THE RESEARCHES OF A. A. FRIDMAN ON THE EINSTEIN THEORY OF GRAVITATION

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AMONG the scientific works of A. A. Fridman, his researches on the Einstein theory of gravitation are in number only a small fraction (less than a tenth) of all his published papers, but in the influence they have had on the development of science they can scarcely be denied first place.

Aleksandr Aleksandrovich Fridman and Vsevolod Konstantinovich Frederiks, at the time professors at Petrograd (now Leningrad) University, were the first to make physicists working in Petrograd acquainted with the theory of gravitation which Einstein had recently created. This was at the very beginning of the twenties, when the blockade of Soviet Russia had just been broken and scientific literature began to come in from abroad. A seminar was formed in the Physical Institute of the University, and Einstein's theory was among the subjects on which reports were given. The members of the seminar were the professors and the students in the advanced program (of whom there were then only a few). The main reports on the theory of relativity were given by V. K. Frederiks* and A. A. Fridman, but Yu. A. Krutkov, V. R. Bursian, and others spoke occasionally. I have a lively recollection of the reports by Frederiks and Fridman. They were in different styles: Frederiks had a deep understanding of the physical side of the theory,† but did not like mathematical calculations, while Fridman put the emphasis not on the physics, but on the mathematics. He strove for mathematical rigor and gave great importance to the complete and precise formulation of the initial hypotheses. The discussions that arose between Frederiks and Fridman were very interesting.

A. A. Fridman wrote his two main papers on relativity theory in 1922 and 1923; they were published in German in Vols. 10 and 21 of Zeitschrift für Physik in 1922 and 1924. Fridman also managed to publish the first paper, "On the Curvature of Space," in Russian, in Journal of the Russian Physico-chemical Society (Vol. 56, in 1924), but the second, "On the Possibility of a World with a Constant Negative Curvature of Space," was not published in Russian. I also remember these papers very well because I translated the second one (and perhaps both of them) into German for A. A. Fridman, and called his attention to one case of a space of negative curvature which he had not analyzed.

Let us briefly consider the contents of these papers. Fridman's initial assumptions are as follows. One introduces a spatial coordinate system, relative to which the matter is assumed to be stationary (comoving coordinates). It is assumed that the time is orthogonal to this space. It is assumed that the space itself has constant curvature (positive or negative) which depends only on the time.* With these assumptions the square of the elementary interval can be written in the form

$$d\tau^2 = -\frac{R^2}{c^2}\,d\sigma^2 + M^2\,dt^2, \tag{1}$$

where $d\sigma^2$ is the square of the line element on the sphere or pseudosphere of unit radius (on the sphere for positive curvature and on the Lobachevskiĭ pseudosphere for negative curvature). The choice of coordinates on the sphere or pseudosphere is unimportant; Fridman assumes for the case of the sphere

$$d\sigma^2 = dx_1^2 + \sin^2 x_1\,dx_2^2 + \sin^2 x_1 \sin^2 x_2\,dx_3^2 \tag{2}$$

and for the case of the pseudosphere

$$d\sigma^2 = \frac{1}{x_3^2}\,(dx_1^2 + dx_2^2 + dx_3^2). \tag{3}$$

Fridman writes the Einstein equations of gravitation in the form

$$R_{ik} - \frac{1}{2}\,g_{ik}\overline{R} + \lambda g_{ik} = -\varkappa T_{ik}. \tag{4}$$

Thus he uses equations containing the cosmological constant λ. Concerning this we must make the following remark. Einstein originally introduced the constant λ into the equations of gravitation in order to get a "stationary" solution (R = const) with a nonvanishing matter density (it is for this reason that λ was called the "cosmological" constant). Later, after Fridman's work, Einstein renounced the term with λ, and for a long time it was generally supposed that this

*The article published by V. K. Frederiks in Uspekhi Fizicheskikh Nauk [Vol. 2, p. 162 (1922)] contains the first exposition of the general theory of relativity in the Russian language, as given in reports presented by the author in Moscow and Petrograd at the beginning of the twenties (Editor's note).

†I remember that to my question "How is the law of motion of many bodies formulated in the general theory of relativity?" Fridman at once replied: "The motion of the bodies is determined by the motion of the singularities of the metric tensor." This was many years before the equations of motion of a system of bodies were derived on this basis (Einstein's papers and my papers of 1938-1939). I also recall that before the new quantum mechanics and the Heisenberg inequalities had appeared Frederiks spoke to me about the necessity of dealing with the problem of observation in atomic physics, and that he was the first to appreciate de Broglie's work of 1923-1924 on matter waves.

*The case of positive curvature is considered in the first paper, and that of negative curvature in the second.

term is not needed and even that the equations (4) with $\lambda \neq 0$ are incorrect. Actually the term λg_{ik} must always be treated along with the term κT_{ik}, and it must be kept in mind that the mass tensor T_{ik} is not uniquely determined from the local conditions, but only up to a term* proportional to g_{ik}. The "cosmological" term λg_{ik} precisely compensates for the ambiguity in the definition of the tensor T_{ik}. Further considerations are necessary for the unique determination of the tensor T_{ik} and the constant λ. In problems of the usual astronomical type we can require that at infinity (i.e., at a sufficiently large distance from the masses under consideration) the space be Euclidean and that the tensor T_{ik} be zero there. Then it is necessary that $\lambda = 0$. In the cosmological problem, on the other hand, one cannot impose conditions at infinity, and any particular choice of λ is a special cosmological hypothesis. Therefore, contrary to a general impression, Fridman's use of the equations (4) is entirely justified.

Solving the equations (4) under the assumptions he made, Fridman studied both the stationary case $R = \text{const}$ and the nonstationary case $R = R(t)$. In the stationary case the expression (2) (positive curvature) leads to somewhat generalized Einstein and de Sitter solutions, which for $\lambda > 0$ correspond to positive density; on the other hand the expression (3) (negative curvature) leads to new solutions, but these correspond to zero or negative density.

Fridman's most remarkable results are those for the nonstationary case. In this case $M = M(t)$, and by using the equation $dt' = M(t)\,dt$ to introduce a new variable instead of t we can without loss of generality reduce the problem to the case $M = 1$. Then for positive curvature Eq. (2) the formula for $R(t)$ is found to be

$$\frac{R\dot{R}^2}{c^2} + R - \frac{\lambda}{3c^2} R^3 = A \tag{5}$$

and for negative curvature [Eq. (3)] it is

$$\frac{R\dot{R}^2}{c^2} - R - \frac{\lambda}{3c^2} R^3 = A, \tag{6}$$

where A is a constant connected with the density ρ by the relation

$$\varkappa \varrho R^3 = 3A. \tag{7}$$

When the constant A is positive we get a solution with positive density.

Thus it is proved that nonstationary solutions are possible, which correspond to a constant (either positive or negative) curvature of the "co-moving" space. This is the main result of Fridman's papers.

It is interesting to note how Einstein reacted to this, to him unexpected, result. Soon after the publication of Fridman's first paper there appeared a note by Einstein, in which he says, somewhat loftily, that Fridman's results seem dubious to him and that he has found a mistake in them, such that when it is corrected Fridman's solution reduces to a stationary solution. At that time (1923) Yu. A. Krutkov was on a trip abroad, and at Fridman's request he saw Einstein in Berlin and with great difficulty (as he told me) convinced him that he was wrong. As a result of the discussions between Krutkov and Einstein there soon appeared a second note by Einstein, in which he fully acknowledges his error and assigns a high value to Fridman's results. This readiness on the part of Einstein—a great scientist, then at the height of his fame—to acknowledge his mistake deserves to be noted.

In conclusion it must be stated that Fridman's results—the proof that a nonstationary (and in particular an expanding) universe is possible—are much more important than he himself supposed. Fridman more than once said that his task was to indicate the possible solutions of Einstein's equations, and that then the physicists could do what they wished with these solutions. Subsequently, however, the solutions that Fridman found were applied in astronomy. This was after the untimely death of Fridman, on the discovery (by Hubble and others) of the red shift in the spectra of distant galaxies, which was interpreted as a Doppler effect from the expansion of the universe. Thus Fridman's work opened the way to the further development of the science of the universe as not only a theoretical but also an observational science, and this is the undiminishing importance of his work.

*Physically this corresponds to an arbitrary constant in the expression for the pressure (the "pressure at infinity").

Translated by W. H. Furry
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