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## *RELATIVISTIC ASTROPHYSICS. I.**


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CONTENTS



## 1. INTRODUCTION

THE main force governing the motion of celestial bodies is that of universal gravitation. This force, together with the pressure forces, determines the structure of individual cosmic bodies.

The law of universal gravitation has been known since Newton's time, and although it has been clear for more than half a century that this law is valid only for weak gravitational fields, with a gravitational potential $\varphi \ll c^2$, and for slow motions with $v \ll c$, astrophysicists have never resorted in their research to Einstein's gravitational theory, for the following reason: Except in cosmology (the study of the universe as a whole), which we do not discuss in this article, astronomers did not know until most recently of any formation with a gravitational potential close to $c^2$.

Interest in the relativity theory of gravitation arose relatively recently in connection with the discovery of cosmic objects of a new type. During the last decade, astrophysicists have diligently tried to find the source of the tremendous energy stored in cosmic rays and magnetic fields of some powerful radio galaxies. This energy, on the order of $10^{60}$ erg, is comparable with the energy of gravitational interaction of stars of giant galaxies. Hoyle and Fowler[1] have suggested that such an energy can be released upon compression of a quasi stellar object of the order of $10^8\ M_\odot$, situated in the center of the galaxy.* Almost at the same time, new objects were discovered, having apparently masses of this order of magnitude, dimensions of about

*The second part of the article is in preparation and will be published in one of the early 1965 issues of UFN.

*$M_\odot$ – mass of the sun, $\approx 2 \times 10^{33}$ g.

$10^6$ cm (light week), and emitting about $10^{46}$ erg/sec in the optical region, which is two orders of magnitude larger than the luminosity of a large galaxy having dimensions a million times larger and containing $10^{11}$ stars.*

A theoretical analysis of the evolution of such a large mass shows that it should be compressed to dimensions such that the potential on its surface is $\varphi \approx c^2$, so that Einstein's gravitation theory becomes essential to the entire problem.

All these results have rekindled interest in the papers of Oppenheimer and his co-workers[2,3], written in the thirties, which were almost forgotten by the astronomers. These and later papers considered models of stars during the last stages of their active life, when the sources of internal energy have already been exhausted and the star has cooled down. It was shown that a sufficiently massive cold non-rotating star will contract unobstructed even if its mass is a modest 2 $M_\odot$, and at a mass of the order of 1.5 $M_\odot$ it will reach the state of a so-called neutron star with a radius on the order of 10 km and with $\varphi \approx 0.2c^2$.

The evolution of non-rotating stellar systems should also lead to their gradual condensation, to an increase in their velocity to $\approx 0.5c$, and then to a relativistic compression.

Thus, it is becoming more and more evident that the massive cosmic bodies and systems of bodies† unavoidably reach a relativistic stage. The processes occurring during such a stage of evolution constitute the subject of relativistic astrophysics.

We begin with a review of the general information on the equilibrium of a star and its evolution, and explain the conditions that bring about the relativistic phase of the star; we then dwell on the properties of a spherically symmetrical gravitational field in vacuum (the Schwartzschild field), after which we turn to an analysis of the objects for which the Einstein gravitational-theory effects play an essential role.

Some questions concerning star stability and individual questions connected with the exposition that follows have been touched upon in a recent review by Chiu[4].

Good expositions of the classical theory of star structure of the nonrelativistic stages of their evolution are contained in[13,25,26,27,41].

The main deductions of Oppenheimer and Volkov[2] and Oppenheimer and Snyder[3] are detailed in the textbooks of L. D. Landau and E. M. Lifshitz[38,48]. Problems in the theory of superdense stellar configurations are considered in[10,8,15,16,70]. Some questions of relativistic astrophysics and the theory of quasars are contained in[32,42,48].

*A review of the experimental data on quasars can be seen in the article by J. Greenstein (UFN 83, 549, 1964 [Scientific American 209 (6), 54 (1963)]).

†At least, weakly rotating bodies. Problems connected with rotation are far from clear, and will be discussed in part II of this review.

## 2. EQUILIBRIUM AND STABILITY OF A STAR AS A WHOLE

A star in its usual state is a gas sphere in hydrodynamic and thermal equilibrium. The hydrodynamic equilibrium is ensured by equality of the gravitational and pressure forces acting on each mass element of the star.

The characteristic time of the hydrodynamic processes in the star is much shorter than the time of the thermal processes and the processes of nuclear fuel conversion. In fact, say for the sun, the characteristic time of the thermal process is determined by the condition

$$t_{T\odot} \approx \frac{E_{T\odot}}{L_\odot} \approx \frac{3kT_\odot}{2m}\frac{M_\odot}{L_\odot} \approx 3\cdot 10^7 \text{ years},$$

where $E_{T\odot}$—thermal energy of the sun, $T_\odot$—temperature of the interior of the sun ($10^7$ deg K), m—molecular weight, $L_\odot$ and $M_\odot$—luminosity and mass of the sun ($L_\odot = 4\times 10^{33}$ erg/sec, $M_\odot = 2\times 10^{33}$ g). The time of nuclear fuel conversion is

$$t_{N\odot} \approx \frac{E_{N\odot}}{L_\odot} \approx \frac{0{,}01c^2 M_\odot}{L_\odot} \approx 10^{11} \text{ years}.$$

Here $E_{N\odot}$—nuclear energy stored in the solar matter and $0.01c^2$—maximum energy of the nuclear reactions per unit mass.

Let us estimate, on the other hand, the time of the hydrodynamic processes. Assume that the force of gravitation on the surface of the star is not fully balanced by the pressure force. Then the acceleration acquired by the matter under the influence of this unbalance force is

$$\frac{d^2r}{dt^2} = a\frac{GM}{R^2}, \tag{2.1}$$

where a—fraction of the unbalance force and R—radius of the star. Let us estimate the time $t_H$ during which the surface is displaced a fraction b of the radius:

$$\frac{d^2r}{dt^2} \approx \frac{\Delta r}{t_H^2} = \frac{bR}{t_H^2},$$

and obtain by comparing with (2.1)

$$t_H = \left(\frac{a}{b}\frac{GM}{R^3}\right)^{-1/2} \approx \left(\frac{GM}{R^3}\right)^{-1/2}, \tag{2.2}$$

if we assume that $b \approx a$ is of the order of unity.

A second approach to the determination $t_H$ is to estimate the time necessary for the sound to cover a distance on the order of the radius of the sun. The condition of equilibrium of the star yields in this approach the same formula (2.2) for $t_H$. In fact, the speed of sound is $v_S = (\partial P/\partial \rho)^{1/2}$. Using for the estimate the averaged equilibrium equation (2.3a) below, we find that

$$v_s = \left(\frac{\partial P}{\partial \varrho}\right)^{1/2} \approx \left(\frac{P}{\varrho}\right)^{1/2} = \left(\frac{GM}{R}\right)^{1/2},$$

from which follows (2.2). For the sun $t_{H\odot} \approx 10^3$ sec, and we see that $t_{H\odot} \ll t_{T\odot}$ and $t_{H\odot} \ll t_{N\odot}$.

Thus, stable hydrostatic equilibrium is necessary for a star to exist in a stationary state. The equilibrium condition is written in the form

$$-\frac{dP}{dr} = \frac{GM(r)\varrho}{r^2}. \tag{2.3}$$

On the left side is the pressure acting on a unit volume, and on the right the force with which the unit volume is attracted to the mass M(r) contained in a sphere of radius r. To analyze the stability and the equilibrium we characterize the total matter in the star by an average density $\bar{\rho}$ and an average pressure $\bar{P}$, and estimate their orders of magnitude (we shall henceforth omit the averaging symbols for brevity). This method is crude, but yields an intuitive insight in the physical nature of the problem. The exact theory of a star model in stable equilibrium confirms the crude estimates[5]. Using the averaged characteristics, we can write for the entire star

$$\frac{P}{R} = \varrho\frac{GM}{R^2}. \tag{2.3a}$$

Since $\rho = M/(4\pi R^3/3)$, we have

$$P = \left(\frac{4}{3}\pi\right)^{1/3} GM^{2/3}\varrho^{4/3}. \tag{2.4}$$

The right side of (2.4) can logically be called the average gravitational pressure, which we denote by q. The condition for the equilibrium of the star is P = q. Let us examine the stability of this equilibrium. To this end we consider the contraction and dilatation of the star as a whole, when both P and $\rho$ vary. The function $P = P(\rho)$ characterizes the equation of state of the matter in the star as a whole. Since thermal processes, as already noted, take a much longer time to evolve in the star than hydrodynamic processes, the hydrodynamic instability must be considered at a constant entropy S (meaning, of course, an "average" entropy that characterizes the star as a whole). Consequently, $P(\rho)$ is the equation of an isentrope. Assume some change in the size of the star, meaning also in its density $\rho$. This leads to changes in P and q. As can be seen from Fig. 1, the condition for stable equilibrium is

$$\frac{dP}{d\varrho} > \frac{dq}{d\varrho}. \tag{2.5}$$

In fact, during equilibrium P = q; this corresponds to the point of intersection of the curves $P(\rho)$ and $q_M(\rho)$. The symbol M denotes the mass of the star. If the $P(\rho)$ curve is steeper than the $q(\rho)$ curve, then P becomes larger than q upon contraction of the star and the uncompensated part $\Delta F_1$ of the pressure of the matter tends to expand the star and return it to equilibrium. During dilatation q > P and the fraction $\Delta F_2$ of the gravitation force (the part uncompensated for by the pressure) compresses the star, again restoring its equilibrium. Thus, (2.5) is the condition for stable equilibrium. From the definition of q it follows that

$$\frac{dq}{d\varrho} = \frac{4}{3}\frac{q}{\varrho}.$$

Using this equation and the equilibrium condition P = q, we rewrite (2.5) in the form

$$\frac{dP}{d\varrho} > \frac{4}{3}\frac{q}{\varrho} = \frac{4}{3}\frac{P}{\varrho}$$

or

$$\frac{d\ln P}{d\ln\varrho} > \frac{4}{3}. \tag{2.6}$$

The quantity

$$\frac{d\ln P}{d\ln\varrho} = \gamma$$

should be called the adiabatic exponent; however, $\gamma$ is not a constant and is itself a function of the density and of the entropy.

Thus, the criterion for hydrodynamic stability is $\gamma > 4/3$. We recall that for an ideal monatomic nonrelativistic gas we have $\gamma = 5/3$.

We shall not consider at present the question of the thermal stability of this star, and assume, in view of the slowness of the thermal processes, that S is constant.

Equation (2.4) can be regarded as an expression for the mass of the star in terms of its average density, if we know the equation of state $P = P(\rho)$. Let us

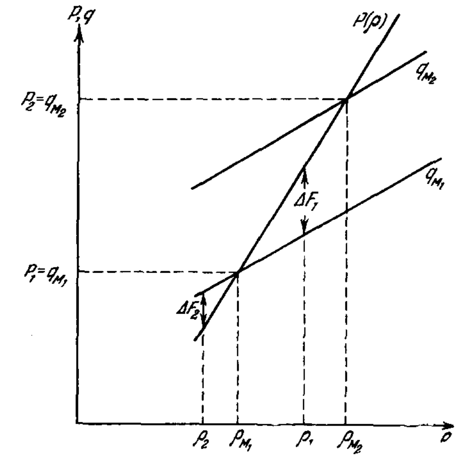


FIG. 1. Condition for equilibrium of a star. $P(\rho)$ – pressure of matter; $q_{M_1}(\rho)$ – gravitational pressure for a star with mass $M_1$, $q_{M_2}(\rho)$ – the same for mass $M_2$. In the stationary state P = q. If the P line is steeper than q, then any small deviation from equilibrium (from the point ($P_1$, $\rho_{M_1}$) for a star with mass $M_1$) gives rise to forces ($\Delta F_1$ or $\Delta F_2$) that return the star to equilibrium.

differentiate (2.4):

$$\frac{2}{3}M^{-1/3}\frac{dM}{d\varrho}=\frac{P\varrho^{-7/3}}{\left(\frac{4}{3}\pi\right)^{1/3}G}\left(\frac{d\ln P}{d\ln\varrho}-\frac{4}{3}\right).$$

The sign of $dM/d\rho$ coincides with the sign of the difference $(\gamma - 4/3)$, as is clearly seen from Fig. 1. Indeed, a larger mass corresponds to larger q, and if the slope of P exceeds that of q (i.e., $\gamma > 4/3$), then the curves cross at larger values of $\rho$.

Let us formulate the result: $dM/d\rho > 0$ when the star is stable, and $dM/d\rho < 0$ when it is unstable. In calculating $dM/d\rho$ we imply a comparison of two star models made of matter having the same equation of state and the same entropy, but unequal masses that differ by dM. This is a natural criterion: in the stable state addition of mass causes compression and an increase in pressure, compensating the increased gravitational force. We note that an analysis of the exact equilibrium equation, rather than the average one, leads to the following stability criterion: $dM/d\rho_c > 0$, where $\rho_c$—central density of the star.[5]

Can the star lose stability because of a strong temperature dependence of the energy-release processes in nuclear reactions? This dependence, for small intervals of P, is given by

$$\varepsilon=\varepsilon_0\varrho T^{\nu}.$$

Here $\epsilon_0$ and $\nu$ are constants. For the proton-proton reaction, for example, $\nu = 4.5$ in the temperature interval $(0.9-1.3) \times 10^7$ deg K. For the carbon cycle $\nu = 20$ at $T = (1.2-1.6) \times 10^7$ deg K.

The process of heat dissipation and radiation of energy from the star into the surrounding space is determined by the conditions of diffusion of radiation from the interior to the outside.* The flux Q of energy to the outside depends on the distribution of the temperature and on the opacity of the stellar matter

$$Q=4\pi r^2 D\frac{dE}{dr},$$

where D—diffusion coefficient, E—density of light energy, proportional to $T^4$. Since usually $\nu > 4$ and consequently the energy release depends more strongly on T than the heat dissipation, one might think that an accidental small excess of the energy release over the process of energy radiation by the star into the surrounding space would lead to an increase in T, meaning also to a sharp increase in the energy release $\epsilon$, so that the perturbation will build up. This phenomenon is analogous to a thermal explosion in a chemical system. Actually, however, the situation is different. We have already emphasized that hydrodynamic processes in a star are much faster than the thermal processes. Therefore an increase in the energy release leads to a deviation from equilibrium: P > q. This causes the star to expand and $\rho$ to decrease. We substitute in (2.4) the equation of state $P = kT\rho/m$ and determine T:

$$T=\left(\frac{4}{3}\pi\right)^{1/3}\frac{m}{k}GM^{2/3}\varrho^{1/3}. \tag{2.7}$$

We see that the decrease in $\rho$ leads to a decrease in T,* meaning to a decrease in $\epsilon$, so that the perturbation will not build up.

The excess of radiation over the energy release leads to the reverse process, and equilibrium is again restored. Thus, the star regulates the power of the nuclear-energy sources, reconciling them with the energy radiated from the surface.

The excess of energy release over heat dissipation leads therefore to a decrease in the temperature of the star. In this sense we can speak of the star having a negative specific heat. The heat capacity in question differs from specific heat at constant pressure or constant volume usually employed in physics. In this case this is "specific heat under stellar equilibrium," i.e., under the condition

$$dP=\frac{4}{3}\frac{P}{\varrho}d\varrho.$$

In the stationary state of the star, the release of nuclear energy exactly compensates the energy lost to radiation. However, a decrease in the concentration of the nuclear fuel leads to violation of the balance: the energy loss exceeds, albeit little, the energy release. This leads to a rise in temperature, such as to ensure the required rate of release of nuclear energy a decreased concentration of the nuclear fuel, or when the fuel is changed (for example from H to He), which necessitates a higher combustion temperature. This constitutes the slow evolution of the star with gradual exhaustion of the nuclear energy reserves.

We note that in accordance with the negative specific heat of the star as a whole, the gradual increase in temperature is accompanied by a decrease in entropy.

In fact, let us write down the equation of state first in terms of the temperature and then in terms of the entropy, and let us rewrite (2.4); we obtain accordingly

$$P=\text{const}\cdot T\varrho, \tag{2.8}$$

$$P=\text{const}\cdot e^{c_1S}\varrho^{5/3}, \tag{2.9}$$

$$P=\text{const}\cdot\varrho^{4/3}. \tag{2.10}$$

From (2.8) and (2.10) it follows that

$$T=\text{const}\cdot\varrho^{1/3}, \tag{2.11}$$

and from (2.9) and (2.10) we get

$$e^{c_1S}=\text{const}\cdot\varrho^{-1/3}. \tag{2.12}$$

Thus, we see from (2.11) and (2.12) that an increase

*Under certain conditions the energy flux is transported not by radiation but by convection, but this does not change the situation.

*In massive stars the pressure is determined principally by the radiation pressure and $P = aT^4/3$. Obviously in this case, too, a decrease in $\rho$ is accompanied by a decrease in T.

in density leads to an increase in T and to a decrease in S.

## 3. EQUATION OF STATE

In the preceding section we have established that the mass of an equilibrium star and the stability of equilibrium depend essentially on the equation of state of the matter. In the interior of ordinary stars, the temperature is so high that the gas is almost completely ionized and constitutes a high-temperature plasma. Under conditions of strong ionization, the separate particles have much smaller dimensions than the atoms and the molecules in the neutral gas, and their interaction is small. Consequently deviations from the ideal-gas equation are negligibly small and we have for the pressure of the matter

$$P = \frac{kT}{m}\varrho. \qquad (3.1)$$

The molecular weight m is equal to the fraction $\mu$ of the proton mass $m_p$ per particle. In a fully ionized hydrogen plasma $m = m_p/2$. The pressure can be expressed also in terms of the entropy S:

$$P = C_2 e^{C_1 S}\varrho^{5/3}, \qquad (3.2a)$$

where $C_1$ and $C_2$ are constants.

In massive stars the temperature is so high that the radiation pressure becomes important. In this case we must add to the right side of (3.1)

$$P_L = \frac{a}{3}T^4.$$

For almost pure radiation the energy is $E \sim T^4V$; here the energy E and the volume V have been calculated per nucleon (although we consider almost pure radiation and there are very few nucleons), since it is precisely the number of nucleons which is conserved. Hence, using the relation $dS = dE/T$, we easily find that for S = const

$$P_L = \text{const}\cdot\varrho^{4/3}. \qquad (3.2b)$$

As can be seen from (3.2a), for an ideal monatomic gas $\gamma = 5/3$. The adiabatic exponent for pure radiation (photon gas), as seen from (3.2b), is 4/3. Matter consisting of non-interacting particles has an adiabatic exponent between 5/3 and 4/3.

However, at very high temperatures, in the presence of statistical equilibrium between the particles, equilibrium reactions can occur, accompanied by absorption of energy. Then the growth in temperature due to contraction slows down, and the adiabatic exponent can become smaller than 4/3. An example of such an endothermic reaction is the process

$$Fe^{56}_{26} = 13\alpha + 4n,$$

the significance of which was pointed out by Hoyle and Fowler[7]. According to calculations by V. S. Imshennik and D. K. Nadezhin, this process causes $\gamma$ to become smaller than 4/3 when $T \gtrsim 5\times 10^9$ at $\rho = 10^7$ g/cm$^3$, reaching at this density a minimum value $\gamma_{\min} = 0.97$ at $T = 6.5\times 10^9$.

Another example of such a process is the production of $e^+$ and $e^-$ pairs. If the density of matter is sufficiently low to make the pressure determined essentially by radiation (or radiation with $e^+$, $e^-$ pairs at $T > 6\times 10^9$), then the adiabatic exponent, as shown by G. V. Pinaeva[45], becomes noticeably smaller than 4/3 in the interval $5\times 10^8 < T < 3.5\times 10^9$.

Thus, at high temperatures there are processes that make $\gamma < 4/3$ and consequently cause instability of the hydrostatic equilibrium in Newtonian theory.

We now turn to the case of low temperatures. During the evolution the star sooner or later exhausts its nuclear-energy reserves. Then the energy radiation, as we have already shown, is accompanied by contraction of the star, which leads in final analysis to the degeneracy of the electron gas. The pressure of the material will be determined essentially just by the pressure of the degenerate particles.

Further radiation of energy and cooling of the star leads ultimately to a situation wherein the temperature drops practically to zero. The stable equilibrium configuration at T = 0, meaning also S = 0 (if such a configuration is possible for the given mass), is the natural final stage of star evolution. Therefore an investigation of the equation of state at S = 0 is of special interest.

Let us see how the state of cold matter and its pressure vary with variation of the density.

At a density larger than $10^2$ g/cm$^3$ the distances between the atoms are smaller than their dimensions, the electrons become collectivized, and no longer belong to individual nuclei, regardless of the temperature of the matter. What is their state then? The Pauli principle establishes that in six-dimensional phase space of the momenta and coordinates, one cell with volume $(2\pi\hbar^3)$, where $\hbar$ is Planck's constant, can contain no more than two electrons with different spin directions. Consequently, at zero temperature and at a specified density $n_e$, the electrons, striving to occupy the lowest possible energy states, fill all the cells of phase-space with momentum from zero to $p_0$, with two in each cell. This state is called degenerate. The limiting Fermi momentum $p_0$ is obviously connected with the electron density:

$$n_e = \frac{1}{3\pi^2\hbar^3}p_0^3.$$

On the right side of this equation is double the number of cells in phase-space volume, being the product of a unit coordinate volume by the volume of a sphere of radius $p_0$ in momentum space. Expressing $n_e$ in terms of the density of matter $\rho$, namely $n_e = \rho Z/Am_p$, where A and Z are the average atomic weight and the atomic charge of the matter, respectively, and $m_p$ is the proton mass, we obtain for the limiting Fermi momentum

$$p_0 = \left(\frac{3\hbar^3 Z\pi^2}{Am_p}\varrho\right)^{1/3}.$$

The pressure is determined by the density of momentum flux. If the particle velocities are nonrelativistic, then $v = p/m$, where m—particle mass. Since the mass of the electron is much smaller than the masses of the atomic nuclei, the electron pressure is much larger than the nuclear pressure. The latter can be disregarded, and the equation of state takes the form*

$$P = 9.9 \cdot 10^{12}\left(\frac{Z\varrho}{A}\right)^{5/3}. \tag{3.3}$$

With increase in density, the momentum of the particles increases. For $\rho \approx A/Z.10^6$ g/cm$^3$, the electron momentum becomes of the order of $m_e c$, and the electron gas becomes relativistic. The adiabatic exponent gradually decreases from 5/3 to 4/3. For an ultrarelativistic gas $\gamma = 4/3$. However, further increase in the density of matter above $\rho = 2 \times 10^6$ g/cm$^3$ causes the electrons on the edge of the Fermi distribution, starting with a certain density, to start to participate in the inverse $\beta$ process with the stable nuclei:

$$(Z, A) + e^- = (Z-1, A) + \nu.$$

The produced neutrinos move freely away from the star. The isolated nucleus $(Z-1, A)$ is unstable and experiences $\beta$ decay:

$$(Z-1, A) \rightarrow (Z, A) + e^- + \tilde{\nu}.$$

However, this process cannot occur in the star at the density in question, since the nuclei are imbedded in the degenerate electron gas and all the phase-space cells corresponding to the momentum of the produced electron are already occupied, so that the electron cannot be created.

The inverse $\beta$ process leads to a decrease in the total number of the electrons per gram of matter and to an increase in the number of neutrons in the stars. This process is called neutronization. The possibility of formation of neutron configurations was indicated in[66-68] and was calculated by L. D. Landau[18]. The density corresponding to the start of neutronization depends on the chemical and isotopic composition of the matter.

We shall show below that for stars neutronization begins when the matter consists of elements of medium atomic weight ($A \approx 24$). Then $\rho_{crit} = 10^9$–$10^{10}$ g-cm$^{-3}$.† The decrease in the number of electrons stops the growth of the pressure, and when $\rho > \rho_{crit}$ the value of $\gamma$ becomes smaller than 4/3. The neutronization process makes the nuclei unstable and leads to their eventual disintegration. When $\rho > 10^{12}$ g/cm$^3$, the pressure (as well as the density) is determined essentially by the degenerate neutron gas. If the neutrons were not to interact with one another, this gas would be ideal and, so long as the gas were still nonrelativistic, the adiabatic exponent would be $\gamma = 5/3$ (and always $\gamma > 4/3$). It is known, however, that attraction forces exist between the neutrons, and although these forces are insufficient to produce nuclei consisting of neutrons, they nevertheless make a negative contribution to the pressure and $\gamma$ is as before smaller than 4/3.

At small distances between baryons, the forces of attraction are replaced by repulsion forces which make a positive contribution to the pressure, so that when $\rho \approx 2 \times 10^{14}$ g/cm$^3$, $\gamma$ again becomes larger than 4/3. According to A. Cameron[8], the equation of state for $\rho \gtrsim 10^{12}$ g/cm$^3$ is

$$P = 5.3 \cdot 10^9 \varrho^{5/3} + 1.6 \cdot 10^{-5} \varrho^{8/3} - 1.4 \cdot 10^5 \varrho^2. \tag{3.4}$$

The second term takes into account the repulsion forces and the third the forces of attraction between the baryons.

Cameron, like many other workers, is inclined to overestimate the accuracy of formulas similar to (3.4). In actuality this is only a crude approximation. At still higher densities ($\rho > 10^{15}$ g/cm$^3$) there should appear in the matter hyperons,* which are stable under these conditions; this is discussed in detail in the papers by V. A. Ambartsumyan, G. S. Saakyan, and their co-workers[9-11], but for the time being we cannot speak with assurance of the exact form of the equation of state in this region.†

In the case of large density, a distinction must be made between the total density of the mass (including the energy density) $\rho$ (g/cm$^3$), and the density of the baryon rest mass $\rho' = mn$(g/cm$^3$), where n(cm$^{-3}$) is the baryon density.

For a long time it was assumed without proof that the pressure should be smaller than or equal to $\rho c^2/3$ (see[38]). Such a dependence is obtained in two cases: for free non-interacting particles (with $P = \rho c^2/3$ in the limit of ultrarelativistic particles) and for an electromagnetic field and particles interacting via the electromagnetic field.

However, Zel'dovich[39] constructed a concrete example of interaction of particles with a field of heavy neutral vector mesons; a relativistically-invariant theory yielded a pressure $P \rightarrow \rho c^2 = \epsilon$ as $\rho \rightarrow \infty$. We note that when $P = \rho c^3/3$ the speed of sound is $v_s = c/\sqrt{3}$ in the new variant, and when $P = \rho c^2$ the

*We use the cgs system throughout.

†A curious exception [58] is the neutronization of He$^3$ (see Sec. 12).

*$\Sigma^-$ hyperons appear in the matter even at a nuclear density $\rho \approx 3 \times 10^{14}$ g/cm$^3$. Their stability is ensured by the presence of degenerate electrons. Under usual conditions there can be no hyperons in a nucleus situated at the center of the atom, since the electrons cancel out the nuclear charge within the volume of the atom only in the mean, and the density of the electrons in the nucleus, even for heavy elements, is one-millionth the density of the electrons when stable hyperons appear in the stellar matter.

†In spite of the production of new particles, the conserved quantity is the baryon charge n; hyperons are also baryons.

Table I. Sign of $\Delta = \lambda - 4/3$ for different values of the density (S = 0)

| Density interval | $\varrho \leqslant 10^9$ g/cm³ | $10^9 \leqslant \varrho \leqslant 10^{14}$ | $10^{14} < \varrho < 10^{15}$ | $\varrho > 10^{15}$ |
|---|---|---|---|---|
| Sign of $\Delta$ | + | − | + | − |

speed of sound tends to that of light, $v_S \to c$, which is aesthetically more satisfying. At any rate, the assumption that $P \leq c^2/3$ always and everywhere has now become a preconceived notion; more than four years elapsed since the publication of[39], but no dissenting opinions have been expressed against it; see also[11,43].

In a recent paper D. A. Kirzhnits and V. L. Polyachenko[40] suggest the possibility of $v_S > c$. Moreover, they construct an example of baryons that interact locally with a pseudoscalar meson field, in which such a case is realized in their opinion. General considerations allow us to state that this example contains an error; the equations for the meson field outside the sources have a value c for the speed of signal propagation; a system of point-like baryons at rest, interacting via mesons, cannot produce in any way a signal velocity larger than c.

Different exponents in the function $\rho = \rho(n)$ correspond to different ratios of $P/\rho$ in that region where $\rho \gg \rho' = nm$:

$$\text{for} \quad P \sim \varrho \sim n^{4/3} \quad P = \frac{1}{3}\varrho c^2;$$

$$\text{for} \quad P \sim \varrho \sim n^2 \quad P = \varrho c^2.$$

We shall show later (Sec. 4) that stars in which densities $\rho \approx 10^{15}$ g/cm³ and more are attained are certainly relativistic, i.e., the decisive influence is exerted on their structure by the effects of general relativity. It turns out in this connection that the absence of reliable information on the equation of state at super-nuclear densities does not prohibit an analysis of the principal problems in the stellar evolution.

We list in conclusion in Table I the signs of $\Delta = \gamma - 4/3$ for S = 0.*

To solve the question of stellar evolution completely it is necessary to be able to compile a similar table for any specified entropy S, i.e., it is necessary to know the equation of state for all values of $\rho$ and T. To write down such an equation of state we must solve the equations of statistical equilibrium for the components of matter for specified $\rho$ and T. In spite of the fact that this problem is in principle trivial for the greater part of the possible values of $\rho$ and T, it has not yet been solved even in crude approximation. Work in this direction has only begun. However, in spite of the lack of numerical calculations at present, we can obviously nevertheless indicate, from considerations of continuity of the function $P(S, \rho)$, the approximate form of $P_{S=\mathrm{const}} = P(\rho)$ for small S, since the function $P = P(0, \rho)$ is known. These considerations will help us in the analysis of stellar evolution.

*See Sec. 8 concerning the singularities that arise when $\rho \gtrsim 10^{15}$ g/cm³.

## 4. MASSES OF STARS IN THE FINAL STAGE OF EVOLUTION

Let us return to the nonrelativistic theory. We calculate the mass of a cold star (S = 0) as a function of the average density. We rewrite formula (2.4) in the form

$$M = \frac{bP^{3/2}}{G^{3/2}\varrho^2}, \tag{4.1}$$

where $b = (3/4\pi)^{1/2}$.

The equilibrium equation (2.3) was integrated in a basic paper by Emden[12] for the case when the equation of state in the entire star has the adiabatic form $P = k\rho^\gamma$, where k and $\gamma$ are constants. The main deductions of his work can be found, for example, in[44]. As a result the expression obtained for M in lieu of (4.1), derived from the averaged equilibrium equation, differs from (4.1) in that the numerical coefficient $b = (3/4\pi)^{1/2}$ is replaced by a factor $b(\gamma)$ which depends on $\gamma$. Frequently $\gamma$ is replaced by the polytrope index $n = 1/(\gamma - 1)$. If $\gamma$ varies from 5/3 to 4/3, then $b(\gamma)$ varies from $b(5/3) \approx 7.4$ to $b(4/3) = 4.6$.

When $\rho \ll 10^6$ g/cm³, the appropriate equation of state for a medium in which the pressure is determined by the degenerate electron gas is (3.3). Substituting (3.3) in (4.1) and putting $b(5/3) = 7.4$, we get

$$M = 6.8 \cdot 10^{-3} \varrho^{1/2} \left(\frac{Z}{A}\right)^{5/2} M_\odot. \tag{4.2}$$

If the electron gas were to become gradually relativistic with increasing $\rho$, but without neutronization of the medium, then it would be necessary to replace (3.3) for $\rho \gg 10^6$ g/cm³ by an equation of state for a medium whose pressure is determined by ultrarelativistic degenerate electrons, and whose density is determined by the atomic nuclei

$$P = 1.23 \cdot 10^{15} \left(\frac{Z\varrho}{A}\right)^{4/3}. \tag{4.3}$$

The atomic nuclei are not yet degenerate and on the whole $P \ll \rho c^2$; therefore Newton's theory is applicable. Substituting this expression in (4.1), using $b(4/3) = 4.6$, and taking account of the fact that the ratio Z/A for heavy elements is of the order of 0.5, we obtain

$$M = 1.45 M_\odot.$$

Thus, for cold matter in equilibrium there exists in Newtonian theory an upper mass limit ("Chandrasekhar limit"[13]), equal to 1.45 $M_\odot$, and attained in the limit as $\rho = \infty$ (when $\gamma$ is exactly equal to 4/3 in all of the star). Actually, however, as we have noted in the preceding section, the neutronization of the medium causes $\gamma = 4/3$ to be reached not at $\rho = \infty$, but at $\rho \approx 10^9$–$10^{10}$ g/cm$^3$. When the average density of the star material is sufficiently high, $\rho \gtrsim 10^7$ g/cm$^3$, the pressure is determined by the relativistically degenerate electrons, and $\gamma$ differs very little from 4/3. A small reduction in $\gamma$ now suffices to make $\Delta$ smaller than zero. When the density at the center of the star exceeds $\rho_c \approx 10^9$ g/cm$^3$, neutronization of matter begins there. As soon as this process is initiated in the center, $\gamma$ becomes smaller than 4/3 for the entire star.* The maximum mass corresponding to this central density is $M_{max} = 1.2\ M_\odot$ (see Fig. 4)[14-16]. This result was obtained by numerically integrating the balance equation (2.3) with account of the change in the equation of state on going from dense stellar interiors to the less dense surface. The value of $M_{max}$ depends (but not very strongly) on the chemical composition, and to an even lesser degree on the spin of the star (up to the limit when the spin causes a strong outflow of matter from the equator of the star). With further increase in density, we get $\gamma < 4/3$ (see Table I). It follows from (4.1) that with increasing $\rho$ the mass decreases and the equilibrium is then unstable ($dM/d\rho < 0$), and consequently stationary cold stars with densities $\rho_c > 10^9$ g/cm$^3$ in the center do not exist (see also[69]).

When the mean density of the neutron core of the star reaches $\rho = 10^{13}$ g/cm$^3$, the density at the center exceeds the nuclear density ($3 \times 10^{14}$ g/cm$^3$) and the average value of $\gamma$ for the star becomes larger than 4/3. Thus, the mass M of an equilibrium star reaches a minimum value when $\rho \approx 10^{13}$ g/cm$^3$. This value $M_{min}$ can be estimated by determining the pressure at $\rho = 10^{13}$ g/cm$^3$ from formula (3.4) and substituting this value in (4.1) with $b \approx 5$:

$$M_{min} \approx 0.05 M_\odot.$$

We recall that this is only an order-of-magnitude estimate, for actually the star cannot consist of neutrons only. In the outer regions the pressure is insufficient for the existence of stable neutrons, and the outer shell consists of nuclei and electrons. More details on masses of cold stars at large densities will be given later (Sec. 8). The details of the calculations can be found in the paper of G. S. Saakyan and Yu. L. Vartanyan[16].

Further increase in density is accompanied by an increase in mass, since $\Delta = \gamma - 4/3 > 0$, meaning that $dM/d\rho > 0$. The equilibrium configurations are then stable. Simultaneously with baryon repulsion at $\rho \approx 10^{15}$ g/cm$^3$, the effects of relativistic gravitational theory come into play. Before we consider these relativistic objects, we must review the properties of strong static gravitational Einstein fields.

*At the critical density, some contribution to the decrease in the effective $\gamma$ is made also by effects due to the difference between the density of the particle rest mass and the energy density (see Sec. 3), and to the change in the gravitational law connected with general relativity [47]. For more details see Sec. 8.

## 5. SCHWARZSCHILD GRAVITATION FIELD

We begin the examination of relativistic effects with the simplest case—strong gravitational field produced by a spherical body in vacuum.

The solution of Einstein's equation for such a field (the Schwarzschild solution[17], 1916*) determines the geometrical properties of the space and the rates of time flow near the body producing the field. It turns out that this field is always constant (even if the material of the central body executes radial oscillations but remains spherically symmetrical), and depends only on the total energy E of the body.

The expression for the four-dimensional interval in the Schwarzschild field is of the form

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) + \left(1 - \frac{2GM}{c^2 r}\right) c^2\, dt^2, \tag{5.1}$$

where $M = E/c^2$. In the expression for $ds^2$ is contained all the information concerning the gravitational field. Let us recall the manner of using this expression for physical deductions. The first three terms in the sum constitute the square of the distance ($dl^2$) between infinitesimally close points, taken with a minus sign, and written in spherical coordinates in curved space. A stationary observer located near the massive body can measure distances in a small vicinity by the usual method, introducing Cartesian coordinates. In these coordinates $dl^2 = dx^2 + dy^2 + dz^2$. If he chooses $dz = r d\theta$ and $dy = r \sin\theta\, d\varphi$, then outside the gravitational field we have in Euclidean space $dx = dr$. Near the massive body, in the Schwarzschild field, we have, as can be seen from (5.1),

$$dx = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2} dr. \tag{5.2a}$$

The factor preceding dr differs from unity, a manifestation of the non-Euclidean nature of the geometry. From this it follows, for example, that the distance between two close circles drawn in a single plane around the central body and having lengths $l_1$ and $l_2$ is not $(l_2 - l_1)/2\pi$, but

$$\frac{l_2 - l_1}{2\pi}\left(1 - \frac{2GM}{rc^2}\right)^{-1/2}.$$

The curvature of space at a given point in a given two-dimensional direction, i.e., for a given orientation

*Concerning the properties of the Schwarzschild solution see, for example, [46].

of the plane (the so-called Riemannian curvature) is measured by the ratio $(\Sigma - \pi)/S$, where $\Sigma$—sum of the angles of a small triangle on this plane, and S—its area. The curvature has the dimension of $cm^{-2}$ and can be either positive or negative. Outside the star, in a direction tangent to its surface, the curvature is negative, while in orthogonal directions it is positive. The curvature averaged over all directions (the so-called Gaussian curvature) is equal to zero.

The last term in (5.1) is the square of the interval of the running time at the given point (multiplied by $c^2$):

$$\Delta\tau = \sqrt{1 - \frac{2GM}{rc^2}}\Delta t. \qquad (5.2b)$$

Far away from the body, as $r \to \infty$, we have $\Delta\tau = \Delta t$. The closer the point of observation to the body producing the field, the slower the course of the time, i.e., the smaller the interval $\Delta\tau$ corresponding to the given time interval $\Delta t$ at infinity.*

Let us find the gravitational force F acting in a Schwarzschild field on a trial mass m with low velocity ($v \ll c$). This force is obviously equal to $F = md^2l/d\tau^2$. The acceleration of free fall $d^2l/d\tau^2$ for the trial particle is written in the form

$$\frac{d^2l}{d\tau^2} = -\frac{GM}{r^2\left(1 - \frac{2GM}{c^2r}\right)^{1/2}},$$

and consequently the gravitational force is

$$F = -\frac{GMm}{r^2\left(1 - \frac{2GM}{c^2r}\right)^{1/2}}. \qquad (5.3)$$

We see that at $r = 2GM/c^2$ the force of gravitation becomes infinite. This singularity is evidence that the central body, if static, cannot have a radius smaller than $2GM/c^2$. The stationary non-deforming spherical system of coordinates used above is applicable likewise only if $r > 2GM/c^2$. This critical radius $r_g = 2GM/c^2$ is called the gravitational radius, and a sphere of radius $r_g$ is called a Schwarzschild sphere. We note that a non-static body can have dimensions smaller than the gravitational radius.

At a distance that is large compared with $r_g$ the Schwarzschild field is the usual gravitational field of Newton's theory, with a gravitational potential $\varphi = GM/r$ and with a force

$$F = -\frac{GMm}{r^2}.$$

The gravitational radius of the sun is 2.96 km and that of the earth 0.443 cm. The radii of the earth and of the sun are much larger than the gravitational radii. Consequently, the gravitational field outside the sun, earth, or other stars or planets is, with tremendous accuracy, a Newtonian field. Schwarzschild solutions cannot be used in matter, and inside the sphere, as will be shown in the next section, there are no singularities of the Schwarzschild-sphere type at all.

## 6. GRAVITATIONAL FIELD INSIDE A STAR

We now consider the properties of a strong gravitational field inside matter at rest. The four-dimensional interval is then customarily written in the form

$$ds^2 = -e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + e^{\nu(r)}c^2dt^2. \qquad (6.1)$$

The two coefficients $e^{\lambda(r)}$, describing the deviation of the geometry from Euclidean, and $e^{\nu(r)}$, describing the change in the rate of time flow, are determined by the distribution of the matter:

$$e^{-\lambda} = 1 - \frac{8\pi G}{rc^2}\int_0^r \varrho r^2\, dr, \qquad (6.2)$$

$$-\nu = \int_r^\infty \left[\frac{8\pi G}{c^4}(\varrho c^2 + P)re^\lambda - \frac{d\lambda}{dr}\right]dr. \qquad (6.3)$$

We recall that $\rho$ is the density of matter and includes not only the sum of the particle masses per unit volume, but also their energy (of motion and of interaction other than gravitational). The coefficients of $dr^2$ in expressions (5.1) and (5.6) should coincide in vacuum outside the star. Hence, using (6.2), we obtain an expression for the mass

$$M = 4\pi\int_0^R \varrho r^2\, dr. \qquad (6.4)$$

We recall that owing to the non-Euclidean nature of the space, the volume element is $dV = 4\pi \exp(\lambda/2)r^2dr \neq 4\pi r^2dr$. The integral of (6.4) contains $4\pi r^2 dr$ and not dV. We shall show that this is connected with the influence of the energy of the gravitational field on the mass of the body. From (6.2) and (6.3) we see that $e^\lambda \geq 1$ and $e^\nu < 1$ (as in the case outside a gravitating mass), and therefore the deviation of geometry from the Euclidean inside the body has the same character as beyond its boundaries, and $dV > 4\pi r^2 dr$, while the time flows more slowly than at infinity.

It follows from (6.2) that $e^\lambda \to 1$ as $r \to 0$, and the metric has in this case a Galilean form. This, of course, does not signify that the space is less curved here than at other points. The point is that we are using spherical coordinates and the condition $r \to 0$ means that we are taking a small vicinity around the center. In the preceding section we have mentioned that the curvature of space has a dimension $cm^{-2}$; consequently, the effects due to the curvature decrease in proportion to the square of the dimension. Therefore as $r \to 0$ the curvature of space does not come into play and $e^\lambda \to 1$.*

Actually, the Gaussian (average) curvature $C_G$ of the space is larger at the center of the star than in

*The simultaneity of events is established by transmission of light signals. Signal transmission in a Schwarzschild field will be considered below.

*We note that solutions with finite mass and with $\rho_c = \infty$ exist. Accordingly, in these solutions the curvature is infinite at the center and $e_c^\lambda \neq 1$. We shall not deal here with these singularities.

other places. The value of $C_G$ is given by (see the paper by A. L. Zel'manov[54]; its conclusions that are of interest to us are contained in[55])

$$C_G = \frac{4}{3}\frac{\pi G \varrho}{c^2}. \tag{6.5}$$

Since the density is maximal at the center of the star, $C_G$ is maximal there, too. Of course, we must not think that (6.5) implies that the space is Euclidean outside the star, where $\rho = 0$, even near its surface in a strong field. Formula (6.5) yields only the average curvature of space in all two-dimensional directions, and this average curvature is indeed equal to zero. However, as indicated in the preceding section, outside the star the Riemannian curvature of space is not equal to zero and can have, depending on the two-dimensional direction, both positive and negative values. At the center of the star all the directions are equivalent; the curvature there is given by formula (6.5) for any orientation and is always positive.

The gravitational field obtained by "joining" the solutions inside the star and outside has no physical singularities of the Schwarzschild-sphere type anywhere, and we have $1 \le e^{\lambda} < \infty$ and $0 < e^{\nu} < 1$ throughout.

## 7. PATHS OF LIGHT RAYS INSIDE AND OUTSIDE A STATIC STAR

Let us see now how rays of light and neutrinos, moving along the radius, will propagate in a spherical gravitational field. Inasmuch as the observer can locally introduce coordinates in which $dx^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$, we obtain, using the principle of constancy (locality) of the velocity of light measured by a local observer ($v_{\text{light}} = c$), an equation for the motion of a particle with zero rest mass: $ds = 0$. Consequently, for $\varphi = \text{const}$ and $\theta = \text{const}$ we have

$$\frac{dr}{dt} = ce^{\frac{\nu-\lambda}{2}}. \tag{7.1}$$

Inside the sphere $\exp[(\nu - \lambda)/2] < 1$ everywhere. Beyond the surface of the sphere, in vacuum, we have

$$e^{\frac{\nu-\lambda}{2}} = 1 - \frac{2GM}{c^2 r} < 1,$$

and this quantity tends to unity as $r \to \infty$. Consequently, for example for a neutrino emitted from the center, the variation of the coordinate r of a remote exterior observer with the time t should have the form shown in Fig. 2. The dashed line in this figure shows the motion of the neutrino in the absence of a gravitational field. We note that as the radius of the star $R \to r_g$, we have in vacuum

$$\frac{dr}{dt} = c\left(1 - \frac{2GM}{c^2 r}\right) \to 0.$$

How do the neutrino energy and the energy of the light quanta, and consequently also the frequency of the corresponding waves, vary as they move in the

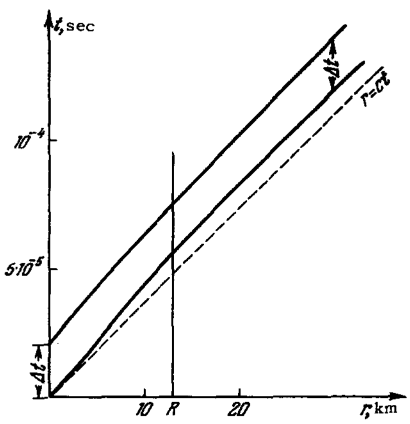


FIG. 2. Plot showing the variation of r as measured by the clock of an external observer t for two neutrinos emitted at t = 0 and t = Δt from the center of a star with mass 0.64 $M_\odot$. R – boundary of the star ($R = 6.9R_g$).

gravitational field? Let us consider the variation of the frequency. Let the emitter in the center of the star produce two flashes separated by an interval Δt. Since $e^{\lambda}$ and $e^{\nu}$ do not depend on t, these flashes will arrive to the remote observer also separated by an interval Δt by his clock, as shown in Fig. 2. However, the interval Δt in a strong field corresponds to a time interval

$$\Delta\tau = e^{\nu/2}\Delta t. \tag{7.2}$$

Consequently, the frequency of the signal received by the observer, which is proportional to $1/\Delta t$, differs from the frequency of the emitted signal $\omega_0 \sim 1/\Delta\tau$:

$$\omega = \omega_0 e^{\nu/2}. \tag{7.3}$$

The signal frequency decreases as the latter emerges from the gravitational field (and increases for motion in the opposite direction). Accordingly, the quantum energy $E = \hbar\omega$ decreases. This phenomenon is called the gravitational red shift. For an observer located on the surface of the star, the emission spectrum of the atoms has exactly the same appearance as in the laboratory on earth. However, the spectrum of the same atoms of the star, observed from the earth, is shifted by this phenomenon towards the red side.*

The gravitational change in the frequency of the quanta demonstrates the amazing orderliness of the general theory of relativity. Indeed, in the framework of Newtonian theory the described phenomenon can be interpreted as loss of energy by the quanta as they leave the gravitational field. However, owing to the connection between the energy and the frequency, $E = \hbar\omega$, the change in energy is connected with the change in frequency, and the latter is proportional to

*The "violet" shift produced in rays arriving from outer space on earth by its gravitational field, amounts only to $\Delta\omega/\omega \approx 10^{-9}$ and will be neglected.

$1/\Delta\tau$. Thus, it follows from this fact that the rate of flow of time changes in a gravitational field, i.e., a change takes place in the properties of the space-time continuum. This already leads directly to Einstein's gravitational theory with the idea of space-time curvature. Numerous attempts to construct in some other manner a modern theory of gravitation were not successful. Einstein's theory is the only orderly gravitation theory, consistent to the end, explaining the entire aggregate of the observed data.

## 8. STRUCTURE OF SUPERDENSE STARS

From the Einstein field equation, as is well known, follow directly the equations of motion and, in the particular case of statics, the equilibrium equations. In the case of spherical symmetry, the equilibrium equation is written in the form*

$$\frac{dP}{dr} = -\frac{1}{2}\frac{d\nu}{dr}(P + \varrho c^2). \tag{8.1}$$

The expression (8.1) together with (6.2), (6.3), and the equation of state determines the hydrostatic equilibrium in relativistic theory, and replaces the equilibrium equation (2.3) of Newton's theory. Integration of this system in conjunction with the equation of state, described in Sec. 3, makes it possible to construct relativistic models of superdense cold stellar configurations, which extend many of the models considered in Sec. 4 into the region of large densities.

The first such calculations were made in the classical paper of Oppenheimer and Volkoff[2] in 1939, using the equation of state of an ideal Fermi gas. The most characteristic feature of the calculated models was the fact that the curve $M = M(\rho)$ has a maximum in the region $\rho \approx 10^{15}$ g/cm$^3$. The unavoidable appearance of a limit on the equilibrium mass was pointed out already in[18,19].

The reasons for the appearance of the maximum are as follows: first, a change takes place in the law of gravitation. We have already noted that the gravitational force (5.3) tends to infinity on the surface of a static star when the radius of the latter approaches the gravitational value (and not zero, as was the case in the Newtonian theory), and consequently the star radius R must exceed $r_g$:

$$R > r_g = \frac{2GM}{c^2}. \tag{8.2}$$

Let us express R in terms of M and the average density $\rho$. According to (6.4), the average density is

$$\varrho = \frac{M}{\frac{4}{3}\pi R^3}.$$

Obtaining from this R and substituting in (8.2), we get

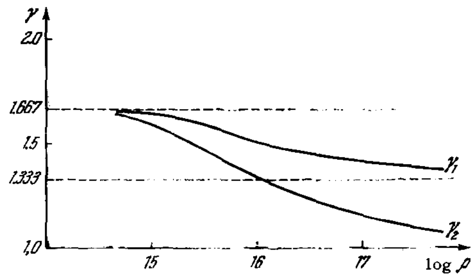


FIG. 3. $\gamma_1 = d \ln P/d \ln n$ and $\gamma_2 = d \ln P/d \ln \rho$ as functions of the density $\rho$ for an ideal cold neutron gas.

$$M < \frac{c^3}{(2G)^{3/2}\left(\frac{4}{3}\pi\right)^{1/2}}\varrho^{-1/2},$$

or, introducing numerical coefficients,

$$M < M_{\odot}\sqrt{\frac{2\cdot 10^{16}}{\varrho}}.$$

We see that no matter what law governs the pressure $P = P(\rho)$, the largest possible value of M must decrease with increasing $\rho$.

Another cause of the maximum of $M(\rho)$ is that at large densities the main contribution to the energy density (and consequently also to the mass density) is no longer made by the rest energy of the particles, but by the energy of their motion and interaction. Let us denote the baryon density by n, and let the quantity $d \ln P/d \ln n$ be defined as the adiabatic exponent $\gamma_1$. The value of $\gamma_1$ so defined is always larger than 4/3 for a degenerate gas, and for repelling particles it can reach in principle a value[39] $\gamma_1 = 2$.

However, the equation contains not n, but the mass density $\rho$ and the pressure P. By virtue of the circumstance noted above, the asymptotic form of the equation of state is $P \sim \rho$, and therefore the effective value of $\gamma_2 = d \ln P/d \ln \rho$ becomes smaller than 4/3 and in the limit $\gamma_2 \to 1$.

The quantity listed in Table I is indeed $\gamma_2$. Figure 3 shows plots of $\gamma_1$ and $\gamma_2$ for an ideal neutron gas. Replacement of $\gamma_2 > 4/3$ by $\gamma_2 < 4/3$ leads, as shown in Sec. 4, to appearance of a maximum in $M(\rho)$. Thus, even regardless of the change in the law of gravitation, $M(\rho)$ should have a maximum. Both effects in question are of the same order and act in the same direction. Numerically these effects, as small corrections to the classical theory, were considered in a little known paper by S. A. Kaplan[47] and later by Fowler[48] (see also[20]). These corrections, in connection with the theory of the construction of very massive stars, will be considered in Part II of the review.

Thus, owing to the factors considered above the mass of the equilibrium configuration can not arbitrarily be large, and in the region of high densities ($\rho \approx 10^{15}$ g/cm$^3$) the $M = M(\rho)$ curve also has a maxi-

*We see from (8.1) that the quantity $\nu/2$ is analogous to the Newtonian potential $\varphi$. In the weak-field approximation $\nu/2 = \varphi/c^2$, and we obtain the Newtonian equilibrium formula (2.3) by recognizing that in this case $\rho c^2 >> P$.

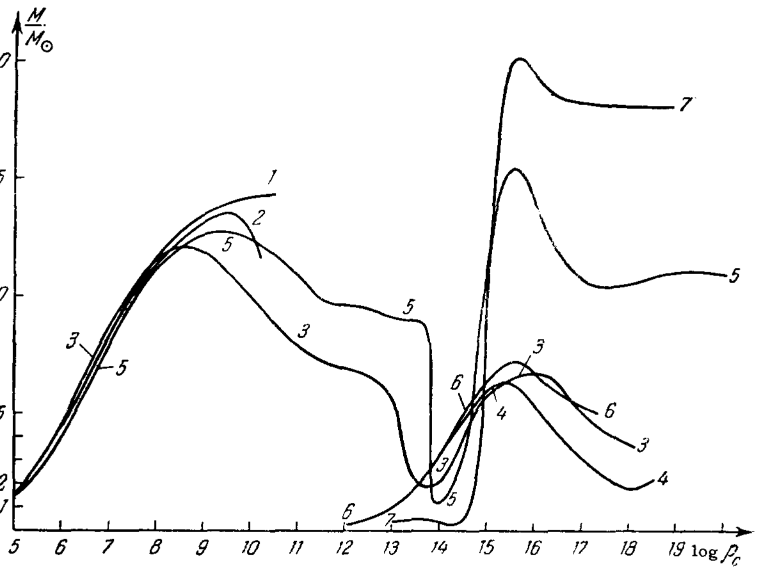


FIG. 4. Masses of cold stars. 1 – Chandrasekhar [13], 1939. 2 – Hamada and Salpeter (for $Mg^{24}$) [15], 1961; 3 – Harrison, Vacano, Wheeler (for a real gas) [14], 1958; 4 – Ambartsumyan and Saakyan (for an ideal gas [9], 1961; 5 – Saakyan and Vartanyan (for a real gas) [16], 1964; 6 – Oppenheimer and Volkoff for an ideal gas) [2], 1939; 7 – Cameron (for a real gas) [8], 1959.

mum. Oppenheimer and Volkoff,[2] using the equation of state of an ideal degenerate neutron gas, obtained $M_{max} \approx 0.72\ M_\odot$. This equation of state, however, as we have seen in Sec. 3, cannot be used at such densities. The latest papers of Cameron[8], Saakyan and Vartanyan[16], with an equation of state of a real gas, yield $M_{max} \approx (1.6-2)\ M_\odot$.

We can now complete the construction of the function $M = M(\rho)$ for equilibrium configurations with $S = 0$. Fig. 4 shows the curve $M = M(\rho_c)$ as calculated by various authors. The abscissas represent here not the average density $\rho$, but the density $\rho_c$ at the center of the configuration, which is a more convenient parameter; in particular, it is precisely $dM/d\rho_c > 0$ which serves as the criterion of star stability for the Newtonian case, as was noted already in Sec. 2. (See Sec. 10 concerning the stability of the models in the relativistic case.)

We see from Fig. 4 that additional information on the properties of matter at high densities has modified somewhat the $M(\rho_c)$ curve, particularly in the region of superhigh densities. This curve has also been calculated differently by different authors, owing to the different assumptions made concerning the chemical composition of the matter (in the region of the first maximum), and owing to different simplifying assumptions used in the calculation. In spite of all these differences, however, the qualitative character of the curve, which is important for the analysis of the fundamental questions of stellar evolution, is the same for all authors. We shall henceforth use for concreteness the curve 5 of Saakyan and Vartanyan[16]. These authors took into account in their calculations the change in the equation of state on going from dense interiors of the star to the envelope. It is curious that in the region of large densities, beyond the maximum, where real equilibrium cold stars can no longer exist, the total mass of the equilibrium star, as shown by N. A. Dmitriev and S. A. Kholin, experiences a periodic attenuating dependence on $\rho_c$ as $\rho_c \to \infty$ (see Fig. 6).

For $\rho_c = \infty$, as already stated in Sec. 6, there is a solution with a finite mass.

We shall call the mass maximum for stars at $\rho_c \approx 10^{15}$ g/cm$^3$ the "OV" (Oppenheimer and Volkoff) maximum $M_{max}^{OV}$, to distinguish it from the Chandrasekhar maximum, which takes place when $\rho_c$ is of the order of $10^9$ g/cm$^3$. On the surface of a star with $M = M_{max}^{OV}$, the quantity $e^\nu$ assumes the minimum possible value for the surface of stars, $e^\nu_{min} = 0.5$. Consequently, the maximum gravitational red shift, which can be observed in principle in the spectrum of an equilibrium star, is

$$\frac{\omega_0}{\omega} = e_{min}^{-\nu/2} \approx 1.4. \tag{8.3}$$

Figure 5 shows plots of $e^{\lambda/2}$ and $e^{\nu/2}$ for two stars

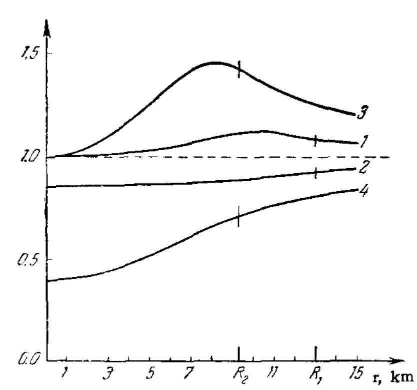


FIG. 5. $e^{\lambda/2}$ and $e^{\nu/2}$ as functions of the radius r for stars with $M_1 = 0.64 M_\odot$ and $M_2 = 1.55 M_\odot$. 1 – $e^{\lambda/2}_{M_1}$; 2 – $e^{\nu/2}_{M_1}$; 3 – $e^{\lambda/2}_{M_2}$; 4 – $e^{\nu/2}_{M_2}$. The radii $R_1$ and $R_2$ correspond to the surfaces of the stars.

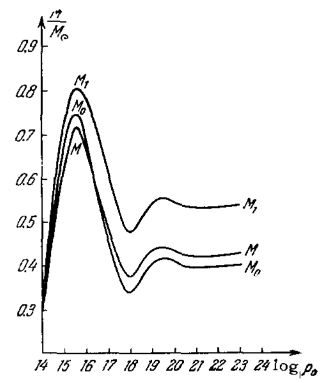


FIG. 6. M, $M_0$, and $M_1$ as functions of $\rho_c$ for a cold ideal Fermi gas of neutrons.

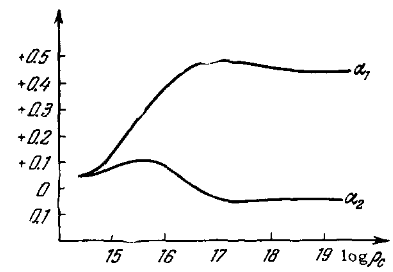


FIG. 7. Dependence of $\alpha_1 = \Delta_1 M/M$ and $\alpha_2 = \Delta_2 M/M_0$ on the density $\rho_c$ at the center of the star.

with $\rho_c = 5.5 \times 10^{14}$ and $3.6 \times 10^{15}$ g/cm$^3$ and with respective masses 0.64 $M_\odot$ and 1.55 $M_\odot$. The $e^{\lambda/2}$ plot characterizes the deviation of the geometry of the space from Euclidean near the star and outside the star. The coordinate radius for these stars, i.e., the values of r for the surface $R = r_{sur} = \sqrt{s/4\pi}$ (where s—area of the surface of the star), is equal to 13 and 9.3 km, respectively. The distance from the center

$$\widetilde{R} = \int_0^{r_{sur}} e^{\lambda/2}\, dr$$

is equal to 13.8 and 11.5 km respectively.

The quantity $e^{\nu/2}$, as already noted, is analogous to the Newtonian potential. It shows directly the slowing down in the rate of time flow as compared with the time at infinity. Unlike $e^{\lambda/2}$, the value of $e^{\nu/2}$ does not tend to unity at the center of the star.

This, of course, is connected with the normalization condition: we have chosen the time coordinate t such that at infinity it always coincides there with the reading of the observer's clock; therefore $(e^{\nu/2})_\infty = 1$, and the ratio of $e^{\nu/2}$ at the center of the star to $(e^{\nu/2})_\infty$ is equal to the ratio of the rate of time flow at infinity to that at the center.

## 9. MASS DEFECT

Let us write down the expression for the total energy E of the star, for the case when the densities are small and Newton's theory is applicable: $E = E_0 + T + \Omega$. Here $E_0 = Nmc^2$—rest energy of the nucleons making up the star, T—energy of motion and interaction of the nucleons, and $\Omega$—potential energy of mutual gravitation. The last term is negative. We put $E_0 + T = E_1$. In the relativistic region we must accordingly distinguish between the following:

$$E = Mc^2 = 4\pi c^2 \int_0^R \varrho r^2\, dr, \tag{9.1}$$

$$E_0 = M_0 c^2 = c^2 \int_V mn\, dV = N mc^2 \tag{9.2}$$

$$E_1 = M_1 c^2 = c^2 \int_V \varrho\, dV, \tag{9.3}$$

where $dV = 4\pi e^{\lambda/2} r^2 dr$. Figure 6 shows plots of M, $M_0$, and $M_1$ as functions of $\rho_c$. The calculations were made for the case of an ideal degenerate neutron gas[21]. We recall that the mass density $\rho$, measured locally, includes not only the rest mass but also the internal energy of motion of the nucleons and the particle interaction energy per cubic centimeter. (Except for the gravitational interaction! The latter is not included in $\rho$, since the forces of gravity are long-range forces and the gravitational energy depends not on the local properties but on the properties of the entire configuration.) The total mass M of the star is not equal to the sum $M_1$ of the masses of the elements of its volume, and since $e^{\lambda/2} \geq 1$, we have

$$M < M_1.$$

The difference $\Delta_1 M = M_1 - M$ will be called the <u>total gravitational mass defect</u>. The origin of $\Delta_1 M$ is obvious: when we unify mass elements $dm = \rho dV$ (which already have a specified density $\rho$) to form a star, we should take into account the energy of gravitational interaction between these elements. This binding energy [which is not taken into account in (9.3), in contrast with (9.1)], and the corresponding mass are negative, therefore $\Delta_1 M > 0$. In the Newtonian approximation $c^2 \Delta_1 M = -\Omega$.

The ratio

$$\alpha_1 = \frac{\Delta_1 M}{M}$$

is called the coefficient of gravitational packing, and characterizes the ratio of the absolute value of the gravitational energy to the total energy. Figure 7 shows the dependence of $\alpha_1$ on $\rho_c$ for stars consisting of a real gas, as given in[16]. $\alpha_1$ is small for small $\rho_c$ and tends to zero as $\rho_c \to 0$. For the densest configurations $\alpha_1 \approx 0.5$.

The difference $\Delta_2 M = M_0 - M = Nm - M$ is called the <u>incomplete mass defect</u> or simply the <u>mass defect</u>. The energy corresponding to $\Delta_2 M$ is precisely the energy that is released when a dense star is formed from the initially rarefied diffuse matter. From the physics of this process it is clear that $\Delta_2 M > 0$ for a stable stationary star.*

In the Newtonian approximation $c^2\Delta_2 M = -(T + \Omega)$. The ratio

$$\alpha_2 = \frac{\Delta_2 M}{M_0}$$

is the ratio of the energy released during star formation to $M_0c^2$. A plot of $\alpha_2(\rho_c)$, calculated for stars made up of a real gas in accordance with the data of[16], is also shown in Fig. 7. For large densities $\alpha_2$ becomes negative. See Sec. 11 on this subject.

The gravitational mass defect is sometimes incorrectly called the gravitational screening. Such a name does not reflect the nature of the phenomenon, which is not at all similar to the effect of a screen. To be sure, when we unite, say, two particles we obtain a system with a mass smaller than the sum of the particle masses but, first, this weakening of gravitation has no direction whatever (as should be the case were the second particle to be a real screen) and, second, any binding force has the same property of reducing the total mass of the particles, and gravitation is no exception in this respect. Indeed, the mass of the deuteron is smaller than the sum of the proton and neutron masses but of course we do not say on this basis that the neutron provides a gravitational screen for the proton.

When particles are united to form a bound system, an energy equal to the mass defect is radiated either in the form of quanta, neutrinos, gravitational waves, etc. A remote observer will notice the mass defect (the decrease in mass) not at the instant when the particles are united, but only after the radiated energy has moved past him (Fig. 8). Until that instant, arbitrary energy transformations will not affect in any manner the star mass as measured by the observer (owing, naturally, to the law of energy conservation).

## 10. STABILITY OF SUPERDENSE STARS

As was indicated in Sec. 2, the criterion for the stability of the equilibrium of a star is $dM/d\rho_c > 0$[5]. However, a proof of this statement is valid only for Newtonian gravitation and is not applicable directly to the relativistic region.

Stable equilibrium denotes a minimum of the star energy for a given entropy and for a given number of particles. Starting from this general premise, we can show that the same criterion is applicable also in the relativistic region for small perturbations.

*We consider here the mass defect only for static configurations. If we forego the static requirements, then the total mass M of a given number of nucleons can in principle be arbitrarily small (see Sec. 16). In particular, M vanishes for a closed cosmological model (see, for example, [46,49]). Especially interesting are the properties of $\Delta M$ for so-called semi-closed worlds [50,51].

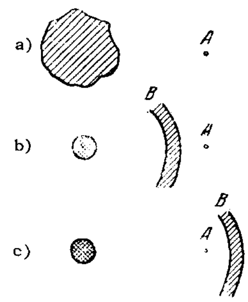


FIG. 8. Variation of the mass of the matter during the formation of a dense star (A – observer). a) Diffuse matter prior to compression into a star. b) The matter has been compressed; the radiated energy (region B) has not yet passed the observer, who does not notice the decrease in the mass of the body. c) Wave has moved past the observer, he notes the decrease in the mass of the body by an amount $\Delta M$.

The concept of the energy of a star will be systematically utilized in Part II of this review. We present here a different proof of the stability criterion[5], without pretending complete mathematical rigor, but satisfactory to physicists.

We consider first the change in the mass of a star in equilibrium when a single particle is added to the star at a radius r from infinity, where its energy was equal to $mc^2$. In other words, we determine dM/dN. When such a particle falls freely in the gravitational field to a radius r, its energy reaches a value*

$$\varepsilon = mc^2 e^{-\frac{\nu(r)}{2}}. \qquad (10.1)$$

The difference $\epsilon(r) - \mu(r)$ (where $\mu(r)$—chemical potential of the particles of the cold star) is radiated, for example in the form of $\gamma$ quanta. Owing to the $\gamma$-quantum energy loss resulting from the gravitational red shift [see (7.3)], an energy

$$\Delta E = (\varepsilon - \mu)\, e^{\nu/2} \qquad (10.2)$$

goes off to infinity. On the other hand, from the equilibrium equation for the cold star it follows that[22,16]

$$\mu(r)\, e^{\frac{\nu(r)}{2}} = \text{const} = mc^2 e^{\frac{\nu(R)}{2}} \qquad (10.3)$$

From (10.1)—(10.3) we obtain

$$\frac{dM}{dN} = me^{\frac{\nu(R)}{2}} = \text{const}.$$

The change in M does not depend on the position where

*The energy is measured by a local observer and does not include the potential energy of the particle in the gravitational field; the total particle energy, of course, does not change during its fall (the radiation of gravitational waves is disregarded, since it is small).

the particle is added to the equilibrium star.* We note that by virtue of (10.2) and of $e^{\nu} < 1$ we always have

$$\frac{dM}{dN} < m. \qquad (10.4)$$

We now return to the question of the stability of the star. We consider the section of the curve $M = M(\rho_c)$ closest to the maximum $M_{max}$. From expression (10.4) it follows that $N(\rho_c)$ has a maximum at the same value $\rho_c = \rho_{crit}$ at which the maximum of $M(\rho_c)$ is attained.

Consequently, on the left and on the right of $\rho_{crit}$ we can choose two different stationary stellar models with different $\rho_{c1}$ and $\rho_{c2}$, but with identical N. Then the solution for one of these models can be represented as a perturbed second solution:

$$\varrho_2(r) = \varrho_1(r) + \delta\varrho. \qquad (10.5)$$

In the most general case the solution for small perturbations is of the form

$$\delta\varrho = \varphi(r) e^{\omega t}. \qquad (10.6)$$

For the particular perturbation $\delta\rho$, which transforms the stationary solution $\rho_1$ into the stationary solution $\rho_2$, naturally, $\delta\rho$ does not depend on the time, i.e., $\omega = 0$.† Thus, at the maximum of the $M(\rho_c)$ curve we have

$$\omega = \omega^2 = 0.$$

Obviously, the case with $\omega^2 = 0$ is on the borderline between $\omega^2 < 0$, where $\omega$ is imaginary, and $\omega^2 > 0$, where $\omega$ is real. The former case corresponds to stability and the latter to instability. We have thus shown that the passage through the maximum corresponds to loss of stability. These considerations are equally applicable to a star model constructed with allowance for general relativity and to the nonrelativistic case. We emphasized that so far we have considered only small perturbations.

## 11. SOLUTIONS WITH NEGATIVE GRAVITATIONAL MASS DEFECT

It was indicated in Sec. 9 that for stable star models the gravitational mass defect is $\Delta_2 M > 0$. However,

*In contradistinction to the erroneous statement by Wheeler [23]. We note that the independence of dM/dN of the place where the particle is added means that if we specify that the particle distribution $\delta n(r)$ is disturbed without changing the total number of particles, i.e., $\delta N = 0$, then in first order we also have $\delta M = 0$, i.e., $\delta M/\delta n|_{n = const} = 0$. This denotes precisely that the equilibrium state corresponds to an extremum of the mass, i.e., an extremum of the total energy of the system (which is perfectly natural). If this extremum is a minimum, then this signifies stability of the state, in accord with the statements made at the beginning of this section.

†The method which we used here was systematically developed in [24]; see also [70].

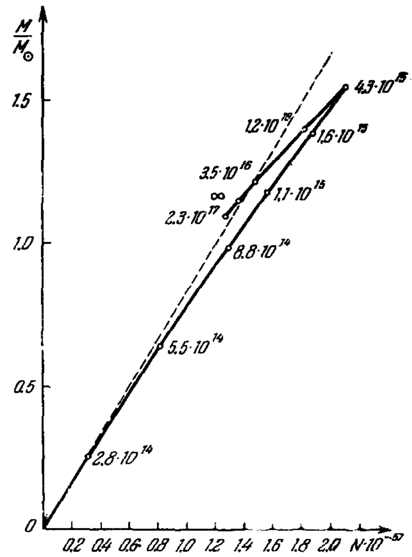


FIG. 9. Dependence of the mass of a cold star on the total number of baryons N. Alongside each circle is indicated the density of the star at the center. Dashed line: $M = Nm_n$.

for unstable configurations corresponding to the descending branch, we cannot make beforehand any definite statements concerning the sign of

$$\Delta_2 M = 4\pi \int_0^R (m n e^{\lambda/2} - \varrho) r^2 \, dr,$$

for, on the one hand, $nm < \rho$ because of the energy of motion and interaction of the nucleons, and on the other hand $e^{\lambda/2} \geq 1$. The sign of $\Delta_2 M$ must be determined from specific calculation of the star models.

Models with negative $\Delta_2 M$ must certainly be unstable; they cannot be realized in nature and are therefore of no particular interest. We dwell on this question only because it is frequently discussed in the literature.

Let us consider the dependence of the star mass M on the number of nucleons in the star. It is clear, first, that this curve passes through zero: $N = 0$, $M = 0$. Furthermore, we have shown in Sec. 10 that $dM/dN < m$. At first glance it might follow that $M < Nm$ and $\Delta_2 M > 0$ always. This is not the case, however.

The $M(\rho_c)$ and $N(\rho_c)$ curves pass through a maximum at the same value of $\rho_{crit}$, while dM/dN is finite everywhere and has no singularities (see Sec. 10). It follows therefore that the dependence of M on N will have a turning point corresponding to the common maximum of M and N. This dependence is shown in Fig. 9, using data from [16], for superdense configurations.* $dM/dN < m$ everywhere on the curve, but there

*In accordance with the remark made in Sec. 8, the $M(\rho_c)$ and $N(\rho_c)$ curves have an unlimited number of maxima at $\rho_c \to \infty$, the amplitudes of M and N attenuate, and the curve $M = M(N)$ has a corresponding number of turning points.

is a section where $Nm > M$ and $\Delta_2 M < 0$. Of course, these configurations are unstable and small perturbations cause the star to compress or expand. When such a star flies apart, the matter will have a nonvanishing kinetic energy at infinity. It is clear that a real star with $\Delta_2 M < 0$ cannot be formed from diffuse matter.

The physical reason for $\Delta_2 M < 0$ consists in the following. At very large density, the energy of motion and repulsion of the baryons is appreciably larger than their rest energy,

$$\varrho > nm.$$

Therefore, in spite of the fact that an account of the negative energy of the gravitational field reduces this difference somewhat, nevertheless

$$\Delta_2 M = 4\pi \int_0^R (mne^{\lambda/2} - \varrho)\, r^2\, dr < 0.$$

It is seen from Fig. 9 that the solutions with $\Delta_2 M < 0$ exist for the equations of state of a real gas.

In Fig. 6, the $M_0$ and M curves for an ideal gas cross, i.e., in this case, too, $\Delta_2 M < 0$ for $\rho_c \gtrsim 5 \times 10^{16}$. We repeat that this circumstance, which is curious in itself, is of no interest in principle.

It is important to note, however, that $\Delta_2 M < 0$ is possible for free particles only in the relativistic theory, not in the Newtonian approximation. Indeed, in Newtonian theory, for a star made up of Maxwell or Fermi gas, the following virial theorem holds true

$$T = -\frac{1}{2}\Omega.$$

On the other hand, in the Newtonian approximation (see Sec. 9) we have

$$c^2 \Delta_2 M = -(T + \Omega).$$

Consequently

$$c^2 \Delta_2 M = -\frac{1}{2}\Omega > 0. \tag{11.1}$$

In a relativistic theory, the virial theorem is no longer applicable, and unstable solutions with $\Delta_2 M < 0$ are already possible.

We emphasize also that all the foregoing holds not only for a gas made up of colliding particles, but also for a gas whose particles do not interact with one another at all except by gravitation.* Indeed, let us take any stationary isothermal solution for a star made up of an ideal gas, and let us "turn off" the particle collisions. Obviously, locally at each point the average distribution of matter, or its energy, remain unchanged, since the collisions cause only an exchange of energy and momenta between the particles. The integral characteristics of the system do not change when the collisions are turned off. Now each particle executes a finite motion in the common gravitational field of all the remaining particles, and its orbit need not be a closed curve. The total energy of each individual particle in the field of the remaining particles (which is conserved during the motion in the orbit) is smaller than $mc^2$, and accordingly the particle cannot go off to infinity.

If the state in question is obtained after "turning off' the collisions from the stationary solution for the cold ideal Fermi gas with $\Delta_2 M < 0$, then, consequently, we shall have $\Delta_2 M < 0$ here, too. Thus, although the total energy of each particle is smaller than $mc^2$, the total energy of the entire system is larger than $Nmc^2$. The possibility of this can be easily understood qualitatively by recalling that in Newtonian theory the energy of one particle is

$$E_i = mc^2 + T_i + \Omega_i.$$

However, the energy of the entire system is

$$E = N\left(mc^2 + \overline{T}_i + \frac{1}{2}\overline{\Omega}_i\right).$$

An essential factor here is the coefficient 1/2 in the third term, which is needed to prevent each pairwise interaction from being taken into account twice. Since $\Omega_i < 0$, we have

$$E = \sum_i E_i - \frac{1}{2}\sum_i \Omega_i > \sum_i E_i.$$

A solution with $E > Nmc^2$ is therefore possible for $E_i < mc^2$. Of course, this necessitates violation of the virial theorem, for this theorem leads to the inequality (11.1).

## 12. EVOLUTION OF A STAR

We can now proceed to an analysis of the evolution of a star. We shall not dwell in detail on all the stages of evolution, referring the reader to the book of S. A. Kaplan[25] and the corresponding monographs[26,27]. We merely make a few remarks that are needed in what follows.

According to generally accepted notions[26], stars are formed from an initially rarefied medium by gravitational condensation of diffuse matter consisting primarily of hydrogen. During the contraction phase, the light of the star is produced at the expense of gravitational energy. This source of stellar energy was already pointed out by Kelvin. The temperatures are still low and the release of nuclear energy is negligibly small. The star is in hydrostatic equilibrium without internal energy sources. The duration of this phase is relatively short and amounts to

$$\tau \approx 5 \cdot 10^7 \left(\frac{M_\odot}{M}\right)^2 \text{ years}.$$

Since the star has negative specific heat, the radiation of energy and the contraction cause an increase in the temperature which rises in the interior of the

*The foregoing is true when the distribution function depends only on $Ee^{\nu/2}$, where E – energy. For an additional exposition see the papers of Finlay-Freundlich [56], cited in the review [57].

star enough to initiate the nuclear fussion of hydrogen into helium. V. A. Ambartsumyan[52] and his coworkers consider a different possible evolutional path prior to the start of the nuclear reactions, namely the formation of protostars not from diffuse matter, but from superdense D-bodies, the nature of which is still unknown. This does not change the subsequent evolution of the star, which is determined by the mass, initial entropy, and initial chemical composition, and also by the angular momentum in the case of a spinning star.* During the start of the nuclear reactions the star is in the state of hydrodynamic and thermal equilibrium. This is the most prolonged period of the active life of the star, the duration of which is determined by the reserves of hydrogen in the core, where the temperature is sufficiently high for the nuclear reactions, and by the rate of conversion of hydrogen into helium. It is obvious that this period $\tau$ is proportional to M/L, where L is the luminosity of the star. Calculations yield for the mass of the core, where the hydrogen is burned up, an order of magnitude 0.1M, from which it follows that

$$\tau = 1.1 \cdot 10^{10} \frac{L_\odot}{L} \frac{M}{M_\odot} \text{ years} \tag{12.1}$$

The inhomogeneity of the chemical composition, due to the burning up of the hydrogen at the center, leads to a change in the structure of the star, its outer envelope expands, while the core contracts.

In sufficiently massive stars ($M > M_\odot$ and $M \sim M_\odot$) the temperature in the core rises to such an extent that triple $\alpha$-particle collisions take place to produce $C^{12}$

$$3He^4 \to C^{12} + \gamma. \tag{12.2}$$

The subsequent fate of the star is very difficult to calculate theoretically. It is possible that the star experiences many radical changes, accompanied by the ejection of part of the outer shells; it is highly improbable, however, that the star would lose the bulk of its mass as a result of such ejection. In one way or another, the general direction of the stellar evolution proceeds along the line indicated at the end of Sec. 2, namely the exhaustion of the reserve of nuclear fuel leads to contraction of the star and to its heating. On the M vs. $\rho$ diagram (Fig. 10), the equilibrium states of the star correspond to a slow displacement from left to right on a horizontal line (M = const). Let us consider first the evolution of stars with $M < 1.2\, M_\odot$ ($M_1$ in Fig. 10). The contraction will continue until the electrons in the main mass of the gas become degenerate. Then the contraction practically terminates, since the pressure depends little on the temperature, and during the entire subsequent evolution P will decrease by approximately 1/2, with the star reaching

*In connection with the possibility of this second manner of star formation, see [60] concerning piezonuclear reactions in cold hydrogen.

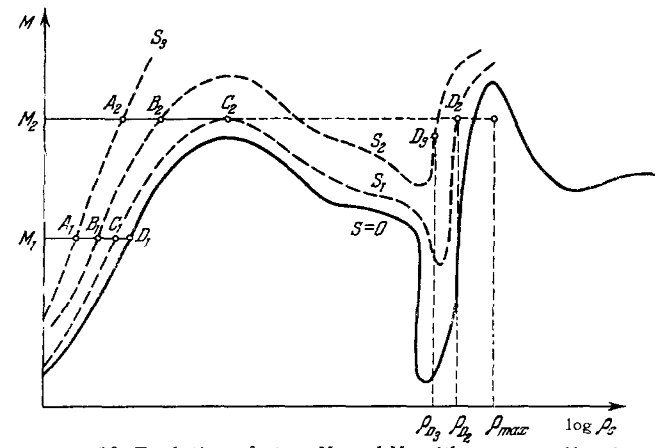


FIG. 10. Evolution of stars $M_1$ and $M_2$ with mass smaller than the "OV" limit.

the point $D_1$. Prior to the start of the degeneracy, the temperature of the star increases upon compression, its specific heat being negative. After electron degeneracy sets in and the compression stops, the star radiates, becomes cooler, and its temperature drops after going through a maximum. The specific heat of the star is then already positive. The maximum of the temperature corresponds in order of magnitude to the degeneracy energy of the electrons in the final state. For stars with mass equal to $M_\odot$, the maximum $T_c$ amounts to $\sim 10^9$ deg K. Subsequently, on cooling, all the thermonuclear reactions with effective energy release cease and the electron degeneracy becomes even stronger. This last stage in the life of a star is called the white-dwarf stage. White dwarfs cool slowly, their radiation being due principally to the thermal energy of the atomic nuclei that are still in the nondegenerate state. The cooling process lasts for billions of years[27]. We see that during the entire evolution the star moves gradually on the M vs. $\rho$ diagram from left to right and tends to the curve corresponding to T = 0 (S = 0).

The final chemical composition of white dwarfs depends on those nuclear reactions which still took place during the stage of their compression and heating, and the possibility of any particular reaction depends in turn on the temperature. In all stars with $M \gtrsim 0.3\, M_\odot$, the temperatures attained during the process of evolution were certainly much higher than $T = 10^7$ deg K, at which the nuclear transformation of H into $He^4$ begins. Let us calculate the maximum temperature which is attained in a star with mass M. We have already said that this is the temperature at which the electron gas becomes degenerate: $T_{max} \approx$ T of degeneracy $\sim \rho^{2/3}$. Using (3.1) and (4.1) we can easily find that $T_{max} \sim \mu^2 M^{4/3}$. Numerical calculations yield for $T_{max}$ in the center of the star[28]

$$\lg T_c = 8.6 + 2 \lg \mu + \frac{4}{3} \lg \frac{M}{M_\odot}, \tag{12.3*}$$

*lg = log.

where $\mu$ — molecular weight. According to an estimate by Opick[28], a temperature close to maximum is retained for $\sim 10^{14}$ sec. We can estimate from this the change in the chemical composition of the star, due to the nuclear helium transformation reaction. If $T > \sim 3 \times 10^8$, then the triple $\alpha$-particle collision process (12.2) leads to the formation of $C^{12}$, but this is not the end of the reaction:

$$\begin{aligned} &C^{12} + He^4 \rightarrow O^{16} + \gamma, \\ &O^{16} + He^4 \rightarrow Ne^{20} + \gamma, \\ &Ne^{20} + He^4 \rightarrow Mg^{24} + \gamma, \\ &Mg^{24} + He^4 \rightarrow Si^{28} + \gamma. \end{aligned} \quad (12.4)$$

According to Opick's calculations white dwarfs with $M > 0.5\ M_\odot$ should consist principally of $Mg^{24}$, and heavier nuclei are not formed because all the helium is exhausted. In stars with mass about (0.4—0.45) $M_\odot$, an appreciable fraction of the helium in the central parts still experiences a similar transformation, but when $M < 0.4\ M_\odot$ the white dwarf should consist essentially of helium.*

Let us trace now the final stages of the evolution of a star with $1.6M_\odot \gtrsim M \gtrsim 1.2M_\odot$. During the course of the decrease in the entropy such a star also moves slowly from left to right via quasi-equilibrium states from $A_2$ to $C_2$ along the line $M_2$ of Fig. 10. After reaching the critical point $C_2$, which is the maximum of the isentrope $M(\rho)$, stability is lost and a catastrophic contraction takes place at a rate on the order of the speed of free fall $dR/dt \sim [GM(R_c - R)/R^2]^{1/2}$. It must be especially emphasized that after reaching the state $C_2$ the speed of further contraction does not depend in any way on the rates of those processes that have brought the star to the critical state during the course of the slow evolution. This is in full accord with the fact that the speed with which a man falls after jumping from a roof does not depend at all on the speed with which he approached the edge of the roof.

We must point out here the following important circumstance. In considering the occurrence of instability of a star and its hydrodynamic contraction, we assume that the processes causing the instability, i.e., making $\gamma < 4/3$, occur within a time much shorter than the hydrodynamic time $t_H$. With respect to such processes, the matter is at all times in the state of equilibrium. These processes are almost adiabatic and have consequently constant entropy. An example of such a process is pair production at high temperature. It is not excluded, however, that neutronization of matter, which we have shown to cause instability at low temperatures, occurs within a time comparable with $t_H$. The neutronization will then lag the equilibrium under these conditions. This is equivalent to the appearance of a large effective viscosity, which leads to an increase in the entropy. To make the picture more detailed it is necessary to consider simultaneously both the hydrodynamics and the neutronization process.

In addition, we have crudely approximated the matter of the star by its average density and average pressure. It is clear that this cannot account for the appearance, during the course of contraction, of shock waves that cause an increase in the entropy with "hydrodynamic" velocity. It is likewise impossible to account for the loss of part of the mass, probably due principally to the emergence of the shock wave to the surface of the star. An account of these phenomena calls for a concrete calculation of the nonstationary processes in the star[71].

The result of all the foregoing phenomena is as follows. After the "collapse" of the star at the point $C_2$, the density increases (horizontal dashed line on Fig. 10) and the star reaches a new stable state (point $D_2$), but the contraction continues by inertia until a certain maximum density is reached ($\rho_{max}$). In the first approximation, damped oscillations about $D_2$ will set in. The damping is caused by the already-described processes of entropy growth; in addition, the star throws off some of its mass. Consequently the star shifts from the isentrope corresponding to the instant of the collapse, to a higher isentrope (from $S_1$ to $S_2$). The horizontal line drops because of the loss in mass, and the star arrives at the point $D_3$ in a state of equilibrium.

This state of the star is usually called the neutron state. This of necessity raises the question: how can this be, when we started the evolution of the star with hydrogen, with protons? The thermonuclear reactions caused the protons to combine into complex nuclei and a tremendous energy, on the order of 0.01 $Mc^2$, was radiated, and at the end of evolution, during the neutron-star stage, we have again a substance consisting of individual baryons (with the neutron rest mass being even larger than the mass of H); where did the energy radiated during the evolution come from? The answer is obvious: gravitation produced a high density, and this has led to neutronization of the matter and caused

*The time of evolution of stars with $M < 0.3\ M_\odot$ exceeds, according to (12.1), the age of the metagalaxy. The smaller M, the smaller the maximum possible temperature. Therefore in stars with sufficiently small M the temperature will ensure in the future the occurrence of nuclear reactions that stop only with $He^3$. It might appear that these stars should consist of $He^3$ at the end of the evolution (at S = 0). However, as recently noted by Parker, Bahcall, and Fowler [58], the neutronization potential of $He^3$ is very low and amounts to only 18 keV. Therefore during the course of the stellar evolution, the $He^3$ will be transformed into tritium, and the latter will be converted into $He^4$ via the usual thermal reaction. Thus, these stars will consist of $He^4$ at the end of the evolution. However, at a mass $M < 0.1\ M_\odot$, the degeneracy occurs before the temperature rises sufficiently for the nuclear reactions to take place. Such stars produce light at the expense of gravitational energy and will consist primarily of hydrogen at the end of evolution. It is curious that a large quantity of $He^3$ was observed in the spectrum of one of the stars [61].

Table II.

| Temperature of core $T_e$, °K | Effective surface temperature $T_e$, °K | Neutron luminosity $L_\nu$ erg/sec | Photon luminosity $L_{ph}$, erg/sec | Effective cooling time $\tau$, years |
|---|---|---|---|---|
| $2 \cdot 10^9$ | $1.3 \cdot 10^7$ | $4 \cdot 10^{39}$ | $2.3 \cdot 10^{37}$ | 10 |
| $1 \cdot 10^9$ | $1 \cdot 10^7$ | $5 \cdot 10^{36}$ | $8 \cdot 10^{36}$ | $10^3$ |
| $5 \cdot 10^8$ | $8 \cdot 10^6$ | $8 \cdot 10^{33}$ | $3 \cdot 10^{36}$ | $10^3$ |
| $2 \cdot 10^8$ | $4.5 \cdot 10^6$ | $1 \cdot 10^{29}$ | $3 \cdot 10^{35}$ | $1.5 \cdot 10^3$ |

the complex nuclei to break up into individual baryons; consequently, the gravitational energy in the final state compensates exactly for the energy radiated by the star into the surrounding space.

The ideas of Kelvin and Helmholz, that the light from the stars is due to gravitational energy, do not hold for the prolonged active stage of the star when its light is due to nuclear reactions. However, in the concluding phase of the stellar evolution, the gravitational energy breaks the nuclear bonds, and in final analysis gravitation is responsible for the entire radiated energy; in this sense the ideas of these great early physicists are correct.

## 13. NEUTRON STARS

The neutron star in position $D_3$ is in a quasi-equilibrium state. The neutron core of the star is surrounded by an envelope consisting of nuclei and degenerate electrons, and the surface layers themselves consist of ordinary plasma. Regardless of the processes that have brought the star to the position $D_3$, the temperature of its interior, as indicated by Chiu[4], cannot exceed several billion degrees, for otherwise the intense production of $(\nu, \bar{\nu})$ pairs which instantaneously leave the star would anyway cool the star.* In the entire interior of the star the thermal conductivity is exceedingly high, since it is determined by the degenerate electrons. Therefore the core of the neutron star is isothermal and a temperature gradient exists only in the outermost envelope. The temperature of the interior is not higher than several billion degrees. The structure of the outer envelope can be easily calculated, since the radius and the mass of the neutron configuration are already known, and a definite interior temperature can be specified. We can then determine the surface temperature and the luminosity of the star. We present the results of Chiu and Salpeter[65] for a neutron star with $M = 0.5\ M_\odot$ and radius $R = 10$ km (Table II).

In spite of the fact that the luminosity of neutron stars is thousands of times larger than the luminosity of the sun, the former are invisible in ordinary telescopes, as noted by A. A. Ambartsumyan and G. S. Saakyan[10]. Indeed, as seen from Table II, their surface temperatures $T_e$ amount to tens of millions of degrees, and the bulk of their energy is radiated in the form of soft x-rays. At a surface temperature $\sim 1.2 \times 10^7$ deg K, the maximum of energy distribution at the center, as a function of $I_\nu d\nu$, lies in the region $\sim 4$ Å. In the optical range, the radiation is negligible and amounts to a millionth of the luminosity of the sun in this range. This, of course, is connected with the negligible surface of the neutron star, which is $2 \times 10^{-10}$ of the surface of the sun.

Are there any neutron stars in the galaxy? Research outside the atmosphere, carried out during the last two years[30] (see also[63,64]) has disclosed several x-ray sources. According to[30], one of these sources is in the direction of the Crab nebula. This nebula is the remnant of a supernova explosion. No optical object is seen at all at the location of another source. We note that it was previously assumed that the weak star of 15th magnitude at the center of the Crab nebula is the remnant of an old supernova explosion. Recent observations by Kraft have shown that *this star cannot be the remnant of a supernova, since* it has the usual spectrum. There are no other stars brighter than 18th magnitude in the center of the nebula.

The x-ray flux from the Crab nebula is $2 \times 10^{-9}$ erg-sec$^{-1}$ cm$^{-2}$ Å$^{-1}$ for $\lambda \approx 5$ Å[30].

If we assume, following Morton[29] or Chiu and Salpeter[65], that the source of radiation is a neutron star, we can easily calculate its effective temperature, since the distance to the nebula is known (1100 psec), and the mass and the radius of the neutron star cannot vary strongly. For $R = 9.25$ km we obtain $T_e = 7.6 \times 10^6$ deg K.

According to the very latest published reports Friedman, using the occultation of the Crab nebula by the moon, found that the dimensions of the x-ray source are $\sim 1/5$ the dimensions of the nebula, i.e., approximately 0.02 psec $= 6 \times 10^{16}$ cm. If this is so, then the visible source is obviously not a neutron star. This makes more probable the hypothesis of V. L. Ginzburg[62], that the x-ray emission has a synchrotron nature. Obviously, the situation is more complicated than was presented by Morton, Chiu, and Salpeter. Some considerations on this subject will be made in Part II of this review.

*The main processes which lead to neutrino production at high temperatures is electron-positron pair annihilation: $e^+ + e^- \to \nu + \bar{\nu}$.

Pontecorvo [72] was the first to note that modern theory of $\beta$ decay leads to the possibility of radiation even without baryon participation (for a review see [73]).

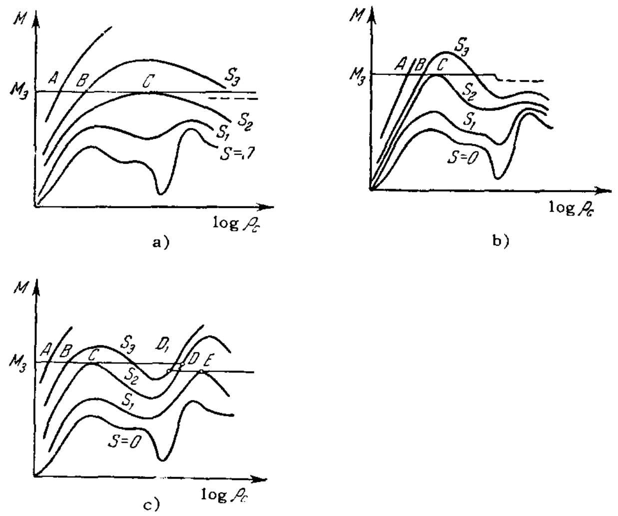


FIG. 11. Different schemes of evolution of a star with $N_3 > M^{OV}$.

According to Table II, neutron stars cool very rapidly and cease to radiate energy. However, this occurs only if the star is in vacuum. The neutron star is a deep potential gravitational well, causing the accretion of matter (i.e., the falling of matter on the star). The gravitational potential at the surface of the star amounts to $\approx 0.2c^2$. The foreign matter falling in this field is accelerated to approximately half the speed of light. Colliding with the surface, the falling matter releases an energy on the order of $0.2mc^2$, where m is the falling mass. As will be shown in the second part of the review, the rate of mass growth during accretion has an order of magnitude

$$\frac{dM}{dt} \approx (2 - 4)\frac{\pi G^2 M^2 \varrho_\infty}{v^3},$$

where $\rho_\infty$—density of the surrounding matter, v—velocity of motion of the matter particles relative to the star at infinity. If we assume that $\rho_\infty = 10^{-24}$ g/cm$^3$ and $v \approx 10^6$ cm/sec, then $dM/dt \approx 2 \times 10^{12}$ g/sec and the luminosity of the neutron star due to accretion is $L \approx 4 \times 10^{32}$ erg/sec. The effective surface temperature is in this case somewhat less than a million degrees, and the Wien-law maximum lies at $\lambda \approx 50$ Å. The time of appreciable increase in the mass of the star is $t \approx 10^{21}$ sec, which is much longer than the age of the metagalaxy.

We see thus that as a result of accretion of interstellar matter the neutron star can exist for a practically unlimited time as a source of soft x-ray emission of luminosity comparable with that of the sun. However, the luminosity due to accretion is many orders of magnitude smaller than the luminosity of a neutron star immediately after its formation (see Table II). Of course, if the star is immersed in a cloud which is much denser than the interstellar diffuse matter (and this should take place if the neutron star is produced immediately after a supernova explosion), its luminosity increases sharply. We shall not deal here with the interaction between the falling matter and the radiation.

## 14. EVOLUTION OF A STAR WITH A MASS LARGER THAN THE "OV" LIMIT

We now consider the last stages of evolution of a star with mass larger than the "OV" limit for superdense configurations ($M \gtrsim 1.6\ M_\odot$). The qualitative difference between this case and the preceding ones lies in the fact that for such large masses there is no equilibrium configuration with S = 0 (and T = 0). This means that without loss of an appreciable part of the mass the cooling massive star cannot reach an equilibrium state. On the other hand, as emphasized by Hoyle, Fowler, and Burbidge[32], there is no reason whatever to cause the mass to eject precisely as much mass as is needed to arrive ultimately at the state of equilibrium with S = 0. Consequently, the concluding part of the evolution of such stars will be essentially nonstationary.

Let us trace the last stages of this evolution (Fig. 11). The star approaches the critical point C slowly, via quasi-equilibrium states. At C the star loses stability and contracts with hydrodynamic velocity. Its further evolution depends on the course of the isentropes on the ρ-M diagram in this region. As already noted in Secs. 3 and 8, when S is small the isentropes qualitatively duplicate the S = 0 curve. For larger S, however, the shape of the isentrope can be essentially different, and there are still no calculations of the equation of state in this region. We shall therefore consider qualitatively the different possibilities represented in Figs. 11a–c. We emphasize that at large densities, owing to relativistic effects, we must have $dM/d\rho < 0$ for any S = const.

In the case shown in Fig. 11a, the horizontal line M = const is tangent to the point C of an isentrope with a single maximum. Then after the slump at the point C, the star will contract without limit, and after a time on the order of $t_H$ it will be contracted to such an extent that the gravitational potential in its surface assumes a value on the order of $c^2$, and effects of the general theory of relativity come into play. Starting with this instant, the star enters the phase of relativistic contraction—collapse. We shall discuss this relativistic stage later (see Sec. 15).

Qualitatively the same will occur if the isentropes have the form shown in Fig. 11b. Here the second maximum lies much lower than the first, and neither the growth of entropy during the contraction process (the equilibrium isentrope corresponding to increase in entropy is no longer $S_2$ but $S_3$ in Fig. 11b), nor the ejection of matter (transition to the dashed horizontal line) will stabilize the star, which will continue to contract without limit. The evolution will proceed differently in the case shown in Fig. 11c. Here the second

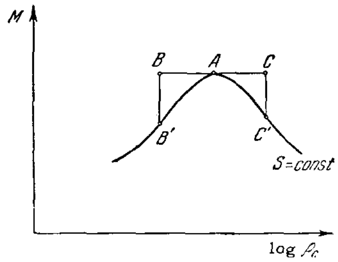


FIG. 12. Star at the point of "slump" A. The deviation from equilibrium to the left (point B) causes an appearance of a force which returns the star to A. A deviation to the right (C) causes a "slump" – hydrodynamic compression of the star.

maximum is higher than the first, and after the slump at the point C, the star will evolve in exactly the same way as in the case leading to formation of a neutron star (Sec. 12). The star will reach a quasi-equilibrium state at the point $D_1$. However, whereas in the case of a neutron star this is the final state of the evolution (the star reaches S = 0 and T = 0), in our case, after reaching $D_1$, a second stage of quasi-equilibrium evolution begins. The star, radiating energy, will again move slowly via quasi-equilibrium states from left to right, from isentrope to isentrope, until the second maximum E is reached, after which the star will again slump and will now be contracted without limit, going over into the stage of relativistic collapse.

We see that the final fate of all stars with $M > M_{max}^{OV}$ is relativistic collapse, which we now proceed to consider.

## 15. RELATIVISTIC COLLAPSE

Let the star be situated at the "slump" point A at the maximum of the isentrope on the $M$-$\rho_c$ diagram (Fig. 12). The perturbations that shift the star to the right or to the left (points B and C) cause it to lose equilibrium. In either case, the equilibrium configurations corresponding to the perturbed density lie lower, and have a lower mass (points B′ and C′). This means that the force of gravitation at B and C exceeds the force of pressure and causes the star to contract and to increase in density. However, whereas the contraction returns the star from position B to the equilibrium A, from position C it takes the star even farther away from equilibrium, and collapse begins. The rate at which the collapse begins at the point A itself is determined by the rate of slow evolution of the star, i.e., the rate with which it arrives at the critical point A and, continuing its motion, passes through this point to the right. However, after some noticeable deviation from the state of equilibrium A, the forces of gravitation exceed the pressure forces already by a finite amount, and the compression acceleration constitutes a finite fraction of the acceleration of free fall. Thus, very soon after the "slump," the star contracts at practically free-fall acceleration, and the pressure forces do not play an appreciable role in the dynamics of the collapse. These considerations are confirmed by a concrete calculation made by M. A. Podurets[33], of the relativistic contraction of a star with counterpressure.

It follows from the foregoing that in the crudest approximation, in the analysis of the dynamics of the collapse of the star as a whole, we can neglect the effects of pressure and put P = 0. Such a collapse was considered in the classical paper by Oppenheimer and Snyder[3] and by Tolman[34].

We shall see below that the qualitative peculiarities of the collapse do not depend on the rate of contraction and are perfectly the same for a speed that amounts to 10% as well as 99% of the speed of free fall.

Let us consider the surface of a collapsing star. During the course of contraction the mass M does not change, and therefore at P = 0 the particle on the surface simply falls under the influence of the gravitation of the mass M. Consequently, in order to explain the character of the collapse, it is sufficient to consider the free fall of a trial particle in the field of the mass M.

For the rate of free fall in a Schwarzschild field (see, for example[35]), we have

$$\frac{dr}{dt}=\left(1-\frac{r_g}{r}\right)\left[1-\frac{1-\frac{r_g}{r}}{1-\frac{r_g}{r_0}}\right]^{1/2}c. \tag{15.1}$$

Here $r_g$—gravitational radius of the central mass, $r_0$—distance from which the fall begins and at which $dr/dt = 0$. At a large distance ($r_0$ and $r \gg r_g$), formula (15.1) goes over into the usual expression of Newtonian theory

$$\frac{dr}{dt}=\sqrt{\frac{2GM}{r}(r_0-r)}.$$

Expression (15.1) shows the rate of change of the coordinate r as measured with the clock of a remote observer. The local stationary observer, situated alongside the falling body, will measure its velocity as follows [see Sec. 5, formulas (5.2a) and (5.2c)]

$$\frac{dx}{d\tau}=\frac{dr}{dt}\frac{1}{1-\frac{r_g}{r}}=\left[1-\frac{1-\frac{r_g}{r}}{1-\frac{r_g}{r_0}}\right]^{1/2}c. \tag{15.2}$$

As the gravitational radius is approached, $dx/d\tau \to c$. The change of $dx/dt$, as measured by the clock t of the remote observer, is quite different. Using (15.1), we get

$$\frac{dx}{dt}=\frac{1}{\left(1-\frac{r_g}{r}\right)^{1/2}}\frac{dr}{dt}\to 0$$

as $r \to r_g$. Of course, the tendency of the velocity $dx/dt$ to zero is due to the slowing down of time near $r_g$. The velocity $v = dx/d\tau$ is a quantity that has a direct physical meaning. It is measured by a co-moving observer.

This is precisely the quantity which enters into the expression for the local energy of the particle according to the formula $E = mc^2/\sqrt{1-(v/c)^2}$ etc. Of course, this velocity increases continuously under the influence of gravitation when the particle falls. The velocity dx/dt, which is expressed in terms of the time of the remote observer, does not have such a direct meaning. Away from the gravitating mass $dx/dt = dx/d\tau = dr/dt$, and dx/dt increases for a falling particle, but near the mass dx/dt decreases and, as shown above, tends to zero as $r \to r_g$. This decrease, however, is not due to "repulsion on the part of the central body," as is stated incorrectly by McVittie ([36], p. 136), but to the connection indicated above between the times $\tau$ and t.

The integral

$$\Delta t = \int_{r_0}^{r} \left(\frac{dr}{dt}\right)^{-1} dr \tag{15.3}$$

diverges at the upper limit if $r = r_g$. Thus, the time t that the particle falls to $r_g$ is always infinite. The value of $\Delta t$ corresponding to attaining $r_g$ is infinite, even for light, the propagation time of which from $r_0$ to r is determined by integration of (7.1) and is equal to

$$\Delta t = \frac{r_0 - r}{c} + \frac{r_g}{c} \ln \frac{r_0 - r_g}{r - r_g}, \tag{15.4}$$

and nothing can move faster than light.

Thus, as measured with the clock of the remote stationary observer, the time to reach $r_g$ is always infinite. Any body, no matter what the acting force, can approach $r_g$ only asymptotically.

What is the time of fall measured by a clock mounted on the falling particle itself. Let us tie the reference frame to that particle. In this system, the clock does not change position, and for it therefore $ds = cdT$, where T is the reading of the clock. Hence $\Delta T = (1/c) \int ds$. But ds is an invariant quantity, which does not change on going over to a different system, and it can be calculated in the Schwarzschild system:

$$\Delta T = \frac{1}{c} \int_{r_0}^{r} \sqrt{\frac{\left(1 - \frac{r_g}{r}\right)c}{\frac{dr}{dt}} - \frac{1}{1 - \frac{r_g}{r}}}\, dr. \tag{15.5}$$

Using for dr/dt expression (15.1) we see that (15.5) converges for any upper limit, including $r = r_g$. In particular, if the particle falls with parabolic velocity (i.e., $dr/dt = 0$ at infinity), then

$$\Delta T = \text{const} \cdot (r_1 - r)^{3/2} \tag{15.6}$$

which coincides with the formula of Newtonian theory. Here $r_1$—position of the particle at the instant of the start of measuring $\Delta T$.

Thus, the time to fall to $r_g$ as measured by the clock on the particle is finite. A time which is infinite when measured by an external observer is finite when measured by the falling observer. What can be a clearer illustration of the relativity of the concept of infinite time.

It remains for us to make one more explanation. With the aid of expression (15.3) we find $r = r(t)$, i.e., the position of the trial particle at the instant t as measured by the clock of the remote observer. But this, of course, is not the same place where this observer sees the particle at the instant t, since it takes the light some time $\Delta_* t$ to cover the path from the particle to the observer. This time can be readily calculated by formula (15.4). Denoting the time of arrival of the light at the observer by $t_*$:

$$t_* = t + \Delta_* t. \tag{15.7}$$

When the particle approaches the gravitational radius $t \to \infty$ and $\Delta_* t \to \infty$, so that $t_*$ certainly tends to infinity. Thus, the observer sees that the particle approaches the gravitational radius only asymptotically in an infinite time. With the aid of the expressions given above we can easily obtain the formula $r = r(t_*)$ for the falling particle, i.e., the law governing the manner in which the observer sees the approach of the particle to the gravitational radius. For $r_g \to r$, the asymptotic form of the formula is:

$$r = r_g + (r_1 - r_g)\, e^{-\frac{c(t_* - t_*^1)}{2r_g}}. \tag{15.8}$$

Here $r_1$—position of the particle at the instant $t_*^1$ $(r_1 - r_g \ll r_g)$.

Let us see now what will be the change in brightness of an emitter falling in a Schwarzschild field, as seen by an external observer. Assume that at some instant the falling source is located near $r_g$ and moves with local velocity $dx/d\tau = v$ along a radius joining the central body with the remote observer A, and for a comoving observer, which falls together with the source, the latter radiates isotropically with constant intensity. Then the flux density at infinity $I_\infty$ will be for the observer A

$$I_\infty = \text{const}\left(1 - \frac{r_g}{r}\right)^2 \left[\frac{1 - \frac{v^2}{c^2}}{\left(1 + \frac{v}{c}\right)^2}\right]^2. \tag{15.9}$$

Here one factor $(1 - r_g/r)$ describes the gravitational red shift, the other factor $(1 - r_g/r)$ is connected with the bending of the ray trajectories in the gravitational field, while one factor $[(1 - v^2/c^2)/(1 + v/c)^2]$ is connected with the Doppler effect, and the other with the aberration. From (15.2) it follows that

$$1 - \frac{v^2}{c^2} = 1 - \frac{r_g}{r} \frac{r_0 - r}{r_0 - r_g},$$

and as $r \to r_g$

$$I_\infty = \text{const} \cdot \left(1 - \frac{r_g}{r}\right)^4. \tag{15.10}$$

The law governing the variation of r with $t_*$ has already been given in (15.8). We thus obtain an expression that shows how a remote observer sees the change in the brightness of a falling source as $r \to r_g$:

$$I_\infty = \text{const} \cdot e^{-\frac{2c}{r_g}(t_* - t_*^1)}. \tag{15.11}$$

Table III. Characteristic time of attenuation of luminosity for objects of different masses

| $M/M_\odot$ | $R_g$, km | $t = R_g/c$, sec |
|---|---|---|
| 1.6 | 4.8 | $1.6\cdot10^{-5}$ |
| 10 | 30 | $10^{-4}$ |
| $10^2$ | $3\cdot10^2$ | $10^{-3}$ |
| $10^5$ | $3\cdot10^5$ | 1 |
| $10^8$ | $3\cdot10^8$ | $10^3$ |
| $10^{11}$ | $3\cdot10^{11}$ | $10^6$ |

The frequency of the light received by the observer tends to zero in accordance with an analogous expression, but the argument of the exponential is one quarter as large in absolute value.

Our calculations of the change in the brightness and the wavelength pertain only to the central point of the visible disc of a contracting star. For the entire disc, the deductions are much more complicated, since it is necessary to consider rays moving at a large angle to the radius, and the paths of such rays near the star are quite complicated. An analysis of this question[37] shows that the formula for the luminosity of the entire star L is similar to (15.11), but with a somewhat different argument of the exponential:

$$L = \mathrm{const}\cdot e^{-\frac{4c}{3\sqrt{3}R_g}(t_*-t_*^1)}, \qquad (15.12)$$

where $R_g$—gravitational radius of the star.

We have considered light sources located on the surface. It is clear that neutrino sources, for example, will be situated in the center of the contracting star. But in this case the radiation is determined by a formula of the type of (15.12) (see[53]).

We can now make the following conclusions. A remote observer sees a catastrophically collapsing star, whose dimensions are still much larger than $R_g$, contracting with hydrodynamic velocity, i.e., very fast. When $R - R_g \sim R_g$, the star itself continues to contract strongly reaching $R_g$ after a finite proper time, and continuing to contract. Owing to the effects analyzed above, the contraction seen by an external observer is strongly slowed down and its radius tends to $R_g$ in accordance with (15.8). The average density of the star tends in this case to

$$\varrho_{\max} = 2\cdot10^{16}\left(\frac{M_\odot}{M}\right)^2 \frac{\mathrm{g}}{\mathrm{cm}^3}. \qquad (15.13)$$

The luminosity of the star decreases rapidly, in spite of the fact that near the instant when $R \approx R_g$ the photons continue to be created in the star at almost the same rate (actually, even at an increased rate). Owing to the gravitational red shift and other effects listed following formula (15.9), the luminosity decreases in accordance with (15.12). The characteristic attenuation time is on the order of $R_g/c$. This time is listed in Table III for objects having different masses.

Thus, seen by the external observer, the star stops to radiate almost instantaneously. He will never know what happened to the star when its radius became smaller than the gravitational radius.

This phenomenon is called gravitational self-closing. No further radiation leaves the star.*

However, the star of course does not "disappear" without trace from our world. During the collapse neither its mass M nor its static gravitational field changes. Such an "extinguished" star interacts with the surrounding bodies by its own gravitational field (which is exceedingly strong near its gravitational radius!).†

We have found the final state of a star with a mass larger than critical, $M > M_{\max}^{OV}$. This state, which is catastrophically nonstationary for the star itself, is asymptotically "stationary" to the external observer in the sense indicated above.

Thus we have resolved the "paradox of large masses" (i.e., the conclusion that a large mass must unavoidably be catastrophically contracted), resulting from the work of Oppenheimer and his co-workers[2,3] and discussed in the literature (see the paper of Wheeler[14] and the review of Chiu[4]). At first glance this paradox is very unpleasant. Indeed, a cooling star with mass $M > M_{\max}^{OV}$ contracts without any limit whatever! What next? Wheeler regards this question as unsolved and he suggests[14] that in a large mass the "excess" nucleons annihilate upon compression, are converted into radiation that leaves the star, so that the remaining mass is always less than critical. This assumption denies a fundamental law of physics—the law of conservation of baryon charge, and for large masses the critical density at which this process should take place is quite moderate. For example, for $M = 10^8\, M_\odot$,‡ we have in accordance with (15.13) $\rho_{cr} = 2\ \mathrm{g/cm}^3$. The temperatures that are attained upon contraction to critical dimensions are also low (see the second part of the review). Under these conditions, which are in no way remarkable, certainly nothing fantastic can take place. The only thing that is unusually large is the gravitational field, but according to the principle of equivalence the gravitational field itself does not produce local changes in the laws that govern the physical processes.

From our point of view there is no paradox whatever. For an external observer the collapse "stops" at $R \to R_g$, and there is no need for fabricating fantastic

*Self-closing brings about a situation wherein the mass of the collapsing star can no longer decrease strongly by radiation of energy [53,32], and the greater part of the gravitational energy is not radiated in the form of light or particles, but is transformed into the kinetic energy of the contracting body.

†We recall that so far we have been talking of nonrotating stars. The role of rotation will be analyzed in the second part of the review.

‡There are grounds for assuming that the so-called quasars have similar masses and are in the stage of relativistic collapse. This will be treated in the second part of the review.

violations of reliably established fundamental laws of physics.

## 16. METASTABILITY OF ANY EQUILIBRIUM STATE

We shall assume for brevity that cold stars consist of an ideal Fermi gas. In Secs. 2 and 10 we considered the stability of the star against small perturbations. It was shown that when the criterion $dM/d\rho_c > 0$ is satisfied the configuration has a minimum energy and is stable against small perturbations.

For stars of cold Fermi gas, having $N < 0.75\ N_\odot$ nucleons, there always exists one or several static solutions. One of these solutions has the smallest total energy. It is stable against small perturbations. Static solutions with the same N, but with smaller total energy (smaller M) do not exist. Does this mean that we must deduce that the nucleons can never be regrouped (without changing their number) in a configuration (which certainly is not static) having a total energy (meaning also a mass M) smaller than the initial energy? We shall show below that such a conclusion would be incorrect, and that the minimum energy corresponding to the stationary state is only a local minimum.

By applying external pressure to the mass we can, in principle, bring its dimension so close to the gravitational radius* that the gravitational forces (which then tend to infinity) exceed the pressure forces (which increase in proportion to $\rho$), thus causing it to contract further independently—to collapse.

This might lead to the conclusion that collapse of a small mass, although possible, is separated from the equilibrium state by a giant energy barrier.

We shall show that this conclusion is also incorrect, and that the energy barrier in this case is infinitesimally small.

We start with the proof of the latter statement.

The smaller the initial mass, the less energy is needed to collapse it, in spite of the fact that the density to which it is necessary to compress the matter beforehand increases with decreasing mass: $\rho = 2 \times 10^{16} \times (M_\odot/M)^2$. Assume that we have a cold configuration in equilibrium. Let us compress its small central part, causing this mass to collapse sufficiently rapidly. Then the layers bordering with the collapsing nucleus will lose their lower support and will start to fall towards the center, dragging more and more of the outer layers with them. The internal layers will, in accordance with the property of relativistic collapse, fall eternally as measured by the clock of the external observer, never reaching a bottom support. Consequently, the outer layers will likewise never stop. Thus, the entire star will be involved in the contraction and will collapse.

The smaller the region of the initially contracted core, the less energy must be consumed in order to start the contraction of the entire star from a stable state.

We have thus proved that the energy barrier that separates collapse from equilibrium is infinitesimally small;* but the perturbations that initiate the collapse of the star are far from small, and the compression of the nucleus before the start of its collapse is the larger, the smaller the required energy. For example, we can cause a star having the same mass as the sun to collapse by compressing in its center a core having a mass equal to the mass of the earth. In order to start the collapse of such a core, it must be compressed to a density

$$\varrho = 2\cdot 10^{16}\left(\frac{M_\odot}{M_\delta}\right)^2 = 2\cdot 10^{27}\ \mathrm{g\cdot cm^{-3}}!$$

Naturally, such "fluctuations" cannot arise spontaneously, either statistically or in quantum fashion. Of course, we cannot discern such a possible transition to a collapse in the linearized theory of small perturbations.

We now show that we can always assemble a specified number of nucleons N such that their total energy is arbitrarily small, i.e., the mass M measured by an external observer will be arbitrarily small[31].

Assume that we are given N baryons. We pack them sufficiently close, so that the expression for ultrarelativistic gas holds:

$$\varrho = \frac{3}{4}\hbar\,(3\pi^2)^{1/3}\frac{1}{c}\,n^{4/3}. \tag{16.1}$$

The formulas for M and N, in the case of matter at rest, are

$$M = 4\pi\int_0^R \varrho(r)\, r^2\, dr, \tag{16.2}$$

$$N = 4\pi\int_0^R n(r)\, e^{\lambda/2} r^2\, dr. \tag{16.3}$$

We specify the distribution

$$\varrho = \frac{a}{r},\ r < R, \text{ and } \varrho = 0,\ r > R, \tag{16.4}$$

where a—arbitrary constant. Using (16.1), (16.2), (16.3), and (16.4) we obtain

$$M = \mathrm{const}\cdot N^{2/3} a^{1/2}\left(1 - \frac{8\pi G}{c^2}a\right)^{1/3}. \tag{16.5}$$

The distribution (16.4) has the following singularities: $\rho \to \infty$ as $r \to 0$, and $\rho$ is discontinuous at $r = R$. It is

*Since we add energy by compression, the mass of the material increases.

*Einstein's gravitational theory is not a quantum theory. We can therefore, starting from dimensionality considerations, indicate a limit of its applicability (Wheeler [14]). From the constants $\pi$, G, and c we can obtain a quantity with the dimension of length, $L^* = (\pi G/c^3)^{1/2} = 1.6 \times 10^{-33}$ cm.

At smaller scales, quantum fluctuations of the metric should become more significant. Consequently, a mass with gravitational radius $r_g = L^*$ is the smallest which can still be compressed to dimensions $r_g$ without resorting to quantum theory. It amounts to $m \approx 10^{-5}$ g. This determines the lower limit of the barrier in question, if this limit depends on quantum effects.

easy to verify that the singularities can always be smoothed out in such a way that the relation (16.15) varies as little as desired. In such a distribution there are no singularities anywhere, either in the metric or in the density.

It follows from (16.5) that for any specified N the mass $M \rightarrow 0$ if $a \rightarrow c^2/8\pi G^2$, q.e.d. Of course, the configuration obtained is not static, for its mass is close to zero and is certainly smaller than static for given N. The nucleons arranged in such a manner are at rest at the initial instant of time, but the acceleration is different from zero and the nucleons will collapse.

We see that in principle a machine can be constructed capable of producing configurations with a mass defect as close to $M_0$ as desired. Thus, this machine extracts from the matter an energy which is almost equal to $M_0c^2$, incomparably larger than the nuclear energy $0.01M_0c^2$.

Of course, the creation of such a machine to work with masses much smaller than $M_{max}^{OV}$ is an impossible task, for it would be necessary to compress the matter to fantastic densities.

For a mass close to the "OV" limit, the corresponding densities are far from fantastic and the transition into collapse is possible, for example, if a star with $M \approx 1.5\ M_\odot$, which "slumps" in the region of the Chandrasekhar maximum, overshoots the stable state during the course of hydrodynamic compression by inertia.

However, the gravitational energy is not radiated in this case to the outside, but is converted into kinetic energy of compressing matter. Therefore the best "gravitational machine" so far is the accretion of matter on a neutron star, with yield on the order of $0.2mc^2$.

We conclude with this section the first part of the review. In the second part we plan to consider possible processes in the vicinity of relativistic objects and relativistic stages of evolution of objects with masses much larger than solar.


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