer, and goes to zero when

$$\frac{\omega a}{u} + m\pi = 2k\pi. \tag{6.21}$$

In particular, the fundamental wave (m = 0) will not be excited if the time for passage of the source through one period is equal to or a multiple of the period of the radiated wave.

From formulas (6.18) and (6.19) we easily see that the field in neighboring plane waveguides, like the currents in neighboring plates, differ by the phase factor $e^{i(\omega a/u)}$. The total flux of energy radiated into the "waveguide" can be found by computing the integral

$$W_\omega = c\int_{na}^{(n+1)a} H_{\omega x}E_{-\omega z}\,dz = \frac{ac}{2k}\sum_m w_m|R_m|^2, \tag{6.22}$$

where the summation extends only over those values of m for which the longitudinal wave number w_m is real at the particular frequency ω.

Now let us look at the radiation field of the charged wire in the free halfspace (y < 0). In this case the radiated field is determined by the poles of the integrand that lie in the lower w halfplane:

$$w = -ik\gamma, \quad \hat{w}_m = \sqrt{k^2 - \left(\frac{2\pi m}{a} - \frac{\omega}{u}\right)^2} \quad (m = \pm 1, \ \pm 2, \ \ldots). \tag{6.23}$$

Calculating the integral (6.16) using residues at the indicated singularities, we get for the radiation field in the free halfspace the expression

$$H_x^1 = -\frac{qce^{-k\gamma b}}{u^2(1+i\gamma)^2}\frac{L_1(-ik\gamma)}{L_1(ik\gamma)}e^{k\gamma y+i\frac{\omega}{u}z} - \frac{2iqk\gamma e^{-k\gamma b}}{u(1+i\gamma)L_1(ik\gamma)}\sum_m \frac{\operatorname{Res} L_1(\hat{w}_m)}{\hat{w}_m^2+k^2\gamma^2}\frac{\frac{\omega}{u}-\frac{2\pi m}{a}}{k-\hat{w}_m}e^{i\hat{w}_m y+i\left(\frac{\omega}{u}-\frac{2\pi m}{a}\right)z}, \tag{6.24}$$

where Res $L_1(\hat{w}_m)$ is the residue of the function $L_1(w)$ at the pole $w = \hat{w}_m$. This quantity can be written in the form

$$\operatorname{Res} L_1(\hat{w}_m) = \lim_{w\to\hat{w}_m}[(w-\hat{w}_m)L_1(w)] = \frac{\hat{w}_m^2+k^2\gamma^2}{2\hat{w}_m L_2(\hat{w}_m)}. \tag{6.25}$$

The expression (6.25) is more convenient for computing radiation intensities, since the square modulus of (6.25) can be written in closed form.

The number of plates does not appear in (6.24), so it represents the field in the lower halfspace for any value of z. The first term, corresponding to the pole $w = -ik\gamma$, determines the surface wave whose electromagnetic field propagates at the speed of the source and damps exponentially as it moves away from the edges of the plates. The terms in the sum over m correspond to the poles $\hat{w}_m$ of (6.23). For real values of $\hat{w}_m$ they describe plane electromagnetic waves radiated by the source in moving along the "comb." For each of these waves the projection of the wave vector on the z axis is given by the equation

$$k_{zm} = \frac{\omega}{u} - \frac{2\pi m}{a} = k\cos\theta_m, \tag{6.26}$$

where θ_m is the angle that the wave vector makes with the z axis. From the last equation we can get the following expression for the frequency ω radiated when one observes at a given angle θ_m:

$$\omega = \frac{2\pi m\frac{u}{a}}{1-\frac{u}{c}\cos\theta_m}. \tag{6.27}$$

This expression has a simple physical meaning. The time for passage of the source through one space period of the structure is T = a/u. We introduce the "frequency of passage" $\Omega = 2\pi/T = 2\pi u/a$. It is easy to see that an integer multiple of the frequency of passage appears in the numerator in (6.27). The denominator contains a characteristic factor giving the Doppler frequency shift of the moving source. During uniform motion of the source along the system of equally spaced plates there is a periodic induction of currents in the plates that are nearest to the source. The induced currents are again the source of periodically varying flashes, where the frequency of the brightness variations is equal to the frequency of passage $2\pi u/a$ or integer multiples of it, while the velocity coincides with velocity of motion of the charged wire.

From the treatment presented it follows that at a given frequency ω, a finite number of waves are radiated, whose directions of propagation make different angles θ_m with the velocity of the source. The number

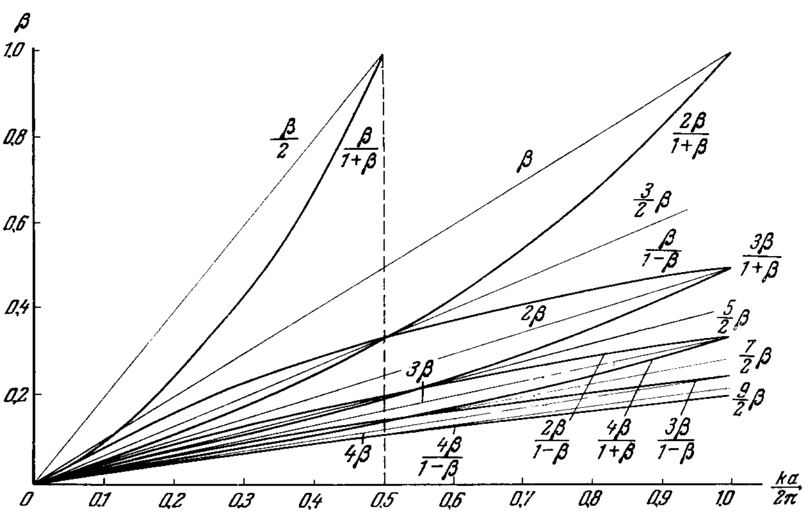

FIG. 9.

of waves radiated can be estimated from the simple requirement

$$-1 < \cos\theta_m = \left(\frac{1}{\beta} - \frac{2\pi m}{ka}\right) < 1, \tag{6.28}$$

from which we get

$$\frac{ka}{2\pi}\left(\frac{1}{\beta}+1\right) > m > \frac{ka}{2\pi}\left(\frac{1}{\beta}-1\right). \tag{6.29}$$

Thus the number of lines Δm radiated at the frequency ω is given by the integer part of the quantity

$$\frac{ka}{\pi} = 2\frac{a}{\lambda}, \tag{6.30}$$

where $\lambda = 2\pi c/\omega$ is the wave length of the wave radiated at the frequency ω. It is interesting to note that at a given frequency ω the number of propagating harmonics radiated into the free space is equal to the number of waveguide modes excited in the resonant region between the plates. If we assign an "order" of the spectral line, m, formulas (6.27)—(6.29) give an inequality which determines the frequency band for a given velocity of the source:

$$\frac{\beta m}{1-\beta} \geqslant \frac{ka}{2\pi} \geqslant \frac{\beta m}{1+\beta}. \tag{6.31}$$

This inequality can be given a simple graphical interpretation. In Fig. 9 the ordinates are the values of $\beta = u/c$, and the abscissas are the values of $ka/2\pi$. The range of values of these parameters for which the radiation of spectral line m is possible is limited by the two curves bearing the label m. From the graph we see that a point corresponding to a fixed value of β and $ka/2\pi$ may be common to the regions of radiation with different values of m.

The simple relations given above, which determine the frequency of the radiation, the number of spectral lines and the characteristic dependence of the properties of the radiation on the velocity of the source (formula (6.27) et seq.) do not depend on the specific form of our model, but are common to all linear periodic media, i.e., media whose properties vary periodically when some one coordinate, say z, is changed. Let us consider the radiation of a charge in such a medium. Suppose that a charge moving in a periodic medium radiates a light quantum with momentum $\hbar\mathbf{k}$ and energy $\hbar\omega$. In the radiation process the periodic medium can take up any integer multiple of the elementary momentum $\hbar\mathbf{k}_0$, where $\mathbf{k}_0$ is a reciprocal lattice vector

$$\mathbf{k}_0 = \frac{2\pi}{a}\mathbf{z}_0, \tag{6.32}$$

a is the period of the lattice, $\mathbf{z}_0$ a unit vector along the z axis. In the radiation process, energy and momentum are conserved:

$$\begin{aligned} \mathbf{p}_1 - \mathbf{p}_2 &= \Delta\mathbf{p} = \hbar\mathbf{k} + n\hbar\mathbf{k}, \\ E_1 - E_2 &= \Delta E = \hbar\omega. \end{aligned} \tag{6.33}$$

We multiply the first of Eqs. (6.33) by **u**. Using the relation $\mathbf{u}\cdot\Delta\mathbf{p} = \Delta E$, which is valid if the velocity of the particle does not change much because of the radiation, we get

$$\mathbf{u}\Delta\mathbf{p} = \Delta E = \hbar\mathbf{k}\mathbf{u} + n\hbar\mathbf{k}_0\mathbf{u} = \hbar\omega. \tag{6.34}$$

If the propagation of the radiated light in the periodic medium can be characterized in terms of some dielectric constant ε averaged over a period,

$$k = \frac{\omega}{c}\sqrt{\varepsilon}, \tag{6.35}$$

and we get

$$\omega = \frac{n(\mathbf{k}_0\mathbf{u})}{1 - \frac{u}{c}\sqrt{\varepsilon}\cos\theta}. \tag{6.36}$$

If the velocity vector **u** of the particle is parallel to the reciprocal lattice vector $\mathbf{k}_0$ and $\epsilon = 1$, we get formula (6.27). Despite the quantum derivation we get a purely classical expression for the radiated frequency.

When $\omega a/u \ll 1$ all the terms in the sum over m in (6.27) are damped faster than the first term, which determines the surface wave. In this case the so-called "impedance approximation" is applicable: the field in the free space can be determined by replacing the "comb" by the plane surface $y = 0$ with an assigned impedance (the ratio of the tangential components of the electric and magnetic fields). The exact formula (6.24) enables us to find the limits of validity of this approximation. From (6.24) one can easily calculate the intensity of the radiation in the free space.

We have considered the radiation from a charged wire (or a plane modulated electron wave), moving along the "comb." The problem of the radiation from a line current (or a plane modulated wave of current) is solved similarly. The physical difference between these two cases is the difference in polarization of the sources (and the radiation fields).[19,20]

In conclusion we consider still another problem of radiation from a linear source, entering or emerging from the comb along the axis of one of the semi-infinite waveguides that form the periodic structure.[21] In this case the source does not possess a periodicity along the structure and there is therefore no resonant radiation in the free space. The transition of the source from the region occupied by the comb into free space is accompanied by a characteristic burst of radiation which is very much like the transition radiation. In the limiting case when the period of the structure becomes much less than the wave length of the radiated field, we get the solution of the problem of transition radiation for a plane surface with anisotropic conductivity.

The geometry of the problem is shown in Fig. 10. The equations of the plates forming the comb are $y > 0$, $z = a(n - ½)$, $n = 0, \pm 1, \pm 2, \ldots$ The source moves into the system along the y axis; its position is given by $y = ut$. We limit our treatment to the case of the charged wire.

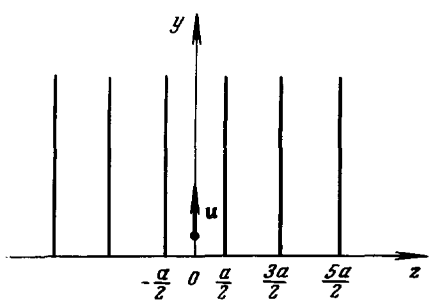

FIG. 10.

Because of the periodicity of the structure along the z axis, all the quantities characterizing the field can be written as a superposition of functions of the form

$$f_{n\mu}(y,\ z) = e^{i\mu n} f_{\mu}(y,\ z), \tag{6.37}$$

where the quantity μ is contained within the limits $-\pi$ to π, f_μ (y, z) depends periodically on z with the period a of the structure, n numbers the cell of the lattice corresponding to a variation of z between the limits

$$\left(n - \frac{1}{2}\right) a < z < \left(n + \frac{1}{2}\right) a. \tag{6.38}$$

The function $f_{n,\mu}$ satisfies the relation

$$f_{n+1,\mu}(y,\ z) = e^{i\mu} f_{n,\mu}(y,\ z). \tag{6.39}$$

Similar relations hold for the currents j induced on the plates during entry or emergence of the source. We shall try to find the current induced on plate n in the form

$$j_n(y) = \int_{-\pi}^{\pi} e^{i\mu n} j_{0\mu}(y)\, d\mu. \tag{6.40}$$

We shall describe the total field by means of the Hertz vector

$$\Pi_{\mu,n} = \Pi^0_{\mu,n} + \Pi^1_{\mu,n}, \tag{6.41}$$

where the symmetry of the problem permits us to choose all the vectors along the y axis; here*

$$\Pi^0_{\mu,n}(y, z) = \frac{1}{2\pi} e^{i\mu n} \Pi^0(y,\ z - an), \tag{6.42}$$

while Π^0 (y, z) is the Hertz vector describing the field of the charged wire in free space:

$$\Pi^0 = \frac{iq}{k\gamma\omega} e^{-k\gamma|z| + i\frac{\omega}{u}y}. \tag{6.43}$$

With such a choice of the "incident wave" the field in the central wave guide $-a/2 \le z \le a/2$ coincides with the field of the source in free space, while the fields in the other waveguides are gotten by periodic continuation of Π^0 along z. Since the function Π^0_n (y, z) which determines the "incident" wave in cell n, satisfies the relation

$$\Pi^0_n(y,\ z) = \int_{-\pi}^{\pi} \Pi^0_{\mu n}(y,\ z)\, d\mu = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\mu n} \Pi^0(y,\ z - an)\, d\mu, \tag{6.44}$$

which is analogous to (6.40), this function is different from zero only for the central waveguide. Thus the assumed form of the incident wave should be regarded as a convenient mathematical device, enabling one to write the incident and scattered fields in a single form.

The Hertz vector of the radiation field is expressed as an integral of the induced currents, where (6.40) enables us to express the Hertz vector Π^1 in terms of the current induced on the zeroth plate:

$$\Pi^1_{\mu,n} = -\frac{2\pi i}{\omega} e^{i\mu n} \int_{-\infty}^{\infty} F_{\mu,0}(w) \times \frac{\sin\left\{v\left[z - a\left(n + \frac{1}{2}\right)\right]\right\} - e^{i\mu} \sin\left\{v\left[z - a\left(n - \frac{1}{2}\right)\right]\right\}}{\cos av - \cos\mu} e^{iwy} \frac{dw}{v}$$

$$\left(\left(n - \frac{1}{2}\right) a \leqslant z \leqslant \left(n + \frac{1}{2}\right) a\right), \tag{6.45}$$

where $F_{\mu,0}$ is the Fourier component of the current induced on the zeroth plate, and the contour of integration over the variable w circles below the point $w = \omega/u$.

The requirements of vanishing of the tangential components of the total electric field on the plates and the absence of current on their extensions leads, as usual, to a pair of integral equations for the functions $F_{\mu,0}$ (w). The solution of this system of equations is obtained by standard methods and has the form

$$F_{\mu,0}(w) = -\frac{q}{8\pi^3 i} \frac{M_2\left(\frac{\omega}{u}\right)}{M_2(w)} \frac{\operatorname{ch}(k\gamma a) - \cos\mu}{\operatorname{sh}(k\gamma a)} \frac{1}{w - \frac{\omega}{u}} e^{-k\gamma\frac{a}{2}}. \tag{6.46}$$

In this formula the function $M_{1,2}$ (w) satisfies the relations

*Writing the zero order field in the form (6.42) is permissible since after summation over μ between the limits $(-\pi, \pi)$ we get (6.43)

$$M_{1,2}(w) = \frac{L_{1,2}(w)}{w \pm \frac{\omega}{u}}, \tag{6.47}$$

while the functions $L_{1,2}(w)$ are defined by the equation

$$\frac{1}{\frac{\mu^2}{a^2} - v^2} \frac{\sin(va)}{va} \frac{\frac{1}{2}(\mu + va)}{\sin\left[\frac{1}{2}(\mu + va)\right]} \frac{\frac{1}{2}(\mu - va)}{\sin\left[\frac{1}{2}(\mu - va)\right]} = \frac{L_1(w) L_2(w)}{w^2 - \frac{\omega^2}{u^2}}. \tag{6.48}$$

The function L_1 is holomorphic and does not vanish in the upper w halfplane, while L_2 has these same properties in the lower w half-plane. The splitting of (6.48) into factors $L_{1,2}$ is very reminiscent of the factoring of the kernel L (w) in (6.10) in the problem of parallel flight of the source past the comb. This is easily verified by making the substitution $a\omega/u \to \mu$ in (6.10). We shall, therefore not give the explicit expressions for the factor functions L_1 and L_2 satisfying the relation (6.48).

The integral representation (6.45)-(6.48) obtained for the solution completely determines the radiation field arising from entry or emergence of the linear source from the periodic structure. Let us note the main features of the radiated field. In the region between the plates (y > 0), the field in each of the waveguides is a superposition of normal waveguide modes, propagating from the open end into the body of the waveguide. Thus, in waveguide number n the magnetic field component H_x can be written in the form

$$H_{xn} = \sum_{m=1}^{\infty} \left\{ P_{nm} \sin\left[\frac{\pi(2m-1)}{a}(z - an)\right] e^{iw_{2m-1}y} + Q_{mn} \cos\left[\frac{2\pi m}{a}(z - an)\right] e^{iw_{2m}y} \right\}, \tag{6.49}$$

where the longitudinal wave numbers are

$$w_m = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2},$$

and P_{nm} and Q_{nm} are the coefficients of excitation of the normal modes. We also note that the amplitudes of the excited waves contain exponential factors exp $(-k\gamma a/2)$, that are characteristic for all problems of diffraction radiation.

The radiation field in the free space (y < 0) at large distances from the periodic structure is in the form of cylindrical waves diverging from the open end of the central waveguide.

We give the expression for the intensity of the radiation accompanying the entry of the charged wire into the periodic structure, in the limit as a → 0. This limiting case is of interest because then the structure becomes equivalent to a certain conducting surface at y = 0. The conductivity of this surface is anisotropic; it is infinite in the x direction and equal to zero along the z axis:

$$I_\omega(\theta) = \frac{2q^2\beta^2}{\pi\omega} \frac{(1-\beta^2)\,\mathrm{ctg}^2\,\theta\,(1-\cos\theta)^2}{(1-\beta^2\cos^2\theta)^2}. \tag{6.50}$$

The angle θ is measured from the y axis. Expression (6.50) gives the intensity of the transition radiation from a charged wire intersecting an anisotropically conducting plate. When $\beta \to 1$ the intensity of the transition radiation (6.50) goes to zero, in contrast to the case when the charge enters an isotropic metal. The explanation is that the electric field of a charged wire moving with relativistic velocity along the normal to the surface has a component only along the direction in which the conductivity of the surface is zero. Thus no currents are excited on the surface and so there is no radiation. We note that formula (6.50) also describes the radiation when the source emerges, if we make the substitution $\beta \to -\beta$. The case where the source of the radiation is a current-carrying wire is treated similarly. Here in the limit as a → 0 the expression for the radiated intensity in the external region is given by the formula

$$I_\omega(\theta) = \frac{2I^2\beta^2c^2}{\pi\omega} \frac{\cos^2\theta}{(1-\beta^2\cos^2\theta)^2}. \tag{6.51}$$

This expression coincides with the formula for the intensity of the transition radiation accompanying the entry or emergence into vacuum of a current-carrying wire from a homogeneous metallic medium. The reason for the identity of the formulas is that the electric field of the current wire has one nonzero component E_x, which is parallel to the axis having infinite conductivity.

We have considered various two-dimensional problems of diffraction radiation that can be solved exactly. This radiation occurs during uniform motion of a source of field past an optical inhomogeneity. As field sources we have considered uniformly charged wires or line currents (or, what is the same thing, plane harmonically modulated current distributions). The optical inhomogeneities considered were ideally conducting surfaces of various forms (plane screens, waveguides and periodic structures). Let us enumerate the characteristic properties of the diffraction radiation that are common to all cases.

1. The energy of the radiation is independent of the mass of the particle, but is determined by its velocity and charge. This property of the radiation is common to all problems where one considers the radiation from particles moving according to some given law, for example, for the Vavilov-Cerenkov radiation or the transition radiation.

2. The character of the diffraction radiation depends on the form of the scattering obstacle. The electromagnetic field of the radiation in free space at distances exceeding the size of the shaping region has the form of cylindrical waves diverging from the edge of the scattering body. If the obstacles are arranged periodically along the direction of motion of the source, the interference of the cylindrical waves diverging from individual inhomogeneities leads to the formation of plane waves of radiation.

If the scattering bodies form a resonant system in which normal modes can be excited, the passing source excites a discrete set of normal modes in the scattering system.

3. The angular and frequency distributions of the radiation are determined by the velocity of the source and the shape of the scattering obstacle. In the ultrarelativistic case the main part of the radiation is at high frequencies. The radiation occurs within narrow angular ranges determined by the direction of motion of the fast-moving source and the direction of motion of its mirror image.

4. The intensity of the radiation at high frequencies falls off as $\exp(-2\omega/u\sqrt{1-\beta^2}\,a)$ where a is a characteristic impact parameter. This factor determines both the limit of the radiation spectrum and the dependence of the radiation on impact parameter.

5. The energy losses in diffraction radiation from line sources are proportional to the first power of the velocity when the velocity is low. In the relativistic

limit the energy loss to radiation depends on the form of the line source and the shape of the obstacle.

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