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THE UNITARY SYMMETRY OF ELEMENTARY PARTICLES

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1. INTRODUCTION

MODERN physics regards its task of investigating the phenomena in the world of elementary particles as completed if it can formulate the regularities of the experimental data in the brief form of conservation laws. Only a few fundamental conservation laws were known to prequantum physics, those of classical mechanics; a large number of such laws has already been accumulated in quantum physics, particularly in elementary-particle physics.

The abundance of conservation laws may be due to the fact that we do not yet know the deeper mechanism that regulates the multitude of processes that occur with particles. It may well turn out in the future that many conservation laws are due to a common cause and are a consequence of some general symmetry of space and time. At present, however, they present themselves as independent, and their study is a major aspect of contemporary research.

The conservation laws in elementary-particle physics and the related symmetries are characterized by the fact that in many cases they are not exact, but only approximate. Owing to this we now have in physics a new possibility for studying phenomena, because the violations of a symmetry are in many cases relatively small in magnitude and rather simple in nature.

An example of such a symmetry is isotopic symmetry; the electromagnetic and weak interactions that break it have been intensively studied. It is no great exaggeration to say that the most interesting results in physics have been achieved precisely in cases when symmetry laws have been found to be violated. This thread can also be traced in astronomical observations. Galileo supposed that the orbits of the planets must naturally be circles. The violation of axial symmetry in these orbits, discovered by Kepler, let to the founding of classical mechanics. A triumph of the general theory of relativity was the discovery of the motion of the perihelion of Mercury, which signalized the breaking of still another symmetry—the closure of the classical mechanical orbit.

In quantum physics the question of new conservation laws arises when one attempts to understand the structure of the bound states of a system. The discovery of the levels of the hydrogen atom led to the Bohr model. The systematics of the levels turned out to be connected with new symmetry properties, which only many years later were formulated by Fock in terms of rotational symmetry in a four-dimensional space.* The possibility of a purely group-theoretical description of the hydrogen atom is a very important fact of atomic physics, which is undeservedly passed over in most courses in quantum mechanics.

Symmetry under permutations and the associated Pauli principle make it possible to understand the level structures of atoms with more than one electron.

In nuclear physics the study of the levels of atomic nuclei has led to the discovery of charge invariance and the associated isotopic space. The isotopic spin of particles and nuclei is now a no less familiar concept than the ordinary spin or the charge of a nucleus. The law of conservation of isotopic spin which was discovered by Wigner in 1937 has revealed its full power in elementary-particle physics.

Up to that time only two heavy particles were known, the proton and the neutron, and the question of new

*The fact that the symmetry reflected by the levels of the hydrogen atom is not one in physical three-dimensional space is very instructive, since it makes the eventual appearance of the isotopic and unitary spaces less unexpected.

quantum numbers did not arise; when hyperons were discovered, there was the question of the cause of their great stability, since on the nuclear scale a lifetime of 10^{-10} sec is a very large time. As in the history of the theory of the atom, the first step in the construction of the theory was the introduction of a "principal quantum number" for the system of baryon levels (as we may call the family of nucleons and hyperons). This principal quantum number was discovered by Gell-Mann and Nishijima as the "strangeness" S (or the hypercharge Y, which is the sum of S and the baryon number B).

At present we do not have the slightest idea as to the connection of this quantum number with any properties of the strongly interacting particles. We do not know whether it is independent of the ordinary properties that are described by quantum mechanics and relativity theory, and will have to be explained by a deeper theory; but it may be, and this seems more natural, that the strangeness is simply a compact description of the interaction of the particles with the not very well understood field which is given the name of "the physical vacuum." However this may be, the discovery of strangeness is one of the most important steps in the development of elementary-particle physics.

In order to include "strangeness" in the apparatus of the theory, it was necessary to broaden the isotopic-spin scheme. The first attempt of this kind was made by Sakata,[A1,A2] who considered the U(3) scheme of a unitary vector (proton, neutron, Λ hyperon), and also by Markov[A4] and by Okun'.[A3] But the singling out of only three from among all the hyperons as the basic ones turned out to be not a radical enough step, and the real success of the theory came with the SU(3) scheme proposed by Gell-Mann and Ne'eman. This scheme, which at first was received with much reserve, has now turned out to be the most effective. A triumph for it was the discovery of the Ω meson which it predicted.[G1]

The scheme of Gell-Mann [C2,3] and Ne'eman[C1] was called the "eightfold way" by the former author.* It is interesting to note that the group SU(3) defines the symmetry of the levels of the three-dimensional harmonic oscillator.

Despite the fact that the mass differences of the hyperons, which are zero in SU(3), are actually large, it was found that extremely simple assumptions about the symmetry of the interaction that breaks the SU(3) symmetry make it possible to describe the actual splitting of the masses. Moreover, the splitting of the isotopic multiplets could also be fitted into a simple scheme. The success of the theory constantly increases. The data obtained lead us to expect great progress in the theory of weak interactions and in the study of reactions between elementary particles.

Already a great number of particles and resonances which two years ago seemed devoid of system have been fitted into a strict scheme of three octets and one decuplet (not counting the antibaryons, which form another octet and a decuplet), so that the "game of solitaire" of the elementary particles now has every prospect of "working out."

The success of a comparatively simple description generates the hope that the description of the interaction of a particle with the vacuum will not be a hopeless problem, and can be realized in a comparatively simple form; the beginning of such a description is given by the description of the mass splitting in terms of a multipole interaction with an effective field.

However this may be, the theory of unitary symmetry is now a necessary apparatus, which should be widely known. From the point of view of the unitary model it is still unclear, however, why there are no particles in nature that correspond to the representation with the fewest dimensions—the three-component spinor, which was the fundamental one in the Sakata theory. The situation is as it would be in quantum mechanics if there were no particles with spin $\frac{1}{2}$. Gell-Mann has made an attempt to introduce this kind of particles (quarks[E1]), but no such particles have been found experimentally.[E2]

A theory including unobservable primitive particles has been developed by Schwinger.[F2] It is still too early, however, to speak of any satisfactory solution of this fundamental problem of the theory.

The present article is to be an elementary introduction to the theory of unitary symmetry. It expounds the tensor algebra associated with the group SU(3).

The exposition is conducted so as to emphasize the analogy with the ordinary tensor algebra associated with the rotation group, or, what is the same thing, the unitary group in the plane—the group SU(2). Therefore the exposition begins with a brief survey of the properties of SU(2). Then, in Secs. 3 and 4, we speak about the tensors of SU(3).

The theory of the unitary symmetry of elementary particles begins in Sec. 5, where the properties of the multiplets are described. Two classes of multiplets are described in this section: 1) fermion multiplets, described by complex matrices—four of these are known: the octet, the decuplet, and their antimultiplets; and 2) boson multiplets, described by Hermitian matrices—of these two are known, and to them we must also add the unitary scalar ω meson.

The formulas for the mass splitting are derived in Secs. 6 and 7. The interval rules are surprisingly reminiscent of the formulas of the elementary Zeeman effect. The analogy with atomic spectroscopy is so obvious that the wish arises to describe the splitting by introducing some quasi-magnetic field to describe ef-

*This name for the group is connected with a legend about the Buddha and the eight ways to the annihilation of suffering: right opinions, right intentions, right words, right actions, right living, right endeavor, right thought, and right concentration.

fectively the interaction of the multiplet with the vacuum. Such a field can be treated as a field of mesons.[F3,F4]

Actually everything said about the mass-splitting formulas could be limited to the derivation of the formulas (7.16), which contain practically all of the results.

These results reduce to three interval formulas (6.7), (7.7), (7.8) for the octet of baryons, one for each type of mesons, (6.19), (6.28), and one (7.14) for the decuplet.

Besides the interval rules there is a large number of other results connected with magnetic moments, form-factors, and reactions.

There are particularly interesting developments in the theory of weak interactions. These topics will require a separate article.

A last comment relates to the literature. Since the number of papers published on unitary symmetry is very large, an attempt has been made to choose a comparatively small number of papers that contain the greater part of the ideas and results published up to June 1, 1964. In these papers the reader will also find further literature references.

2. ISOTOPIC SPIN

A consequence of the charge invariance of nuclear forces is that it is convenient to classify the states of systems of nucleons and other fundamental particles by means of an isotopic spin. If we neglect the electromagnetic field and the weak interaction, the properties of a system are determined by the magnitude T of the isotopic spin alone, and do not depend on the projection T_3.

The electromagnetic field and the weak interaction lead to a "splitting" of the levels, so that the properties of the system depend also on the projection T_3 of the isotopic spin.

The wave function of the proton and neutron is described by a two-component spinor function*

$$\Psi \equiv \begin{pmatrix} \Psi_{-1/2} \\ \Psi_{1/2} \end{pmatrix}. \tag{2.1}$$

The isotopic spin projection $T_3 = +\frac{1}{2}$ corresponds to the charged state p (proton), and the projection $T_3 = -\frac{1}{2}$ to the neutral state n (neutron):

$$\Psi = \begin{pmatrix} n \\ p \end{pmatrix}. \tag{2.2}$$

We shall denote the adjoint function with a bar

$$\overline{\Psi} \equiv (\overline{\Psi}_{-1/2}, \ \overline{\Psi}_{+1/2}) \tag{2.3}$$

or

$$\overline{\Psi} = (\bar{n}, \ \bar{p}). \tag{2.4}$$

The function (2.1) can be subjected to a linear transformation by means of a matrix U:

$$\Psi' = U\Psi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \Psi_{-1/2} \\ \Psi_{+1/2} \end{pmatrix}. \tag{2.5}$$

If there is isotopic invariance, then the components of the new function Ψ' can be regarded as functions describing the two states of charge, equally as well as the components of the original function Ψ. For this to be so, however, the new functions must be orthogonal and normalized. This requirement will be satisfied if the matrix U is unitary; that is, if the reciprocal matrix is equal to the Hermitian adjoint matrix:

$$U^{-1} = U^{+}, \quad UU^{+} = 1. \tag{2.6}$$

These relations will hold if the matrix U is of the form

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \tag{2.7}$$

with

$$|\alpha|^2 + |\beta|^2 = 1. \tag{2.8}$$

The condition (2.8) also means that the determinant of the matrix is equal to 1.

The matrix is still unitary if we multiply it by exp $(i\varphi)$; the determinant is still equal to unity if $\varphi = 0$ or π.

If Ψ is transformed with the matrix U, then $\overline{\Psi}$ gets transformed with the matrix U^+, and this multiplication with the matrix is from the left:

$$\Psi' = U\Psi, \tag{2.9}$$

$$\overline{\Psi}' = \overline{\Psi}U^{+}. \tag{2.10}$$

If we write the indices out explicitly, then for spinors that are transformed according to the law (2.9) one puts the index above, and the spinors are called contravariant:

$$\Psi'^{\alpha} = U^{\alpha}_{\beta}\Psi^{\beta} \tag{2.11}$$

(summation over equal indices!).

Spinors that transform according to the law (2.10) are marked with indices below and are called covariant:

$$\overline{\Psi}'_{\beta} = (U^{+})^{\alpha}_{\beta}\, \overline{\Psi}_{\alpha}. \tag{2.12}$$

Since the indices are written out explicitly, the order of the factors in the right member of the equation is immaterial.

From these formulas it follows that the transformation (2.9), (2.10) does not change the scalar product:

$$(\overline{\Psi}', \ \Psi') = (\overline{\Psi}U^{+}, \ U\Psi) = \overline{\Psi}\Psi. \tag{2.13}$$

This relation (2.13) is the definition of unitary transformation.

Let us introduce the antisymmetric matrix

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.14}$$

and its inverse

$$\varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.15}$$

*Each of the components in turn depends both on the coordinates and on the spin, which we shall not introduce explicitly here.

Then to each contravariant spinor we can assign a corresponding covariant spinor by lowering the index:

$$\Psi_\alpha = \varepsilon_{\alpha\beta}\Psi^\beta. \tag{2.16}$$

Conversely, by raising the index, we can turn a covariant spinor into a contravariant spinor:

$$\Psi^\alpha = \varepsilon^{\alpha\beta}\Psi_\beta. \tag{2.17}$$

It follows that the difference between contravariant and covariant components is a purely formal one, and

$$\Psi_1 = -\Psi^2, \tag{2.18}$$

$$\Psi_2 = \quad \Psi^1. \tag{2.19}$$

The matrix $\epsilon^{\alpha\beta}$ remains unchanged under unitary transformations:

$$\varepsilon = U\varepsilon U^{-1}. \tag{2.20}$$

This equation is easily verified; it is simply a consequence of the fact that the determinant of U is unity.

Accordingly, if

$$\Psi^\alpha = \begin{pmatrix} n \\ p \end{pmatrix}, \tag{2.21}$$

then

$$\Psi_\alpha = \begin{pmatrix} -p \\ n \end{pmatrix}. \tag{2.22}$$

Besides this, we note that

$$\Psi^\alpha \Psi_\alpha = 0 \tag{2.23}$$

and that the matrix $\epsilon^{\alpha\beta}$ plays the role of a metric tensor.

Just as in ordinary tensor algebra one introduces tensors which depend on several indices. A mixed second-rank tensor

$$A_b^a = \begin{pmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \end{pmatrix} \tag{2.24}$$

transforms like the product of two spinors, one contravariant and the other covariant. The trace of this tensor,

$$\mathrm{Sp}\, A = A_1^1 + A_2^2, \tag{2.25}$$

obviously remains unchanged under transformations, and is a scalar.

The tensor

$$A_b^a = \begin{pmatrix} A_1^1 & A_2^1 \\ A_1^2 & -A_1^1 \end{pmatrix} \tag{2.26}$$

is already irreducible. This tensor is equivalent to a vector in three-dimensional space. The connection between the components of the three-vector and those of the tensor (2.26) is made by means of the Pauli matrices. Taking the scalar product of the three-vector **A** and the Pauli vector $\sigma(\sigma_1, \sigma_2, \sigma_3)$, we get

$$A_b^a = \begin{pmatrix} A_3 & A_1 - iA_2 \\ A_1 + iA_2 & -A_3 \end{pmatrix}. \tag{2.27}$$

We shall not give the proof that the unitary transformations of the tensor A_b^a are equivalent to rotations of the vector **A** in three-dimensional space.

We shall not need here any further information about higher-rank tensors.

Let us now take the electromagnetic field into account. In the isotopic space the axis 3 is now singled out, since the projection along this axis determines the charge of the state. In this case the interaction is no longer isotopically invariant. The transformation matrices become diagonal and can be written in the form

$$U = \begin{pmatrix} \exp\left(-\frac{i\varphi}{2}\right) & 0 \\ 0 & \exp\frac{i\varphi}{2} \end{pmatrix}.$$

They describe rotations in a plane around the z axis. We get the one-parameter subgroup of two-dimensional rotations R(2). In the tensor algebra corresponding to this group the difference between upper and lower indices disappears, and the only transformation that remains is multiplication by a phase factor. It may be helpful to recall that everything said about the properties of the matrices can be illustrated with the model of particles with spin, in which the spherical symmetry is broken by a magnetic field directed along the z axis.

An important case of a spin-tensor is the operator (vector) T of the isotopic spin. In accordance with (2.27) we write

$$T = \begin{pmatrix} T_3 & T_- \\ T_+ & -T_3 \end{pmatrix}. \tag{2.28}$$

The elements of this matrix are the components of the *isotopic spin, which in themselves are also matrices.* The form of these matrices (the number of rows and columns) depends on the representation of the group, i.e., on the magnitude of the spin of the particle on whose wave function these matrices are to act.

For the nucleon, T = ½, the components of T are the matrices

$$T_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad T_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

For example, by means of these matrices we can write the current (vector and axial-vector) which occurs in the weak interaction. The vector current corresponding to the β decay of the neutron is the J_+ component of the isotopic current

$$J_{\alpha+} = \langle p \,|\, \gamma_\alpha (1 + \lambda\gamma_5)\, T_+ \,|\, n\rangle.$$

It is indicated here that the matrix element is taken between the initial state n and the final state p, for the weak-interaction operator $\gamma_\alpha(1 + \lambda\gamma_5)$ ($\lambda \approx 1.25$, γ_α and γ_5 are Dirac matrices), which acts on the ordinary spin indices, and for the isotopic operator T_+, which converts neutron into proton.

The current component J_- associated with the operator T_- describes positron decay, and according to the theory of universal interaction the neutral component J_3 enters into the electrodynamical current.

Thus, for example, the vector current can be written in the matrix form

$$V=\begin{pmatrix} V_3 & V_- \\ V_+ & -V_3 \end{pmatrix}.$$

Here we are to understand with each element the unwritten index of the component in the ordinary Minkowski four-space. Returning to (2.28), we note that Sp T = 0, and the determinant of this matrix is

$$\operatorname{Det} D=\frac{1}{2}(T_+T_-+T_-T_+)+T_z=T^2$$

(in calculating the determinant we have symmetrized it in the elements T_- and T_+, since these matrices do not commute). The last formula shows that the determinant gives a matrix invariant of the transformation —the square of the isotopic spin. It is clear that this formula does not depend on the representation; that is, it is true for any value of the spin.

For completeness we shall also describe briefly an extension of the group. If we drop the unitarity condition and keep only the unimodularity condition Det U = 1, the matrices will describe the Lorentz transformation $L_6(4)$.* Since in this case $U^+ \neq U^{-1}$, in the tensor algebra of the Lorentz group there are not just two types of spinors, but four, which transform with the matrices U, U^{-1}, U^+ and U^{+-1}. For the description of these types one introduces a further index with a dot, so that the transformations are written as follows:

$$\left.\begin{aligned} \Psi'^{\alpha} &= U^{\alpha}_{\beta}\Psi^{\beta}, \\ \Psi'_{\alpha} &= (U^{-1})^{\beta}_{\alpha}\,\Psi_{\beta}, \\ \Psi^{\dot\alpha\prime} &= (U^+)^{\dot\alpha}_{\dot\beta}\,\Psi^{\dot\beta}, \\ \Psi'_{\dot\alpha} &= (U^{+-1})^{\dot\beta}_{\dot\alpha}\,\Psi_{\dot\beta}. \end{aligned}\right\} \qquad (2.29)$$

In the tensor algebra of L(4), however, there is an operation of raising and lowering indices, just as in the algebra of SU(2). Therefore in the Lorentz group a representation is characterized by two numbers—the number of dotted indices and the number of undotted ones. In this algebra a tensor is written by means of four Pauli matrices σ_0, σ_1, σ_2, σ_3, where σ_0 is the unit matrix. The tensor

$$A^{\dot\alpha}_{\dot\beta} = \begin{pmatrix} t-z, & x-iy \\ x+iy & t+z \end{pmatrix}$$

corresponds to a four-vector; its transformations are equivalent to Lorentz transformations. Since one cannot sum over a dotted and an undotted index (i.e., such a summation does not give an invariant), the trace of the tensor (2.30) (sic) cannot be regarded as equal to zero in all coordinate systems.

To go over from the Lorentz group $L_6(4)$ to the group of three-dimensional rotations R(3) it suffices to drop the distinction between dotted and undotted indices. Then a scalar, the trace, can be separated out from the tensor. If in the matrices of the representations we keep only the diagonal components, we go over to the group R(2). Thus the reduction of the group

$$L(4)\supset R(3)\supset R(2)$$

is accompanied by a corresponding simplification of the tensor algebra.

In conclusion we shall show how the components of a tensor are placed in correspondence with the values of the projection of the isotopic spin.

A tensor with p upper indices corresponds to the isotopic spin p/2 = T, since the tensor has p + 1 components. If all of the indices are equal to 1, we agree to assign to this tensor component the value $T_3 = -T = -p/2$. Then, if a component of the tensor has p(1) indices equal to 1 and p(2) = p − p(1) indices equal to 2, for this component

$$T_3=-T+p(2)=-\frac{p(1)+p(2)}{2}+p(2)=\frac{p(2)-p(1)}{2}. \qquad (2.30)$$

In an analogous way we get for a tensor with q(1) lower indices equal to 1 and q(2) lower indices equal to 2 the value

$$T_3=-\frac{q(2)-q(1)}{2}. \qquad (2.31)$$

For a mixed tensor with p upper and q lower indices and with trace zero we get (cf. Eq. (4.14))

$$T_3=\frac{1}{2}(m_2-m_1)\equiv m_1-\frac{m}{2}, \qquad (2.32)$$

where

$$\left.\begin{aligned} m_2 &= p(2)-q(2), \\ m_1 &= p(1)-q(1), \\ m &= m_1+m_2. \end{aligned}\right\} \qquad (2.33)$$

We can say that the number $\frac{1}{2}(p+q) = T$ characterizes the representation, and the number $m_2 - m_1$ characterizes the subgroup of rotations around the 3 axis —i.e., the value of the projection of T along this axis.

3. THE UNITARY GROUP

The algebra of the unitary group of complex third-order matrices is constructed in the same way as that of SU(2). Third-order matrices will also be denoted by U, or by U^a_b (a, b = 1, 2, 3) if it is necessary to show the components explicitly. The matrices U are chosen so that

$$UU^+=1, \qquad (3.1)$$

$$\operatorname{Det} U=1. \qquad (3.2)$$

A spinor in this space has three complex components:

$$\Psi^{\alpha}=\begin{pmatrix} \Psi^1 \\ \Psi^2 \\ \Psi^3 \end{pmatrix}. \qquad (3.3)$$

This contravariant spinor, a spinor with upper index, transforms by means of the matrix U. There also exists a covariant spinor, with lower indices,

*Transformations of four-dimensional space which depend on six parameters.

$$\Phi=(\Phi_1,\ \Phi_2,\ \Phi_3). \tag{3.4}$$

The spinor Φ transforms by means of the matrix U^+. It is obvious that the spinor adjoint to (3.3) transforms like Φ.

The scalar product, which is unchanged under unitary transformations, is defined by

$$(\Phi,\ \Psi)=\Phi_\alpha\Psi^\alpha. \tag{3.5}$$

In SU(3) the operation of lowering an index has a different appearance from that in SU(2). We introduce two antisymmetric tensors, which in three-dimensional space have three indices:

$$\varepsilon^{abc}=\varepsilon_{abc}=\begin{cases} 1 & \text{(even permutations a, b, c)}, \\ -1 & \text{(odd permutations a, b, c)}, \\ 0 & \text{(two indices equal)}. \end{cases} \tag{3.6}$$

Since Det U = 1, the values of the components of these tensors remain unchanged under the transformations of the group.

The action of the tensor ϵ_{abc} can be demonstrated with the example of a second-rank tensor in SU(3).

Second-rank tensors can be of different types.

1) Ψ^a_b—the tensor with one upper and one lower index. The trace of such a tensor is a scalar of the group, and therefore an irreducible tensor (analogous to the quadrupole moment tensor in electrostatics) must have trace zero:

$$\mathrm{Sp}\,\Psi=\Psi^\alpha_\alpha=0. \tag{3.7}$$

2) Ψ^{ab} —the tensor with two upper indices. We break this tensor up into two tensors, symmetric (in a and b) and antisymmetric:

$$\Psi^{ab}=\Psi^{[ab]}+\Psi^{\{ab\}}, \tag{3.8}$$

$$\Psi^{[ab]}=\frac{1}{2}(\Psi^{ab}-\Psi^{ba}), \tag{3.9}$$

$$\Psi^{\{a,\,b\}}=\frac{1}{2}(\Psi^{ab}+\Psi^{ba}). \tag{3.10}$$

This separation of the tensor is covariant under unitary transformations. Now we can use the tensor ϵ_{abc}. Multiplying it with the tensor (3.9), we get a spinor with one lower index,

$$\Psi_a=\varepsilon_{abc}\Psi^{[bc]}. \tag{3.11}$$

Thus in this algebra an antisymmetric tensor of the second rank is equivalent to a spinor. On the other hand the symmetric tensor (3.10) cannot be simplified. Its product with ϵ is zero. Therefore we shall use Ψ^{ab} to denote the symmetric tensor, and shall not use the antisymmetric tensor at all.

3) Similar arguments can be repeated for tensors with two lower indices, Ψ_{ab}; for this case we use the tensor ϵ^{abc}. The result is that we can also take this two-index symbol to mean a symmetric tensor.

All of this can be repeated for any pair of upper or of lower indices. Therefore the general case of an irreducible tensor can be characterized by two numbers—the number of lower indices and the number of upper ones. Furthermore all traces (sums over any pair of one lower and one upper index) must be equal to zero. Irreducible tensors (and the corresponding representations) are denoted by the symbol D(p, q), where p is the number of upper indices and q is the number of lower indices. Thus we arrive at the following classification of tensors in SU(3):

D(0, 0)—scalar (one component)
D(1, 0)—contravariant spinor (three components)
D(0, 1)—covariant spinor (three components)
D(1, 1)—mixed tensor (eight components)
D(2, 0)—contravariant tensor (six components)
D(0, 2)—covariant tensor (six components), and so on.

We now give some formulas for computing the number of components. A symmetric tensor with k indices (all upper or all lower) has a number of components given by

$$N(k,\ 0)=N(0,\ k)=\frac{(k+1)(k+2)}{2}. \tag{3.12}$$

A tensor with equal numbers of upper and lower indices has a number of components given by

$$N(k,\ k)=(k+1)^3. \tag{3.13}$$

The first of these formulas is simply the number of ways in which k can be made up of three integers (the numbers of ones, twos, and threes among the indices). The second is obtained from the well known formula for the sum of the cubes of a sequence of integers,

$$\frac{1}{4}(k+1)^2(k+2)^2=\sum_{s=1}^{k+1}s^3.$$

We note that the left member is the square of the number (3.12); that is, it is the total number of components of a tensor which has the same number of upper and lower indices, but with nonvanishing traces. Therefore we can interpret this last formula as giving the decomposition of such a tensor into irreducible tensors with smaller numbers of indices (the proof is easily obtained by induction).

For the case of a tensor with p upper and q lower indices the number of components is

$$N(p,\ q)=\frac{1}{2}(p+1)(q+1)(p+q+2). \tag{3.14}$$

This is obtained at once if we note that before setting the traces equal to zero the number of components can be found from (3.12) and is $\frac{1}{4}(p+1)(q+1)(p+2)(q+2)$. The condition that the traces be zero is equivalent to the vanishing of a tensor D(p − 1, q − 1) with $\frac{1}{4}pq(p+1)(q+1)$ components. The difference of these two numbers gives the result (3.14).

We can now formulate the rule for the composition of vectors. In the rotation group this rule is that from two tensors with $2j_1+1$ and $2j_2+1$ components there will come tensors with numbers of components $2J+1$, where J runs through all integer (or half-integer) values from $|j_1 - j_2|$ to j_1+j_2. In SU(3) the rule is in general more complicated, and it is simplest to use a direct procedure.

For example, from two spinors Ψ^a and Ψ_b we can form a scalar Ψ and a tensor Ψ^a_b with eight components. We shall write this symbolically

$$D(1,\ 0)\times D(0,\ 1)=D(0,\ 0)+D(1,\ 1) \qquad (3.15)$$

or with simply the numbers of components

$$3\times 3=1+8. \qquad (3.16)$$

Let us consider the product of two tensors $\Psi^a_b\Phi^c_d$. To find the irreducible parts, we proceed as follows. We take the double sum, and get a scalar

$$X=\Psi^a_b\Phi^b_a \quad \text{(one component)}. \qquad (3.17)$$

Summing only once, in two ways, and setting the traces equal to zero, we get two tensors:

$$X^a_d=\Psi^a_b\Phi^b_d, \quad X^c_b=\Psi^a_b\Phi^c_a$$
$$\text{(eight components each)}. \qquad (3.18)$$

If by means of the tensor ϵ we raise the lower indices, and then symmetrize in all the upper indices, we get a third-rank tensor with upper indices

$$X^{abc}. \qquad (3.19)$$

In analogous fashion we construct a tensor with three lower indices

$$X_{abc}. \qquad (3.20)$$

There still remains a fourth-rank tensor with two upper and two lower indices and with traces equal to zero. This tensor has $(2+1)^3 = 27$ components

$$X^{ab}_{cd} = \Psi^{\{a}_{\{c}\Phi^{b\}}_{d\}}. \qquad (3.21)$$

This result is written formally

$$D(1,\ 1)\times D(1,\ 1)=D(0,\ 0)+D(1,\ 1)+D(1,\ 1)$$
$$+D(3,\ 0)+D(0,\ 3)+D(2,\ 2) \qquad (3.22)$$

or

$$8\times 8=1+8+8+10+\overline{10}+27. \qquad (3.23)$$

The notation records the fact that the two octets (second-rank tensors) are equivalent—i.e., transform in the same way—and the two decuplets (third-rank tensors) transform with matrices that are each other's adjoints.

We give also the formula for the product of two decuplets; from (3.13) we can write at once (k = 2)

$$10\times\overline{10}=1+8+27+64 \qquad (3.24)$$

or

$$D(3,\ 0)+D(0,\ 3)=D(0,\ 0)+D(1,\ 1)$$
$$+D(2,\ 2)+D(3,\ 3). \qquad (3.25)$$

We give several further formulas without proof:

$$3\times\bar{3}=1+8,$$
$$D(1,\ 0)\times D(0,\ 1)=D(0,\ 0)+D(1,\ 1),$$
$$3\times 3=\bar{3}\times 6,$$
$$D(1,\ 0)\times D(1,\ 0)=D(0,\ 1)+D(2,\ 0),$$
$$\bar{3}\times\bar{3}=3+\bar{6},$$
$$D(0,\ 1)\times D(0,\ 1)=D(1,\ 0)+D(0,\ 2),$$
$$6\times 3=8+10,$$
$$D(2,\ 0)\times D(1,\ 0)=D(1,\ 1)+D(3,\ 0),$$
$$\bar{6}\times 3=\bar{3}+15,$$
$$D(0,\ 2)\times D(1,\ 0)=D(0,\ 1)+D(1,\ 2),$$
$$6\times\bar{3}=3+\overline{15},$$
$$D(2,\ 0)\times D(0,\ 1)=D(1,\ 0)+D(2,\ 1),$$
$$\bar{6}\times 6=1+8+27,$$
$$D(0,\ 2)\times D(2,\ 0)=D(0,\ 0)+D(1,\ 1)+D(2,\ 2),$$
$$6\times 6=6+\overline{15}+15',$$
$$D(2,\ 0)\times D(2,\ 0)=D(0,\ 2)+D(2,\ 1)+D(4,\ 0),$$
$$\bar{6}\times\bar{6}=6+15+\overline{15}',$$
$$D(0,\ 2)\times D(0,\ 2)=D(2,\ 0)+D(1,\ 2)+D(0,\ 4) \text{ etc.}$$

We see that only the simplest representations can be described with a number alone. The two tensors D(2,1) and D(4,0) both have 15 components, and to distinguish them we have written 15 and 15′.

The calculation of the coefficients in the composition formulas (the Clebsch-Gordan coefficients) is more lengthy. Since we shall not have need for them, we shall not discuss them here (they are given in [B2]).

4. UNITARY SPIN

Just as in SU(2) a representation was characterized by the magnitude of the isotopic spin, we can also introduce in the algebra of SU(3) an analogous characteristic, the unitary spin, which we shall denote by U. The components of the isotopic spin, which were written in the form of a 2×2 matrix,

$$T=\begin{pmatrix} T_3 & T_- \\ T_+ & -T_3 \end{pmatrix}, \qquad (4.1)$$

are generators of a rotation group. This means that the matrix for a rotation through the angle $\delta\varphi$ around the 3 axis is of the form

$$M=1+\frac{i}{2}\delta\varphi T_3. \qquad (4.2)$$

There are analogous formulas for rotations around the 1 and 2 axes; these contain the respective matrices

$$T_1=T_++T_-, \quad T_2=\frac{1}{i}(T_+-T_-). \qquad (4.3)$$

The unitary spin is introduced as a 3×3 matrix:

$$U=\left(\begin{array}{cc:c} Q & T_- & L_- \\ T_+ & Y-Q & K_- \\ \hdashline L_+ & K_+ & -Y \end{array}\right). \qquad (4.4)$$

The four elements in the upper left-hand corner form a matrix of the type of the matrix (4.1), but with nonvanishing trace [i.e., this matrix is reducible in SU(2)].

This matrix can be written as the sum of two matrices, one of which has zero trace:

$$\begin{pmatrix} Q & T_- \\ T_+ & Y-Q \end{pmatrix} = \begin{pmatrix} Q-\frac{1}{2}Y & T_- \\ T_+ & \frac{1}{2}Y-Q \end{pmatrix} + \frac{1}{2}\begin{pmatrix} Y & 0 \\ 0 & Y \end{pmatrix}. \qquad (4.5)$$

Comparing this with (4.1) we see that

$$Q-\frac{1}{2}Y=T_3 \qquad (4.6)$$

or

$$Q=T_3+\frac{1}{2}Y. \qquad (4.7)$$

If we take the eigenvalues of Q to be values of charge, then Y is the matrix corresponding to the hypercharge S + B (S is strangeness and B is baryon number). The remaining elements (the elements L_+, L_-, K_+, K_-) are matrices of the same form as the isotopic spin matrices T_1, T_2, and T_3, but defined in the subspaces (2, 3) and (1, 3). Since the group SU(3) consists of transformations that conserve the quadratic form $|X|^2 + |Y|^2 + |Z|^2$, in each of the two-dimensional subspaces there exists a subgroup which conserves the sum of two squares of absolute values, that is, a subgroup equivalent to SU(2). From the structure of the matrix (4.4) it can be seen that one may choose a system of commuting matrices in different ways. Taking as a subgroup the matrices (4.5), we get the system of matrices Y, T^2, and T_3. If we take as subgroup the 2×2 matrix in the lower right-hand corner of (4.4), we get the matrix

$$\begin{pmatrix} Y-Q & K_- \\ K_+ & -Y \end{pmatrix} = \begin{pmatrix} Y-\frac{1}{2}Q & K_- \\ K_+ & -Y+\frac{1}{2}Q \end{pmatrix} - \frac{1}{2}\begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}. \qquad (4.8)$$

In this case the commuting matrices will be the matrices Q, K^2, and K_3, where

$$K_3=Y-\frac{1}{2}Q. \qquad (4.9)$$

With this choice of the commuting operators the charge of the particle will be determined by one of the quantum numbers. Such a representation is convenient in dealing with problems of the weak and electromagnetic interactions.

Let us return to the choice of the system of commuting matrices Y, T^2, T_3. We consider an arbitrary tensor with p upper and q lower indices. We denote it, and also the associated representation, by D(p, q). Each of the p + q indices can take the values 1, 2, 3. We introduce the following notations:

$$\begin{array}{ll} p(1) & \text{number of upper indices equal to } 1, \\ p(2) & \text{number of upper indices equal to } 2, \\ p(3) & \text{number of upper indices equal to } 3, \\ q(1) & \text{number of lower indices equal to } 1, \\ q(2) & \text{number of lower indices equal to } 2, \\ q(3) & \text{number of lower indices equal to } 3. \end{array} \qquad (4.10)$$

The connection between the eigenvalues of Y, T^2, and T_3 and the components of a tensor D(p, q) is established if we prescribe Y for the components of any one tensor. This is where the choice of representation occurs for the description of the actual particles. In the model of Gell-Mann and Ne'eman the octet D(1, 1) is taken as the basis.

We assign the hypercharges for the components of this octet:

$$\left.\begin{array}{ll} \Psi^a_b\,(a,\ b=1,\ 2,\ 3): & Y=0, \\ \Psi^3_b\,(b=1,\ 2,\ 3): & Y=1, \\ \Psi^a_3\,(a=1,\ 2): & Y=-1, \\ \Psi^3_3: & Y_a=0. \end{array}\right\} \qquad (4.11)$$

We recall that the components Ψ^a_b comprise a mixture of an isotopic vector and an isotopic scalar (a, b = 1, 2):

$$\left.\begin{array}{ll} \Psi^a_b-\frac{1}{2}\delta^a_b\,\mathrm{Sp}\,\Psi, & T\text{-vector}, \\ \mathrm{Sp}\,\Psi, & T\text{-scalar}, \\ \Psi^a_3,\ \Psi^3_b, & T\text{-spinor}, \\ T^3_3, & T\text{-scalar} \end{array}\right\} \qquad (4.12)$$

From octets we shall form higher-order tensors. By multiplying octets together we obtain tensors in which the numbers of upper and lower indices are equal, but the traces are not equal to zero. We denote such a tensor by D′(k, k), indicating with the prime the fact that Sp D′ ≠ 0. The hypercharge corresponding to a component of the tensor will be determined by the number of indices that are equal to 3, i.e., by the numbers p(3) and q(3). Since according to (4.11) each upper index contributes the amount 1 to Y, and each lower index contributes −1, for the components of a tensor D′(k, k) the hypercharge is p(3) − q(3). It can be seen from this that among the components of this tensor Y varies over the range $-k \le Y \le k$.

We now separate out from the tensor its irreducible parts. To do this we first symmetrize the tensor in the p upper indices, and separately from this in the q lower indices. We get an irreducible tensor D(k, k) for which Y varies over the same range $-k \le Y \le k$.

If p or q ≥ 2, then by means of ϵ_{abc} we can lower two indices on D′(k, k), turning two upper indices into one lower one. Similarly, with the tensor ϵ^{abc} we can raise two lower indices and turn them into one upper one. Thus we can convert the tensor D′(k, k) into a tensor D′(k − 2, k + 1) or into a tensor D′(k + 1, k − 2). After this we must symmetrize the resulting tensors to tensors D(k − 2, k + 1) or D(k + 1, k − 2).

The tensors ε are antisymmetric, so that if the tensor D(p, q) has all p indices equal to 3, multiplication by ϵ_{abc} gives zero. Similarly, if the tensor D(p, q) has all lower indices equal to 3, multiplication by ϵ^{abc} gives zero. From this we can conclude that the number of values of Y for the tensor D(k − 2, k + 1) is one less than for the original tensor D(k, k); it has no component with Y = k. Thus the tensor D(k − 2, k + 1) has components with Y in the range $-k \le Y \le k-1$. We can also conclude that the tensor D′(k + 1, k − 2) has components with Y in the range $-k+1 \le Y \le k$. If we lower 2s upper indices, converting them into s lower ones, we find by a similar argument that for the new tensor $D(k-2s, k+s)$ the hypercharge varies over the range $-k \le Y \le k-s$. So also, for the tensor $D(k+s, k-2s)$ we have $-k+s \le Y \le k$.

Then for an arbitrary tensor $D(p,q)$, when we set $k = (p+2q)/3$ and $s = -(p-q)/3$ $(p = k-2s, q = k+s)$, we find that the hypercharge varies over the range

$$-\frac{p+2q}{3} \leqslant Y \leqslant \frac{2p+q}{3}. \tag{4.13}$$

The hypercharge of a component of the tensor $D(p,q)$ is determined by the number of indices equal to 3. We set

$$Y = p(3) - q(3) + a, \tag{4.14}$$

where a is a constant. We find this constant by noting that for $p(3) = p$ and $q(3) = 0$ the hypercharge Y takes its maximum value, $(2p+q)/3$. Therefore $a = -(p-q)/3$.

With the notation $p(3) - q(3) = m_3$, $p-q = m$, we get

$$Y = m_3 - \frac{1}{3}m. \tag{4.14'}$$

In a similar way we can also get a formula for the charge:

$$Q = -m_1 + \frac{1}{3}m, \tag{4.15}$$

where $m_1 = p(1) - q(1)$. Its form is determined by the symmetry of the determinant of (4.4) relative to the interchange ($Y \rightleftarrows -Q$ and $1 \rightleftarrows 2$). If we write the further formula $Y - Q = -m_2 + m/3$, then we can note that the three numbers m_1, m_2, m_3 play the roles of "magnetic" quantum numbers. In order to make the sum of the eigenvalues of the three operators that are on the diagonal of (4.4) equal to zero, one puts in a term $m/3 = (m_1+m_2+m_3)/3$. This is the algebraic reason for the appearance of the coefficient $1/3$ in the formulas [cf. Eq. (2.31)].

From (4.13) we can get an expression for the "width" of the unitary multiplet:

$$Y_{\max} - Y_{\min} = p + q. \tag{4.16}$$

In analogy with the isotopic spin, we can call half of the total number of indices the magnitude of the unitary spin U,

$$U = \frac{p+q}{2}, \tag{4.17}$$

so that the number of different values of Y is $2U+1$. For the "center of gravity" of the multiplet we get

$$Y_{\max} + Y_{\min} = \frac{p-q}{3}. \tag{4.18}$$

The formulas (4.14)—(4.17) completely describe the hypercharge structure of the multiplet.

These formulas give integer values for Y, but only for tensors for which the difference $|p-q| = 3n$, where n is a positive integer. In the scheme we adopted for constructing tensors from a tensor $D(k,k)$ we get tensors $D(k+s, k-2s)$ and $D(k-2s, k+s)$ which satisfy this condition. Tensors with $|p-q| \neq 3n$ cannot be obtained in this way. This recalls the situation in the rotation group, where by means of vectors one can construct only tensors with integer values of the isotopic spin; spinors with half-integer isotopic spins must be introduced independently. In the group SU(3) there also occur spinors with non-integer values of the hypercharge, which are multiples of $1/3$. If we retain the rules we have obtained also for tensors with $|p-q| \neq 3n$, we find, for example, that for a spinor with one upper index, $D(1,0)$, the assignment of values of Y to the components is:

$$\begin{aligned} &\Psi^1: \quad p=0, && Y=-\tfrac{1}{3}, \\ &\Psi^2: \quad p=0, \quad \tfrac{1}{3}(p-q)=-\tfrac{1}{3}, && Y=-\tfrac{1}{3}, \\ &\Psi^3: \quad p=1, && Y=\tfrac{2}{3}. \end{aligned} \tag{4.19}$$

Similarly, a tensor Ψ_a has components with $Y = 1/3$, $1/3$, $-2/3$. Thus fractional values of the hypercharge occur in SU(3). Attempts to detect particles corresponding to such representations (quarks, in Gell-Mann's terminology) have so far been unsuccessful (see Introduction).

We can now continue the classification of the components of a tensor and go on to the values of the isotopic spin and its projections. If in the tensor $D(p,q)$ we set some number of indices equal to 3, the remaining indices, which take the values 1 and 2, form a tensor in isotopic space with $p(1) + p(2) = p(1,2)$ upper indices and $q(1) + q(2) = q(1,2)$ lower indices. The resolution of such a tensor in SU(2) into irreducible tensors is done with the usual procedure of summing over the values 1 and 2 of one upper and one lower index. Since a tensor with $p(1,2)$ indices that take the values 1, 2 has $p(1,2) + 1$ components, the isotopic spin of such a tensor is

$$T_p = \frac{1}{2}p(1,2). \tag{4.20}$$

Similarly for a tensor with $q(1,2)$ lower indices

$$T_q = \frac{1}{2}q(1,2). \tag{4.21}$$

A tensor with $p(1,2)$ upper indices and $q(1,2)$ lower indices can be resolved into irreducible tensors whose spins are

$$T_p + T_q, \quad T_p + T_q - 1, \ \ldots, \quad T_p - T_q. \tag{4.22}$$

Finally, we determine the value of the component of isotopic spin according to Eq. (2.32):

$$T_3 = \frac{1}{2}(m_2 - m_1) = \frac{1}{2}[p(2) - q(2) - p(1) + q(1)], \tag{4.23}$$

and this agrees with the definition $T_3 = Q - 1/2\,Y$ and with (4.14) and (4.15).

Thus we arrive at the following classification of the components of unitary tensors. The components of an irreducible unitary tensor are characterized by five quantum numbers:

1) the number p of upper indices,
2) the number q of lower indices,
3) the isotopic spin T,
4) the hypercharge Y [Eq. (4.14)],
5) the isotopic spin projection T_3 [Eq. (4.23)].

Instead of p and q we can introduce:

1′) the unitary spin $U = \frac{1}{2}(p+q)$ [Eq. (4.17)],
2′) the "center of gravity" of the multiplet, $C = (p-q)/3$ [Eq. (4.18)].

Instead of C, Y, and T_3 we can introduce the quantum numbers

$$m_s = p(s) - q(s), \quad s = 1,\ 2,\ 3. \tag{4.24}$$

Then

$$\left.\begin{aligned} C &= \frac{1}{3}(m_1 + m_2 + m_3), \\ T_3 &= \frac{1}{2}(m_2 - m_1), \\ Y &= \frac{1}{3}[2m_3 - m_1 - m_2]. \end{aligned}\right\} \tag{4.25}$$

The numbers m_1, m_2, m_3 together with U and T are another set of five quantum numbers which describe a unitary multiplet.

We further note a useful formula for tensors which have indices of only one type—tensors D(p, 0) and D(0, q). According to (4.14) the hypercharge of a component of a tensor D(p, 0) is $Y = p(3) - p/3$. The isotopic spin of components with this value of Y is obviously $\frac{1}{2}[p - p(3)]$, since $p - p(3)$ is the number of indices equal to 1 or 2. From this we see that for tensors of the type D(p, 0) there is a relation

$$Y + 2T = \frac{2}{3}p \tag{4.26}$$

and similarly for tensors D(0, q)

$$-Y + 2T = \frac{2}{3}q. \tag{4.27}$$

Thus for such tensors T is determined when Y is given. Let us now consider several multiplets.

Table I. Multiplets in SU(3)

I. Octet $D(1,1)$, $U=1$

	Y	T
Ψ_b^a $(a, b = 1, 2)$	0	1 и 0
Ψ_b^3 $(b = 1,2)$	1	1/2
Ψ_3^a $(a = 1,2)$	−1	1/2

II. Covariant decuplet $D(0, 3)$, $U = \frac{3}{2}$

	Y	T
Ψ_{abc} $(a, b, c = 1, 2, 3)$	1	3/2
Ψ_{ab3} $(a, b = 1, 2, 3)$	0	1
Ψ_{a33} $(a = 1, 2, 3)$	−1	1/2
Ψ_{333}	−2	0

$\Psi_3^3 = -\Psi_1^1 - \Psi_2^2$ and therefore is not included in the table.

III. Contravariant decuplet $D(3, 0)$, $U=\frac{3}{2}$

	Y	T
Ψ^{abc} $(a,\ b,\ c = 1, 2, 3)$	−1	3/2
Ψ^{ab3} $(a,\ b = 1, 2, 3)$	0	1
Ψ^{a33} $(a = 1, 2, 3)$	1	1/2
Ψ^{333}	2	0

IV. 27-et $D(2, 2)$, $U = 2$

	Y	T
Ψ_{cd}^{33} $(c, d = 1, 2, 3)$	2	1
Ψ_{cd}^{3b} $(b, cd = 1, 2, 3)$	1	3/2 and 1/2
Ψ_{cd}^{ab} $(a, b, c, d = 1, 2, 3)$	0	2, 1 and 0
Ψ_{3d}^{ab} $(a, b, d = 1, 2, 3)$	−1	3/2 and 1/2
Ψ_{33}^{ab} $(a, b = 1, 2, 3)$	−2	1

5. MESONS AND BARYONS

Let us now put actual particles in correspondence with the unitary tensors. The octet of baryons consists of the nucleon and the hyperons Λ, Σ, and Ξ. In accordance with the rules of the preceding section it can be written in the form of a matrix (upper index for rows and lower for columns):

$$(B_b^a) = \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & \Sigma^- & \Xi^- \\ \Sigma^+ & -\frac{1}{2}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & \Xi^0 \\ -p & n & -\sqrt{\frac{2}{3}}\Lambda \end{pmatrix}. \tag{5.1}$$

The coefficients in the matrix are chosen so that in the expression

$$\mathrm{Sp}\bar{B}B = \bar{p}p + \bar{n}n + \bar{\Sigma}^+\Sigma^+ + \bar{\Sigma}^0\Sigma^0 + \bar{\Sigma}^-\Sigma^- + \bar{\Xi}^0\Xi^0 + \bar{\Xi}^-\Xi^- \tag{5.2}$$

all of the coefficients are equal to 1, and so that $\mathrm{Sp}\,\Psi = 0$.

In the matrix (5.1) the proton appears with the minus sign. This is in accordance with the definition of the covariant components of a spinor by Eq. (2.22). In the literature a definition of the octet is used which differs from (5.1) by interchange of rows and columns with the minus sign for Ξ^0.

Mesons (and resonances) form two known octets.

Table II. Components of the baryon decuplet

Serial No.	Component	Y	T	T_3	Resonance	Serial No.	Component	Y	T	T_3	Resonance
1	333 (1)	−2	0	0	Ω^-	6	322 (3)			1	$\frac{1}{\sqrt{3}}\Sigma^{*+}$
2	331 (3)	−1	1/2	−1/2	$\frac{1}{\sqrt{3}}\Xi^{*-}$	7	111 (1)	1	3/2	−3/2	Δ^-
3	332 (3)			1/2	$\frac{1}{\sqrt{3}}\Xi^{*0}$	8	112 (3)			−1/2	$\frac{1}{\sqrt{3}}\Delta^0$
4	311 (3)	0	1	−1	$\frac{1}{\sqrt{3}}\Sigma^{*-}$	9	122 (3)			1/2	$\frac{1}{\sqrt{3}}\Delta^+$
5	312 (6)			0	$\frac{1}{\sqrt{6}}\Sigma^{*0}$	10	222 (1)			3/2	Δ^{++}

The octet of pseudoscalar mesons consists of the π, η, and K mesons. The matrix of these mesons is formed in analogy with the matrix (5.1) and is

$$(P_b^a)=\begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0+\frac{1}{\sqrt{6}}\eta & \pi^- & K^- \\ \pi^+ & -\frac{1}{\sqrt{2}}\pi^0+\frac{1}{\sqrt{6}}\eta & \bar{K}^0 \\ -K^+ & K^0 & -\sqrt{\frac{2}{3}}\eta \end{pmatrix}. \tag{5.3}$$

The octet of vector mesons ρ, K* (Kπ resonance), and φ forms the matrix

$$(V_b^a)=\begin{pmatrix} \frac{1}{\sqrt{2}}\varrho^0+\frac{1}{\sqrt{6}}\varphi & \varrho^- & K^{*-} \\ \varrho^+ & -\frac{1}{\sqrt{2}}\varrho^0+\frac{1}{\sqrt{6}}\varphi & \bar{K}^{*0} \\ -K^{*+} & K^{*0} & -\sqrt{\frac{2}{3}}\varphi \end{pmatrix}.$$

The meson octets differ from the baryon octet in that particles occur in them along with their antiparticles. These octets are described by Hermitian matrices

$$P^+=P, \quad V^+=V. \tag{5.4}$$

Besides the octet (5.1) there is also known a baryon decuplet, which is headed by the famous Ω^- particle. This decuplet is described by a tensor with three lower indices, Ψ_{abc}, which has 10 distinct components. These components can be arranged in a table. After the number of the component (lower indices!) we indicate the number of identical components of the tensor Ψ_{abc} that can be obtained by rearrangements of the indices. This number determines the coefficient in the last column.

The normalization is chosen so that in the quadratic expression $\bar{\Psi}\Psi$ all of the particles appear with the same coefficient unity.

6. MASS SPLITTING OF THE MULTIPLETS

The multiplets of particles are completely degenerate; that is, all of the components would have exactly equal masses if the particles were not in interaction with anything. The existence of the strong interaction with virtual particles (we shall say "with the vacuum" for short) leads to a splitting of the multiplets. This splitting can be described in a way very similar to the usual description in quantum mechanics of the Zeeman splitting of atomic levels.

We shall assume that the interaction of the multiplet with the vacuum is described by a constant effective field whose properties are those of a real tensor D(1,1).* We denote the field by $H \equiv H_b^a$ (see page 652). For a more exact description of the splitting one can also introduce "fields" of higher ranks, H_{cd}^{ab}, H_{def}^{abc}, and so on; as we shall see later, however, comparison with experiment shows that the corresponding terms in the interaction are small. In the free (unperturbed) Lagrangian of the system the mass of the particles occurs in the combinations

$$\left.\begin{array}{l} m\,\mathrm{Sp}\,\bar{\Psi}\Psi \text{ for baryons}, \\ m^2\,\mathrm{Sp}\,\bar{\Psi}\Psi \text{ for mesons}. \end{array}\right\} \tag{6.1}$$

The perturbation (the interaction with the field) will add a term to the mass for baryons and to the square of the mass for mesons.† To calculate this added term we find the analogs of multipoles of the system. From the function Ψ and its adjoint $\bar{\Psi}$ we can form‡

$$\left.\begin{array}{l} \text{0-pole: } \mathrm{Sp}\,\bar{\Psi}\Psi, \\ \text{8-poles: } \bar{\Psi}_c^a\Psi_b^c \pm \bar{\Psi}_b^c\Psi_c^a. \end{array}\right\} \tag{6.2}$$

The remaining components form a 27-pole.

The average value of the 0-pole (scalar) determines the unperturbed mass; the 8-pole leads to a perturbation proportional to H_b^a, and the 27-pole to a perturbation proportional to H_{cd}^{ab}.

If Ψ is Hermitian (bosons), the two 8-poles are

*If we consider reactions and decays in this scheme it will be necessary to take into account the energy dependence of the tensor components.

†Since the splitting is comparatively small for baryons, we can suppose that for them also one gets relations for the squares of the masses.

‡For brevity we shall call a multipole with k components a k-pole.

equal, since for Hermitian matrices $\bar{\Psi}\Psi = \Psi\bar{\Psi}$; therefore we can form two 8-poles from the components of a baryon octet and only one 8-pole from those of a meson octet.

In an analogous way we can form from the components of a decuplet:

$$\left.\begin{array}{l} \text{0-pole: } (\bar{\Psi}, \Psi), \\ \text{8-pole: } \bar{\Psi}^{abc}\Psi_{abf}, \\ \text{27-pole: } \bar{\Psi}^{abc}\Psi_{aef}, \\ \text{64-pole: } \bar{\Psi}^{abc}\Psi_{def} \end{array}\right\} \tag{6.3}$$

(with all traces made to vanish). As we have already stated, in the derivation of the formulas we shall include only the interaction with the 8-pole.

We begin with the baryon octet. The interaction with the field H is described by two terms of the forms (C_1 and C_2 are constants)

$$C_1 \ \mathrm{Sp}\, H\bar{\Psi}\Psi \text{ and } C_2 \ \mathrm{Sp}\, H\Psi\bar{\Psi}. \tag{6.4}$$

We now choose H. In first approximation it is natural to neglect the mass differences within an isotopic multiplet, on the assumption that T is still a good quantum number. At the end of the article we shall also consider the splitting of an isotopic multiplet.

We choose H so that the component different from zero is H_3^3. Then the interaction with the field, Eq. (6.4), can be written in the form

$$\Delta M = a\bar{\Psi}_r^3\Psi_3^r + b\bar{\Psi}_3^r\Psi_r^3. \tag{6.5}$$

The constants a and b determine the strengths of the interaction of the field with the two 8-poles. If we now go back to (5.1), we get the following values for the masses of the baryons (here m_0 is the mass of the unperturbed octet):

$$\left.\begin{array}{l} m(\Xi) = m_0 + a, \\ m(N) = m_0 + b, \\ m(\Sigma) = m_0, \\ m(\Lambda) = m_0 + \frac{2}{3}(a+b). \end{array}\right\} \tag{6.6}$$

From this there follows the Gell-Mann–Okubo mass formula

$$\frac{1}{2}[m(\Xi) + m(N)] = \frac{1}{4}[m(\Sigma) + 3m(\Lambda)]. \tag{6.7}$$

If we take the mass of Σ as a reference level,* then from the known masses of the baryons we can compose Table III (for Ξ and N we take half the sum of the masses of the two components).

The formulas (6.6) can be given a different form. It can be shown from (4.5) that after the symmetry is lowered to SU(2) the remaining invariants of the group are the trace of the 2×2 matrix and its determinant. The determinant is equal to

$$QY - Q^2 + \frac{1}{2}(T_+T_- + T_-T_+).$$

*If we use the symbol of a baryon to denote the difference between its mass and that of Σ^0, Eq. (6.7) takes the form $\Xi + N = (3/2)\Lambda$.

Table III.
Intervals in the baryon octet

	$m - m(\Sigma^0)$, MeV
Ξ	125
N	−253
Σ^0	0
Λ	−77
$\frac{\Xi + N}{2}$	−64
$\frac{3}{4}\Lambda$	−58

It follows from (4.6), however, that

$$Q^2 - QY = T_3^2 - \frac{1}{4}Y^2.$$

From this it follows that the quantum numbers that characterize the split multiplet will be

$$Y \text{ and } T(T+1) - \frac{1}{4}Y^2. \tag{6.8}$$

Therefore to first order the mass of the baryon is given by

$$M = M_0 + M_1 Y + M_2\left[T(T+1) - \frac{1}{4}Y^2\right], \tag{6.9}$$

where M_0, M_1, and M_2 are new constants. In this form the formula can be applied to any baryon multiplet. The connection between a, b and M_0, M_1, and M_2 can be found easily.

The question naturally arises as to whether it is legitimate to use the formulas of first-order perturbation theory. It is clear that the ratio of the mass splitting to the mass of the unperturbed multiplet cannot be used as a measure of the smallness of the perturbation; one must look for a different explanation, and strictly speaking there is none at present.

We may suppose that in the baryon multiplet there are no admixtures whatever of higher multipole order, just as the deuteron has no electric moments higher than the quadrupole. This means that there is no other baryon multiplet with nearly the same mass, which could introduce a perturbation of lower symmetry. In any case, the pronounced manifestation of the interaction of lowest multipole order is a major factor causing success of the entire scheme of the breaking of unitary symmetry.

If, on the other hand, we include in the mass calculation a field H_{cd}^{ab} (the component H_{33}^{33}), then we must add to (6.5) the term*

*The 27-pole has the components: $\Psi_{cd}^{ab} = \bar{\Psi}_{\{c}^{\{a}\Psi_{d\}}^{b\}} - \frac{1}{3}\bar{\Psi}_c^{\varepsilon}\Psi_{\varepsilon}^{b}\delta_d^a - \frac{1}{3}\bar{\Psi}_{\varepsilon}^{a}\Psi_a^{\varepsilon}\delta_c^b + \bar{\Psi}_{\varkappa}^{\varepsilon}\Psi_{\varepsilon}^{\varkappa}\delta_d^a\delta_c^b$. Multiplication of the 27-pole by H_{33}^{33} leads to a shift of the remaining masses. This shift, however, reduces to a change of the meanings of the constants, so that we need consider explicitly only the first term in the formula as written.

$$c\overline{\Psi}^3_3\Psi^3_3, \qquad (6.10)$$

which shifts the mass of Λ in the formulas (6.6) by the amount $2c/3$. It is clear that this term will determine the deviation from the result (6.7), so that we have

$$c = -\frac{3}{2}[2m(\Xi) + 2m(N) - m(\Sigma) - 3m(\Lambda)]. \qquad (6.11)$$

It follows from this that $c \approx 36$ MeV, and this gives the amount of coupling of the 27-pole with H^{33}_{33}.

We now consider the decuplet. In this case the mass correction is given by

$$\Delta M = d\overline{\Psi}^{ab3}\Psi_{ab3}. \qquad (6.12)$$

By means of Table II we get the following values of the masses:

$$\left.\begin{aligned} m(\Omega) &= m_0 + d, \\ m(\Xi^*) &= m_0 + \frac{2}{3}d, \\ m(\Sigma^*) &= m_0 + \frac{1}{3}d, \\ m(\Delta) &= m_0. \end{aligned}\right\} \qquad (6.13)$$

Thus in the decuplet the levels are equidistant, with the separation d/3. From experiment the separation is 145 MeV, so that

$$d = 435 \text{ MeV}. \qquad (6.14)$$

The equidistant pattern can also be obtained from (6.9), if we use the fact that for the decuplet (4.20) gives

$$T = \frac{1}{2}Y + 1, \qquad (6.15)$$

from which we have

$$M = (M_0 + 2M_2) + \left(M_1 + \frac{3}{2}M_2\right)Y, \qquad (6.16)$$

where in general the constants are not the same as for the octet.

The formula for meson octets is obtained from that for the baryon octet if we set a = b = e [or if in (6.9) we set $M_1 = 0$] and take these formulas as written not for the masses but for their squares. For the octet of pseudoscalar mesons we get, in analogy with (6.6):

$$\Delta m^2 = e\,\mathrm{Sp}(H\overline{P}P), \qquad (6.17)$$

$$\left.\begin{aligned} m^2(K) &= m_0^2 + e, \\ m^2(\pi) &= m_0^2, \\ m^2(\eta) &= m_0^2 + \frac{4}{3}e. \end{aligned}\right\} \qquad (6.18)$$

The m_0 here is of course not the same as the constant m_0 in the formula for the baryons.

From (6.18) we get a relation similar to (6.7):

$$m^2(K) = \frac{1}{4}[m^2(\pi) + 3m^2(\eta)]. \qquad (6.19)$$

The squares of the masses of the pseudoscalar mesons are collected in Table IV.

The relation between the differences of squares of masses

$$m^2(K) = \frac{3}{4}m^2(\eta) \qquad (6.20)$$

is rather well satisfied.

Table IV. Squares of masses of pseudoscalar mesons (intervals)

	$(\text{Mass})^2 - m^2(\pi)$, $(\text{GeV})^2$
η	0.28
K	0.22
π	0

Table V. Squares of masses of vector mesons (intervals)

	$(\text{Mass})^2 - m^2(\rho)$, $(\text{GeV})^2$
φ	0.46
K^*	0.21
ω	0.03
ρ	0

For the octet of vector mesons the agreement is much poorer. This is evidently to be explained by the fact that within the octet there lies one other vector meson ω (Y = T = 0). Naturally this meson can perturb the octet (Table V).

The description of the perturbation that the ω meson produces in the vector octet would seem to be both beyond the framework of the group SU(3) and outside the scheme of symmetry breaking that we have described. The fact that two mesons (the ω meson and the φ meson) have been found in nature with nearly equal masses and with the same quantum numbers evidently indicates the existence of a higher symmetry, whose breaking is demonstrated in the splitting of the ω and φ mesons, as the breaking of the SU(3) symmetry leads to the splitting of the masses of the φ and ρ^0 mesons. A simple extension of the group, however, for example to SU(4), leads to an increase of the number of components of the multiplet, so that the solution of the puzzle must be more subtle than this.* We must also note a second puzzle which the meson octets present. It can be seen from the data that have been given that the first intervals in the two octets are equal:

$$m^2(K) - m^2(\pi) = m^2(K^*) - m^2(\rho). \qquad (6.21)$$

It is clear that this sort of relation cannot follow from the SU(3) symmetry. The point here is that the coupling of different meson octets with the field H is the same (a universal interaction). If the equality is not accidental, its explanation must be associated with the breaking of a higher symmetry. It can also be noted that if we replace the squares of the masses of φ and ω by their half sum, the scheme so obtained practically coincides with the scheme for the pseudoscalar octet.

We turn to a possibility for describing the splitting which has been pointed out by Schwinger.†

Indeed, let us assume that the ninth meson occurs in the octet, which will then have trace different from zero. Such an octet is described by the matrix

*As an example of a similar situation, we can refer to the extension of the rotation group to the Lorentz group. As is well known, when this extension is made there is a mixing of states with a given spin with states with smaller spins.

†Mixing of ω and φ has been considered in a paper by Sakurai.[F4]

$$\begin{pmatrix} \frac{1}{\sqrt{2}}\varrho^0+\frac{1}{\sqrt{6}}\varphi+\frac{1}{\sqrt{3}}\omega & \varrho^- & K^{-*} \\ \varrho^+ & -\frac{1}{\sqrt{2}}\varrho^0+\frac{1}{\sqrt{6}}\varphi+\frac{1}{\sqrt{3}}\omega & \overline{K}^0 \\ -K^+ & K^0 & -\sqrt{\frac{2}{3}}\varphi+\frac{1}{\sqrt{3}}\omega \end{pmatrix}. \tag{6.22}$$

As the interaction which breaks the symmetry we take two terms: one of the ordinary type, Sp $H\overline{V}V$, and another simpler one which splits the masses of ω and φ. We write the second term in the simple form $h\overline{\omega}\omega$. Our interaction will now mix the φ and ω mesons, since it contains the square of the element that appears in the lower right-hand corner of (6.22). Thus the interaction we consider is

$$\Delta M = g\overline{V}_\alpha^3 V_3^\alpha + h\overline{\omega}\omega. \tag{6.23}$$

This interaction leads to the following meson masses [m_0 is a new constant not connected with the constant in (6.18)]:

$$\left.\begin{aligned} m^2(K) &= m_0^2+g, \\ m^2(\varrho) &= m_0^2, \\ m^2(\varphi) &= m_0^2+\frac{4}{3}g, \\ m^2(\omega) &= m_0^2+\frac{2}{3}g+h, \\ m^2(\overline{\omega}\varphi) &= \frac{2\sqrt{2}}{3}g; \end{aligned}\right\} \tag{6.24}$$

$m^2(\overline{\omega}\varphi)$ denotes the matrix element that mixes the original φ and ω. As in the theory of the Zeeman effect, the actual levels—the masses of the actual φ and ω—are described by the roots of an eigenvalue equation:

$$\left.\begin{aligned} \left(m_0^2+\frac{4}{3}g\right)m^2(\varphi)+\frac{2\sqrt{2}}{3}gm^2(\omega) &= \lambda m^2(\varphi), \\ \frac{2\sqrt{2}}{3}gm^2(\varphi)+\left(m_0^2+\frac{2}{3}g+h\right)m^2(\omega) &= \lambda m^2(\omega). \end{aligned}\right\} \tag{6.25}$$

The determinant of this equation is

$$\begin{vmatrix} m_0^2+\frac{4}{3}g & \frac{2\sqrt{2}}{3}g \\ \frac{2\sqrt{2}}{3}g & m_0^2+\frac{2}{3}g+h \end{vmatrix}. \tag{6.26}$$

As is well known, the sum of the roots of the eigenvalue equation is equal to the trace of the determinant (6.26):

$$m^2(\varphi)+m^2(\omega) = 2m_0^2+2g+h.$$

Comparing this with (6.24), we get

$$h = m^2(\varphi)+m^2(\omega)-2m^2(K^*).$$

The product of the roots of (6.25) is equal to the determinant (6.26). Instead of a formula for the masses we get a formula for the intervals (choosing the mass of ρ as the starting point). This means that we set the constant m_0^2 in (6.24) equal to zero. When we now use the symbols for the particles themselves to mean the squares of the masses on this scale, we find that the determinant (6.26) is equal to $4hg/3$. Since g is the mass of K*, and h is defined above, we have

$$\omega\varphi = \frac{4}{3}K^*(\omega+\varphi-2K^*) \tag{6.27}$$

or, in terms of squares of masses,

$$[m^2(\omega)-m^2(\varrho)][m^2(\varphi)-m^2(\varrho)] = \frac{4}{3}[m^2(K^*)-m^2(\varrho)][m^2(\varphi)+m^2(\omega)-2m^2(K^*)]. \tag{6.28}$$

To within experimental error this relation is satisfied by the experimental values of the masses. It is clear that the procedure we have described is based on assumptions of which we have little understanding. Formally we would have to consider an interaction of the general type $h'(\overline{\omega}\varphi+\overline{\varphi}\omega)$, introducing two new constants, h and h′. In this case comparison with experiment would lead to some relation between h and h′, but obviously would not give rise to any relation between the masses. The Schwinger solution corresponds to the choice $h' = 2^{3/2}g/3$. Whether or not this choice has any deeper meaning will be shown by the further development of the theory.

7. SPLITTING OF ISOTOPIC MULTIPLETS

The scheme used for describing the splitting of unitary multiplets can be extended so as to include also a description of the splitting of the charge multiplets, which by the conditions of the problem remain degenerate in the fields H_3^3 and H_{33}^{33}.

The simplest generalization uses the symmetry of the unitary multiplet under replacement of charge by hypercharge. Once more we write down the matrix of the baryon octet

$$\left(\begin{array}{c|cc} \frac{1}{\sqrt{2}}\Sigma^0+\frac{1}{\sqrt{6}}\Lambda^0 & \Sigma^- & \Xi \\ \hline \Sigma^+ & -\frac{1}{\sqrt{2}}\Sigma^0+\frac{1}{\sqrt{6}}\Lambda_0 & \Xi^0 \\ -p & n & -\sqrt{\frac{2}{3}}\Lambda \end{array}\right). \tag{7.1}$$

We consider the subgroup corresponding to the matrix marked out in the lower right-hand corner. Obviously its structure is analogous to that of the isotopic-spin matrix; its quantum number is called the K-spin. The components of a K multiplet are defined in the same way as those of a T multiplet (isotopic multiplet). The trace of the matrix multiplied by $6^{1/2}/2$ gives the state with K = 0 (compare this with the way Λ is obtained from the matrix of the T multiplet)

$$\Phi_0 = \frac{1}{2}(\sqrt{3}\Sigma^0 + \Lambda^0). \quad (7.2)$$

When we subtract half of the trace from each diagonal element of the 2×2 matrix, we get the function with K = 1:

$$\Phi_1 = \begin{pmatrix} -\frac{1}{2\sqrt{2}}\Sigma^0 + \sqrt{\frac{3}{8}}\Lambda, & \Xi^0 \\ n & \frac{1}{2\sqrt{2}}\Sigma^0 - \sqrt{\frac{3}{8}}\Lambda \end{pmatrix}. \quad (7.3)$$

The elements of the matrix (7.3) are made up of the three components of Φ_1: n, $+\frac{1}{2}(-\Sigma^0 + 3^{1/2}\Lambda)$, Σ^0, in analogy with the way the isotopic matrix Σ is made up up of the components Σ^+, Σ^0, Σ^-.

Finally, there are two functions with K-spin ½:

$$\Phi^{(1)}_{1/2} = (\Sigma^-, \Xi^-), \qquad \Phi^{(2)}_{1/2} = (\Sigma^+, -p). \quad (7.4)$$

In each of these functions the components have the same charge, just as the components of an isotopic multiplet have the same hypercharge.

Perturbations that conserve charge are introduced by the field components H^1_1 and H^{11}_{11}. Here we have no reason to neglect the field H^{11}_{11}, since one of the processes leading to the splitting of isotopic multiplets is the emission or absorption of a photon; the matrix element for such a process transforms like the square of H^1_1 or, what is the same thing, like H^{11}_{11}.

The components of the perturbing field have the same quantum numbers as the corresponding components of the baryon or meson multiplet (two types of field!), since the classification obviously has no connection with the concrete choice of the particles. The field H^{11}_{11} transforms like the corresponding component of the 27-plet. The components important in the theory of weak interactions will be H^1_3 and H^3_1, which possess charge and strangeness (they transform like K^+ and K^-). One can also consider fields which transform like the decuplet. The only component of the decuplet that changes neither charge nor hypercharge is the component that transforms like Σ^{0*}. We shall consider it separately.

Accordingly we take for the terms added to the mass an expression of the type

$$\Delta M = \alpha\overline{\Psi}^1_\alpha \Psi^\alpha_1 + \beta\overline{\Psi}^\alpha_1\Psi^1_\alpha - 2\gamma\overline{\Psi}^1_1\Psi^1_1. \quad (7.5)$$

For this together with the old splitting we get:

$$\left.\begin{aligned} m(\Xi^-) &= m_0 + a + \alpha, \\ m(\Xi^0) &= m_0 + a, \\ m(\Sigma^-) &= m_0 + \alpha, \\ m(\Sigma^0) &= m_0 + \frac{1}{2}(\alpha+\beta) - \gamma, \\ m(\Sigma^+) &= m_0 + \beta, \\ m(\Lambda) &= m_0 + \frac{2}{3}(a+b) + \frac{1}{6}(\alpha+\beta) - \frac{1}{3}\gamma, \\ m(n) &= m_0 + b, \\ m(p) &= m_0 + b + \beta, \\ m(\Lambda\Sigma) &= \frac{1}{2\sqrt{3}}(\alpha+\beta) - \frac{2}{\sqrt{3}}\gamma. \end{aligned}\right\} \quad (7.6)$$

The coefficient γ can be evaluated from the relation

$$\gamma = m(\Sigma^0) - \frac{1}{2}[m(\Sigma^+) + m(\Sigma^-)]. \quad (7.7)$$

The value that follows from the experimental values of the masses is $\gamma = -0.95 \pm 0.20$. We shall set $\gamma = 0$. We then get from (7.6) one further relation connecting three mass differences:

$$[m(\Xi^-) - m(\Xi^0)] - [m(p) - m(n)] = m(\Sigma^-) - m(\Sigma^+), \quad (7.8)$$

which is in comparatively good agreement with experiment:

$$\left.\begin{aligned} m(\Xi^-) - m(\Xi^0) &= 6.5 \pm 1.0, \\ m(p) - m(n) &= -1.3, \\ m(\Sigma^-) - m(\Sigma^+) &= 7.7 \pm 0.3. \end{aligned}\right\} \quad (7.9)$$

We now turn to the perturbation of the decuplet type. If as before we require that the hypercharge remain unchanged, the only term of the decuplet with Q = Y = 0 is H^{123}, which transforms like Σ^{0*}. The contribution to the mass from the decuplet is of the form

$$(\Delta M)_g = \varepsilon_{abc}H^{ade}\overline{\Psi}^b_d\Psi^c_e + \varepsilon^{ade}H_{abc}\overline{\Psi}^b_d\Psi^c_e.$$

Keeping only the terms proportional to H^{123} and H_{123}, and dropping those whose effect reduces to changing the constants in (7.5), we get

$$(\Delta M)_d = \delta_1(\overline{\Psi}^1_1\Psi^2_2 - \overline{\Psi}^1_2\Psi^2_1) + \delta_2(\overline{\Psi}^2_2\Psi^1_1 - \overline{\Psi}^2_1\Psi^1_2). \quad (7.10)$$

From this we get a new correction of $-\frac{1}{2}(\delta_1 + \delta_2) = -\frac{1}{2}\delta$ to the mass of Λ^0, one of $\delta/6$ to the mass of Σ^+, one of $-\delta_1$ to the mass of Σ^+, and one of $-\delta_2$ to the mass of Σ^-. Without incurring any contradiction with the experimental data we can set $\delta_1 = \delta_2 = 0$. Let us now consider the mixing of Λ and Σ. Because of the electromagnetic interaction Λ and Σ, which were eigenstates of the isotopic spin, become mixed, while remaining eigenvalues of the isotopic spin projection $T_3 = 0$. (An effect analogous to the mixing of a spin singlet and a spin triplet in a magnetic field.) Setting $\gamma = 0$ in (7.6), we get

$$m(\Lambda\Sigma) = \frac{1}{\sqrt{3}}[m(\Xi^-) - m(\Xi^0) - m(n) + m(p)]. \quad (7.11)$$

From the experimental values of the masses we find

$$m(\Lambda\Sigma) = 1.5 \pm 0.4 \text{ MeV} \quad (7.11')$$

From this we get for the physical (mixed) states (Dalitz[D7])

$$\left.\begin{aligned} \Lambda_{\text{phys}} &= \Lambda\cos\alpha + \Sigma^0\sin\alpha, \\ \Sigma_{\text{phys}} &= -\Lambda\sin\alpha + \Sigma^0\cos\alpha, \end{aligned}\right\} \quad (7.12)$$

where

$$\tan 2\alpha = -\frac{m(\Lambda\Sigma)}{m(\Sigma) - m(\Lambda)} = -0.019 \pm 0.006. \quad (7.13)$$

The angle α characterizes the degree of mixing of the states with T = 0 and T = 1. Since the neutral components of these states have different parities relative to charge symmetry (interchanges $p \rightleftarrows n$ and $\pi^+ \rightleftarrows \pi^-$), Eq. (7.13) characterizes the degree of deviation from charge symmetry of the Λ hyperon. An example of reactions which can be used for the experimental measurement of the angle α is $\pi^+ + d \to \Lambda + p + K^+$ and $\pi^- + d \to \Lambda + n + K^0$.

We can treat the baryon decuplet in a way analogous to the treatment of the baryon octet. If here also we confine ourselves to the lowest multipole interaction, then in analogy with the unitary splitting we find that the levels are equidistantly separated, so that

$$m(\Delta^{++}) - m(\Delta^{+}) = m(\Delta^{+}) - m(\Delta^{0}) = m(\Delta^{0}) - m(\Delta^{-}) = m(\Sigma^{*+}) - m(\Sigma^{*0}) = m(\Sigma^{*0}) - m(\Sigma^{*-}) = m(\Xi^{*0}) - m(\Xi^{*-}). \quad (7.14)$$

A deviation from the linear dependence will indicate that there is an admixture of an interaction of higher multipole order.

In the case of meson octets the relation (7.10) becomes an identity. In the pseudoscalar octet, however, the question arises as to the nature of the mass difference between π^0 and $\pi^{\pm}$, which must be zero in the 8-pole approximation. It must be ascribed to an electromagnetic interaction of the type H_{11}^{11}.

The formula (7.16) can be written in a simpler form in which the mass of Σ^0 remains undisplaced.

The field H_b^a must have quantum numbers $Q = Y = 0$; it can be written in the form of a diagonal matrix with the diagonal elements A, −A, B, −B, where A and B are arbitrary real numbers.

From this matrix we subtract the product of the unit matrix and the number $\frac{1}{2}$B. This operation is simply a shift of the origin from which mass is measured, since it does not produce any splitting. We shall see that in this way we have fixed the mass of Σ^0 as the reference level.

Writing $A - \frac{1}{2}B = \kappa$ and $\frac{3}{2}B = -\lambda$, we find that the field can be written in the form of the matrix

$$H_b^a = \begin{pmatrix} \varkappa & 0 & 0 \\ 0 & -\varkappa & 0 \\ 0 & 0 & \lambda \end{pmatrix}. \quad (7.15)$$

We denote the products of the field components by the magnitude of the multipole in Eq. (6.4) by

$$\alpha = \varkappa C_1, \quad \beta = \varkappa C_2, \quad a = \lambda C_1 \quad b = \lambda C_2.$$

Then by means of the matrix (7.15) (keeping the condition $\gamma = 0$) we get a new set of formulas for the baryon masses:

$$\left.\begin{aligned} m(\Xi^-) &= m_0 + a + \alpha, \\ m(\Xi^0) &= m_0 + a - \alpha, \\ m(\Sigma^-) &= m_0 + \alpha - \beta, \\ m(\Sigma^0) &= m_0, \\ m(\Sigma^+) &= m_0 - \alpha + \beta, \\ m(\Lambda) &= m_0 + \frac{2}{3}a + \frac{2}{3}b, \\ m(n) &= m_0 + b + \beta, \\ m(p) &= m_0 + b - \beta. \end{aligned}\right\} \quad (7.16)$$

The formulas obtained so far for the mass splitting (there have been three of them, since the masses of 8 baryons have been described by 5 parameters) have not depended on the model. If, however, we take the field seriously, then from the relations given it follows that $\alpha/\beta = a/b$, or (cf. Coleman and Galshow [D1])

$$\frac{m(\Xi^0) - m(\Xi^-)}{m(n) - m(p)} = \frac{m(\Xi^0) + m(\Xi^-) - 2m(\Sigma^0)}{m(n) + m(p) - 2m(\Sigma^0)}. \quad (7.17)$$

The agreement with this relation is poor. The left side is equal to −5, and the right side to −0.5. This means that there is a large contribution to the isotopic mass difference from purely electromagnetic corrections. There is another relation which is poorly fulfilled. The quantity

$$\frac{m(\Xi^-) - m(\Xi^0) + m(p) - m(n)}{\frac{1}{2}[m(\Xi^-) + m(\Xi^0) + m(p) + m(n)] - 2\Sigma^0} = 2\frac{\alpha + \beta}{a - b} \quad (7.18)$$

characterizes the ratio of the field components κ and λ. From the known values of baryon masses we find the value 0.038 for this quantity.

A similar quantity can be calculated from the squares of the masses of the pseudoscalar mesons. We find

$$\frac{m^2(K^+) - m^2(K^0)}{\frac{1}{2}[m^2(K^+) + m^2(K^0)] - m^2(\pi^2)} = -0.017. \quad (7.19)$$

We can understand the meaning of Eqs. (7.18) and (7.19) if we note that when we measure the masses from the center of the multiplet, i.e., from the masses of Σ^0 and π^0, respectively, we can write the two formulas in the form

$$\frac{\Delta m(\Xi) + \Delta m(N)}{2m_{av}(\Xi, N)} = 2\frac{\Delta m(K)}{m_{av}(K)}, \quad (7.20)$$

where Δm means the mass differences (not using squares!) of the respective doublets, and m_{av} the average displacements from Σ^0 and π^0. It can be seen from (7.20) that we are concerned with a universal character of the isotopic splitting which is less accurate than the universal character of the unitary splitting.

8. "TADPOLES" AND "QUARKS"

"There on unknown ways
Are tracks of beasts not seen..."
—A. Pushkin

Our task has not included the exposition of all of the ideas and results of the unitary theory, and we have left to one side, for example, such an important but as yet not clearly delineated development as that in the theory of weak interactions; nevertheless it is not without point to note certain new and still speculative ideas, since they indicate that the fauna of the unitary world may be extremely unusual. The mass splitting has shown that the interaction between the particles and the vacuum can be successfully described by a field H_b^a, whose neutral components, H_3^3 and H_1^1, are respectively responsible for the unitary and isotopic splittings. It is natural to wish to give to this field the meaning of an actual physical field, by assigning to the components of this unitary field a new unitary meson multiplet. A meson octet of this sort has been considered in a paper by Glashow and Coleman. [F3]

The idea of a tadpole has been put forward in papers by Schwinger [Ann. Phys. 2, 407 (1957)] and by Salam and Ward [Phys. Rev. Letters 5, 390 (1960) and Revs. Modern Phys. 33, 428 (1961)]. A meson octet has been introduced by Sakurai. [F4]

Let us suppose that the field H^a_b is a field of scalar mesons. If we write the matrix of this field in analogy with the matrix of the pseudoscalar mesons, the diagonal elements are two neutral particles which, to emphasize that the unitary properties are identical, we denote by π'^0 and η' (the remaining components of the octet, $\pi'^\pm$, $K'^\pm$, K'^0, $\bar{K}'^0$, are associated with change of charge or hypercharge and do not contribute to the splitting, like the corresponding components of the field H^a_b).

Since the quantum numbers of π'^0 and η' are the same as those of vacuum, they can be annihilated without any trace, provided that their masses are zero—for example, they could be converted into an unobservable bound state of proton and antiproton with total mass zero!

This means that formally there exists a process of emission of a neutral meson which in the virtual state has zero energy and is converted into an unobservable state; since in a diagram such a process is represented by a line with a "blot" on the end, such a meson is called a "tadpole."*

Obviously this scheme formally coincides with the scheme of the field H^a_b. If we add to what has been said the assumption that the interaction of a tadpole with all multipoles is described by a universal constant, we get a model in which the interval rules connecting different multiplets receive a natural interpretation.

A scalar meson from which one constructs a tadpole in a free state can also have a nonvanishing mass. In this case the authors of the model point out the possibility of identifying it with the resonances $K' \to \kappa$ (730 MeV), $\pi' \to \zeta$ (570 MeV), and η' with mass ~ 770 MeV near ρ^0; for these three components the squares of the masses are in good agreement with the interval rule:

$$m^2(K') - m^2(\pi') = 0.22, \qquad m^2(\eta') - m^2(\pi') = 0.28. \tag{8.1}$$

The very existence of these resonances and their quantum numbers are not well established, however, and the assignment cannot as yet be taken seriously.*

One can also try a different explanation of "tadpoles," using Gell-Mann's idea of unitary spinors or "quarks," which in Russian should apparently be called "besy (demons)."

The field H^a_b can be represented as the product of two unitary spinors $\bar{\psi}_b$ and φ^a:

$$H^a_b = \bar{\psi}_b \varphi^a. \tag{8.2}$$

The unitary spinor φ^a has components with charges $-e/3$, $-e/3$, and $2e/3$, and the unitary spinor $\bar{\psi}_b$ has components with charges $e/3$, $e/3$, $-2e/3$. The hypercharge of the spinors has similar fractional values.

The appearance of a fractional charge is due to the fact that in the group SU(3) the matrices of charge Q and hypercharge Y are diagonal and have zero trace. If we normalize the eigenvalues of Q and Y so that they take the values 0 and ± 1 for the components of an octet, then since these values are sums of the charges of the components of the spinors $\bar{\psi}_b$, φ^a all of these requirements are satisfied if the charges are multiples of $1/3$. In this case $1/3 + 1/3 - 2/3 = 0$, and from the charges of $\bar{\psi}_b$ and φ^a we can make only the charges 0 and ± 1. As before, only the neutral components of the product of $\bar{\psi}_b$ and φ^a take part in the interaction.

If we assume (8.2), then the action of the field H^a_b can be described as the emission and absorption of a "quark" at the same point of the diagram (or the emission of a pair $\bar{\psi}_b$ and φ^a and its subsequent annihilation). Such a loop leads to a splitting of the masses and is equivalent to a tadpole. If, however, $\bar{\psi}_b$ and φ^a can be produced in free states we arrive at Gell-Mann's scheme, but this has not been confirmed by experiment.

The search for particles responsible for the breaking of unitary symmetry reminds us of the search for the neutrino, which had given clues of itself in the form of energy nonconservation. How the new search will end, the future will show.

CONCLUSION

The formulas for particle masses, or, as we can call them in spectroscopic terminology, the interval rules, by no means exhaust the applications of the scheme of unitary symmetry and the breaking of this symmetry. These formulas are especially significant, however.

The unitary scheme has for the first time enabled

*The reader has of course noticed that the tadpole belongs to the family of "spurions" introduced by various authors to describe processes of symmetry breaking. Tadpole diagrams can obviously also exist for ordinary π^0 and η mesons, but if their interaction is unitary-invariant, such tadpoles do not lead to any splitting.

*Glashow and Coleman refer to the following experiments: G. Alexander et al., Phys. Rev. Letters 8, 447 (1962); D. H. Miller et al., Physics Letters 5, 279 (1963); S. G. Wojcicki et al., Physics Letters 5, 283 (1963); ζ: D. B. Lichtenberg, Stanford Linear Accelerator Report No. 13 (1963) (unpublished), page 53; η': V. Hagopian and W. Selove, Phys. Rev. Letters 10, 533 (1963); Z. Guiragossian, Phys. Rev. Letters 11, 85 (1963).

us to consider the masses of particles from a unified, albeit still very imperfect, point of view. Hitherto the differences between the masses had been regarded only as a vexing violation of the symmetry, and it seemed that only in the high-energy region, in which the differences may be unimportant, could attempts at theoretical schemes be made.

In the scheme of SU(3) it was unexpectedly discovered that the breaking of symmetry has simple properties and can be described very naturally in the scheme of unitary multiplets. The question arises as to whether from the nature of the symmetry breaking one can figure out the properties of the interaction between the multiplets and the vacuum. This is a natural question if we recall that the breaking of isotopic symmetry is caused by the electromagnetic field (the interaction of particles with the electromagnetic field of the vacuum); by studying the deviations from isotopic symmetry in various reactions, one could in principle obtain a large amount of information about this interaction (although in this case the splitting is small). Of course there is no need to do this, since we have better methods for studying the electromagnetic field. The situation is different in the case of the interaction with the field H_3^3 that breaks the SU(3) symmetry.

This interaction, which fortunately is comparatively large, cannot be reduced to any known field. Therefore the study of decays and reactions from the point of view of the group SU(3) is a good source of information about the interaction of particles with the vacuum.

The simplicity of the resulting scheme allows us to expect that in this very direction there will be important progress in the understanding of the laws of strong interaction.

APPENDIX

1. THE OCTET AS AN EIGHT-VECTOR

In the algebra of SU(2) the components of a second-rank tensor form a three-dimensional vector. The connection between the two representations is accomplished with the Pauli matrices in Eq. (2.27). In the same way we can put an eight-vector in correspondence with the components of a unitary octet. In the notation of Gell-Mann[G2,3] the octet A_b^a is written in the form

$$A_b^a=\begin{pmatrix} A_3+\frac{1}{\sqrt{3}}A_8, & A_1-iA_2, & A_4-iA_5 \\ A_1+iA_2, & -A_3-\frac{1}{\sqrt{3}}A_8, & A_6-iA_7 \\ A_4+iA_5, & A_6+iA_7, & -\frac{2}{\sqrt{3}}A_8 \end{pmatrix}.$$

Obviously any octet can be written in this form. We remark that it is customary to denote the components of the unitary spin (8.8) by F_α ($\alpha = 1, 2, \ldots, 8$).

The relations between A_b^a and A_α can be written most simply if we introduce seven matrices λ_α which here play a role analogous to that of the Pauli matrices. The forms of the matrices λ_α are

$$\lambda_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\quad \lambda_2=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix},\quad \lambda_3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},$$

$$\lambda_4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\quad \lambda_5=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix},$$

$$\lambda_6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad \lambda_7=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\quad \lambda_8=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}.$$

The pairs of matrices (λ_1, λ_2), (λ_4, λ_5), and (λ_6, λ_7) are the Pauli matrices σ_1 and σ_2 in three two-dimensional subspaces. Here there are two matrices of the type of σ_3, since there is a supplementary condition $\mathrm{Sp}\,\lambda_\alpha = 0$.

The matrices λ_α satisfy the following conditions:

$$\mathrm{Sp}\,\lambda_\alpha\lambda_\beta=2\delta_{\alpha\beta},$$
$$[\lambda_\alpha, \lambda_\beta] \equiv \lambda_\alpha\lambda_\beta-\lambda_\beta\lambda_\alpha=2if_{\alpha\beta\gamma}\lambda_\gamma,$$
$$\{\lambda_\alpha, \lambda_\beta\} \equiv \lambda_\alpha\lambda_\beta+\lambda_\beta\lambda_\alpha=2d_{\alpha\beta\gamma}\lambda_\gamma+\frac{4}{3}\delta_{\alpha\beta}.$$

The values of the "structural" factors—the antisymmetric $f_{\alpha\beta\gamma}$ and the symmetric $d_{\alpha\beta\gamma}$—are given by the formulas

$$f_{\alpha\beta\gamma}=\frac{1}{4i}\mathrm{Sp}\,[\lambda_\alpha\lambda_\beta]\,\lambda_\gamma:\ f_{123}=1,$$
$$f_{147}=f_{246}=f_{257}=f_{345}=f_{516}=f_{367}=\frac{1}{2},$$
$$f_{458}=f_{678}=\frac{\sqrt{3}}{2};$$
$$d_{\alpha\beta\gamma}=\frac{1}{4i}\mathrm{Sp}\,\{\lambda_\alpha, \lambda_\beta\}\,\lambda_\gamma:\ d_{118}=d_{228}=d_{338}=-d_{888}=\frac{1}{\sqrt{3}},$$
$$d_{146}=d_{157}=-d_{247}=d_{256}=d_{344}=d_{355}=-d_{366}=-d_{377}=\frac{1}{2},$$
$$d_{448}=d_{558}=d_{668}=d_{778}=-\frac{1}{2\sqrt{3}}.$$

The remaining nonvanishing components are obtained by permuting the indices, with appropriate sign changes in the case of the antisymmetric $f_{\alpha\beta\gamma}$.

It is obvious that

$$A_b^a=\sum_{\alpha=1}^{7} A_\alpha(\lambda_\alpha)_b^a,$$
$$2A_\alpha=\mathrm{Sp}\,A\lambda_\alpha \equiv A_b^a(\lambda_\alpha)_a^b.$$

The coefficients f and d enable us to write out products of octets. The formula for multiplying octets

$$(X^\pm)_b^a=\frac{1}{2}(A_c^aB_b^c \pm A_b^cB_c^a)$$

is now rewritten:

$$(X^\pm)_\alpha=i\begin{pmatrix}d_{\alpha\beta\gamma}\\f_{\alpha\beta\gamma}\end{pmatrix}A_\alpha B_\beta.$$

The symmetric product is sometimes called the D-product, and the antisymmetric product the F-product.

2. $\mathscr{C}$-PARITY

Let us consider two transformations of an octet: 1) the R-transformation, which is interchange of the rows and columns of the octet, and 2) the charge conjugation C. It is not hard to see that the product of these two transformations

$$\mathscr{C} = RC$$

leaves all elements of meson (Hermitian) octets in their original places and can multiply them all by +1 or else by −1 (since $\mathscr{C}^2 = 1$). Accordingly we have a new quantum number, the $\mathscr{C}$-parity, characterizing Hermitian octets.

It is obvious that the $\mathscr{C}$-parity is determined by the charge parity of the particles that are on the diagonal and keep their places in the R-transformation. Therefore the octet of pseudoscalar mesons has $\mathscr{C} = +1$, and the octet of vector mesons has $\mathscr{C} = -1$.

From the definition of the components of the eight-vector it is clear that the components A_1, A_3, A_4, A_6, A_8 have the same charge parity, which is opposite to the charge parity of the components A_2, A_5, A_7.

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