PROBLEMS OF RELATIVISTIC COSMOLOGY

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I. FEATURES OF COSMOLOGICAL SOLUTIONS OF THE GRAVITATIONAL EQUATIONS

1. Introduction

THAT general relativity theory provides in principle a new insight into the properties of the world as a whole was first indicated by Einstein in 1919. Subsequent progress in relativistic cosmology is connected principally with the solution of Einstein's gravitational equations, first obtained by A. A. Fridman in 1922.

As is well known, this solution is based on the assumption that the distribution of matter in space is completely homogeneous and isotropic ("isotropic cosmological model"). Two cases are possible here, corresponding to a space of constant positive curvature (so-called "closed model") or a space of constant negative curvature ("open model"). The main property of these solutions is that they are not stationary. The resultant notion of an expanding universe has found, as is well known, brilliant confirmation in the red shift effect, discovered by E. Hubble, and by now it can be supposed that the isotropic model gives, in general outline, a representative description of the modern state of the universe.

At the same time it is clear that the assumption that a real world is homogeneous can be justified at best only approximately. Even if the distribution of matter density can be regarded as homogeneous when averaged over distances that are large compared with metagalactic distances, this homogeneity vanishes at any rate on going over to smaller scales. On the other hand, this assumption is very far reaching from the mathematical point of view, for it imparts to the solution a high degree of symmetry, which can result in specific properties that disappear on going over to the more general case.

This raises the question: how general is the second property of the isotropic model, namely its possession of a space-time metric with a singularity with respect to time? The presence of such a singular point denotes boundedness of the time. In the open isotropic model there is, as is well known, one singular point and the time is bounded in it only on one side, while the closed model has two singular points and the time is bounded in both directions.

Naturally, a question important to all of cosmology is the degree to which this important property is general: is the presence of a singularity a general property of cosmological solutions, a property not connected with any of the specific assumptions (about the character of the distribution of matter and of the gravitational field) on which some particular solution of the gravitational equations is based?

By now it is known that there are, in addition to the isotropic solution, quite a few other exact (that is, valid in all of space and during all of time) solutions of the equations of gravitation. The determination of such solutions can, of course, be of appreciable interest from the point of view of clarifying various properties of such an exceedingly complicated system of nonlinear differential equations as are Einstein's gravitational equations. However, the accumulation of exact solutions cannot by itself answer the question raised above. Each of these particular solutions is connected with some rather specific assumptions with respect to their form, and the fact that a solution has or has not a singular point cannot lead to any conclusions with respect to the behavior of the solution in the most general case.* Furthermore, these special assumptions are unavoidably very far reaching, and are usually governed only by the requirement that the solution of the equations be made as exact as possible; they are therefore usually purely mathematical in character (limitation on the number of independent variables, separation of the variables, diagonality of the metric tensor, etc.) and have no direct physical meaning.

A more accurate formulation of the problem of interest to us is to ascertain whether there is a singularity in the general solution of the gravitational equations, that is, the solution which admits of a perfectly arbitrary specification of the conditions (the distribution of matter and of the gravitational fields) at any instant of time chosen to be the initial time.

A criterion for the generality of the solution is the number of arbitrary functions of the spatial coordinates it contains. It must be borne in mind here, however, that among the arbitrary functions contained in any solution of the equations of gravitation there are, generally speaking, such whose arbitrariness is connected simply with the arbitrariness of the choice of the reference frame for the equation.† We, on the other hand, should obviously be interested only in the number of "physically arbitrary" functions, which cannot be reduced by any choice of reference frame. The number of such functions can be established for the general case readily from physical considerations. The arbitrary initial conditions should specify the initial spatial distribution of the density of matter, the three components of its velocity, and also the four quantities which determine the free gravitational field (that is, the field not connected with matter). One can arrive at the last number by considering, for example, weak gravitational waves: by virtue of their transversality, their field is determined by two independent quantities (components of metric tensor); these quantities satisfy a second-order equation (wave equation), and therefore the initial conditions for them should be specified by four functions. Thus, the general solution of the equations of gravitation should contain eight different physically arbitrary functions of the spatial coordinates.*

The determination of the general solution in exact form is of course an insoluble problem. There is no need for such a solution, however, to answer the question of interest to us. It is sufficient to investigate the form of the solution near the singularity.

We thus arrive at the following formulation of the problem: assuming the singularity to exist, it is required to find near it the form of the broadest class of solutions of the equation of gravitation, so as to judge, from the number of the arbitrary functions it contains, whether this solution is general.

This program has been the subject of the authors' papers [2-4], and is described in detail in Secs. 2—5. In order not to clutter up the exposition, many calculations and some secondary problems are relegated to appendices.

The entire investigation is based on Einstein's equations in their classical form, in which they follow logically from the general principles of relativity, without the "cosmological term," for the introduction of which there exist no theoretical or astronomical grounds whatever at the present time.

2. General Solution with Fictitious Singularity

Of primary significance in the investigation of questions connected with general relativity is a successful choice of the reference frame, appropriate to the problem under consideration.

We shall show below that the most general properties of the cosmological solutions with respect to their singularities do not depend on the presence or absence of matter. In this connection it is not necessary to employ in the investigation of these properties the so-called "co-moving" reference frame, that is, the system moving at each point together with the matter contained in it, as is frequently the custom in cosmology.

A natural choice of the reference frame is in this case the coordinate system obeying the conditions†

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*Incidentally, the overwhelming majority of the known exact solutions have singularities.

†The greatest possible number of arbitrary functions in the solution of the gravitational equations in an arbitrary reference frame is 20 (see [1], Sec. 95).

*A formal mathematical proof of this statement is given in appendix A.

†We use the notation of the book [1] throughout. In particular, Latin subscripts run through the values 0, 1, 2, and 3 while Greek subscripts run through the three spatial values 1, 2, and 3. The square of the interval element is written as $-ds^2 = g_{ik}dx^i dx^k$, so that the matrix of the quantities g_{ik} has a signature $-+++$.

In addition, we use everywhere a system of units in which the velocity of light and the Einstein gravitational constant are equal to unity.

$$g_{0\alpha}=0, \quad g_{00}=-1. \tag{2.1}$$

As is well known (see, for example, [1], Sec. 98a), the vanishing of the components $g_{0\alpha}$ of the metric tensor is the condition permitting synchronization of clocks in different points of space. If, in addition, $g_{00}=-1$, then the time coordinate $x^0=t$ represents the proper time at each point of space. A reference system satisfying these conditions will be called synchronous. The interval element in such a system is given by the expression

$$-ds^2=-dt^2+dl^2, \quad dl^2=g_{\alpha\beta}\,dx^\alpha\,dx^\beta. \tag{2.2}$$

The three-dimensional tensor $g_{\alpha\beta}$ represents here the spatial metric.

The equations of the gravitational field in the synchronous reference system have the following form (see [1], Sec. 99):

$$R_0^0=\frac{1}{2}\frac{\partial}{\partial t}\varkappa_\alpha^\alpha+\frac{1}{4}\varkappa_\alpha^\beta\varkappa_\beta^\alpha=T_0^0-\frac{1}{2}T, \tag{2.3}$$

$$R_\alpha^0=\frac{1}{2}(\varkappa_{\beta;\,\alpha}^\beta-\varkappa_{\alpha;\,\beta}^\beta)=T_\alpha^0, \tag{2.4}$$

$$R_\alpha^\beta=P_\alpha^\beta+\frac{1}{2\sqrt{-g}}\frac{\partial}{\partial t}(\sqrt{-g}\,\varkappa_\alpha^\beta)=T_\alpha^\beta-\frac{1}{2}\delta_\alpha^\beta T. \tag{2.5}$$

Here $\kappa_{\alpha\beta}$ denotes the three-dimensional tensor

$$\varkappa_{\alpha\beta}=\frac{\partial g_{\alpha\beta}}{\partial t}, \tag{2.6}$$

and all further operations of raising and lowering the indices and covariant differentiation are carried out in three-dimensional space with metric $g_{\alpha\beta}$; we note that

$$\varkappa_\alpha^\alpha=g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial t}=\frac{\partial}{\partial t}\ln(-g), \tag{2.7}$$

where g — determinant of the tensor g_{ik} (which differs from the determinant $|g_{\alpha\beta}|$ by a factor $g_{00}=-1$). The tensor $P_{\alpha\beta}$ in Eq. (2.5) is the three-dimensional Ricci tensor, expressed in terms of the three-dimensional metric tensor $g_{\alpha\beta}$ in the same way as R_{ik} is expressed in terms of g_{ik}; it contains only spatial (and not time) derivatives of $g_{\alpha\beta}$.

L. D. Landau has indicated long ago that the determinant g of the metric tensor should vanish in the synchronous reference frame within a finite time, regardless of what assumptions are made concerning the distribution, motion, or equation of state of the matter or the character of the gravitational field (this circumstance was recently noted also by Komar[5]*).

It is easy to arrive at this conclusion with the aid of Eq. (2.3), noting that the right side of this expression is negative for any distribution of matter (or is equal to zero in the case of empty space†). Therefore

$$R_0^0=\frac{1}{2}\frac{\partial}{\partial t}\varkappa_\alpha^\alpha+\frac{1}{4}\varkappa_\alpha^\beta\varkappa_\beta^\alpha\leqslant 0.$$

By virtue of the algebraic inequality*

$$\varkappa_\alpha^\beta\varkappa_\beta^\alpha\geqslant\frac{1}{3}(\varkappa_\alpha^\alpha)^2,$$

we get from this

$$\frac{\partial}{\partial t}\varkappa_\alpha^\alpha+\frac{1}{6}(\varkappa_\alpha^\alpha)^2\leqslant 0,$$

or

$$\frac{\partial}{\partial t}\frac{1}{\varkappa_\alpha^\alpha}\geqslant\frac{1}{6}. \tag{2.8}$$

Assume, for example, that at some instant of time $\kappa_\alpha^\alpha>0$. Then $1/\kappa_\alpha^\alpha$ decreases and has everywhere a finite (nonvanishing) derivative when t decreases, so that it must vanish (on the positive side) after some finite time. In other words, κ_α^α goes to $+\infty$ and by virtue of (2.7) this means that the determinant g vanishes (and, in accordance with inequality (2.8), not faster than as t^6). On the other hand, if we have at the initial instant $\kappa_\alpha^\alpha<0$, the same is obtained for increasing time.

This result, however, does not prove at all the inevitability of the existence of a true physical singularity in the metric. A physical singularity is only one that is specific for space-time as such, and is not connected with the character of the chosen reference frame. Such a singularity is characterized by the blowing up of the scalar quantities—density of matter and invariants of the curvature tensor†.

Yet the singularity which we have proved to be inevitable in the synchronous reference frame may turn out to be fictitious and vanish on going over to a different reference frame. The possibility of such a situation is evident even from the fact that the foregoing proof remains valid in the case when the non-Galilean nature of the metric is due merely to the use of curvilinear coordinates in flat space-time, when it is obvious beforehand that the singularity of the metric is fictitious.

Simple geometrical considerations show that this singularity which is inevitable in the synchronous system is actually fictitious in the general case. To this end, we call attention to the geometrical properties of the synchronous reference frame, in which the time lines are geodesics in four-space. Indeed, the four-vector $u^i=dx^i/ds$ of the tangent to the world line $x^1, x^2, x^3=$ const, has components $u^\alpha=0$, $u^0=1$

*An analogous result was obtained also by Raychaudhuri[6] for the case of "dustlike" matter (equation of state p = 0), moving without rotation — limitations which actually are not at all obligatory.

†Indeed, for the energy-momentum tensor of matter

$$T_{ik}=(p+\varepsilon)\,u_iu_k+pg_{ik}$$

we have

$$T_0^0-\frac{1}{2}T=-\frac{1}{2}(\varepsilon+3p)-(p+\varepsilon)\,u_\alpha u^\alpha,$$

so that this quantity is obviously negative (p — pressure, ε — energy density of the matter).

*Its correctness can be verified by diagonalizing the tensor $\varkappa_\alpha^\beta$ (at any specified instant of time).

†The invariants of the curvature tensor R_{iklm} are obtained as is well known, by reducing it to the Petrov canonical form.

and satisfies automatically the geodesic equations

$$\frac{du^i}{ds} = \Gamma^i_{kl} u^k u^l = \Gamma^i_{00} = 0,$$

since the Christoffel symbols Γ^{α}_{00} and Γ^{0}_{00} vanish identically under conditions (2.1).

It is also easy to see that these lines are normal to the hypersurfaces t = const. Indeed, the four-vector of the normal $n_i = -\partial t/\partial x^i$ to such a hypersurface has covariant components $n_\alpha = 0$ and $n_0 = -1$. The corresponding contravariant components, under conditions (2.1) are $n^\alpha = 0$ and $n^0 = 1$, that is, they coincide with the components of the four-vector u_i tangent to the time lines.

Conversely, these properties can be used for a geometrical construction of a synchronous reference frame in any space-time. To this end we start with some chosen space-like hypersurface, that is, a hypersurface the normal to which has a time-like direction at each point (the normal lies inside the light cone with vertex at the same point); all the interval elements on such a hypersurface are spacelike*. If we now choose these lines as coordinate lines for the time, and define the time coordinate t as the length of the geodesic line reckoned from the initial hypersurface, then we obtain a synchronous reference frame.

It is clear that such a construction, and by the same token the choice of the synchronous reference frame, is always possible in principle. Moreover, this choice is still not unique: a metric of the type (2.2) admits of any transformation of the three spatial coordinates which does not concern the time, and also a transformation corresponding to arbitrariness in the choice of the initial hypersurface in the indicated geometric construction†.

However, the geodesic lines of an arbitrary family, generally speaking, cross one another on certain envelope hypersurfaces, which are four-dimensional analogs of the caustic surfaces of geometrical optics. On the other hand, crossing of the coordinate lines produces, of course, a singularity in the metric in the given coordinate system. Thus, the appearance of a singularity has a geometrical reason which is connected in an obvious fashion with the specific properties of the synchronous system and consequently has no physical character.

The arbitrary metric of four-space admits also, generally speaking, the existence of non-intersecting families of time-like geodesic lines. The unavoidability, on the other hand, of the vanishing of the determinant of g in the synchronous system denotes that the curvature properties of real space-time, which are admitted by the equations of gravitation (a property expressed by the inequality $R^0_0 \leq 0$), exclude the possibility of existence of such families, so that the time lines must unavoidably cross one another in any synchronous frame.*

From the analytic point of view this means that the equations of gravitation have in the synchronous reference frame a general solution with a fictitious time singularity; in an arbitrary synchronous frame, such a solution should contain 12 arbitrary functions of the coordinates, namely, the 8 "physically arbitrary" functions and in addition 4 arbitrary functions connected with the aforementioned ambiguity in the choice of the synchronous reference frame.

The character of the fictitious singularity of the metric is clear beforehand from geometrical considerations. First, the caustic hypersurface should have a timelike character, since it contains, in any case, timelike intervals—elements of the length of geodesic lines at the points of their tangency with the caustic.

Further, one of the principal values of the metric tensor vanishes on the caustic, in accordance with the vanishing of the distance between two neighboring geodesics that cross each other at the point of their tangency to the caustic (the corresponding principal direction lines, obviously, along the normal to the caustic). This distance vanishes in proportion to the first power of the distance to the point of intersection. Therefore the principal value of the metric tensor, and with it the entire determinant g, vanishes like the square of this distance.

It can be shown that under suitable choice of the spatial coordinates, the first terms of the expansion in the spatial metric can be represented near the singularity in the form

$$dl^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta = a_{ab}\, dx^a\, dx^b + (t-\varphi)^2 a_{33}\, dx_3^2 + 2\,(t-\varphi)\, a_{a3}\, dx^a\, dx^3 \tag{2.9}$$

(the indices a and b run through values 1 and 2; the quantities a_{ab}, a_{a3}, a_{33}, and φ are functions of all three coordinates†).

*On the other hand, if the directions of the normal to the hypersurface lie outside the light cones, then the interval elements in the surface can be both time-light and space-light. We shall arbitrarily speak of such hypersurfaces as having a timelike character, although the terminology is not quite appropriate in this case.

†The admissibility of the latter transformation is particularly clear analytically in the infinitesimal case (see end of Appendix I).

*We disregard, of course, the trivial exception – beams of parallel straight lines in flat four-space.

†A complete analytic construction of the entire solution with fictitious singularities for empty space is given in [3].

The spatial metric (2.9) admits also of an arbitrary transformation $x^{3\prime} = x^{3\prime}(x^1, x^2, x^3)$, which reduces to a redesignation of the quantities a_{03}, a_{33}, and the higher terms of the expansion of the component g_{ab}. It is possible to use this transformation to convert the function φ, which gives the form of the caustic surface, into $\varphi = x^3$, after which only transformations of the two coordinates x^1 and x^2 in terms of each other remain admissible. With such a choice of coordinates, the solution should contain merely five arbitrary functions (of three coordinates): the four functions necessary to specify

The singularity in the metric (2.9) is not simultaneous—different spatial points reach it at different instants of time $t = \varphi$. It is easy to see, however, that it is always possible to construct also a synchronous reference frame such that the singularity (fictitious) is attained simultaneously in all of space. It is clear that such a singularity cannot be located on the hypersurface that is tangent to the time lines at the points of their intersection, since the existence of timelike intervals in such a surface would certainly exclude the simultaneity of the singularities. Therefore the time lines should cross on a "manifold of points" which has fewer dimensions than the hypersurface, that is, which is some two-dimensional surface in four-space; it can be called the focal surface of the corresponding family of geodesic lines. By choosing arbitrarily the focal surface, by constructing from each of its points all possible normal directions to the surface (all the directions in the two-dimensional plane normal to the focal surface), and by drawing the geodesic lines in these directions, we construct by the same token a synchronous reference frame having the required property.

Thus, the general solution of the equations of gravitation can also be represented (by suitable choice of the synchronous frame) in a form in which the singularity is simultaneous for all of space. In such a form it contains, of course, the same eight physically arbitrary functions (of three spatial coordinates), which are sufficient for the specification of the arbitrary initial conditions. Compared with the solution in the form (2.9), it contains one less arbitrary function: if we construct the synchronous reference system starting with some initial hypersurface, then a hypersurface which is far from arbitrary can lead to the focusing of the geodesic lines constructed along the normals to it.*

As was already indicated, the fictitious nature of the singularity of the solution under consideration is obvious already from the method used for its construction. The singularity can be eliminated by transforming the reference systems, but only at the expense of foregoing their synchronous nature.

For the same reason it is obvious that the qualitative character of this solution does not depend on the presence or absence of matter, and that the density of the latter has no singularity whatever and remains finite. This becomes particularly clear if it is noted that the matter moves (in the synchronous reference frame) along world lines which do not coincide with the time lines and which are not even geodesic.

The latter circumstance denotes that the reference frame cannot, generally speaking, be chosen such as to be synchronous and at the same time co-moving, with the world lines of the matter coinciding with the time lines. The only possible exception is "dustlike" matter (pressure $p = 0$), which moves along geodesic lines. Therefore in this case the "co-moving" condition for the reference frame of the matter does not contradict the condition for its being "synchronous." This, however, is still not enough—not all families of timelike geodesic lines have the property of being normal to a spacelike hypersurface, something necessary to make the reference system synchronous. This condition is satisfied if the matter moves "without rotation," that is, if the curl of its velocity vanishes everywhere*. In the "synchronous—co-moving" system, which we can construct in this case, the density of the matter becomes infinite on the caustic—simply as a result of the crossing of the particle trajectories. It is clear, however, that this density singularity has likewise no physical character and is eliminated merely by assigning to the matter a pressure which is arbitrarily small but different from zero.

Thus, the singularity in the general solution of the gravitational equations, the necessity for the existence of which follows in the synchronous system from the inequality $R_0^0 \leq 0$, turns out to be unphysical. By the same token, there are no further grounds for the existence of a singularity of another type, which would be true and at the same time specific to the general solution. These results, however, do not exclude the possibility of existence of narrower classes of cosmological solutions of the gravitational equations, with true singularities. Their determination is treated in Secs. 3—4. In addition to the independent interest that

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the initial conditions for the field in vacuum, and one function connected with the remaining arbitrariness in the choice of the synchronous reference frame (the choice of the initial hypersurface, from which the time coordinate is reckoned). These five arbitrary functions are contained in the six quantities a_{ab}, a_{a3}, and a_{33}, which turn out to be interrelated by a single equation.

*In some sense this solution corresponds to a vanishing function φ in the solution (2.9). On the singularity ($t = 0$) the square of the interval $-ds^2 = -dt^2 + dl^2$ reduces to the quadratic form $-ds^2 = a_{ab}dx^a dx^b$ of only two differentials. We emphasize, however, that the expansion of the metric near such a singularity cannot be obtained at all by merely putting $\varphi = 0$ in the formulas pertaining to a solution of type (2.9). We also point out that such a system does not encompass all of space-time. This is clear from the fact that in each hypersurface t = const all the points lie at equal time distances from the spatial focal surface, that is, these hypersurfaces are completely contained in the region of the absolute future or the absolute past with respect to the focal surface.

*The necessity of this condition is obvious from the following considerations. In the co-moving reference system the contravariant components of the four-velocity are $u^\alpha = 0$, $u^0 = 1$. If this reference system is also synchronous, then we also have the covariant components $u_\alpha = 0$ and $u_0 = -1$, so that its four-curl is

$$u_{i;k} - u_{k;i} \equiv u_{i,k} - u_{k,i} = 0.$$

But this tensor equation should then be valid in any other reference frame. Thus, in a synchronous but not co-moving system we obtain from this the condition curl $\mathbf{v} = 0$ for the three-dimensional velocity.

can be attached to the investigation of possible types of singularities of the solutions of the gravitation equations, by constructing these solutions and by ascertaining the degree of their generality we confirm the conclusion that the general solution has no true singularity.

3. Anisotropic Solution with Singularity

The solutions of the gravitational equations can have on the hypersurface $t = \varphi(x^\alpha)$, a (true) singularity which can be both spacelike and nonspacelike*. In the former case it is always possible to choose a reference frame, without violating the conditions of its synchronism, in such a way as to convert this hypersurface into a "hyperplane" t = const; in other words, in this case there exists a synchronous reference frame in which the singularity "sets in" simultaneously in all of space. It can be said that such a singularity is timelike. To the contrary, in the second case no choice of the reference system can make the singularity simultaneous in all of space; it can be said that it is spacelike.

From the cosmological point of view, the singularities of major interest are those with a time character. In particular, when searching for a general solution with a true singularity, it would be natural to think that if any singularity were inevitable, it would be precisely a timelike one. We shall consider time singularities.†

We assume that by suitable choice of the reference system the singularity has been reduced in all of space to a single instant of time, which we choose to be $t = 0$. This condition, together with the synchronism conditions, establishes the choice of the time coordinate, so that the ambiguity of the synchronous reference system reduces merely to the admissibility of arbitrary transformations of the three spatial coordinates in terms of one another.

The equations of the gravitational field in empty space have a simple particular exact solution

$$-ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2, \tag{3.1}$$

where p_1, p_2, and p_3 are arbitrary three numbers, interrelated by the two equations

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 \tag{3.2}$$

*Inasmuch as the metric becomes singular when $t = \varphi$, the manifold defined by this equation is, strictly speaking, not a hypersurface (it can, in particular, reduce to a manifold with fewer dimensions). When referring to its character, we imply the character of a hypersurface that is arbitrarily close to, but not coincident with, the singular metric.

†Along with the solution with time singularity, which will be considered in this section, there exist solutions with an analogous space singularity. In addition, space singularities which do not exist for the timelike case are also admissible (see Appendix B). It is essential, however, that even such singularities lead to solutions that are less inclusive than those required of the general solution.

(this solution was apparently first pointed out by Kasner[7]).

The numbers connected by relations (3.2) will play an important role in what follows; we therefore indicate here some of their properties. Since the three numbers p_1, p_2, and p_3 are connected by two relations, only one of them is independent. The three numbers p_1, p_2, and p_3 can never have the same value, and two of them can be equal only in the triplets 0, 0, 1 and $-1/3$, $2/3$, $2/3$.* In all other cases these numbers are different, only one being negative with the other two positive; we arrange them in sequence

$$p_1 < p_2 < p_3. \tag{3.3}$$

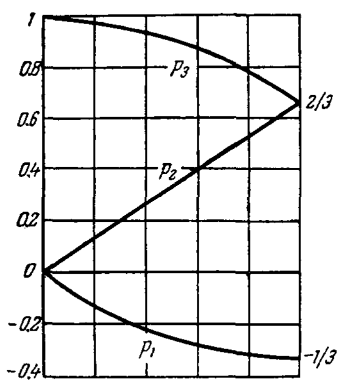
The numbers p_1, p_2, and p_3 run through values in the intervals

$$-\frac{1}{3} \leqslant p_1 \leqslant 0, \quad 0 \leqslant p_2 \leqslant \frac{2}{3}, \qquad \frac{2}{3} \leqslant p_3 \leqslant 1. \tag{3.4}$$

They can be represented in parametric form as

$$p_1 = \frac{-s}{1+s+s^2}, \quad p_2 = \frac{s(1+s)}{1+s+s^2}, \quad p_3 = \frac{1+s}{1+s+s^2}, \tag{3.5}$$

with the parameter s ranging from 0 to 1. In the figure, any two of the numbers p_1, p_2, or p_3 can be determined from specified values of the third (the three values lie on one vertical line).



Although the solution (3.1) is itself very particular, it has a simple and lucid physical character, corresponding to a completely homogeneous (but anisotropic) space. It is natural to expect such a solution to be a particular case of some broad class of solutions.

We shall seek a space metric near the singularity, in first approximation (the principal terms of the expansion in powers of t), in the form

$$g_{\alpha\beta} = t^{2p_1}l_\alpha l_\beta + t^{2p_2}m_\alpha m_\beta + t^{2p_3}n_\alpha n_\beta, \tag{3.6}$$

where **l**, **m**, and **n** are three-dimensional vectors which are functions of the coordinates; the exponents

*When $(p_1, p_2, p_3) = (0, 0, 1)$ the metric (3.1) can be transformed to a Galilean one by the transformation $t \sinh z = \zeta$, $t \cosh z = \tau$; that is to say, we are actually dealing here with flat space-time.

p_1, p_2, and p_3, which are related by (3.2), are now also functions of the coordinates.

The determinant of the tensor (3.6) is

$$-g = (\mathbf{l}\,[\mathbf{mn}])^2 t^2. \quad (3.7)^*$$

The tensor $g^{\alpha\beta}$, which is the inverse of (3.6), can be represented in the form

$$g^{\alpha\beta} = t^{-2p_1} l^\alpha l^\beta + t^{-2p_2} m^\alpha m^\beta + t^{-2p_3} n^\alpha n^\beta. \quad (3.8)$$

the letters l^α, m^α, and n^α with superior indices denote here the components of the vectors†

$$\widetilde{\mathbf{l}} = \frac{[\mathbf{mn}]}{(\mathbf{l}\,[\mathbf{mn}])}, \quad \widetilde{\mathbf{m}} = \frac{[\mathbf{nl}]}{(\mathbf{l}\,[\mathbf{mn}])}, \quad \widetilde{\mathbf{n}} = \frac{[\mathbf{lm}]}{(\mathbf{l}\,[\mathbf{mn}])}, \quad (3.9)$$

which are "inverse" to the vectors l, m, and n, so that

$$l_\alpha l^\alpha = 1, \quad l_\alpha m^\alpha = l_\alpha n^\alpha = 0, \ \ldots \quad (3.10)$$

Differentiating the tensor (3.6) with respect to the time, we obtain

$$\varkappa_{\alpha\beta} = \sum 2p_1 t^{2p_1 - 1} l_\alpha l_\beta \quad (3.11)$$

and, raising the indices, we get

$$\varkappa_\alpha^\beta = \frac{1}{t} \sum 2p_1 l_\alpha l^\beta, \quad (3.12)$$

where the summation sign denotes henceforth summation over the cyclic permutations of the vectors **l**, **m**, and **n** and the numbers p_1, p_2, and p_3.

The representation of $g_{\alpha\beta}$ in the form (3.6) is in correspondence with the fact that the time variation of the linear distances follows different laws in the three different directions (defined by the vectors **l**, **m**, and **n**) at each point of space. It must be emphasized, however, that the vectors **l**, **m**, and **n** cannot, generally speaking, be chosen as a reference frame for a spatial coordinate system. In order for the direction, say, of the vector $\mathbf{l}(x^1, x^2, x^3)$ (specified in terms of its covariant components l_α) to be able to serve at each point of space as a direction of one of the coordinate lines $(x^{1'})$ it is necessary that the sum $l_\alpha dx^\alpha$ be proportional to the total differential $l_\alpha dx^\alpha = \psi d\varphi$ (ψ, φ — two scalar functions); then the surfaces φ = const will be the surfaces $x^{1'}$ = const. Thus, the choice of the coordinate lines along the directions **l** is possible only for a vector of the form $\mathbf{l} = \psi \nabla\varphi$, which reduces to merely 2 (in place of 3) independent functions.

It is easy to verify that the singularity possessed by the metric (3.6) is actually a true singularity for all values of the exponents, with the exception of (0, 0, 1); when t = 0 the invariants of the curvature tensor of this metric become infinite. For the values (0, 0, 1) the singularity of the metric becomes fictitious and can be eliminated by transforming the reference frame (see the preceding footnote); we eliminate these values from further consideration.

*[mn] = m × n.

†Here and throughout all symbols for vector operations (vector products, the curl and gradient operators, etc.) must be regarded in pure formal fashion as operations on components (covariant) of the vectors **l**, **m**, and **n**, as if the coordinates x^1, x^2, and x^3 were Cartesian.

a) Case of empty space. We consider first the case of empty space. Then the gravitation equations (2.3)—(2.5) are

$$R_0^0 = -\frac{1}{2}\frac{\partial}{\partial t}\varkappa_\alpha^\alpha + \frac{1}{4}\varkappa_\alpha^\beta \varkappa_\beta^\alpha = 0, \quad (3.13)$$

$$R_\alpha^0 = \frac{1}{2}(\varkappa_{\beta;\,\alpha}^\beta - \varkappa_{\alpha;\,\beta}^\beta) = 0, \quad (3.14)$$

$$R_\alpha^\beta = P_\alpha^\beta + \frac{1}{2\sqrt{-g}}\frac{\partial}{\partial t}(\sqrt{-g}\,\varkappa_\alpha^\beta) = 0. \quad (3.15)$$

Upon substitution of (3.12), Eq. (3.13) is satisfied automatically by virtue of the relation $p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2$. The second term of (3.15) vanishes identically, since $\kappa_\alpha^\beta \sim 1/t$ and $\sqrt{-g} \sim t$. This term is "potentially" of order t^{-2}. Therefore in order to satisfy (3.15) (in its principal terms), it remains to stipulate that the tensor P_α^β contain no terms of order t^{-2} or larger terms. Let us clarify the conditions that ensure the absence of such terms.

Inasmuch as the time dependence of the metric is essentially different along the directions **l**, **m**, and **n**, it is convenient to "project" all the tensors on these directions. Denoting the corresponding projections by the indices l, m, and n, we define them in the following fashion:

$$P_{ll} = P_{\alpha\beta} l^\alpha l^\beta, \quad P_{lm} = P_{\alpha\beta} l^\alpha m^\beta, \ \ldots \quad (3.16)$$

In this notation we have, in particular,

$$g_{ll} = t^{2p_1}, \quad g_{mm} = t^{2p_2}, \quad g_{nn} = t^{2p_3}. \quad (3.17)$$

The "mixed" components of the tensor are defined accordingly as

$$P_l^l = \frac{P_{ll}}{g_{ll}} = t^{-2p_1} P_{ll}, \quad P_l^m = \frac{P_{lm}}{g_{mm}} = t^{-2p_2} P_{lm}, \ \ldots \quad (3.18)$$

The general formulas for the tensor components $P_{\alpha\beta}$ defined in this manner are given in Appendix C. It is seen from these formulas that the highest-order term in the diagonal components of the tensor is

$$P_l^l = -P_m^m = -P_n^n = \frac{(\mathbf{l}\,\mathrm{rot}\,\mathbf{l})^2}{2\,(\mathbf{l}\,[\mathbf{mn}])^2}\, t^{-2(p_2 + p_3 - p_1)}. \quad (3.19)^*$$

Inasmuch as $p_1 < 0$, we have $2(p_2 + p_3 - p_1) = 2(1 - 2p_1) > 2$, so that this term is of higher order than t^{-2}, and in order to satisfy (3.15) it is necessary, in any case, that this term be missing, that is, we must have

$$\mathbf{l}\,\mathrm{rot}\,\mathbf{l} = 0. \quad (3.20)$$

According to the foregoing, this condition (which is equivalent to $\mathbf{l} = \psi\nabla\varphi$) signifies geometrically that the direction of the vector **l** can be chosen at each point of space as a direction of one of the coordinate lines.

If condition (3.20) is satisfied, the terms in the tensor components $P_{\alpha\beta}$ turn out to have the orders of magnitude

*rot = curl.

$$P_l^l \sim P_m^m \sim P_n^n \sim t^{-2p_3}(\ln t)^2, \qquad P_{lm} \sim t^{2(p_2-p_3)}\ln t,$$

$$P_{ln} \sim P_{mn} \sim (\ln t)^2$$

and do not affect the principal terms in (3.15)*.

It remains to satisfy Eq. (3.14). The largest terms in these equations could have an order $t^{-1}\ln t$: such terms appear when the exponents in the derivatives of $g_{\beta\gamma}$ are differentiated with respect to the coordinates entering into the expression

$$\varkappa^\beta_{\alpha;\beta} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\beta}(\sqrt{-g}\,\varkappa^\beta_\alpha) - \frac{1}{2}\varkappa^{\beta\gamma}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha}. \qquad (3.21)$$

Calculating these terms, we get

$$\varkappa^{\beta\gamma}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha} \approx 4\sum p_1 t^{-2p_1-1} l^\beta l^\gamma \sum t^{2p_1}\ln t\,\frac{\partial p_1}{\partial x^\alpha} l_\beta l_\gamma$$

$$= 4\frac{\ln t}{t}\sum p_1\frac{\partial p_1}{\partial x^\alpha} = 2\frac{\ln t}{t}\frac{\partial}{\partial x^\alpha}\sum p_1^2, \qquad (3.22)$$

and by virtue of (3.2) these terms cancel identically.

Thus, the principal terms in (3.14) turn out to be those proportional to 1/t. Since $\kappa^\beta_\beta \approx 2/t$ and does not depend on the coordinates, in this approximation we have $\kappa^\beta_{\beta;\alpha} = 0$. To calculate (3.21), we write

$$\varkappa^\beta_{\alpha;\beta} = \frac{2}{t(\mathbf{l}[\mathbf{mn}])}\sum\left\{\frac{\partial}{\partial x^\beta}(p_1 l_\alpha[\mathbf{mn}]_\beta) - \frac{1}{2}\frac{p_1[\mathbf{mn}]_\beta[\mathbf{mn}]_\gamma}{(\mathbf{l}[\mathbf{mn}])}\frac{\partial}{\partial x^\alpha}l_\beta l_\gamma\right\}$$

$$= \frac{2}{t(\mathbf{l}[\mathbf{mn}])}\sum\left\{l_\alpha\frac{\partial}{\partial x^\beta}(p_1[\mathbf{mn}]_\beta)\right.$$

$$\left.+ p_1[\mathbf{mn}]_\beta\left(\frac{\partial}{\partial x^\beta}l_\alpha - \frac{\partial}{\partial x^\alpha}l_\beta\right)\right\}$$

$$= \frac{2}{t(\mathbf{l}[\mathbf{mn}])}\sum\{l_\alpha([\mathbf{mn}]\nabla p_1)$$

$$+ l_\alpha p_1\operatorname{div}[\mathbf{mn}] - p_1[[\mathbf{mn}]\operatorname{rot}\mathbf{l}]_\alpha\}.$$

Expanding the vector expressions and rearranging the terms in the sum, we obtain

$$R^0_\alpha = -\frac{1}{2}\varkappa^\beta_{\alpha;\beta} = -\frac{1}{t(\mathbf{l}[\mathbf{mn}])}\sum l_\alpha\{[\mathbf{mn}]\nabla p_1 + (p_3-p_1)\,\mathbf{m}\operatorname{rot}\mathbf{n}$$

$$+ (p_1-p_2)\,\mathbf{n}\operatorname{rot}\mathbf{m}\} = 0. \qquad (3.23)$$

Projecting this equation on the directions **l**, **m**, and **n**, we obtain the three relations

$$\left.\begin{aligned}&(\mathbf{l}[\mathbf{mn}])\,p_{1,l} + (p_3-p_1)\,\mathbf{m}\operatorname{rot}\mathbf{n} + (p_1-p_2)\,\mathbf{n}\operatorname{rot}\mathbf{m} = 0,\\ &(\mathbf{l}[\mathbf{mn}])\,p_{2,m} + (p_1-p_2)\,\mathbf{n}\operatorname{rot}\mathbf{l} + (p_2-p_3)\,\mathbf{l}\operatorname{rot}\mathbf{n} = 0,\\ &(\mathbf{l}[\mathbf{mn}])\,p_{3,n} + (p_2-p_3)\,\mathbf{l}\operatorname{rot}\mathbf{m} + (p_3-p_1)\,\mathbf{m}\operatorname{rot}\mathbf{l} = 0\end{aligned}\right\} \qquad (3.24)$$

(the letters l, m, and n following the commas in the subscripts denote differentiation along the corresponding directions in accordance with definition C.3).

The next expansion terms [following (3.6)] of the metric tensor are expressed in terms of quantities contained in (3.6); the corresponding calculations are given in Appendix D.

Expression (3.6) contains only ten different functions of the coordinates; three components of each of the vectors **l**, **m**, and **n**, and one function in the exponents of t [any one of the functions p_1, p_2, or p_3, which are related by Eq. (3.2)]. These ten functions are connected by the four relations (3.20) and (3.24). In addition, the reference system used by us admits of arbitrary transformation of the three spatial coordinates in terms of one another. Therefore the solution obtained contains merely $10 - 4 - 3 = 3$ physically arbitrary functions of the three space coordinates. This is one less than is needed to specify the arbitrary initial conditions for the gravitational field in vacuum*.

By some specific choice of the spatial coordinates we can recast the metric (3.6) in various simpler forms, for example:

$$dl^2 = l_1^2 t^{2p_1}dx^2 + m_2^2 t^{2p_2}dy^2 + n_3^2 t^{2p_3}dz^2$$

$$+ 2m_1 m_2 t^{2p_2}dx\,dy + 2n_1 n_3 t^{2p_3}dx\,dz. \qquad (3.25)$$

The five quantities l_1, m_1, m_2, n_1, n_3 (and the exponents p_1, p_2, p_3) are connected by three relations, which can be readily obtained from (3.24); on the other hand, condition (3.20) has already been used in choosing **l** as the directions for the x coordinate lines. In (3.25) the coordinates y and z can be also subjected to transformations of the type $y \to f(x,y)$ and $z \to g(x,z)$; such transformations do not affect the principal terms of the expansion of the metric, given by (3.25).

We note that the foregoing solution is in principle anisotropic: the exponents p_1, p_2, and p_3, which determine the variation of the linear distances along the three different directions in space, cannot be the same. We also call attention to the mathematical peculiarity of this solution—one of the arbitrary functions enters in it as the power of the time.

b) Solution in space filled with matter. We now show that the presence of matter does not change the character of the obtained "anisotropic" solution, and the initial conditions for the distribution and motion of the matter can be specified in completely arbitrary fashion.

In considering a solution of the gravitational equations near a singular point at which the pressure p and the energy density ϵ of the matter become infinite, it is necessary, of course, to use for its equation of state the ultrarelativistic relation

$$p = \frac{\varepsilon}{3}. \qquad (3.26)$$

Then the energy-momentum tensor of the matter becomes

$$T_{ik} = (p+\varepsilon)\,u_i u_k + p g_{ik} = \frac{\varepsilon}{3}(4u_i u_k + g_{ik}), \quad T_i^i = 0. \qquad (3.27)$$

*When the next terms of the expansion of the vector $\mathbf{l} = \mathbf{l}^{(0)} + \mathbf{l}^{(1)} + \ldots$ are taken into account, the product **l** curl **l** ceases to be equal to zero, but the correction terms that result from (3.19) are of smaller order than t^{-2p_3}, and are therefore small compared with those written out (see end of Appendix D).

*In Appendix E we present arguments which explain more lucidly the reasons why one arbitrary function is "lost" in this solution.

The gravitation equations (2.3)—(2.5) assume the form

$$R_0^0 = \frac{1}{2}\frac{\partial}{\partial t}\varkappa_\alpha^\alpha + \frac{1}{4}\varkappa_\alpha^\beta\varkappa_\beta^\alpha = \frac{\varepsilon}{3}(4u_0u^0+1), \quad (3.28)$$

$$R_\alpha^0 = \frac{1}{2}(\varkappa_{\beta;\,\alpha}^\beta - \varkappa_{\alpha;\,\beta}^\beta) = \frac{4\varepsilon}{3}u_\alpha u^0, \quad (3.29)$$

$$R_\alpha^\beta = P_\alpha^\beta + \frac{1}{2\sqrt{-g}}\frac{\partial}{\partial t}(\sqrt{-g}\varkappa_\alpha^\beta) = \frac{\varepsilon}{3}(4u_\alpha u^\beta + \delta_\alpha^\beta). \quad (3.30)$$

To estimate the orders of magnitude of the material density and velocity it is convenient to use the hydrodynamic material equations of motion, which are contained, as is well known, in the gravitational equations (the equations $T_{i;k}^k = 0$):

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}(\sqrt{-g}\,\sigma u^i) = 0, \quad (3.31)$$

$$(p+\varepsilon)\,u^k\left\{\frac{\partial u_i}{\partial x^k} - \frac{1}{2}u^l\frac{\partial g_{kl}}{\partial x^i}\right\} = -\frac{\partial p}{\partial x^i} - u_iu^k\frac{\partial p}{\partial x^k} \quad (3.32)$$

(see, for example, [8], Sec. 125). Here σ — entropy density; for the ultrarelativistic equation of state (3.26) the entropy is $\sigma \sim \epsilon^{3/4}$.

We now make an assumption, to be confirmed by the result, that the principal terms in (3.31)—(3.32) are those containing time derivatives. Then Eq. (3.31) and the spatial components of (3.32) (the time component yields nothing new) give

$$\frac{\partial}{\partial t}(\sqrt{-g}u_0\varepsilon^{3/4}) = 0,\ 4\varepsilon\frac{\partial u_\alpha}{\partial t} + u_\alpha\frac{\partial\varepsilon}{\partial t} = 0,$$

hence

$$tu_0\varepsilon^{3/4} = \text{const}, \quad u_\alpha\varepsilon^{1/4} = \text{const},$$

where "const" stands for the time-independent quantities. In addition, from the identity $u_iu^i = -1$ we have (taking into account that all the covariant components u_α are of the same order)

$$u_0^2 \approx u_nu^n = u_n^2t^{-2p_3};$$

we again use the components along **l**, **m**, and **n**, that is, we represent the three-dimensional vector **u** in the form

$$\mathbf{u} = u_l\mathbf{l} + u_m\mathbf{m} + u_n\mathbf{n},$$

with $u_l = \widetilde{\mathbf{ul}}, \ldots$

From the relations written out we get

$$\varepsilon \sim t^{-2(1-p_3)}, \quad u_0^2 \sim t^{-(3p_3-1)}, \quad u_\alpha \sim t^{(1-p_3)/2}, \quad (3.33)$$

after which we can readily check that the terms left out of (3.31)—(3.32) are actually small compared with those retained.

We now estimate the components of the energy-momentum tensor in the right halves of (3.28)—(3.30). In (3.28) we have

$$T_0^0 \sim \varepsilon u_0^2 \sim t^{-(1+p_3)}.$$

Inasmuch as $p_3 < 1$, this quantity is of lower order in $1/t$ than the principal terms in the left half of the equation ($\sim t^{-2}$). The same applies to (3.30): the spatial components of the tensor T_i^k "projected" along the directions **l**, **m**, and **n**, have orders of magnitude

$$T_l^l \sim \varepsilon \sim t^{-2(1-p_3)},\ T_m^m \sim \varepsilon u_mu^m \sim t^{-(1+2p_2-p_3)},$$

$$T_n^n \sim \varepsilon u_nu^n \sim t^{-(1+p_3)}, \quad (3.34)$$

which are all smaller than t^{-2}.

On the other hand, in (3.29) we have

$$T_\alpha^0 \sim \varepsilon u_\alpha u^0 \sim \frac{1}{t},$$

that is, the same order of magnitude as of the left half of the equation. But this circumstance likewise does not change the character of the solution. Indeed, in accordance with (3.33), we write

$$\varepsilon = \varepsilon^{(0)}t^{-2(1-p_3)}, \quad u_\alpha = u_\alpha^{(0)}t^{(1-p_3)/2}$$

for the first terms of the expansion of these quantities; here

$$u_0^2 \approx u_n^{(0)2}t^{-(3p_3-1)}.$$

Equating the expression (3.23) for R_α^0 to the quantity $T_\alpha^0 = 4\epsilon u_\alpha u^0/3$, we obtain in place of (3.24)

$$(\mathbf{l}[\mathbf{mn}])\,p_{1,\,l} + (p_3-p_1)\,\mathbf{m}\,\text{rot}\,\mathbf{n} + (p_1-p_2)\,\mathbf{n}\,\text{rot}\,\mathbf{m}$$

$$= -\frac{4}{3}\varepsilon^{(0)}u_l^{(0)}u_0^{(0)},\ldots \quad (3.35)$$

Thus, the only change is in the connection between the functions involved in (3.6), which now contains also the new functions $\epsilon^{(0)}$ and $\mathbf{u}^{(0)}$.

A change occurs also in the form of the higher terms of the expansion of the metric tensor, with the terms immediately following (3.6) being precisely the terms connected with the presence of matter (see Appendix D).

Thus, the obtained anisotropic solution of the gravitational equations is a very broad class of solutions with singularity. It contains seven arbitrary functions of the coordinates: in addition to the three functions present already in the absence of matter, it contains also the function $\epsilon^{(0)}$ and the three functions $u_\alpha^{(0)}$. This, however, is one less than required for the general case, so that this solution is not general.*

The character of variation of the metric near the singularity ($t \to 0$) does not depend in this solution on the presence or absence of matter (and by the same token on its equation of state). It is such that at each point of space the linear distances along two directions decrease (as t^{p_2} and t^{p_3}) and increase along the third (as $t^{-|p_1|}$); the volumes decrease here

*In the particular case when $(p_1, p_2, p_3) = (-1/3, 2/3, 2/3)$ the matter can be "written in" in the metric (3.6) in still another manner, such that its velocity tends to zero as $t \to 0$. Then, however, the matter introduces only two and not four arbitrary functions, that is, the initial conditions for it must have some particular character. For the class of solutions obtained in this way see [2]. This class includes, in particular, the general solution for a centrally-symmetrical collapse of matter.

in proportion to t. The laws governing these variations (that is, the values of p_1, p_2, and p_3) vary in space and are determined by the specification of the initial conditions.

The density of matter becomes infinite at each point of space like $\epsilon \sim t^{-2(1-p_3)}$. This fact in itself is an obvious indication that the singularity has a physical (not fictitious) character.

As $t \to 0$, the velocity of the matter tends in this solution (in the reference system under consideration) to the velocity of light. Indeed, the three-dimensional scalar $u^\alpha u_\alpha \approx u_n u^n$ tends as $t \to 0$ to infinity like $t^{-(3p_3-1)}$. This means that the matter moves at each point essentially along the **n** direction, and the absolute value of its ordinary three-dimensional velocity v ($v^2 = v_\alpha v^\alpha$) tends to unity like

$$\sqrt{1-v^2} \sim t^{(3p_3-1)/2}. \tag{3.36}$$

The proper time τ of the moving matter is connected with the time t through $d\tau = dt\sqrt{1-v^2}$. Therefore

$$\tau \sim t^{(3p_3+1)/2}. \tag{3.37}$$

In the co-moving reference system, the energy density therefore becomes infinite like

$$\varepsilon \sim \tau^{-\frac{4(1-p_3)}{3p_3+1}}. \tag{3.38}$$

4. Quasi-isotropic Solution

The solution considered in the preceding section is in principle anisotropic: inasmuch as the exponents p_1, p_2, and p_3 cannot be identical in this solution, the "contraction" of space occurs in anisotropic fashion.

It is therefore natural that this solution does not contain the isotropic (Fridman) solution. We shall show that the latter is indeed a particular case of a second class of solutions, in which the contraction of space occurs in "quasi-isotropic" manner—the linear distance changes in all directions with the same power of the time. As in the completely isotropic case, this solution exists only for space filled with matter.*

*In vacuum the gravitational equation can be satisfied by a quasi-isotropic metric of the form $g_{\alpha\beta} = t^2 a_{\alpha\beta}$, where $a_{\alpha\beta}$ – functions of the coordinates.

Equation (3.13) is then satisfied identically ($\varkappa_\alpha^\beta = 2\delta_\alpha^\beta/t$), while Eq. (3.15) yields $P_\alpha^\beta = -2\delta_\alpha^\beta$, where the tensor $P_{\alpha\beta}$ is calculated with the simple metric $a_{\alpha\beta}$; but such a form of $P_{\alpha\beta}$ denotes that the space has a constant negative curvature. The corresponding space-time metric can be written with the aid of the four-dimensional spherical coordinates χ, θ, and φ in the form

$$-ds^2 = -dt^2 + t^2 [d\chi^2 + \mathrm{sh}^2 \chi (d\theta^2 + \sin^2\theta\, d\varphi^2)],$$

but the transformation

$$r = t \sinh \chi, \quad \tau = t \cosh \chi$$

reduces such a metric to the Galilean one

$$-ds^2 = -d\tau^2 + dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2).$$

The isotropic model, as is well known, is formulated in most natural fashion in the co-moving reference system. In this system there appear in explicit form the isotropy and homogeneity of space, by virtue of which the quantities $g_{0\alpha}$ vanish automatically (so that the reference system is at the same time synchronous), and the singularity takes place simultaneously in all of space. The specific law governing the dependence of the metric on the time depends in this solution on the equation of state of the matter. For the ultrarelativistic equation $p = \epsilon/3$, the metric has as $t \to 0$ the form $g_{\alpha\beta} \approx a_{\alpha\beta} t$, where $a_{\alpha\beta}$ are completely defined functions of the coordinates, corresponding to a constant curvature of space. As functions of the time, the $g_{\alpha\beta}$ are expanded in integer powers of t.

We shall formulate the quasi-isotropic solution in the synchronous system, which, however, is no longer strictly co-moving. The spatial metric will be sought in the form

$$g_{\alpha\beta} = t a_{\alpha\beta} + t^2 b_{\alpha\beta} + \ldots, \tag{4.1}$$

where now $a_{\alpha\beta}$ — arbitrary functions of the coordinates. The tensor inverse to (4.1) is

$$g^{\alpha\beta} = t^{-1} a^{\alpha\beta} - b^{\alpha\beta}, \tag{4.2}$$

where the tensor $a^{\alpha\beta}$ is the inverse of $a_{\alpha\beta}$; all the operations of lowering and raising the indices and of covariant differentiation are carried out on the other tensors everywhere in this section with a time-independent metric $a_{\alpha\beta}$ (for example, $b_\alpha^\beta = a^{\beta\gamma} b_{\alpha\gamma}$, etc.).

Calculating the left halves of (3.28) and (3.29) respectively accurate to two and to one principal order in $1/2$, we obtain

$$-\frac{3}{4t^2} + \frac{1}{2t} b = \frac{\varepsilon}{3}(-4u_0^2 - 1), \tag{4.3}$$

$$\frac{1}{2}(b_{;\alpha} - b_{\alpha;\beta}^\beta) = -\frac{4\varepsilon}{3} u_\alpha u_0, \tag{4.4}$$

where $b = b_\alpha^\alpha$. If we compare the right halves of these equations and take into account the identity

$$-1 = u_i u^i \approx -u_0^2 + \frac{1}{t} u_\alpha u_\beta a^{\alpha\beta},$$

we see readily that $\epsilon \sim t^{-2}$ and $u_\alpha \sim t^2$; then, by virtue of the indicated identity, $u_0^2 - 1 \sim t^3$. From Eq. (4.3) we now obtain the first two terms of the expansion of the energy density

$$\varepsilon = \frac{3}{4t^2} - \frac{b}{2t}, \tag{4.5}$$

and from (4.4) we get the first term of the expansion of the velocity

$$u_\alpha = \frac{t^2}{2}(b_{;\alpha} - b_{\alpha;\beta}^\beta). \tag{4.6}$$

The three-dimensional Christoffel symbols, and with them also the tensor $P_{\alpha\beta}$, do not depend on the time in the first approximation in $1/t$; $P_{\alpha\beta}$ coincides here with the expression obtained in calculations with a time-independent metric $a_{\alpha\beta}$. Taking this circumstance into account, we now find that the terms of order t^{-2} cancel out automatically in (3.30), while the terms proportional to t^{-1} yield

$$P^{\beta}_{\alpha} + \frac{3}{4} b^{\beta}_{\alpha} + \frac{5}{12} \delta^{\beta}_{\alpha} b = 0.$$

Hence

$$b^{\beta}_{\alpha} = -\frac{4}{3} P^{\beta}_{\alpha} + \frac{5}{18} \delta^{\beta}_{\alpha} P. \qquad (4.7)$$

We see that actually the six functions $a_{\alpha\beta}$ remain completely arbitrary. From the specified $a_{\alpha\beta}$ we determine with the aid of (4.7) the coefficients $b_{\alpha\beta}$ of the next term of the expansion, and with them also the coefficients of the first terms of expansions (4.5) and (4.6) of the matter density and velocity. We note that as $t \to 0$ the distribution of matter becomes more homogeneous, and its density tends to a value independent of the coordinates. As regards the velocity distribution (4.6), it can be transformed by taking into account the relation

$$b^{\beta}_{\alpha;\beta} = \frac{7}{9} b_{;\alpha},$$

which is a consequence of the relation

$$P^{\beta}_{\alpha;\beta} - \frac{1}{2} P_{;\alpha} = 0,$$

which is satisfied, as is well known, by any Ricci tensor. We then have

$$u_{\alpha} = \frac{t^2}{9} b_{;\alpha}, \qquad (4.8)$$

that is, in this approximation the velocity is a gradient of some function and its curl vanishes (a nonvanishing curl appears, however, in the next terms of the expansion).

The metric (4.1) admits also of the possibility of arbitrary transformation of the three space coordinates (the choice of time is completely determined by the condition $t = 0$ at the singular point); these transformations can be used, for example, to reduce the tensor $a_{\alpha\beta}$ to a diagonal form. Therefore the solution obtained contains merely $6 - 3$, that is, three different physically arbitrary functions of the coordinates.

The isotropic model corresponds to the particular case of completely defined functions $a_{\alpha\beta}$ —those corresponding to a space of constant curvature (in this case $P^{\beta}_{\alpha} = \text{const} \cdot \delta^{\beta}_{\alpha}$).

5. General Conclusions Concerning Singularities of Cosmological Solutions

The main conclusion of the foregoing results is that the presence of time singularities is not an obligatory property of cosmological models of the general theory of relativity, and that the general case of arbitrary distribution of matter and of the gravitational field does not lead to the appearance of singularities.

On the other hand, solutions which have physical singularities have a degree of generality which is not sufficient to take into account arbitrary initial conditions specified at any instant of time. The most extensive among these solutions is the anisotropic solution, which contains seven arbitrary functions of the coordinates. Although this is merely one less than the maximum possible, this is sufficient, of course, for the initial conditions admitted by this solution to have "zero measure" compared with the entire manifold of possible initial conditions.

The insufficient degree of generality of the solution denotes that it describes an unstable mode; there exist small perturbations of a type such that their superposition leads to a violation of the solution and by the same token to the vanishing of the singularity. Without loss of generality, we can always subject an arbitrary perturbation to conditions that prevent it from violating the synchronism of the reference system. Since the singularity cannot vanish at all in the synchronous reference system, this means that it should go over into a fictitious singularity as a result of the perturbation.

The considerations advanced in Sec. 2 regarding the fictitious character of the inevitable singularity in the synchronous reference system pertain to an equal degree to empty space and to space filled with matter with any equation of state. We have seen also in Sec. 3 that the presence of matter does not change the qualitative properties of the anisotropic solution with a true singularity. All this offers evidence that the most general time-singularity properties of the cosmological solutions are already manifest in the case of empty space, and that matter does not change these properties qualitatively. This result is natural if we note that the gravitational properties of the "wave packets" made up of short-wave gravitational waves can imitate the gravitational properties of matter (with an equation of state $p = \epsilon/3$).

An exceptional position is occupied in this respect by the isotropic model, as well as by the quasi-isotropic solution that generalizes it (Sec. 4)—these solutions exist only for space filled with matter. This exception, however, has a simple explanation, which merely confirms the general rule. It is connected precisely with the high symmetry (homogeneity) of distribution of matter, which is characteristic of this solution, and which cannot be imitated by any aggregate of transverse gravitational waves.

An assumption frequently made in the literature is that the time singularity is obligatory in the absence of "spin" of the matter filling the space, but can vanish in models which take the spin into account.* It becomes clear from the foregoing that actually the char-

*The reason for this assumption is that the term connected with the rotation (in the nonsynchronous system) in the 00 component of the gravitational equation has a sign which, so to speak, slows down the decrease of the determinant.

acter of motion of the matter has no direct bearing whatever on the time singularities of cosmological solutions.

We have spoken everywhere of the approach to the singularity having the same direction as the decrease in time. Actually, by virtue of the symmetry of the gravitational equations under time reversal, we can speak equally well of approaching the singularity in the direction of increasing time. Physically, however, in view of the physical non-equivalence of the future and the past, there is an essential difference between these two cases with respect to the formulation of the problem itself. A singularity in the future can have a physical meaning only if it is admissible under perfectly arbitrary conditions, specified in some preceding instant of time; clearly, there are no grounds for the matter and the field to attain at some instant, during the evolution of the universe, a distribution corresponding to the specific conditions necessary for realization of a particular solution (of the gravitational equations) with a true singularity. Moreover, if we even admit the existence, for some reason, of such a distribution at some instant of time, then it will unavoidably be disturbed in the future, if for no other reason than the unavoidable thermodynamic (and quantum) fluctuations. Therefore the results presented exclude the possibility of the existence of a singularity in the future and denote that any contraction of the world (if it should occur at all) must ultimately again give way to expansion.

As to the existence of singularities in the past, an investigation based on the gravitational equations alone cannot yield a definite answer. The requirement that the singularity occur for an arbitrary distribution of matter and field is not a priori essential in this case. In this form it will be equivalent to the patently unacceptable assumption that the real universe is described by some purely random solution of the gravitational equation.

Actually there is no doubt that the choice of the solution corresponding to the real world is in fact unique and is connected with some deep physical requirements, which cannot be established on the basis of the existing gravitational theory alone, and which can be explained only as a result of further synthesis of physical theories. Only after these requirements are established will it be possible to state unambiguously whether the specific solution of the gravitational equation satisfying these requirements has a singularity.

Doubts may arise as to how correct it is in general to consider the question of the "singular state" of the world on the basis of existing gravitational theory, since we do not know the extent to which its equations are applicable for an arbitrarily high density of matter. It must be stated in this respect, first, that although the physical applicability of these equations under the indicated conditions can be clarified only in the future theory, it is important that gravitational theory itself does not lose its logical cohesion (that is, that its equations not lead to any internal contradictions) for any density of matter. In other words, this theory is not limited as such to any of the conditions that follow from the theory itself, and which could make its application logically invalid and contradictory for arbitrary density; the limitations can occur in the future synthesis of the physical theories as a result of factors which are "extraneous" with respect to the gravitational theory itself. This circumstance makes it formally legitimate to consider the question of singularities in gravitational theory. As regards the physical interpretation of the results obtained thereby, it is determined by the fact that although the equations can actually turn out to be inapplicable for arbitrarily large densities, there are at any rate no grounds for doubting their applicability even for densities on the order of nuclear density, that is, tremendously large compared with modern average density of matter in the universe. Therefore, for example, if the equations of gravitation were to lead to the result that a singularity occurs upon contraction of the world, then, although this would not of necessity mean that the density would become infinite, it would denote at any rate contraction to densities of the order of nuclear. From the physical point of view, even such a state of the world would be sufficiently "singular." From this point of view the consideration of the singularities of the solutions of the gravitational equations has likewise a fully physical meaning.

Finally, let us stop to discuss the purely mathematical aspect of the results obtained. In this aspect it may be of interest to consider the question of the classification of all possible types of true singularities of cosmological solutions of the gravitation equations, independently of the degree of broadness of these solutions. A clarification of this question by systematic scanning of all possibilities would be very cumbersome*. However, the extensive searches which we have made for solutions with singularities give grounds for assuming that these types are restricted to those to which we arrive naturally by the method developed in Secs. 3—4 and in Appendices B and F. These types include, in particular, the singularities that are possessed by all known exact solutions of the gravitational equations (see Appendix G).

II. GRAVITATIONAL STABILITY OF THE ISOTROPIC WORLD

6. Initial Model and Equations of Small Perturbations

The Fridman solution occupies a special position in relativistic cosmology, because its premises are physically clear and natural. There are all grounds for as-

*We are guided by the proposition expressed by Landau in a different connection: "In view of our short life we cannot allow ourselves the luxury of engaging in problems which do not promise any new results."[9]

suming that it gives an adequate description of the contemporary state of the world, when viewed on a large scale. At the same time, the exceptional nature of the homogeneous distribution of matter gives a priori grounds for expecting that it is precisely this solution that can turn out to be that exceptional solution which should describe the initial stages of the expansion of the real world (homogeneity of the density at this stage would occur in this case also on a microscopic scale).

In this connection it is of great interest to consider the behavior of small perturbations in the isotropic model, that is, its gravitational stability; we present below a general investigation of this question*. Gravitational instability phenomena can play a role during the evolution of the world—the decay of matter into galaxies and stars, etc.; however, we shall not concern ourselves with this aspect of the problem at all†.

For convenience, we write down here certain known formulas pertaining to the isotropic model (see, for example, [1], Secs. 104–107).

The metric of the isotropic world is defined by the expression

$$-ds^2 = -dt^2 + a^2(t)\,dl^2, \tag{6.1}$$

where $a(t)$ is the "radius of curvature" of space and dl is the element of spatial distance, measured in units of a. In the case of a space with constant positive curvature (closed model) we have

$$dl^2 = d\chi^2 + \sin^2\chi\,(\sin^2\theta\,d\varphi^2 + d\theta^2), \tag{6.2a}$$

and for a space with constant negative curvature (open model) we have

$$dl^2 = d\chi^2 + \mathrm{sh}^2\chi\,(\sin^2\theta\,d\varphi^2 + d\theta^2), \tag{6.2b ‡}$$

where χ, φ, θ —"spherical" spatial coordinates. Expression (6.2a) corresponds mathematically to the geometry on the surface of a hypersphere (of unit radius) in four-dimensional Euclidean space, while (6.2b) corresponds to the geometry on the surface of a four-dimensional "pseudosphere" of imaginary radius.

In place of the time t it is convenient to use an auxiliary variable η, defined by the relation

$$dt = a\,d\eta, \tag{6.3}$$

Then ds^2 is written in the form

$$-ds^2 = a^2(\eta)\,(-d\eta^2 + dl^2). \tag{6.4}$$

The time coordinate x^0 will be taken to mean henceforth precisely this variable η.

In the case of "dustlike" matter, the pressure of which can be neglected ($p = 0$), the function $a(t)$ is defined by the parametric equations

$$a = a_0(1 - \cos\eta), \qquad t = a_0(\eta - \sin\eta), \tag{6.5a}$$

$$a = a_0(\mathrm{ch}\,\eta - 1), \qquad t = a_0(\mathrm{sh}\,\eta - \eta). \tag{6.5b*}$$

where a_0 is a constant [formulas (6.5a) pertain to the closed model and (6.5b) to the open one]. The dependence of the density ϵ on the time is determined by the equation

$$\varepsilon = \frac{6a_0}{a^3}. \tag{6.6}$$

At the earlier stages (small times t, that is, small η), we have the inverse limiting case of very dense matter with ultrarelativistic equation of state $p = \epsilon/3$. Then

$$a = b_0\sin\eta \approx b_0\eta, \quad t = b_0(1 - \cos\eta) \approx \frac{1}{2}b_0\eta^2, \tag{6.7a}$$

$$a = b_0\,\mathrm{sh}\,\eta \approx b_0\eta, \quad t = b_0(\mathrm{ch}\,\eta - 1) \approx \frac{1}{2}b_0\eta^2 \tag{6.7b}$$

(b_0 — a second constant), and the dependence $\epsilon(t)$ is determined by the formula

$$\varepsilon = \frac{3b_0^2}{a^4}. \tag{6.8}$$

We note that the metrics of the closed and open models go over into each other upon making the substitution

$$\eta \to i\eta, \quad \chi \to i\chi, \quad a \to ia. \tag{6.9}$$

Therefore all the equations for one model can be obtained from the equations for the other model by means of the same substitution.

Since the reference system in which the model is isotropic is a co-moving system, the components of the four-velocity of matter are

$$u^\alpha = 0, \quad u^0 = \frac{1}{a}. \tag{6.10}$$

An arbitrary small perturbation of the isotropic model is described by changes in the metric tensor δg_{ik} (which we shall denote by h_{ik} —see Appendix I), in the four-velocity of the matter δu^i, and in the energy density $\delta\epsilon$. Without loss of generality we impose on the quantities h_{ik} the four additional conditions

$$h_{00} = 0, \qquad h_{0\alpha} = 0, \tag{6.11}$$

that is, we use as before a synchronous reference system. However, it will no longer be co-moving (as it was prior to the perturbation), that is, the δu^α are different from zero.

In the linear approximation the small perturbations satisfy the equations

$$\delta R_i^k - \frac{1}{2}\delta_k^i\,\delta R = \delta T_i^k, \tag{6.12}$$

where the δR_i^k are determined by the formulas obtained in Appendix I, and the perturbation of the energy-momentum tensor is

$$\delta T_i^k = (p + \varepsilon)(u_i\delta u^k + u^k\delta u_i) + (\delta p + \delta\varepsilon)u_i u^k + \delta_i^k\,\delta p.$$

The components of the perturbation of the four-velocity δu^i are related with one another by

*The content of this part is based on work by Lifshitz.[10]

†Many ideas on these questions were advanced recently by Zel'dovich.[12]

‡sh = sinh.

*ch = cosh.

$$h_{ik}u^i u^k + g_{ik}(u^i\,\delta u^k + u^k \delta u^i) = 0,$$

which is obtained by varying the identity $g_{ik}u^i u^k = -1$. Having in mind the unperturbed values of the velocity (6.10), we obtain from this, subject to conditions (6.11),

$$\delta u^0 = 0. \tag{6.13}$$

Therefore the components δT_i^k are

$$\delta T_\alpha^\beta = \delta_\alpha^\beta\,\delta p,\ \delta T_0^\alpha = -a\,(p+\varepsilon)\,\delta u^\alpha,\quad \delta T_0^0 = -\delta\varepsilon. \tag{6.14}$$

In view of the smallness of δp and $\delta\epsilon$, we can write $\delta p = (dp/d\epsilon)\delta\epsilon$, and we obtain

$$\delta T_\alpha^\beta = -\delta_\alpha^\beta \frac{dp}{d\varepsilon}\,\delta T_0^0. \tag{6.15}$$

In the investigation that follows we confine ourselves to an analysis of perturbations in only relatively small regions of space—regions with linear parameters, which are small compared with the radius of curvature a. Such an assumption greatly simplifies all the calculations; it is found at the same time that an account of the perturbations in regions of dimensions comparable with a does not contribute anything that is principally new to the character of the behavior of the perturbations.

In each small region of space the metric can in first approximation be assumed to be Euclidean. Accordingly, the spatial metric (6.2) is replaced by the metric

$$dl^2 = dx^2 + dy^2 + dz^2, \tag{6.16}$$

where x, y, z —Cartesian coordinates in the given region of space, measured in units of the radius a.

The expressions for δR_i^k can be obtained, as was already indicated, with the aid of formulas (I.10—I.12). It must be borne in mind here that in these formulas the differentiation (designated by a dot) is with respect to t; it is connected with the differentiation with respect to η (which we designate here by a prime) by the relation $\partial/\partial t \rightarrow \partial/a\,\partial\eta$. In particular, we have

$$\varkappa_{\alpha\beta} = \dot{g}_{\alpha\beta} = \frac{2a'}{a^2}\,g_{\alpha\beta},\quad \varkappa_\alpha^\beta = \frac{2a'}{a^2}\,\delta_\alpha^\beta,$$

which can be readily verified by noting that the time dependence of the components $g_{\alpha\beta}$ is contained in the factor a^2. In the case of the Euclidean spatial metric (6.16), all the covariant differentiations in (I.10)—(I.12) reduce to simple derivatives with respect to the coordinates x^α (on the other hand, contravariant differentiations reduce to an additional division by a^2). Finally, the three-dimensional tensor P_α^β vanishes for the metric (6.16). Bearing all this in mind, we obtain after simple calculation the following expressions:

$$\left.\begin{aligned}
\delta R_\alpha^\beta &= \frac{1}{2a^2}\left(h_{\alpha,\gamma}^{\gamma,\beta} + h_{\gamma,\alpha}^{\beta,\gamma} - h_{\alpha,\gamma}^{\beta,\gamma} - h_{;\alpha}^{;\beta}\right) + \frac{1}{2a^2}\,h_\alpha^{\beta''} + \frac{a'}{a^3}\,h_\alpha^{\beta'} + \frac{a'}{2a^3}\,h'\delta_\alpha^\beta,\\
\delta R_0^0 &= \frac{1}{2a^2}\,h'' + \frac{a'}{2a^3}\,h',\\
\delta R_\alpha^0 &= \frac{1}{2a^2}\left(h'_{,\alpha} - h_{\alpha,\beta}^{\beta'}\right),\\
\delta R &= \frac{1}{a^2}\left(h_{\alpha,\beta}^{\beta,\alpha} - h_{;\alpha}^{;\alpha}\right) + \frac{1}{a^2}\,h'' + \frac{3a'}{a^3}\,h'.
\end{aligned}\right\} \tag{6.17}$$

Both the lower and upper indices following the comma denote here simple differentiation with respect to the corresponding coordinates in a space with metric (6.16) (to make the notation uniform we continue to write the upper and lower indices, although there is no difference between them in the case of Euclidean dl^2).

The final equations for the perturbation h_α^β of the metric tensor are obtained by substituting in (6.15) the components δT_i^k expressed in terms of δR_i^k in accordance with (6.12). It is convenient to choose for these equations those obtained from (6.15) with $\alpha \neq \beta$ and with simplification with respect to the indices α and β; they have the form

$$\left(h_{\alpha,\gamma}^{\gamma,\beta} + h_{\gamma,\alpha}^{\beta,\gamma} - h_{;\alpha}^{;\beta} - h_{\alpha,\gamma}^{\beta,\gamma}\right) + h_\alpha^{\beta''} + 2\,\frac{a'}{a}\,h_\alpha^{\beta'} = 0,\ \alpha \neq \beta, \tag{6.18}$$

$$\frac{1}{2}\left(h_{\gamma,\delta}^{\delta,\gamma} - h_{;\gamma}^{;\gamma}\right)\left(1 + 3\frac{dp}{d\varepsilon}\right) + h'' + h'\frac{a'}{a}\left(2 + 3\frac{dp}{d\varepsilon}\right) = 0. \tag{6.19}$$

The perturbations in the density and velocity of matter can be determined from the known h_α^β with the aid of the formulas

$$\left.\begin{aligned}
\delta\varepsilon &= -\delta T_0^0 = -\delta R_0^0 + \frac{1}{2}\,\delta R,\\
a\delta u^\alpha &= -\frac{1}{p+\varepsilon}\,\delta T_0^\alpha = -\frac{1}{p+\varepsilon}\,\delta R_0^\alpha.
\end{aligned}\right\} \tag{6.20}$$

At low velocities the components u^α of the four-velocity coincide with the components of the three-dimensional velocity. But for our choice of the spatial coordinates x, y, and z, the length elements correspond not to the differentials dx^α themselves, but to the products $a\,dx^\alpha$. Therefore the ordinary three-dimensional velocity δv^α, which occurs upon perturbation, corresponds not to the δu^α themselves, but to the products $a\,\delta u^\alpha$.

Substituting (6.17) in (6.20) we obtain for the relative change in density

$$\frac{\delta\varepsilon}{\varepsilon} = \frac{1}{2\varepsilon a^2}\left(h_{\alpha,\beta}^{\beta,\alpha} - h_{,\alpha}^{,\alpha} + \frac{2a'}{a}\,h'\right) \tag{6.21}$$

and for the velocity perturbation

$$\delta v^\alpha = \frac{1}{2\,(\varepsilon + p)\,a^2}\left(h^{,\alpha} - h_\beta^{\alpha,\beta}\right)'. \tag{6.22}$$

Among the solutions of (6.18)—(6.19) are some which can be eliminated by simple transformation of the reference system [compatible with conditions (6.11)], and therefore represent no real physical change in the metric. The form of such solutions can be established beforehand with the aid of formulas (I.13)—(I.14), which are derived in Appendix I (we recall again that in these formulas the index 0 pertains to the time coordinate t, and not to η). Recognizing that the time dependence of the unperturbed metric tensor $g_{\alpha\beta}$ reduces to a factor a^2, we can easily obtain from the indicated formulas the following expression for the fictitious perturbations of the metric:

$$h_\alpha^\beta = f_{0;\alpha}^{;\beta}\int\frac{d\eta}{a} + \frac{a'}{a^2}\,f_0\delta_\beta^\alpha + \left(f_\alpha^{,\beta} + f^\beta_{,\alpha}\right), \tag{6.23}$$

where f_0 and f_α are arbitrary (small) functions of the coordinates.

7. Expansion in Plane Waves

Inasmuch as we consider small regions of space where the metric is assumed Euclidean, an arbitrary perturbation in each such region can be expanded in plane waves.* Taking x, y, and z to be Cartesian coordinates measured in units of the radius a, we can write the spatial periodic factor of the plane waves in the form $\exp(i\mathbf{n}\cdot\mathbf{r})$, where $\mathbf{n}$ —dimensionless vector representing a wave vector measured in units of 1/a (wave vector $\mathbf{k} = \mathbf{n}/a$). If we have a perturbation in a region of space with dimensions $\sim l$, then its expansion will contain essentially waves with lengths $\lambda = 2\pi a/n \sim l$. Confining ourselves to perturbations in regions with dimensions $l \ll a$, we assume by the same token that the number n is sufficiently large ($n \gg 2\pi$).

Gravitational perturbations can be divided into three types. This classification reduces to a determination of the possible types of plane waves, in the form of which it is possible to represent the symmetric second-rank tensor $h_{\alpha\beta}$. Thus we obtain the following classification.

1. The scalar function

$$Q = e^{i\mathbf{nr}} \tag{7.1}$$

can be used to set up the tensors

$$Q_\alpha^\beta = \frac{1}{3}\delta_\alpha^\beta Q, \quad P_\alpha^\beta = \left(\frac{1}{3}\delta_\alpha^\beta - \frac{n_\alpha n^\beta}{n^2}\right)Q \tag{7.2}$$

(these tensors are defined such that $Q_\alpha^\alpha = 1$ and $P_\alpha^\alpha = 0$). With the aid of the same function Q we can set up a vector

$$P_\alpha = \frac{n_\alpha}{n} Q. \tag{7.3}$$

Such plane waves correspond to perturbations in which the velocity and the density of matter experience changes along with the gravitational field, that is, we deal with perturbations that are accompanied by the occurrence of condensation or rarefaction of matter. The perturbation h_α^β is expressed in this case in terms of the tensors Q_α^β and P_α^β, the velocity perturbation δv^α is expressed in terms of the vector P^α, while the density perturbation $\delta\epsilon$ is expressed in terms of the scalar Q.

2. The transverse vector wave

$$S_\alpha = s_\alpha e^{i\mathbf{nr}}, \quad s_\alpha n^\alpha = 0 \tag{7.4}$$

can be used to set up the tensor

$$S_\alpha^\beta = \frac{1}{n}(n^\beta S_\alpha + n_\alpha S^\beta); \tag{7.5}$$

The corresponding scalar does not exist, since $S_\alpha n^\alpha = 0$. These waves correspond to perturbations in which the velocity experiences a change along with the gravitational field, but not the density of matter. The perturbation h_α^β is expressed in this case in terms of the tensor S_α^β, while the perturbation δv^α is in terms of the vector S^α.

3. Transverse tensor wave:

$$G_\alpha^\beta = \gamma_\alpha^\beta e^{i\mathbf{nr}}, \quad \gamma_{\alpha\beta} n^\beta = 0. \tag{7.6}$$

It can be used to set up neither a vector nor a scalar (since $G_\alpha^\beta n_\beta = 0$ and $G_\alpha^\beta n^\alpha n_\beta = 0$). These waves correspond to perturbations of the gravitational field, under which the matter remains stationary and uniformly distributed in space. In other words, these are gravitational waves in an isotropic world.

We shall consider below perturbations of each of the foregoing three types. To be specific, we shall write out all the formulas for the open model. We have already indicated that the changeover to the closed model is by means of substitution (6.9). In the Euclidean metric (6.16) the substitution $\chi \to i\chi$ corresponds to the substitution $x, y, z \to ix, iy, iz$. To conserve the wave character of the functions introduced above, it is necessary along with this substitution of the coordinates to replace n by in. Therefore the changeover to the closed model is realized in the formulas considered below by means of the substitution

$$a \to ia, \quad \eta \to i\eta, \quad n \to in. \tag{7.7}$$

8. Perturbations with Variation of Density of Matter

We begin with perturbations of the first type and assume

$$h_\alpha^\beta = \lambda(\eta) P_\alpha^\beta + \mu(\eta) Q_\alpha^\beta, \quad h = \mu Q. \tag{8.1}$$

From (6.21)—(6.22) we obtain for the relative change in density

$$\frac{\delta\varepsilon}{\varepsilon} = \frac{1}{3\varepsilon a^2}\left[n^2(\lambda+\mu) + 3\frac{a'}{a}\mu'\right]Q \tag{8.2}$$

and for the velocity

$$\delta v^\alpha = \frac{in}{3a^2(p+\varepsilon)}(\mu' + \lambda')P^\alpha. \tag{8.3}$$

The equations for the functions λ and μ are obtained by substituting (8.1) in (6.18)—(6.19):

$$\lambda'' + 2\frac{a'}{a}\lambda' - \frac{n^2}{3}(\lambda+\mu) = 0,$$

$$\mu'' + \mu'\frac{a'}{a}\left(2 + 3\frac{dp}{d\varepsilon}\right) + \frac{n^2}{3}(\lambda+\mu)\left(1 + 3\frac{dp}{d\varepsilon}\right) = 0. \tag{8.4}$$

These equations have, first, the following two particular integrals, corresponding to those fictitious changes of the metric (6.23) which can be eliminated by transforming the reference system

$$\lambda = -\mu = \text{const}, \tag{8.5}$$

$$\lambda = -n^2\int\frac{d\eta}{a} \equiv \lambda_0, \quad \mu = n^2\int\frac{d\eta}{a} - \frac{3a'}{a^2} \equiv \mu_0 \tag{8.6}$$

(the first is obtained from (6.23) by choosing $f_0 = 0$ and $f_\alpha = P_\alpha$, and the second by choosing $f_0 = Q$ and

*In the general case of perturbations in regions of arbitrary size, including those commensurate with a, the perturbations must be expanded in four-dimensional spherical functions. Such an investigation is given in [10]; these calculations are presented in somewhat greater detail in [11].

$f_\alpha = 0$). With the aid of these integrals it is possible to reduce the order of Eqs. (8.4). To this end we take the sum and the difference of these equations, in which we make the substitution

$$\left.\begin{aligned} \lambda+\mu &= (\lambda_0+\mu_0)\int \xi\, d\eta, \\ \lambda'-\mu' &= (\lambda_0'-\mu_0')\int \xi\, d\eta + \frac{\zeta}{a}. \end{aligned}\right\} \tag{8.7}$$

After simple transformations we obtain as a result the following system of equations for the new unknown functions $\xi(\eta)$ and $\zeta(\eta)$:

$$\xi' + \xi\left[\frac{2a''}{a'} + \frac{a'}{a}\left(-2+\frac{3}{2}\frac{dp}{d\varepsilon}\right)\right] + \frac{1}{2}\frac{dp}{d\varepsilon}\zeta = 0, \tag{8.8}$$

$$\zeta' + \zeta\frac{a'}{a}\left(1+\frac{3}{2}\frac{dp}{d\varepsilon}\right) + \xi\left(-2n^2 + \frac{3a''}{a} - \frac{6a'^2}{a^2} + \frac{9}{2}\frac{a'^2}{a^2}\frac{dp}{d\varepsilon}\right) = 0. \tag{8.9}$$

The arbitrariness in the choice of the two integration constants in the determination of λ and μ by formulas (8.7) corresponds to the arbitrariness in the choice of the reference system.

Let us start with the earliest stages of the expansion of the world, when the matter is described by an equation of state $p = \epsilon/3$. Inasmuch as such a compression can be considered meaningful only for very small times t, we can confine ourselves to an investigation of the equations with $\eta \ll 1$. We have then for the radius of curvature $a = b_0 \sinh \eta \approx b_0\eta$ (6.7).

The principal terms in (8.8) yield

$$\zeta = -6\xi' + \frac{9}{\eta}\xi, \tag{8.10}$$

and from (8.9) we get

$$\zeta' + \frac{3}{2\eta}\zeta - \xi\left(2n^2 + \frac{9}{2\eta^2}\right) = 0.$$

Substituting ζ from (8.10) in the last equation we obtain the following simple equation for ξ:

$$\xi'' + \frac{n^2}{3}\xi = 0,$$

hence

$$\xi = \text{const}\cdot\exp\left(\frac{in}{\sqrt{3}}\eta\right), \tag{8.11}$$

where the constant is complex.

The subsequent investigation is best carried out separately for two limiting cases, depending on the mutual relationship between the two large quantities n and $1/\eta$.

We assume first that the number n is not too large (or that η is sufficiently small), so that $n\eta \ll 1$. Expanding (8.11) in powers of $n\eta$ and separating the real and imaginary parts, we obtain ξ in the form

$$\xi = -C_1 b_0\left(1-\frac{n^2}{6}\eta^2\right) - \frac{4}{3}C_2 b_0\eta\left(1-\frac{n^2}{18}\eta^2\right),$$

where C_1, C_2 — real constants; ζ is then calculated by formula (8.10), while λ and μ are given by (8.7).

The arbitrary integration constants must be chosen in the calculation of λ and μ such as to cause the principal terms of the expansion to vanish where possible (in this case the term $\sim \eta^{-2}$ in μ and the term $\sim$ const in $\lambda - \mu$ vanish). By simple calculation we obtain

$$\lambda = \frac{3C_1}{\eta} + C_2\left(1+\frac{n^2}{9}\eta^2\right), \quad \mu = -\frac{2n^2}{3}C_1\eta + C_2\left(1-\frac{n^2}{6}\eta^2\right)$$

[we have written out here those terms of the expansions of λ and μ which are needed for the calculation of $\delta\epsilon/\epsilon$ and δv^α in accordance with (8.2)–(8.3)]. The final expressions for the principal terms of the expansion in the perturbations of the metric, density, and velocity are:

$$p = \frac{\varepsilon}{3}, \quad \eta \ll 1/n \quad \left\{\begin{aligned} & h_\alpha^\beta = \frac{3C_1}{\eta}P_\alpha^\beta + C_2(Q_\alpha^\beta + P_\alpha^\beta), \\ & \frac{\delta\varepsilon}{\varepsilon} = \frac{n^2}{9}(C_1\eta + C_2\eta^2)Q, \quad \delta v^\alpha = -\frac{in}{12}\left(3C_1 + C_2\frac{n^2}{9}\eta^3\right). \end{aligned}\right. \tag{8.12}$$

The constants C_1 and C_2 must satisfy certain conditions which express the smallness of the perturbation at the instant of its occurrence t_0. The mixed components of the perturbation h_α^β of the metric tensor must be compared with the unperturbed values $g_\alpha^\beta = \delta_\alpha^\beta$; from this we get the conditions $\lambda \ll 1$ and $\mu \ll 1$. In addition, we must have $\delta\epsilon/\epsilon \ll 1$ and $\delta v^\alpha \ll 1$. When applied to the perturbations (8.12), these conditions lead to the inequalities $C_1 \ll \eta_0$ and $C_2 \ll 1$, where η_0 ($\eta_0 \ll 1$) is the value of η corresponding to the instant of time t_0.

Expressions (8.12) contain terms which increase, in an expanding world, as different powers of the radius of curvature $a \approx b_0\eta$. However, this increase does not cause the perturbation to become large, that is, loss of stability: if we employ (8.12) as order-of-magnitude formulas with $\eta \sim 1/n$, we find (by virtue of the inequalities obtained above for C_1 and C_2) that the perturbations remain small even at the upper limit of applicability of these formulas.

We note also that the existence of a solution $\lambda = \mu = C_2$, in which the perturbation of the metric remains constant in time, corresponds precisely to the possibility already indicated in Sec. 4, of generalizing the Fridman solution. The relative change in the energy density in this solution is proportional to $\eta^2 \sim t$, in accordance with (4.5).

Assume now that n is sufficiently large so that $n\eta \gg 1$. With the aid of (8.11) we now find from (8.10) and (8.7) that the principal terms in λ and μ are of the form*

$$\lambda = -\frac{\mu}{2} = \frac{3\sqrt{3}}{inb_0\eta^2}\xi$$

(the integration constants in (8.7) are chosen such that λ and μ contain no terms without a periodic factor).

*We have corrected here the error made in formulas (4.10) of [16], namely the superfluous terms $\pm 2\eta$ in λ and μ.

Calculating also the perturbations of the density and the velocity, we obtain the following final expressions:

$$p=\frac{\varepsilon}{3},\quad \frac{1}{n}\ll\eta\ll 1 \quad \begin{cases} h_{\alpha}^{\beta}=\frac{C}{n^2\eta^2}\left(P_{\alpha}^{\beta}-2Q_{\alpha}^{\beta}\right)e^{in\eta/\sqrt{3}},\\ \frac{\delta\varepsilon}{\varepsilon}=-\frac{C}{9}e^{in\eta/\sqrt{3}}Q,\quad \delta v^{\alpha}=\frac{C}{12\sqrt{3}}e^{in\eta/\sqrt{3}}P^{\alpha},\end{cases} \tag{8.13}$$

where C is a complex constant satisfying the condition $|C|\ll 1$. The presence of a periodic factor in these expressions is perfectly natural. In the case of large n we deal with a perturbation whose spatial periodicity is determined by a large wave vector $k = n/a$. Such perturbations should propagate like sound waves, with a velocity $u = \sqrt{dp/d\epsilon} = 1/\sqrt{3}$; accordingly the temporal part of the phase is determined, as is assumed in geometrical acoustics, by the large integral $\int ku\,dt = n\eta/\sqrt{3}$. The amplitude of the relative change in the density remains, as we have seen, constant while the amplitudes of the metric perturbations decrease like a^{-2} as the world expands.

We consider further the later stages of the expansion of the world, when the matter is already rarefied enough so that this pressure can be neglected ($p = 0$); in place of the energy density ϵ it is more natural to speak here of the mass density ρ, which coincides with it.

Equations (8.8)—(8.9) with $p = 0$ and $a = a_0 \times (\cosh\eta - 1)$ can be totally integrated in terms of elementary functions; from the first we determine ξ, and then from the second we determine ζ:*

$$\xi=-C_1 a_0\,\mathrm{th}^2\frac{\eta}{2},$$

$$\zeta=-\frac{2n^2C_1a_0}{\mathrm{ch}\,\eta-1}\left(\mathrm{sh}\,\eta-3\eta+4\,\mathrm{th}\frac{\eta}{2}\right)-\frac{4n^2C_2a_0}{\mathrm{ch}\,\eta-1}.$$

Calculation with the aid of (8.7), (8.2), and (8.3) then yields the following expressions:

$$\left.\begin{aligned} \lambda+\mu&=-C_1(\varphi-1)-A\psi,\\ \lambda-\mu&=\frac{2n^2}{3}C_1\varphi-2n^2C_2\left(\mathrm{cth}\frac{\eta}{2}-\frac{1}{3}\mathrm{cth}^3\frac{\eta}{2}\right)+\\ &\quad+A\psi+\frac{4n^2}{3}A\,\mathrm{cth}\frac{\eta}{2}+B,\\ \frac{\delta\varrho}{\varrho}&=\frac{n^2}{6}(C_1\varphi+C_2\psi)Q+\frac{A}{2}\psi Q,\\ \delta v^{\alpha}&=\frac{in}{6}(A-C_2)\frac{1}{\mathrm{sh}^2\frac{\eta}{2}}P^{\alpha}.\end{aligned}\right\} \tag{8.14}$$

We have introduced here the functions

$$\varphi(\eta)=\frac{3}{\mathrm{sh}^2\frac{\eta}{2}}\left(1-\frac{\eta}{2}\mathrm{cth}\frac{\eta}{2}\right)+1,\quad \psi(\eta)=\frac{\mathrm{ch}\frac{\eta}{2}}{\mathrm{sh}^3\frac{\eta}{2}};\quad (8.15)\dagger$$

A and B are integration constants, the arbitrariness of which is connected with the arbitrariness in the choice of the reference system.

It was noted at the end of Sec. 2 that in the case of "dustlike" matter ($p = 0$) the reference system can be chosen such as to be simultaneously synchronous and co-moving. It is seen from (8.14) that it is actually possible to cause δv^{α} to vanish by suitable choice of the constant A ($A = C_2$). Such a choice of the reference system is the most natural, and the perturbation $\delta\rho$ pertains in this case to the intrinsic density of the matter. Putting also $B = 0$, we obtain ultimately

$$p=0\left\{\begin{aligned} \lambda+\mu&=-C_1(\varphi-1)-C_2\psi,\\ \lambda-\mu&=\frac{2n^2}{3}(C_1\varphi+C_2\psi),\\ \frac{\delta\varrho}{\varrho}&=\frac{n^2}{6}(C_1\varphi+C_2\psi)Q,\quad \delta v^{\alpha}=0.\end{aligned}\right\} \tag{8.16}$$

In order to investigate these expressions, let us consider them in two limiting cases—small and large η. Small η ($\eta \ll 1$) correspond to the stage of the expansion of the world when the radius of curvature is still very small compared with its contemporary value, but all the matter is already sufficiently rarefied that its pressure can be neglected.* On the other hand, the values $\eta \gg 1$ correspond to later stages of expansion, when the metric approaches Galilean.

The terms with the constant C_2 in (8.16) yield†

$$\left.\begin{aligned}p&=0,\\ \eta&\ll 1\end{aligned}\right\}\ h_{\alpha}^{\beta}=\frac{8n^2C_2}{3\eta^3}\left(P_{\alpha}^{\beta}-Q_{\alpha}^{\beta}\right),\quad \frac{\delta\varrho}{\varrho}=\frac{4n^2C_2}{3\eta^3}Q, \tag{8.17}$$

$$\left.\begin{aligned}p&=0,\\ \eta&\gg 1\end{aligned}\right\}\ h_{\alpha}^{\beta}=\frac{4C_2}{3}e^{-\eta}\left(P_{\alpha}^{\beta}-Q_{\alpha}^{\beta}\right),\quad \frac{\delta\varrho}{\varrho}=\frac{n^2C_2}{3}e^{-\eta}Q. \tag{8.18}$$

When $\eta \ll 1$ we have $a \approx a_0\eta^2/2$, $t \approx a_0\eta^3/6$, and for $\eta \gg 1$ we have $a \approx a_0e^{\eta}/2$ and $t \approx a_0e^{\eta}/2$. We therefore see that these perturbations attenuate as the world expands, first like $a^{-3/2}$ and then like $1/a$; in terms of time, both laws correspond to $1/t$.

On the other hand, in terms with the constant C_1 we distinguish (for $\eta \ll 1$) between the cases $n\eta \ll 1$ and $n\eta \gg 1$. In the first case we obtain

$$\left.\begin{aligned}p&=0,\\ \eta&\ll\frac{1}{n}\end{aligned}\right\}\ h_{\alpha}^{\beta}=\frac{C_1}{2}\left(P_{\alpha}^{\beta}+Q_{\alpha}^{\beta}\right),\quad \frac{\delta\varrho}{\varrho}=\frac{n^2C_1}{60}\eta^2Q. \tag{8.19}$$

Although the relative change in density increases, nevertheless it does not become large here even for $\eta \sim 1/n$, by virtue of the condition $C_1 \ll 1$. In the case $n\eta \gg 1$, on the other hand, we get

$$\left.\begin{aligned}p&=0,\\ \frac{1}{n}&\ll\eta\ll 1\end{aligned}\right\}\ h_{\alpha}^{\beta}=\frac{C_1n^2}{30}\left(P_{\alpha}^{\beta}-Q_{\alpha}^{\beta}\right),\quad \frac{\delta\varrho}{\varrho}=\frac{C_1n^2}{60}\eta^2Q. \tag{8.20}$$

*th = tanh.

†cth = coth.

*The contemporary value of η can be obtained from the contemporary values of the average density of matter ρ and the Hubble constant h (for the open model $\cosh(\eta/2) = h\sqrt{3/8\pi G\rho}$, where G is the gravitational constant). Such a determination can, however, be made at the present time only quite tentatively, in view of the large uncertainty in the values of h and particularly ρ. Putting $h = 0.25 \times 10^{-17}$ sec^{-1} (25 km/sec in 10^6 light years) and introducing for ρ Oort's estimate $\rho = 3 \times 10^{-31}$ g/cm^3 [13], we obtain $\eta = 5.0$. If we put $\rho = 10^{-30}$ g/cm^3, we get $\eta = 6.1$.

†For $\eta \ll 1$ we have

$$\varphi\approx\frac{\eta^2}{10},\quad \psi\approx\frac{8}{\eta^3}.$$

These perturbations disclose a true instability. When $\eta \sim 1$ the relative change in the density becomes of the order of $C_1 n^2$, whereas the smallness of the initial perturbation necessitates merely that we have $C_1 n^2 \eta_0^2 \ll 1$. Thus, although the increase in the perturbations is slow (proportional to a, that is, $t^{2/3}$), the overall increase may be appreciable and as a result the perturbation may become relatively large*.

For $\eta \gg 1$ we have

$$\left.\begin{array}{l} p=0, \\ \eta \gg 1 \end{array}\right\} \; h_\alpha^\beta = C_1 n^2 \left(\frac{1}{3} - 2\eta e^{-\eta}\right)(P_\alpha^\beta - Q_\alpha^\beta), \quad \frac{\delta \varrho}{\varrho} = \frac{n^2}{6} C_1 Q. \tag{8.21}$$

We see that the increasing relative perturbation of the density tends to a constant limit. The constant term in the perturbation of the metric (in which $\lambda = -\mu = \text{const}$) can be eliminated by transforming the reference system (which does not involve the density); the second term in h_α^β attenuates in proportion to $(\ln a)/a$.

Finally, let us consider the case of an equation of state that is intermediate between $p = 0$ and $p = \epsilon/3$. Namely, we consider an expansion stage in which the derivative $dp/d\epsilon$ is small but still cannot be set equal to zero. The quantity

$$u = \sqrt{\frac{dp}{d\varepsilon}}$$

is the "velocity of sound" in the matter filling the world (measured in units of the velocity of light); we assume, consequently, that this quantity is small: $u \ll 1$. The inverse influence of the finite pressure on the law governing the expansion of the world can be neglected here, that is, we can use the same function $a(\eta)$ as for $p = 0$, and we assume that we still have $\eta \ll 1$, so that $a \approx a_0 \eta^2/2$.

The behavior of the perturbations depends essentially in this case on the value of $nu\eta$. When $nu\eta \ll 1$ an estimate of the terms in (8.8)—(8.9) shows that all the terms containing u can be left out, so that we return to the already investigated case $p = 0$.

To the contrary, when $nu\eta \gg 1$, the terms containing u become essential. Equations (8.8)—(8.9) assume the form

$$\xi' - \frac{2}{\eta}\xi + \frac{1}{2}u^2\zeta = 0, \quad \zeta' + \frac{2}{\eta}\zeta - 2n^2\xi = 0.$$

Eliminating ζ we obtain, with the same accuracy, the equation

$$\xi'' - \frac{2u'}{u}\xi' + n^2u^2\xi = 0,$$

hence

$$\zeta = \text{const}\sqrt{u}\,\Phi, \quad \Phi = \exp\left(in\int u\,d\eta\right); \tag{8.22}$$

we put below const = $3a_0C/in$. We obtain further with the aid of the first formula of (8.7)

$$\lambda + \mu = \frac{36C}{n^2\eta^3\sqrt{u}}\Phi.$$

According to the second formula

$$\lambda' - \mu' \approx -\frac{2n^2}{a}\int \xi\,d\eta + \frac{\zeta}{a} = -\frac{2n^2}{a}\int \xi\,d\eta - \frac{2}{au^2}\xi' + \frac{4\xi}{au^2\eta}.$$

Inserting (8.22) and integrating in the first term twice by parts, and then integrating the entire expression with respect to $d\xi$ (which reduces to division by inu), we obtain

$$\lambda - \mu = \frac{12C}{n^2\eta^2u^{5/2}}\left(\frac{u'}{u} - \frac{2}{\eta}\right)\Phi.$$

Finally, calculating also $\delta\rho/\rho$ and δv^α in accordance with (8.2)—(8.3), we obtain the following final expressions, in which we retain only the principal terms:

$$\left.\begin{array}{r} nu\eta \gg 1, \\ u \ll 1, \\ \eta \ll 1 \end{array}\right| \begin{array}{l} h_\alpha^\beta = \dfrac{6C}{n^2\eta^2u^{5/2}}\left(\dfrac{u'}{u} - \dfrac{2}{\eta}\right)\Phi(P_\alpha^\beta - Q_\alpha^\beta), \\ \dfrac{\delta\varrho}{\varrho} = \dfrac{C}{\eta\sqrt{u}}\Phi Q, \quad \delta v^\alpha = -\dfrac{C\sqrt{u}}{\eta}\Phi P^\alpha. \end{array} \tag{8.23}$$

The constant C must satisfy the inequality $|C|/\eta_0\sqrt{u_0} \ll 1$.

Expressions (8.23) correspond to sound waves propagating with velocity u, and we are in the region of applicability of "geometrical acoustics" (the phase $\int nu\,d\eta$ is large). The velocity u decreases with expansion of the world, and thereby slows down the decrease in the wave amplitude. Nonetheless, the amplitude of the relative change in density does not increase, generally speaking. If we estimate the dependence of u on the time, regarding the matter as being an adiabatically expanding monatomic ideal gas, then $p \sim \rho^{5/3}$ and $u \sim \rho^{1/3}$; inasmuch as $\rho \sim a^{-3} \sim \eta^{-6}$, we have $u \sim \eta^{-2}$. Then $\eta\sqrt{u} = \text{const}$, so that the amplitude of $\delta\rho/\rho$ remains constant. In the case of slower decrease in u, $\delta\rho/\rho$ attenuates in time.

All the foregoing results, which we have formulated for the open model, can be directly transferred to the closed model by means of the transformation (7.7) $\eta \to i\eta$, $n \to in$. This transformation does not change at all any of the conclusions concerning the character of the time variation of the perturbations during those stages of the expansion of the world when we still have $\eta \ll 1$. When $\eta \sim 1$, when the expansion in the closed model slows down, ultimately turning into contraction, the formulas of course change (on the other hand, the case $\eta \gg 1$ does not exist at all). They are obtained from (8.15)—(8.16) by the already mentioned transformation and some rearrangement of the terms (with redesignation of the constants):

$$\left.\begin{array}{l} \lambda + \mu = C_1(\varphi + 1) + C_2\psi, \\ \lambda - \mu = \dfrac{2n^2}{3}(C_1\varphi + C_2\psi), \\ \dfrac{\delta\varrho}{\varrho} = \dfrac{n^2}{6}(C_1\varphi + C_2\psi), \quad \delta v^\alpha = 0, \\ \varphi = \dfrac{3}{\sin^2\frac{\eta}{2}}\left(1 - \dfrac{\eta - \pi}{2}\operatorname{ctg}\dfrac{\eta}{2}\right) - 1, \quad \psi = \dfrac{\cos\frac{\eta}{2}}{\sin^3\frac{\eta}{2}}. \end{array}\right\} \text{(8.24)*}$$

*Thus, for an expansion in which the average density of the matter changes from nuclear ($\sim 10^{14}$ g/cm^3) to the contemporary value ($\sim 10^{-30}$) the value of $a(\eta)$ increases by $(10^{14}/10^{-30})^{1/3} = 5 \times 10^{14}$ times.

*ctg = cot.

We note that the time variation of the perturbations is represented here in the form of a sum of two functions, one (with constant C_1) even and the other (with constant C_2) odd with respect to the instant $\eta = \pi$, that is, with respect to the substitution $\eta \to 2\pi - \eta$. The instant $\eta = \pi$ corresponds to the maximum of the radius $a(\eta)$ in the closed model, so that the indicated property denotes that during the compression stage each of the two parts of the perturbations duplicates (apart from the sign) the variation during the expansion stage, but in reverse order.

Summarizing the results obtained, we can state that the expansion of the world exerts a stabilizing influence on the development of the perturbations. In long-wave perturbations ($un\eta \ll 1$) the change in density of matter increases with time. During the earlier stages of the world expansion (with an ultrarelativistic equation of state $p = \epsilon/3$, $u^2 = 1/3$) this increase cannot cause the perturbation to become large. This can occur, however, at later stages of the expansion, when the pressure of the matter becomes negligibly small; but here, too, the increase in the perturbation of the density is slow ($\sim t^{2/3}$). On the other hand, the short-wave perturbations ($un\eta \gg 1$) represent hydrodynamic sound waves in which the amplitude of density perturbation attenuates with time.

To the contrary, a contracting world would be essentially unstable, and perturbations in it must ultimately become large. Further behavior of the model cannot be traced, of course, with the aid of perturbation theory. But the general conclusions made in Chapter I of the present article signify that the increase in the perturbation should lead in final analysis to a cessation of the overall contraction of the world and eventual expansion. It is sensible to attempt here an estimate of the maximum attainable contraction, assuming that it is determined by the instant when the perturbation $\delta\rho/\rho$ becomes of the order of unity. Assume that some perturbation $\delta\rho/\rho = \Delta$ exists in the closed model at some instant $\eta_0 \sim 1 < \pi$ (during the expansion stage). Inasmuch as $\delta\rho/\rho$ is the sum of an even and an odd function of $\eta - \pi$, by the instant $\eta = 2\pi - \eta_0$ (during the compression stage) we again have $\delta\rho/\rho \sim \Delta$. With further contraction of the world, $\delta\rho/\rho$ will increase like $(2\pi - \eta)^{-3}$ for small $2\pi - \eta$; the value of $\delta\rho/\rho \sim 1$ will be attained when $\eta = \eta_1$, where $2\pi - \eta_1 \sim \eta_0\Delta^{1/3}$. Inasmuch as the average density of the matter in the contracting world increases like $a^{-3} \propto (2\pi - \eta)^{-6}$, the density reaches by the instant η_1 of maximum compression a value

$$\varrho_1 \sim \varrho_0\Delta^{-2}, \tag{8.25}$$

where ρ_0 is the density at the instant η_0 of the initial perturbation.

In the entire investigation of the present section it was tacitly assumed that the perturbations were adiabatic, that is, they occurred at constant entropy, and all dissipative processes were neglected. Although the role of these processes is quite negligible for the expansion of the world itself, we can not exclude a priori the possibility that these small effects may lead to some new instability. An investigation of this question calls for an analysis of nonadiabatic perturbations, in which a change takes place also in the entropy of the matter, and it is necessary to take into account heat-conduction and viscosity processes (for the general equations necessary for this purpose see [8], Sec. 126). We shall not present the corresponding calculations here, and indicate merely that the net result is that the dissipative processes have no essential influence on the stability properties of the expanding world.

In conclusion we point out that Bonnor[14] has proposed a brilliant method, with the aid of which some of the results presented above can be obtained on the basis of Newtonian gravitation theory. This method is applicable to perturbations in regions whose linear dimensions are sufficiently small compared with the world's radius of curvature ($n \gg 1$); the idea consists in the following.

If we isolate a small spherical part in an isotropic world (filled with dust-like matter), then the surrounding matter will not exert a gravitational influence on the spherical part, the motion of matter inside of which can be considered with the aid of the Newtonian theory of gravitation. It is therefore clear that the law of expansion of the isotropic model of the general theory of relativity should coincide with the law of expansion of the homogeneous gravitational sphere in the Newtonian theory (this circumstance was first noted by Milne and McCrea). It follows in turn that the behavior of the perturbations in small regions of an isotropic world should coincide with their behavior in an expanding Newtonian sphere, and they can be considered with the aid of ordinary classical hydrodynamic equations with Newtonian gravitation as the volume forces*. The zeroth approximation in the solution of the hydrodynamic equations is in this case radial motion in a uniformly expanding sphere; the small perturbation superimposed on it (with a wavelength that is small compared with the radius of the sphere) can be sought in the form of a plane wave.

In such a hydrodynamic approach, the characteristic quantity determining the behavior of the perturbations is naturally the ratio of the length of the perturbation λ to the length $u/\sqrt{\rho G}$, a function of the density of the matter ρ and the velocity of sound in it u (as well as the gravitational constant G); these quantities are regarded here as functions of the time, varying in accordance with the general expansion of the medium.

*This method can probably be extended also to the case of an ultrarelativistic equation of state $\rho = \epsilon/3$, if suitable account is taken in the hydrodynamic equations of the relativistic gravitational pressure effect.

It is easy to see that this criterion $(u/\lambda\sqrt{\rho G})$ coincides with the criterion $n\eta u/c$ which was used in the foregoing calculations*.

9. Rotational Perturbations

We proceed to an analysis of the perturbations of the second type considered in Sec. 7. In these perturbations changes occur in the metric and in the velocity, but not in the density of the matter; the ensuing motion of the matter has a rotational character.

We put

$$h_\alpha^\beta = \sigma(\eta) S_\alpha^\beta. \tag{9.1}$$

Equation (6.19) is satisfied identically, since $h = 0$. On the other hand, Eq. (6.18) yields following the substitution (9.1) the following simple equation for the function $\sigma(\eta)$:

$$\sigma'' + 2\frac{a'}{a}\sigma' = 0; \tag{9.2}$$

We note that it does not contain the wave vector n. Hence

$$\sigma = \text{const} \int \frac{d\eta}{a^2}. \tag{9.3}$$

The constant part of this solution (the integration constant) corresponds to a fictitious change of the metric, consisting of a transformation of the coordinates (obtained from (6.23) by choosing $f_0 = 0$, $f_\alpha = S_\alpha$). For the velocity perturbation, calculation by means of (6.22) yields

$$\delta v^\alpha = -\frac{in\sigma'}{2(\varepsilon + p)a^2} S^\alpha. \tag{9.4}$$

During the early stage of the expansion ($\eta \ll 1$), with an equation of state $p = \epsilon/3$, (9.3) and (9.4) yield

$$\sigma = -\frac{C}{\eta}, \quad \delta v^\alpha = -\frac{inC}{8} S^\alpha. \tag{9.5}$$

For "dustlike" matter ($p = 0$) we obtain

$$\sigma = C\left(\operatorname{cth}\frac{\eta}{2} - \frac{1}{3}\operatorname{cth}^3\frac{\eta}{2} - \frac{2}{3}\right), \quad \delta v^\alpha = -\frac{inC}{12(\operatorname{ch}\eta - 1)} S^\alpha. \tag{9.6}$$

In the two limiting cases we have

$$\eta \ll 1: \ \sigma = -\frac{8C}{3\eta^3}; \qquad \eta \gg 1: \ \sigma = -4Ce^{-\eta}. \tag{9.7}$$

Thus, the perturbations of the metric attenuate with time in all cases. On the other hand, the perturbations of the velocity remain constant [in (9.5)] or decrease as $1/a$ [in (9.6)]*.

*To this end it is necessary to use the following estimates (in the usual units): the expansion law corresponding to dustlike matter $a \sim a_0\eta^2$, the density of matter $\rho \sim a_0c^2/Ga^3$, and the wavelength $\lambda \sim a/n$.

A similar relation between criteria exists also in the case of an ultrarelativistic equation of state ($p = \epsilon/3$, $u = c/\sqrt{3}$). In this case the expansion law is $a \sim b_0\eta$, and the energy density varies as $\epsilon \sim b_0^2c^4/Ga^4$. From this we readily find that

$$\frac{u}{\lambda\sqrt{G\varepsilon/c^2}} \sim n\eta,$$

that is, we return again to the characteristic quantity $n\eta$, used in the analysis above.

10. Gravitational Waves

Finally, in perturbations of the third type, in which

$$h_\alpha^\beta = \nu(\eta) G_\alpha^\beta, \tag{10.1}$$

only the metric changes; the matter remains stationary ($\delta v^\alpha = 0$) and uniformly distributed in space ($\delta\epsilon = 0$).

For $\nu(\eta)$ we obtain from (6.18) the following expression:

$$\nu'' + 2\frac{a'}{a}\nu' + n^2\nu = 0. \tag{10.2}$$

Both solutions of this equation correspond to real changes of the metric, which cannot be eliminated by coordinate transformations (inasmuch as in this case there exists neither scalar nor vector capable of being substituted for f_0 and f_α in (6.23)).

With the required accuracy, the solution of (10.2) is

$$\nu = C\frac{e^{in\eta}}{a}, \tag{10.3}$$

where C is a complex constant. The periodic factor corresponds here to gravitational waves propagating with the velocity of light (wave vector $k = n/a$, so that the temporal part of the phase is $\int k\,dt = n\eta$). The amplitude of the gravitational waves attenuates like $1/a$. The energy density of these waves ($\sim k^2(h_\alpha^\beta)^2$) decreases in proportion to a^{-4}, as it should.

During all the stages of the investigations reported here, we were continuously supported by our teacher and friend L. D. Landau, discussions with whom were of inestimable help to us, and to whom we wish to express here our deep gratitude.

APPENDICES

A. EXPANSION OF THE SOLUTION OF THE GRAVITATIONAL EQUATIONS NEAR A REGULAR POINT

Let us consider the expansion of the equations of a gravitational field in vacuum in a synchronous reference system near a nonsingular point which is regular in time†.

Choosing an arbitrarily considered point as the time reference, we have a metric tensor in the form

$$g_{\alpha\beta} = a_{\alpha\beta} + tb_{\alpha\beta} + t^2c_{\alpha\beta} + \ldots, \tag{A.1}$$

*The indicated law of variation of the velocity perturbation is directly connected (as pointed out by Ya. B. Zel'dovich) with momentum conservation. The momentum of the small portion of the matter, in which the rotational perturbation took place, has an order of magnitude $\epsilon l^3 \cdot l \cdot v$, where l – linear dimensions of the section. When the world expands l increases in proportion to a, and ϵ decreases as a^{-3} (in the case when $p = 0$) or as a^{-4} (when $p = \epsilon/3$); in the former case we get $v \sim 1/a$ and in the latter $v \sim$ const.

†This question is considered also in the book by Petrov,[15] Sec. 40.

where $a_{\alpha\beta}$, $b_{\alpha\beta}$, and $c_{\alpha\beta}$ are functions of the spatial coordinates. In the same approximation, the inverse tensor is

$$g^{\alpha\beta}=a^{\alpha\beta}-tb^{\alpha\beta}+t^2(b^{\alpha\gamma}b_\gamma^\beta-c^{\alpha\beta}),$$

where $a^{\alpha\beta}$ is a tensor inverse to $a_{\alpha\beta}$, and the raising of the indices in the remaining tensors is carried out with the aid of $a^{\alpha\beta}$. We further have

$$\varkappa_{\alpha\beta}=b_{\alpha\beta}+2tc_{\alpha\beta},$$
$$\varkappa_\alpha^\beta=b_\alpha^\beta+t(2c_\alpha^\beta-b_{\alpha\gamma}b^{\beta\gamma}).$$

The field equations (3.13)—(3.15) lead to the following relations

$$R_0^0=c-\frac{1}{4}b_\alpha^\beta b_\beta^\alpha=0, \tag{A.2}$$

$$R_\alpha^0=\frac{1}{2}(b_{;\alpha}-b^\beta_{\alpha;\beta}) +t\left[c_{;\alpha}-\frac{3}{8}(b_\beta^\gamma b_\gamma^\beta)_{;\alpha}-c^\beta_{\alpha;\beta})-\frac{1}{4}b_\alpha^\beta b_{;\beta}+\frac{1}{2}(b_\alpha^\gamma b_\gamma^\beta)_{;\beta}\right]=0, \tag{A.3}$$

$$R_\alpha^\beta=P_\alpha^\beta+\frac{1}{4}b_\alpha^\beta b-\frac{1}{2}b_\alpha^\gamma b_\gamma^\beta+c_\alpha^\beta=0 \tag{A.4}$$

($b\equiv b_\alpha^\alpha$, $c\equiv c_\alpha^\alpha,\ldots$). The covariant differentiation operation is carried out here in three-dimensional space with metric $a_{\alpha\beta}$; the tensor $P_{\alpha\beta}$ is determined with the same metric.

Using (A.4), the coefficients $c_{\alpha\beta}$ are fully determined from the coefficients $a_{\alpha\beta}$ and $b_{\alpha\beta}$. Then (A.2) yields the relation

$$P+\frac{1}{4}b^2-\frac{1}{4}b_\alpha^\beta b_\beta^\alpha=0. \tag{A.5}$$

From the zero-order terms in (A.3) we have

$$b^\beta_{\alpha;\beta}=b_{;\alpha}. \tag{A.6}$$

On the other hand, terms proportional to t vanish in this equation identically if (A.5) and (A.6) (as well as the identities $P^\beta_{\alpha;\beta}=\frac{1}{2}P_{;\alpha}$) are used.

We see that the 12 quantities $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are related by (A.5) and the three relations (A.6). This leaves therefore a total of eight arbitrary functions of the three spatial coordinates, in accordance with the calculation made in the text*.

B. SOLUTIONS THAT DEPEND ON ONE VARIABLE

Let us consider the exact solutions of the gravitational equations in vacuum, in which (in the synchronous frame) the metric depends only on one variable. We assume first that this variable is the time.

For a metric that does not depend on the spatial coordinates, Eq. (3.14) is satisfied identically, and from (3.15) we get

$$\varkappa_\alpha^\beta=\frac{2}{t}\lambda_\alpha^\beta, \tag{B.1}$$

where λ_α^β are constants, with

$$\lambda_\alpha^\alpha=1 \tag{B.2}$$

(here $\dot{g}/g\equiv\kappa_\alpha^\alpha=2/t$ and $-g=\text{const}\cdot t^2$). Substitution of (B.1) in (3.13) gives one more relation

$$\lambda_\alpha^\beta\lambda_\beta^\alpha=\lambda_\alpha^\alpha, \tag{B.3}$$

which relates the constants λ_α^β.

Omitting the index β, we rewrite (B.1) in the form of a system of ordinary differential equations for $g_{\alpha\beta}$:

$$\dot{g}_{\alpha\beta}=\frac{2}{t}\lambda_\alpha^\gamma g_{\gamma\beta}. \tag{B.4}$$

Various cases can occur here, depending on the roots of the characteristic equation of the metrics of the coefficients λ_α^β (the equation $|\lambda_\alpha^\beta-\lambda\delta_\alpha^\beta|=0$).

a) The characteristic equation has three different real roots (p_1, p_2, p_3); by virtue of (B.2) and (B.3) they are related by

$$p_1+p_2+p_3=p_1^2+p_2^2+p_3^2=1. \tag{B.5}$$

By means of a suitable linear transformation of the quantities $g_{1\beta}$, $g_{2\beta}$, $g_{3\beta}$ (or, what is equivalent, of the coordinates x^1, x^2, x^3), the matrix λ_α^β reduces in this case to a diagonal form, and we obtain the solution (3.1) already indicated in Sec. 3

$$-ds^2=-dt^2+t^{2p_1}dx^2+t^{2p_2}dy^2+t^{2p_3}dz^2. \tag{B.6}$$

b) The characteristic equation has one real root (p_3) and two complex roots ($p_{1,2}=p'\pm ip''$); the numbers p_1, p_2, and p_3 satisfy as before the relations (B.5), with either $p_3<-\frac{1}{3}$ or $p_3>1$. After diagonalizing the matrix λ_α^β we introduce, in order to make the form of the metric real, new coordinates defined as $x^{1,2}=x\pm iy$ and find a solution in the form

$$-ds^2=-dt^2+t^{2p'}\cos\left(2p''\ln\frac{t}{\alpha}\right)(dx^2-dy^2)+t^{2p_3}dz^2 +2t^{2p'}\sin\left(2p''\ln\frac{t}{\alpha}\right)dx\,dy \tag{B.7}$$

(α is a constant). However, the determinant of the metric tensor $g=g_{00}|g_{\alpha\beta}|=t^2$ does not satisfy the necessary condition $g<0$, so that the metric (B.7) cannot correspond to physical space-time.

c) Two of the roots of the characteristic equation coincide ($p_2=p_3$)*; in this case the pair of number p_1, p_2 can have values 1 and 0 or $-\frac{1}{3}$ and $\frac{2}{3}$.

As is known from general theory of linear differential equations, in this case the matrix λ_α^β can be reduced to the form

$$\begin{pmatrix} p_1 & 0 & 0\\ 0 & p_2 & 0\\ 0 & \lambda & p_2\end{pmatrix}.$$

If $\lambda=0$, we return to solution (B.6). On the other hand, if $\lambda\neq 0$, then the solution of (B.4) (without ac-

*By virtue of the differential nature of relations (A.6), arbibitrary functions of a smaller number of variables can appear in the solution. We leave aside the question of the geometrical origin of these functions.

*Equality of all three roots is excluded by the conditions (B.2) and (B.3).

count of the symmetry conditions $g_{23} = g_{32}$) leads to the metric

$$-ds^2 = -dt^2 + t^{2p_1}\,dx^2 + 2t^{2p_2}\,dy\,dz \pm t^{2p_2}\ln\frac{t}{\alpha}\,dz^2. \tag{B.8}$$

In this case, too, the determinant $g = t^2$ does not satisfy the condition $g < 0$.

Thus, (B.6) is the only solution in which the metric depends only on the time. On the other hand, if the one variable on which the metric depends is a space coordinate (x), then it becomes possible to have solutions of all three types. The transition to this case is by means of a corresponding reversal of the signs in the obtained solutions:

$$-ds^2 = -x^{2p_1}\,dt^2 + dx^2 + x^{2p_2}\,dy^2 + x^{2p_3}\,dz^2, \tag{B.9}$$

$$-ds^2 = dx^2 + x^{2p'}\cos\left(2p''\ln\frac{x}{\alpha}\right)(d\xi^2 - d\eta^2) + 2x^{2p'}\sin\left(2p''\ln\frac{x}{\alpha}\right)d\xi\,d\eta + x^{2p_3}\,dz^2, \tag{B.10}$$

$$-ds^2 = dx^2 + 2x^{2p_2}\,d\xi\,d\eta \pm x^{2p_2}\ln\frac{x}{\alpha}\,d\eta^2 + x^{2p_1}\,dz^2. \tag{B.11}$$

All these metrics satisfy the condition $g < 0$. The value $x = 0$ is a singular point of these solutions, with the exception of the case $(p_1, p_2, p_3) = (0, 0, 1)$ in the metric (B.9) (which reduces to Galilean in this case) and the case $(p_1, p_2) = (1, 0)$ in the metric (B.11), in which the singularity turns out to be fictitious.

Returning again to the solution (B.6), we show that it can be transformed also to the form

$$-ds^2 = 2d\eta\,d\zeta + \eta^{2s_1}\,dx^2 + \eta^{2s_2}\,dy^2 + \lambda\eta^{2s_3}\,d\zeta^2, \tag{B.12}$$

where the numbers s_1, s_2, s_3 are connected by the relations

$$s_1 + s_2 = s_1^2 + s_2^2,\quad s_3 = \frac{1}{2}(1 - s_1 - s_2); \tag{B.13}$$

λ is an arbitrary constant, which can be eliminated (if it differs from zero) by suitable change of the scale of the coordinates. The transformation of the metric (B.12) to the form (B.6) is then made by the substitution

$$\eta = \frac{1}{1+p_3}\,t^{1+p_3},\quad \zeta = z - \frac{t^{1-p_3}}{1-p_3}, \tag{B.14}$$

with the numbers p_1, p_2, and p_3 connected with the numbers s_1, s_2, s_3 by means of

$$s_1 = \frac{p_1}{1+p_3},\quad s_2 = \frac{p_2}{1+p_3},\quad s_3 = \frac{p_3}{1+p_3} \tag{B.15}$$

(the relative magnitudes of the numbers p_1, p_2, and p_3 are not specified at all).

If we put in (B.12) $\lambda = 0$, we obtain the solution

$$-ds^2 = 2d\eta\,d\zeta + \eta^{2s_1}\,dx^2 + \eta^{2s_2}\,dy^2. \tag{B.16}$$

This metric is transformed to the synchronous form by the transformation

$$\eta = \frac{t}{z\sqrt{2}},\quad \zeta = -\frac{zt}{\sqrt{2}},$$

$$-ds^2 = -dt^2 + \left(\frac{t}{z}\right)^{2s_1}dx^2 + \left(\frac{t}{z}\right)^{2s_2}dy^2 + \left(\frac{t}{z}\right)^{2}dz^2, \tag{B.17}$$

but in this case it depends not only on t, but also on one of the spatial coordinates.

Thus, the form (B.12) turns out to be broader than (B.6). It includes as a particular case the metric (B.17) which is not contained in (B.6). More general aspects of this circumstance are considered in Appendix F.

C. THREE-DIMENSIONAL RICCI TENSOR $P_{\alpha\beta}$

We present here general expressions for the components of the three-dimensional Ricci tensor $P_{\alpha\beta}$, calculated with a metric in the form

$$g_{\alpha\beta} = a^2 l_\alpha l_\beta + b^2 m_\alpha m_\beta + c^2 n_\alpha n_\beta, \tag{C.1}$$

where both the vectors **l**, **m**, **n** and the scalars a, b, c can be functions of the coordinates.

We write down expressions for the components along the directions of the vectors **l**, **m**, **n** in accordance with the definitions (3.16)–(3.18) in which we must write a, b, c in place of t^{p_1}, t^{p_2}, t^{p_3}):*

$$\begin{aligned}P_{ll} = \frac{a^2}{\Delta^2}\Big\{&\frac{1}{2}(a\mathbf{l}\operatorname{rot}a\mathbf{l})^2 - \frac{1}{2}(b\mathbf{m}\operatorname{rot}b\mathbf{m})^2 - \frac{1}{2}(c\mathbf{n}\operatorname{rot}c\mathbf{n})^2 - (c\mathbf{n}\operatorname{rot}b\mathbf{m})^2\\ &- (b\mathbf{m}\operatorname{rot}c\mathbf{n})^2 - (b\mathbf{m}\operatorname{rot}a\mathbf{l})^2 - (c\mathbf{n}\operatorname{rot}a\mathbf{l})^2\\ &+ (c\mathbf{n}\operatorname{rot}c\mathbf{n})(b\mathbf{m}\operatorname{rot}b\mathbf{m}) + (c\mathbf{n}\operatorname{rot}a\mathbf{l})(a\mathbf{l}\operatorname{rot}c\mathbf{n})\\ &+ (a\mathbf{l}\operatorname{rot}b\mathbf{m})(b\mathbf{m}\operatorname{rot}a\mathbf{l})\Big\} + a^2\Big\{\frac{1}{b}\left(\frac{c\mathbf{n}\operatorname{rot}a\mathbf{l}}{\Delta}\right)_{,m}\\ &- \frac{1}{a}\left(\frac{c\mathbf{n}\operatorname{rot}b\mathbf{m}}{\Delta}\right)_{,l} + \frac{1}{a}\left(\frac{l\mathbf{m}\operatorname{rot}c\mathbf{n}}{\Delta}\right)_{,l} - \frac{1}{c}\left(\frac{b\mathbf{m}\operatorname{rot}a\mathbf{l}}{\Delta}\right)_{,n}\Big\},\\ P_{lm} = \frac{ab}{\Delta^2}\Big\{&(a\mathbf{l}\operatorname{rot}a\mathbf{l})(b\mathbf{m}\operatorname{rot}a\mathbf{l}) + (b\mathbf{m}\operatorname{rot}b\mathbf{m})(a\mathbf{l}\operatorname{rot}b\mathbf{m})\\ &+ (a\mathbf{l}\operatorname{rot}c\mathbf{n})(b\mathbf{m}\operatorname{rot}c\mathbf{n}) - \frac{1}{2}(c\mathbf{n}\operatorname{rot}c\mathbf{n})[(a\mathbf{l}\operatorname{rot}b\mathbf{m})\\ &+ (b\mathbf{m}\operatorname{rot}a\mathbf{l})] + \frac{1}{2}(b\mathbf{m}\operatorname{rot}c\mathbf{n})(c\mathbf{n}\operatorname{rot}a\mathbf{l})\\ &+ \frac{1}{2}(a\mathbf{l}\operatorname{rot}c\mathbf{n})(c\mathbf{n}\operatorname{rot}b\mathbf{m})\Big\} + \frac{ab}{2}\Big\{\frac{1}{b}\left(\frac{b\mathbf{m}\operatorname{rot}c\mathbf{n}}{\Delta}\right)_{,m}\\ &- \frac{1}{a}\left(\frac{a\mathbf{l}\operatorname{rot}c\mathbf{n}}{\Delta}\right)_{,l} - \frac{1}{c}\left(\frac{b\mathbf{m}\operatorname{rot}b\mathbf{m}}{\Delta}\right)_{,n} + \frac{1}{c}\left(\frac{a\mathbf{l}\operatorname{rot}a\mathbf{l}}{\Delta}\right)_{,n}\Big\}.\end{aligned} \tag{C.2}$$

Here we put

$$\Delta = \sqrt{-g} = abc\,(\mathbf{l}\,[\mathbf{mn}]),$$

and the letters *l*, m, and n following the commas in the indices denote differentiation in the corresponding directions, in accordance with the definition

$$f_{,l} = l^\alpha\frac{\partial f}{\partial x^\alpha},\ \ldots \tag{C.3}$$

We note also that in the products which we write for the sake of symmetry in the form al curl al) (with two identical vectors), the scalars a, ... can, of course, be taken outside the curl sign: (al curl al) $= a^2$(**l** curl **l**).

The remaining components are obtained from those written out by cyclic permutation of the letters **l**, **m**, **n** and a, b, c.

*rot = curl.

D. NEXT TERMS OF THE EXPANSION OF THE ANISOTROPIC SOLUTION

The next higher terms in the expansion of the anisotropic solution obtained in Sec. 3, in powers of t, could be represented in the form of an expansion of the vectors **l**, **m**, **n**. It is simpler, however, to seek them directly as small corrections $h_{\alpha\beta}$ in the metric tensor

$$g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + h_{\alpha\beta}, \tag{D.1}$$

where $g^{(0)}_{\alpha\beta}$ is given by Eq. (3.6) with constant (time-independent) vectors **l**, **m**, **n**.

The correction terms in the gravitation equations are calculated with the aid of expressions (I.10)–(I.12) for the changes δR_i^k. It must be noted here that the quantities $\varkappa^{(0)\beta}_{\alpha}$ are proportional to 1/t, while $\varkappa^{(0)\gamma}_{\gamma} = 2/t$. In δR_α^β we can neglect the contribution from δP_α^β since the largest terms in it are proportional to $t^{-2p_3}h_\alpha^\beta$, that is, they are small compared with the terms proportional to $t^{-2}h_\alpha^\beta$. Omitting, in accordance with the notation of the Appendix I, the index 0 in the zeroth-approximation quantities, we obtain for the first-approximation corrections the following equations:

$$\frac{1}{2}\left(\ddot{h} + \varkappa_\alpha^\beta \dot{h}_\beta^\alpha\right) = T_0^0, \tag{D.2}$$

$$\frac{1}{2}\left(\ddot{h}_\alpha^\beta + \frac{1}{t}\dot{h}_\alpha^\beta + \frac{1}{2}\varkappa_\alpha^\beta \dot{h} - \varkappa_\alpha^\gamma \dot{h}_\alpha^\gamma + \varkappa_\gamma^\beta \dot{h}_\alpha^\gamma\right) = -P_\alpha^\beta + T_\alpha^\beta. \tag{D.3}$$

In the right halves of these equations are the components of the energy-momentum tensor and of the tensor P_α^β, calculated in the zeroth approximation metric (see also the remark at the end of the present section).

Inasmuch as the equations contain derivatives of h_α^β only with respect to the time (and not with respect to the coordinates), we can change over directly in them to projections along the directions **l**, **m**, **n**. Recognizing that only the "diagonal" components differ from zero

$$\varkappa_l^l = \frac{2p_1}{t}, \quad \varkappa_m^m = \frac{2p_2}{t}, \quad \varkappa_n^n = \frac{2p_3}{t},$$

we obtain the following equations:

$$\frac{1}{2}\ddot{h} + \frac{1}{t}\left(p_1\dot{h}_l^l + p_2\dot{h}_m^m + p_3\dot{h}_n^n\right) = T_0^0, \tag{D.4}$$

$$\frac{1}{2}\left(\ddot{h}_l^l + \frac{1}{t}\dot{h}_l^l + \frac{p_1}{t}\dot{h}\right) = -P_l^l + T_l^l, \tag{D.5}$$

$$\frac{1}{2}\left(\ddot{h}_l^m + \frac{1+2p_2-2p_1}{t}\dot{h}_l^m\right) = -P_l^m + T_l^m, \ldots \tag{D.6}$$

(the equations not written out are obtained from those written out by cyclic permutation of the letters l, m, n and p_1, p_2, p_3.)

In the case of empty space there is no energy momentum tensor and only P_α^β remain in the right halves of Eqs. (D.5)–(D.6). With the aid of (C.2) we find that the highest-order terms in this tensor, which remain after the terms (3.19) are made to vanish by the condition (3.20), are

$$P_l^l = \frac{1}{2}p_{1,n}\,p_{3,n}\,t^{-2p_3}\ln^2 t, \quad P_m^m = \frac{1}{2}p_{2,n}\,p_{3,n}\,t^{-2p_3}\ln^2 t,$$

$$P_n^n = -\frac{1}{4}\left(p_{1,n}^2 + p_{2,n}^2 + p_{3,n}^2\right)t^{-2p_3}\ln^2 t,$$

$$P_l^n = \frac{1}{2}\left(p_{3,l}\,p_{2,n} + p_{1,n}\,p_{2,l} - p_{2,n}\,p_{2,l}\right)t^{-2p_3}\ln^2 t,$$

$$P_m^n = \frac{1}{2}\left(p_{2,n}\,p_{1,m} + p_{3,m}\,p_{1,n} - p_{1,m}\,p_{1,n}\right)t^{-2p_3}\ln^2 t.$$

We shall not stop to write out the resultant expressions for the components h_α^β. We merely point out that they have the following orders of magnitude:

$$h_l^l \sim h_m^m \sim h_n^n \sim h_l^n \sim h_m^n \sim t^{2(1-p_3)}\ln^2 t \tag{D.7}$$

(The component h_{lm} is of higher order of smallness and in this sense is part of the next approximation.)

In the presence of matter, the principal quantity in the right halves of the "diagonal" equations (D.5) is the component of the energy-momentum tensor

$$T_n^n = \frac{4}{3}\varepsilon u_n u^n \sim t^{-(1+p_3)},$$

which contains the highest power of 1/t. Compared with this component it is possible to neglect T_l^l and T_m^m, and also all the P_l^l, P_m^m, and P_n^n. On the other hand, in the "nondiagonal" equations (D.6) we can leave out $P_l^n, \ldots$ compared with $T_l^n, \ldots$ As a result we obtain

$$\left.\begin{aligned} &h_l^l = -\frac{p_1}{1-p_3}h, \quad h_m^m = -\frac{p_2}{1-p_3}h, \quad h_n^n = 2h, \\ &h = \frac{8\varepsilon^{(0)} u_n^{(0)\,2}}{3(1-p_3)(2-p_3)}t^{1-p_3}, \\ &h_l^n = \frac{8\varepsilon^{(0)} u_l^{(0)} u_n^{(0)}}{3(1-p_3)(1+p_3-2p_1)}t^{1-p_3}, \quad h_m^n = \frac{8\varepsilon^{(0)} u_m^{(0)} u_n^{(0)}}{3(1-p_3)(1+p_3-2p_2)}t^{1-p_3} \end{aligned}\right\} \tag{D.8}$$

(The component $h_{lm} \sim t^{1+p_3}$ again turns out to be of relatively higher order of smallness).

Equation (D.4) is satisfied by expressions (D.8) identically. The equations $R_\alpha^0 = T_\alpha^0$, which we did not write out, were necessary only for the determination of the next higher expansion terms of the velocity and energy.

Finally, we must make one more remark to complete the justification of the calculations made. We have left out from (D.3) the terms derived from the tensor $P_{\alpha\beta}$ as a result of the corrections $h_{\alpha\beta}$ to the metric tensor; it must be verified that these terms are actually sufficiently small. Namely, this should be verified for the correction terms that are due to the "large" terms in $P_{\alpha\beta}$, the "zero" part of which is made to vanish by the condition **l** curl **l** = 0.

These are the terms of (3.19) in the diagonal components P_l^l, and the terms

$$P_{lm} = 2\,\frac{(\mathbf{l}\,\mathrm{rot}\,\mathbf{l})\,p_{1,n}}{(\mathbf{l}\,[\mathbf{mn}])}\ln t\cdot t^{2(p_1-p_3)},$$

$$P_{ln} = -2\,\frac{(\mathbf{l}\,\mathrm{rot}\,\mathbf{l})\,p_{1,m}}{(\mathbf{l}\,[\mathbf{mn}])}\ln t\cdot t^{2(p_1-p_2)}$$

in the off-diagonal components. Writing $\mathbf{l} = \mathbf{l}^{(0)} + \lambda$, we find that the corrections (D.8) in the metric tensor correspond to the corrections of the following orders in the vector **l**:

$$\lambda_l \sim t^{1-p_3}, \ \lambda_m \sim \lambda_n \sim t^{1+p_3-2p_1} \tag{D.9}$$

Therefore

$$\mathbf{l}\,\mathrm{rot}\,\mathbf{l} \sim t^{1+p_3-2p_1} \tag{D.10}$$

and it is easy to check that the discarded terms are actually small compared with the terms retained in (D.5) and (D.6).

E. STABILITY OF ANISOTROPIC SOLUTION

As was already indicated in Sec. 3, the metric

$$dl^2 = t^{2p_1}\,dx^2 + t^{2p_2}\,dy^2 + t^{2p_3}\,dz^2 \tag{E.1}$$

(with constant numbers $p_1 < p_2 < p_3$) is an exact solution of the gravitation equations in vacuum. Let us consider the behavior of the arbitrary small perturbations of the gravitational field in this homogeneous but anisotropic space. Such an investigation makes more lucid the origin of the "loss" of one arbitrary function on going over to the general anisotropic solution (3.6).

The equations of the small perturbations are obtained by equating to zero the expressions (I.10)—(I.12) for the changes δR_i^k. Inasmuch as the spatial metric (E.1) is Euclidean at any instant of time, the three-dimensional coordinate derivatives in these equations reduce to ordinary derivatives. On the other hand, the unperturbed tensor κ_α^β is diagonal, with $\kappa_1^1 = 2p_1/t$.

In view of the homogeneity of space we can expand an arbitrarily small perturbation in a spatial Fourier integral and consider the individual expansion component. Then all the $h_\alpha^\beta \sim \exp(i\mathbf{k}\cdot\mathbf{r})$, and we obtain the following system of ordinary differential equations:

$$\delta R_0^0 = \frac{1}{2}\sum_\alpha\left(\ddot{h}_\alpha^\alpha + \frac{1}{t}\,2p_\alpha \dot{h}_\alpha^\alpha\right) = 0,$$

$$\delta R_\alpha^0 = \frac{\dot{h}k}{2}\left\{\dot{h}k_\alpha + \sum_\beta\left[-\dot{h}_\alpha^\beta k_\beta - \frac{1}{t}k_\alpha(p_\alpha - p_\beta)\,h_\beta^\beta + \frac{2}{t}(p_\alpha - p_\beta)\,h_\alpha^\beta k_\beta\right]\right\},$$

$$\delta R_\alpha^0 = \frac{i}{2}\left\{\ddot{h}_\alpha^\alpha + \frac{1}{t}\,\dot{h}_\alpha^\alpha + \frac{p_\alpha}{t}\,\dot{h} + t^{-2p_\alpha}\left[hk_\alpha^2 - 2k_\alpha\sum_\gamma h_\alpha^\gamma k_\gamma\right] + h_\alpha^\alpha\sum t^{-2p_\gamma}k_\gamma^2\right\} = 0,$$

$$\delta R_\alpha^\beta = \frac{1}{2}\left\{\ddot{h}_\alpha^\beta + \frac{1}{t}\,\dot{h}_\alpha^\beta + \frac{2}{t}(p_\beta - p_\alpha)\,\dot{h}_\alpha^\beta + t^{-2p_\beta}\left[hk_\alpha k_\beta - \sum_\gamma(h_\alpha^\gamma k_\beta k_\gamma + h_\beta^\gamma k_\alpha k_\gamma)\right] + h_\alpha^\beta\sum_\gamma t^{-2p_\gamma}k_\gamma^2\right\} = 0, \quad \alpha \neq \beta \tag{E.2}$$

(no summation over repeated indices is implied here and below).

Among the solutions of these equations there are such changes in the metric h_α^β, which can be eliminated by transforming the reference system. According to formulas (I.13)—(I.14), we find that the general form of such "fictitious" perturbations is

$$\left.\begin{aligned} h_\alpha^\alpha &= 2k_\alpha C_\alpha + 2C_0\left(\frac{k_\alpha^2}{1-2p_\alpha}\,t^{1-2p_\alpha} - \frac{2p_\alpha}{t}\right), \\ h_{\alpha\beta} &= t^{2p_\alpha}k_\beta C_\alpha + t^{2p_\beta}k_\alpha C_\beta + C_0\frac{2k_\alpha k_\beta(1-p_\alpha-p_\beta)}{(1-2p_\alpha)(1-2p_\beta)}\,t, \quad \alpha \neq \beta, \end{aligned}\right\} \tag{E.3}$$

where C_α and C_0 are constants (we leave out the exponent $\exp(i\mathbf{k}\cdot\mathbf{r})$ from now on, for the sake of brevity).

The general solution of (E.2) can be represented in the form of a series of increasing powers of t. The first terms of these expansions are as follows:

$$\left.\begin{aligned} h_\alpha^\alpha &= A_\alpha + B_\alpha \ln t, \\ h_{12} &= t^{2p_1}\left[C_{12} + \frac{k_3(k_2C_{13} - k_3C_{12})}{8p_1(p_2+p_3)}\,t^{2(p_1+p_2)}\right] + t^{2p_2}C_{21}, \\ h_{13} &= t^{2p_1}\left[C_{13} + \frac{k_2(k_3C_{12} - k_2C_{13})}{8p_1(p_2+p_3)}\,t^{2(p_1+p_3)}\right] + t^{2p_3}C_{31}, \\ h_{23} &= t^{2p_2}C_{23} + t^{2p_3}C_{32}, \end{aligned}\right\} \tag{E.4}$$

With the constants A_α, B_α, $C_{\alpha\beta}$ ($C_{\alpha\beta} \neq C_{\beta\alpha}$) connected by the relations

$$\sum_\alpha B_\alpha = \sum p_\alpha B_\alpha = 0, \tag{E.5}$$

$$k_\alpha\left[B_\alpha + \sum_\beta(p_\alpha - p_\beta)\,A_\beta\right] + 2\sum_\beta k_\beta C_{\beta\alpha}(p_\beta - p_\alpha) = 0. \tag{E.6}$$

We have already left out of (E.4) the terms that can be excluded by transforming the coordinates and correspond to the coefficient C_0 in (E.3). All the remaining omitted expansion terms are those known not to become large as $t \to 0$, and the coefficients in them are expressed in terms of the constants contained in (E.4). The criteria for the smallness of the perturbations are the conditions

$$h_\alpha^\alpha \ll 1, \qquad h_{\alpha\beta} \ll \sqrt{g_{\alpha\alpha}g_{\beta\beta}}. \tag{E.7}$$

We shall arrange the three arbitrary constants C_α in (E.3) in a way as to exclude where possible the largest terms in (E.4). Namely, we put

$$C_{12} = C_{31} = C_{23} = 0. \tag{E.8}$$

so that h_{23} satisfies the condition (E.7), but h_{12} and h_{13} still contain terms not satisfying this condition as $t \to 0$. In other words, these perturbations experience a relative increase, that is, the solution (E.1) is unstable with respect to them. To eliminate this instability it is sufficient to put in addition

$$k_3C_{12} - k_2C_{13} = 0, \tag{E.9}$$

after which the coordinate transformation which causes C_{12} to vanish will cause C_{13} to vanish, too. The increase of the logarithmic terms in the diagonal component h_α^α as $t \to 0$ is only an apparent instability. These terms correspond actually to merely a small change in the exponents in the metric (E.1): the numbers p_α are replaced by $p_\alpha + B_\alpha$, and the previous relations between them remain in force by virtue of the conditions (E.5).

The arbitrary constant in the Fourier component of the perturbation denotes the presence of an arbitrary function (of three space coordinates) in the perturbation itself. On the other hand, the presence of arbitrary functions in the perturbations, which do not lead to instability of the main solution, denotes the possibility of expanding the latter. Altogether (E.4) contains three independent arbitrary parameters (the 12 parameters $C_{\alpha\beta}$, A_α, and B_α are connected by the nine conditions (E.5)—(E.9)). They correspond to the three arbitrary functions in the anisotropic solution (3.6).

We see that in order to ensure stability of the metric (E.1) it becomes necessary to impose on the arbitrary perturbation one additional condition (E.9). This condition corresponds precisely to the additional condition $\mathbf{l}\,\mathrm{curl}\,\mathbf{l} = 0$ (3.20), which brought about the "loss" of one arbitrary function in the anisotropic solution.

F. ORIGIN OF OTHER TYPES OF SINGULARITIES

In Sec. 2 we described a geometrical procedure for the construction of a synchronous reference system. This construction begins with an arbitrary spacelike hypersurface, chosen as the initial hypersurface.

On the other hand, if we choose as the initial hypersurface the "null" hypersurface (that is, the hypersurface the normals to which are null vectors), then we can obtain by the same construction a reference system in which the metric has the following form (see [15], Sec. 7)*:

$$-ds^2 = 2d\eta\, d\zeta + g_{ab}\, dx^a\, dx^b + 2g_{a3}\, dx^a\, d\zeta + g_{33}\, d\zeta^2, \tag{F.1}$$

that is, $g_{00} = g_{0\alpha} = 0$, $g_{03} = 1$ (the indices a and b run through the values 1 and 2, while the indices 0, 1, 2, and 3 correspond to the four coordinates η, x, y, ζ).

The solution (B.12) indicated in Appendix B pertains precisely to such a reference system. The remark made at the end of that appendix suggests that if the anisotropic solution obtained in Sec. 3 is transformed to the synchronous reference frame, some particular cases may drop out of the solution formulated in the form of the metric of the type (E.1). Let us show briefly how this solution is constructed (in vacuum).

We seek the components of the metric tensor near the singular point $\eta = 0$ in the form

$$g_{ab} = \eta^{2s_1} l_a l_b + \eta^{2s_2} m_a m_b, \quad g_{a3} = \eta^{2s_3} n_a, \quad g_{33} = \eta^{2s_3} g, \tag{F.2}$$

with

$$s_1^2 + s_2^2 = s_1 + s_2, \qquad s_3 = \frac{1}{2}(1 - s_1 - s_2). \tag{F.3}$$

The two-dimensional vectors l_a, m_a, n_a, the scalar q, and the numbers s_1, s_2, s_3 are all functions of the coordinates x, y, ζ. The components of the inverse tensor are

$$\left.\begin{array}{l} g^{ab} = \eta^{-2s_1} l^a l^b + \eta^{-2s_2} m^a m^b, \quad g^{a0} = -g_{b3} g^{ab}, \quad g^{a3} = 0, \\ g^{00} = -g_{33} + g^{ab} g_{a3} g_{b3}, \quad g^{03} = 1, \quad g^{33} = 0; \end{array}\right\} \tag{F.4}$$

here l^a, m^a — components of the two-dimensional vectors connected with l_a, m_a by the relations $l_a l^a = m_a m^a = 1$, $l_a m^a = 0$. The metric determinant is

$$-g = |g_{ab}| = \eta^{2(s_1+s_2)} (l_1 m_2 - l_2 m_1)^2. \tag{F.5}$$

Let us agree that $s_2 > s_1$. The relative magnitude of the numbers s_3 and s_1 or s_2 is not defined. We assume first that $s_3 > s_2$ (it is easy to see that in this case $1/5 < s_1 < 0$, $0 < s_2 < 2/5$, and $2/5 < s_3 < 1/2$). In estimating the different terms in the gravitational equations in this case, it is important that the expression

$$g_{a3} g_{b3} g^{ab} = \eta^{4s_3} n_a n_b (\eta^{-2s_1} l^a l^b + \eta^{-2s_2} m^a m^b) \tag{F.6}$$

contains higher powers of η than $g_{33} \sim \eta^{2s_3}$.

Let us show how it is possible to satisfy a metric of the type (F.2) by the principal terms of the gravitational equations. These terms are

$$R_0^3 = -\frac{1}{2} \varkappa^a_{a,0} - \frac{1}{4} \varkappa^b_a \varkappa^a_b = 0, \tag{F.7}$$

$$R_a^3 = \frac{1}{2\sqrt{-g}} [\sqrt{-g}\,(g_{a3,0} - g_{b3}\varkappa^b_a)]_{,0} = 0, \tag{F.8}$$

$$R_a^b = \frac{1}{2\sqrt{-g}} [\sqrt{-g}\, g_{33} \varkappa^b_a]_{,0} - \frac{1}{2\sqrt{-g}} [\lambda^b_a \sqrt{-g}]_{,0} = 0, \tag{F.9}$$

$$R_3^3 = \frac{1}{2\sqrt{-g}} [\sqrt{-g}\, g_{33,0}]_{,0} - \frac{1}{4} \varkappa^b_a \lambda^a_b = 0, \tag{F.10}$$

$$R_a^0 = -\frac{1}{2\sqrt{-g}} [(g_{a3,0} - g_{b3}\varkappa^b_a)\sqrt{-g}]_{,3} - \frac{1}{2} (g_{33}\varkappa^b_a)_{;b} + \frac{1}{2\sqrt{-g}} [g_{33,a}\sqrt{-g}]_{,0} + \frac{1}{2} g_{33}\varkappa^b_{b;a} = 0, \tag{F.11}$$

$$R_3^0 = \frac{1}{2\sqrt{-g}} [(g_{33,3}\sqrt{-g})_{,0} - (g_{33,0}\sqrt{-g})_{,3}] + \frac{1}{4} g_{33}\varkappa^b_a \lambda^a_b + \frac{1}{2} g_{33}\varkappa^b_{b,3} = 0. \tag{F.12}$$

The indices ,0 and ,3 denote here simple differentiations with respect to η and ζ, respectively, while the indices ;a denote covariant differentiation in two-dimensional space with metric g_{ab}. κ_{ab} and λ_{ab} denote the two-dimensional tensors

$$\varkappa_{ab} = g_{ab,0}, \quad \lambda_{ab} = g_{ab,3}, \quad \varkappa^b_a = \varkappa_{ac} g^{bc}, \quad \lambda^b_a = \lambda_{ac} g^{bc}.$$

Equations (F.7) and (F.8) are satisfied by the metric (F.2) identically. On the other hand, only the first terms of Eqs. (E.9) and (E.10) vanish. Yet the second terms are potentially principal ones; they contain a power $(1/\eta)^{1+2s_2-2s_1}$ higher than $(1/\eta)^{2-2s_3}$, to which the first terms are formally proportional. Therefore to satisfy these equations it is necessary to impose on the metric an additional condition, which forbids the

*In Petrov's book[15] this system is called isotropic semigeodesic, to distinguish it from the synchronous system, which is called merely semigeodesic.

appearance of such large terms. It is easy to see that such a condition is

$$l_{a,\,3}m^a = 0 \tag{F.13}$$

(which causes the terms $\sim \eta^{2(s_1 - s_2)}$ to vanish in the quantities λ_a^b). Finally, substitution of the metric (F.2) with the condition (F.13) in (F.11)—(F.12) leads to the appearance of terms of order $\eta^{2s_3 - 1} \ln \eta$ and η^{2s_3-1}. The first of them cancel identically, by virtue of the relations (F.3) between the numbers s_1, s_2, and s_3. On the other hand, the terms $\sim \eta^{2s_3-1}$ in these equations yield three relations (equal to the number of equations), which connect the functions of the coordinates x, y, and ζ contained in (F.2).

We therefore have, together with (F.13), four relations between the eight functions (q, two components each of the vectors l_a, m_a, n_a, and one of the numbers s_1, s_2, or s_3). In addition, the metric (F.1)—(F.2) admits of one more transformation (which contains one arbitrary function of the coordinates x, y, z) leaving its form invariant; it is implied here that the permissible transformation should retain the situation wherein the singularity of the metrics is situated at $\eta = 0$, and g_{33} contains a power higher than in g_{ab}. This transformation, for example, can be used to turn the coefficient q in g_{33} equal to unity. Thus, the metric (F.1)—(F.2) contains only three physically independent functions of the coordinates x, y, ζ.

Investigation of the cases when s_3 is not the largest of the three numbers s_1, s_2, and s_3 reduces to that made above. Let $s_1 < s_3 < s_2$. The metric (F.1)—(F.2) admits in this case also of one arbitrary transformation. This transformation can no longer make q equal to unity, but can cause the vector m_a [which is the coefficient of the highest power of η in (F.2)] to become "perpendicular" to the vector n^a, that is, to make $n_a m^a = 0$. Then expression (F.6) will again be small compared with g_{33}, and the principal terms in the gravitational equations remain the same as in (F.7)—(F.12).

The obtained solution is in general equivalent to the *anisotropic solution* (3.6), into which it can be transformed by changing over to the synchronous reference system. The exponents p_1, p_2, and p_3 are then connected with the exponents s_1, s_2, s_3 by (B.15), and the "superfluous" condition (F.13) corresponds to the additional condition $\mathbf{l}\,\mathrm{curl}\,\mathbf{l} = 0$ (3.20), which must be superimposed on the coefficients of the solution (3.6).

However, a search for the solutions in the reference system of (F.1) leads in natural fashion to solutions with singularities also of a type not contained in the solution (3.6). This type occurs in a special case when the coefficient in (F.2) is $q = 0$, so that the solution (near the singularity) has the form

$$-ds^2 = 2d\eta\, d\zeta + (l_a l_b \eta^{2s_1} + m_a m_b \eta^{2s_2})\, dx^a\, dx^b + 2n_a t^{2s_3}\, dx^a\, d\zeta, \tag{F.14}$$

with $s_1 + s_2 = s_1^2 + s_2^2$. In the synchronous reference system this solution would be characterized by exponents $(s_1, s_2, 1)$ for the variable t, not contained in the set of numbers (p_1, p_2, p_3); such a representation of this solution seems to be, however, less natural for the investigation of its properties than the representation in the form (F.14) [see (B.16) and (B.17)].

A solution of the type (F.14) contains apparently less than three physically arbitrary functions of the three variables x, y, and ζ. An establishment of this number and a clarification of the limitations that must be imposed on the quantities contained in (F.14) necessitate, however, a special investigation with account of the terms of the next higher orders [beyond those written out in (F.7) and (F.12)] in the gravitational equations, and possibly the next higher terms [following (F.14)] in the expansion of the components of the metric tensor.

G. EXAMPLES OF SINGULARITIES IN EXACT SOLUTIONS

We present several examples from among the known exact solutions of the gravitational equations in vacuum, demonstrating by means of these examples singularities of different types.

1. The metric

$$-ds^2 = -z^{\frac{12}{7}}\, dt^2 + t^{-\frac{2}{3}} z^{-\frac{4}{7}} dx^2 + t^{\frac{4}{3}} z^{\frac{6}{7}}\, dy^2 + t^{\frac{4}{3}}\, dz^2 \tag{G.1}$$

is obtained by obvious transformations from one of the exact solutions obtained by Harrison[16] (solution I-A-1 in his notation).

Transformation to the synchronous reference system near the singular point t = 0 is conveniently carried out by the following iteration method. By means of the substitution $\sqrt{-g_{00}(x^\alpha)}\,dt \to dt$ we make the new $-g_{00}$ equal to unity, but we obtain instead nonvanishing components $g_{0\alpha}$ in the form $g_{0\alpha} = tf_\alpha(x^1, x^2, x^3)$. They are eliminated by the transformation $x^\alpha \to x^\alpha + t^{2-p\alpha}\varphi^\alpha(x^1, x^2, x^3)$ and by suitable choice of the functions φ^α ($t^{2p\alpha}$ — time factor contained in $g_{\alpha\alpha}$). At the same time, a small ($\sim t^{2-2p\alpha}$) addition to g_{00} appears, and is eliminated by the next transformation, etc; on going over to higher-order terms, the form of the transformations becomes naturally more complicated. As a result, we can shift the deviations from synchronism to small quantities of arbitrarily high order; the components $g_{\alpha\beta}$ are obtained in this case as expansions of t.*

We thus find that near the singular point t = 0 the metric (G.1) is equivalent to a metric whose first expansion terms are

$$-ds^2 \approx -dt^2 + t^{-\frac{2}{3}}\, dx^2 + t^{\frac{4}{3}}\, z^{-\frac{1}{2}}\, (dy^2 + dz^2), \tag{G.1a}$$

that is, we have a singularity of the type

*An exact transformation to the synchronous reference system (which can be carried out, for example, by the method indicated in [1], Sec. 98a), usually entails very cumbersome calculations.

$$(p_1, p_2, p_3)=\left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right).$$

Changing the time variable in (G.1) by means of a complex transformation, we obtain the metric

$$-ds^2=-z^{\frac{4}{3}}dt^2+t^{-\frac{4}{7}}z^{-\frac{2}{3}}dx^2+t^{\frac{6}{7}}z^{\frac{4}{3}}dy^2+t^{\frac{12}{7}}dz^2. \qquad (G.2)$$

Near $t = 0$ it is equivalent to the metric

$$-ds^2\approx -dt^2+t^{-\frac{4}{7}}z^{-\frac{2}{7}}dx^2+t^{\frac{6}{7}}z^{\frac{16}{21}}dy^2+t^{\frac{12}{7}}z^{-\frac{24}{21}}dz^2, \qquad (G.2a)$$

That is, we have a singularity of the type $(-2/7, 3/7, 6/7)$.

2. The metric

$$-ds^2=\xi^{\frac{4}{3}}du\,d\zeta+\xi^{-\frac{2}{3}}u^{-\frac{2}{5}}dx^2+u^{\frac{6}{5}}d\xi^2 \qquad (G.3)$$

(Harrison's solution III-2) is transformed by means of the substitution $u = \eta y^{-20/3}$, $\xi = y^5$ (with some change in the scale of the coordinates) into the form

$$-ds^2=2d\eta\,d\zeta+\eta^{-\frac{2}{5}}y^{-\frac{2}{3}}dx^2+\eta^{\frac{6}{5}}dy^2-\frac{40}{3}\frac{\eta}{y}dy\,d\zeta, \qquad (G.3a)$$

that is, we have a solution of the type (F.14) with numbers $(s_1, s_2) = (-1/5, 3/5)$.

3. The metric

$$-ds^2=-(x-t)^{1+\sqrt{2}}t^{7-5\sqrt{2}}z^2dt^2+(x-t)^{1+\sqrt{2}}t^{3-\sqrt{2}}z^{\sqrt{2}}dx^2$$
$$+(x-t)^{-\sqrt{2}}t^{-2+\sqrt{2}}z^{-\frac{1}{\sqrt{2}}}dy^2+\frac{1}{8}(x-t)^{2+\sqrt{2}}t^{8-5\sqrt{2}}dz^2 \qquad (G.4)$$

(Harrison's solution I-B-3) has singular points at $t = 0$ and at $t = x$.

Near the point $t \to 0$ we reduce g_0 by means of the substitution $t^{(9-5\sqrt{2})/2} \to t$ to a form that depends (in first approximation) only on the spatial coordinates, after which we proceed as in Example 1. Without writing out the form of the metric near the singularity, we point out merely that the latter belongs to the type

$$(p_1, p_2, p_3)=\left(-\frac{2-\sqrt{2}}{9-5\sqrt{2}}, \frac{8-5\sqrt{2}}{9-5\sqrt{2}}, \frac{3-\sqrt{2}}{9-5\sqrt{2}}\right).$$

Near the singularity $t = x$ the reduction of the metric to the standard form is by means of the following succession of transformations. We substitute $t - x \to t$, and then eliminate t from g_{00} by means of a transformation of the type $t^s \to t$. This gives rise to a non-vanishing g_{01}, which is eliminated by transformation of the type $x \to x + \varphi(x, z, t)$ with a suitably chosen function φ; continuing further as in Example 1, we obtain as a result a metric with singularity of the type

$$(p_1, p_2, p_3)=\left(-\frac{\sqrt{2}}{3+\sqrt{2}}, \frac{1+\sqrt{2}}{3+\sqrt{2}}, \frac{2+\sqrt{2}}{3+\sqrt{2}}\right).$$

With the aid of a complex transformation we can obtain from (G.4) the solution

$$-ds^2=-\frac{1}{8}(z-y)^{2+\sqrt{2}}z^{8-5\sqrt{2}}dt^2+(z-y)^{-\sqrt{2}}z^{-2+\sqrt{2}}t^{-\frac{1}{\sqrt{2}}}dx^2$$
$$+(z-y)^{1+\sqrt{2}}z^{3-\sqrt{2}}t^{\sqrt{2}}dy^2+(z-y)^{1+\sqrt{2}}z^{7-5\sqrt{2}}t^2dz^2. \qquad (G.5)$$

To ascertain the type of singularity possessed by this metric at $t = 0$, we assume to simplify the calculations, that $z \gg y$; then

$$-ds^2\approx t^2z^{10-4\sqrt{2}}\left(\frac{dz^2}{z^2}-\frac{dt^2}{8t^2}\right)+z^{-2}t^{-\frac{1}{\sqrt{2}}}dx^2+z^4t^{\sqrt{2}}dy^2.$$

We make first the substitution

$$u=zt^{-\frac{1}{2\sqrt{2}}}, \quad v=zt^{\frac{1}{2\sqrt{2}}},$$

and then $v^5 = \eta$, $u^{5-4\sqrt{2}} = \zeta$, and some change in the coordinate scale. Ultimately we get

$$-ds^2=2d\eta\,d\zeta+\eta^{-\frac{2}{5}}dx^2+\eta^{\frac{4}{5}}dy^2, \qquad (G.5a)$$

that is, a singularity of the type (F.14), with $(s_1, s_2) = (-1/5, 2/5)$.

4. The solution

$$-ds^2=-\gamma_1^2\gamma_2\,dt^2+\gamma_1\,dx^2+(\gamma_1\sin^2 x+\gamma_2\cos^2 x)\,dy^2$$
$$+2\gamma_2\cos x\,dy\,dz+\gamma_2\,dz^2, \quad \gamma_1=\operatorname{ch} t/4\operatorname{ch}^2\frac{t}{2}, \quad \gamma_2=\frac{1}{\operatorname{ch} t} \qquad (G.6)$$

(obtained by Taub[17]) has a singularity at $t \to \infty$. Near the singularity this metric can be reduced by means of the substitution $e^{-t/2} \to t$ to the form

$$-ds^2\cong -dt^2+\frac{1}{2}dx^2+\frac{1}{2}\sin^2 x\,dy^2+2t^2(dz^2+2\cos x\,dy\,dz), \qquad (G.6a)$$

that is, a singularity of the type $(p_1, p_2, p_3) = (0, 0, 1)$. But this type of singularity is fictitious, so that the metric (G.6) actually has no physical singularity.

I. EQUATIONS OF SMALL PERTURBATIONS OF THE GRAVITATIONAL FIELD

Let the metric $g_{ik}^{(0)}$ represent some solution of the gravitational equations, on which a small perturbation δg_{ik} is imposed. Let us calculate the quantities necessary to set up the equations for these perturbations.

We introduce the notation $\delta g_{ik} = h_{ik}$ for the perturbation of the covariant components of the metric tensor, and to simplify the formulas we denote the unperturbed metric simply by g_{ik}, leaving out the index (0).

The tensor h_{ik} will be regarded below as a tensor in the space of the unperturbed metric g_{ik}, so that all further operations of raising the indices of h_{ik}, and also all the operations of covariant differentiation, are carried out with the aid of the metric g_{ik}. Then, accurate to small quantities of first order, $\delta g_{ik} = -h^{ik}$. Thus, we should make in the gravitational equations the substitution

$$g_{ik}\to g_{ik}+h_{ik}, \quad g^{ik}\to g^{ik}-h^{ik}. \qquad (I.1)$$

The change in the determinant is $\delta g = gg^{ik}h_{ik} = gh$, where $h \equiv h_i^i$, so that

$$g\to g(1+h). \qquad (I.2)$$

The corrections to the Christoffel symbols are expressed in terms of h^{ik} by means of

$$\delta\Gamma_{kl}^i=\frac{1}{2}\left(h_{k;l}^i+h_{l;k}^i-h_{kl}^{;i}\right), \qquad (I.3)$$

as can be verified directly. With their aid we can obtain for the perturbation of the curvature tensor

$$\delta R^{i}_{klm} = \frac{1}{2}\left(h^{i}_{k;m;l} + h^{i}_{m;k;l} - h_{km};{}^{i}{}_{;l} - h^{i}_{k;l;m} - h^{i}_{l;k;m} + h_{kl};{}^{i}{}_{;m}\right), \tag{I.4}$$

from which we get for the corrections to the Ricci tensor

$$\delta R_{ik} = \delta R^{l}_{ilk} = \frac{1}{2}\left(h^{l}_{i;k;l} + h^{l}_{k;i;l} - h_{ik};{}^{l}{}_{l} - h_{;i;k}\right). \tag{I.5}$$

From the relation

$$R^{k}_{i} + \delta R^{k}_{i} = (R_{il} + \delta R_{il})\,(g^{kl} + \delta g^{kl})$$

we obtain for the change in the mixed components R^k_i

$$\delta R^{k}_{i} = g^{kl}\delta R_{il} - h^{kl}R_{il}. \tag{I.6}$$

If the unperturbed metric is specified in the synchronous reference frame and the perturbation does not violate the synchronism (this can always be attained by means of suitable small transformation of the coordinates), then

$$h_{00} = 0, \quad h_{0\alpha} = 0. \tag{I.7}$$

The changes δR^k_i are best calculated in this case by varying the quantities in (2.3)—(2.5), using at the same time formulas (I.5)—(I.6) for the determination in the change of $\delta P^{\beta}_{\alpha}$. Obviously, the change in the three-dimensional Ricci tensor P^{β}_{α} is determined by formulas of the same type as for the four-dimensional tensor R^k_i, and all the tensor operations are carried out in three-dimensional space with unperturbed metric $g_{\alpha\beta}$:

$$\delta P^{\beta}_{\alpha} = \frac{1}{2}\left(h^{\gamma}_{\alpha}{}^{;\beta}{}_{;\gamma} + h^{\beta}_{\gamma}{}_{;\alpha}{}^{;\gamma} - h^{\beta;\gamma}_{\alpha;\gamma} - h_{;\alpha}^{;\beta}\right) - h^{\beta}_{\gamma}P^{\gamma}_{\alpha}. \tag{I.8}$$

For the change in the tensor $\kappa_{\alpha\beta}$ we have

$$\delta\varkappa_{\alpha\beta} = \dot{h}_{\alpha\beta}, \quad \delta\varkappa^{\beta}_{\alpha} = \dot{h}^{\beta}_{\alpha} - \varkappa^{\gamma}_{\alpha}h^{\beta}_{\gamma} + \varkappa^{\beta}_{\gamma}h^{\gamma}_{\alpha}, \tag{I.9}$$

where the dot denotes differentiation with respect to t (this operation, of course, does not commute with the operations of raising or lowering the indices).

The final formulas for the changes δR^k_i are of the form

$$\delta R^{0}_{0} = \frac{1}{2}\left(\ddot{h} + \varkappa^{\beta}_{\alpha}\dot{h}^{\alpha}_{\beta}\right), \tag{I.10}$$

$$\delta R^{0}_{\alpha} = \frac{1}{2}\dot{h}_{;\alpha} - \frac{1}{2}\left(\dot{h}^{\beta}_{\alpha} + h^{\gamma}_{\alpha}\varkappa^{\beta}_{\gamma} - h^{\beta}_{\gamma}\varkappa^{\gamma}_{\alpha}\right)_{;\beta} + \frac{1}{4}\left(\varkappa^{\beta}_{\gamma}h^{\gamma}_{\beta;\alpha} - \varkappa^{\beta}_{\alpha}h_{;\beta}\right); \tag{I.11}$$

$$\delta R^{\beta}_{\alpha} = \delta P^{\beta}_{\alpha} + \frac{1}{2}\Big\{\ddot{h}^{\beta}_{\alpha} + \frac{1}{2}\varkappa^{\beta}_{\alpha}\dot{h} - \varkappa^{\gamma}_{\alpha}\dot{h}^{\beta}_{\gamma} + \varkappa^{\beta}_{\gamma}\dot{h}^{\gamma}_{\alpha} + \dot{\varkappa}^{\beta}_{\gamma}h^{\gamma}_{\alpha} - \dot{\varkappa}^{\gamma}_{\alpha}h^{\beta}_{\gamma} + \frac{1}{2}\varkappa\left(\dot{h}^{\beta}_{\alpha} - \varkappa^{\gamma}_{\alpha}h^{\beta}_{\gamma} + \varkappa^{\beta}_{\gamma}h^{\gamma}_{\alpha}\right)\Big\}. \tag{I.12}$$

In solving the small-perturbation equations it is always necessary to bear in mind that the obtained solutions contain some that can be eliminated by transforming the reference system and therefore represent no real physical change in the metric. The point is that the conditions (I.7) still do not determine the choice of the reference system uniquely. Indeed, under the transformation $x^i \rightarrow x^i + \xi^i$ (where ξ^i are small quantities), the tensor g_{ik} receives an increment $h_{ik} = \xi_{i;k} + \xi_{h;i}$ or, expanding the covariant derivatives,

$$h_{00} = 2\dot{\xi}_0, \quad h_{0\alpha} = \frac{\partial\xi_0}{\partial x^{\alpha}} + \dot{\xi}_{\alpha} - \varkappa^{\beta}_{\alpha}\xi_{\beta} = -\frac{\partial\xi^0}{\partial x^{\alpha}} + \dot{\xi}^{\beta}g_{\alpha\beta},$$
$$h_{\alpha\beta} = \xi_{\alpha;\beta} + \xi_{\beta;\alpha} - \varkappa_{\alpha\beta}\xi_0. \tag{I.13}$$

Conditions (I.7) give a sum of four equations for the permissible values of ξ_0 and ξ_{α}. The general solution of these equations is

$$\xi^0 = f^0(x^1, x^2, x^3), \quad \xi^{\alpha} = \frac{\partial f^0}{\partial x^{\beta}}\int g^{\alpha\beta}\,dt + f^{\alpha}(x^1, x^2, x^3). \tag{I.14}$$

It contains, as expected, four arbitrary (small) functions of the spatial coordinates f^0 and f^{α} (see footnote on page 498).

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