537.212 + 538.122

EVOLUTION OF THE CONCEPTS OF MAGNETIC AND ELECTRIC LINES OF FORCE

M. A. LEONTOVICH

Usp. Fiz. Nauk **84**, 715-721 (December, 1964)

As we know, Faraday pictured magnetic and electric fields as an assemblage of lines of force ("tubes"), which for him were material formations of special nature. Modern physics describes the electromagnetic field as the field of two vectors, **E** and **B**, and lines of force are only a graphical representation of these fields. Lines of force constructed in the usual way do not represent a relativistically invariant geometric form, and therefore no materialization of them has meaning.

For electromagnetic fields of a particular type, however, for which $(\mathbf{E}\cdot\mathbf{B}) = 0$, it is possible to construct relativistically invariant moving lines of force of the magnetic field (and in vacuum also of the electric field). These invariant lines of force can be "materialized" as chains of charges or as chains of particles of an ideally conducting fluid. In the conclusion of the article, consideration is given to the possibility of constructing relativistically invariant forms for fields of a more general type, when the condition $(\mathbf{E}\cdot\mathbf{B}) = 0$ is not satisfied.

* * *

The picture of lines of force of an electric and magnetic field is extremely useful not only for graphical representation of the field, but also for determination of the forces that act on bodies in these fields. The concept of magnetic and electric lines of force originated with Faraday, but since then both the meaning and the role of lines of force in electrodynamics have changed significantly.

As we know, Faraday, when he was creating the bases of a theory of electromagnetic phenomena, did not use any complete mathematical description of an electric and magnetic field. His ideas about what we now call the electromagnetic field were based on lines of force. Faraday's lines of force ("tubes") were completely material; he explained the forces that act on a body located in an electric and magnetic field by tension and compression of the lines of force.

Maxwell (about a hundred years ago) created the modern theory of electromagnetic phenomena (electrodynamics). On the basis of the results and ideas of Faraday, Maxwell developed that theory in already modern mathematical form. Since then the basic characteristic of the electromagnetic field has been two vectors—the vector intensities of the electric and magnetic fields. The lines of force became a means of graphical representation of those fields. Attempts to unite Maxwellian electrodynamics (and electron theory) with "material" lines of force, in the spirit of Faraday, were made by J. J. Thomson. They remained sterile.

The impossibility of introducing lines of force into electrodynamics as material ("real") objects, in the earlier sense, became clear after the creation of the special theory of relativity. Although deductions from the theory of relativity embrace the whole of physics, it originated in questions connected with electrodynamics. Not without reason was Einstein's fundamental work of 1905 called "On the Electrodynamics of Moving Bodies". In that work it was shown that the laws of electrodynamics are valid in an arbitrary inertial system of observation. This was a fundamental accomplishment of the orderly theory that we now call classical electrodynamics (the term "classical theory" is now used in physics to indicate that this is a theory that does not take account of quantum phenomena).

Why is it impossible to represent the lines of force as material or "real"? This follows from the fact that a line of force defined in the usual manner depends on the system of observation. A line of force of a given magnetic (electric) field is a line whose tangent at each point is directed along the magnetic (electric) field at that point. If the field depends on time, then it is necessary to construct a line of force for each instant, by taking the value of the field everywhere simultaneously. (In this case the form and position of a line of force will depend on the time.) If we carry out this construction in another system of observation (moving rectilinearly and uniformly with respect to the first), we get, in general, altogether different lines, which do not represent the previous lines moving with respect to the new system of observation with the velocity of relative motion of these two systems! But this last should be the situation, if we could visualize a line of force as a material formation.

One can convince oneself of this dependence of a line of force on the system of observation (noninvariance) by considering two important aspects of the theory of relativity; namely:

1. Events at different points, simultaneous in one system of observation, will not be simultaneous in another.

2. The separation of an electromagnetic field into an electric and a magnetic depends on the system of observation. Only the combination of electric and magnetic fields, which does not depend on the system of observation, has meaning. (Example. In system K

the charges are at rest and give an electric (electrostatic) field. In system K′, in motion with respect to K, the charges will be moving (there will be a current) and will produce not only an electric but, like all currents, also a magnetic field.) Therefore it can be established that no kind of material object can correspond to lines of force (in the sense of the definition given above).

How can we reconcile with this proof the fact, for example, that the magnetic lines of force completely determine the trajectory of a charged particle moving in a magnetic field? Actually, it is known that the trajectory of an electric charge in a stationary (not changing with time) magnetic field (and in the absence of an electric field) is a spiral line, winding around a magnetic line of force as axis of the spiral.

This picture of the motion is approximate, but it corresponds to the facts the more accurately, the stronger the magnetic field and the smaller the speed of rotation of the charge about the line of force. The better these conditions are satisfied, the closer the spiral line is to the line of force.

This characteristic of the motion of charges in magnetic fields is often used in laboratories for clarification of details of the distribution of the magnetic field in apparatus in which strong magnetic fields (1 kilogauss and higher) are applied: in particle accelerators, apparatus for containment of plasma, and "electromagnetic separators," that is apparatus for electromagnetic separation of isotopes. For this purpose a gas (neon, argon) at low pressure (of order 10^{-3} mm mercury) is admitted to the apparatus; from an electron gun (source of electrons), located at various places, electrons are started (accelerated in the gun with a voltage of several kilovolts) in a direction approximately along the magnetic field. Under these conditions the electrons excite gas atoms along their path and cause them to luminesce. The path of an electron is then a luminous line, graphically exhibiting a magnetic line of force, thus "materialized" by a series of luminous atoms.

If the charge moves in a region where, besides the magnetic field, there is also a weak electric field, so that the ratio of the field intensities E/B is small in comparison with unity (here it is understood that the intensities of the electric field E and the magnetic field B are expressed in the same units, for example both in gausses or both in volts per meter!), then the charge not only winds around the magnetic line of force but also moves perpendicularly to it and in the direction of the electric field with a velocity

$$w = c\frac{E}{B},$$

where c is the speed of light. The charge now "drifts" across the magnetic field.

We consider another phenomenon: rapid motion of a highly conducting fluid or of an ionized, highly conducting gas (or solid body) in a magnetic field. In the phenomena under consideration, the resistivity of the conducting medium may be neglected, since the effects connected with it play no role.

In motion of a conducting medium in a magnetic field, electric currents develop in it. The forces acting on these currents in a magnetic field, in their turn, influence the motion of the medium. The motion occurs in such a way that <u>the same particles of fluid</u> remain always on <u>the same</u> magnetic line of force. A magnetic line of force moves with the material. This moving magnetic line of force is marked by individual particles of the material.

This deduction is a direct consequence of Faraday's law of induction for a <u>moving</u> conducting circuit. In fact, the flux of magnetic induction (flux of magnetic field) across a surface bounded by a circuit that goes through a constant set of particles of the material must remain always unchanged, since otherwise an infinitely large current would arise in the medium under consideration, with its negligibly small resistivity. But conservation of the flux of the magnetic field through a material circuit likewise implies motion of magnetic lines of force with the material.

Upon motion of a highly conducting material in a magnetic field, an electric field necessarily appears in it. This electric field must balance the Lorentz force that acts, because of the magnetic field, on each charge (current carrier) that is moving with the medium; otherwise there would be an infinitely large current, and this is impossible. Hence it is not difficult to deduce that the velocity u of the medium is equal, both in magnitude and in direction, to the "drift velocity":

$$u = c\frac{E}{B} = w.$$

We see that in both the cases considered—motions of a charge and of a conducting medium in a magnetic field—the lines of force can be materialized. Each such line is marked in the first case by a definite chain of electrons, and in the second by definite particles of the fluid. It is obvious that this <u>"materialization" of lines of force is quite different from that that Faraday imagined</u>: he considered that the lines of force were in themselves material; the material of which they consisted, to him, was not a material of electrons and atoms, but a material of special nature. To Faraday the lines of force played the role of the "ether."

How, now, are we to reconcile the role of magnetic lines of force as trajectories of charges in a magnetic field and in motion of a conducting medium with what was said about their "relativistic noninvariance," that is with the fact that the lines of force constructed will be different when their construction is performed in different (inertial) systems of observation? For the trajectory of a charge and a chain of particles of a conducting medium cannot depend on the system of observation.

The answer to this question is found by investigation of it within the framework of the theory of relativity, and it consists in the following.

For electromagnetic fields of special type, for which $(\mathbf{E}\cdot\mathbf{B}) = 0$, that is either when $\mathbf{E} = 0$ or $\mathbf{B} = 0$ or when $\mathbf{E}$ and $\mathbf{B}$ are mutually perpendicular, it is possible to construct a geometrical (kinematic) form, namely a system of moving magnetic lines of force, independent of the system of observation. The tangents to these lines are directed at each point along the magnetic vector, and they move perpendicularly to their direction and perpendicularly to the electric vector (at the given point) with the "drift velocity" $w = cE/B$. These lines move even in the case in which the fields are stationary! Construction of such magnetic lines of force, independent of the system of observation ("relativistically invariant"), is possible only for fields of the special type mentioned, when $(\mathbf{E}\cdot\mathbf{B}) = 0$ (this condition itself, as is known, is independent of the system of observation).

In fact, as follows from what has been said, a line of force can be a relativistically invariant geometric form, provided it is moving. This means that in four-dimensional space (of the coordinates and of time) it is a two-dimensional manifold. The intersection of this manifold with the plane (in four-dimensional space) t = const gives a one-dimensional manifold — a line in the usual three-dimensional space.

The moving, relativistically invariant magnetic lines of force are obtained from the system of equations for their elements (dr, dt)

$$[d\mathbf{r},\ \mathbf{B}] + c\mathbf{E}\,dt = 0, \qquad (1)$$

$$(\mathbf{E},\ d\mathbf{r}) = 0. \qquad (2)$$

This system of equations is relativistically invariant; it can be written in four-dimensional tensor form

$$F_{ik}dx^k = 0,$$

where F_{ik} is the electromagnetic field tensor, and where dx^k is the tensor dx, dy, dz, cdt.

The system of four equations (1) and (2) for dx, dy, dz, and dt has nonvanishing solutions for them, provided

$$(\mathbf{E},\ \mathbf{B}) = 0. \qquad (3)$$

Under this condition, of the four equations for four differentials dx, dy, dz, dt, only two are algebraically independent. Thus this system can define a two-dimensional manifold. For this, however, there is still the requirement of satisfaction of the condition for integrability of the system of two equations for four total differentials. This integrability condition has the form

$$\left[\mathbf{B}, \operatorname{curl}\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\right] - \mathbf{E}\operatorname{div}\mathbf{B} = 0 \qquad (4)$$

and is satisfied by virtue of Maxwell's equations.

From Eq. (1) it follows that when dt = 0, dr is parallel to B; consequently dr is an element of a line of force. On multiplying (1) vectorially by B, we get

$$d\mathbf{r} = \frac{\mathbf{B}(\mathbf{B},\ d\mathbf{r})}{B^2} + c\frac{[\mathbf{E},\ \mathbf{B}]}{B^2}dt,$$

whence it is clear that the components of velocity perpendicular to B are equal to

$$\mathbf{w} = c\frac{[\mathbf{E},\ \mathbf{B}]}{B^2}.$$

This means, by virtue of the condition (3), that w = cE/B. This derivation was given by Newcomb[1] (1958). We add that if the condition (3) is not satisfied (E and B not perpendicular to each other), it can be shown that it is in general impossible to construct invariant magnetic lines of force.

*[dr, B] = dr × B.

In our example of the motion of a very highly conducting medium in a magnetic field, the electromagnetic field does in fact belong to the class in which the electric field in the medium is perpendicular to the magnetic. The material particles of the medium move while remaining on these "relativistically invariant" magnetic lines of force.

In the case, analyzed above, of motion of a charge in a stationary magnetic field (in the absence of an electric field), the value of $(\mathbf{E}\cdot\mathbf{B})$ is likewise obviously equal to zero in the laboratory system of observation under consideration. In this system the charge, as we have already said, moves along a magnetic line of force. In another inertial system of observation, besides the (changed) magnetic field $\mathbf{B}'$, there will be also an electric field $\mathbf{E}'$, and the same line of force (= trajectory of the charge) will, with respect to the new system of observation, move with the drift velocity $w = cE'/B'$, which is exactly equal to the velocity of the old system of observation with respect to the new.

The situation will be similar if the charge moves in magnetic and electric fields that are everywhere perpendicular to each other. With satisfaction of the conditions mentioned (strong field, small charge velocity), the trajectory of the charge will coincide approximately with a drifting magnetic line of force.

To this point the subject was magnetic lines of force. It is not at all accidental that attention centers on them: magnetic lines play a greater role in physics than do electric. Just what is the situation with regard to electric lines of force, which give a graphical picture for the other (electric) component of the electromagnetic field? At first glance it may seem that here there should be a complete analogy with magnetic lines. But we must keep in mind that there is no symmetry between magnetic and electric fields. Magnetic charges do not exist! (Searches for these magnetic charges, that is for free magnetic poles, continued into recent years, with use of all the possibilities of modern experimental technique. People tried to obtain them as a result of interactions of elementary particles in accelerators, sought them in the cosmos, in meteorites; but in vain!) Therefore as regards electric lines of force the situation is not quite the same as with magnetic, but is as follows.

For electromagnetic fields of the special type $(\mathbf{E}\cdot\mathbf{B}) = 0$, it is possible to construct a system of moving electric lines of force, independent of the system of observation. The electric lines of force drift perpendicularly to their own direction (the direction of **E**) and perpendicularly to the direction of the magnetic vector **B** with velocity $w_* = cB/E$. In this case the construction of electric lines of force is possible not at all points of the electromagnetic field (and herein lies a difference from magnetic lines!), but only where there are no charges or currents, or at least the currents are parallel to the electric field and there are no charges.

Invariant electric lines of force can be defined in a manner quite similar to that used for magnetic. For this purpose it is necessary in all the equations to replace B by E and E by −B. Instead of Eqs. (1) and (2) we get

$$[d\mathbf{r}, \mathbf{E}] - c\mathbf{B}\, dt = 0,$$
$$(\mathbf{B}, d\mathbf{r}) = 0.$$

This also is a relativistically invariant system of equations. It can be written in the form $F^*_{ik}dx^k = 0$, where F^*_{ik} is the dual field tensor. Again satisfaction of condition (3) is necessary. The integrability condition is obtained from (4) by replacement of B by E and of E by −B; it has the form

$$\left[E, \operatorname{curl}\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right] + \mathbf{B} \operatorname{div} \mathbf{E} = 0. \tag{5}$$

But these conditions are not satisfied at an arbitrary point of the field; by virtue of the second pair of Maxwell's equations, they reduce to the condition

$$[\mathbf{E}, \mathbf{j}] + c\mathbf{B}\rho = 0 \tag{6}$$

where ρ and j are the charge and current densities. Only under conditions (6) – which are obviously satisfied everywhere where there are no charges or currents – is it possible to construct invariant electric lines of force. On replacing E by B in the expression for the drift velocity of magnetic lines, we get the drift velocity of electric lines:

$$\mathbf{w}_* = -c\frac{[\mathbf{E}, \mathbf{B}]}{E^2}, \tag{7}$$

and this means that $w_* = cB/E$.

We mention also that from these expressions for the drift velocity of magnetic ($w = cE/B$) and the drift velocity of electric lines of force ($w_* = cB/E$) it follows that $w < c$ when $E < B$ and that $w_* < c$ when $B < E$ (the conditions are independent of the system of observation). Particles can move only with velocities not exceeding the velocity of light c. Therefore particles can move with the magnetic lines of force only if $E < B$. If under some conditions or other some kind of particles or other moved with the electric lines of force, then this would be possible only for fields in which $E > B$.

In the case when $(\mathbf{E}\cdot\mathbf{B}) \neq 0$, it is not possible to construct either invariant magnetic or invariant electric lines of force. It is possible, however, to seek more general geometric (kinematic) forms; specifically, certain "hybrid" lines of force, which (in a very special case, it is true) determine the motion of charges in a field.

In fact, it is possible to seek lines of force of the vector

$$\mathbf{P} = \mathbf{B} + \lambda\mathbf{F}.$$

The corresponding four-dimensional tensor will be $F_{ik} + \lambda F^*_{ik}$; and we must consider, besides P (the generalization of B), the vector

$$\mathbf{Q} = -\lambda\mathbf{B} + \mathbf{E},$$

which is the generalization of E and the complement of P. Instead of (1) and (2) we write the differential equations of the "hybrid" lines of force in the form

$$[d\mathbf{r}, \mathbf{P}] + c\mathbf{Q}\, dt = 0,$$
$$(\mathbf{Q}, d\mathbf{r}) = 0.$$

The condition for existence of nonvanishing solutions will be $(\mathbf{P}, \mathbf{Q}) = 0$; that is,

$$\lambda^2 + \frac{B^2 - E^2}{(\mathbf{EB})}\lambda - 1 = 0, \tag{8}$$

which gives two values $\lambda = \lambda(x, y, z, t)$.

The integrability condition will be

$$\left[\mathbf{P}, \operatorname{curl}\mathbf{Q} + \frac{1}{c}\frac{\partial \mathbf{P}}{\partial t}\right] - \mathbf{Q} \operatorname{div} \mathbf{P} = 0; \tag{9}$$

it imposes on the field distribution a still more complicated condition than the condition (6) for electric lines when $(\mathbf{E}\cdot\mathbf{B}) = 0$.

Nevertheless a trivial case can be immediately pointed out, in which condition (9) is satisfied. This is the case of a uniform stationary magnetic field and a uniform electrostatic field that are not perpendicular to each other. In this case we can always choose a system of observation K such that in it, these fields are parallel to each other. If we direct the z axis along their common direction, we have in this system of observation

$$(\mathbf{EB}) = E_zB_z,\quad B^2 - E^2 = B_z^2 - E_z^2,$$

and the roots of Eq. (8) will be

$$\lambda = \frac{E_z}{B_z} \text{ and } \lambda = -\frac{B_z}{E_z}.$$

The first value gives

$$P_z = \frac{B_z^2 + E_z^2}{B_z},\quad P_x = P_y = 0,\quad \mathbf{Q} = 0.$$

Consequently, in the system K we have nonmoving lines of force of the vector **P**. In an arbitrary other system of observation, they will be in motion, and it will be easily possible to find their drift velocity. These are hybrid magneto-electric lines (the other root gives electro-magnetic lines). The lines of force of the hybrid vector **P** can be "materialized." In fact, let us imagine that in the system K a narrow beam of electrons is sent along the direction z (that is, along the direction of the vectors E and B). They will preserve the direction of their motion, and thus a chain of electrons will mark the hybrid magneto-electric line of force.

[1] W. A. Newcomb, Ann. Phys. (New York) **3**, 347 (1958).

Translated by W. F. Brown, Jr.