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CUMULATION OF ENERGY AND ITS LIMITS

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INTRODUCTION

MANY cumulation phenomena are known in which the energy density per unit volume increases spontaneously, and in such a way as to attain arbitrarily large values. They include the collapse of bubbles in liquids, convergent shock waves (e.g., spherical waves), and also electromagnetic shock waves (cylindrical and conical). These phenomena arouse in physicists an understandable interest and a natural tendency to find the causes which ultimately limit the increase of energy density in them.

There is as yet no complete answer to this question, in spite of the fact that we can write exact equations for each of these phenomena and in principle solve them as precisely as we like.

This prepares the ground for the appearance of hypotheses.

A review is given below of some cases of infinite cumulation, and it is shown that dissipation phenomena (viscosity, heat conductivity) sometimes do not act as limiting factors. However, whenever one can carry out a thorough stability analysis, it indicates that cumulation is limited by instability.

On this basis, the hypothesis has been advanced that every infinite cumulation is necessarily unstable, and in such a way that it does not simply change in form owing to instability, but completely breaks down (becomes finite).

I. A REVIEW OF CUMULATION PHENOMENA

1. Collapse of Bubbles in a Liquid

The collapse or cavitation of spherical bubbles in an incompressible liquid is evidently the first studied case of infinite cumulation. Rayleigh discussed it in 1917.[1]

In the focusing stage in this phenomenon, the centripetal velocity of the bubble surface increases without limit near the center, and also the pressure nearby.

A real liquid will be appreciably compressed by the large pressure. This will change the nature of the motion, and possibly decrease the acceleration toward the center.

It did not change the result qualitatively[2,3] to take the compressibility into account by using a power-function equation of state of the form $p = \text{const} \cdot \rho^{\gamma}$ for any $\gamma > 1$.

As before, the pressure in the liquid becomes infinite at the focus, and also its velocity and density, although with a lower order of infinity: whereas the velocity of the bubble surface increases as $r^{-3/2}$ in an incompressible liquid, it increases only as $r^{-0.411}$ when $\gamma = 3$.* The temperature remains everywhere finite. Thus, accumulation occurs even in a compressible liquid, and remains infinite. Near the center, the liquid undergoes strong and rapid shear deformations, which could lead to a strong energy dissipation due to viscosity.

It has been possible to take the viscosity into account exactly in the incompressible-liquid problem,[4] but even this did not alter the result: if the initial radius of the bubble is large enough (more exactly, if $R \geq 8.4\sqrt{p/\rho}$), cumulation occurs even in a viscous liquid, without even lowering the order of infinity. The role of the viscosity is reduced merely to a certain diminution in the effective energy of the motion. Only bubbles having dimensions less than critical cavitate slowly; in this case their motion up to the focusing point completely "forgets" the initial dimensions, and independently of them, the velocity varies here according to the law:

$$u = -\frac{p}{4\nu\varrho} r.$$

Thus, neither compressibility nor viscosity alone eliminates cumulation in bubbles, at least in certain cases.

The atomic nature of matter sets an unconditional limit to cumulation, since wherever the dimensions of the bubble are comparable to an atom, the liquid ceases to resemble a continuous medium, and cumulation ceases. However, this limit is very ample and not at all fundamental. For example, for a one-centimeter bubble in water, it does not formally prevent one from obtaining a velocity of the order of 10^9 cm/sec. However, even this ample limit can be raised by increasing the initial dimensions of the bubble or the pressure of the liquid.

2. Convergent Shock Waves

Convergent spherical and cylindrical waves are also accompanied by infinite cumulation upon focusing. Landau and Stanyukovich[5] have shown this analytically for waves in an ideal gas for $\gamma = 3$, and

*Here and below the fractional exponents were found from the exact solution describing the neighborhood of the center about the instant of focusing.

Guderley[6] has done this for $\gamma = 1.4$.

In these cases, the density of the material is everywhere finite, but the velocity, pressure, and temperature are infinitely large: $T \sim r^{-1.14}$ when $\gamma = 3$, and $T \sim r^{-0.79}$ when $\gamma = 1.4$.

The temperature gradients near the center are infinitely large, and this could lead to a high energy dissipation due to heat conductivity. Because of this, the wave front becomes diffuse, and the temperature attained in the center actually becomes finite. This problem has been discussed in[7].

In the convergence stage in this case, a thermal wave front precedes, but when $\gamma < 3$, it is followed by a second wave, or isothermal density discontinuity, which is a convergent shock wave having an amplitude growing to infinity. Thus, heat conductivity has only eliminated the infinite temperature, but infinite density has appeared. That is, infinite energy density still exists. Thus, the cumulation has changed in form but has not disappeared.

The limit on the attainable temperature due to heat conductivity is not fundamental, like that due to the atomic nature of matter: its maximum value is determined by the scale of the phenomenon (the dimensions of the wave at unit amplitude), and it can be made as large as one wishes by increasing the scale.

3. Convergent Electromagnetic Shock Waves

An electromagnetic wave can have a steplike front, i.e., it can be a shock wave. It can arise when an ordinary shock wave emerges from a conductor to a surface where there is a parallel magnetic field. The concept of this type of wave and certain examples of cumulation in them, showing qualitatively new characteristics, are discussed in[8,9].

a) Cylindrical wave. Let a cylindrical cavity in a conductor contain a longitudinal magnetic field. In addition, let a convergent cylindrical shock wave pass through the material of the conductor, simultaneously emerging at the surface of the cavity. This will give rise to a convergent electromagnetic shock wave in the cavity having an arbitrarily sharp front. As was shown in[8], the width of the front in vacuo upon emerging from copper amounts to ~10 cm. However, it is 0.1 cm wide on emerging from copper cooled to 20°K (this involves the improvement in the conductivity upon refrigeration).

The front of the convergent wave brings about an increase in the longitudinal magnetic field and the appearance of an annular electric field.

If the front is sharp, then the fields on it increase as $r^{-1/2}$ as it approaches the axis, i.e., it increases without limit.

An additional interesting feature was found: the amplitude reflected from the axis of the wave proved to be infinite not only at the reflection site but also at a finite distance from the axis. That is, the infinitely strong wave of the field, upon arising at the axis, then passes through all the points of a finite volume.

This characteristic was found in studying the self-similar solution of the equations for the wave near the axis[8]: the series describing the field near the reflected wave diverges logarithmically as one approaches it from any side.

Ya. B. Zel'dovich[10] has obtained the same result by another method by considering a cylindrical wave as a superposition of plane waves.

The same characteristic is shown by weak cylindrical shock waves (acoustic waves). That is, it does not involve the physical nature of the shock wave, but its geometric form. (Evidently, the equations of acoustics, which describe only weak waves, are valid only when the density variations due to the wave are small).

b) Conical wave. The concept of an electromagnetic shock wave has made it possible to construct a curious example of steady-state cumulation, apparently the first of its kind, as described in[9].

A convergent conical wave in a material does not produce an infinite cumulation, since the vertex of the cone is unavoidably truncated by the intensification and acceleration of the wave near the axis. (This situation is well-known as a theorem of the impossibility of certain conical flow patterns.) Hence it has been impossible to construct an example of steady-state cumulation for ordinary shock waves.

This hindrance does not exist for the shock wave of a field whose velocity is constant (and equal to the velocity of light), and it can be a converging conical wave as far as the vertex of the cone, where an infinite cumulation occurs, and furthermore, as a steady-state process. The fields at the front near the axis increase here as $r^{-1/2}$. Just as with the cylindrical wave, the amplitude reflected from the axis of the wave proved to be infinite not only on the axis but also over the whole front.

The extent of cumulation of field shock waves encounters no physical limitations as long as Maxwell's equations hold.

Owing to the constancy of velocity of all field waves, a diffuse front remains diffuse, and in distinction from a wave in matter, a discontinuity does not develop in it. Diffuseness of the front, due to non-ideality of the conductor that "set off" the field and gave rise to the wave, persists as the wave converges on the axis, and finally limits its cumulation. However, this limitation is also not absolute, and can be overcome by increasing the scale of the phenomenon.

II. ON INSTABILITY OF CUMULATION

The reason for limitation of cumulation can be instability, i.e., loss of symmetry of the phenomenon.

We can find out by studying the behavior of small perturbations whether a fundamental pattern of cumulation is stable. However, we still do not know what it changes into when it is unstable. Cumulation can completely break down (the infinite energy density can disappear), or it can merely change in form. For example, a collapsing bubble can be deformed and change into a torus. In collapsing, the latter in turn could continue to show cumulation, but in a different form.

Thus, there are two different questions: is the fundamental process stable, and what does it change into when unstable? and does infinite cumulation continue to exist at all?

The second question is especially interesting, but cannot be solved by an analysis of small perturbations. Since we want to pay especial attention to it in particular, we shall discuss below only those cases in which we can investigate the behavior not only of small, but also large perturbations. We can follow the behavior of perturbations throughout all stages of the cumulation process, and exactly besides, in one special case that has proved highly instructive.

Let a thin cylindrical shell of an ideal liquid move toward its axis while rotating slowly. The shell thickens as it converges, and the rotation of the inner layers accelerates. Owing to the centrifugal force, the shell does not reach the axis, but begins to expand again. There is no infinite cumulation in this case.

The additional degree of freedom (rotation), no matter how weakly it is excited, gradually takes up more and more energy from the main form of motion until it has taken over all the energy. Upon expansion, the energy is again transformed into radial motion.

We find from the law of conservation of angular momentum that the tangential velocity is proportional to r^{-1}. We can easily see that at the instant that the radial motion stops, the energy per unit length of the cylinder is

$$E = \pi\varrho v^2 r^2 \ln\frac{R}{r},$$

where r and R are the inner and outer radii of the cylinder, and v is the tangential velocity at the inner boundary, i.e., the maximum velocity reached throughout the process.

A simple calculation shows that the angular momentum is

$$Q = mrv,$$

where m is the mass of the cylinder (this and other quantities are per unit length).

We obtain from these equations

$$v = \frac{Q}{mR} e^{\frac{m^2 E}{\pi\rho Q^2}},$$

i.e., the value of the velocity for any finite Q is not infinitely great, but finite.

In this example, a small perturbation Q rules out infinite cumulation, but the smaller this perturbation is, the greater the energy density reached.

This result is natural, and fully explains the limitation of cumulation in the considered case.

The question might arise whether cumulation might not reappear if we introduce viscosity, which will hinder the rotation of the inner layers with respect to the outer, and will diminish the centrifugal force. However, as it turns out, viscosity here rules out cumulation per se, even without rotation; a viscous cylindrical shell stops before it reaches the axis. Thus, cumulation has not been reintroduced by viscosity in this case. However, the decay of certain "dangerous" perturbations through dissipation is not ruled out in other cases, of course.

Another example of a complete breakdown of cumulation due to small perturbations is given by field shock waves. The appearance of a singularity at the center of a convergent cylindrical field wave involves the simultaneous arrival there of the front from all sides. This becomes especially clear if we use Zeldovich's[10] conception of a cylindrical wave as being the sum of a large number of plane waves. However, if the front is not completely cylindrical (or more precisely, if its surface nowhere strictly coincides with a cylinder), the simultaneous arrival at the center will not take place, owing to the constancy of velocity, and the singularity will disappear, i.e., cumulation will break down.

Of course, this also holds for a conical convergent wave. Just as for the rotating shell, the attainable energy density at the center is greater for smaller initial perturbations.

III. THE HYPOTHESIS OF THE INSTABILITY OF CUMULATION

It seems natural to expect that cumulation is limited by something besides the atomic nature of matter. As the examples have shown, dissipation does not provide these limitations, at least in certain cases.

On the other hand, certain perturbations of the main motion lead to the breakdown of cumulation, as could be thoroughly investigated in two cases. On these grounds, we can advance the following hypothesis: every infinite cumulation is unstable. Attempts to prove this have not yet succeeded. A contrary example would be interesting, but this also has not yet been found.

We note that the proposed proof must be very general, and not involve any concrete form of the equations of the process, since these equations differ in different cases (the equations of gas dynamics for various processes and various equations of state, and the equations of electrodynamics for waves of various configurations).

A special case of cumulation is the collapse of the universe, whose possibility is seemingly implied by the equations.

E. M. Lifshitz and I. M. Khalatnikov[11] have studied this problem; they showed that collapse can occur only for a symmetrical initial state, while the existence of asymmetry rules it out.

The hypothesis advanced here resembles this theorem (being a generalization of it), and this enhances its plausibility.

The hypothesis can bear upon certain phenomena in stars. A rapid and very strong compression of a star or its inner part can evidently occur only when its initial state is sufficiently symmetrical, in particular, when it shows hardly any rotation.

If this is so, then the evolution of a star and its possible cataclysms are not unequivocally determined by such parameters as its mass, dimensions, etc., but can depend on how symmetrical the star is, or in particular, how rapidly it is rotating.

In technology, the limitation of cumulation can be of interest in studying the pinch effect in pulsed discharges in gases.

CONCLUSION

The phenomenon of cumulation is encountered in nature and in technology. It is natural to expect that it is somehow limited. In a number of cases, it has not been possible to point out physical limitations, but in other cases it has been shown that cumulation completely breaks down owing to instability. This has provided grounds for the hypothesis that every cumulation process is unstable, and that instability not only alters the fundamental process of infinite cumulation, but rules it out completely.

It would be of interest either to prove this hypothesis, or to reject it through some example.

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