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DYNAMICAL SYMMETRY OF STRONGLY INTERACTING PARTICLES

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CONTENTS

1. QUANTUM NUMBERS OF HADRONS

1. Baryon Number

AFTER Einstein established the equivalence of energy and mass it became incomprehensible why ordinary matter was stable. Why doesn't it annihilate, say by being converted to radiation?

If any process is not realized, it is natural to attribute this to a selection rule imposed by some conservation law. The laws of conservation of energy, momentum and angular momentum do not forbid the annihilation of matter. The stability of matter indicates the existence of another class of conservation laws.

One such conservation law has been known for a long time. This is the law of conservation of charge. This law enables us to understand qualitatively the nature of the stability of the electron. We argue as follows. The spectrum of masses of particles is discrete, like the energy levels of a quantum mechanical system. Among these levels there is always a lowest level, and this state is stable. But the electron has the smallest mass of all the charged particles.

But the conservation of charge does not forbid the conversion of a proton into a positron with the emission of a photon or the annihilation of a neutral atom. The proton retains its stability if we attribute to it, in addition to charge, another conserved quantity which is not possessed by the electron. We call this quantity the baryonic charge or baryon number. We shall denote the baryon number by A. Suppose that the proton has $A = 1$ (just as the electric charge is expressed in units of the charge on the electron). If the proton has the smallest mass of all the baryons (baryonically charged particles), it will be stable. For nuclei A will obviously coincide with the mass number. These two conservation laws, for the charge Q and the baryon number A, already guarantee the possible existence of our universe consisting of nuclei and electrons.

The baryon number does not exhaust the "charges" that have had to be introduced in the physics of elementary particles. We now know two other similar rigorous conservation laws. But these apply to leptons, and will not be needed in what follows.

2. Gauge Groups

After having introduced the baryon number, it is useful to find a place for it in the formal scheme of quantum mechanics.

In quantum mechanics every conservation law expresses a particular symmetry of the system. This symmetry appears as a group of transformations that can be applied to the wave function of the system. The laws of conservation of momentum, angular momentum and energy reflect the symmetry properties of space and time. (They may therefore be said to be geometric.[1]). For example, the homogeneity of space leads to the group of transformations $\psi \to e^{ikx}\psi$, where x is the displacement of the system (or the displacement of the coordinate origin, taken with the opposite sign), and $\hbar k$ is its momentum. The fact that such a transformation is admissible at any moment of time expresses, from this point of view, the law of conservation of momentum. In fact, suppose for example that there is a collision of two particles. Suppose that before the collision the particles had momenta $\hbar k_1$ and $\hbar k_2$, and that their wave functions were ψ_1 and ψ_2. Then in the displacement the wave function $\psi = \psi_1\psi_2$ was subjected to the transformation $\psi \to e^{ikx}\psi$, where $k = k_1 + k_2$. The properties of the particles and their number after the collision can have changed arbitrarily, but the form of this transformation must not be changed, i.e., k = const.

Now we imagine that the properties of the particles and their interactions are such that the wave function of the system can be subjected to an analogous transformation with a phase factor: $\psi \to e^{iG\alpha}\psi$, where α, like x, plays the role of the parameter of the transformation, while G, like k, characterizes the state of the system. If we assume the parameter α to be universal, i.e., like the displacement of the origin of coordinates, it is the same for all systems, and that the transformation is admissible at any time, then this will signify the presence of an additional conservation law, G = const.

We must emphasize that the parameter α is no longer directly related to the properties of space and time. In contrast to the geometrical symmetries, such a symmetry might therefore be called dynamical. (In treating collisions of particles, one customarily regards all conservation laws as kinematical.)

The transformation $\psi \to e^{iG\alpha}\psi$ represents the general form of transformation admissible in quantum mechanics, depending on one parameter. This follows from the fact that a transformation should not change the square modulus of the wave function, leaving $|\psi|^2$ invariant (the unitarity property).

This group is called the gauge group. In this case, where we know the equations of motion, the gauge group should of course be contained in them. This state of affairs exists in electrodynamics, whose equations are invariant under the transformation $\psi \to e^{iQ\alpha_q}\psi$, where Q is the charge of the system. Introducing the baryon number, we postulate the existence of another gauge group $\psi \to e^{iA\alpha_a}\psi$, where A is the baryon number.

The introduction of a gauge group is in a way equivalent to introducing an additional degree of freedom of the system. For the same dependence on all other variables, states of the system are distinguished by different values of the quantum number G. We may say that the wave function depends on an additional continuous variable, the group parameter α. For an infinitesimal change of the parameter, the wave function ψ changes by

$$\delta\psi = \frac{\partial\psi}{\partial\alpha}\,\delta\alpha = i\delta\alpha\, G\psi,$$

where $G = -i\partial/\partial\alpha$ is a hermitian operator whose eigenvalue is the quantum number G. We have here the usual connection between an infinitesimal transformation (generator) of the group and a physical quantity. G and α are canonically conjugate quantities.

Let us give an example in which α is used as an additional coordinate. Let us consider two particles (proton and neutron), differing only in their charge Q = 1, 0, which are in the same state, i.e., the dependence of their wave functions on space and spin is the same; we denote the wave function by φ. Then the proton wave function is $p = \varphi e^{i\alpha}$, and the neutron function $n = \varphi$. Their different dependence on α makes the functions orthogonal:

$$\int p^* n \, d\alpha = 0.$$

(The last integral can be taken either between infinite limits or over an interval of 2π; cf. later in par. 4.)

If a particle has charge Q and baryon number A, the corresponding antiparticle should be assigned the quantum numbers $-Q$ and $-A$. This follows immediately from the fact that the antiparticle is described by a wave function that is the complex conjugate of the wave function for the particle. Thus the gauge transformations for them differ in the sign of the phase.

3. Hypercharge

A rather large number of baryons are known, i.e., particles for which the baryon number is A = 1. Among these the proton has the least mass. All the other baryons are unstable, and in the course of time transform into a proton and a number of leptons and photons guaranteeing the fulfilment of the other conservation laws. A further classification of the baryons and the study of their symmetry properties is possible only by using approximations associated with dividing their interactions into three types: strong, electromagnetic, and weak. We shall be interested in the particles possessing the property of strong interaction. We call them hadrons.[2]

If we consider processes in which we can neglect the weak interaction ("switch it off"), the hadrons are subjected to two additional conservation laws, to which there correspond two gauge groups.

First, only the weak interaction can change the electric charge of a system of hadrons, transferring leptons (electrons and muons) to it. Thus when the weak interaction is switched off, we have not only conservation of the total charge Q of the closed system, but we have separate conservation of the leptonic charge Q_l and the charge of the hadrons, which we shall denote by Z $(Q = Q_l + Z)$. The law of conservation of the charge of the leptons splits further into two separate conservation laws: for the charge of the electrons Q_e and the charge of the muons Q_μ. We may say that we are dealing not with one gauge group $\psi \rightarrow e^{iQ\alpha}\psi$, but with three independent groups $\psi \rightarrow e^{iZ\alpha_Z}, e^{iQ_e\alpha_e}, e^{iQ_\mu\alpha_\mu}\psi$. Switching on the weak interaction reduces the symmetry, which manifests itself in the "synchronization" of the three parameters: $\alpha_Z = \alpha_e = \alpha_\mu$; at the same time the three conservation laws are coalesced into one.

Thus when the weak interaction is switched off, we may consider the law of conservation of the charge Z of the hadrons independently of the presence of other particles.

Secondly, there also exist hadrons (strange particles) which become stable when the weak interaction is switched off, even though all the conservation laws, including the conservation of A and Z, do not forbid their decay (in particular, there exist baryons for which $A = Z = 1$, as for the proton). The way to describe this stability is as above. We must introduce still another additive quantum number; it is called the hypercharge, and is denoted by Y (the difference $Y - A = S$ is called the strangeness). To this conservation law there corresponds the gauge group $\psi \rightarrow e^{iY\alpha_Y}\psi$.

It appears that we can assign to each of the hadrons known at present a value of Y so that the conservation of the total hypercharge together with the conservation of Z and A can explain the allowed or forbidden nature of reactions which are caused by strong and electromagnetic interactions. In particular, for nuclei $Y = A$.

4. Combining of Hadrons

When we say that a "nucleus consists of nucleons", we mean the following two properties of nuclei. First, the nucleus with quantum numbers A and Z can be gotten from Z protons and A—Z neutrons without using weak interactions; secondly, the mass defect of the nucleus is small.

There are systems for which the first assumption is satisfied but not the second. In this case, instead of the word "consists of" we shall say "can be built up from" or "is composed of." We thus obtain, in particular, a class of particles, the mesons, which are hadrons with $A = 0$. The nonstrange $(Y = 0)$ mesons are built up from nucleons and antinucleons. The unstable baryons (baryon resonances) are made up of a nucleon and a meson.

Since $Y = A$ for nucleons, this relation is preserved for all composites of nucleons. Strange particles can be gotten from "nucleonic" particles by adding only particles of the type of the Λ hyperon $(A = 1, Z = 0, Y = 0)$ and its antiparticle $\overline{\Lambda}$. All the hadrons known at present are composed of the appropriate number of p, n and Λ (the "Sakata triplet") and their antiparticles.

This scheme contains no model aspects. It merely expresses a definite property of the hadrons—that they are composite. We know that the quantum numbers A, Z and Y are integers. These quantum numbers are the eigenvalues of the generators of the corresponding gauge groups. Thus the gauge groups are cyclic, like the group of rotations about an axis: the transformations corresponding to the parameters α and $\alpha + 2\pi n$ are identical. We have no way of interpreting this property other than the simple scheme of composites. Here the three "fundamental" particles, p, n, and Λ, are not distinguished from the other stable particles. Instead of them we can, for example, use others, say p, Λ and Σ^+. An example of the use of such a composition of the particles is given in Sec. 3.

5. The U_2 and SU_2 Groups

The masses of the two nucleons differ very little from one another $(\Delta M/M \simeq 0.0014)$. This gave rise to a fruitful idea: all the differences between proton and neutron are caused solely by their different electromagnetic properties. Actually the electromagnetic interaction is characterized by a small parameter of this same order of magnitude, $e^2/\hbar c = 0.0073$.

Let us switch off the electromagnetic interaction, just as earlier we switched off the weak interaction. We then get a typical picture of quantum mechanical degeneracy. There are two states (p and n) differing only in one quantum number $(Z = 1, 0)$, which plays no role in the interaction, just like two atomic states which differ only in their magnetic quantum number, in the absence of an external field. Any superposition $\psi^1 p + \psi^2 n$ has the same properties as the neutron or the proton.

The set of two quantities $\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$ is called an isospinor. We shall also call it a t-spinor or (when there is no danger of a misunderstanding) simply a spinor. The equivalence of different superpositions means that a spinor can be subjected to the transformation $\psi \rightarrow V\psi$, where V is a 2-by-2 matrix. The transformation must preserve the normalization and orthogonality of two linearly independent states (like p and n). This means that the matrix V must be unitary:

$$VV^+ = 1$$

(where V^+ is the hermitian conjugate of V).

The collection of such transformations forms a group, which is called the group U_2 (the group of unitary transformations of a two-dimensional linear complex space).

From the unitarity of V it follows that the modulus of its determinant is equal to unity, i.e.,

$$\det V = e^{2i\alpha}.$$

Thus every matrix V can be written in the form

$$V = Ue^{i\alpha},$$

where U is a unitary matrix with unit determinant (unimodular):

$$UU^+ = 1, \qquad \det U = 1.$$

Thus each transformation of the group U_2 can be split into two (commutative) transformations. The first reduces to multiplication by an arbitrary phase factor $e^{i\alpha}$. These transformations form a gauge group. Groups of this sort corresponding to the quantum numbers A, Z, and Y were already treated earlier.

We need therefore only look at the second transformation, which is accomplished by the matrix U. The set of such (unitary unimodular) transformations form a group that is called SU_2. This is precisely the group that expresses the identity of proton and neutron with respect to the strong interaction. We shall also call it the isospin group (isogroup).

In a purely mathematical sense, the group SU_2 is equivalent (isomorphic) to the group of rotations of three-dimensional (real) space. On the one hand this is a fortunate circumstance, since the formal apparatus for SU_2 is just the familiar algebra of three-dimensional vectors, spinors, and quantum mechanical angular momentum operators. On the other hand, this coincidence has its psychologically bad points. Rotations in the unphysical "isospace" are operations whose physical meaning is unclear. But the primary operations are not these rotations but rather the unitary transformations. Their meaning is clear—they reflect the degeneracy of the nucleon. We still do not understand the nature of this degeneracy.

Those properties of particles that are related to the isogroup are widely used and well known. Nevertheless we shall devote the next paragraph to SU_2; this will enable us to present the properties of SU_3 economically.

2. THE ISOSPIN GROUP

1. The Generators of SU_2 and the Nucleon Diagram.

A matrix U which is a transformation of the group SU_2, like any unitary matrix can be written in the form

$$U = e^{iH},$$

where H is a hermitian matrix. Since det U = 1, Sp H = 0.

Under infinitesimal transformations,

$$U = 1 + i\delta H, \quad \delta\psi = i\delta H\psi.$$

There are altogether three $(2 \times 2 - 1)$ linearly independent 2×2 hermitian matrices with zero trace. These are the Pauli matrices σ_i (i = 1,2,3). Thus the Pauli matrices are the generators of SU_2. This last statement means that

$$\delta\psi = i \sum_{i=1}^{3} \delta\alpha_i \sigma_i \psi.$$

We shall also call these matrices isospin (or simply spin) operators. We see that the group SU_2 contains three real parameters α_i (like the group of 3-dimensional rotations).

We shall assume that the Pauli matrices are normalized so that $\sigma_i^2 = 1/4$. Let p and n be eigenfunctions of σ_3. The eigenvalues of σ_3 are $\pm 1/2$. This quantum number (the "isospin projection") distinguishes p from n. It therefore serves the same function as Z, but more symmetrically. Obviously $Z = 1/2 + \sigma_3$. Thus the charge gauge group appears as a subgroup of SU_2 $(\alpha_z = \alpha_3)$.

The other two generators transform one of the nucleons into the other. If we introduce the matrices

$$\sigma_{\pm} = \sigma_1 \pm i\sigma_2,$$

then

$$\sigma_+ n = p, \quad \sigma_- p = n, \quad \sigma_+ p = \sigma_- n = 0.$$

All the properties of the generators are illustrated by the "nucleon diagram" (Fig. 1). It shows an axis on which the two points $(\pm 1/2)$ correspond to the nucleons (eigenstates of σ_3); the arrows show the effect of the operators $\sigma_\pm$. It is easy to obtain the commutation properties of the operators by using the diagram. For example, let us find $[\sigma_+\sigma_-]$. Suppose we apply this operator to p:

$$(\sigma_+\sigma_- - \sigma_-\sigma_+)p = \sigma_+\sigma_- p = \sigma_+ n = p = 2\sigma_3 p.$$

Thus

$$[\sigma_+\sigma_-] = 2\sigma_3.$$

Instead of the matrices σ_i or $\sigma_\pm$ it is sometimes convenient to consider four matrices $\sigma^j{}_k$ (j,k = 1,2) defined as follows:

$$\sigma_1^1 = -\sigma_2^2 = \sigma_3, \quad \sigma_2^1 = \sigma_-, \quad \sigma_1^2 = \sigma_+,$$

i.e., so that

$$\sum_j \sigma_j^j = 0, \quad \sigma_k^j = \sigma_j^{k+}.$$

One can give a general expression for the matrix elements of $\sigma^j{}_k$ (we use the convention of labelling the rows of the matrix by a superscript and the columns by a subscript):

$$(\sigma_k^j)_\beta^\alpha = \delta_\beta^j \delta_k^\alpha - \frac{1}{2} \delta_k^j \delta_\beta^\alpha,$$

from which we find the expression for the commutators:

$$[\sigma_k^j \sigma_m^l] = \delta_m^j \sigma_k^l - \delta_k^l \sigma_m^j.$$

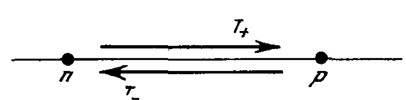


FIG. 1. The nucleon diagram. The arrows show the effect of the operators $T_\pm$ $(\sigma_\pm)$; the horizontal axis is the axis of T_3; the points are the nucleon states: p $(T_3 = 1/2)$ and n $(T_3 = -1/2)$.

An infinitesimal transformation can then be written as

$$\delta\psi = i\sum_{j,\,k}\delta\alpha_j^k\sigma_k^j\psi,$$

where the parameters α_j^k form a hermitian matrix with zero trace:

$$\alpha_k^j = \alpha_j^{k*},\quad \sum_i \alpha_j^j = 0.$$

2. The Doublet of Antinucleons.

The wave function of an antinucleon is the complex conjugate of the wave function for the nucleon. Since the transformations of SU_2 are complex, the quantities that are the complex conjugates of the components of a spinor transform differently from the components themselves.

These transfomations (by means of the matrix U*) form another group. Though it is in one-to-one correspondence (isomorphic) to SU_2, it is not identical with it. This group is called the contragradient SU_2 group. It is easy to see that its generators (which we denote by $\bar{\sigma}_k^j$) are related to the generators of SU_2 by the relation.

$$\bar{\sigma}_k^j = -\sigma_j^k.$$

We can write the wave function of the antinucleon as a superposition

$$\varphi_1 p^* + \varphi_2 n^*.$$

The quantities φ_α (where we write the indices as subscripts) transform like $\psi^{\alpha*}$, we call them covariant spinors in contrast to the contravariant ψ^α.

But we can avoid having to deal with two types of spinors. The components ψ^α can be written as linear combinations forming the spinor ψ_α. These are $\epsilon_{\alpha\beta}\,\psi^\beta$ (where we sum over β), and where $\epsilon_{\alpha\beta}$ is the unit antisymmetric tensor ($\epsilon_{12} = \epsilon_{21} = 1$, $\epsilon^{11} = \epsilon^{22} = 0$).

This last result follows from the existence of two quadratic forms that are invariant under these transformations. Let ψ and φ be two spinors. Then the form $\psi^\alpha\varphi^{\alpha*}$ is invariant because of the unitarity, and the form $\epsilon_{\alpha\beta}\psi^\alpha\varphi^\beta$ because of the unimodularity of the transformations ($\epsilon_{\alpha\beta}U_{\alpha'}^{\alpha}U_{\beta'}^{\beta} = \epsilon_{\alpha'\beta'}$). Comparison of these two invariants proves our assertion:

$$\varepsilon^{\alpha\beta}\varphi_\beta = \varphi^\alpha,\quad \varepsilon_{\alpha\beta}\psi^\beta = \psi_\alpha\quad (\varepsilon_{\alpha\beta} = \varepsilon^{\beta\alpha}).$$

Thus the wave function of the antinucleon has the form

$$\varphi^1 n^* - \varphi^2 p^*.$$

3. Isomultiplets.

The wave function of a system consisting of p particles (nucleons and antinucleons) is a product (or a linear combination of products) of wave functions of the particles. Each of these is an isospinor ψ^α. Thus the wave function of the system is an isospinor of rank p, $\psi^{\alpha_1\cdots\alpha_p}$, i.e., it transforms like a spinor with respect to each index α_r. We shall sometimes denote a spinor by its rank: (p). The spinor (p) has 2^p components which are transformed in a definite way by each transfomation of SU_2. This fact is expressed by the statement: the transformations of the spinor (p) form a group which is a representation of the group SU_2.

An infinitesimal transformation of the spinor (p) = ψ has the form

$$\delta\psi = i\sum_{i=1}^{3}\delta\alpha_i T_i\psi,$$

or

$$\delta\psi = i\sum_{j,k=1}^{2}\delta\alpha_j^k T_k^j\psi,$$

where α_i or α_j^k are the parameters of SU_2, while T_i or T_k^j are the generators of the representation:

$$T_i = \sum_{r=1}^{p}\sigma_i^{(r)},\quad T_k^j = \sum_{r=1}^{p}\sigma_k^{j(r)}.$$

It is understood that each operator $\sigma^{(r)}$ acts only on its own index (degree of freedom) α_r in the spinor $\psi^{\cdots\alpha_r\cdots}$.

We then see that the generators T have the same commutation properties as the corresponding matrices α.

$$[T_k^j T_m^l] = \delta_m^j T_k^l - \delta_k^l T_m^j.$$

In general the spinor (p) is reducible. By this we mean that we can form linear combinations of its components that form a spinor of lower rank. Reducibility is related to the existence of the invariant $\epsilon_{\alpha\beta}\,\psi^{\alpha\beta}$, which was already discussed above. The quantities $\epsilon_{\alpha\beta}\,\psi^{\cdots\alpha\cdots\beta\cdots}$ form a spinor of rank p-2. (We note that by multiplying a spinor by $\epsilon_{\alpha\beta}$ we can «lower» one of its indices:

$$\varepsilon_{\alpha_1\beta}\psi^{\alpha_1\alpha_2\cdots} = \varphi_\beta^{\alpha_2\cdots}.)$$

Irreducible spinors are of special importance. These are spinors that are symmetric in all indices; multiplication by $\epsilon_{\alpha\beta}$ reduces them to zero. (If we write an irreducible spinor as $\varphi_{\beta\cdot\cdot}^{\alpha\cdot\cdot}$, then $\varphi_{\alpha\cdot\cdot}^{\alpha\cdot\cdot} = 0$.) We shall denote an irreducible spinor of rank p by [p]. We agree to write the rank of an irreducible spinor as p = 2T; T denotes the isospin of the state.

It is easy to see that the number of independent components of an irreducible spinor is p + 1. (The number of indices that are equal to unity can vary from 0 up to p, while the others are equal to two.) Correspondingly, there are p + 1 linearly independent states. They form an isomultiplet (t-multiplet). All these states (components of the isomultiplet) are equivalent with respect to strong interactions.

The components of an isomultiplet can be uniquely classified according to the eigenvalues of the generator T_3 (the spin projection). Since $T_3 = \Sigma\sigma_3^{(r)} = \Sigma(\pm\frac{1}{2})$, T_3 takes on values from -T to T in unit steps (a total of 2T + 1 different values).

It is easy to see that the action of the generators $T_\pm$ and $T^2 = \Sigma T_i^2 = \frac{1}{2}\sum_{jk} T_k^i T_j^k$ is analogous to the action of σ_t and σ^2, i.e., if we denote the state defined by given values of T and T_3 by $|T, T_3\rangle$, then

$$T_\pm|T,\,T_3\rangle \sim |T,\,T_3\pm 1\rangle,\quad T_+|T,\,T\rangle = T_-|T,-T\rangle = 0,$$
$$T^2|T,T_3\rangle = T(T+1)|T,\,T_3\rangle.$$

Isospinors can be divided into two classes. Spinors of even rank (integral T) are called tensors of rank T. The spinors of odd rank are called true spinors. The two classes of spinors differ in their behavior under the «center» of the SU_2 group, i.e., under the two transformations given by the matrices $U = \sqrt{1} = \pm 1$. One of them is the identity transfomation, the other gives multiplication by $(-1)^p$, i.e., it is the identity for tensors and changes the sign of spinors. (In the three-dimensional rotation group these are the rotations through 0 or 2π.)

4. Expansion of Products.

Irreducible spinors play a special role in the theory because they determine a new quantum number, the isospin T, a quantity that is conserved in strong ineractions. Suppose that the initial state of the system ("before the collision") is characterized by definite values of T and T_3. The collision matrix is invariant under the transformations of the isogroup. Therefore the final state has these same values of the quantum numbers T and T_3.

Since T_3 can change under the transformations of the isogroup, (each component of the isomultiplet goes over into a linear combination of components of the same isomultiplet), the scattering matrix in general is independent of T_3. It does depend on the value of T, but is diagonal in it: the different isomultiplets behave independently of one another. We may therefore assign a definite isospin to all bound states of the nucleon system (particles, resonances), that are described by the poles of the scattering matrix. The mass and width of a resonance (for decays caused by strong interaction) is the same for all terms of an isomultiplet. (This was the starting point for the introduction of the isogroup in the case of the nucleon doublet.)

The wave function of a system of two particles $\psi = [p_1][p_2]$ does not have a definite isospin – the corresponding spinor is reducible. It can, however, be written as the sum of irreducible spinors. This expansion is called the Clebsch-Gordan series.

It is quite simple to explain the structure of this series, if we don't attempt to find the coefficients of the series.

$[p_1][p_2]$ is the spinor $(p_1 + p_2) = \psi^{\alpha_1 \ldots \alpha_{p_1}, \beta_1 \ldots \beta_{p_2}}$, which is symmetric separately in the p_1 first and the p_2 last indices. By symmetrizing it completely we get the irreducible spinor of rank $p_1 + p_2$. Multiplying the initial spinor by $\epsilon_{\alpha\beta}$ and symmetrizing the result, we get $[p_1 + p_2 - 2]$. Repeating the process until we exhaust the smaller of the two numbers p_1 or p_2, we get the Clebsch-Gordan series

$$[p_1][p_2] = [p_1 + p_2] + [p_1 + p_2 - 2] + \ldots + [|p_1 - p_2|].$$

We note that all the terms of the series are either tensors (for integral $T_1 + T_2$) or true spinors (for half-integral $T_1 + T_2$), and that a given value of p occurs only once in the series.

5. Tensor Operators.

The scattering matrix (in the approximation that includes only the strong interaction) is an example of an invariant operator, not changing the isospin structure of the system wave function. All other operators ("operators on the isogroup") can be constructed from the Pauli matrices σ_k^j.

Let us consider the action of σ_k^j on the (1st rank) spinor ψ^α:

$$(\sigma_k^j)_\beta^\alpha \psi^\beta = \delta_k^\alpha \psi^j - \frac{1}{2}\delta_k^j \psi^\alpha.$$

Then

$$\psi^* \sigma_k^j \psi = \psi^{k*}\psi^j - \frac{1}{2}\delta_k^j \psi^*\psi.$$

Since $\psi^{k*} \sim \psi_k$, this means that

$$\psi^*\sigma_k^j\psi = \varphi_k^j, \quad \varphi_\alpha^\alpha = 0,$$

forms an irreducible spinor of rank two (a vector).

Thus σ_k^j (or σ_i) may be called a vector operator. This property is preserved in its action on a spinor of arbitrary rank (p), i.e., $\langle p|\sigma_k^j|p\rangle = \varphi_k^j$ (where the operator σ_k^j acts on one of the spinor indices). Furthermore, one can form an arbitrary linear combination of Pauli matrices

$$R_k^j = \sum_r c_r \sigma_k^{j(r)}$$

and this is also a vector operator: $\langle p|R_k^j|p\rangle = \varphi_k^j$. It is easy to obtain the commutation relations of an arbitrary vector operator with the generators of the group. They have the standard form

$$[T_k^j R_m^l] = \delta_m^j R_k^l - \delta_l^k R_m^j.$$

Let us check that a vector operator behaves like a vector in the sense of the Clebsch-Gordan series, i.e., that

$$R_k^j \,|\, p\rangle = [p+2] + [p] + [p-2].$$

In fact, the spinor [p] is symmetric in all its indices. The operator $\sigma_k^{j(r)}$, acting on the index α_r, makes it unsymmetric relative to all the others. By means of symmetrization and antisymmetrization (i.e., multiplication by ϵ), we get two irreducible spinors, [p] and [p − 2]. Furthermore, each of the spinors contains an arbitrary number of indices antisymmetric in pairs ($\psi^{\alpha_1 \ldots \alpha_p} \sim \psi^{\alpha_1 \ldots \alpha_p}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}\ldots$). If the operator acts on one of these indices, the antisymmetry is destroyed and we have a reducible spinor (p + 2), which is symmetric in p indices. Reducing it, we get [p + 2] and [p]. Thus, $R_k^j \,|\, p\rangle$ contains the three terms listed above.

We can construct products of the operators $\sigma_k^{j(r)}$ (with different r). They form tensor operators (the total number of indices j,k is always even). They can be made irreducible by treating the indices j,k in the same way as for spinors. The components of an irreducible tensor can be denoted by $O[P]$. This means that $\langle p_1|O[P]|p_1\rangle = [p]$.

The effect of the irreducible tensor operator $O^{[P]}$ on the wave function $[p_1]$ will be given by the corresponding Clebsch-Gordan series:

$$O^{[p]}\,|\,p_1\rangle = \sum_{p_r}[p_2].$$

From this we obtain, first of all, a selection rule: the matrix elements $\langle p_2|O[P]|p_1\rangle$ differ from zero only for $T_1 + T_2 \geq T \geq |T_1 - T_2|$. Secondly, since each value of p appears once in the Clebsch-Gordan series, the matrix elements of operators of the same rank differ only by an invariant factor. In particular, the diagonal matrix elements of all vector operators differ from the matrix elements of the generators only by a scalar factor:

$$\langle p\,|R_k^j|\,p\rangle = c\,\langle p\,|T_k^j|\,p\rangle.$$

Suppose that we have to form an invariant expression $\mathscr{L}$ from ψ, ψ^*, and π, where ψ is an arbitrary isospinor, while π is an isovector ($\mathscr{L}$ is the Yukawa interaction Lagrangian). Such an invariant is unique (i.e., for combining states of the same isomultiplet):

$$\mathscr{L} = (\psi^* T_\beta^\alpha \psi)\,\pi_\alpha^\beta.$$

6. Displaced Isomultiplets

Thus, all the nucleonic (nonstrange) hadrons can be assigned to definite isomultiplets. At present we know the nonstrange baryon doublet N and quadruplet Δ, the meson singlet η and triplet π, etc.

The strange hadrons are obtained from the nucleonic ones by adding Λ particles. If the wave function of the Λ is an isoscalar (a t-singlet), we can form the same kind of isomultiplet of strange particles from each nucleonic isomultiplet. Since Y = 0 for the Λ particle, the particles belonging to the corresponding isomultiplet will differ from the original particles only in their baryon number. We thus get the mesonic doublets $N + \overline{\Lambda} \rightarrow \kappa$, the baryon triplet $\pi + \Lambda \rightarrow \Xi$, a doublet $\overline{\kappa} + \Lambda \rightarrow \Xi$, a singlet $\overline{\kappa} + \Xi \rightarrow \Omega$, etc. If we compare the multiplets for a given baryon number A, they will be displaced in the hypercharge Y.

We have already remarked before that the group of the charge Z is contained as a subgroup in the isogroup. Thus the charge Z is uniquely related to the quantum number T_3. For the nucleon, $Z = \frac{1}{2} + T_3$. Thus, because of the additivity of these quantum numbers, we have for any nucleonic system

$$Z = \frac{A}{2} + T_3,$$

or, since A = Y,

$$Z=\frac{Y}{2}+T_3.$$

Since the strange multiplets differ from the corresponding nonstrange ones only in the number A, with no change in Y, T and T_3, the last relation remains valid for them too.

From it we get a definite relation between Y and T. Since Z is an integer and since $2T_3$ has the same parity as 2T, Y and 2T are integers with the same parity:

$$Y=2T\ (\mathrm{mod}\ 2).$$

We note that this relation is not connected with the isogroup, since Y is a quantum number foreign to the isogroup. The connection of Y with T is a consequence of compositeness, as is the integral nature of A and Z.

We average the equation $Z = Y/2 + T_3$ over the multiplet. Since $\overline{T}_3 = 0$ (where the dash denotes an average),

$$\overline{Z}=\frac{Y}{2}.$$

The center of charge of the multiplet is independent of A. If we compare multiplets with the same A and different Y, the center of charge is displaced for the strange particles:

$$\overline{Z}(S)-\overline{Z}(0)=\frac{S}{2},$$

where $S = Y - A$ is the strangeness.

7. The Electromagnetic Interaction

The electromagnetic interaction does not possess the symmetry of the isogroup. It is therefore, strictly speaking, not possible to classify particles according to isomultiplets, when we consider electromagnetic processes. But if the electromagnetic interaction is treated approximately as a weak-perturbation, the scattering amplitude can be expressed as the matrix element of some operator with respect to the unperturbed functions, i.e., the components of the isomultiplets.

The breakdown of the isosymmetry manifests itself in the fact that such an operator is no longer an invariant; it can be expressed in terms of certain definite components of a tensor operator. For first order processes (radiation, scattering of an electron by a hadron), this is the vertex operator $\Gamma^{(e)}$ (Fig. 2). It is a function of the four-momentum q transferred by the field. With respect to space-time properties $\Gamma^{(e)}$ is a polar vector.

We shall be interested only in the isostructure of the operator $\Gamma^{(e)}$. We can find out its character by noting that for $q = 0$ the form factor of particle a, i.e., $\langle a\,|\,\Gamma^{(e)}(0)\,|\,a\rangle \equiv \langle \Gamma^{(e)}(0)\rangle_a$ (more precisely, its time component, which is the only nonvanishing one), reduces to the charge:

$$\langle\Gamma^{(e)}(0)\rangle\sim Z=\frac{Y}{2}+T_3.$$

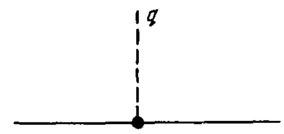


FIG. 2. Vertex for interaction of a hadron with the electromagnetic field. Solid line – hadron; dashed line – field.

In the sense of the isogroup, this expression consists of an isoscalar and the components of an isovector.

It is natural to assume that the isotensor character of the operator $\Gamma^{(e)}$ is independent of q. Then

$$\Gamma^{(e)}=e\,(\mathrm{v}^{(0)}+\mathrm{v}_3),$$

where $\mathrm{v}^{(0)}$ is an isoscalar, and v is an isovector operator (e is the electron charge, introduced for purposes of normalization). This constitutes the basic hypothesis concerning the character of the interaction of hadrons with the electromagnetic field (the "hypothesis of minimal interaction"). It is also assumed that $\mathrm{v}^{(0)}$ and v are invariants under the group of Y, i.e., the electromagnetic interaction does not change the strangeness of particles.

The picture for justifying this hypothesis is the following. In the last analysis the interaction with the electromagnetic field is produced via the charges of the "bare" proton or antiproton, for which

$$\Gamma_p^{(e)}\sim Z_p=\frac{1}{2}+\sigma_3,\quad \Gamma_{\bar{p}}^{(e)}\sim Z_{\bar{p}}=-\frac{1}{2}+\sigma_3.$$

Because of the strong (isoinvariant) interaction, each hadron gives rise to a collection of particles, among which some are charged (protons and antiprotons). The interaction should be taken into account once, i.e., we must add the individual vertices with arbitrary weights. This can give nothing other than $\mathrm{v}^{(0)} + \mathrm{v}_3$.

From the form of the operator it follows that the form factors of particles have the structure

$$\langle\Gamma^{(e)}(q)\rangle=c_1(q)+c_2(q)\,T_3,$$

where c_1 and c_2 are fixed functions for a given multiplet.

This formula gives definite relations when the number of components of the multiplet exceeds two, i.e., when $T > 1/2$. For example, for the triplet of Σ hyperons it gives[3]

$$\langle\Gamma^{(e)}\rangle_{\Sigma^+}+\langle\Gamma^{(e)}\rangle_{\Sigma^-}=2\,\langle\Gamma^{(e)}\rangle_{\Sigma^0}.$$

In particular, such a relation should hold between the magnetic moments of these particles.

The probability amplitude for radiation (i.e., for decay of hadron a into hadron b and a photon) has the form $\langle b\,|\,\Gamma^{(e)}\,|\,a\rangle$. From the structure of the operator $\Gamma^{(e)}$ we get the selection rule[4] $\Delta T = 0, \pm 1$.

An operator describing a second order process must have the isotensor structure of the product of two vertex operators, i.e., be expressed in the form

$$\mathrm{c}+\mathrm{R}_3+\mathrm{L}_{33},$$

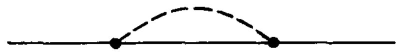


FIG. 3. Feynman diagram for the electromagnetic correction to the hadron mass. The solid line is the hadron, the dashed line the radiation and absorption of a virtual photon.

where c is an isoscalar, R an isovector, and L an isotensor of second rank. In particular this is the structure of the amplitude for the Compton effect on a hadron and the electromagnetic mass ΔM, which is given by the diagram of Fig. 3. It follows that

$$\Delta M = c_1 + c_2 T_3 + c_3 T_3^2.$$

This formula obviously gives definite relations if $T > 1$. Thus, for the Δ quadruplet[5]

$$\Delta^{++} - \Delta^{-} = 3(\Delta^{+} - \Delta^{0})$$

(where the masses are denoted by the symbols for the corresponding particles).

8. The Weak Interaction

In general the weak interaction conserves neither Z nor Y. Therefore four types of processes are possible. First, there are two types of processes in which Z changes. Since the total charge Q is conserved, leptons arise in such processes. One such process is leptonic decay without change of Y, the other with change of Y. Secondly there are two types of processes in which Z does not change and no leptons are emitted. One of these, with change of Y, is the nonleptonic decay of hadrons; the other, without change of Y, may appear on the background of strong and electromagnetic interactions because of nonconservation of parity (parity nonconserving nuclear forces).

One can find a natural place for leptonic processes without strangeness change on the nucleon diagram (cf. Fig. 1): $n \rightleftarrows p$ is such a transition. The decay is described by the vertex diagram (cf. Fig. 3), in which the leptons play the role of the photon. For concreteness we shall assume that the leptons carry negative charge $Q_l = -1$. Then the change in charge of the hadron is $\Delta Z = 1$ ($n \to p$), and the "weak" vertex operator for the nucleon will have the form $\Gamma^{(w)} \sim \sigma_+$. (The process with $\Delta Z = -1$ will obviously be determined by the operator $\Gamma^{(w)+} \sim \sigma_-$.) The fundamental hypothesis about the character of the weak interaction is that in general the weak vertex operator for a hadron is constructed additively from nucleon vertices, just as in the case of electromagnetic interaction. Then $\Gamma^{(w)}$ must be a component ((+)-component) of some isovector.

The vertex $\Gamma^{(w)}$ is a sum of two terms, differing in their space inversion properties: one is a polar 4-vector and the other an axial 4-vector. Each of these two terms must have the isovector property. Thus

$$\Gamma^{(w)} = g_\beta (v_+ + a_+),$$

where v and a are isovectors (v is a polar 4-vector, and a is axial), while g_β is a constant. One of the most important properties of the weak interaction is the identity of the isovector v with the isovector that enters in the electromagnetic vertex $\Gamma^{(e)}$.

The parity nonconserving nuclear forces (processes with $\Delta Y = \Delta Z = 0$) are expressed by an operator that is diagonal in T_3. We can find the isostructure of this operator if we suppose that it contains the product of two vertices $\Gamma^{(w)}$ and $\Gamma^{(w)+}$ ("tied in" by the strong interaction). The product of two isovector operators gives $O^{[2]}O^{[2]} = O^{[0]} + O^{[2]} + O^{[4]}$. Thus the isospin selection rules will be

$$\Delta T = 0, \ \pm 1, \ \pm 2.$$

There is no place for the other two types of weak process on the nucleon diagram. The corresponding operators cannot be expressed as operators of the isogroup.

We note that for processes with $\Delta Y \neq 0$, the isospin changes by a half-integral number. But within the framework of the isogroup one can construct only tensor operators. There are no proper spinor operators. It is therefore impossible, within the framework of the isogroup, to formulate the principle of the universal weak interaction. Later (cf. Sec. 7) we shall see that the SU_3 group gives a natural description also for processes with $\Delta Y \neq 0$.

3. THE SECOND ISOSPIN GROUP

1. u-Multiplets

Our whole presentation reduces to the following. To each hadron one must ascribe, in addition to the rigorously conserved baryon number A, two other similar types of additive quantum numbers: the charge Z and the hypercharge Y. The conservation laws associated with them are approximate, being violated by the weak interaction, which we assume to have been switched off. Moreover, we have not dealt on the same footing with both Z and Y. Calling attention to the fact that one of them, Y, plays the more important role in strong interactions, we switched off the electromagnetic interaction. After that the hadrons were distributed over isomultiplets (t-multiplets), the main quantum numbers were Y and T, while Z, which is uniquely related to T_3, became an index distinguishing the identical components of the multiplet. The significance of Z as the electric charge appears only after the electromagnetic interaction is switched on, destroying the symmetry under the isogroup.

Let us now try to interchange the roles of charge and hypercharge. We shall try to justify the similarity in the naming and description of the two quantum numbers. For this purpose we momentarily forget about the smallness of the electromagnetic interaction. Suppose that the quantum number Z is essential and that the proton and neutron do not form a t-multiplet, but are simply two different hadrons.

There is another pair of particles that replace the pair of nucleons for us. These are the proton and the Σ^+ hyperon. For both particles, Z = 1 (just as for the nucleons Y = 1). Their difference is in their different values of Y = 1 and 0 (just as, for nucleons, Z = 1, 0). We shall call these two baryons (since we lack a term analogous to "nucleon") B^+ particles.

The two nucleons have a small mass difference, and this was the origin of the idea of degeneracy in Z. The two B^+ particles also do not differ very much in mass, $\Delta M/M \simeq 0.2$. This number is not so small as for the case of the nucleons, but it still small enough to justify the following hypothesis, which is of fundamental importance for the following.

Strong interactions can be divided into two classes: the truly strong (i.e., the "very strong," which we shall from now on call simply the "strong" or s-interaction) and the medium strong (we call it the medium or m-interaction). The medium interaction is approximately as much weaker than the strong as the mass difference of Σ^+ and p is smaller than their average mass. The strong interaction is the same for the two B^+ particles. It for the most part determines their masses. The medium interaction splits this doublet. Thus the s-interaction is independent of Y; only the m-interaction depends on Y.

Now we can completely interchange the roles of Y and Z. Switching off the medium interaction, we retain the electromagnetic interaction. Then p and Σ^+ are two degenerate states, differing in the quantum number Y, which plays no role in the interaction. This symmetry is that of SU_2. Everything that was said about this group can be taken over to this new basis. It is sufficient to replace p, n by p, Σ^+ on the nucleon diagram. In order to distinguish this new isogroup from the earlier one, we shall call it the u-isogroup.[6] Correspondingly its generators will be called u-spin operators and will be denoted by U_i (i = 1, 2, 3) or U^j_k (j, k = 1, 2). The rank of an irreducible spinor will be denoted by 2U, and the corresponding multiplet will be called a u-multiplet. The components of a u-multiplet will be classified according to the eigenvalues of the generator U_3, which we shall call the projection of the u-spin ($U_3 = U, \ldots, -U$).

Just as we can construct all nonstrange t-multiplets (Y = A) from the fundamental t-doublets (nucleons N and antinucleons $\overline{N}$), we construct all those u-multiplets for which Z = A from the fundamental u-doublets B^+ and $\overline{B}^+$. We can construct all the known hadrons if, as in the t-multiplet scheme, we introduce in addition the baryonic u-singlet Λ (A = 1, Z = Y = U = 0). The connection between the value of Y and the u-spin projection U_3 for a given multiplet will be

$$Y = \frac{Z}{2} + U_3,$$

while the average value of Y over a u-multiplet ("the center of hypercharge") will be

$$\overline{Y} = \frac{Z}{2}.$$

Both these formulas are exactly the same as for t-multiplets, except that the roles of Z and Y are interchanged. From them it follows that only those u-multiplets are possible for which Z and 2U have the same parity:

$$2U = Z \pmod 2.$$

In particular, singly charged particles can have only half integral u-spin.

We give some examples of composition of u-multiplets.

1) The meson doublet (A = 0, Z = 1, U = 1/2):

$$B^+ + \overline{\Lambda} \to M^+ = \begin{pmatrix} K^+ \\ \pi^+ \end{pmatrix}.$$

2) $p\pi^+$-resonance, singlet (A = 1, Z = 2, U = 0):

$$B^+ + B^+ + \overline{\Lambda} \to \Delta^{++}.$$

3) Negatively charged baryon doublet (A = 1, Z = −1, U = 1/2):

$$\overline{B}^+ + \Lambda + \Lambda \to B^- = \begin{pmatrix} \Sigma^- \\ \Xi^- \end{pmatrix}.$$

(This doublet could have been used as the starting point in place of B^+.)

4) Negatively charged baryon quartet (A = 1, Z = −1, U = 3/2):

$$B^+ + \overline{B}^+ + B^- \to B^{-*} = \begin{pmatrix} \Delta^- \\ \Sigma^{-*} \\ \Xi^{-*} \\ \Omega^- \end{pmatrix}.$$

We note that before the discovery of the Ω^- hyperon only three baryons were known with Z = −1 and angular momentum J = 3/2. But from the relation 2U = Z (mod 2) they could not constitute a u-triplet. At least one additional component was needed, i.e., the Ω^-.

2. The Medium Strong Interaction

If we extend the analogy between the t-spin and u-spin schemes, the medium interaction should lead to a splitting of u-multiplets just as the electromagnetic interaction splits a t-multiplet. The use of perturbation theory should give cruder but still reasonable results. The medium interaction can be expressed as a tensor operator of the u-group.

If we use the analogy quite literally, the hypercharge Y (like the charge Z) should characterize the interaction with some field. It is natural to represent the vertex for this interaction in the form

$$\Gamma^{(m)} = m^0 + m_3,$$

where m^0 is a u-scalar and m_3 a vector. If the splitting of the mass of a u-multiplet is determined by a diagram of the type of Fig. 3, the formula for the masses of components of a u-multiplet should have the form

$$\Delta M = c_1 + c_2 U_3 + c_3 U_3^2.$$

Applying this formula to the quartet B^{-*}, one can express the mass of the Ω^- hyperon in terms of the masses of the other three particles:

$$\Omega^- = \Delta^- + 3(\Xi^{-*} - \Sigma^{-*})$$

(the masses being denoted by the symbols for the particles). The value of the constant is

$$2c_3 = (\Delta - \Sigma^{-*}) - (\Sigma^{-*} - \Xi^{-*}).$$

We know that the experimental data, to good accuracy, satisfy the first relation and lead to the value $c_3 = 0$, i.e., the masses are equidistant:

$$\Omega^- - \Xi^{-*} = \Xi^{-*} - \Sigma^{-*} = \Sigma^{-*} - \Delta^-.$$

The situation is as if the mass splitting were directly determined by a vertex operator $\Gamma^{(m)}$ (cf. the diagram of Fig. 4 instead of Fig. 3). This question will be discussed in Sec. 5. In this case the same relation also holds for the triplet

$$B^{0*} = \begin{pmatrix} \Delta^0 \\ \Sigma^{0*} \\ \Xi^{0*} \end{pmatrix},$$

namely,

$$\Xi^{0*} - \Sigma^{0*} = \Sigma^{0*} - \Delta^0.$$

3. Electromagnetic Interaction

The classification of particles into u-multiplets is most natural for treating electromagnetic processes. Since such processes, by definition, are impossible when the electromagnetic interaction is switched off, the absolute smallness of the latter compared to the medium interactions is here unimportant. We can therefore in first approximation neglect the m-interaction.

Then the electromagnetic vertex of the hadron is a u-scalar: all terms of a u-multiplet are equivalent. Consequently the electromagnetic form factors are the same for all components of a u-multiplet.[7-9] For example,

$$\langle\Gamma\rangle_{\Sigma^+} = \langle\Gamma\rangle_p, \quad \langle\Gamma\rangle_{\Sigma^-} = \langle\Gamma\rangle_{\Xi^-},$$
$$\langle\Gamma\rangle_{\Omega^-} = \langle\Gamma\rangle_{\Xi^{-*}} = \langle\Gamma\rangle_{\Sigma^{-*}} = \langle\Gamma\rangle_{\Delta^-}.$$

This applies in particular to the values of magnetic moments of the particles.

Obviously these same relations will be valid for the interaction with the electromagnetic field in second order. For example, these same relations

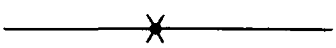


FIG. 4. Feynman diagram describing the correction to the hadron mass due to the medium strong interaction. Solid line – hadron; cross – contribution of the medium interaction, depending on the hypercharge.

will be valid for the amplitude for the Compton effect (pictured as a two-photon vertex; Fig. 5). We shall not write them out in detail; it is sufficient to understand by $\langle\Gamma\rangle_a$ the amplitude for the Compton effect on particle a.

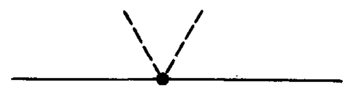


FIG. 5. Schematic diagram for the Compton effect on a hadron. Dashed lines – incident and scattered photon, solid line – hadron.

Similar relations will also hold for the amplitude for decay of particle a into particle b with emission of a photon, if a and b belong to similar u-multiplets.[9] An example is the meson doublet $M^{+*} = \binom{K^{+*}}{\rho^+}$, decaying into the doublet $M^+ = \binom{K^+}{\pi^+}$:

$$\langle\pi^+ | \Gamma | \rho^+\rangle = \langle K^+ | \Gamma | K^{+*}\rangle.$$

If the u-spins of particles a and b are different, such decays are forbidden.

One can include the medium interaction as a perturbation. If we represent the electromagnetic vertex as an operator applied to unperturbed states, $\Gamma^{(e)}$ will have the same u-tensor structure as the mass operator. The formulas of par. 2 will therefore be valid, if we understand the symbols for particles to stand for their electromagnetic form factors or their Compton amplitudes, etc.[10]

4. Extension of a Group

We should point out that the basis on which the scheme of u-multiplets was constructed is not very satisfactory. Switching off the medium interaction while keeping the electromagnetic is not very good. The symmetry of the t-group is still better satisfied than that of the u-group. The proton, for example, differs more in mass from its partner in a u-doublet, the Σ^+ hyperon, than from the neutron which is in a different u-multiplet. Thus the splitting within a u-multiplet is of the same order as the distance between multiplets and, strictly speaking, the classification according to u-spin is meaningless.

In the real examples that were given above, we chose those u-multiplets for which the distance to the nearest multiplets with the same value for Z and the other quantum numbers is largest. But this is not always the case.

For example, we have introduced the u-singlet Λ. But we cannot say what relation it bears to the real Λ particle. In fact in our scheme there is a u-triplet

$$B^+ + \overline{B}^+ + \Lambda \to B^0 = \begin{pmatrix} n \\ \Sigma^0 \\ \Xi^0 \end{pmatrix}.$$

Its components $U_3 = \pm 1$ can be identified with the real neutron and Ξ^0 hyperon. But the component having $U_3 = 0$ will combine with the u-singlet. These superpositions will represent both the real Σ^0 and Λ particles. There is no possibility of identifying the coefficients of this superposition within the framework of the group considered here.

It will be more consistent to switch off both the medium and the electromagnetic interactions. Then the two quantum numbers Z and Y become equivalent, as a result of which we get a higher symmetry characterizing the truly strong interaction. What will this symmetry be? The corresponding group should obviously include as subgroups the two SU_2 groups already considered: the t-group and the u-group, each of which contains in turn as a subgroup one of the gauge groups of Z or Y.

A general picture of this symmetry can be gotten from the experimental data on the mass spectrum of the baryons. There are eight baryons with masses the same to within about 20% and with the same values for their other quantum numbers (angular momentum J and parity P). This octet is shown in Fig. 6. The axes are labelled Z and Y, and the points are labelled by the names of the members of the octet. The horizontal lines contain t-multiplets, the vertical lines contain the members of u-multiplets. Both resolutions are equivalent; in both cases there is one singlet and two doublets.

All the members of the octet are equivalent with respect to the s-interaction. We are dealing with a higher degeneracy than that considered earlier. The properties of the baryons are expressed by the superposition $\sum_{i=1}^{8} c_i\psi_i$, and the group should express the equivalence of all such superpositions.

But it is by no means necessary that this be the group SU_8. The latter would lead to the appearance of too large a set of quantum numbers, which are still not required by the experiments. We want the group to contain the two additive quantum numbers

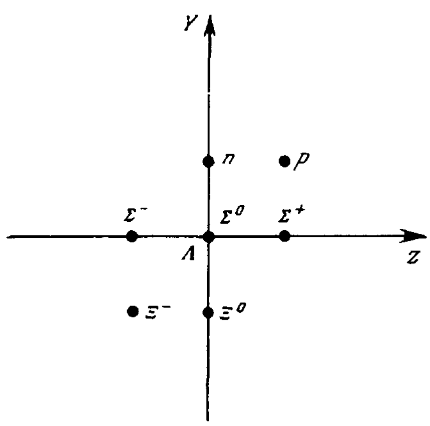


FIG. 6. The octet of (½)$^+$ baryons. The abscissa gives the charge, the ordinate the hypercharge; the points are labelled by the particle symbols.

Y and Z. We have seen that the whole octet, as a t- or a u-multiplet, can be constructed from three fundamental particles bearing different sets of values of Y and Z. It is therefore natural to assume that the symmetry of the s-interaction is expressed by the group SU_3.

We point out that the triplet of baryons whose use as fundamental constituents made possible the composition of all the hadrons was not the same for the cases of composition in the t- and u-isogroup bases. It appears to be impossible to choose three real particles which would form the fundamental triple of the group SU_3 so that they can be used to build up all the hadrons in the same way as the nonstrange hadrons are constructed using the nucleon doublet of the SU_2 group. By "real" we here mean not only particles known at present, but any particles with integral quantum numbers A, Y and Z.

We shall return to the consideration of this question later, but now we proceed to consider the properties of SU_3.

4. THE SU_3 GROUP

1. The Generators of SU_3 and the Quark Diagram

And so we shall consider three states (particles) that are equivalent with respect to the strong interaction. We shall denote the corresponding wave functions by "p", "n" and "Λ". The quotation marks are to remind us that we do not mean the real particles denoted by these symbols. These hypothetical particles we shall call quarks.[11]

We can consider the superposition

$$\psi^1 \text{«p»} + \psi^2 \text{«n»} + \psi^3 \text{«}\Lambda\text{»}.$$

The set of three quantities

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \end{pmatrix}$$

we shall call a superspinor or an f-spinor (and when there is no basis for a misunderstanding, simply a spinor). The equivalence of difference superpositions means that an f-spinor can be subjected to unitary transformation. Separating out the (baryonic) gauge subgroup, we can restrict ourselves to the unimodular transformations $\psi \rightarrow U\psi$:

$UU^+ = 1$, $\det U = 1$ (U is a 3×3 matrix). These transformations form the group SU_3.

If we write U as e^{iH}, where H is a hermitian matrix with zero trace, the infinitesimal transformations take the form

$$\delta\psi = i\sum_{i=1}^{8} \delta\alpha_i\lambda_i\psi.$$

Here λ_i is a set of linearly independent 3×3 hermitian matrices with zero trace; it is clear that there are eight of them ($3 \times 3 - 1$). Correspondingly the group contains eight real parameters α_i. We can also write the formula for $\delta\psi$ in the form[12]

$$\delta\psi = i \sum_{j,k=1}^{3} \delta\alpha_j^k \lambda_k^j \psi,$$

where the parameters α_j^k form a hermitian matrix with zero trace, while the λ_k^j are nine real matrices related by the conditions

$$\sum_j \lambda_j^j = 0, \quad \lambda_k^j = \lambda_j^{k+}.$$

These matrices and their commutation relations can be written explicitly (as in Sec. 2, we use the convention of labelling rows by a superscript and columns by a subscript):

$$(\lambda_k^j)_\beta^\alpha = \delta_\beta^j \delta_k^\alpha - \frac{1}{3}\delta_k^j \delta_\beta^\alpha$$

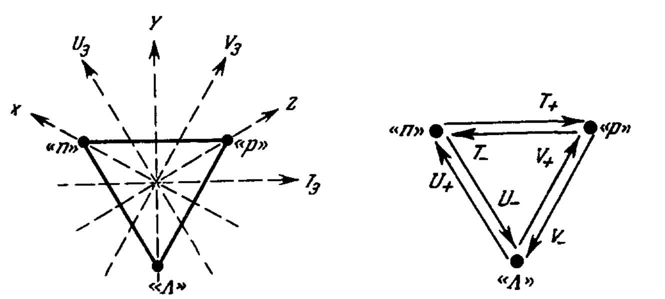
and

$$[\lambda_k^j \lambda_m^l] = \delta_m^j \lambda_k^l - \delta_k^l \lambda_m^j.$$

All these formulas are repetitions of the corresponding formulas of Sec. 2.1. They differ only in having the index run over three values instead of two. The generators λ_i or λ_k^j will be called superspin or f-spin operators.

The structure of the SU_3 group can be understood very easily using the "quark diagram" shown in Fig. 7. Points corresponding to the quarks are at the vertices of an equilateral triangle. This is the only symmetrical arrangement of points, and from it there already follows that each is characterized by two coordinates, i.e., each of the particles is labelled by two quantum numbers. On the other hand it is easy to see that of the eight matrices λ_i, two can be taken to be simultaneously diagonal, and the two quantum numbers regarded as eigenvalues of these operators.

Such a choice can be made in different ways. On the quark diagram the natural origin of coordinates is the center of the triangle, and one can draw six natural axes; three passing through the vertices (denoted by Z, Y, X) and three parallel to the edges (denoted by U_3, T_3, V_3). Any pair of these axes determines the coordinates of the points, i.e., the quantum numbers of the quarks. The values for the others are expressed in terms of these two. It is most convenient to use one of the three pairs of cartesian axes (YT_3, ZU_3 or XV_3). It is also convenient to



a) b)

FIG. 7. a) Quark diagram. The quark symbols are at the vertices of the triangle. The axes passing through the vertices are the "charges" Z,Y,X; the axes parallel to the edges are the isospin projections U_3, T_3, V_3. Each quark is determined by any pair of quantum numbers (coordinates in the plane). b) Quark diagram. The arrows give the action of the six nondiagonal components of the f-spin. They break up into three pairs, isospin components of the three types: $T_\pm$, $U_\pm$, $V_\pm$.

choose the natural scale: let the unit along the Z, Y and X axes be the altitude of the triangle, and along the U_3, T_3 and V_3 axes, the side of the triangle.

Relative to the axes Y, T_3, for example, the coordinates of the quarks have the following values:

$$\overset{\text{«p»}}{\left(\tfrac{1}{3},\ \tfrac{1}{2}\right)} \qquad \overset{\text{«n»}}{\left(\tfrac{1}{3},\ -\tfrac{1}{2}\right)} \qquad \overset{\text{«Λ»}}{\left(-\tfrac{2}{3},\ 0\right)}.$$

We see that relative to these axes the three quarks split into a pair "p" and "n" that are symmetric with respect to the Y axis, and a singlet "Λ" on the Y axis. We may say that the pair form a t-doublet, and the third particle a singlet. Actually, the difference in values for the doublet and singlet is equal to unity, the difference in Z values for the two terms of the doublet is also unity, i.e., the quantum number differences in the triplet "p", "n", "Λ" are the same as for the Sakata triplet p, n, Λ. The quantum numbers T_3 are the eigenvalues of the operator T_3 ($T_3 = \pm 1/2$ for $T = 1/2$; $T_3 = 0$ for $T = 0$). We complete the separating of the t-group from the f-group by introducing the two operators $T_\pm$, whose actions are shown on the diagram by arrows: T_+"n" = "p", T_-"p" = "n", $T_\pm$"Λ" = 0; we shall regard Y as the eigenvalue of the corresponding (hypercharge) operator Y.

Similarly the axes Z, U_3 divide the quark triplet into a doublet:

$$\overset{\text{«n»}}{\left(-\tfrac{1}{3},\ \tfrac{1}{2}\right)} \quad \overset{\text{«Λ»}}{\left(-\tfrac{1}{3},\ -\tfrac{1}{2}\right)} \text{ and a singlet } \overset{\text{«p»}}{\left(\tfrac{2}{3},\ 0\right)}.$$

Their coordinates are eigenvalues of the operators for the charge Z and the u-spin projection U_3. We note that now the difference in values of Z between the doublet and singlet is unity and the difference in values of U_3 for the two terms of the doublet is unity, i.e., the relations in the quark triplet are the same as in the triple B^-, Λ (cf. Sec. 3). Also introducing the operators $U_\pm$, we separate out of the f-group the second isogroup, the u-group. Thus the quark triplet combines within itself the possibilities of composition of hadrons both according to t-spin and u-spin. But this is achieved at the expense of having a fractional charge for the quarks. This question will be discussed in more detail later (cf. par. 7 of this section).

Finally one can have a third resolution into a doublet ("p", "Λ") and a singlet "n." This will be a third isogroup, which we shall call the v-group. To it there correspond operators V_3 and $V_\pm$.

We have thus introduced six nondiagonal operators $T_\pm$, $U_\pm$, $V_\pm$. Together with the two diagonal operators they fill out the set of eight generators of the SU_3 group. This permits us to investigate its content completely.

We remark that in addition to the t- and u-isogroups, whose appearance as subgroups we worked to achieve, we also found the v-group. Other groups which, like SU_3, give two additive quantum numbers (groups of rank two) contain a larger number of analogous subgroups.

By means of the quark diagrams one can also follow the action of the operators and find their commutation relations. It is clear that the relations within a given isogroup (between U_3, $U_\pm$ and similar sets) are the same as for the Pauli matrices. For example, let us find $[T_+, V_-]$. To do this we use the quark diagram and consider

$$(T_+V_- - V_-T_+)\,«n» = -V_-\,«p» = -«\Lambda» = -U_-\,«n»,$$

i.e.,

$$[T_+V_-] = -U_-.$$

It is easy to see, in particular, that all the operators leaving the same vertex or arriving at the same vertex of the diagram commute, for example,

$$[U_+V_+] = 0, \quad [U_+T_-] = 0.$$

It also easy to see that each of the "charge operators" Z, Y, X commutes with the components of the corresponding ("perpendicular") spin, i.e.,

$$[YT_\pm] = [ZU_\pm] = [XV_\pm] = 0,$$
$$[YT^2] = [ZU^2] = [XV^2] = 0.$$

In conclusion, we give the relations between the various diagonal generators:

$$T_3 + U_3 = V_3, \qquad X = Y - Z,$$
$$Y = \frac{2}{3}(U_3 + V_3), \qquad Z = \frac{2}{3}(T_3 + V_3)$$

and the expressions for the generators λ^j_k in terms of the "charges" and isospins:

$$\lambda^1_1 = Z, \quad \lambda^2_2 = X, \quad \lambda^3_3 = -Y,$$
$$\lambda^2_1 = T_+, \quad \lambda^3_2 = U_+, \quad \lambda^3_1 = V_+,$$
$$\lambda^1_2 = T_-, \quad \lambda^2_3 = U_-, \quad \lambda^1_3 = V_-.$$

(The mnemonic is: the operator λ^j_k transforms quark number j into quark number k, if the particles are labelled in the order "p", "n", "Λ".)

The corresponding expressions for the operators λ_i have the form

$$\lambda_1 \pm i\lambda_2 = T_\pm, \quad \lambda_4 \pm i\lambda_5 = V_\pm, \quad \lambda_6 \pm i\lambda_7 = U_\pm,$$
$$\lambda_3 = T_3, \qquad \lambda_8 = \frac{\sqrt{3}}{2}Y, \qquad Z = \lambda_3 + \frac{\lambda_8}{\sqrt{3}}$$

(where the normalization is such that $\sum_i \lambda_i^2 = \frac{1}{2}\sum_{j,k} \lambda^j_k \lambda^k_j = 1$).

2. Antiquarks

To the antiquarks there obviously corresponds the superposition of the complex conjugate wave functions

$$\psi_1\,«p»^* + \psi_2\,«n»^* + \psi_3\,«\Lambda»^*.$$

The superposition coefficients ψ_α (with subscript $\alpha = 1, 2, 3$) form a (covariant) spinor, transforming according to the contragredient group SU_3. We shall write it as a row $\psi = (\psi_1, \psi_2, \psi_3)$. Here an essential difference between SU_3 and SU_2 appears. We cannot form a linear combination of the components $\psi_\alpha \sim \psi^{\alpha *}$ which transforms like ψ^α. The transformations of SU_3 leave only one quadratic form invariant, namely $\psi^{\alpha *}\psi^\alpha$. This is guaranteed by the unitarity of the transformation. The invariant associated with the unimodularity is now not a quadratic but a cubic form, since the transformation matrix and its determinant are of third order.

Thus we must consider the existence of two independent types of spinors ψ_α and ψ^α ($\psi_\alpha \sim \psi^{\alpha *}$). If we write the infinitesimal transformation of the spinor ψ_α in the form

$$\delta\psi_\beta = i\sum_{jk} \delta a^k_j \psi_\alpha (\bar{\lambda}^j_k)^\alpha_\beta$$

(keeping our convention relating position of indices to labels of rows or columns), then it easy to see from a comparison of the matrices U and U* that

$$\bar{\lambda}^j_k = -\lambda^{j+}_k.$$

Thus the quantum numbers for the antiquarks are opposite to those for the corresponding quarks. This is illustrated by the antiquark diagram (Fig. 8). It is obtained from the quark diagram of Fig. 7 by inversion of the triangle in its center, while the directions of the axes are maintained as before. We shall denote the nondiagonal generators by the same symbols $T_\pm$, $U_\pm$, $V_\pm$, understanding them to be the matrices λ^j_k when they act on spinors ψ^α and the matrices $\bar{\lambda}^j_k$ when they act on spinors ψ_α. Thus the operators in the diagram of Fig. 8 change the sign of the corresponding states (for example, T_+"p"* = −"n"*).

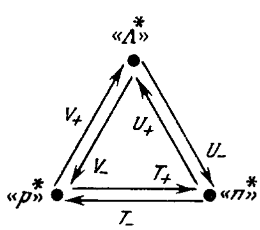


FIG. 8. Antiquark diagram. We show that these are not reducible to quarks. The triangle has a different arrangement of vertices compared to Fig. 7, but the same directions for the coordinate axes. The action of the generators is accompanied by a change of sign.

3. Supermultiplets.

The wave function of a system consisting of p quarks and q antiquarks is obviously described by a superspinor of higher rank $\psi^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q}$. The meaning of this notation is clear: relative to each superscript ψ behaves under transformations of SU_3 like the spinor ψ^{α}, but relative to the subscripts it behaves like the covariant spinor ψ_{β}. The rank of a superspinor is thus determined by two numbers p and q; this is the essential difference between SU_3 and SU_2, and is related to the essential difference in the transformation properties of quarks and antiquarks. We shall sometimes denote a superspinor by its rank symbols: $\binom{p}{q}$.

The group of transformations of the f-spinor $\binom{p}{q}$ is a representation of the SU_3 group. The generators of this group are the operators

$$F_k^j=\sum_{r=1}^{p}\lambda_k^{j(r)}+\sum_{r=1}^{q}\bar{\lambda}_k^{j(r)}.$$

It then follows that the commutation properties of the F_k^j coincide with the properties of the λ_k^j:

$$[F_k^j F_m^l]=\delta_m^j F_k^l-\delta_k^l F_m^j.$$

The symmetry properties of the generators are also preserved:

$$F_k^j=F_j^{k+},\quad \sum_{j=1}^{3}F_j^j=0.$$

Just as for the F_k^j, we can separate out of the components of the generators F_k^j the isospin operators T,U,V and the charges Z,Y,X. We shall denote isospins and charges by these same symbols for spinors of any rank. We shall not repeat the relations between them and the F_k^j, since it is sufficient to replace λ_k^j by F_k^j in the formulas of section 1. We can similarly introduce eight generators F_i corresponding to the matrices λ_i.

It is obvious that we still have the commutativity of each charge with the "perpendicular" isospin.

In general the f-spinor $\binom{p}{q}$ is reducible. There are two operations of contraction relative to the SU_3 group (whereas there was only one for SU_2). The first, which is related to the unitarity of the transformations, is the contraction of one superscript with one subscript, or multiplication by δ_{α}^{β}:

$$\delta_{\alpha_1}^{\beta_1}\psi_{\beta_1\beta_2\ldots}^{\alpha_1\alpha_2\ldots}=\varphi_{\beta_2\ldots}^{\alpha_2\ldots},$$

i.e.,

$$\binom{p}{q}\rightarrow\binom{p-1}{q-1}.$$

The second operation is related to the unimodularity of the transformation. The quantities $\epsilon_{\alpha\beta\nu}\psi^{\alpha\beta\nu}$ or $\epsilon^{\alpha\beta\nu}\psi_{\alpha\beta\nu}$, where $\epsilon_{\alpha\beta\nu}$ and $\epsilon^{\alpha\beta\nu}$ are unit antisymmetric symbols, are invariants. Thus

$$\varepsilon_{\beta\alpha_1\alpha_2}\psi_{\beta_1\ldots}^{\alpha_1\alpha_2\alpha_3\ldots}=\varphi_{\beta\beta_1\ldots}^{\alpha_3\ldots},$$

i.e.,

$$\binom{p}{q}\rightarrow\binom{p-2}{q+1}$$

or, analogously,

$$\binom{p}{q}\rightarrow\binom{p+1}{q+2}.$$

It is obvious that an f-spinor will be irreducible if all three of the multiplications by δ_{α}^{β}, $\epsilon^{\alpha\beta\nu}$ and $\epsilon_{\alpha\beta\nu}$ make it vanish. In other words, an irreducible spinor is symmetric in both its upper and lower indices (separately), and any trace (on one upper and one lower index) vanishes. We shall sometimes denote an irreducible spinor of rank p,q by the symbol $\left[{p \atop q}\right]$.

Let us count the number of independent components of the irreducible spinor $\left[{p \atop q}\right]$. Because of the symmetry, only those components are different which differ in one of the three numbers p_1, p_2 and p_3, which give the numbers of ones, twos and threes in the superscripts (for fixed subscripts), without regard to order. Since $p_1+p_2+p_3=p$, for fixed p_1 the number p_2 can vary from 0 to $p-p_1$ (while p_3 varies from $p-p_1$ to 0). This gives $p-p_1+1$ different components. Now varying the value of p_1 from 0 up to p, we get the total number of different components with fixed subscripts to be $N_p=\sum_{p_1=0}^{p}(p-p_1+1)=\frac{1}{2}(p+1)(p+2)$. Similarly, the number of different components with fixed upper indices will be N_q. We thus get altogether N_pN_q components, but these are not independent, since they are related by the zero trace conditions. The trace is a spinor $\binom{p-1}{q-1}$. This means that among the components of $\left[{p \atop q}\right]$ there are still $N_{p-1}N_{q-1}$ relations. Finally the number of independent components of the spinor $\left[{p \atop q}\right]$ is equal to $N_{pq}=N_pN_q-N_{p-1}N_{q-1}$, i.e.,

$$N_{pq}=\frac{1}{2}(p+1)(q+1)(p+q+2).$$

Some values of $N_{pq}=N_{qp}$ are given in Table I.

Table I. Number of particles in a supermultiplet, N_{pq}

q \ p	0	1	2	3	4
0	1	3	6	10	15
1		8	15	24	35
2			27	42	60
3				64	90
4					125

The number N_{pq} is sometimes used as the symbol for the corresponding spinor. For $p\geq q$, we can denote $\left[{p \atop q}\right]$ by $\{N_{pq}\}$, while for $p<q$, we use $\{\bar{N}_{pq}\}$; for example, $\left[{3 \atop 0}\right]=\{10\}$, $\left[{0 \atop 3}\right]=\{\overline{10}\}$.

If the wave function of a state is an irreducible spinor $\left[{p \atop q}\right]$, there are N_{pq} degenerate states. These form a supermultiplet. The supermultiplet is denoted by the same symbol as the corresponding spinor. Each component of a supermultiplet is identical to any other with respect to s-interactions. This means that the scattering matrix is independent of the quantum numbers that distinguish the components, and is diagonal with respect to the pairs of numbers p,q that determine the rank of the irreducible f-spinor.

A supermultiplet can be characterized by another pair of numbers, which can be expressed in terms of p and q. These are the quadratic and cubic invariants which can be formed from the components of the generators of the irreducible representation F_k^i, just as the rank of an isospinor can be given by the square of the spin operator T^2. These are the invariants $\Sigma F_k^j F_j^k$ (which we shall meet later) and $\Sigma F_k^j F_l^k F_j^l$. They are called the Casimir operators.

We distinguished isospinors of two types: tensors and true spinors. Similarly there are superspinors of three types, which differ in their behavior under the transformations of the center of SU_3. The center of SU_3 is made up of the three matrices $U=\sqrt[3]{1}$. We denote them by $U_\nu=e^{i2\pi\nu/3}$ ($\nu=0,\pm 1$). Under these transformations, a spinor of rank p,q is multiplied by $e^{i2\pi(p-q)\nu/3}$. From this it is clear that

$$\text{for } p-q=3n \qquad U_\nu=1,$$
$$\text{for } p-q=3n\pm 1 \qquad U_\nu=e^{\pm i2\pi\nu/3}.$$

Spinors that do not change under the transformations of the center will be called tensors (supertensors, f-tensors), spinors which are multiplied by a factor $e^{\pm i2\pi\nu/3}$ will be called true spinors. We shall call the tensor $[^1_1]$ a vector. The number of its components (eight) is equal to the number of parameters of the group.

4. The Isomultiplet Diagram

The components within a supermultiplet should differ in three quantum numbers. Actually, for given p and q, the components of an irreducible spinor are determined by four numbers, for example p_1, p_2, q_1, q_2 (the numbers of ones and twos among the superscripts and subscripts), where these are still connected by the zero trace conditions. It is convenient to characterize each component by the eigenvalues of three commuting operators—one of the charges and the corresponding isospin, for example, Y, T, T_3 or Z, U, U_3.

Let us see how many isomultiplets are contained within a given supermultiplet and which ones they are. If we picture a supermultiplet by points in the T_3, Y plane, some of the points will be multiple, since we have seen that two quantum numbers are not sufficient for a unique characterization of the components. We can therefore form superpositions of them corresponding to definite values of T. To find the values of T, we need not find the superposition coefficients explicitly. It will be sufficient merely to find the multiplicity of the corresponding points.

The values of T_3 and Y can be found by summing the contribution from each of the spinor indices. It is easy to see, for example, from the quark diagram, that

$$T_3=\frac{1}{2}(p_1-p_2)-\frac{1}{2}(q_1-q_2),\quad Y=\frac{1}{3}(p_1+p_2-2p_3)-\frac{1}{2}(q_1+q_2-2q_3).$$

First we find the maximum value of T_3; this will occur for $p_1 = p$, $q_2 = q$ (i.e., $\psi^{11\cdots 1}_{22\cdots 2}$):

$$T_3=\frac{p+q}{2}.$$

This value occurs once; the point is nondegenerate, and can correspond only to the maximum value of $T = T_0$:

$$T_0=\frac{p+q}{2}.$$

To this isomultiplet there corresponds a unique value of the hypercharge

$$Y_0=\frac{p-q}{3}.$$

We have obtained the first, farthest right, point in Fig. 9. The abscissa is the value of 2T, and the ordinate is the value of Y. For concreteness, we shall suppose that $p > q$. (In the converse case the diagram is reflected in the Y axis.)

Now we find the components with values $Y = Y_0$, $T_3 = T_0 - 1$. In general there will be three of these, which are obtained from the preceding one, if: 1) one superscript is replaced by a two, or 2) one subscript is replaced by a one, or 3) a superscript and a subscript are replaced by threes. But since the sum of these three components is zero (zero trace), only two independent components remain. One of the combinations of components must be assigned to the isomultiplet $T = T_0$, the second forms the start of a new isomultiplet $T = T_0 - 1$. On the diagram this second point appears on the same horizontal line.

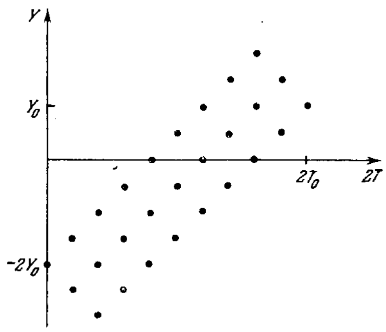


FIG. 9. Isomultiplet diagram. Each isomultiplet appearing in a given supermultiplet $[^p_q]$ is shown by a point in the (2T,Y) plane. The points fill a rectangle which can be constructed if one knows $T_0 = (p + q)/2$, $Y_0 = (p - q)/3$

Starting from these two components we find three more independent components with values $T_3 = T_0 - 2$. They enter into the previous isomultiplets $T = T_0$, $T_0 - 1$ and give the start of a new isomultiplet $T = T_0 - 2$. Continuing the process, we shift each time by one unit to the left along the horizontal in Fig. 9 until the number of components ceases to increase. This will occur when all the subscripts become ones ($q_1 = q$) for the value $T_3 = T_0 - q$. Thus, for $Y = Y_0$, one can have values $T = T_0, T_0 - 1, \ldots, \frac{p-q}{2}$.

Now we consider the initial component $\psi^{11\cdots 1}_{22\cdots 2}$. We shall change the indices so as to change the value of Y. The minimal change occurs if we replace one index by a three. If we change a superscript ($p_1 = p - 1$, $p_3 = 1$), we get $Y = Y_0 - 1$, $T_3 = T_0 - 1/2$, while if we change a subscript ($q_2 = q - 1$, $q_3 = 1$) we get $Y = Y_0 + 1$, $T_3 = T_0 - 1/2$. Each of these components belongs to a definite isomultiplet $T = T_0 - 1/2$. Starting from each of these two points, we find in the same way as before all the points located to their left along these two horizontal lines:

$$\text{for } Y=Y_0+1,\quad T=T_0-\frac{1}{2},\quad T_0-\frac{3}{2},\ \ldots,\ \frac{p-q}{2}+\frac{1}{2};$$

$$\text{for } Y=Y_0-1,\quad T=T_0-\frac{1}{2},\quad T_0-\frac{3}{2},\ \ldots,\ \frac{p-q}{2}-\frac{1}{2}.$$

Such a procedure gives all the points shown in the diagram. It is now easy to formulate a simple rule for its construction.[13] The diagram shows a region bounded by a rectangle. The vertices are the following points (2T, Y):

$$1)\ (2T_0,\ Y_0),\quad 2)\ (0,\ -2Y_0),$$
$$3)\ \left(p,\ T_0-\frac{Y_0}{2}\right),\quad 4)\ \left(q,\ -T_0-\frac{Y_0}{2}\right).$$

The sides are at an angle of 45° to the coordinate axes, so that the first two points are sufficient for the construction. The points showing the isomultiplets are at all points of the boundary having integral coordinates, and in the interior, so that the distance between neighboring points on the same horizontal (or vertical) line is equal to 2.

We mention two important cases where the isomultiplet diagram has an especially simple form. For p = q the rectangle becomes a square with its vertex at the origin and symmetric about the T axis. For q = 0 or p = 0 the rectangle degenerates into a line. Then Y and T are uniquely related (in the plane T_3, Y, there are then no multiple points).

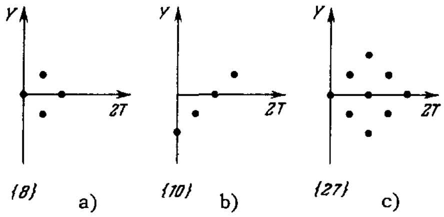


FIG. 10. Isomultiplet diagrams. a) octet; b) decuplet; c) supermultiplet $\{27\} = \left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]$ on the figure:

Figure 10 gives isomultiplet diagrams for the octet $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, the decuplet $\left[\begin{smallmatrix}3\\0\end{smallmatrix}\right]$ and the supermultiplet $\{27\} = \left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]$.

It is easy to go over from the diagram in the 2T, Y plane to the one in the $2T_3$, Y plane. How to do this is clear from Fig. 11; we must reflect all the points in the Y axis and fill the horizontals with points separated from one another by two units. The presence of points in the interior of the rectangle of Fig. 10 leads to multiple points in Fig. 11 (indicated by circles).

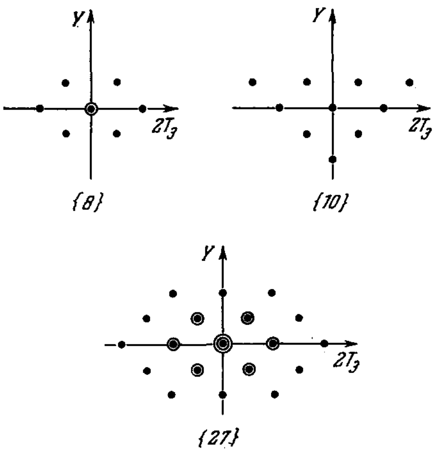


FIG. 11. Structure of supermultiplets in the ($2T_3$, Y) plane. Each point corresponds to a particle. Multiple points are ringed by the appropriate number of circles.

We have used a classification of components according to values of Y and T. It is obvious that one could do exactly the same thing in the quantum numbers Z, U or X, V. The diagrams would have exactly the same form, but with the direction of the Y axis changed. This difference is related to the definition of the Y axis (cf. Fig. 7).

5. Expansion of a Product.

Let us now try to construct the Clebsch-Gordan series for superspinors, i.e., find the structure of the expansion

$$\begin{bmatrix}p_1\\q_1\end{bmatrix}\begin{bmatrix}p_2\\q_2\end{bmatrix}=\sum\begin{bmatrix}p\\q\end{bmatrix}.$$

As in the case of isospinors, we shall not be interested in the explicit form of the coefficients, but only in the values of p and q which enter in the sum. However, here a new situation arises as compared to SU_2. As we shall see, each value of p,q can be multiple, i.e. may appear more than once in the expansion.

To construct the expansion, we shall use the contraction operations which already have served us earlier. There are three such operations: 1) symmetrization, 2) contraction, i.e., multiplication by δ^{β}_{α}, 3) raising and lowering of indices, i.e., multiplication by $\epsilon_{\alpha\beta\gamma}$ and $\epsilon^{\alpha\beta\gamma}$. We explain the simple algorithm, [14, 15] which quickly gives the result, on a numerical example. Suppose that we want to multiply $\left[\begin{smallmatrix}5\\2\end{smallmatrix}\right]$ by $\left[\begin{smallmatrix}4\\4\end{smallmatrix}\right]$:

$$\begin{array}{rcl}
\begin{bmatrix}5\\2\end{bmatrix}\begin{bmatrix}4\\4\end{bmatrix}=\binom{5+4}{2+4} & \rightarrow & \begin{bmatrix}9\\6\end{bmatrix}, \\
\swarrow\quad\searrow & & \\
\binom{4+4}{2+3}+\binom{5+3}{1+4} & \rightarrow & \begin{bmatrix}8\\5\end{bmatrix}_2, \\
\swarrow\quad\searrow\quad\swarrow\quad\searrow & & \\
\binom{3+4}{2+2}+\binom{4+3}{1+3}+\binom{5+2}{0+4} & \rightarrow & \begin{bmatrix}7\\4\end{bmatrix}_3, \\
\swarrow\quad\searrow\quad\swarrow\quad\searrow\quad\swarrow & & \\
\binom{2+4}{2+1}+\binom{3+3}{1+2}+\binom{4+2}{0+3} & \rightarrow & \begin{bmatrix}6\\3\end{bmatrix}_3, \\
\swarrow\quad\searrow\quad\swarrow\quad\searrow\quad\swarrow & & \\
\binom{1+4}{2+0}+\binom{2+3}{1+1}+\binom{3+2}{0+2} & \rightarrow & \begin{bmatrix}5\\2\end{bmatrix}_3, \\
\searrow\quad\swarrow\quad\searrow\quad\swarrow & & \\
\binom{1+3}{1+0}+\binom{2+2}{0+1} & \rightarrow & \begin{bmatrix}4\\1\end{bmatrix}_2, \\
\searrow\quad\swarrow & & \\
\binom{1+2}{0+0} & \rightarrow & \begin{bmatrix}3\\0\end{bmatrix}
\end{array}$$

The meaning of this diagram is the following: a reducible spinor is indicated by two terms, each giving the number of indices symmetric among themselves. Arrows to the right denote symmetrization, after which the spinor becomes irreducible without changing its rank. Arrows downward denote contraction, which can be done only "crisscross," since contractions of the initial (irreducible spinors are necessarily zero. The contraction process is continued until all the superscripts or subscripts are exhausted. The diagram shows clearly how the multiplicity arises (the multiplicity is indicated at the right by a subscript). We note the "trapezoid rule": the multiplicity increases by unity with each contraction (starting from unity) until the smallest term in one of the two pairs is exhausted. The multiplicity increases symmetrically from both ends, and remains constant in the center.

Thus we have obtained the first group of terms of the series (it is now clear how to write them):

$$\binom{5+4}{2+4}\rightarrow\begin{bmatrix}9\\6\end{bmatrix}+\begin{bmatrix}8\\5\end{bmatrix}_2+\begin{bmatrix}7\\4\end{bmatrix}_3+\begin{bmatrix}6\\3\end{bmatrix}_3$$
$$+\begin{bmatrix}5\\2\end{bmatrix}_3+\begin{bmatrix}4\\1\end{bmatrix}_2+\begin{bmatrix}3\\0\end{bmatrix}.$$

Next raise subscripts:

$$\binom{5+4}{2+4}\rightarrow\binom{5+4+1}{1+3}+\binom{5+4+2}{0+2}.$$

Each of the subscript terms is reduced by unity, while a one is added to the top as a separate term. This last term does not participate in the further contractions, since it appeared because of antisymmetrization relative to the subscripts.

We carry out symmetrization and contraction with each of the spinors obtained:

$$\binom{5+4+1}{1+3}\rightarrow\begin{bmatrix}10\\4\end{bmatrix}+\begin{bmatrix}9\\3\end{bmatrix}_2+\begin{bmatrix}8\\2\end{bmatrix}_2+\begin{bmatrix}7\\1\end{bmatrix}_2+\begin{bmatrix}6\\0\end{bmatrix},$$
$$\binom{5+4+2}{0+2}\rightarrow\begin{bmatrix}11\\2\end{bmatrix}+\begin{bmatrix}10\\1\end{bmatrix}+\begin{bmatrix}9\\0\end{bmatrix}.$$

For the last group of terms remaining we «lower» indices:

$$\binom{5+4}{2+4} \to \binom{4+3}{2+4+1} + \binom{3+2}{2+4+2} + \binom{2+1}{2+4+3} + \binom{1+0}{2+4+4}$$

Again there follows symmetrization and contraction. We shall carry it out only for the first term:

$$\binom{4+3}{2+4+1} \to \begin{bmatrix} 7 \\ 7 \end{bmatrix} + \begin{bmatrix} 6 \\ 6 \end{bmatrix}_2 + \begin{bmatrix} 5 \\ 5 \end{bmatrix}_3 + \begin{bmatrix} 4 \\ 4 \end{bmatrix}_3 + \begin{bmatrix} 3 \\ 3 \end{bmatrix}_3 + \begin{bmatrix} 2 \\ 2 \end{bmatrix}_2 + \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(There is no need to go further, since the first two terms in the lower row are already exhausted, the remaining one belongs to the third term. And so we go on.)

This example shows how to proceed in the general case. We can use the equation $N_{p_1q_1}N_{p_2q_2} - \Sigma N_{pq}$ as a check. We give the results for some simple cases that are important in the applications:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}_2 + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix},$$

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

6. Tensor Operators.

As in the case of isospinors, it is easy to see that we can form a vector from the spinor ψ^α and its complex conjugate by means of the matrices λ_k^j:

$$\psi^*\lambda_k^j\psi = \psi^{k*}\psi^j - \frac{1}{3}\delta_k^j\psi^*\psi = \varphi_k^j \qquad (\varphi_\alpha^\alpha = 0).$$

If ψ is an irreducible spinor of arbitrary rank, the result will be the same as above if we assume that the operator λ_k^j acts on one of its superscripts. If we replace the operator λ_k^j by the operator $\Lambda_k^j = \Sigma_r c_r \lambda_k^{j(r)}$, then because of the symmetry of the spinor we get the same vector φ_k^j except for a factor:

$$\left\langle {p \atop q} \middle| \Lambda_k^j \middle| {p \atop q} \right\rangle = \varphi_k^j.$$

We note that the vector operator Λ_k^j acts only on the superscripts of the spinor. The subscripts were not used in constructing the vector, but were only contracted. Now we can construct a vector on the lower indices.

If ψ_β is a covariant spinor,

$$\psi\bar\lambda_k^j\psi^* = -\psi_j^*\psi_k + \frac{1}{3}\delta_k^j\psi\psi^* = -\Phi_k^j \qquad (\Phi_\alpha^\alpha = 0).$$

By using the operator

$$\bar\Lambda_k^j = \sum_r c_r \bar\lambda_k^{j(r)}$$

we can form the vector

$$\left\langle {p \atop q} \middle| \bar\Lambda_k^j \middle| {p \atop q} \right\rangle = -\Phi_k^j.$$

Thus in general we can construct two linearly independent vectors from the irreducible spinor $\left[{p \atop q}\right]$ and its complex conjugate (i.e., the spinor $\left[{q \atop p}\right]$). This also follows from the general Clebsch-Gordan series as applied to the product $\left[{p \atop q}\right]\left[{q \atop p}\right]$. To these two vectors there correspond two vector operators.

If we construct the vector using the generator of the respresentation

$$F_k^j = \sum_r \lambda_k^{j(r)} + \sum_r \bar\lambda_k^{j(r)},$$

it is obvious from the foregoing that

$$\left\langle {p \atop q} \middle| F_k^j \middle| {p \atop q} \right\rangle = \varphi_k^j - \Phi_k^j.$$

A second vector can be gotten using the operator $D_k^j = \sum_{r=1}^{p} \lambda_k^{j(r)} - \sum_{r=1}^{q} \bar\lambda_k^{j(r)}$:

$$\left\langle {p \atop q} \middle| D_k^j \middle| {p \atop q} \right\rangle = \varphi_k^j + \Phi_k^j.$$

From the definition of D_k^j we easily find its commutation properties:

$$[F_k^j D_m^l] = \delta_m^j D_k^l - \delta_k^l D_m^j,$$

$$[D_k^j D_m^l] = \delta_m^j F_k^l - \delta_k^l F_m^j;$$

the first of these relations is obviously valid for an vector operator G_k^j:

$$[F_k^j G_m^l] = \delta_m^j G_k^l - \delta_k^l G_m^j.$$

We note an essential difference as compared to SU_2. There there was one vector, constructed bilinearly from the components of a spinor of arbitrary rank, and to it there corresponded one vector operator: the generator of the representation T_k^j. In SU_3 we have two vectors and two «natural» operators: one of them is the generator F_k^j, the other is D_k^j. The operator D_k^j also has nonzero matrix elements $\left\langle {p_2 \atop q_2} \middle| D_k^j \middle| {p_1 \atop q_1} \right\rangle$, unlike the generator F_k^j. But from now on we shall understand the operator D_k^j to mean only its matrix elements «diagonal» in p and q.

Such a matrix D_k^j can be expressed in terms of the generator F_k^j of the representation without explicitly using the operators λ_k^j and $\bar\lambda_k^j$. In fact, any quadratic combination of the F_k^j that forms a vector must be a linear combination of F_k^j and D_k^j (since there exist only the two vector matrices $\left\langle {p \atop q} \middle| F_k^j \middle| {p \atop q} \right\rangle$ and $\left\langle {p \atop q} \middle| D_k^j \middle| {p \atop q} \right\rangle$). If we require symmetric action on superscripts and subscripts and zero trace, we get an expression differing from D_k^j only by a normalization factor C (depending on p and q):

$$C^{-1}D_k^j = \frac{1}{2}\sum_l (F_l^j F_k^l + F_k^l F_l^j) - \frac{1}{3}\delta_k^j \sum_{lm} F_m^l F_l^m.$$

We also give the explicit form of the components of D_k^j, expressed in terms of charges and isospins. We shall use the same notation as for the corresponding components of F_k^j, but with the index d:

$$D_1^1 = Z^d = C\left(\frac{1}{3}F^2 + \frac{Z^2}{4} - U^2\right),$$

$$D_2^2 = X^d = C\left(\frac{1}{3}F^2 + \frac{X^2}{4} - V^2\right),$$

$$D_3^3 = -Y^d = C\left(\frac{1}{3}F^2 + \frac{Y^2}{4} - T^2\right),$$

$$D_2^1 = T_-^d = C\left(YT_- + \frac{1}{2}\{U_+V_-\}\right),$$

$$D_3^2 = U_-^d = C\left(-ZU_- + \frac{1}{2}\{T_+V_-\}\right),$$

$$D_3^1 = V_-^d = C\left(-XV_- + \frac{1}{2}\{T_-U_-\}\right),$$

Here $\{\ \}$ denotes an anticommutator. The expressions for U_+^d, V_+^d, T_+^d are gotten from those for U_-^d, V_-^d, T_-^d by replacing U_-, V_-, T_- by U_+, V_+, T_+; F^2 is the square of the generator

$$F^2=\frac{1}{2}\sum_{lm} F_m^l F_l^m = U^2+V^2+T^2-\frac{1}{4}(X^2+Y^2+Z^2).$$

It is easy to find the eigenvalues of F^2 from the expression for this operator. Since it is an invariant we can consider that component of the f-multiplet for which $T_3 = (p+q)/2$, i.e., where only the component $\psi^{11\ldots 1}_{22\ldots 2}$ is different from zero. From the diagram for the supertriplet we see that then $T_+\psi = V_+\psi = U_-\psi = 0$. Therefore

$$F^2=\frac{1}{2}(X^2+Y^2+Z^2)+2(T_3+V_3-U_3)=\frac{1}{3}(p^2+pq+q^2)+p+q.$$

We note that the classification of the supermultiplet according to (Y,T) isomultiplets is equivalent to classification according to eigenvalues of the operators F_3^3 and D_3^3.

The existence of the two f-vector matrices F_k^j and D_k^j has the consequence that the general expression for the «Yukawa Lagrangian» for the strong interactions will contain two constants. Let ψ be an arbitrary f-spinor and π^{α}_{β} a vector. Then

$$\mathscr{L}=\psi^*(g_1F_k^j+g_2D_k^j)\psi\cdot\pi_j^k.$$

By definition, the matrices F_k^j acting on the components of an f-multiplet keep it within the same multiplet. The general vector operators Λ_k^j and Λ_k^j do not have this property. It is obviously possible to construct an arbitrary irreducible tensor operator of rank p,q: $O\left[{p \atop q}\right]$. Such an operator has matrix elements $\left\langle {p_2 \atop q_2} \middle| O^{\left[{p \atop q}\right]} \middle| {p_1 \atop q_1} \right\rangle$ for states appearing in the Clebsch-Gordan series.

7. The Problem of Composition

All the presently known supermultiplets are of the tensor type. These are the two meson octets, the octet (possible not single) and decuplet of baryons, and also, apparently, singlets (mesonic and baryonic). This is understandable within our scheme of composition, based on the quarks—particles with fractional charge and hypercharge. We have seen (cf. par. 4 of this section) that the hypercharges and charges of particles in the supermultiplet $\left[{p \atop q}\right]$ are equal to $(p-q)/3$ + an integer. Consequently they are integral only for tensor multiplets.

Now we shall discuss the question whether one can replace the quarks by a triplet of more "normal" particles.[11] First let us convince ourselves that we must ascribe a fractional baryonic charge to the quarks. Suppose this charge is A_0. Then for a hadron composed from ν quarks and $\bar{\nu}$ antiquarks, $A = A_0(\nu - \bar{\nu})$. If this hadron belongs to the supermultiplet $\left[{p \atop q}\right]$, then $\nu - \bar{\nu} = p - q + 3n_1$ (where n_1 is an integer), while for tensor multiplets, $p - q = 3n_2$. Thus, $A = 3nA_0$ (with n an integer). We see that only $A_0 = 1/3n$ guarantees an integral value for A. Since baryons are fermions, n must be odd (the simplest case being $n = 1$).

From the fact that the baryon number is fractional, we deduce the stability of the quark (or the analogous formation, for example, of a "diquark" with $A = 2/3$), since it cannot transform into any collection of particles with integral A.

As for the charge and hypercharge of the quarks, we may still arbitrarily identify them with the numbers for the corresponding generators Z and Y, the coordinates on the quark diagram. No group or electrodynamic requirements will be contradicted by this shift of coordinates on the quark diagram. Suppose the electric charge is not Z, but Q_h:[15]

$$Q_h = Z + c/3,$$

where c is some constant. If we choose $c = 1$ for quarks and $c = -1$ for antiquarks, their electric charges will coincide with the charge of the correspondingly named real particles p, n, Λ (and their antiparticles). The number c, like the baryon number, will be additive. It may be called the supercharge. Since $A = 3A_0n$ for any hadron in a tensor supermultiplet, $c = 3A$.

Here, however, we come into contradiction with the properties of the baryon multiplets. The average value of Z over a supermultiplet is zero. This is directly seen from the quark diagram: the sum of the vectors drawn from the center of the triangle to its vertices is equal to zero. Because of additivity, every supermultiplet will also have this property, i.e., $\bar{Z} = 0$ and consequently $\bar{Q}_h = c/3 = A$. But for the real baryon supermultiplets $\bar{Q}_h = 0$, i.e., $c = 0$. This means that also for the quarks $c = 0$, i.e., the quarks have a fractional electric charge.

A search has been instituted for quarks as charged particles differing from ordinary particles in their ionization properties. If quarks were found, this would mean the discovery of the "truly primary" elementary particles. Even if they do not occur, the corresponding fields could still represent the primary fields. One could say that the Lagrangian for hadrons should be constructed from just these three fields. The one- or two-particle states of such a field with fractional A and Z may correspond to very large (even infinite) masses, but the multiparticle bound states are just the known hadrons. Since the present field theories can neither prove or disprove this assertion, it is difficult to say whether this contains more than a statement of the fact that the hadrons have the SU_3 symmetry, but that only tensor representations of the group are realized in nature.

This last thesis is the content of the "eightfold way" of Gell-Mann and Ne'eman,[16,17] which is the starting point for the successful application of SU_3. We can take the baryon octet as the basis for composition. From it (together with the antibaryon octet) we can obviously construct all the tensor multiplets. But we then lose the basis for introducing the SU_3 group, which was the threefold degeneracy of the fundamental particles.

From now on we shall not essentially go beyond the framework of the eightfold way, although for convenience we may sometimes use the term "quark" as a shorthand way of expressing the symmetry properties of the hadrons.

There is another possibility for composition of hadrons that have the SU_3 symmetry.[18,11] We may choose as basis instead of the quarks a triplet of "normal" particles (i.e., ones with integer A and Z like the Sakata triplet) but introduce a fourth fundamental particle, regarding it as a supersinglet. The fourth particle enables us to eliminate the contradiction described above.

Suppose that we ascribe to the fundamental triplet the baryonic number $A_0 = 1$ and supercharge $c = 1$ (with $c = -1$ for the antiparticles). The construction of the mesons is not changed. For them, independent of the values of A_0 and c,

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A = c = 0.$$

The baryons are obtained from the mesons by adding the baryon singlet, for which $A = 1$, $c = 0$ (this baryon has spin $J = 1/2$).

Thus the required properties of the known supermultiplets are assured: tensorial character and zero value of the supercharge. We avoid the quarks at the expense of first, introducing a fourth fundamental particle, and secondly having the possibility of a class of particles for which $c \neq 0$ (supercharged particles).[15] If it turns out that such particles exist, this will, on the one hand, give a pictorial meaning to the SU_3 symmetry and, on the other hand, will pose a series of new problems. Is c rigorously conserved, like A, or approximately, like Y? Doesn't there exist still another step in the hierarchy of interactions, combining the supermultiplets into "ultramultiplets", just as the introduction of s- and m-interactions combined isomultiplets into a supermultiplet? Shouldn't we change from the triangle of the SU_3 group to the tetrahedron of the SU_4 group,[9] in order to absorb the quantum number c, just as Y was absorbed before? We shall not discuss these questions.

5. THE DECUPLET AND OCTET

1. The Decuplet Diagram

In the preceding section we presented almost all the algebra related to the SU_3 group. Only the expressions for the Clebsch-Gordan coefficients are lacking for carrying out all operations.

However, since the supermultiplets known at present correspond to the simplest f-tensors: $[\begin{smallmatrix}1\\1\end{smallmatrix}]$ and $[\begin{smallmatrix}3\\0\end{smallmatrix}]$, most of the results obtained to date do not require the general expressions for the Clebsch-Gordan coefficients. The situation so far is like that in the usual algebra of three-dimensional vectors. So long as our operations are restricted to vector multiplication or operations with second rank tensors (symmetric and antisymmetric), we can avoid the use of the theory of the rotation group.

The wave function of a particle that is a member of the decuplet is the f-tensor $[\begin{smallmatrix}3\\0\end{smallmatrix}]$. Of the two numbers p and q, giving the rank of the tensor, one is zero. Therefore the states within the decuplet are characterized by only two quantum numbers, instead of the three needed for the general case. This means that if we draw their points in a plane, each point uniquely determines a state. There are no multiple points.

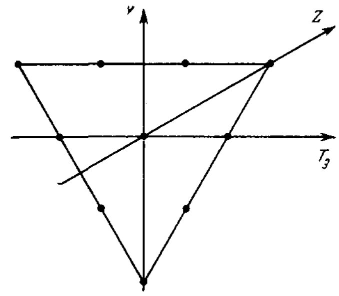


FIG. 12. Decuplet diagram in symmetric form. The triangle is similar to the triangle in the quark diagram.

The decuplet diagram is shown in Fig. 12. It is similar to the quark diagram, being an equilateral triangle. The three axes passing through the vertices of the triangle correspond to the quantum numbers X, Y, Z, and the three axes parallel to the sides of the triangle, to the quantum numbers V_3, T_3, U_3. The charges X, Y, Z are measured in the scale; altitude of the triangle equals three ($p = 3$); the spin projections are measured in the scale: side of triangle equals three. The isomultiplets are arranged along a line parallel to the corresponding side.

Each point (state) can be characterized by any pair of these quantum numbers. All the other quantum numbers are uniquely related to any given pair; for example, if we choose Y and Z, then

$$T_3 = Z - \frac{Y}{2}, \quad U_3 = Y - \frac{Z}{2}.$$

An important point is that we can also associate with each state a definite value of any of the three isospins, i.e., it is a simultaneous eigenstate of T^2, U^2 and V^2. The isospin value is uniquely related to the corresponding charge:

$$T = 1 + \frac{Y}{2}, \quad U = 1 - \frac{Z}{2}.$$

These relations follow from the isomultiplet diagrams (Fig. 10b).

Each state can be represented as a tensor basis function which is the product of three quarks. To the center of the triangle there correspond the basis states "p," "n," and "Λ," to the vertices the states "p"^{3}, "n"^{3}, "Λ"^{3}, etc. We give the relation between the basis states and the symbols of the particles of the $(3/2)^+$ baryon decuplet:

$$\Delta^{++} = «p^3», \ \Delta^+ = «p^2»«n», \ \Delta^0 = «p»«n^2», \ \Delta^- = «n^3»,$$
$$\Sigma^{+*} = «p^2»\Lambda, \ \Sigma^{0*} = «p»«n»«\Lambda», \ \Sigma^{-*} = «n^2»«\Lambda»,$$
$$\Xi^0 = «p»«\Lambda^2», \ \Xi^- = «n»«\Lambda^2»,$$
$$\Omega^- = «\Lambda^3».$$

Since we are dealing with the f-spinor $\psi^{\alpha\beta\gamma}$ which has no subscripts, the two f-vector operators F^j_k and D^j_k coincide. The action of the vector operator F^j_k on each of the basis states is easily found, either directly from the quark diagram or by using the properties of the isospin operators. We can freely use all three isospins T, U, V, since we can at our convenience assign any one of them to each state.

2. Vector Basis Functions

The octet diagram (for the example of the $(\frac{1}{2})^+$ baryons) was shown in Fig. 6, where we chose Z and Y as cartesian axes. One point in the diagram (Y = Z = 0) was double. This expresses the fact that when Y = 0 we have two isomultiplets: T = 0 and T = 1, or, when Z = 0, we have U = 0 or U = 1 (cf. the isomultiplet diagram, Fig. 10a).

Because of this fact (which is related to having $q \neq 0$), the properties of the f-vector $[^1_1]$ are somewhat more complicated than those of the tensor $[^3_0]$. But they are still quite simple.

We introduce nine "vector basis functions," each of which is a composite of one quark and one antiquark. We use the notation given in Table II.

Table II. Scheme for composition of f-vectors from quarks and antiquarks

quark \ antiquark	«p»*	«n»*	«Λ»*
«p»	ν_1	π^+	$\varkappa^+$
«n»	π^-	ν_2	$\varkappa^0$
«Λ»	$\varkappa^-$	$\bar{\varkappa}^0$	ν_3

Figure 13 repeats the diagram of Fig. 6 in different notation corresponding to the labelling of the basis states (which were chosen to agree with the symbols for the (0^-) mesons). The nondiagonal boxes in Table II give the nondegenerate states located on the perimeter of the hexagon in the diagram of Fig. 13. We can assign to each of them definite values for any of the isospins (T, U, V). The two states located at the center of the diagram and labelled π^0 and η represent the two linearly independent combinations of the three basis states ν_i

$$c_1\nu_1 + c_2\nu_2 + c_3\nu_3,$$

satisfying the irreducibility condition

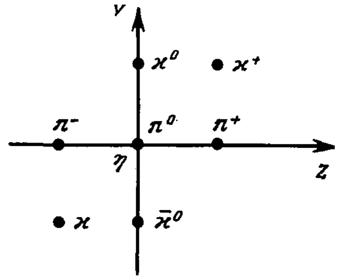


FIG. 13. Octet diagram in coordinates Z,Y. The points show the basis states of the f-vector.

$$c_1 + c_2 + c_3 = 0.$$

These superpositions can be chosen in different ways. If we require the states to have a definite t-spin, these will be

$$\pi^0_t = \frac{1}{\sqrt{2}}(\nu_1 - \nu_2) \qquad (T = 1),$$
$$\eta_t = \frac{1}{\sqrt{6}}(\nu_1 + \nu_2 - 2\nu_3) \quad (T = 0).$$

If we require definite values of the u-spin, then

$$\pi^0_u = \frac{1}{\sqrt{2}}(\nu_2 - \nu_3) \qquad (U = 1),$$
$$\eta_u = \frac{1}{\sqrt{6}}(\nu_2 + \nu_3 - 2\nu_1) \quad (U = 0).$$

The structure of these expressions is obvious; π^0_t together with $\pi^\pm$ forms a t-triplet made up from the two doublets $\binom{\text{“p”}}{\text{“n”}}$ and $\binom{\text{“n”}^*}{-\text{“p”}^*}$; similarly π^0_u, κ^0, $\bar{\kappa}^0$ form a u-triplet built up from $\binom{\text{“n”}}{\text{“Λ”}}$ and $\binom{\text{“Λ”}^*}{-\text{“n”}^*}$. π^0_u and η_u are the corresponding orthogonal superpositions.

From these expressions we get the relations

$$\pi^0_u = \frac{1}{\sqrt{2}}(-\pi^0_t + \sqrt{3}\eta_t),$$
$$\eta_u = -\frac{1}{\sqrt{2}}(\sqrt{3}\pi^0_t + \eta_t).$$

If we use these relations the effect of all the generators F^j_k can be reduced to the action of the isospin operators; each time we need only choose the corresponding basis element. For example, (cf. Fig. 13) $T_+\pi^- \rightarrow \pi^0_t$, while $U_+\bar{\kappa}_0 \rightarrow \pi^0_u$. To determine $U_+\pi^0_t$ or $T_+\pi^0_u$ one must carry out the resolution of the first basis in terms of the second. It is simplest to use the combinations of the basis states ν_i. Table III,a shows the action of the generators $F^j_k = \lambda^j_k + \bar{\lambda}^j_k$ on the basis states, as obtained directly from the diagrams for quarks and antiquarks. The components F^j_k are written as charge and isospin operators.

One can also find the action of the second vector operator $D^j_k = \lambda^j_k - \bar{\lambda}^j_k$. The action of its components is given in Table III,b. The components have the same notation as for F^j_k (with the same j, k), but have an

Table III

a) Action of operator F

(The boxes of the table contain the result of action of the operators at the left on the basis states in the top row)

	ν_1	ν_2	ν_3	π^+	π^-	$\varkappa^+$	$\varkappa^-$	$\varkappa^0$	$\bar{\varkappa}^0$
Y	0	0	0	0	0	$\varkappa^+$	$-\varkappa^-$	$\varkappa^0$	$-\bar{\varkappa}_0$
Z	0	0	0	π^+	$-\pi^-$	$\varkappa^+$	$-\varkappa^-$	0	0
X	0	0	0	$-\pi^+$	π^-	0	0	$\varkappa^0$	$-\bar{\varkappa}_0$
T_+	$-\pi^+$	π^+	0	0	$\nu_1-\nu_2$	0	$-\bar{\varkappa}^0$	$\varkappa^+$	0
T_-	π^-	$-\pi^-$	0	$\nu_2-\nu_1$	0	$\varkappa^0$	0	0	$-\varkappa^-$
U_+	0	$-\varkappa^0$	$\varkappa^0$	$-\varkappa^+$	0	0	π^-	0	$\nu_2-\nu_3$
U_-	0	$\bar{\varkappa}^0$	$-\bar{\varkappa}^0$	0	$\varkappa^-$	$-\pi^+$	0	$\nu_3-\nu_2$	0
V_+	$-\varkappa^+$	0	$\varkappa^+$	0	$-\varkappa^0$	0	$\nu_1-\nu_3$		π^+
V_-	$\varkappa^-$	0	$-\varkappa^-$	$\bar{\varkappa}^0$	0	$\nu_3-\nu_1$	0	$-\pi^-$	0

b) Action of the operator D

Y^d	$\frac{2}{3}\nu_1$	$\frac{2}{3}\nu_2$	$-\frac{4}{3}\nu_3$	$\frac{2}{3}\pi^+$	$\frac{2}{3}\pi^-$	$-\frac{1}{3}\varkappa^+$	$-\frac{1}{3}\varkappa^-$	$-\frac{1}{3}\varkappa^0$	$-\frac{1}{3}\bar{\varkappa}_0$
Z^d	$\frac{4}{3}\nu_1$	$-\frac{2}{3}\nu_2$	$-\frac{2}{3}\nu_3$	$\frac{1}{3}\pi^+$	$\frac{1}{3}\pi^-$	$\frac{1}{3}\varkappa^+$	$\frac{1}{3}\varkappa^-$	$-\frac{2}{3}\varkappa^0$	$-\frac{2}{3}\bar{\varkappa}^0$
X^d	$-\frac{2}{3}\nu_1$	$\frac{4}{3}\nu_2$	$-\frac{2}{3}\nu_3$	$\frac{1}{3}\pi^+$	$\frac{1}{3}\pi^-$	$-\frac{2}{3}\varkappa^+$	$-\frac{2}{3}\varkappa^-$	$\frac{1}{3}\varkappa^0$	$\frac{1}{3}\varkappa^0$
T_+^d	π^+	π^+	0	0	$\nu_1+\nu_2$	0	$\bar{\varkappa}^0$	$\varkappa^+$	0
T_-^d	π^-	π^-	0	$\nu_1+\nu_2$	0	$\varkappa^0$	0	0	$\varkappa^-$
U_+^d	0	$\varkappa^0$	$\varkappa^0$	$\varkappa^+$	0	0	π^+	0	$\nu_2+\nu_3$
U_-^d	0	$\bar{\varkappa}^0$	$\bar{\varkappa}^0$	0	$\varkappa^-$	π^+	0	$\nu_2+\nu_3$	0
V_+^d	$\varkappa^+$	0	$\varkappa^+$	0	$\varkappa^0$	0	$\nu_1+\nu_3$	0	π^+
V_-^d	$\varkappa^-$	0	$\varkappa^-$	$\bar{\varkappa}^0$	0	$\nu_1+\nu_3$	0	π^-	0
	ν_1	ν_2	ν_3	π^+	π^-	$\varkappa^+$	$\varkappa^-$	$\varkappa^0$	$\bar{\varkappa}^0$

added index d. We note that for the octet the coefficient in the expression for D^j_k (p. 163) is $C = 2/3$.

3. Real Vectors and Charge Conjugation

Just as the basis vectors in ordinary three-dimensional space can be chosen to be real (cartesian) or complex (spiral), the f-vector basis can be chosen to be real (the group SU_3 being defined by eight real parameters) or "spiral." The basis vectors used above were "spiral," as one sees from their definition. Six of the basis vectors are complex conjugate in pairs (while the ν_i or π^0 and η are real).

One can introduce real basis vectors. We call them π_i, κ_i, κ_i^0 ($i = 1, 2$) and define them by the following relations:

$$\pi^{\pm} = \frac{1}{\sqrt{2}}(\pi_2 \pm i\pi_1), \quad \varkappa^0 = \frac{1}{\sqrt{2}}(\varkappa_2^0 + i\varkappa_1),$$
$$\varkappa^{\pm} = \frac{1}{\sqrt{2}}(\varkappa_2 \pm i\varkappa_1), \quad \bar{\varkappa}^0 = \frac{1}{\sqrt{2}}(\varkappa_2^0 - i\varkappa_1).$$

The actual wave functions of the particles may be either real or complex vectors. An example of sets of real vectors are the meson octets.

Let us consider the 0^- mesons (for which the intrinsic angular momentum, the spin in the usual sense of the word, is $J = 0$, parity $-$). The particles are described by the correspondingly named spiral basis vectors: π, κ (or K), η. The reasons for the distinction of the spiral vectors is that they are eigenfunctions of the charge Z and hypercharge Y (when $Y = Z = 0$ we have the real basis vectors π^0, η^0). Because of the strict conservation of electric charge, we always deal with states of definite Z. The situation is different for the hypercharge. When $Z = 0$, in various problems we have to consider states with undetermined Y, described by real basis vectors. These are the states κ_1^0 and κ_2^0, which have a definite charge parity.

Charge conjugation is the operation of replacing the quarks by antiquarks (reflection of Table II in the diagonal). From the definition of our basis vectors it follows that under charge conjugation each of the spiral basis vectors is transformed into its complex conjugate. The real vectors π^0, η^0, π_2, κ_2, κ_2^0 do not change, while π_1, κ_1, κ_1^0 change sign.[20]

The behavior of the 0^- meson octet under charge conjugation coincides with that of the correspondingly named basis vectors. We shall by convention say that it has positive C_S-parity ($C_S = +1$). By this it is understood that for π^0, η, κ_2^0, $C = +1$, while for κ_1^0, $C = -1$.

This is not the case in general. The wave function for a particle may contain, in addition to the appropriate basis vector, a factor having definite properties under charge conjugation (just as it may have with respect to other transformations not included in SU_3). It is important that this factor is invariant under SU_3, and the same for all members of the octet. An example of an octet with negative C_S-parity is the 1^- meson octet. The symbols for these mesons are related as follows to the basis vectors:

$$\varrho \longrightarrow \pi, \quad K^* \longrightarrow \varkappa, \quad \varphi \longrightarrow \eta.$$

The fact that $C_S = -1$ means that for ρ^0, φ and K_2^{0*}, $C = -1$, while for K_1^{0*}, $C = +1$.

The wave functions for members of the baryon octets are complex vectors. The invariant factor multiplying the corresponding basis vector is essentially complex in this case. It depends on baryon number, and under charge conjugation the latter changes sign. The baryon octet goes over into the octet of antibaryons. We cannot ascribe a quantum number C_S to a complex vector. Only the spiral basis vectors have a real meaning.

For the baryon octet the relation between the particles and the basis vectors is the following:

$$\Sigma \longrightarrow \pi, \quad \Lambda \longrightarrow \eta, \quad p \longrightarrow \varkappa^+, \quad \Xi^0 \longrightarrow \overline{\varkappa}^0,$$
$$n \longrightarrow \varkappa^0, \quad \Xi^- \longrightarrow \varkappa^-.$$

6. MEDIUM AND ELECTROMAGNETIC INTERACTION

1. Vector Perturbations

The most important applications of the theory of supersymmetry of hadrons are not to relations valid when the symmetry is rigorously observed, but rather to the regularities in violations of the symmetry associated with perturbations having lower symmetry. If these perturbations are small, we can retain the classification of states according to quantum numbers of the supermultiplets and assign a definite f-tensor structure to the perturbation.

In Secs. 2 and 3 we studied the isostructure of the medium and electromagnetic interactions. It was made clear that they can be expressed in terms of isoscalars and isovectors. Under analogous assumptions we can write our perturbations as components of f-tensor operators.

To determine the form of the electromagnetic vertex we can start from the same "principle of minimal interaction." But the result will depend on which model we take as our basis, i.e., which particles we choose as fundamental.

If these are the quarks, their electric charge coincides with Z. From combining of charges of all the quarks, taking account of their strong interaction (i.e., the invariant s-interaction) only that same component of some vector can appear, so that

$$\Gamma^{(e)} = e\mathrm{v}_1^1,$$

where v is an f-vector (and a polar 4-vector).

If this triplet is of the Sakata type, for which the charge is $Z + 1/3$, the vertex will contain an additional f-scalar $\mathbf{v}^0$:

$$\Gamma^{(e)} = e(\mathrm{v}^0 + \mathrm{v}_1^1).$$

Limiting ourselves only to transformations of the t-group or u-group, it is obvious that we come back to our earlier expressions (Sec. 2.7, Sec. 3.3). In fact, with respect to the u-group v_1^1 is a scalar, while for the t-group it is the sum of a scalar and the third component of a vector (cf. the explicit expressions for λ_k^j and D_k^j in Sec. 3).

The purely phenomenological expression $\Gamma^{(e)} = e(v^0 + v_1^1)$ is the general expression having the required isostructure.

On the other hand, the medium interaction is a scalar with respect to the t-group and a combination of a scalar and the third component of a vector relative to the u-group. The general expression for the vertex $\Gamma^{(m)}$ will be

$$\Gamma^{(m)} = \mathrm{m}_3^3,$$

where m is an f-vector. The scalar does not appear here. By definition it belongs to the s-interaction.

2. The Gell-Mann—Okubo Mass Formula

To treat the question of the splitting of the energy levels of the multiplet, i.e., the masses of particles, it is obviously necessary to include the larger of the two perturbations, i.e., the medium interaction. (Then the masses of particles within a supermultiplet will differ by amounts depending on T and Y.) Several questions then arise.

The first concerns the energy scale, which should serve as a criterion of the smallness of the perturbation. It was already mentioned in Sec. 3 that one can use as a basis for introducing the concept of medium strong interaction the fact that the mass differences within the baryon octet are relatively small compared to the mass of their center of gravity (say, the Λ hyperon). Thus, in the absence of more rigorous criteria, our energy scale will be the baryon mass of 1 GeV, and smallness will mean smallness compared to this number.

At the same time we are not forced to require that the splitting of any multiplet be small compared to the center of gravity of that same multiplet. In quantum mechanics, in applying perturbation theory one requires only smallness of the splitting compared to the distance to the next unperturbed level (with suitable quantum numbers). We shall therefore also apply the perturbation method to the meson multiplets, for which the splitting is the same in absolute value as for the baryons, i.e., is much larger compared to the center of gravity of the meson multiplet.

In this connection it is irrelevant which quantity we apply the perturbation formulas to—the mass M or its square M^2. Actually good results are obtained if we apply them to M^2. As a justification for this we may use the argument that all physical quantities (scattering amplitudes) are functions of the relativistic invariant $p^2 = M^2$ (the square of the 4-momentum), so that M^2, and not M, is the more natural quantity.[21] In the absence of more detailed pictures of the mechanism of the splitting, we shall stop with this argument and write the splitting formula in the form $M^2 = M_0^2 + \delta M^2$. If $\delta M^2 \ll M^2$, this is equivalent to the formula $M = M_0 + \delta M$. But this is not so for the case of the mesons.

Finally the third question concerns the expression for δM^2 in terms of the perturbation operator. In Sec. 3 we saw that although the natural (field theory) picture should lead to the Feynman diagram of Fig. 3, i.e., to a second order perturbation, the first order formula corresponding to the diagram of Fig. 4 gives good results.

A "clear" picture of such a perturbation can be gotten from the quark model. Suppose that one of the three quarks, say the "Λ", is different from the others right from the start. Its lines on Feynman diagrams will contain "mass interactions" of the same type as appear in the theory of mass renormalization (cf. Fig. 4). The set of diagrams containing various strong interactions, in which at one point a "cross" for medium interaction is included (cf. Fig. 4) leads to the first order splitting formula

$$\delta M^2 = \langle \Gamma^{(m)} \rangle = \langle m_3^3 \rangle,$$

where the brackets denote the average value of the operator.

After these remarks it is easy to get the splitting formula.[16,12] In fact, we saw (cf. Sec. 4.6) that any f-vector (i.e., its diagonal elements) is expressible in terms of two vectors F and D. Therefore,

$$m_3^3 = c_1 F_3^3 + c_2 D_3^3.$$

Since $F_3^3 = Y$, while the expression for D_3^3 in terms of T and Y was given in Sec. 4.6,

$$M^2 = c_0 + c_1 Y + c_2 \left[T(T+1) - \frac{Y^2}{4} \right].$$

The masses of the particles of known multiplets are given well by this formula. For the baryon octet it gives the relation

$$3\Lambda + \Sigma = 2(N + \Xi),$$

where the particle symbols stand for the values of M^2 (or M).

For the mesons, replacement of Y by $-Y$ for a given T means charge conjugation, in which the mass is unchanged. Thus for the meson octets $c_1 = 0$. For the octet of 0^- mesons we get

$$\pi + 3\eta = 4K,$$

where the particle symbols mean the values of M^2.

For the 1^- mesons this gives

$$\varphi = \frac{1}{3}(4K^* - \rho) = 0.86\ \text{GeV}^2.$$

Actually $\varphi = 1.040\ \text{GeV}^2$. The discrepancy can be attributed to the nearness of the singlet $\omega = 0.613\ \text{GeV}^2$, which spoils the conditions for the applicability of perturbation theory.

For the decuplet, T is expressed in terms of Y: $T = 1 + Y/2$. Thus the mass formula takes the form

$$M^2 = c_1 + c_2 Y.$$

This is the equal spacing formula, obtained in Sec. 3 for the special case of particles with the same charge:

$$\Omega^- - \Xi^* = \Xi^* - \Sigma^* = \Sigma^* - \Delta.$$

3. Electromagnetic Mass Splitting

In Sec. 2 we found formulas for the splitting of the mass within a multiplet. They were based on the properties of the electromagnetic vertex under the t-group. Now, using the properties of the vertices for electromagnetic and medium interaction under the f-group

$$\Gamma^{(m)} = m_3^3, \quad \Gamma^{(e)} = e(v^0 + v_1^1),$$

we can get a formula relating the electromagnetic splittings of different t-multiplets contained in the same supermultiplet.

Let us consider the interaction in the approximation in which the two perturbations add. Actually this will be a good approximation if we treat the electromagnetic interaction in first approximation (i.e., according to second order perturbation theory) and the medium interaction in second or even third order. Then the splitting formula will be the sum of two functions, each of which depends on two (different) variables:

$$\delta M^2 = f^{(m)}(Y, T) + f^{(e)}(Z, U).$$

Finding the mass difference for particles in one t- or u-multiplet, we get the desired relations, which should thus have high accuracy.

Let us consider the six particles forming the hexagon of Fig. 6 (excluding the interior points). They enter into three t-multiplets (N, Σ, Ξ) and three u-multiplets (B^+, B^0, B^-). Denoting the contribution of the medium interaction by m, and of the electromagnetic by e, we get

$$\begin{aligned} p &= m_N + e_+, & \Sigma^+ &= m_\Sigma + e_+, & \Xi^0 &= m_\Xi + e_0, \\ n &= m_N + e_0, & \Sigma^- &= m_\Sigma + e_-, & \Xi^- &= m_\Xi + e_- \end{aligned}$$

(here the particle symbols denote the corresponding masses, the subscripts on m correspond to the t-multiplets, and those on e to the u-multiplets), from which[18]

$$(p - n) - (\Sigma^+ - \Sigma^-) + (\Xi^0 - \Xi^-) = 0.$$

For the decuplet the analogous formula has the form of a sum of two functions of one variable,

$$\delta M^2 = f^{m}(Y) + f^{(e)}(Z),$$

since Y is related uniquely to T, and Z to U. This gives various relations, for example,

$$\Delta^0 - \Delta^+ + \Sigma^{+*} - \Sigma^{0*} = 0,$$
$$\Delta^- - \Delta^0 + \Sigma^{0*} - \Sigma^{-*} = 0.$$

4. The Case of Multiple Points

All the relations obtained above were based on the fact that $\Gamma^{(m)}$ is a t-scalar, while $\Gamma^{(e)}$ is a u-scalar (cf. Secs. 2 and 3). The SU_3 group gives a connection between properties relative to the t- and u-groups; $\Gamma^{(m)}$ is a combination of a u-scalar and a component of a u-vector, while $\Gamma^{(e)}$ is a combination of a scalar and the component of a t-vector. These relations represent differences taken along the directions U = const or T = const on the diagram for a given supermultiplet.

Multiple points lead to specific relations which cannot be gotten within the framework of the isogroups alone. Consider the u-triplet B^0 (Z = 0) in the baryon octet.[7,8] The relation for the electromagnetic form factors (neglecting the m-interaction) has the form

$$\langle \Gamma^{(e)} \rangle_u = \langle \Gamma^{(e)} \rangle_{\Xi^0} = \langle \Gamma^{(e)} \rangle_{\Sigma_u^0}.$$

Σ_u^0 does not represent a real particle because the medium strong interaction mixes the u-triplet Σ_u and the u-singlet Λ_u. The real particle is closer to the state Σ_t, in which we neglect only the electromagnetic corrections. Thus we must use the relation between the u- and t-basis vectors (cf. Sec. 5.2), i.e.,

$$\Sigma_u^0 = \frac{1}{2}(\sqrt{3}\Lambda - \Sigma^0).$$

Thus,

$$\langle \Gamma^{(e)} \rangle_{\Sigma_u^0} \equiv \langle \Sigma_u^0 | \Gamma^{(e)} | \Sigma_u^0 \rangle = \frac{3}{4}\langle \Gamma^{(e)} \rangle_\Lambda + \frac{1}{4}\langle \Gamma^{(e)} \rangle_\Lambda - \frac{\sqrt{3}}{2}\langle \Lambda | \Gamma^{(e)} | \Sigma^0 \rangle.$$

The last term is the matrix element for the decay of the Σ^0 into a Λ with emission of a photon. (All these quantities are functions of the momentum q transferred to the field; it is obvious that a definite q corresponds to real emission; in addition, in the formula it is understood that $\langle \Lambda | \Gamma^{(e)} | \Sigma^0 \rangle$ is real.) From the orthogonality of the states $|\Sigma_u^0\rangle$ and $|\Lambda_u\rangle$, we get

$$\langle \Lambda | \Gamma^{(e)} | \Sigma^0 \rangle = \frac{\sqrt{3}}{2}(\langle \Gamma^{(e)} \rangle_{\Sigma^0} - \langle \Gamma^{(e)} \rangle_\Lambda).$$

Actually the decay $\Sigma^0 \to \Lambda + \gamma$ is a magnetic dipole transition, so the quantities appearing on the right of this formula are magnetic form factors. Since q is relatively small, we may suppose that the corresponding form factors coincide with the magnetic moments of the particles.

Eliminating $\langle \Gamma^{(e)} \rangle_{\Sigma_u^0}$ and $\langle \Lambda | \Gamma^{(e)} | \Sigma^0 \rangle$ from the last three relations between the form factors, we get

$$\langle \Gamma^{(e)} \rangle_n = \frac{3}{2}\langle \Gamma^{(e)} \rangle_\Lambda - \frac{1}{2}\langle \Gamma^{(e)} \rangle_{\Sigma^0}.$$

These relations supplement those obtained in Secs. 2 and 3.

They can be obtained from the general expression for $\Gamma^{(e)} = v^0 + v_1^1$. Since $\langle v_1^1 \rangle$ reduces to a combination of F_1^1 and D_1^1, using the explicit expressions for these (Sec. 4.6) (where $F^2/3 = 1$ for the octet), we get

$$\langle \Gamma^{(e)} \rangle = c_1 + c_2 Z + c_3 \left[1 + \frac{Z^2}{4} - U(U+1) \right].$$

If $\Gamma^{(e)}$ does not contain a scalar term (which corresponds to the quark model, cf. par. 3), we get still another important relation from the last expression; it can help explain the nature of the electromagnetic interaction. Set $c_1 = 0$ in the formula, and apply it to Λ_u (Z = U = 0) and (Z = 0, U = 1), i.e.,

$$\langle \Gamma^{(e)} \rangle_{\Lambda u} = c_3, \quad \langle \Gamma^{(e)} \rangle_n = \langle \Gamma^{(e)} \rangle_{\Sigma_u^0} = -c_3.$$

Then, having expressed $\langle \Gamma^{(e)} \rangle_{\Lambda_u}$ in terms of

$\langle \Gamma^{(e)} \rangle_\Lambda$ and $\langle \Gamma^{(e)} \rangle_{\Sigma^0}$ using the relation (Sec. 5.2),

$$\Lambda_u = -\frac{1}{2}(\sqrt{3}\Sigma^0 + \Lambda),$$

which gives

$$\langle \Gamma^{(e)} \rangle_{\Lambda u} = \frac{1}{4}\langle \Gamma^{(e)} \rangle_\Lambda + \frac{3}{4}\langle \Gamma^{(e)} \rangle_\Sigma + \frac{\sqrt{3}}{2}\langle \Lambda | \Gamma^{(e)} | \Sigma^0 \rangle,$$

we get

$$\langle \Gamma^{(e)} \rangle_\Lambda = -\langle \Gamma^{(e)} \rangle_{\Sigma^0}.$$

Substituting this result in the preceding relation, we get[7,8]

$$\langle \Gamma^{(e)} \rangle_n = 2\langle \Gamma^{(e)} \rangle_\Lambda.$$

Using the relation between Σ_u^0, Λ_u^0 and Σ_t^0, Λ_t^0 and the formulas for the electromagnetic mass splitting, one can get an interesting relation expressing the mixing coefficient for the t-triplet and t-singlet in terms of the real particles Λ and Σ^0, which we regard approximately as states with definite t-spin. These coefficients are expressed in terms of the nondiagonal elements $\langle \Sigma^0 | e | \Lambda \rangle$ of the same matrix e whose diagonal elements $\langle e \rangle = f^e(Z, U)$ gave the electromagnetic mass corrections. If $|\Sigma^0\rangle = |\Sigma_t^0\rangle + |\Lambda_t\rangle \gamma$, then

$$\gamma = \frac{\langle \Lambda_t | e | \Sigma_t^0 \rangle}{\Sigma^0 - \Lambda}$$

(the particle symbols in the denominator denote the masses).

To determine this matrix element we shall use the same method and the same approximation as in par. 3. We first consider the earlier expressions for the masses $p = m_N + e_+$, $n = m_N + e_0$, and $\Sigma^+ = m_\Sigma + e_+$, and add to them

$$\Sigma_t^0 = m_\Sigma + \langle \Sigma_t^0 | e | \Sigma_t^0 \rangle.$$

To find the last matrix element we use the fact that the matrix e is diagonal with respect to states with definite u-spin:

$$e_0 = \langle \Sigma_u^0 | e | \Sigma_u^0 \rangle = \frac{3}{4}\langle \Lambda_t | e | \Lambda_t \rangle + \frac{1}{4}\langle \Sigma_t^0 | e | \Sigma_t^0 \rangle - \frac{\sqrt{3}}{2}\langle \Lambda_t | e | \Sigma_t^0 \rangle,$$

and the relation

$$\langle \Lambda_t | e | \Sigma_t \rangle = \frac{\sqrt{3}}{2}(\langle \Sigma_t^0 | e | \Sigma_t^0 \rangle - \langle \Lambda_t | e | \Lambda_t \rangle).$$

Eliminating $\langle \Lambda_t | e | \Lambda_t \rangle$ from these relations, we get

$$\Sigma_t^0 = m_\Sigma + e_0 + \sqrt{3}\langle \Lambda | e | \Sigma^0 \rangle.$$

From the four expressions for the masses there now follows

$$\sqrt{3}\langle \Lambda_t | e | \Sigma_t^0 \rangle = p - n + \Sigma^0 - \Sigma^-.$$

7. THE WEAK INTERACTION

1. The Cabibbo Parameter

In Sec. 2.8 the weak interaction processes for hadrons were discussed from the point of view of the isogroup. This permitted us to treat only processes with no change in strangeness. From the point of view of the SU_3 symmetry, both types of processes, $\Delta Y = 0$ and $\Delta Y \neq 0$ can be studied on an equal footing.

Let us again consider the vertex $\Gamma^{(w)}$, responsible for the leptonic decay of the hadron with increase in charge ($\Delta Z = 1$).

We look at the quark diagram. Figure 14 shows the two directions corresponding to the transitions of interest. The first is "n" → "p", which is described by one of the t-spin operators T_+. It determines the decays without change in strangeness ($\Delta Y = 0$), which were discussed in Sec. 2.8. The second is "Λ" → "p", which is described by one of the v-spin operators V_+. It determines decays with change in strangeness ($\Delta Y = 1$).

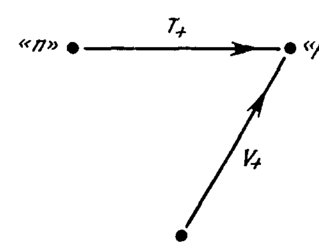


FIG. 14. Structure of the weak interaction. The two directions on the quark diagram that are responsible for leptonic decays are shown: without change (T_+) and with change (V_+) of strangeness.

Both operators T_+ and V_+ are components of the superspin (generator of SU_3) λ, which is an f-vector:

$$T_+ = \lambda_1^2 = \lambda_1 + i\lambda_2, \quad V_+ = \lambda_1^3 = \lambda_4 + i\lambda_5$$

(for the notation cf. Sec. 4.1). It is natural to assume that, in general, leptonic decays of the $\Delta Y = 1$ type are determined by the same components ($^2_1$ and $^3_1$) of some f-vector. If we picture the hadrons as being made up from quarks (or other particles forming a fundamental triplet), it is impossible to construct anything else in first approximation. In fact, aside from single action of the operator λ_k^j, there are only the invariant strong interactions which do not change the vector character of the vertex.

Then the general expression for this vertex will be

$$\Gamma^{(w)} = \alpha R_1^2 + \beta R_1^3.$$

Since CP-invariance requires the coefficients α and β to be real, while the overall normalization of the vector R is as yet arbitrary, this expression can be written in the form

$$\Gamma^{(w)} = \cos\vartheta\, R_1^2 + \sin\vartheta\, R_1^3.$$

We shall call ϑ the Cabibbo parameter.

The formula for the leptonic decay vertex in the form of components of an f-vector containing the parameter ϑ is perhaps the most interesting application of the hypothesis of supersymmetry (SU_3)

for hadrons. First, as we have seen, for this symmetry both types of leptonic decay ($\Delta Y = 0$ and $\Delta Y \neq 0$) are a priori on the same footing. The f-vector properties of the operator R_1^3 are such that for decays with change of hypercharge one gets the selection rule

$$\Delta Y = \Delta Z, \quad \Delta T = \frac{1}{2},$$

which is confirmed by experiment. Among the components of the vector operator of the group SU_3, it turns out that there is an operator $R_1^3 \sim V_+$ having the properties of a spinor relative to the t-group (cf. Sec. 2.8).

Secondly, despite the a priori equal status of the two directions "n" → "p" and "Λ" → "p" on the quark diagram, we see that the general expression for $\Gamma^{(w)}$ contains a parameter ϑ. This leads to a new formulation of the principle of universality of the weak interaction.

The expression for $\Gamma^{(w)}$ can be interpreted as follows, comparing it with quantities characterizing other interactions. The strong interaction is invariant under SU_3, and therefore the amplitudes given by it are scalars. All three interactions that destroy this symmetry can be characterized by their "axes," i.e., their vector basis functions. The vertices for the medium and electromagnetic interactions can be written in the form (cf. Secs. 6.1 and 5.2)

$$\Gamma^{(m)} = (\mathrm{m}\nu^{(m)}), \quad \Gamma^{(e)} = e\nu^0 + e(\nu\nu^{(e)}),$$

where the parentheses contain scalar products of f-vectors, where

$$\nu^{(m)} = \nu_3, \quad \nu^{(e)} = \nu_1$$

(the notation for the vector basis functions is that of Sec. 5.2, Table II).

The vertex for leptonic decay $\Gamma^{(w)}$ can also be written as

$$\Gamma^{(w)} = (\mathrm{R}\nu^+),$$

where

$$\nu^+ = \pi^+ \cos\vartheta + \varkappa^+ \sin\vartheta$$

(π^+, κ^+ are the vector basis functions, cf. Table II). Since κ^+ and π^+ form a u-doublet (for κ^+ "the spin is along the U_3 axis", for π^+ it is "opposite to the U_3 axis"), the expression for ν^+ gives the expression for a rotation in u-spin space ($\nu^+ = e^{i\alpha U_2}\kappa^+$, $\alpha = 2\vartheta - \pi$, i.e., rotation of the U_3 axis through angle α; Fig. 15).

In other words, ν^+ is a superposition of basis vectors with the same value of Z ($Z = 1$) and different values of U_2 (i.e., $Y = 1, 0$). The meaning of this is simple. The leptons carry a definite charge, and the total charge is conserved, so ν^+ has a definite Z. But the hypercharge is not conserved, and the weak interaction "doesn't know" this quantum number. Therefore there is no reason why ν^+ should coincide

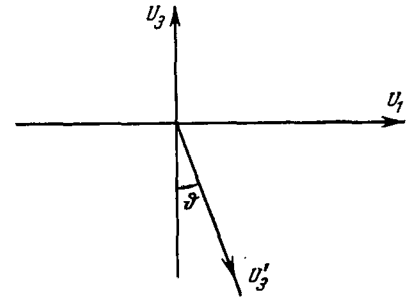


FIG. 15. The Cabibbo parameter in the plane of the u-spin vector. U_3 is the axis given by the strong interaction; U_3' is the axis determining the weak interaction.

with π^+ (this would correspond to the value $\vartheta = 0$ and would forbid decays with change of strangeness) or with κ^+ (this would correspond to $\vartheta = \pi/2$ and would forbid decays with no change of strangeness) or with the bisector of these directions ($\cos\vartheta = \sin\vartheta$).

This last case would mean equal probability for the two types of leptonic decay ($\Delta Y = 0$ and $\Delta Y = 1$). This equality of probability was used earlier as a formulation of the universality of the weak interaction. We see, however, that such a requirement would mean requiring invariance of the weak interaction under the u-group, (no dependence of $\Gamma^{(w)}$ on U_3, i.e., on Y), for which there is no basis.

Let us turn again to the quark diagram. For the strong (truly strong) interactions, there is no difference between "p", "n" and "Λ." We can place any of their orthonormal superpositions "p′", "n′", "Λ′" at the vertices of the triangle. The choice of the Z axis sets the law of conservation of charge. Among the three particles, "p" is distinguished. The choice of the other two vertices of the triangle is still arbitrary. It is determined by which perturbations we want to include. If the medium interaction is included, the quantum number Y is the one (or $U_3 = Y - Z/2$) and "n" and "Λ" appear at vertices of the triangle. But if the weak interaction is included (or if we switch off the medium interaction) it selects its axes (U_3' or Y') and its superpositions of "n" and "Λ"—"n′" and "Λ′" having the property that the "n′" decays while the "Λ′" is stable.

$$«\mathrm{n}'» = \cos\vartheta\, «\mathrm{n}'» + \sin\vartheta\, «\Lambda»,$$

$$\nu^+ = «\mathrm{n}'»^* «\mathrm{p}».$$

The quark diagram for the leptonic decay takes the form shown in Fig. 16. The direction "n′" → "p" determines the leptonic decay, and to it corresponds the operator $T_+' = (\lambda\nu^+)$.

2. Leptonic Decays

Let us consider some consequences of the general expression for the leptonic decay vertex $\Gamma^{(w)}$.[23]

The simplest leptonic decay process is the decay

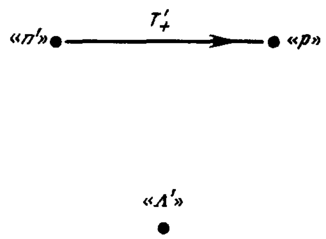


FIG. 16. Structure of the weak interaction. After rotation of the axes, only one direction remains (that of the isooperator T'_+).

of a meson into two leptons. The decay amplitude reduces to the matrix element of $\Gamma^{(w)}$ for the transition of the meson into vacuum. Let us compare the decays $K^- \to \mu^- + \bar{\nu}$ and $\pi^- \to \mu^- + \bar{\nu}$. For the first, $\Delta T = 1/2$, $\Delta Y = 1$, and it is determined by the operator R_1^3. For the second, $\Delta T = 1$, $\Delta Y = 0$; it is given by the operator R_1^2. Since both mesons belong to the same octet,

$$\langle 0 | R_1^3 | K^- \rangle = \langle 0 | R_1^2 | \pi^- \rangle.$$

Thus the ratio of amplitudes is

$$\frac{\langle 0 | \Gamma^{(w)} | K^- \rangle}{\langle 0 | \Gamma^{(w)} | \pi^- \rangle} = \tan \vartheta.$$

We see that the measurement of the decay probabilities $K_{\mu 2}$ and $\pi_{\mu 2}$ permits a direct determination of ϑ. From these data, $\vartheta \approx 0.26$.

So far we have not paid attention to the space time properties (in the sense of the Lorentz group) of the quantity R. For decays with strangeness conservation, it is well established that $\Gamma^{(w)}$ consists of two terms: a polar vector term and an axial vector. This property must now also be extended to decays with change in strangeness, so that R must have the form

$$R = g(v + a),$$

where v is a polar 4-vector, and a an axial 4-vector, and both are f-vectors. The corresponding components of these f-vectors form the isovectors considered in Sec. 2.8, and

$$g_\beta = g \cos \vartheta.$$

The isovector v, appearing at the vertex for leptonic decay with conservation of strangeness, as pointed out in Sec. 2.8, coincides with the isovector giving the electromagnetic vertex. Therefore the f-vector **v** introduced above must coincide with the one introduced in Sec. 6.1 in the expression for the electromagnetic vertex.

Most of the known leptonic decay processes are processes of transformation of one hadron, belonging to a superoctet, into another belonging to the same octet. Then the quantities v and a, which are in general functions of the momentum q transferred to the leptons, should be taken as constants, corresponding to their values when q = 0. But then (in the rest system of the decaying particle) the polar 4-vector v can have only a time component v_0, and the axial 4-vector a only its space components **a**, proportional to the intrinsic angular momentum operator **J**.

One can determine v_0 from a comparison with the electromagnetic vertex. Since we are treating transitions within a given octet, v, like any f-vector, can be expressed in the form (cf. Sec. 3.6)

$$v = c_1 F + c_2 D.$$

For the electromagnetic vertex this gives

$$\Gamma^{(e)} = e v_1^1 = e(c_1 Z + c_2 Z^d).$$

But, when q = 0 the electromagnetic vertex (more precisely, its time component) coincides with the electric charge eZ. Consequently

$$c_1 = 1, \quad c_2 = 0.$$

Thus the time component of the vertex $\Gamma^{(w)}$ is equal to

$$\Gamma_0^{(w)} = g(\cos \vartheta T_+ + \sin \vartheta V_+).$$

This expression determines the β decays of the π and K mesons (transitions without parity change of particles with intrinsic angular momentum J = 0):

$$\langle \pi^0 | \Gamma^{(w)} | \pi^- \rangle = g \cos \vartheta \langle \pi^0 | T_+ | \pi^- \rangle = \sqrt{2}\, g \cos \vartheta,$$
$$\langle \pi^0 | \Gamma^{(w)} | K^- \rangle = g \sin \vartheta \langle \pi^0 | V_+ | K^- \rangle = \frac{1}{\sqrt{2}} g \sin \vartheta$$

(the matrix elements can be found from Table IIIa, using the relation $\pi^0 = (1/\sqrt{2})(\nu_1 - \nu_2)$).

The first of these was used for an experimental test of the basic hypothesis of the identity of the isovectors for β decay and for the electromagnetic vertex. We see that the ratio of the probabilities for these two decays is determined only by the Cabibbo parameter. The existing experimental data agree with the value of ϑ given above.

The value of the constant g is found from comparison of leptonic decays of hadrons with the leptonic decay of the lepton (muon). By definition, for the muon decay vertex (free of strong interactions)

$$\Gamma^{(\mu)} = g.$$

This is the formulation of the universal weak interaction. The appearance of the Cabibbo parameter in the ratio of $\Gamma^{(\mu)}$ and $\Gamma^{(w)}$ shows that universality does not necessarily mean equality of the constants for μ and β decay. (The latter is more precisely determined from the O^{14} decay). The value of ϑ given above leads to a difference between these two constants of about 2%. Such a difference actually exists, but it is difficult to say what it should be ascribed to, in view of the uncertainty about radiative corrections for hadrons.

For the axial vector **a** we can write the general expression in the form

$$\mathbf{a} = \mathbf{J}(\alpha D + \beta F).$$

Correspondingly the spatial components of the vertex are equal to

$$\Gamma^{(w)} = g\mathbf{J}\{\alpha T_+^d + \beta T_+)\cos\vartheta + (\alpha V_+^d + \beta V_+)\sin\vartheta\}.$$

The two constants appearing here can be found from studies of the β decays of baryons without change of strangeness. For the β decay of the neutron (according to Table III, where $n \rightarrow \kappa^0$, $p \rightarrow \kappa^+$)

$$\langle p|\Gamma^{(w)}|n\rangle = g\cos\vartheta[\beta\langle p|\mathbf{J}T_+|n\rangle + \alpha\langle p|\mathbf{J}T_+^d|n\rangle]$$

$$= g\cos\vartheta(\alpha+\beta)\langle\mathbf{J}\rangle,$$

where $\langle\mathbf{J}\rangle$ is a quantity depending on the polarization of the particles. From the experimental data, $\alpha + \beta = 1.15$.

For the decay $\Sigma^- \rightarrow \Lambda$ (cf. Table III, $\Sigma^- \rightarrow \pi^-$, $\Lambda \rightarrow (1/\sqrt{6})(\nu_1 + \nu_2 - 2\nu_3)$)

$$\langle\Lambda|T_+|\Sigma^-\rangle = 0,\quad \langle\Lambda|T_+^d|\Sigma^-\rangle = \sqrt{\frac{2}{3}},$$

$$\langle\Lambda|\Gamma^{(w)}|\Sigma^-\rangle = \sqrt{\frac{2}{3}}\, g\cos\vartheta\alpha\langle\mathbf{J}\rangle.$$

From the experimental data, $\alpha \simeq 0.7$.

A large number of other decays can be analyzed on the basis of the expressions for the vertices, with the already known constants. The smallness of $\sin^2\vartheta$ gives a natural explanation of the small probability for β decay of hyperons (with change of strangeness) as compared to the ordinary β decays.

3. Nonleptonic Decays and Weak Nuclear Forces

It was already mentioned in Sec. 2.8 that there are two types of processes of weak interaction occurring without participation of leptons and, consequently, with conservation of charge of the hadrons ($\Delta Z = 0$). These are the nonleptonic decays ($\Delta Y \neq 0$) and processes caused by weak nuclear forces not conserving parity. Both these processes differ essentially in character from leptonic processes. For leptonic decays, the presence of part of the system which has no strong interactions reduced the scattering matrix to the vertex operator. This cannot be done for processes proceeding without emission of leptons. Therefore, strictly speaking one cannot obtain quantitative consequences for the processes with $\Delta Z = 0$ from the properties of the leptonic decays.

This applies also to the f-tensor character of the scattering matrix determining these processes. It is natural, however, to assume that the scattering matrix (cf. Sec. 2.8) contains a sort of product of two vertices (for the "decaying" and "emitted" hadrons). Then its transformation properties are expressed by the product of the two vertices $\Gamma^{(w)}\Gamma^{(w)+}$. In this restricted form, we can then apply the principle of universality of the weak interaction to processes without leptons.

The empirical rules are known for nonleptonic decays: $\Delta Y = 1$, $\Delta T = 1/2$, i.e., the same as for leptonic decays with change of strangeness. Obviously both rules will be satisfied if the scattering matrix (like the leptonic decay vertex) is a component of a supervector. But this is not a consequence of the principle of universality in the formulation given above, since the product of two f-vectors contains, in particular, the tensor $[\,^2_2]$, and the corresponding operator will allow transitions with $\Delta Y = 2$ and $\Delta T = 3/2$. Thus further hypotheses are needed, reducing, for example, to the requirement that the properties of strong interactions should particularly enhance the f-vector part of the scattering matrix.[24]

In any case we shall assume that the scattering matrix for processes of weak interaction without leptons is determined by some f-vector operator. We call it $S^{(h)}$.

$S^{(h)}$ should consist of two terms, differing in their properties under space inversion. One of them should be a 4-scalar, the other a 4-pseudoscalar. The first gives processes with conservation of P-parity, the second with nonconservation of P-parity. Since the combined CP-parity must be conserved, the first term must be C-even, and the second C-odd. But, as shown in Sec. 5.3, the components of an f-vector have different C-parity; namely, for the components corresponding to projections along ν_i, π_2, κ_2, κ_2^0 (i.e., for the components $S_i^{(h)}$, with $i = 1, 3, 4, 6, 8$), $C = C_S$; for projections along π_1, κ_1, κ_1^0 ($i = 2, 5, 7$) $C = -C_S$, where $C_S = \pm 1$ is the same number for all of the f-vector.

Let us explain how the components $S^{(h)}$ can enter into the amplitudes for weak interaction processes. First let us consider the weak nuclear forces. Since both Z and Y are conserved for these processes, the scattering matrix contains only components of $S^{(h)}$ commuting with $Z = F_1^1$ and $Y = F_3^3$, i.e., $S_1^{(h)1}$ and $S_3^{(h)3}$, or in the other notation, $S_3^{(h)}$ and $S_8^{(h)}$.

If we assume that these components satisfy the requirement of conservation of CP-parity, then we must automatically assume that the f-vector $S^{(h)}$ has no projections along π_1, κ_1, κ_1^0 ($S_i^{(h)} = 0$ for $i = 2, 5, 7$). With respect to the t-group the first is the component of an isovector and the second an isoscalar. There then follows the selection rule $\Delta T = 0, 1$.

It is worthy of note that the selection rule becomes more rigorous if we assume that the vector $S^{(h)}$ is the product of two vector vertices.[25] Let $S_\beta^{(h)\alpha} = R_\gamma^\alpha R_\beta^\gamma - \frac{1}{3}\delta_\beta^\alpha R^2$, where R is a vector having components $R_3^1 \sim \sin\vartheta$ and $R_1^2 \sim \cos\vartheta$, where ϑ is the Cabibbo parameter (the other components being zero). Then

$$S_1^{(h)1} = \frac{1}{3},\quad S_3^{(h)3} = \sin^2\vartheta - \frac{1}{3},\quad S_3^{(h)} = \frac{1}{2}\sin^2\vartheta.$$

Because of the smallness of $\sin^2\vartheta$ we have the inequality $S_3^{(h)} \ll S_8^{(h)}$. Thus the isosinglet component is the main thing, and we have $\Delta T = 0$. Thus the experimental investigation of selection rules for the weak nuclear forces is of particular interest for the whole theory of the weak interaction.

4. θ-Decay

Now we turn to the nonleptonic decays. In such processes Z is conserved, while Y changes. This means that the corresponding components of the vector $S^{(h)}$ are nondiagonal and commute with $Z = F_1^1$. Since the u-spin operators commute with Z and $U_1 = F_6$ and $U_2 = F_7$ (the directions not changing Z on the quark diagram: "n" $\rightleftarrows$ "Λ"), the nonleptonic processes can be determined by the components $S_6^{(h)}$ and $S_7^{(h)}$. But the latter, as we have seen, is excluded by the requirement of conservation of CP-parity.

Thus the nonleptonic decays are given by the basis vector κ_2^0 (the component $S_6^{(h)}$).[20]

If we picture the vector $S^{(h)}$ as the product of two vector vertices $S^{(h)} \sim RR^+$, we can use the transformation by the rotation in u-space (cf. part. 1) that takes ν^+ into $\pi^{+\prime}$, i.e., $\Gamma^{(w)} \sim T'_+$. Then $S^{(h)}$ will contain the component T'_3 of a t′-vector or a t′ scalar. This means, if we look at the diagram in Fig. 16, that in nonleptonic processes there are no transformations of one particle into another "Λ′" does not participate at all in the weak interaction (cf. par. 1).

The amplitude for nonleptonic decay should thus be regarded as an isoscalar, or the third component of an isovector, with respect to the transformed states. Let us consider the decay $K_1^0 \rightarrow 2\pi$ (θ-decay).[20] This process proceeds with change of P and change of C ($C_{K_1^0} = -1$). The CP conservation law allows this decay and forbids the decay $K_2^0 \rightarrow 2\pi$ (since $C_{K_2^0} = +1$). But all the t′ scalars formed from three mesons, of the type $(K'K)\eta'$, $(K'\sigma_i K')\pi'_i$, and t′ vectors of the type $(K'\sigma_3 K')\eta'$ etc. are C-even. Thus θ-decay is forbidden by the symmetry properties of the strong interaction.

The fact that the observed θ-decay should be regarded as forbidden is very important in comparing the probability for this decay with the $K^+ \rightarrow 2\pi$ decay. The latter is forbidden by the $\Delta T = 1/2$ rule, so that it is natural to assume that the probability contains the small coefficient $(1/137)^2$ (because of the electromagnetic interaction). But the experimental ratio of these two decays is 700. If the decay probability for $K_1^0 \rightarrow 2\pi$ itself contains a small coefficient $(1/5)^2$, we get just the required magnitude.

CONCLUDING REMARKS

The symmetry theory of hadrons, based on SU_3, contains quite a number of results: first, the classification of particles into supermultiplets and formulas for their splitting; second, the properties of electromagnetic and weak interactions. (To date, the results concerning the strong interactions are less definite.)

The relations predicted by the theory are confirmed to the accuracy to which the theory can pretend. For a number of relations, such as those between magnetic moments and decay probabilities, the accuracy of the experiments is still not good enough. Undoubtedly the predictions of the theory will stimulate appropriate experiments.

All the comparisons so far refer to corrections of first order in the medium interaction. Since the corresponding expansion parameter is not very small ($\sim 1/5$), and the physical picture of this interaction is not entirely clear, the existing agreement does not yet give one a great feeling of security. Effects in which one can test the predictions of the theory to high accuracy will therefore be especially important. Of particular importance is the result[25] for leptonic decays with change in strangeness, produced by polar vector interactions. In Sec. 7.2 it was shown that such interactions are described by the same f-vector as for the electromagnetic interaction. This property is conserved, not only in zeroth, but also in first order in the medium interaction. The deviation should consequently already be of order $(1/5)^2$.

Finally, further experiments on the spectrum of elementary particles will be particularly important. There are particles known at present whose assignment to supermultiplets is still not clear.

If we look carefully at the table of elementary particles in the region of the established supermultiplets, we note the fact that the two meson octets and the two baryon supermultiplets (octet and decuplet) are not well separated from one another, while the intervals within them are approximately the same. Thus, for example, $\rho - \pi \approx K^* - K$ (the symbols denote the squares of the masses) and $\Xi - \Sigma \approx \Xi^* - \Sigma^*$.

This leads to the idea of a possible extension of the symmetry group to include the spin angular momentum of the particles. The development of this idea leads to various interesting results. We are now dealing with the group SU_6, which contains SU_3 and SU_2 (the space of rotations associated with the transformation of the spin angular momentum). A similar symmetry under SU_4, which includes the isogroup SU_2 and the group of the spin angular momentum SU_2, was considered long ago by Wigner[27] for describing the properties of the lightest nuclei. For the SU_6 group a vector is a quantity ψ_β^α ($\alpha, \beta = 1, 2, \ldots, 6$, $\psi_\alpha^\alpha = 0$), having 35 components. If we assume that they are all on an equal footing in zeroth approximation, while in first approximation they split into the supermultiplets of SU_3 with definite J, this splitting will be the following:

$$35 = 8 \times 1 + 8 \times 3 + 1 \times 3,$$

where each term denotes $N(2J + 1)$ (where N is the number of particles in the SU_3 supermultiplet). Thus we have a 0^- octet, a 1^- octet and a 1^- singlet (all with the same parity). It is extremely interesting that this higher symmetry predicts, in addition to the 1^- superoctet, a 1^- supersinglet, i.e., two isosinglets 1^- (φ and ω) with almost the same mass, so that the real particles φ and ω are superpositions of particles of the superoctet and supersinglet, and the theory enables us to find the coefficients of this superposition. We note that as before the 0^- mesons form only an octet (the 0^- meson (supersinglet) now known has a mass of 960 MeV, which is far from the center of the octet).

For the baryons, the simplest quantity transforming according to SU_6 is the quantity $\psi^{\alpha\beta\gamma}$ ($\alpha, \beta, \gamma = 1, 2, \ldots, 6$), symmetric in all indices. It has 56 components and splits as follows:

$$56 = 10 \times 4 + 8 \times 2,$$

i.e., a 3/2 decuplet and a 1/2 octet—just the known supermultiplets of baryons.

The SU_6 theory allows one to obtain the mass relations given earlier. We shall limit ourselves to these few remarks about this very promising, but not unique, direction for generalization of the hadron symmetry. It is still not clear whether one can reconcile such a symmetry with the requirements of relativistic invariance.

A peculiarity of the new types of symmetry is that they are actually seriously violated by the real interactions. We are trying to establish the "original" symmetry, not knowing precisely what the word "original" means. Illusions are therefore possible, and perhaps unavoidable.

Because of the low precision of its results, the symmetry theory does not have the required clear foundation. Everything would be understandable if quarks existed. Then the developing theories would be similar to the nuclear shell theory. But there are no quarks, and it is already strange to assume the possible existence of such particles. They are stable and yet are never found by us in ordinary matter.

If supercharged particles exist, the theory should be generalized accordingly. The corresponding higher symmetry has not yet manifested itself.

We may say that the symmetry theory based on SU_3 and its generalizations leads to an understanding of new nontrivial aspects of the structure of elementary particles.

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