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DYNAMICAL THEORY OF DISLOCATIONS

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INTRODUCTION

THE concept of a dislocation in a crystal, together with the different dislocation representations and dislocation models associated with this concept, has been used very extensively during the last decade for a theoretical interpretation of many different physical phenomena observed in studies of plasticity and strength of solids. The great popularity of the dislocation concept among physicists and engineers, who discuss "from a unified point of view" a tremendous amount of experimental facts, is apparently due to a considerable degree to the fact that the concept of dislocation is very broad, and admits of many varied microscopic models of the structure and properties of this singularity in the crystal. The latter makes it very easy to adapt dislocation models in a crystal of any symmetry to the description of many phenomena of inelastic deformation, which at first glance do not have a common physical nature.

However, the greater part of the significant physical properties of the dislocations is not connected with their microscopic models, and can be described phenomenologically within the framework of macroscopic elasticity theory, to the same degree that elasticity theory is capable of describing the propagation of sound waves of not too short a wavelength in a crystalline body. A dislocation theory based on the notion of a continuous medium and founded on elasticity theory is customarily called the continual theory of dislocations. Continual dislocation theory has by now been sufficiently well developed and can be formulated and expounded in the same form as other branches of theoretical physics. In particular, there exists a complete system of equations which relates in the general case the elastic fields with the distributions and fluxes of the dislocations. There exist rigorous equations for obtaining the plastic deformation of a body from the known dislocation motion.

The present review is an attempt at an exposition of the principles of the continual theory of dislocations, that is, to an explanation of the position occupied by dislocations in the general scheme of elasticity theory, and at a description of dislocation properties that do not depend on microscopic models. It is not our purpose to review the application of dislocation theory to concrete problems in plasticity and strength of solids, since this is covered in a large number of surveys, monographs, and collections (see, for example, [1-7]).

The appended bibliography can serve only as a brief index on dislocation theory. A rather complete list of papers on dislocation theory can be found in the monographs [5,8] and reviews [6,9-13], as well as in the corresponding section of the bibliographic index on dislocations.* In addition, we emphasize that when we mention some statement or result, we refer as a rule not to the source where this result was first formulated or proved, but to the latest publication in which, in our opinion, the result is described in the simplest and most general fashion.

1. DEFINITION OF DISLOCATION. DISLOCATION DENSITY

A dislocation in a continuous medium is defined as a singularity of the displacement-vector field (and consequently of the strain and stress field), extending along a certain line and having the following property (see, for example, [11]): If we trace a closed contour $\mathscr{L}$ that circles the dislocation line $\mathscr{D}$ once (Fig. 1), then the vector of elastic displacement of the medium, u, is increased after circuiting around this contour by the definite amount

$$\oint_{\mathscr{L}} dx_i \nabla_i u_k = -b_k, \quad \nabla_i \equiv \frac{\partial}{\partial x_i}, \tag{1.1}$$

where x_k (k = 1, 2, 3) are rectangular coordinates and b is a specified vector (Burgers vector). In formula (1.1), as well as in all that follow, we sum over repeated indices from 1 to 3.

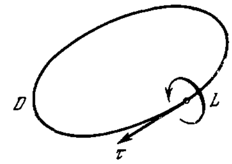

FIG. 1

The direction along the contour $\mathscr{L}$ is that of a right-hand screw relative to the chosen direction along the dislocation loop $\mathscr{D}$, that is, relative to the direction of the unit vector τ tangent to the dislocation line (Fig. 1).

Condition (1.1) means that in the presence of a dislocation in the medium the elastic displacement vector **u** is not a single-valued function of the coordinates, since it receives a definite increment on going around

L. V. Matveeva, Dislokatsii v kristallakh (Dislocations in Crystals), Bibliographic index edited by V. L. Indenbom, Moscow, 1960 the dislocation line.

From (1.1) follow two important corollaries, the proof of which is obvious [9]:

1) The dislocation line is always either closed or has both ends on the surface of the solid.

2) The vector **b** has the same value along the entire line of dislocation $\mathscr{D}$. The latter corollary is sometimes called the law of conservation of the Burgers vector along the dislocation.

A cylindrical surface with generators parallel to the vector **b** and with a directrix along the dislocation line is called the dislocation slip surface. This surface is the envelope of a family of slip planes of all the dislocation elements. By slip plane of a dislocation element is meant the plane tangent to the corresponding line element, a plane specified by the vectors $\boldsymbol{\tau}$ and **b**.

Returning to the initial definition (1.1), we rewrite it in the form

$$\oint_{\mathscr{L}} dx_i u_{ik} = -b_k, \quad u_{ik} = \nabla_i u_k. \tag{1.2}$$

The tensor u_{ik} in (1.2) is called the elastic-distortion tensor in dislocation theory. Its symmetrical part yields the ordinary elastic strain tensor ϵ_{ik}.

In addition to the condition (1.1), which determines the type of dislocation singularity, it is assumed [9,11] that in the presence of dislocations in the medium the elastic distortion tensor u_{ik} is a unique function of the coordinates, continuous and differentiable over all of space.† In dislocation theory, the distortion tensor is usually regarded to be an independent quantity describing the deformation of the crystal.

The definition (1.2) can be generalized by representing the Burgers vector **b** in the form of an integral over the surface S bounded by the contour $\mathscr{L}$:

$$b_i = \int_S dS_k \alpha_{ki}, \tag{1.3}$$

where α_{ik}—Burgers-vector density tensor (dislocation density tensor)[8-12].

We then have in place of (1.2)

$$\oint_{\mathscr{L}} dx_k u_{ki} = -\int_S dS_k \alpha_{ki}. \tag{1.4}$$

In the case of an individual linear dislocation, the tensor α_{ik}, in accordance with (1.3), takes the form

$$\alpha_{ik} = \tau_i b_k \delta(\xi), \tag{1.5}$$

where τ—unit vector tangent to the dislocation line (Fig. 1), $\delta(\xi)$—two-dimensional δ-function, and ξ—two-dimensional radius vector measured from the dislocation axis in a plane perpendicular to the vector τ at the given point.

Inasmuch as the contour $\mathscr{L}$ and the surface S bounded by it are arbitrary in (1.4), it follows from (1.4) that

$$e_{ilm} \nabla_l u_{mk} = -\alpha_{ik}, \tag{1.6}$$

where e_{ikl}—unit antisymmetrical tensor of third rank.

As seen from (1.6), the tensor α_{ik} should satisfy the condition

$$\nabla_i \alpha_{ik} = 0, \tag{1.7}$$

which in the case of a linear dislocation expresses the law of conservation of the Burgers vector along the dislocation line.

Equations (1.4) and (1.6) admit of a natural generalization to the case of a continuous distribution of dislocations in the medium [8-12]. To effect such a generalization, it is necessary to assume that the Burgers-vector density α_{ik} is some continuous function of the coordinates, satisfying the condition (1.7).

2. DISLOCATION FLUX DENSITY TENSOR

Condition (1.1) and Eq. (1.6), which are the fundamental relations that introduce dislocations in elasticity theory, do not depend on whether the dislocations are stationary or moving. However, it is obvious that in the dynamical case the time variation of the distortion tensor should be determined essentially by the character of motion of the dislocations.

If the dislocations remain stationary when the elements of the medium are displaced, we have the obvious equality

$$\frac{\partial u_{ik}}{\partial t} = \nabla_i v_k, \tag{2.1}$$

where $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$—velocity of displacement of an element with coordinate **r** at the instant of time t.

On the other hand, if the dislocations move and the dislocation density varies with time, then (2.1) is not compatible with (1.6), and we shall therefore replace it with [14]

$$\frac{\partial u_{ik}}{\partial t} = \nabla_i v_k + j_{ik}, \tag{2.2}$$

in which the tensor j_{ik} should be chosen such as to make (1.6) and (2.2) compatible.

*We note that in a medium with an individual dislocation loop $\mathfrak{D}$ we can always introduce a uniquely defined vector **u** in lieu of the ambiguous elastic displacement vector by drawing an imaginary cut through the medium along an arbitrary surface S bounded by the line $\mathfrak{D}$, and by specifying in lieu of condition (1.1) the difference of the values of **u** on opposite edges of the cut:

$$\delta\mathbf{u} \equiv \mathbf{u}_+ - \mathbf{u}_- = \mathbf{b},$$

where $\mathbf{u}_+$ and $\mathbf{u}_-$—values of **u** on the "upper" and "lower" edges of the cut, respectively. The "upward" (positive) direction is defined in the right-hand system by the direction of motion along the line $\mathfrak{D}$.

†If we confine ourselves to the requirement that the strain tensor ϵ_{ik}, and not the distortion tensor u_{ik}, be unambiguous, continuous, and differentiable, then we can take into consideration also dislocations of a more general type than those defined by the property (1.1) (see, for example, [9]).

The condition for the compatibility of (1.6) and (2.2) takes the form of an equation

$$\frac{\partial \alpha_{ik}}{\partial t} + e_{ilm}\nabla_l j_{mk} = 0. \tag{2.3}$$

It is easy to verify that (2.3) is the differential form of the law of conservation of the Burgers vector in the medium. Indeed, let us consider some stationary closed line $\mathscr{L}$ in the medium. We take an arbitrary surface bounded by the line $\mathscr{L}$, and introduce in the formula (1.3) the total Burgers vector b of the dislocations that are "coupled" with this surface, that is, enclosed by the line $\mathscr{L}$. Then we can obtain from (2.3) in elementary fashion

$$\frac{db_k}{dt} = -\oint_{\mathscr{L}} dl_i j_{ik}. \tag{2.4}$$

From the meaning of relation (2.4) it follows that the integral in the right side of (2.4) determines the magnitude of the Burgers vector "flowing" per unit time through the contour $\mathscr{L}$, that is, carried away by the dislocations that cross the line $\mathscr{L}$. Therefore the tensor j_{ik} can naturally be called the dislocation flux density, and (2.3) is the equation for the continuity of the dislocation flux. An equation coinciding in form with (2.3) was derived by Hollander [15], but it expresses the relationship between somewhat different quantities.

The definition of the tensor j_{ik} becomes unambiguous if we note that the dislocation flux density determines directly the rate of plastic deformation of the medium. To verify this, we note that the vector **v** is the velocity of the total geometric displacement of an element of the medium, and consequently determines the velocity of the total geometrical distortion U_{ik}:

$$\frac{\partial U_{ik}}{\partial t} = \nabla_i v_k. \tag{2.5}$$

With the aid of (2.5) we rewrite (2.2) in the form

$$\frac{\partial}{\partial t}(U_{ik} - u_{ik}) = -j_{ik}. \tag{2.6}$$

The difference $U_{ik} - u_{ik}$ defines that part of the total distortion not connected with the elastic stresses, usually called the plastic distortion of the body. Denoting this quantity by u_{ik}^{pl}, we obtain

$$\frac{\partial u_{ik}^{pl}}{\partial t} = -j_{ik}. \tag{2.7}$$

Thus, the change in the tensor of plastic distortion at a certain point of the medium after a short time δt is equal to

$$\delta u_{ik}^{pl} = -j_{ik}\delta t. \tag{2.8}$$

If we write a relation similar to (2.8) for the plastic deformation tensor ϵ_{ik}^{pl}, its form will be

$$\delta\varepsilon_{ik}^{pl} = -J_{ik}\delta t, \quad J_{ik} = \mathrm{Sy}\{j_{ik}\}, \tag{2.9}$$

where the symbol Sy{...} denotes that we are taking the symmetrical part of the second-rank tensor.

A relation equivalent to (2.7)—(2.9), but in a different notation, was indicated by Kroner and Rider [16].

Starting from the physical meaning of the dislocation flux density, we can easily establish a relation between the tensors j_{ik} and α_{ik} for a known dislocation velocity.

We consider first a linear dislocation with a Burgers vector **b**, each point of the loop of which moves with a velocity $\mathbf{V} = \mathbf{V}(\mathbf{r}, t)$, and we calculate the Burgers-vector flux produced by such dislocations when some line $\mathscr{L}$ is crossed (Fig. 2). If dl—element of arc of the line $\mathscr{L}$, and $\boldsymbol{\tau}$—unit vector tangent to the dislocation loop in the vicinity of the point where the line $\mathscr{L}$ is crossed, then crossing of the line $\mathscr{L}$ by the dislocation loop with transport of the Burgers vector will occur only when there is a velocity component **V** perpendicular to both dl and τ. It is obvious that the number of such crossings of the element dl by "parallel" dislocation loops per unit time is given by the quantity

$$N\,[d\mathbf{l}, \boldsymbol{\tau}]\,\mathbf{V},$$

where N is the number of dislocations passing per unit area of the plane perpendicular to $\boldsymbol{\tau}$.

Therefore, the flux of the Burgers-vector component b_k through the line $\mathscr{L}$ is equal to

$$\oint_{\mathscr{L}} dl_i j_{ik} = \oint_{\mathscr{L}} N\,[d\mathbf{l}, \boldsymbol{\tau}]\,\mathbf{V} b_k. \tag{2.10}$$

It follows from (2.10) that in the case of linear dislocations with identical velocities at the point of space in question we have

$$j_{ik} = N e_{ilm}\tau_l b_k V_m. \tag{2.11}$$

In the case of a continuous distribution of the dislocations, *the tensor* j_{ik} *is a continuous function of* the coordinates, satisfying the condition (2.3). The tensor j_{ik} is significant in itself and is a fundamental characteristic of dislocation motion.

Special interest is attached to the connection between the trace of the tensor j_{ik} ($j_0 \equiv j_{kk}$) and the continuity equation of a continuous medium [14]. The trace j_0 enters in the following equation derivable from (2.2):

$$\mathrm{div}\,\mathbf{v} - \frac{\partial \varepsilon_{kk}}{\partial t} = -j_0. \tag{2.12}$$

It is easy to explain the physical meaning of (2.12). Indeed, the trace ϵ_{kk} is the relative elastic change in the volume element of the medium, connected in obvious fashion with the corresponding relative

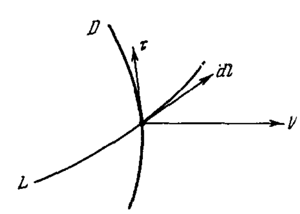

FIG. 2

change in its density

$$\varepsilon_{kk} = -\frac{\delta\varrho}{\varrho} \quad (2.13)$$

(ρ—density of the medium).

Substituting (2.13) in (2.12) and using the linearity of the theory, we arrive at the relation

$$\frac{\partial\varrho}{\partial t} + \operatorname{div}\varrho\mathbf{v} = -\varrho j_0. \quad (2.14)$$

If the medium elements are displaced during the motion of dislocation without loss of continuity then, by virtue of the continuity equation, the left side of (2.14) vanishes and

$$j_0 \equiv j_{kk} = 0. \quad (2.15)$$

Comparing (2.15) with (2.9), we see that the statement that j_0 vanishes is equivalent the statement that the corresponding plastic deformation is not connected with the change in the volume of the body.

For linear dislocations, the condition (2.5) has a simple meaning. Indeed, in the case of a relative linear dislocation the trace j_0 is proportional to $[\mathbf{b} \times \boldsymbol{\tau}] \cdot \mathbf{V}$ [this can be seen even from (2.11)], that is, it is proportional to the projection of the dislocation velocity on the direction perpendicular to the vectors $\boldsymbol{\tau}$ and $\mathbf{b}$, in other words, on the direction perpendicular to the slip plane of the dislocation. Thus, (2.15) signifies that when the medium remains continuous the dislocation velocity vector $\mathbf{V}$ always lies in its slip plane, and consequently mechanical motion of the dislocation can occur only in this plane[9-11].

On the other hand, if the dislocation motion is accompanied by formation of certain discontinuities, for example, macroscopic accumulation of vacancies along a section of the dislocation line, then the left side of (2.14) differs from zero and is equal to the rate of relative elastic increase in the mass of a certain volume element of the medium (or, accordingly, an increase in its specific volume). We denote by $q(\mathbf{r}, t)$ the relative increase of the specific volume of the medium at the point $\mathbf{r}$ per unit time; then

$$j_0 = q(\mathbf{r}, t). \quad (2.16)$$

It follows from (2.16) that a dislocation can move in a direction perpendicular to its slip plane ("climb") as a result of formation (or annihilation) of a certain chain of vacancies or interstitial atoms.

3. ELASTICITY EQUATIONS WITH MOVING DISLOCATIONS. AVERAGING THE EQUATIONS. DISLOCATION MOMENT

Let us set up a complete system of differential equations, describing the dynamics of an elastic body in which there are moving dislocations. We assume that the displacement of the dislocation is not accompanied by transport of mass, and take account of the fact that no additional distribution of the concentrated volume force is associated with the dislocation line[11]. Then the equation of motion of the elastic medium in an approximation that is linear in the displacement rate should be written in the form

$$\varrho\frac{\partial v_i}{\partial t} = \nabla_k \sigma_{ki}, \quad (3.1)$$

where the elastic stress tensor σ_{ik} is connected by Hooke's law with the elastic strain tensor

$$\sigma_{ik} = \lambda_{iklm}\varepsilon_{lm} = \lambda_{iklm}u_{lm}. \quad (3.2)$$

Relation (3.2) contains the usual moduli of elasticity λ_{iklm}.

Equation (3.1) together with (1.6) and (2.2) constitutes a system of elasticity-theory equations for the dynamic deformation field in a medium with moving dislocations.

To simplify the notation, we shall denote any second-rank tensor A_{ik} by the symbol $\hat{A}$ and introduce the notation [8]

$$\operatorname{Rot}\hat{\mathbf{A}} \equiv (e_{ilm}\nabla_l A_{mk}),\ \operatorname{Div}\mathbf{A} \equiv (\nabla_i A_{ik}),\ \operatorname{Grad}\mathbf{a} \equiv (\nabla_i a_k). \quad (3.3)^*$$

Then the system (1.6), (2.2), and (3.1) can be represented in the form [14]

$$\operatorname{Rot}\hat{\mathbf{u}} = -\hat{\boldsymbol{\alpha}}, \quad (3.4)$$

$$\operatorname{Div}\hat{\boldsymbol{\sigma}} = \varrho\frac{\partial\mathbf{v}}{\partial t}, \quad \sigma_{ik} = \lambda_{iklm}u_{lm}, \quad (3.5)$$

$$\operatorname{Grad}\mathbf{v} = \frac{\partial\hat{\mathbf{u}}}{\partial t} - \hat{\mathbf{j}}. \quad (3.6)$$

If the tensors $\hat{\alpha}$ and $\hat{\mathbf{j}}$, are specified, i.e., for specified dislocation densities and dislocation fluxes, the system (3.4)—(3.6) is complete. The conditions for the compatibility of this system are the "conservation laws" (1.7) and (2.3):

$$\operatorname{Div}\hat{\boldsymbol{\alpha}} = 0, \quad \frac{\partial\hat{\boldsymbol{\alpha}}}{\partial t} + \operatorname{Rot}\hat{\mathbf{j}} = 0. \quad (3.7)$$

The system (3.4)—(3.6) enables us to find $\hat{\mathbf{u}}$ (or $\hat{\sigma}$) and $\mathbf{v}$ from any known distribution of dislocations and their fluxes. A general solution of this system was indicated for the case of an isotropic medium in a paper by the author [14].

Since we have assumed Hooke's law (3.2) with an elastic-modulus tensor having the usual symmetry properties, the elastic stresses are determined only by the symmetrical part of the distortion tensor (strain tensor). In this connection, it is sometimes convenient to represent the system (3.4)—(3.6) in such a notation which contains only the symmetrical part of the distortion tensor. The latter can be done by representing the tensor $\hat{\alpha}$ in the form of a sum of two tensors

$$\hat{\boldsymbol{\alpha}} = -\hat{\boldsymbol{\alpha}}^0 - \hat{\boldsymbol{\alpha}}', \quad (3.8)$$

the first of which ($\hat{\alpha}^0$) generates the Kroner symmetrical incompatibility tensor $\hat{\eta}$ [8]

$$\eta_{ik} \equiv e_{ilm}\nabla_l\alpha^0_{mk} = e_{klm}\nabla_l\alpha^0_{mi} \quad (3.9)$$

*Rot = Curl.

and defines the strain tensor by means of an equation analogous to (3.4) while the second part ($\hat{\alpha}'$) determines the curvature of the lattice [6,8].

Equation (3.5) contains the symmetrical tensor $\hat{\sigma}$, and equation (3.6) can be symmetrized in trivial fashion, so that the "symmetrized" form of (3.4)—(3.6) is

$$\mathrm{Rot}\,\hat{\varepsilon}=\hat{\alpha}^0, \tag{3.10}$$

$$\mathrm{Div}\,\hat{\sigma}=\varrho\frac{\partial \mathbf{v}}{\partial t}, \tag{3.11}$$

$$\mathrm{Def}\,\mathbf{v}=\frac{\partial\hat{\varepsilon}}{\partial t}-\hat{\mathbf{J}},\quad (\mathrm{Def}\,\mathbf{v})_{ik}\equiv\frac{1}{2}(\nabla_i v_k+\nabla_k v_i). \tag{3.12}$$

It is essential to note that (3.10)—(3.12) do not contain the tensor $\hat{\alpha}'$ and the antisymmetrical part of the dislocation flux density tensor (the latter determines the rate of change of the lattice-rotation tensor).

It must be borne in mind, however, that (3.10) is a purely formal equation. Indeed, the introduction of the initial tensor $\hat{\alpha}$ was connected with the definition of the dislocations, and the dislocation density described by this tensor can be determined experimentally. In the absence of dislocations, $\hat{\alpha}\equiv 0$. As to the tensor $\hat{\alpha}^0$, *it is uniquely connected with the* dislocation density and, in particular, it can be different from zero if there are no dislocations at all in the crystal but the sample is subject to torsion.

A system of equations similar to (3.10)—(3.12), but still different from that given above, was proposed by Hollander [15].

It is noted above that the tensor densities contained in (3.4)—(3.6) can describe both a system of discrete (linear) dislocations and a continuous distribution of dislocations.

Naturally, when we speak of a continuous distribution of dislocations, we have in mind physical conditions under which the system of dislocations can be described by certain averaged (macroscopic) characteristics. These conditions arise in those problems, in which we are not interested in an exact distribution of the field between the different dislocations and in which the theory operates with physical quantities that are averaged over the elements of a volume containing a large number of dislocations. However, inasmuch as the dislocations are extended crystal defects, which frequently have macroscopic linear dimensions, the question of introducing averaged quantities, that is, of choosing "a physically infinitesimally small" volume, is not trivial. The choice of the volume element over which the averaging is carried out is naturally related with the choice of the ratio of the linear dimensions of this volume to the linear dimensions of the dislocation lines. It is obvious that this relation can be different in various problems, and therefore the averaged quantities can be described by different distribution functions.

The simplest case of spatial arrangement of the dislocations occurs when the average dimension of the dislocation loops r_m is much larger than the average distance h between the different dislocations: $r_m \gg h$. If we are interested in this case in the variation of the elastic field over distances that are smaller than or of the order of r_m, then it is natural to define the linear dimension l_0 by the condition

$$h\ll l_0\ll r_m. \tag{3.13}$$

If we now take into account the fact that the vector **b** can have only a fixed number of fully defined directions in the crystal, then the average dislocation density $\hat{\alpha}$ (the average value will henceforth be denoted by the same letter) can in this case be written in the form [6,14]

$$\alpha_{ik}(\mathbf{r})=\sum_\beta\int\tau_i b_k^\beta\varrho^\beta(\boldsymbol{\tau};\ \mathbf{r})\,dO, \tag{3.14}$$

where β—numbers of the possible directions of the vector b, the integration is carried out over the complete solid angle, and $\rho^\beta(\tau;\ \mathbf{r})$—scalar density of the distribution of the vectors **b** and τ over the possible directions; $\rho^\beta(\tau;\ \mathbf{r})\,dO$ is the number of dislocations having a Burgers-vector direction β passing through a unit area perpendicular to the vector τ, and located inside a solid angle dO around the direction of $\boldsymbol{\tau}$.

In the dynamic case the scalar distribution function $\rho(\mathbf{r})$ depends, naturally, on the time. The average dislocation flux density can be expressed in terms of the same scalar distribution function $\rho^\beta(\boldsymbol{\tau})$ which enters in (3.14):

$$j_{ik}(\mathbf{r})=e_{ilm}\sum_\beta\int\tau_l b_k^\beta V_m^\beta(\boldsymbol{\tau};\ \mathbf{r})\,\varrho^\beta(\boldsymbol{\tau};\ \mathbf{r})\,dO, \tag{3.15}$$

where $V_\beta(\tau)$—average velocity of the dislocation-length element having corresponding directions of **b** and τ.

Another simple case of averaging corresponds to a situation in which we are interested in the change of the field over distances considerably larger than the dimensions of the dislocation loops and the distance between them. In such a case the averaging volume should be chosen to satisfy the condition

$$l_0\gg r_m,\ h, \tag{3.16}$$

and will contain a large number of closed dislocation loops.

The form of the equations with such an averaging will be considered in the next section. However, before we proceed to do so, we introduce a new quantity with which it is convenient to describe a system of closed dislocation loops, namely the quantity [14]

$$D_{ik}=\frac{1}{2}\int e_{ilm}x_l\alpha_{mk}\,d\Omega, \tag{3.17}$$

where $d\Omega$—volume differential, and the integration is carried out over the volume occupied by the dislocation system.

The meaning of the quantity $\hat{\mathbf{D}}$ can be readily explained by going over in (3.17) to summation over the dislocation loops $\mathscr{D}$:

$$D_{ik}=\frac{1}{2}e_{ilm}\sum b_k\oint_{\mathscr{D}} x_l\,dx_m=\sum s_i b_k; \qquad (3.18)$$

s—axial vector with components equal to the areas bounded by the projections of the dislocation loop $\mathscr{D}$ on the plane perpendicular to the corresponding coordinate axes:

$$s_i=\frac{1}{2}e_{ikl}\oint_{\mathscr{D}} x_k\,dx_l,\quad \oint_{\mathscr{D}} x_i\,dx_k=e_{ikl}s_l.$$

Thus, the tensor $\hat{\mathbf{D}}$ is expressed directly in terms of the areas of the dislocation loops and their Burgers vectors. In analogy with a similar definition in electrodynamics, we call the tensor $\hat{\mathbf{D}}$ the tensor of the dislocation moment of the system.*

Since

$$\frac{ds_i}{dt}=\oint_{\mathscr{D}} e_{ikm}V_k\,dx_m, \qquad (3.19)$$

where V—velocity of motion of the element of the dislocation loop, it follows from (3.18) and (2.11) that

$$\frac{d\hat{\mathbf{D}}}{dt}=-\int \hat{\mathbf{j}}\,d\Omega. \qquad (3.20)$$

4. PLASTIC POLARIZATION OF SOLID. DEFORMATION FIELD PRODUCED BY A SYSTEM OF CONTINUOUSLY DISTRIBUTED DISLOCATION LOOPS IN AN UNBOUNDED MEDIUM

The simplest but most frequently encountered dislocation distribution is one in which, first, every macroscopically small volume element of the body contains a large number of dislocation loops, and, second, the resultant (total) Burgers vector of all the dislocations in the body in question vanishes.† The latter condition has the simple mathematical form

$$\int_S dS_i\alpha_{ik}=0, \qquad (4.1)$$

where S—arbitrary transverse surface in the body.

By virtue of the arbitrariness of the surface S, it follows from (4.1) that the dislocation density can be written in the form

$$\hat{\alpha}=\operatorname{Rot}\hat{\mathbf{P}}, \qquad (4.2)$$

where $\hat{\mathbf{P}}$ is some second-rank tensor, equal to zero outside the body. Indeed, integrating (4.2) over the surface outlined by a contour $\mathscr{L}$ on the exterior surface of the body, we obtain

$$\int_S dS_i\alpha_{ik}=\int_S dS_i e_{ilm}\nabla_l P_{mk}=\oint_{\mathscr{L}} dl_i P_{ik}=0.$$

To clarify the physical meaning of the tensor $\hat{\mathbf{P}}$, let us consider the total dislocation moment of the dislocation loops in the body, calculating the integral (3.17) over the volume of the entire body. We use (4.2), carry out an elementary transformation of the integral (3.17):

$$D_{ik}=\frac{1}{2}\int e_{ilm}e_{mpq}x_l\nabla_p P_{qk}\,d\Omega$$
$$=\frac{1}{2}\int x_m(\nabla_i P_{mk}-\nabla_m P_{ik})\,d\Omega=\int P_{ik}\,d\Omega,$$

and verify that

$$\hat{\mathbf{D}}=\int \hat{\mathbf{P}}\,d\Omega. \qquad (4.3)$$

Thus, the tensor $\hat{\mathbf{P}}$ is equal to the density of the dislocation moment of the deformed medium. Apparently this tensor was introduced for the first time by Kroupa [18]. We shall call $\hat{\mathbf{P}}$ the plastic polarization tensor.

In exactly the same way that the magnetization vector of the body determines the current surface density in electrodynamics, the value of the tensor $\hat{\mathbf{P}}$ on the surface of the body determines the surface dislocation density $\hat{\alpha}^s$:

$$\alpha^s_{ik}=e_{iml}P_{mk}n_l; \qquad (4.4)$$

where n is the normal to the surface of the body.

Comparing (3.20) with (4.3), we can conclude that

$$\hat{\mathbf{j}}=-\frac{\partial\hat{\mathbf{P}}}{\partial t}. \qquad (4.5)$$

Inasmuch as the dislocation flux density tensor determines, in accordance with (2.8), the variation of the plastic distortion of the body, we get

$$\delta\hat{\mathbf{u}}^{\mathrm{pl}}=\delta\hat{\mathbf{P}}. \qquad (4.6)$$

Consequently, in the absence of a resultant Burgers vector in the body, the change in the plastic distortion tensor at each point of the medium is equal to the change in the dislocation polarization tensor at the same point. It must be recalled, however, that there is a fundamental difference between the tensors $\hat{\mathbf{P}}$ and $\hat{\mathbf{u}}^{\mathrm{pl}}$. Whereas the tensor $\hat{\mathbf{P}}$ is a function of the state of the body, the plastic-distortion tensor is not a function of the state of the body, but depends on the process through which the body reached the given state.

We now return to the system (3.4)—(3.6) and ascertain the simplifications that arise from the condition (4.1) or from relation (4.2). Substituting (4.2) in (3.4) we obtain

$$\operatorname{Rot}\hat{\mathbf{U}}=0, \qquad (4.7)$$

where we have introduced a new quantity $\hat{\mathbf{U}}$, defined by

$$\hat{\mathbf{U}}=\hat{\mathbf{u}}+\hat{\mathbf{P}}. \qquad (4.8)$$

It follows from (4.7) that the new distortion tensor can be represented in the form

$$\hat{\mathbf{U}}=\operatorname{Grad}\mathbf{U}. \qquad (4.9)$$

*The dislocation moment of a closed loop does not depend on the choice of the origin.

†The vanishing of the complete dislocation Burgers vector denotes the absence of macroscopic plastic bending of the solid sample.

If we agree to assume that the body is undeformed if its state is that possessed when $\hat{P} = 0$ and in the absence of other forces ($\hat{u} = 0$), then the vector $\hat{U}$ can be set equal to the vector of the total geometric displacement of the element of the medium. Then the tensor $\hat{U}$ determines the complete geometrical distortion of the body, produced by both the elastic deformation of the medium and by the displacement of the dislocations [6]. The rate of displacement is in this case connected in the usual fashion with the time derivative of the vector U

$$\mathbf{v} = \frac{\partial \mathbf{U}}{\partial t}. \qquad (4.10)$$

We substitute (4.10) in the equation of motion (3.5) of the elastic medium and reduce this equation to the usual form

$$\varrho \frac{\partial^2 U}{\partial t^2} = \operatorname{Div} \hat{\boldsymbol{\sigma}}. \qquad (4.11)$$

We now use Hooke's law and, combining (4.11) with (4.8), obtain an equation for the determination of the vector $\hat{U}$:

$$\varrho \frac{\partial^2 U_i}{\partial t^2} - \lambda_{iklm}\nabla_k\nabla_l U_m = -\lambda_{iklm}\nabla_k P_{lm}. \qquad (4.12)$$

Thus, the question of finding the elastic field produced by dislocations with zero total Burgers vector reduces to a known problem of elasticity theory. When the plastic polarization tensor $\hat{P}$ of the body is specified, the vector $\hat{U}$ can be determined from the dynamical elasticity equation expressed in terms of displacements, in which the role of the density of the volume force is assumed by the vector $p_i = -\lambda_{iklm}\nabla_k P_{lm}$.

Let us rewrite the definition of the vector p, using relations (3.18) and (4.3):

$$p_i = \nabla_k Q_{ik}, \quad Q_{ik} = -\lambda_{iklm} P_{lm} = -\lambda_{iklm} \sum_{1\,\text{un. vol.}} s_l b_m. \qquad (4.13)$$

The tensor $\hat{Q}$ has the meaning of the density of the dipole force. Consequently, with respect to producing the displacement field, the system of dislocation loops is equivalent to a definite dipole-force distribution.

Let us consider the displacement field produced by a system of dislocation loops in an unbounded medium. The solution of (4.12) for this case can be represented in the form [17,19]

$$U_i(\mathbf{r}, t) = \int d\Omega' \int_{\infty}^{t} d\tau G_{ik}(\mathbf{r}-\mathbf{r}', t-\tau)\, p_k(\mathbf{r}', \tau), \qquad (4.14)$$

where $G_{ik}(\mathbf{r}, t)$ is the Green's tensor of the dynamical elasticity-theory equation for unbounded space.

Formula (4.14) in conjunction with (4.8) and (4.10) completely solves the problem of finding the displacement fields, the elastic strains and stresses, and the displacement velocities.

In the static case, when the retardation of the sound waves can be completely neglected, the expression for the vector of the total geometrical displacement can be written in the form

$$U_i^0(\mathbf{r}) = \int G_{ik}^0(\mathbf{r}-\mathbf{r}')\, p_k(\mathbf{r}')\, d\Omega', \qquad (4.15)$$

where $G_{ik}^0(\mathbf{r})$—static Green's tensor of elasticity theory.

Accordingly, the static elastic distortion tensor is described by the formula

$$u_{ik}^0(\mathbf{r}) = \int \nabla_i G_{kl}^0(\mathbf{r}-\mathbf{r}')\, p_k(\mathbf{r}')\, d\Omega' - P_{ik}(\mathbf{r}). \qquad (4.16)$$

Expressions for the fields $\mathbf{U}_0$ and $\hat{u}_0$ produced by the continuously distributed dislocation loops were discussed by Kroupa [18]. However, Kroupa chose a very unwieldy notation for $\hat{u}_0$, and consequently the integrals representing the elastic distortion of the medium turned out to have nonphysical singularities. The relation given above for $\hat{u}_0$ is more convenient, for when the functions $p_i(\mathbf{r})$ are bounded and this relation is used the question of the convergence of the integrals does not arise.

5. DISPLACEMENT AND DEFORMATION FIELD IN THE APPROXIMATION LINEAR IN THE DISLOCATION VELOCITY [17,28]

Assume that the dislocations occupy a certain part of the elastic medium, moving with small velocities V in a region with linear dimensions L. We shall assume that $V \ll c$, where c is the speed of sound in the medium, or, what is the same, $L \ll \lambda$, where λ is the characteristic length of the sound wave generated by the motion of the dislocations, and expand all the quantities of interest to us in powers of the delay time of the sound waves inside the dislocation system, confining ourselves to the first terms of the expansion. Such an expansion is valid both inside the dislocation system and at the distances $R \sim L$ from the system.

In an isotropic medium (4.12) becomes simpler and formula (4.14) for the vector U takes the form [20]

$$\begin{aligned} U_i(\mathbf{r}, t) = {} & \frac{1}{4\pi\varrho} \int \nabla_i\nabla_k \left(\frac{1}{R}\right) \left\{ \int_{R/a}^{R/c} \tau p_k(\mathbf{r}', t-\tau)\, d\tau \right\} d\Omega' \\ & - \frac{1}{4\pi\mu} \int (n_i n_k - \delta_{ik})\, p_k\left(\mathbf{r}', t - \frac{R}{c}\right) \frac{d\Omega'}{R} \\ & + \frac{1}{4\pi\varrho a^2} \int n_i n_k p_k \left(\mathbf{r}', t - \frac{R}{a}\right) \frac{d\Omega'}{R}, \end{aligned} \qquad (5.1)$$

where

$$R = |\mathbf{r} - \mathbf{r}'|, \; R\mathbf{n} = \mathbf{r} - \mathbf{r}', \; c^2 = \mu/\varrho, \; a^2 = (\lambda + 2\mu)/\varrho,$$

and the vector p is equal to

$$\mathbf{p} = -2\mu \operatorname{Div} \hat{\mathbf{P}}^S - \lambda \operatorname{grad} P_{kk}, \quad \hat{\mathbf{P}}^S = \operatorname{Sy}\{\hat{\mathbf{P}}\}$$

(μ and λ—elastic Lame moduli for the isotropic medium).

We write down the vector of the total displacement in the form $\mathbf{U} = \mathbf{U}_0 + \mathbf{U}_1$, where the subscript zero denotes the expansion terms obtained when the delay is completely neglected (quasistatic terms), and the subscript 1 denotes terms proportional to the time derivatives of the vector **p**. In the calculation of the main terms of the expansion we take account of the fact that by virtue of the definition (4.13) of the vector **p**, and by virtue of the finite dimensions of the dislocation system, we have

$$\int \mathbf{p}\, d\Omega = 0. \tag{5.2}$$

Using the last condition, we get

$$\mathbf{U}_0 = \frac{1}{8\pi\mu}\left\{(1-\gamma^2)\int \mathbf{n}(\mathbf{np})\frac{d\Omega}{R} + (1+\gamma^2)\int \mathbf{p}\frac{d\Omega}{R}\right\}, \tag{5.3}$$

$$\mathbf{U}_1 = \frac{1}{32\pi\mu c^2} \times \frac{\partial^2}{\partial t^2}\left\{(\gamma^4 - 1)\int \mathbf{n}(\mathbf{np})\, R\, d\Omega + (3+\gamma^4)\int \mathbf{p} R\, d\Omega\right\}, \tag{5.4}$$

where **n**—unit vector drawn from the volume element in question to the observation point, and $\gamma^2 = c^2/a^2$.

Formula (5.3) for the quasistatic displacement vector coincides, of course, after suitable transformation, with the formula obtained by Kroupa [18] for a system of stationary dislocation loops.

The term $\mathbf{U}_1$, defined by (5.4), is due to the motion of dislocations and describes essentially dynamic effects.

With the aid of expressions (5.3) and (5.4) for **U**, it is easy to construct the elastic strain tensor of the medium, which will also be represented in the form of two terms, the first ($\hat{\epsilon}_0$) defining the quasistatic deformation field and the second ($\hat{\epsilon}_1$) describing the dynamical effect. The quasistatic term will be written in the form

$$\varepsilon^0_{ik} + P^S_{ik} = \frac{1}{8\pi\mu}\left\{(1-\gamma^2)\int(\delta_{ik} - n_i n_k)(\mathbf{np})\frac{d\Omega}{R^2} - \gamma^2\int(n_i p_k + n_k p_i)\frac{d\Omega}{R^2}\right\}, \tag{5.5}$$

and will remain expressed in terms of the vector **p**.

The term $\hat{\epsilon}_1$, on the other hand, will be transformed to express it in terms of integrals of the tensor $\hat{\mathbf{P}}$, and we shall use the connection (4.5) between the tensor $\hat{\mathbf{P}}$ and the flux density tensor $\hat{\mathbf{j}}$:

$$\varepsilon^1_{ik} = \frac{1}{16\pi c^2}\frac{\partial}{\partial t}\left\{(1-\gamma^4)\int(\delta_{ik} - 3n_i n_k)\, n_l J_{lm} n_m \frac{d\Omega}{R} - 2\gamma^4\int(n_i J_{kl} n_l + n_k J_{il} n_l)\frac{d\Omega}{R} + 2(1+\gamma^4)\int J_{ik}\frac{d\Omega}{R} - (1-2\gamma^2+3\gamma^4)\int(\delta_{ik} - n_i n_k) J_{mm}\frac{d\Omega}{R}\right\}. \tag{5.6}$$

Expressions (5.5) and (5.6) for the elastic strain tensor enable us to construct, using Hooke's law, the stress tensor $\hat{\sigma}$, namely $\sigma_{ik} = 2\mu\epsilon_{ik} + \lambda\delta_{ik}\epsilon_{ll}$, and the components of the trace of the tensor $\hat{\epsilon}$ are

$$\varepsilon^0_{ll} + P_{ll} = -(\gamma^2/4\pi\mu)\int(\mathbf{np})\frac{d\Omega}{R^2}, \tag{5.7}$$

$$\varepsilon^1_{ll} = \frac{1}{4\pi a^2}\frac{\partial}{\partial t}\left\{-\gamma^2\int n_i J_{ik} n_k \frac{d\Omega}{R} + (1-\gamma^2)\int J_{ll}\frac{d\Omega}{R}\right\}. \tag{5.8}$$

We note that the last terms in (5.6) and (5.8) vanish if the motion of dislocations takes place without a discontinuity occurring in the medium, when $J_{ll} = 0$.

We shall not write out in explicit form the quasistatic part of the stress tensor, since it is treated widely in the literature [18,21] and can be derived by obvious operations from (5.5) and (5.7). We note, however, that this part of the stress tensor can be represented in the form

$$\sigma^0_{ik} = \frac{1}{4\pi}\int q_{ikl}(\mathbf{n})\, p_l \frac{d\Omega}{R^2}, \tag{5.9}$$

where the fully defined tensor function $q_{ikl}(\mathbf{n})$ contains only the components of the vector **n** and dimensionless vectors of the order of unity.

As to the dynamical part of the tensor $\hat{\sigma}$, we represent it in abbreviated form as

$$\sigma^1_{ik} = \frac{\varrho}{4\pi}\int \beta_{iklm}(\mathbf{n})\, \dot{J}_{lm}\frac{d\Omega}{R^2}, \tag{5.10}$$

where the dot denotes differentiation with respect to time, and in an isotropic medium the tensor $\beta_{iklm}(\mathbf{n})$ is connected with the components of the vector **n** by the relation

$$\begin{aligned}\beta_{iklm}(\mathbf{n}) = {} & \frac{1}{2}[(1-2\gamma^2+3\gamma^4)\delta_{ik} - 3(1-\gamma^4)n_i n_k]\, n_l n_m \\ & - \frac{1}{2}\gamma^4[(n_i\delta_{km} + n_k\delta_{im})n_l + (n_i\delta_{kl} + n_k\delta_{il})n_m] \\ & + \frac{1}{2}(1+\gamma^4)(\delta_{il}\delta_{km} + \delta_{im}\delta_{kl}) - \Big[\frac{1}{2}(1-2\gamma^2+3\gamma^4)(\delta_{ik} - n_i n_k) \\ & - \Big(\frac{\lambda}{\mu}\Big)\gamma^2(1-\gamma^2)\delta_{ik}\Big]\delta_{lm}.\end{aligned} \tag{5.11}$$

In an anisotropic medium formula (5.10), naturally, remains in force and the connection between the tensor $\beta_{iklm}(\mathbf{n})$ and the elastic moduli of the medium has been determined for the general case in [28].

6. DEFORMATION FIELD IN AN ISOTROPIC MEDIUM AT LARGE DISTANCES FROM THE DISLOCATION SYSTEM [14]

Let us consider the deformation produced in an unbounded isotropic medium by a system of moving dislocation loops at large distances R considerably in excess of the dimensions of the system L. We assume that the speed V of the dislocations is small compared with the speed of sound c in the solid. Then the ratios L/R and V/c are small parameters, in powers of which we can expand the quantities characterizing the deformation field. We confine ourselves to the first nonvanishing terms of such an expansion, assuming the smallness of both parameters to be of the same order of magnitude.

We choose the origin somewhere inside the system of dislocations and denote by R_0 the distance from the origin to the point of observation ($\mathbf{n}_0$—unit vector in the appropriate direction). Then by virtue of the smallness of the dimensions of the system, we can put in (5.1) $R \approx R_0 - \mathbf{r}'\mathbf{n}_0$ and expand the integrand in powers of $\mathbf{r}'\mathbf{n}$.

For simplicity we confine ourselves to the case $J_{ll} = j_0 = 0$. Then, by virtue of (3.20), the trace D_{ll} does not depend on the time, and we can separate the part of the vector $\hat{\mathbf{U}}$ which contains this trace as a distinct statistical term. Therefore the vector $\mathbf{U}$ is best written in the form

$$\mathbf{U} = \mathbf{U}_{st} + \widetilde{\mathbf{U}} + \mathbf{u}_t + \mathbf{u}_l, \qquad (6.1)$$

where the individual components are determined in the following fashion.

The statistical term $\mathbf{U}_{st}$, as already noted, is that part of $\mathbf{U}$ which contains the trace D_{ll}:

$$U_i^{st}(\mathbf{r}) = \int G_{ik}^0(\mathbf{r}-\mathbf{r}')\,\lambda\nabla_k' P_{ll}(\mathbf{r}')\,d\Omega', \qquad (6.2)$$

where $G_{ik}^0(r)$ is the static Green's tensor of elasticity theory. Elementary transformations of (6.2) and an explicit formulation of the approximation employed by us enable us to rewrite the formula for U_{st} in the form

$$U_i^{st} = \lambda\nabla_k G_{ik}^0(\mathbf{R}_0)\,D_{ll}. \qquad (6.3)$$

The term $\widetilde{\mathbf{U}}$ in (6.1) depends on the behavior of the dislocation moment in the time interval $(t - R_0/a,\ t - R_0/c)$:

$$\widetilde{U}_i = -\frac{c^2}{2\pi}\Big(\int\limits_{R_0/a}^{R_0/c} \tau D_{kl}^S(t-\tau)\,d\tau\Big)\nabla_i\nabla_k\nabla_l\Big(\frac{1}{R_0}\Big),$$

$$\hat{\mathbf{D}}^S = \mathrm{Sy}\{\hat{\mathbf{D}}\}. \qquad (6.4)$$

The last two terms in (6.1) are determined by the dislocation moment of the system at the instants of time $t = R_0/c$ and $t - R_0/a$, connected with the delays of the shear waves ($\mathbf{u}_t$) and the compression waves ($\mathbf{u}_l$) in the solid, respectively:

$$u_i^t = -\frac{1}{2\pi}\Big[t_{ik}\nabla_l\frac{D_{kl}^S}{R_0} - N_{ikl}\frac{D_{kl}^S}{R_0^2}\Big], \qquad (6.5)$$

$$u_i^l = -\frac{\gamma^2}{2\pi}\Big[n_i n_k\nabla_l\frac{D_{kl}^S}{R_0} + N_{ikl}\frac{D_{kl}^S}{R_0^2}\Big]_a. \qquad (6.6)$$

In (6.5) and (6.6) we introduced the tensor $t_{ik} = \delta_{ik} - n_i n_k$, the multiplication of which by a vector gives the "transverse" component of this vector (relative to the direction of $\mathbf{n}$), and we put

$$N_{ikl} = t_{ik}n_l + n_i\delta_{kl} + n_k\delta_{il} - 4n_i n_k n_l.$$

The differentiation in (6.3)—(6.6) is with respect to the coordinates of the point of observation.

The square brackets in (6.5) enclose quantities taken at the instant of time $t - R_0/c$. The square brackets with subscript a in (6.6) contain quantities taken at the instant of time $t - R_0/a$.

It is easy to verify that $\mathrm{div}\,\mathbf{u}_t = 0$ and $\mathrm{curl}\,\mathbf{u}_l = 0$, and therefore $\mathbf{u}_t$ and $\mathbf{u}_l$ contribute to the transverse and longitudinal parts of the displacement field, respectively.

Formulas (6.5) and (6.6) together with (6.3) and (6.4) determine the displacement vector $\mathbf{U}$ in the approximation in question.

At distances R_0 that are large compared with the wavelength of the emitted elastic waves ($R_0 \gg cL/V$), the expressions obtained determine the vector of elastic displacement of the radiation field. In the latter case, all the formulas become much simpler and we have for the displacement vector in the radiated sound wave

$$\mathbf{u} = \frac{1}{2\pi c R_0}\{\gamma^3\mathbf{n}\,[(\mathbf{n}\dot{\mathbf{D}})]_a + [\dot{\mathbf{D}} - \mathbf{n}\,(\mathbf{n}\dot{\mathbf{D}})]\}. \qquad (6.7)$$

The vector $\mathbf{D}$ in (6.7) denotes the "projection" of the symmetrical part of the tensor $\hat{\mathbf{D}}$ on the direction of $\mathbf{n}$:

$$D_i = n_k D_{ki}^S \equiv \frac{1}{2}n_k(D_{ik} + D_{ki}).$$

With the aid of (6.7), using the well-known formula for the flux density of sound energy [20], we calculate the differential intensity of radiation dI (radiation in a solid-angle element dO):

$$dI = \frac{\varrho}{4\pi^2 c}\{\gamma^5\,\overline{[(\mathbf{n}\ddot{\mathbf{D}})]_a^2} + \overline{[\ddot{\mathbf{D}}]^2} - \overline{[(\mathbf{n}\ddot{\mathbf{D}})]^2}\}\,dO. \qquad (6.8)$$

The bar in (6.8) denotes averaging over a region of space whose dimensions exceed the radiation wavelength.

The total elastic-wave energy radiated by the system of moving dislocations is equal to

$$I = \frac{\varrho}{5\pi c}\Big\{\frac{2}{3}\gamma^5\overline{[\ddot{D}_{ik}^S]_a^2} + \overline{[\ddot{D}_{ik}^S]^2}\Big\}. \qquad (6.9)$$

The radiation of elastic waves by the system of moving dislocations in an anisotropic medium can also be considered in general form [17].

7. FIELD NATURE OF THE EQUATION OF MOTION OF THE DISLOCATIONS. LAGRANGE FUNCTION FOR THE DISLOCATIONS AND FOR THE ELASTIC FIELD.

The system (3.4)—(3.6) formulated in Sec. 3 determines the elastic distortion tensor $\hat{\mathbf{u}}$ and the material displacement velocity vector $\mathbf{v}$ from the known distribution of the dislocations and their fluxes. The tensors $\hat{\alpha}$ and $\hat{\mathbf{j}}$, meaning also the motion of the dislocations, are assumed specified. In order for the system (3.4)—(3.6) to be completely closed and descriptive of the self-consistent evolution of the dislocations and of the elastic field, it is necessary to show how the densities of the dislocation densities and of their fluxes vary under the influence of elastic fields. In other words, it is necessary to indicate an equation of motion for the dislocations.

By virtue of the definition given above for the dislocation (in particular, by virtue of the absence of an additional volume force and the absence of mass transport on the dislocation line when the dislocation moves), the equation of motion of the dislocation should be derivable from a field. We shall derive the equation of dislocation motion in analogy with the Lorentz derivation of the equation of motion of the electron. That is to say, we start from the notion of self-action of the dislocation. The field-theoretic aspects of the equations of motion and of the corresponding masses are universally known and will not be developed here.

It is convenient to derive the equation of dislocation motion in the Lagrangian form. In the presence of linear (discrete) dislocations, the Lagrange function for the elastic field is usually written

$$L=\int \mathscr{L}_0\, d\Omega, \quad \mathscr{L}_0=\frac{1}{2}(\varrho v^2-\sigma_{ik}\varepsilon_{ik}), \tag{7.1}$$

where the "field intensities" $\mathbf{v}$ and $\hat{\epsilon}$ (or $\hat{\sigma}$) are expressed in usual fashion in terms of the derivatives of the elastic displacement vector $\mathbf{u}$, which plays the role of the deformation-field potential.

Since the presence of discrete dislocations admits of the introduction of a vector $\mathbf{u}$ and does not violate the uniqueness of its derivatives, formula (7.1) describes completely the elastic field between the dislocations. However, the choice of the elastic displacement vector $\mathbf{u}$ as the field potential deprives us of the possibility of introducing a Lagrangian density for the interaction of the elastic field with the dislocations, since the vector $\mathbf{u}$ is meaningless in those sections of the elastic medium, where the dislocations are located. It is therefore more convenient to introduce for the field potential quantities other than $\mathbf{u}$.

Since the Lagrange function is expressed in terms of the tensor $\hat{\epsilon}$ (the symmetrical part of the distortion tensor), we choose the new potentials by starting from the system (3.10)—(3.12), such as to satisfy (3.11) identically:

$$\hat{\sigma}=\operatorname{Rot}\hat{\varphi}+\frac{\partial\hat{\psi}}{\partial t}, \quad \mathbf{v}=\frac{1}{\varrho}\operatorname{Div}\hat{\psi}, \tag{7.2}$$

where $\hat{\varphi}$ and $\hat{\psi}$ are some second-rank tensors. In the static case the potential $\hat{\varphi}$ coincides with the potential introduced by Kroner[8].

It will be convenient in what follows to choose a symmetrical tensor ($\psi_{ik}=\psi_{ki}$); then the tensor curl $\hat{\varphi}$ will also be symmetrical.

We now write down the Lagrange function for the field of elastic stresses and dislocations in terms of the potentials $\hat{\varphi}$ and $\hat{\psi}$. This Lagrange function, which replaces (7.1), should lead to Eqs. (3.10)—(3.12), and should go over into (7.1) for a free elastic field, after transformations in which total derivatives with respect to the time and surface integrals over infinitely remote surfaces are left out.* The Lagrange function of interest to us is of the form

$$L=\int \mathscr{L}\, d\Omega, \quad \mathscr{L}=\frac{1}{2}(\sigma_{ik}\varepsilon_{ik}-\varrho v^2)-\alpha^0_{ik}\varphi_{ik}+J_{ik}\psi_{ik}, \tag{7.3}$$

where $\hat{\sigma}$ and $\mathbf{v}$ are expressed in terms of the potentials $\hat{\varphi}$ and $\hat{\psi}$ by means of (7.2), and the tensor $\hat{\epsilon}$ is related to $\hat{\sigma}$ by Hooke's law.

It is easy to verify that the Lagrange function (7.3) leads to (3.10) and (3.12).

The Lagrange function can be divided in natural fashion into two parts: the Lagrange function of the elastic field with Lagrangian density

$$\mathscr{L}_{\mathrm{el}}=\frac{1}{2}(\sigma_{ik}\varepsilon_{ik}-\varrho v^2)$$

and the Lagrange function of the interaction between the dislocation and the elastic field

$$L_{\mathrm{int}}=\int (J_{ik}\psi_{ik}-\alpha^0_{ik}\varphi_{ik})\, d\Omega. \tag{7.4}$$

To obtain the equation of motion of the dislocation using the Lagrangian formalism, it is necessary to have the Lagrange function for the dislocations in the field of the elastic deformations. Since the dislocation does not have any Lagrange function "of its own," other than that connected with the elastic field, the Lagrange function of interest to us coincides with the function L_{int}, defined by (7.4).

Bearing in mind that in order to derive the equation of motion we shall have to express the Lagrange function in terms of the coordinates of the dislocation, we transform (7.4) by expressing L_{int} directly in terms of the dislocation density $\hat{\alpha}$. This can be easily done, since the dislocations in an unbounded medium satisfy, by virtue of the symmetry of Curl $\hat{\varphi}$, the equation

$$-\int \alpha^0_{ik}\varphi_{ik}\, d\Omega=\int \alpha_{ik}\varphi_{ik}\, d\Omega. \tag{7.5}$$

Let us substitute (7.5) in (7.4), and obtain the Lagrange function for the dislocations in the elastic field, which we shall denote by $L_{\mathscr{D}}$:

$$L_{\mathscr{D}}=\int (\alpha_{ik}\varphi_{ik}+j_{ik}\psi_{ik})\, d\Omega. \tag{7.6}$$

The first term in (7.6) coincides formally with the expression for the energy (taken with the negative sign) of interaction between the dislocation and the external stress field in the static case. We shall show below that in the general case the Hamiltonian of the interaction between the dislocation and the external stress field does indeed have the following form

$$\mathscr{H}^{\mathrm{int}}=-\int \alpha_{ik}\varphi_{ik}\, d\Omega. \tag{7.7}$$

Let us consider a linear dislocation loop $\mathscr{D}$, each element of which moves with a certain velocity $\mathbf{V}$, and let us express $L_{\mathscr{D}}$ in terms of the coordinates and velocities of the loop. To this end, we go over from the volume integrals in (7.6) to contour integrals along the dislocation line. This transition is

*In order to be able to omit these surface integrals, it is sufficient to confine oneself to the case when the conditions $\psi_{ik}=0$ and $\varphi_{ik}=0$ are satisfied on the external surfaces of the body. Simultaneous satisfaction of these conditions means that transitional motion and rotation of the body as a whole are excluded.

realized by making the formal substitution

$$\alpha_{ik}\, d\Omega \to \tau_i b_k\, dl, \quad j_{ik}\, d\Omega \to e_{ilm}\tau_l b_k V_m\, dl, \tag{7.8}$$

where τ—unit vector tangent to the dislocation line, b—Burgers vector, dl—dislocation line length element, and V—its velocity.

Substituting (7.8) in (7.6), we arrive at the following formula for $L_{\mathscr{D}}$:

$$L_{\mathscr{D}} = \oint\limits_{\mathscr{D}} \mathscr{L}_1\, dl,$$

where $\mathscr{L}_1$—"linear density" of the Lagrange function of the dislocation

$$\mathscr{L}_1 = \tau_i \varphi_{ik} b_k + e_{ilm}\tau_l V_m \psi_{ik} b_k. \tag{7.9}$$

It follows from (7.9) that the "linear density" of the Hamiltonian of the interaction between the dislocation and the external stress field is of the form

$$\mathscr{H}_1^{\text{int}} = V_k \frac{\partial \mathscr{L}_1}{\partial V_k} - \mathscr{L}_1 = -\tau_i \varphi_{ik} b_k. \tag{7.10}$$

Equation (7.10) leads in natural fashion to (7.7).

Knowing the Lagrangian density (7.9) or the Hamiltonian density (7.10), we can derive in the usual manner the equation of motion of a dislocation element with coordinates $X_i = X_i(l, t)$, where l—length measured from a certain point along the instantaneous position of the loop (for a fixed instant of time t). We present only the final result of the derivation [22]:

$$e_{ikl}\tau_k \sigma_{lp} b_p + \varrho\, (\mathbf{vb})\, [\boldsymbol{\tau}, \mathbf{V}]_i = 0. \tag{7.11}$$

The tensor $\hat{\sigma}$ and the vector $\mathbf{v}$ contained in (7.11) include both the external elastic fields and the self-fields of the dislocation loop. Thus, (7.11) relates the dislocation self-field, meaning also the dislocation-loop motion that generates this field, to the external fields. Inasmuch as the external fields in (7.11) are taken at the same point of space, $X_i = X_i(l, t)$, at which the dislocation element under consideration is situated at the given instant of time, (7.11) is an implicit equation of motion for the dislocation.

The expression

$$f_i(\hat{\sigma}) = e_{ikl}\tau_k \sigma_{lm} b_m + \varrho\, (\mathbf{vb})\, [\boldsymbol{\tau}, \mathbf{V}]_i, \tag{7.12}$$

which when set equal to zero constitutes the equation of motion (7.11), determines the total force exerted on a unit length of the dislocation by the external field. In the static case, when $\mathbf{v} \equiv 0$, an analogous expression was derived for the force $\mathbf{f}(\hat{\sigma})$ by Peach and Kohler [21].

In order for the equation of motion (7.11) to have the usual explicit form, that is, for it to relate the instantaneous coordinates and velocity of the dislocation with the instantaneous values of the external stress field, it is necessary to express the self-field of the moving dislocation at some instant of time in terms of the coordinates and velocity of the elements of its loop at the same instant of time. We know that this can always be done in the approximation that is linear in the dislocation velocity. The next section will be devoted to such a calculation.

However, before we proceed to derive explicit equations of motion and to introduce the effective dislocation mass, we must call attention to the fact that the equation of motion (7.11) includes only forces of elastic origin. A real dislocation moving in a crystal is also acted upon by certain definite forces of inelastic origin, which are "extraneous" with respect to the elasticity-theory forces. These include, first, the deceleration force due to the discrete nature of the structure of the crystal and the atomic character of the structure of the dislocation nucleus. This force, which describes the resistance of the crystal to the displacement of the dislocation, consists of two parts. There is a static component, analogous to some degree to the "dry friction" force (Peierls force) [1], and a component proportional to the velocity of the dislocation [23,24], equivalent to ordinary friction of motion. The magnitude and direction of the deceleration force depend on the form of the dislocation, and to a considerable degree on the model of the dislocation nucleus. Second, a partial dislocation, which is a contour of some planar stacking fault of the crystal, is acted upon by the surface tension connected with the surface of the defect. This force is always directed in the plane of the defect normal to the dislocation line at the given point.

8. EXPLICIT FORM OF THE EQUATION OF MOTION OF A DISLOCATION IN A MEDIUM. EFFECTIVE MASS OF A DISLOCATION [22,28].

In order for the equation of motion of the dislocation to have the usual form, it is necessary to separate explicitly in (7.11) the force of interaction of the dislocation, and to exclude the singularities of the dislocation self-field, similar to what is done in the derivation of the field mass and the equation of motion of an electron [25].

As already mentioned in the end of the last section, the instantaneous self-action of a dislocation, defined by its coordinates and velocities at the given instant of time, can be introduced only in the approximation that is linear in V/c (V—dislocation velocity). However, since $v < V$ always, the equation of motion in question is meaningful only in the approximation linear in the velocity. Analyzing the order of magnitude of the terms in (7.11), we can easily note that the second term is quadratic in the velocities, and an account of this term is beyond the limits of accuracy of our equation of motion. Therefore the second term in (7.11) should be omitted, and we should write

$$f_i(\hat{\sigma}) \equiv e_{ikl}\tau_k \sigma_{lm} b_m = 0. \tag{8.1}$$

To take the interaction into account, we represent the field $\hat{\sigma}$ in (8.1) in the form $\hat{\sigma} = \hat{\sigma}^e + \hat{\sigma}^s$, where $\hat{\sigma}^e$—external field, and $\hat{\sigma}^s$—self-field of the dislocation stresses. We transform the self-force $f(\hat{\sigma}^s)$ in

(8.1), assuming the dislocation loop to be not infinitesimally thin, but of some small thickness with a Burgers vector which is "smeared" over this thickness*:

$$f_i(\hat{\sigma}^s) = e_{ikl}\tau_k b_m \int \sigma^s_{lm} g(\xi)\, d\xi, \qquad (8.2)$$

where ξ—two-dimensional vector measured from the dislocation axis in a plane perpendicular to τ, and $g(\xi)$—some function that differs from zero in a small vicinity of the dislocation line and has the obvious property

$$\int g(\xi)\, d\xi = 1.$$

We now use for the tensor $\hat{\sigma}^S$ an expansion that follows from Hooke's law and formulas (5.7) and (5.10), and represent the tensor as a sum of two terms: $\hat{\sigma}^S = \hat{\sigma}^S_0 + \hat{\sigma}^S_1$, the first of which ($\hat{\sigma}^S_0$) determines the quasistatic stress field, and σ^S_1 the stresses proportional to the acceleration of the dislocation.

Part of the force (8.2), generated by the term $\hat{\sigma}^S_0$, characterizes the linear self-tension of the static dislocation loop. This force can be easily calculated by substituting in (8.2) the "static" expressions for $\hat{\sigma}^S_0$. Since this operation is perfectly obvious, and since the quasistatic tension force of the dislocation loop has itself no bearing on the question of interest to us, that of the effective mass of the dislocation, we shall not write down an explicit expression for this force. We note, however, that the self-tension force corresponds to a definite self-energy of the dislocation at rest.

$$E_s = \frac{1}{2} \oint\oint_{\mathscr{D}} \mathscr{E}(l, l')\, dl\, dl',$$

where $\mathscr{E}(l, l')$ has the meaning of non-local density of the dislocation self-energy. An expression for $\mathscr{E}(l, l')$ follows directly from the corresponding formulas in [8,13].

The second part of the self-action force (8.2) is determined by the component $\hat{\sigma}^S_1$, which is connected with the flux density by formula (5.10). It is necessary to go over in (5.10), with the aid of (7.8), to integration over the dislocation loop, and then the tensor $\hat{\sigma}^S_1$ will be expressed in terms of the dislocation acceleration.

Let us write down with the aid of (5.11), (5.10), and (7.8) for a dislocation in an isotropic medium, the trace $\sigma^1_{ik} b_k$ which enters in (8.2);

$$\sigma^1_{ik} b_k = \frac{\varrho b^2}{8\pi} \int \{\Phi_i(\mathbf{n}, \boldsymbol{\beta})[\mathbf{n}, \boldsymbol{\tau}]\dot{\mathbf{V}} + n_i(1 - 2\gamma^2 + 2\gamma^4)\cos\theta\, [\boldsymbol{\beta}, \boldsymbol{\tau}]\dot{\mathbf{V}} + (1 + \gamma^4 \sin^2\theta)[\boldsymbol{\tau}, \dot{\mathbf{V}}]_i + 2(1-\gamma^2)^2 \beta_i [\boldsymbol{\beta}, \boldsymbol{\tau}]\dot{\mathbf{V}}\} \frac{g(\xi)\, d\xi\, dl}{R}, \qquad (8.3)$$

where

$$\cos\theta = \mathbf{n}\boldsymbol{\beta},$$
$$\Phi_i(\mathbf{n}, \boldsymbol{\beta}) = \beta_i(1 - 2\gamma^2 + 2\gamma^4)\cos\theta - n_i[3(1-\gamma^4)\cos^2\theta + \gamma^4],$$

and β—unit vector directed along the Burgers vector: $\mathbf{b} = b\boldsymbol{\beta}$.

It follows from (8.3) that only the dislocation velocity component perpendicular to the dislocation line at the given point contributes to the stress. This result is obvious, since the displacement of a dislocation line along itself has no physical meaning.

Since the contraction given above must be calculated for points lying on the dislocation line or in a small vicinity of this line, and should be substituted in (8.2), the integral (8.3) can be broken up into two groups of essentially different components. For a linear dislocation whose line is smooth everywhere, the components in one group, which contain under the integral sign in the numerator either the vector product $\mathbf{n} \times \tau$ or the vector $\mathbf{n}$, remain finite upon substitution in (8.2). (For a transition to a linear dislocation it is necessary to put $g(\xi) = \delta(\xi)$, where $\delta(\xi)$ is the two-dimensional δ function.) The components of the second group have in this case a logarithmic singularity. Since we are interested in the limiting case of a linear dislocation, we should consider only the last type of component. In such an analysis we can separate the main part of the force $f(\overline{\sigma}^S_1)$ only with logarithmic accuracy, which is perfectly sufficient for us. Then, substituting in (8.2) the indicated part of $\sigma^1_{ik} b_k$, we can write for the corresponding part of the self-action force

$$f_i(\hat{\sigma}^s_1) = -\oint_{\mathscr{D}} \mu_{ik}(l, l') W_k(l')\, dl', \quad \mathbf{W}(l) = \dot{\mathbf{V}}(l), \qquad (8.4)$$

where

$$\mu_{ik}(l, l') = \frac{1}{2}\varrho b^2 \{\delta_{ik}(\boldsymbol{\tau}\boldsymbol{\tau}' - \tau'_i \tau_k)(1 + \gamma^4 \sin^2\theta) + [\boldsymbol{\beta}, \boldsymbol{\tau}]_i [\boldsymbol{\beta}, \boldsymbol{\tau}']_k\} G(l, l'),$$

$$G(l, l') = \frac{1}{4\pi} \int\int g(\xi) g(\xi') \frac{d\xi\, d\xi'}{R}, \quad \boldsymbol{\tau} = \boldsymbol{\tau}(l), \quad \boldsymbol{\tau}' = \boldsymbol{\tau}(l'). \qquad (8.5)$$

We note that in (8.5) the vector $\mathbf{n}$ is taken outside the sign of integration with respect to ξ and ξ'. Allowance for $\mathbf{n}$ in such an integration leads to a small correction to (8.4), if the ratio of the "thickness" of the dislocation line to the characteristic linear dimension of the entire loop is small. The latter ratio will be assumed to be very small.

The expression for $\mu_{ik}(l, l')$ in an anisotropic medium was obtained by the author and Natsik in [28].

Substituting (8.4) in (8.1) and taking into account the presence of forces of inelastic origin, which were mentioned at the end of Sec. 7, and which act on the dislocation, we write down the final form of the equation of motion of the element of dislocation loop:

$$\oint_{\mathscr{D}} \mu_{ik}(l, l') W_k(l')\, dl' = f^0_i(l) + e_{ikm}\tau_k(l)\sigma^e_{mp}(l) b_p + S_i(l, \mathbf{V}), \qquad (8.6)$$

*It is obvious that, as in the case of static dislocations [8] the self-action of moving strictly-linear dislocations (that is, dislocations with zero thickness), is described by divergent integrals.

where $\mathbf{W}(l)$—acceleration of the dislocation element, $\mathbf{f}_0$—the already-mentioned quasistatic dislocation self-tension, and $\mathbf{S}$—force of inelastic origin, which, naturally, depends on the dislocation velocity.

We can conclude from (8.6) that $\mu_{ik}(l, l')$ has the meaning of nonlocal effective field mass density of a dislocation line. The physical meaning of the tensor function $\mu_{ik}(l, s)$ is that it establishes the contribution made by the k-th component of the velocity of the element with coordinate s on the dislocation loop to the i-th component of the momentum of an element of unit length with coordinate l on the same loop.

The physical meaning of the quantity $\mu_{ik}(l, l')$ can be described from a somewhat different point of view by writing down the kinetic energy of the dislocation loop

$$E_{\text{kin}} = \frac{1}{2} \oint\oint_{\mathscr{D}} \mu_{ik}(l, l') V_i(l) V_k(l') \, dl \, dl'. \tag{8.7}$$

From (8.7) follows the obvious symmetry of the tensor $\mu_{ik}(l, s)$ which, naturally, is implied in its definition (8.5):

$$\mu_{ik}(l, s) = \mu_{ki}(s, l). \tag{8.8}$$

It is important to note that the inertial term in the equation of motion, that is, the left side of (8.6), plays an important role only in the case of sharply nonstationary motion of the dislocation, when its acceleration is very large. If the acceleration of the dislocation is small, then the major role is played by the deceleration forces (forces of inelastic origin), which include the dissipative forces. It is the magnitude and the dependence on the dislocation velocity of these forces which determine essentially the character of the almost-stationary motion of the dislocation.

If we compare $\mu_{ik}(l, l')$ with the expression for the density of the dislocation self-energy $\mathscr{E}(l, l')$, we can easily verify that $\mu(l, l')$ and $\mathscr{E}(l, l')$ have a similar structure, in the sense of functional dependence on the points on the dislocation loop, and that in order of magnitude

$$\mu(l, l') c^2 \sim \mathscr{E}(l, l').$$

The concrete expressions, however, and even the tensor dimensionalities, are different and therefore the dynamical nonlocal mass density does not reduce in the general case to a nonlocal "rest mass" density. Only for a straight-line screw dislocation do we have

$$\mu_{ik}(l, l') c^2 = \delta_{ik} \mathscr{E}(l, l').$$

We call attention to the fact that the component in the first line of (8.5) determines the mass connected with the mechanical motion of the dislocation in the slip plane, while the component in the second line determines the mass connected with the displacement of the dislocation along the normal to the slip plane.

The second form of motion has in the majority of cases (except for the case of dynamical development of a fissure) low velocity and low accelerations, so that the corresponding inertial terms do not play any role. Bearing this in mind, let us consider the mass of a dislocation moving in the slip plane. We introduce at each point of the dislocation line a right-hand triplet of unit vectors $(\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\kappa})$, $\boldsymbol{\nu}$ being the vector normal to the slip plane and $\boldsymbol{\kappa}$ the vector normal to the dislocation line in the slip plane.

As already noted above, a contribution to the self-action force is made only by the dislocation velocity (or acceleration) component perpendicular to the dislocation line. In the case in question, these components are equal to

$$\mathbf{V}_\perp = \boldsymbol{\varkappa} V_\perp, \quad V_\perp = (\boldsymbol{\varkappa}, \mathbf{V}), \quad \mathbf{W}_\perp = \boldsymbol{\varkappa} W_\perp, \quad W_\perp = (\boldsymbol{\varkappa}, \mathbf{W}).$$

Therefore, multiplying (8.6) by $\boldsymbol{\kappa}$, we obtain

$$\oint_{\mathscr{D}} \mu(l, l') W_\perp(l') \, dl' = f_0(l) + \nu_m(l) \sigma^e_{mp}(l) b_p + S(l, V_\perp), \tag{8.9}$$

where for an isotropic medium we get with logarithmic accuracy

$$\mu(l, l') = \frac{1}{2} \varrho b^2 (\boldsymbol{\tau}\boldsymbol{\tau}')(\boldsymbol{\varkappa}\boldsymbol{\varkappa}')(1 + \gamma^4 \sin^2\theta) G(l, l'). \tag{8.10}$$

An expression for $\mu(l, l')$ in lieu of (8.10) in an anisotropic medium was obtained in [28].

Equation (8.9) describes the mechanical motion of a dislocation loop in an external stress field.

Along with the equation of motion of the dislocation element (8.6), we can consider the averaged equation of motion of the entire dislocation loop, by integrating (8.6) over the loop

$$\oint_{\mathscr{D}} m_{ik}(l) W_k(l) \, dl = F^e_i, \quad F^e_i = e_{ikm} b_p \oint_{\mathscr{D}} \tau_k \sigma^e_{mp} \, dl + \oint_{\mathscr{D}} S_i \, dl, \tag{8.11}$$

where $\mathbf{F}^e$—total external force acting on the dislocation

$$m_{ik}(l) = \oint_{\mathscr{D}} \mu_{ik}(l', l) \, dl'. \tag{8.12}$$

The first term in the right side of (8.6) drops out after integration over the entire loop, since the total self-action force of a dislocation at rest is equal to zero.

It follows from (8.11) that when the motion of the entire dislocation loop is considered $m_{ik}(l)$ plays the role of an effective mass per unit length of dislocation. It is obvious, however, that the effective mass per unit length of dislocation, introduced in this manner, is not a local property of the point in question on the dislocation loop. It depends on the dimensions and on the shape of the entire loop.

Using expression (8.12) and the definition of the tensor μ_{ik} (8.5), let us estimate the order of magnitude of m_{ik}. For such an estimate we shall assume that the function $g(\xi)$ is equal to a constant value inside a tube of small radius r_0, described around the dislocation line, and equal to zero outside this tube. We can then readily obtain the estimate

$$m \sim \frac{\varrho b^2}{4\pi} \ln \frac{r_m}{r_0}, \quad r_m \gg r_0, \tag{8.13}$$

where r_m—characteristic radius of curvature of the dislocation line at the point under consideration (in the case of a straight dislocation r_m is its length). According to the existing atomic dislocation models, r_0 should be of the order of the magnitude of the displacement that gives rise to the dislocation, that is, $r_0 \sim b$. This estimate is also natural from the point of view of elasticity theory, since it places a lower-limit on the distance over which the dislocation theory considered here is valid.

The estimate obtained for m_{ik} justifies our assumption of a pure field mass for the dislocation. The point is that when a real dislocation moves in a crystal, it sets in motion also some of the atoms in the vicinity of the dislocation axis, at distances on the order of r_0 from the axis. This produces an additional dislocation inertia, connected with the ordinary mass of these atoms. The order of magnitude of the mass of the atoms inside a tube of radius $r_0 \sim b$ can be estimated at $\rho r_0^2 \sim \rho b^2$ per unit length of the dislocation. Comparing this estimate with (8.13) we see that when $r_m \gg r_0$, an account of the masses of the moving atoms near the dislocation line does not change noticeably the dislocation inertia, and the dislocation mass can be actually regarded with logarithmic accuracy as a field mass.

Finally, let us consider the motion of a dislocation loop as a unit,

$$M_{ik} W_k^0 = F_i^e, \tag{8.14}$$

by introducing the average acceleration $\mathbf{W}_0$ of the dislocation loop and the tensor of its total mass M_{ik}, defined by the formula

$$M_{ik} W_k^0 = \oint_{\mathscr{D}} m_{ik}(l)\, W_k(l)\, dl, \quad M_{ik} = \oint_{\mathscr{D}} m_{ik}(l)\, dl. \tag{8.15}$$

For a straight-line dislocation, when the vectors τ and $\mathbf{b}$, and also $\sin^2\theta$, are constant along the dislocation line, the total mass tensor M_{ik} is described by an expression analogous to the expression for the self-inductance coefficient of a straight-line conductor. Using the well-known expression for the self-induction coefficient, we obtain for the total mass of a straight-line dislocation

$$M_{ik} = \frac{\varrho b^2 r_m}{4\pi} \{(\delta_{ik} - \tau_i \tau_k)(1 + \gamma^4 \sin^2\theta) + [\boldsymbol{\beta}, \boldsymbol{\tau}]_i [\boldsymbol{\beta}, \boldsymbol{\tau}]_k 2(1-\gamma^2)^2\} \ln \frac{r_m}{r_0}, \tag{8.16}$$

where r_m denotes the length of the dislocation ($r_m \gg r_0$). In the case of a screw dislocation, relation (8.16) leads to the expression previously derived for the screw-dislocation mass by Frank [26] and Eshelby [27].

Using the average acceleration of the entire dislocation, obtained from (8.14), let us describe the relative motion of the individual elements of the dislocation line. To set up the appropriate equation of motion, we represent the acceleration of the element at the point l on the dislocation in the form

$$\mathbf{W}(l) = \mathbf{W}_0 + \mathbf{w}(l), \tag{8.17}$$

where $\mathbf{w}(l)$ is the acceleration of the dislocation element relative to the "center of mass" of the dislocation loop. We then get from (8.6) and (8.14)

$$\oint_{\mathscr{D}} \mu_{ik}(l, l')\, w_k(l')\, dl' = f_i^0(l) - m_{ik}(l)\, M_{km}^{-1} F_m^e + e_{ikn} b_p \tau_k(l)\, \sigma_{mp}^e(l) + S_i(l, \mathbf{V}), \tag{8.18}$$

where M_{ik}^{-1}—is the inverse of the total dislocation mass tensor.

The equations of motion (8.14) and (8.18) enable us to separate the study of the motion of the dislocation loop as a whole from the study of the relative motion of its elements.

9. EQUATIONS OF MOTION OF CONTINUOUSLY DISTRIBUTED DISLOCATIONS

Having derived the equation of motion of the individual dislocation, we have, in principle, a complete system of equations defining the evolution of a set of dislocation loops and the elastic field in the solid. However, the dislocation equations of motion in this system are equations of motion of discrete formations. Yet the entire theory of the dynamic elastic field produced by the dislocations, and also the Lagrange function for the dislocations and the field, have been formulated in terms of a continuous dislocation distribution. A natural way of recasting this system of equations in a unified form is to average the equations of motion of a large number of individual dislocations and to transform them into equations of motion of continuously distributed dislocations. In any attempt to go over to the equations of motion of continuously distributed dislocations, we unavoidably return to the question raised in Sec. 3, that of the method used to average the spatial distribution of the dislocations. Noting the importance of this question, let us analyze the averaging of the equations of motion only in the two simplest cases singled out in Sec. 3, and let us verify that the transition to the equations of motion of uniformly distributed dislocation loops leads in different cases to different physical results. In particular, we shall show that the effective mass density cannot have the same form under different averaging conditions.

We confine ourselves to examination of a purely elastic interaction of the dislocations with one another and with the external fields, without discussing the question of production, multiplication, and mutual transformation of the dislocations. We take into account here, of course, the fact that in a real case the evolution of a system of dislocations should be described by some analog of a kinetic equation which takes into account all these processes and their like. We are interested primarily, however, in the introduction of the effective-mass density of continuously distributed dislocations that execute purely mechanical motions and do not participate in the aforementioned processes.

We begin with a dislocation distribution defined by the conditions (3.13). We assume in addition that the characteristic linear dimension λ, which determines the spatial variation of the external field (for example the wavelength of the external field), satisfies the conditions

$$h \ll \lambda \ll r_m. \qquad (9.1)$$

Then, in going to the macroscopic equations, the linear dimensions l_0 of the volume over which the averaging takes place must be chosen in accordance with the condition

$$h \ll l_0 \ll \lambda \ll r_m. \qquad (9.2)$$

In this case, as already noted in Sec. 3, the distributions of the dislocations and their fluxes are described by a scalar density $\rho^{\beta}(\tau; \mathbf{r})$, connected with the tensors $\hat{\boldsymbol{\alpha}}$ and $\hat{\mathrm{j}}$ by formulas (3.14) and (3.15).

Assume that the dislocations execute mechanical motion in their slip planes. We separate at some point $\mathbf{r}$ an element of the elastic medium with volume Ω_0, in the form of a cylinder of length l_0 and with an axis parallel to some vector $\boldsymbol{\tau}$, and we consider dislocations with specified identical vectors $\mathbf{b}$, which have inside the volume Ω_0 specified parallel vectors $\boldsymbol{\tau}$.

Then the averaging of any arbitrary local dislocation characteristic over the cross section of the cylinder Ω_0 reduces to multiplication of its average value by the scalar density $\rho(\mathbf{r})$. Averaging of a local characteristic over the height of the cylinder Ω_0 chosen above is equivalent to multiplication by unity. Therefore, the averaging of the external forces acting on the dislocation and entering in the right side of (8.6) entails no difficulty:

$$\overline{f_i^e} = e_{ilm}\tau_l b_k \sigma^e_{mk}(r)\,\varrho(\boldsymbol{\tau};\ \mathbf{r}), \qquad (9.3)$$

$$\overline{S_i} = \varkappa_i S_0(\boldsymbol{\tau};\ \mathbf{V})\ \varrho(\boldsymbol{\tau};\ \mathbf{r}), \qquad (9.4)$$

where $\hat{\sigma}^{e}$ is the field external relative to all dislocations, and κ, as before, is a unit vector normal to the dislocation line and lying in the slip plane. The magnitude of the force S_0 and its dependence on the directions of the vectors τ and $\mathbf{b}$ should be concretely specified in each separate case.

As regards the averaging of the force of the static self-action, we carry it out with account of the fact that after averaging this force should describe the quasistatic interaction with all the continuously distributed dislocations:

$$\overline{f_i^0} = e_{ilm}\tau_l b_k \sigma^0_{mk}(r)\,\varrho(\boldsymbol{\tau};\ \mathbf{r}), \qquad (9.5)$$

where the tensor $\hat{\sigma}^0(\mathbf{r})$ is connected with the continuous dislocation density $\hat{\boldsymbol{\alpha}}(\mathbf{r})$, at the same instant of time, by the general formula

$$\sigma^0_{ik} = \frac{\mu}{4\pi}\int \gamma_{iklm}(\mathbf{n})\,\alpha_{lm}\frac{d\Omega}{R}. \qquad (9.6)$$

The tensor function $\gamma_{iklm}(\mathbf{n})$ of the unit vector $\mathbf{n}$, a function that depends on the elastic moduli of the medium, contains a factor of the order of unity in each of the components n_i. In the isotropic case the tensor $\gamma_{iklm}(\mathbf{n})$ can be constructed on the basis of the well-known expressions [21,22]:

$$\gamma_{iklm}(\mathbf{n}) = 2\{3n_i n_k - \gamma^2(\delta_{ik} + 3n_i n_k)\,n_p e_{plm}\} + (2\gamma^2 - 1)(n_i e_{klm} + n_k e_{ilm}) + e_{ipl}\delta_{km}n_p + \delta_{im}e_{kpl}n_p. \qquad (9.7)$$

In the two-dimensional case, when all the dislocations are linear and of infinite length along one axis, the integral (9.6) must be taken in the sense of principal value.

Finally, we proceed to average the left side of (8.6), that is, the term

$$\oint_{\mathscr{D}} \mu_{ik}(l,\ l')\,W_k(l')\,dl', \qquad (9.8)$$

in which the point l of the dislocation loop is located in the volume element Ω_0.

We write down the averaging of (9.8) in the form

$$\frac{1}{\Omega_0}\int_{l_0} dl \sum \oint_{\mathscr{D}} \mu_{ik}(l,\ l')\,W_k(l')\,dl', \qquad (9.9)$$

where the summation is carried out over all the dislocations having identical vectors $\mathbf{b}$, possessing inside the volume Ω_0 a tangent vector $\boldsymbol{\tau}$, and crossing the base of the cylinder Ω_0.

We break down the contour integral in (9.9) into two parts:

$$\frac{1}{l_0}\int_{l_0} dl\,\varrho(r)\int_{l_0} \mu_{ik}(l,\ l')\,\overline{W_k(l')}\,dl' + \frac{1}{\Omega_0}\int_{l_0} dl \sum \int_{\mathscr{D}-l_0} dl'\,\mu_{ik}(l,\ l')\,W_k(l'), \qquad (9.10)$$

in the first of which all the quantities pertain to a dislocation element with specified vectors $\boldsymbol{\tau}$ and $\mathbf{b}$, and therefore their averaging is elementary. $\overline{W}_k$ is the average acceleration of the corresponding dislocation elements. In the second term the integration $\int dl' \ldots$ is carried out over the entire length of the dislocation, after subtracting the straight-line section which crosses the volume Ω_0.

Let us transform the second term in (9.10), recalling the relation (8.4), with which we introduce formally the definition of the dislocation effective mass:

$$\oint_{\mathscr{D}} \mu_{ik}(l,\ l')\,W_k(l')\,dl' \equiv -f_i^1(r) = -e_{ikl}\tau_k b_m \sigma^1_{lm}, \qquad (9.11)$$

where $\hat{\sigma}_1$—dynamic part of the stress field produced by the dislocation motion.

The sum in (9.10) takes into account the fields of all the dislocations of a given type passing through the volume element Ω_0. However, since we are now considering some arbitrary aggregate of continuously distributed dislocations, we should be interested in the interaction between the elements of the loops inside Ω_0 and the remaining dislocation lines. Therefore we must write for $\hat{\sigma}_1$, which determines the second term in (9.10), an expression of the type (5.10), in which the integral extends over all space with the exception of the volume Ω_0. However, since an integral such as (5.10) has no singularities whatever, and when calculated over an infinitesimally small volume it vanishes with this volume, we can use for the tensor $\hat{\sigma}_1$ of interest to us formula (5.10) as it stands, with the integration carried out over the entire volume.

However, the replacement of the second term in (9.10) by the quantity $-\mathbf{f}_1(\mathbf{r})$ defined by (9.11) above does not yield with the same accuracy the contribution made to the dynamic self-action force by the near and far sections of the discrete dislocation. The contribution of the far parts of the dislocation line itself ($|\mathbf{r} - \mathbf{r}'| \gg l_0$) does not differ at all from the contribution from other dislocation-loop elements located the same distance away from the volume Ω_0. Therefore, the motion of the far sections of the dislocation loop, and their influence on the motion of the dislocation in the volume element Ω_0, are given just as rigorously and correctly by expression (9.11) for the other ("foreign") dislocation lines.

We must remember, however, that the dynamic interaction between different parts of a dislocation decreases slowly with the distance between them. Consequently, the sections of individual dislocations of the type considered here, adjacent to the volume Ω_0, make an appreciable contribution to the effective mass per unit length of the dislocation element in the volume Ω_0. Their influence on the effective mass is typically discrete, and the individuality of the dislocation elements that are closest to the volume Ω_0 is not "dissolved" completely in the continuous distribution of the dislocations, as in the case of large distances.

Therefore, relation (9.11), which describes with sufficient accuracy the smooth character of the average dynamic field of the dislocation-loop sections closest to the volume Ω_0, does not take into account their individual contribution to the mass of "their own" dislocation lines. Consequently, when we replace the second term of (9.10) by $-\mathbf{f}_1(\mathbf{r})$, we must simultaneously "correct" the dislocation mass per unit length in the first term of (9.10). This correction consists in having the mass per unit dislocation length in the volume Ω_0 made up of dislocation sections that are identical in form for all the dislocations of the type considered. Such a dislocation section is the one which moves like a straight-line dislocation moving in translation. The length of this dislocation section is of the order of λ. Indeed, λ is the distance over which, in accordance with condition (9.1), the dislocation line can be regarded as a straight line. On the other hand, λ is the largest distance over which the acceleration of the dislocation line can be approximately regarded as constant.

The contribution of the more remote elements of different dislocation loops to the mass per unit length is in no way correlated, so that it drops out after averaging. This signifies that the role of the remote sections of the dislocations is correctly accounted for by (9.11).

Bearing all the foregoing in mind, we can write in lieu of (9.10)

$$\varrho(\mathbf{r})\, m_{ik}(\lambda)\, W_k(\mathbf{r}) - e_{ikl}\tau_k b_m \varrho(\mathbf{r})\, \sigma^1_{lm}(\mathbf{r}), \qquad (9.12)$$

where $\mathbf{W}(\mathbf{r})$—average acceleration of the dislocation element of the given type at the point $\mathbf{r}$, and $m_{ik}(\lambda)$—effective mass per unit length of the straight-line dislocation length λ. In an isotropic medium, the mass of $m_{ik}(\lambda)$ a gliding dislocation is given by a formula derivable from (8.16):

$$m_{ik}(\lambda) = \frac{\varrho b^2}{4\pi}(\delta_{ik} - \tau_i\tau_k)(1 + \gamma^4 \sin^2\theta)\ln\frac{\lambda}{r_0}. \qquad (9.13)$$

Only the order of magnitude of λ is determined, so that one might think that the relations containing this quantity yield only rough qualitative estimates. However, relation (9.13) shows that the dependence on λ is very weak (logarithmic), so that although the foregoing analysis is qualitative to a considerable degree, we can hope that the formal result is sufficiently accurate.

Thus, the averaged equation of motion of continuously distributed dislocations, for dislocations of each type (for dislocations which have at the point $\mathbf{r}$ definite vectors τ and $\mathbf{b}$) can be written in the form

$$m_{ik}(\lambda)\, W_k = e_{ilm}\tau_l b_k \{\sigma^e_{mk} + \sigma^0_{mk} + \sigma^1_{mk}\} - \varkappa_i S_0(\tau, \mathbf{V}), \qquad (9.14)$$

where $\hat{\sigma}_0$ and $\hat{\sigma}_1$ are connected by formulas (9.6) and (5.10) with the dislocation density and dislocation flux tensors, the latter being expressed in terms of the scalar density $\rho^\beta(\tau; \mathbf{r})$ and the average dislocation velocities by relations (3.14) and (3.15). In addition, the tensors $\hat{\alpha}$ and $\hat{\jmath}$ are connected by the continuity equations of the dislocation flux (3.7).

The equation of motion of continuously distributed dislocations acquires a form different from (9.14) in the other limiting case, when the dimensions of the dislocation loops and the distances between them are small compared with the dimension λ:

$$\lambda \gg r_m,\ h. \qquad (9.15)$$

In this case the linear dimensions l_0 of the volume over which the averaging must be carried out should be smaller than λ, and must satisfy at the same time the requirement (3.16); that is to say, the following inequalities must hold:

$$r_m,\ h \ll l_0 \ll \lambda. \tag{9.16}$$

We shall make some definite supplementary model assumptions to simplify the analysis of the equations. First, we assume that the dislocations are sufficiently far from one another, so that we can disregard phenomena connected with their intersections (collisions).

Second, we assume that all the dislocation loops are plane and lie in their slip planes. Then the possible orientations of the loops are determined by the set of directions of the Burgers vector **b** and the directions of the vector normal to the slip plane $\boldsymbol{\nu}$. These directions will be labeled by the indices γ and β, respectively.

Third, the elastic properties of the medium will be assumed to be such that the free dislocation loop in a homogeneous external field has a circular form (in the case of an isotropic medium these properties reduce to the vanishing of the Poisson coefficient). Then, as the dislocation moves in a homogeneous field, its form will differ little from circular. Indeed, if the sections of the dislocation with smaller masses begin to move more rapidly than the neighboring sections, then the curvature of the corresponding section of the line will change and an additional force of a quasistatic linear attraction will arise, tending to impart a circular form to the dislocation. Therefore no considerable change can occur in the shape of the dislocation loop as the latter moves. Inasmuch as we are interested primarily in a clarification of the qualitative aspect of the difference between the second limiting case and the first, we shall carry out the analysis by assuming for simplicity that all the dislocations are circular. If we use this simplification, then for specified ν and **b** the loops are characterized by still another parameter—their radius, which we shall denote here by r_m.

Under such simple conditions, the continuous distribution of the dislocations can be described by a density $\rho^{\gamma\beta}(r_m; \mathbf{r})$, which indicates the average number of dislocations of the type (γ, β) and radius r_m per unit volume with coordinate r. The plastic polarization tensor $\hat{\mathrm{P}}(\mathrm{r})$ is expressed in terms of the indicated density in the following manner:

$$P_{ik} = \pi \sum_{\gamma, \beta} \nu_i^\gamma b_k^\beta \int \varrho^{\gamma\beta}(r_m)\, r_m^2\, dr_m, \tag{9.17}$$

where the integration is over all possible loop radii.

By virtue of conditions (9.15), the stress fields are almost homogeneous over the extent of a single loop, and therefore the total elastic force F, acting on the dislocation and entering in (8.11), is determined by the elastic-stress gradient

$$F_i = s\nabla_i \sigma_{mk} \nu_m b_k, \tag{9.18}$$

where s is the area bounded by the dislocation loop: $s = \pi r_m^2$.

Bearing (9.18) and an expression of the type (9.4) in mind, we write down the equation for the motion of the loop as a unit

$$M_{ik} W_k^0 = s\nabla_i \sigma_{mk} \nu_m b_k - \oint_{\mathscr{D}} \varkappa_i S_0(\mathrm{V}_0)\, dl, \tag{9.19}$$

where $\mathrm{W}_0(\mathrm{r})$ and $\mathrm{V}_0(\mathrm{r})$ are respectively the average acceleration and the average velocity of the loop situated at the point r, while M_{ik} is the total mass of the loop. In an isotropic medium the mass M_{ik} is given by a formula which follows from (8.5) and (8.15):

$$M_{ik} = \frac{1}{2}\varrho b^2 \oiint_{\mathscr{D}} (1 + \gamma^4 \sin^2\theta)\,[\delta_{ik}(\boldsymbol{\tau\tau}') - \tau_i \tau_k']\, G(l, l')\, dl\, dl'. \tag{9.20}$$

The tensor σ_{ik} in (9.19) describes all the fields that are external with respect to the dislocation in question, that is, both the field of the external forces $\hat{\sigma}^e$ and the field produced by all the dislocations except the one under consideration. The moving dislocation loops generate a field which we shall represent, as before, by the sum of the quasistatic field $\hat{\sigma}_0$ and the purely dynamic field $\hat{\sigma}_1$:

$$\hat{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}}^e + \hat{\boldsymbol{\sigma}}_0 + \hat{\boldsymbol{\sigma}}_1. \tag{9.21}$$

The tensor $\hat{\sigma}_0$ is expressed by formula (5.9) in terms of the spatial derivatives of the plastic polarization tensor $\hat{\mathbf{P}}$, while the tensor $\hat{\sigma}_1$ can be expressed in terms of the time derivative of $\hat{\mathbf{P}}$, if we substitute (4.5) in formula (5.10).

Consequently, as in the first case, the right side of the equation of motion (9.19) is determined by the distribution of the dislocations, and by their average velocities and accelerations.

If we now represent the acceleration of the loop element in the form of the sum (8.17), then the acceleration w relative to the center of mass will have, in accordance with the assumed model, only a radial component: $\mathrm{w} = \kappa \mathrm{w}$, where $\mathrm{w} = \dot{\mathrm{v}}_m$, $\mathrm{v}_m = \dot{\mathrm{r}}_m$.

In describing the loop motion connected only with the change in the loop dimensions, we can neglect completely the inhomogeneity of the elastic fields and write (8.9) in the form

$$m^{\gamma\beta}(r_m)\, w^{\gamma\beta}(r_m) = -f_0^{\gamma\beta}(r_m) + \nu_i^\gamma b_k^\beta \sigma_{ik} - S_0^{\gamma\beta}(v_m), \tag{9.22}$$

where the elastic field is determined by formula (9.21) and is taken at the center of the loop.

In (9.22), $m(r_m)$ is the average mass of a unit length of a circular dislocation with radius r_m. In an isotropic medium, the average mass can be determined from a formula obtained by averaging (8.10):

$$m(r_m) = \frac{\varrho b^2}{4\pi r_m} \oiint_{\mathscr{D}} (1 + \gamma^4 \sin^2\theta)(\boldsymbol{\tau\tau}')\, G(l, l')\, dl\, dl'. \tag{9.23}$$

The quantity $f_0(r_m)$ defines the dislocation self-tension, which is constant along the dislocation length and which leads to a force that tends to reduce the loop dimensions (the sign of the corresponding term in (9.22) takes account of the direction of this force).

The force $f_0(r_m)$ can be obtained by calculating the static self-energy $E_s(r_m)$ of a circular loop of radius r_m and using the obvious formula

$$f_0(r_m) = \frac{1}{2\pi r_m} \frac{\partial E_s}{\partial r_m} . \qquad (9.24)$$

With logarithmic accuracy, formula (9.24) yields for an isotropic medium (if we set the Poisson coefficient equal to zero)

$$f_0(r_m) = \frac{\mu b^2}{4\pi r_m} \ln \frac{r_m}{r_0} . \qquad (9.25)$$

The velocities $V_0(r)$, $v(r_m)$, and the function $\rho(r_m, r)$ are connected by the continuity equation in usual space and in a space having dimensions:

$$\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial \mathbf{r}} (\varrho \mathbf{V}_0) + \frac{\partial}{\partial r_m} (\varrho v_m) = 0. \qquad (9.26)$$

Thus, in the limiting case under consideration, the equations of motion of continuously distributed dislocations are actually equations of motion of a unique "gas of dislocation loops," interacting via their elastic fields, experiencing no collisions, having dimensions that change in time.

The extreme limiting cases analyzed above confirm the statement made at the beginning of the section, that, depending on the formulation of the problem and on the dislocation properties of the medium, there exist different methods of averaging, and the effective masses entering in the averaged equations have different physical meanings.

[1] A. H. Cottrell, Dislocations and Plastic Flow in Crystals, Clarendon Press, Oxford, 1953.

[2] G. Schoeck, Phys. Rev. 102, 1458 (1956).

[3] Dislokatsiya i mekhanicheskie svoĭstva kristallov (Dislocation and Mechanical Properties of Crystals) (Translation Collection), M., IL, 1960.

[4] V. L. Indenbom, Some Problems in the Physics of Plasticity, Itogi nauki (Science Summaries), no. 3, 1960, p. 117.

[5] H. G. Van Bueren, Imperfections in Crystals, Amsterdam, 1960.

[6] V. L. Indenbom and A. N. Orlov, UFN **76**, 557 (1962), Soviet Phys. Uspekhi **5**, 272 (1962).

[7] Atomnyĭ mekhanizm razrusheniya (Atomic Mechanism of Failure). Report of a Conference, M., Metallurgizdat, 1963.

[8] E. Kröner, Kontinuumstheorie der Versetzungen und Eigenspannungen, Springer, Berlin, 1958.

[9] F. R. Nabarro, Advances Phys. **1**, 269 (1952).

[10] A. Seeger, Handb. Phys., Bd. 7(1), 1955, p. 383.

[11] J. D. Eshelby, Solid State Phys. **3**, 79 (1956).

[12] B. A. Bilby, Progr. Solid. Mech. **1**, 329 (1960).

[13] R. de Wit, Solid State Phys. **10**, 249 (1960).

[14] A. M. Kosevich, JETP **42**, 152 (1962), Soviet Phys. JETP **15**, 108 (1962).

[15] E. F. Hollander, Czech. Phys. J. **B10**, 409, 479, 551 (1960).

[16] E. Kröner and G. Rider, Zs. Phys. **145**, 424 (1956).

[17] A. M. Kosevich, and V. D. Natsik, FTT **6**, 228 (1964), Soviet Phys. Solid State **6**, 181 (1964).

[18] F. Kroupa, Czechosl. Phys. J. **B12**, 191 (1962).

[19] T. Mura, Phil. Mag. 8, 843, 1963; Int. J. Engng. 1, 371 (1963).

[20] A. E. H. Love, A Treatise on the Mathematical Theory of Elasticity, Dover, 1945.

[21] W. Peach and J. S. Köhler, Phys. Rev. **80**, 436 (1950).

[22] A. M. Kosevich, JETP **43**, 637 (1962), Soviet Phys. JETP **16**, 455 (1963).

[23] J. D. Eshelby, Proc. Roy. Soc. **A197**, 396 (1949).

[24] F. R. Nabarro, Proc. Roy. Soc. **A209**, 279 (1951).

[25] D. D. Ivanenko and A. A. Sokolov, Klassicheskaya teoriya polya (Classical Field Theory), Gostekhizdat 1949, p. 188.

[26] F. G. Frank, Proc. Phys. Soc. **A62**, 131 (1949).

[27] J. D. Eshelby, Phys. Rev. **90**, 248 (1953).

[28] A. M. Kosevich and V. D. Natsik, FTT **7**, 33 (1965), Soviet Phys. Solid State **7**, (in press).

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