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METHOD OF GEOMETRICAL OPTICS IN THE ELECTRODYNAMICS OF AN INHOMOGENEOUS PLASMA

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THE electrodynamics of media with spatial dispersion has been developing quite vigorously in recent years, especially as a result of theoretical investigations of plasma properties. Most hitherto considered problems in the electrodynamics of media with spatial dispersion were devoted to electromagnetic waves in unbounded homogeneous media or to effects connected with the presence of a distinct surface which also bounds a homogeneous medium [1]. Very recently many papers appeared devoted to the theory of the electromagnetic properties of a weakly inhomogeneous plasma. The progress attained in the development of such a theory is connected primarily with the application of the method of geometrical optics to electrodynamics of media with spatial dispersion. In the present review we present the principles of the method of geometrical optics as applied to media with spatial dispersion. By considering a specific problem involving the oscillations of a weakly inhomogeneous plasma confined by a strong magnetic field, we shall present several results which characterize the spectra of the natural oscillations of the plasma. Such spectra are described by the Bohr and Sommerfeld phase integrals ("quasiclassical quantization rules") [2]. With the aid of such "quasiclassical quantization rules" we analyze the spectra of the oscillations and obtain the conditions for the instability of a weakly inhomogeneous plasma.

1. METHOD OF GEOMETRICAL OPTICS IN THE ELECTRODYNAMICS OF MEDIA WITH SPATIAL DISPERSION AND THE DIELECTRIC TENSOR OF A WEAKLY INHOMOGENEOUS PLASMA CONFINED BY A STRONG MAGNETIC FIELD

As is well known, a plasma is a medium whose electromagnetic properties are connected in many respects not only with the frequency dispersion but also with the spatial dispersion of the dielectric constant. The concept of spatial dispersion was introduced earlier for homogeneous media. In the language of the material equations, the presence of spatial dispersion is manifest in the fact that the kernel $\hat{\varepsilon}_{ij}$ in the integral relation

$$D_i'(\mathbf{r}, t) = \int_{-\infty}^{t} dt' \int d\mathbf{r}' \hat{\varepsilon}_{ij}(t-t', \mathbf{r}-\mathbf{r}') E_j(\mathbf{r}', t'), \quad (1.1)$$

between the induction and the electric field depends not only on the time* but also on the spatial variables. For homogeneous media, such a dependence results only from the difference between the coordinates of the points in which the field and the induction are considered. The dielectric tensor is then defined by the relation

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \int_0^{\infty} dt \int d\mathbf{r} e^{-i\mathbf{kr}} \hat{\varepsilon}_{ij}(t, \mathbf{r}). \quad (1.2)$$

In an arbitrary inhomogeneous medium we can no longer write relation (1.1). Consequently the dielectric tensor (1.2) should be replaced by a function that depends on two vectors, making the theory of electromagnetic properties of the inhomogeneous media much more cumbersome and more complicated [1]. On the other hand, it is clear that in weakly inhomogeneous media, if we consider wavelengths that are much shorter than the characteristic dimensions of the inhomogeneity, we can hope for an appreciable simplification of the theory, as is usually the case when using the method of geometrical optics.

For a weakly inhomogeneous medium the material equation, which supplements the field equation

$$\left.\begin{aligned} \operatorname{div} \mathbf{D}' &= 4\pi\varrho_0, & \operatorname{rot} \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \\ \operatorname{rot} \mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{D}'}{\partial t} + \frac{4\pi}{c}\mathbf{j}_0, & \operatorname{div} \mathbf{B} &= 0, \end{aligned}\right\} \quad (1.3)\dagger$$

can be written in the form [3]

$$D_i'(\mathbf{r}, t) = \int_{-\infty}^{t} dt' \int d\mathbf{r}' \hat{\varepsilon}_{ij}(t-t', \mathbf{r}-\mathbf{r}', \mathbf{r}) E_j(\mathbf{r}', t'). \quad (1.4)$$

When speaking of a weak inhomogeneity we assume here that the dependence of the kernel of the integral equation (1.4) on $\mathbf{r} - \mathbf{r}'$ is stronger than the dependence on $\mathbf{r}$, since it is precisely the latter which is connected with the spatial inhomogeneity. This form of the material equation is quite convenient when the geometrical optics approximation is used.

We represent the electric field in the form [4]

$$\mathbf{E} = \mathbf{E}_0 e^{i\Psi(\mathbf{r}, t)}, \quad (1.5)$$

where the principal and strongest dependence of the field on the coordinates and on the time is determined by the eikonal Ψ, and the amplitude $\mathbf{E}_0$ is assumed to be a slowly varying function. For media which do

*The corresponding dependence determines the temporal or frequency dispersion of the dielectric constant.
†rot = curl

not depend on the time, it is convenient to use the abbreviated eikonal*

$$\Psi_1(\mathbf{r}) = \Psi + \omega t. \qquad (1.6)$$

In the zeroth approximation of geometrical optics, in the absence of external sources, the field equations (1.3) and the material equation (1.4) can be written in the form

$$\left.\begin{array}{l}\mathbf{k}\mathbf{D}' = 0,\ [\mathbf{k}\mathbf{B}] = -\frac{\omega}{c}\mathbf{D}', \quad [\mathbf{k}\mathbf{E}] = \frac{\omega}{c}\mathbf{B}, \quad \mathbf{k}\mathbf{B} = 0, \\ D_i'(\mathbf{r}, t) = e^{-i\omega t + \Psi_1(\mathbf{r})}\varepsilon_{ij}(\omega, \mathbf{k}, \mathbf{r})E_{0j}.\end{array}\right\} (1.7)\dagger$$

Here $\mathbf{k} = \nabla\Psi_1(\mathbf{r})$ is the local wave vector, and (see [3])

$$\varepsilon_{ij}(\omega, \mathbf{k}, \mathbf{r}) = \int_0^\infty dt \int d\mathbf{r}'\varepsilon_{ij}'(t, \mathbf{r}', \mathbf{r})\, e^{i\omega t - i\mathbf{k}\mathbf{r}'}. \qquad (1.8)$$

Formula (1.8) is analogous to (1.2). This is precisely why we call the quantity defined by (1.8) the dielectric tensor of the weakly inhomogeneous medium. In writing down (1.7) and (1.8) we have neglected the dependence of $\mathbf{E}_0$ and $\mathbf{k}$ on the coordinates. For this to be permissible it is necessary to have a large wave-vector component in the direction of the spatial variation of the material properties. Specifically, the product of this component by the characteristic distance of the variation of the material properties should be much larger than unity:

$$k_L L \gg 1, \qquad (1.9)$$

where L is the characteristic dimension of the inhomogeneity of the medium and k_L is the projection of the wave vector on the direction of the inhomogeneity [see also condition (2.3)].

The condition for the solvability of (1.7) is of the form [3]

$$\left| k^2\delta_{ij} - k_i k_j - \frac{\omega^2}{c^2}\varepsilon_{ij}(\omega, \mathbf{k}, \mathbf{r}) \right| = 0. \qquad (1.10)$$

Equation (1.10) is the eikonal equation. The difference between (1.10) and the corresponding equation for media without spatial dispersion lies in the dependence of ε_{ij} on $\mathbf{k}$. In the general case Eq. (1.10) is therefore transcendental in $\mathbf{k}$ and not a power function, as in geometrical optics of media without spatial dispersion.

The electromagnetic field can be frequently assumed to be potential ($\mathbf{E} = -\nabla\Phi$, curl $\mathbf{E} = 0$). Then the eikonal equation is written in the form

$$k_i k_j k^{-2}\varepsilon_{ij}(\omega, \mathbf{k}, \mathbf{r}) \equiv \varepsilon(\omega, \mathbf{k}, \mathbf{r}) = 0. \qquad (1.11)$$

In the sections that follow we shall use both (1.10) and (1.11), and consider concrete spectra of plasma oscillations. The eikonal equation in itself, of course, does not determine the spectrum of the field oscillations, and the determination of the spectra of the natural frequencies will be treated in the next section. In this section we derive the dielectric tensor of a collisionless weakly-inhomogeneous plasma confined by a strong magnetic field. We obtain here, first, concrete expressions which serve as illustrations of the formulas written out above. Second, we are able to obtain initial relations for a subsequent analysis of the spectra of plasma oscillations and for an analysis of the stability of a plasma confined by a strong magnetic field.

To describe a collisionless plasma we use, as is customary, the kinetic equation in the self-consistent-field approximation

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} + e\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}\mathbf{B}]\right)\frac{\partial f}{\partial \mathbf{p}} = 0. \qquad (1.12)$$

It is necessary to start with an analysis of the equilibrium state of the plasma. We assume that all quantities characterizing the fundamental state of the plasma depend only on one coordinate x. The constant magnetic field B_0 with straight force lines is assumed parallel to the z axis. Assuming that in the equilibrium state there is no electric field*, we obtain from (1.12) the following equation for the particle distribution function in an equilibrium plasma:

$$v_\perp \cos\varphi \frac{\partial f_0}{\partial x} - \Omega\frac{\partial f_0}{\partial \varphi} = 0. \qquad (1.13)$$

Here $\Omega = eB_0/mc$ is the Larmor frequency of the particles, $v_\perp$ the absolute value of the projection of the particle velocity on the plane perpendicular to the magnetic field, and φ the polar angle characterizing the position of the velocity projection in this plane.

Our first concrete assumption concerning the weak inhomogeneity of the plasma is that the variation of the quantities characterizing the equilibrium state of the plasma is small over distances on the order of the Larmor radius of the particle. Namely, we assume henceforth that the following inequality is satisfied

$$\frac{v_T}{\Omega L} \ll 1, \qquad (1.14)$$

where $v_T = \sqrt{T/m}$ is the thermal velocity of the particles and L is the characteristic dimension of the inhomogeneity of the fundamental state of the plasma. Accurate to terms of first order in this small parameter, the solution of (1.13) can be written in the form

$$f_0(\mathbf{v}, x) = \left(1 + \frac{v_\perp \sin\varphi}{\Omega}\frac{\partial}{\partial x}\right)F(v_\perp, v_z, x), \qquad (1.15)$$

where $F(v_\perp, v_z, x)$ is an arbitrary function of the velocities $v_\perp$ and v_z and of the coordinate x.

*Since the question of media with slow time variation is investigated in the theory of electromagnetic properties of media with frequency dispersion, we shall concentrate out attention below on spatial inhomogeneity and spatial dispersion.

†$[\mathbf{k}\ \mathbf{B}] = \mathbf{k} \times \mathbf{B}$

*In the low pressure plasma in which we are interested, when the inhomogeneity of the magnetic field can be neglected (see below), we can always choose a coordinate system in which there is no constant electric field.

Substituting (1.5) in the field equations, we obtain the following equations for the plasma equilibrium:

$$\sum e \int d\mathbf{p} F = \sum eN = 0, \tag{1.16}$$

$$\sum e \int d\mathbf{p} v_z F = 0, \tag{1.17}$$

$$\frac{d}{dx}\left(\frac{B_0^2}{8\pi} + P_\perp\right) = 0, \tag{1.18}$$

where

$$P_\perp = \sum \int d\mathbf{p}\, \frac{m v_\perp^2}{2} F, \tag{1.19}$$

and the summation extends over all the charged-particle species in the plasma. We confine ourselves to a low-pressure plasma, when

$$\beta = \frac{8\pi P_\perp}{B_0^2} \ll 1. \tag{1.20}$$

In this case, writing Eq. (1.18) in the form

$$\frac{d}{dx} \ln \frac{B_0^2}{8\pi} + \beta \frac{d}{dx} \ln P_\perp = 0, \tag{1.21}$$

we see that the characteristic scale of the variation of the constant inhomogeneous magnetic field turns out to be much larger than the characteristic scale of variation of the inhomogeneous distribution of the particles of the equilibrium plasma. This circumstance makes it possible to neglect the spatial variation of $\mathbf{B}_0$ in the first approximation.

Proceeding now to a derivation of the dielectric tensor of a weakly inhomogeneous plasma, let us consider a small deviation from the equilibrium state, which we describe by means of the small addition to the equilibrium distribution function δf. From (1.12) we then have

$$\frac{\partial}{\partial t}\delta f + \mathbf{v}\frac{\partial}{\partial \mathbf{r}}\delta f - \Omega \frac{\partial}{\partial \varphi}\delta f = -e\left(\mathbf{E} + \frac{1}{c}[\mathbf{vB}]\right)\frac{\partial f_0}{\partial \mathbf{p}}. \tag{1.22}$$

Here $\mathbf{E}$ and $\mathbf{B}$ are the nonequilibrium electric and magnetic fields. To obtain the dielectric tensor we can assume that the distribution function is proportional to $\exp(-i\omega t)$, where ω has an infinitesimally small positive imaginary increment [1]. Bearing in mind also the spatial homogeneity of the distribution of the fundamental state along the y and z axes, we seek a solution of (1.22) in the form

$$\delta f(x)\, e^{-i\omega t + i k_y y + i k_z z}. \tag{1.23}$$

We assume a similar dependence on the time and on the coordinates y and z for nonequilibrium fields, too. We can then write the following solution of (1.22) (periodic in φ):

$$\delta f(x) = e \int_\infty^\varphi \frac{d\varphi'}{\Omega(x')}\left\{\mathbf{E}(x') + \frac{1}{c}[\mathbf{v}(\varphi')\mathbf{B}(x')]\right\}\frac{\partial f_0(x', \varphi')}{\partial \mathbf{p}(\varphi')}$$
$$\times \exp\left(i\int_\varphi^{\varphi'} d\varphi'' \frac{\omega - k_y v_\perp \sin\varphi'' - k_z v_z}{\Omega(x'')}\right). \tag{1.24}$$

The quantity x' in the right side of (1.24) is connected with x, φ, and the integration variable φ' by the characteristic equation*

$$x' + \frac{v_\perp}{\Omega}\sin\varphi' = x + \frac{v_\perp}{\Omega}\sin\varphi = \text{const.} \tag{1.25}$$

Substituting (1.24) in the relation that defines the nonequilibrium current density

$$\delta \mathbf{j} = \sum e \int d\mathbf{p} \mathbf{v} \delta f$$

and bearing in mind the definition [1]

$$\frac{\partial \mathbf{D}'}{\partial t} = \frac{\partial \mathbf{E}}{\partial t} + 4\pi\delta\mathbf{j}, \tag{1.26}$$

we can obviously write

$$D_i'(\omega, x, k_y, k_z)$$
$$= \int dx' \hat{\varepsilon}_{ij}(\omega, x - x', x, k_y, k_z) E_j(\omega, x', k_y, k_z)$$
$$= \int dk_x e^{ik_x x} \varepsilon_{ij}(\omega, \mathbf{k}, x) E_j(\omega, k). \tag{1.27}$$

In order for the induction to be expressed only in terms of the electric field, we use, as is customary, Maxwell's equation

$$i\omega \mathbf{B} = c \operatorname{rot} \mathbf{E}.$$

Relation (1.27) is a material equation of the type (1.4), written for the Fourier components with respect to the time and the variables y and z, something which can be done in our case of a uniform inhomogeneity along the x axis only. The dielectric tensor is of the form

$$\varepsilon_{ij}(\omega, \mathbf{k}, x)$$
$$= \delta_{ij} + \sum \frac{i\pi e^2}{\omega^2}\int d p v_i \int_\infty^\varphi \frac{d\varphi'}{\Omega(x')}\frac{\partial f_0(x', \varphi')}{\partial p_l(\varphi')}\{\delta_{lj}[\omega - \mathbf{kv}(\varphi')]$$
$$+ k_l v_j(\varphi')\}\exp\left(i\int_\varphi^{\varphi'} d\varphi'' \frac{\omega - k_y v_\perp \sin\varphi'' - k_\perp v_\perp \cos\varphi'' - k_z v_z}{\Omega(x'')}\right). \tag{1.28}$$

Here, as in (1.24), the quantities x' and x'' are connected with x by the characteristic equation (1.25). Equation (1.28) takes into account the inhomogeneity of the magnetic field and is therefore convenient for a plasma with finite pressure.

Taking into consideration the formula (1.15) for the equilibrium distribution function, we obtain from (1.28)[5]

$$\varepsilon_{ij}(\omega, \mathbf{k}, x) = \delta_{ij} - \left(\delta_{ij} - \frac{B_i B_j}{B^2}\right)\sum \frac{2\pi e^2}{\omega^2}\int d p v_\perp \frac{\partial F}{\partial p_\perp}$$
$$- \frac{B_i B_j}{B^2}\sum \frac{4\pi e^2}{\omega^2}\int d p v_z \frac{\partial F}{\partial p_z} + \sum \sum_n \frac{4\pi e^2}{\omega^2}$$
$$\times \int dp \frac{F_i^n(\mathbf{k}) F_j^{(n)}(\mathbf{k})}{\omega - \frac{k_y v_\perp^2}{2\Omega^2}\frac{d\Omega}{dx} - n\Omega - k_z v_z}$$

*This characteristic equation is approximate and is obtained from the exact equation $v_\perp \sin\varphi + \int^x dx\, \Omega(x) = \text{const}$ in an approximation in which the coordinate dependence of the constant field is neglected, which is reasonable in the case of a low pressure plasma when inequality (1.20) is satisfied.

$$\times\left(k_z\frac{\partial}{\partial p_z}-\frac{n\Omega}{v_\perp}\frac{\partial}{\partial p_\perp}+\frac{ck_y}{eB}\frac{\partial}{\partial x}\right)F, \quad (1.29)$$

where

$$\left.\begin{aligned} F_x^{(n)}(\mathbf{k})&=\frac{v_\perp}{k_\perp}\left[ik_yJ_n'\left(\frac{k_\perp v_\perp}{\Omega}\right)-k_x\frac{n\Omega}{k_\perp v_\perp}J_n\left(\frac{k_\perp v_\perp}{\Omega}\right)\right],\\ F_y^{(n)}(\mathbf{k})&=\frac{v_\perp}{k_\perp}\left[-ik_xJ_n'\left(\frac{k_\perp v_\perp}{\Omega}\right)-k_y\frac{n\Omega}{k_\perp v_\perp}J_n\left(\frac{k_\perp v_\perp}{\Omega}\right)\right],\\ F_z^{(n)}(\mathbf{k})&=v_zJ_n\left(\frac{k_\perp v_\perp}{\Omega}\right).\end{aligned}\right\} \quad (1.30)$$

We assume further for the function F a Maxwellian distribution of the particles, with inhomogeneous density and temperature

$$F=\frac{N(x)}{(2\pi mT(x))^{3/2}}e^{-\frac{mv^2}{2T(x)}} \quad (1.31)$$

and neglect the inhomogeneity of the magnetic field (low pressure plasma, $\beta \ll 1$). As a result (1.29) takes the form

$$\varepsilon_{ij}(\omega,\ \mathbf{k},\ x)=\delta_{ij}+\sum\frac{1}{T}\left[1-\frac{k_yv_T^2}{\omega\Omega}\left(\frac{\partial N}{\partial x}\frac{\partial}{\partial N}+\frac{\partial T}{\partial x}\frac{\partial}{\partial T}\right)\right]$$
$$\times T\left[\varepsilon_{ij}^{(0)}(\omega,\ \mathbf{k},\ x)-\delta_{ij}\right]. \quad (1.32)$$

Here $\varepsilon_{ij}^{(0)}(\omega,\ \mathbf{k},\ x)$ is the dielectric tensor of one species of particles and coincides with the corresponding tensor of the inhomogeneous plasma, except that N and T are now assumed to depend on the coordinate x. The summation is over the electrons and the ions, which we assume to be singly charged; and in addition, we confine ourselves to only one species of ions.

The expression frequently given in the literature (see, for example, [1]) for the dielectric tensor of an homogeneous plasma is written in a coordinate frame in which $\mathbf{k} = (k_\perp,\ 0,\ k_z)$. We need a similar expression, but in an arbitrary reference frame, where $\mathbf{k} = (k_x,\ k_y,\ k_z)$. We can easily show that its form is

$$\varepsilon_{ij}^{(0)}=\begin{pmatrix}\varepsilon_1^{(0)}\cos^2\alpha+\varepsilon_{22}^{(0)}\sin^2\alpha, & \sin\alpha\cos\alpha\,(\varepsilon_{11}^{(0)}-\varepsilon_{22}^{(0)})+\varepsilon_{12}^{(0)}, & \varepsilon_{13}^{(0)}\cos\alpha-\varepsilon_{23}^{(0)}\sin\alpha\\ \sin\alpha\cos\alpha\,(\varepsilon_{11}^{(0)}-\varepsilon_{22}^{(0)})-\varepsilon_{12}^{(0)}, & \varepsilon_{11}^{(0)}\sin^2\alpha+\varepsilon_{22}^{(0)}\cos^2\alpha, & \varepsilon_{23}^{(0)}\cos\alpha+\varepsilon_{13}^{(0)}\sin\alpha\\ \varepsilon_{13}^{(0)}\cos\alpha+\varepsilon_{23}^{(0)}\sin\alpha, & -\varepsilon_{23}^{(0)}\cos\alpha+\varepsilon_{13}^{(0)}\sin\alpha, & \varepsilon_{33}^{(0)}\end{pmatrix}, \quad (1.33)$$

where $k_x = k_\perp\cos\alpha$, $k_y = k_\perp\sin\alpha$, and the six components of the dielectric tensor have in the system $\mathbf{k} = (k_\perp,\ 0,\ k_z)$ the form [6,7]

$$\left.\begin{aligned}\varepsilon_{11}^{(0)}&=1-\sum_n\frac{n^2\omega_L^2}{\omega(\omega-n\Omega)}\frac{A_n(z)}{z}J_+(\beta_n),\\ \varepsilon_{22}^{(0)}&=\varepsilon_{11}^{(0)}+2\sum_n\frac{\omega_L^2}{\omega(\omega-n\Omega)}zA_n'(z)J_+(\beta_n),\\ \varepsilon_{13}^{(0)}&=\sum_n\frac{n\omega_L^2}{\omega\Omega}\frac{A_n(z)}{z}\frac{k_\perp}{k_z}[1-J_+(\beta_n)],\\ \varepsilon_{23}^{(0)}&=-i\sum_n\frac{\omega_L^2}{\omega\Omega}A_n'(z)\frac{k_\perp}{k_z}[1-J_+(\beta_n)],\\ \varepsilon_{12}^{(0)}&=-i\sum_n\frac{n\omega_L^2}{\omega(\omega-n\Omega)}A_n'(z)J_+(\beta_n),\\ \varepsilon_{33}^{(0)}&=1+\sum_n\frac{\omega_L^2(\omega-n\Omega)}{\omega k_z^2v_T^2}A_n(z)[1-J_+(\beta_n)].\end{aligned}\right\} \quad (1.34)$$

We have used here the notation:

$$A_n(z)=e^{-z}I_n(z),\quad J_+(\beta)=\beta e^{-\beta^2/2}\int_{i\infty}^{\infty}d\tau e^{\tau^2/2},$$
$$\beta_n=\frac{\omega-n\Omega}{|k_z|v_T},\quad z=\frac{k_\perp^2v_T^2}{\Omega^2}.$$

In addition, $\omega_L = (4\pi e^2N/m)^{1/2}$ is the Langmuir frequency and $I_n(z)$ is a Bessel function of imaginary argument.

This expression for the dielectric tensor of the weakly inhomogeneous plasma is much simpler than the corresponding expressions in [8], where terms of higher order in the small parameter defined by relation (1.14) are taken into account. An actual analysis has shown that such higher-order terms are unimportant for the applications considered below*.

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*The expression in[8] contains a whole string of terms which add up to a dielectric tensor different from ours. However, whereas the eikonal equations (1.10) and (1.11) are sufficient for the use of our tensor, the tensor given in[8] can be used only in conjunction with a much more complicated procedure, involving an analysis of the field equations (1.3).

It can be readily shown that an exact expression for the dielectric tensor of an inhomogeneous plasma, with the dependence of the constant magnetic field on the coordinate neglected, has the form

$$\varepsilon_{ij}(\omega,\ \mathbf{k},\ x)=\delta_{ij}-i\sum\frac{4\pi e^2}{\omega\Omega T}\left\{\left(1-\frac{k_yv_T^2}{\omega\Omega}\frac{\partial}{\partial x}\right)\int d\mathbf{p}v_if_0(\Omega x+v_y,\ \varepsilon)\right.$$
$$\times\int_\infty^\varphi d\varphi' v_j\exp\left[\frac{i}{\Omega}\int_\varphi^{\varphi'}d\varphi''(\omega-\mathbf{kv})\right]$$
$$-\delta_{yj}\frac{v_T^2}{\omega\Omega}\frac{\partial}{\partial x}\int d\mathbf{p}v_if_0(\Omega x+v_y,\ \varepsilon)$$
$$\left.\times\int_\infty^\varphi d\varphi'(\omega-\mathbf{kv})\exp\left[\frac{i}{\Omega}\int_\varphi^\varphi d\varphi''(\omega-\mathbf{kv})\right]\right\}.$$

Here

$$f_0(\Omega x+v_y,\ \varepsilon)=\left(1+\frac{v_y}{\Omega}\frac{\partial}{\partial x}+\ldots+\frac{1}{n!}\frac{v_y^n}{\Omega^n}\frac{\partial^n}{\partial x^n}+\ldots\right)F(x,\ \varepsilon),$$

[$\varepsilon = mv^2/2$ and $F(x, \varepsilon)$ is given by (1.31)] is the equilibrium distribution function of the particles in the inhomogeneous plasma. We note that if we retain only the first term of this expansion we

Let us write out an expression for the potential dielectric constant of a weakly inhomogeneous plasma confined by strong magnetic field:

$$\varepsilon(\omega, \mathbf{k}, x) = 1 + \sum \frac{\omega_L^2}{k^2 v_T^2}\left\{1 - \sum_n \frac{\omega}{\omega - n\Omega} \times \left[1 - \frac{k_y v_T^2}{\omega\Omega}\left(\frac{\partial \ln N}{\partial x} + \frac{\partial T}{\partial x}\frac{\partial}{\partial T}\right)\right] A_n(z) J_+(\beta_n)\right\}. \quad (1.35)$$

It must be noted that the dielectric-constant terms which contain the spatial derivatives of the particle number and of the temperature play an important role when the absolute magnitude of the ratio ω/k_y is small compared with the particle drift velocity $v_d \sim v_T^2/\Omega L$. In the opposite case, when the frequency of the oscillations greatly exceeds the drift frequency $\omega_d \sim k_y v_d$, we can neglect terms that contain spatial derivatives, in which connection the dielectric tensor of a weakly inhomogeneous plasma coincides precisely in form with the analogous tensor for the spatially homogeneous plasma. Such an expression for the dielectric constant can be obtained also for a high pressure plasma. Moreover, it becomes possible to write down an expression for the dielectric tensor of an inhomogeneous plasma with neglect of the constant external magnetic field. In the latter case we have

$$\varepsilon_{ij}(\omega, \mathbf{k}, x) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\varepsilon^{tr}(\omega, k, x) + \frac{k_i k_j}{k^2}\varepsilon^l(\omega, k, x), \quad (1.36)$$

where [1]

$$\varepsilon^{tr}(\omega, k, x) = 1 - \sum \frac{\omega_L^2}{\omega^2} J_+\left(\frac{\omega}{k v_T}\right),$$

$$\varepsilon^l(\omega, k, x) = 1 + \sum \frac{\omega_L^2}{k^2 v_T^2}\left[1 - J_+\left(\frac{\omega}{k v_T}\right)\right] \quad (1.37)$$

are the transverse and longitudinal (potential) dielectric constants of the inhomogeneous isotropic plasma.

The formulas obtained above are directly applicable to the case of cylindrical geometry, which is frequently realized in practical conditions, when a radially inhomogeneous plasma cylinder is confined by a strong longitudinal magnetic field B_0 directed along the z axis (plasma cylinder axis). Along with condition (1.14), it is necessary in the cylindrical case also to stipulate satisfaction of the condition

$$v_T/r\Omega \ll 1. \quad (1.38)$$

Without presenting the corresponding derivation (which is analogous in many respects to the one given above for the case of planar geometry) we point out that the dielectric tensor of a cylindrically inhomogeneous plasma coincides in form with (1.28) [see also (1.29)—(1.35)], provided we make in it the formal substitutions

$$x \to r, \quad k_x \to k_r, \quad k_y \to l/r, \quad (1.39)$$

where l are integers.* Taking this circumstance into account, we confine ourselves throughout to results for a planar geometry only. The transition to cylindrical geometry by means of (1.39) entails no difficulty.

To conclude this section we note that the expression presented above for the dielectric tensor can be readily generalized to include the case when the plasma contains, in addition to the field B_z, also a small, generally speaking inhomogeneous transverse magnetic field $B_\perp$ (small compared with B_z in the direction transverse to the z axis and to the inhomogeneity plasma; in the case of planar geometry $B_\perp$ is directed along the y axis, while in the case of cylindrical geometry it is directed along the azimuth). To obtain the dielectric tensor of an inhomogeneous plasma in this case it is necessary to make in (1.28)—(1.35) the substitution

$$k_y \to k_y \frac{B_z}{B_0} - k_z \frac{B_\perp}{B_0}, \quad k_z \to k_z \frac{B_z}{B_0} + k_y \frac{B_\perp}{B_0}. \quad (1.40)$$

An analogous substitution is made also in the case of cylindrical geometry.

2. QUASICLASSICAL QUANTIZATION RULES AND SPECTRUM OF OSCILLATIONS OF AN ISOTROPIC INHOMOGENEOUS PLASMA

In this section we determine the spectrum of the natural oscillations of a weakly inhomogeneous plasma in the case when there is no external magnetic field. The method used here to obtain the dispersion relations will be used in the following sections also for more complicated cases of a magnetoactive inhomogeneous plasma.

We begin with the simplest case of transverse oscillations with phase velocities larger than the thermal velocities of the plasma particles. Getting a little ahead of ourselves, we note that the result will show that the phase velocities of the transverse oscillations, as in the case of a homogeneous plasma, exceed the velocity of light. Therefore the imaginary part of the transverse dielectric constant (1.37) is equal to zero. Taking this circumstance into account, we obtain from (1.36), (1.37), and (1.10) the

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get (1.32). Since the operator $\frac{k_y v_T^2}{\omega\Omega}\frac{\partial}{\partial x}$ is of the order of unity for the drift oscillations considered below, it is not sensible to obtain the dielectric tensor in the approximation of a specified number of spatial derivatives. This is precisely why only the necessary first-order terms in the spatial derivatives are retained in (1.32), just as not all the second-order terms have been retained in.[8] Expansion of the equilibrium distribution function shows that the small parameter is actually the quantity defined by (1.14).

*In the derivation of the dielectric tensor for a cylindrically inhomogeneous plasma it becomes necessary also to stipulate satisfaction of the condition $|l| \gg 1$. This condition is used in the elimination of the magnetic field of the wave with the aid of Maxwell's equations. The expression for the potential dielectric constant (1.35) is valid also without this limitation.

following eikonal equation for the high-frequency transverse oscillations of the field in an inhomogeneous isotropic plasma:

$$k^2 - \frac{\omega^2}{c^2}\left(1 - \frac{\omega_{Le}^2}{\omega^2}\right) = 0. \qquad (2.1)$$

On the other hand, if (1.37) is substituted in (1.27), then we obtain in this case from Maxwell's equations (1.3) the following equation for the transverse field:

$$\Delta E^{tr} + \frac{\omega^2}{c^2}\left(1 - \frac{\omega_{Le}^2}{\omega^2}\right) E^{tr} = 0. \qquad (2.2)$$

Equation (2.1) is the eikonal equation corresponding to the field equation (2.2) in the zeroth approximation of the geometrical-optics method. The small parameter of geometrical optics is the quantity in the left side of the inequality

$$\frac{1}{k_x}\frac{d}{dx}\ln k_x \ll 1. \qquad (2.3)$$

In the case under consideration the function

$$k_x^2 = -k_y^2 - k_z^2 + \frac{\omega^2}{c^2}\left(1 - \frac{\omega_{Le}^2}{\omega^2}\right)$$

is then purely real. From this condition it follows that $k_x L \gg 1$, that is, the wavelength of the oscillations along the inhomogeneity of the plasma is much shorter than the characteristic dimension of the inhomogeneity [see condition (1.9)].

Equation (2.2) is similar to the Schrödinger equation with a real potential $k_x^2(\omega, x)$. It is well known from quantum mechanics that from the uniqueness of the solution of Eq. (2.2) (if condition (2.3) is satisfied) follows the rule for quasiclassical quantization, which determines the spectrum of the eigenvalues of this equation,

$$\int dx k_x(\omega, x) = \int dx \left(-k_y^2 - k_z^2 + \frac{\omega^2}{c^2} - \frac{\omega_{Le}^2}{c^2}\right)^{1/2} = \pi n, \qquad (2.4)$$

where n is an integer much larger than unity. The integral in (2.4) is taken over the region of "transparency" of the plasma ($k_x^2(\omega, x) \geq 0$), located between the transition points defined by the relation

$$k_x^2(\omega, x) = 0. \qquad (2.5)$$

If there is no transition point in the physical region of values of x (region occupied by the plasma), then the integration extends over the entire range of variation of x. [It is necessary to stipulate here that nondissipative boundary conditions be satisfied for Eq. (2.2).] Of greatest interest is the case when transition points exist in the physical region of variation of x, for then the plasma oscillations are locked in the region of "transparency" between the transition points, that is, they are locked inside the plasma. If there are several pairs of transition points, such that the transparency regions are separated by a distance larger than the wavelength of the oscillations along the inhomogeneity, then the integration in (2.4) must be carried out over any "transparency" region included between two neighboring transition points. These arguments are general in character, and will not be repeated from now on.

From the quantization condition (2.4) we see that the transverse oscillations of the isotropic inhomogeneous plasma are stable, that is, $\omega^2 > 0$, and the phase velocity of the oscillations exceeds the velocity of light. Therefore such oscillations of the inhomogeneous plasma, as also in the case of the homogeneous plasma, are undamped.

We now turn to the more complicated case of longitudinal oscillations of an inhomogeneous isotropic plasma in the frequency region $\omega \gg kv_{Te}$, when the phase velocities of the oscillations exceed the thermal velocities of the particles in the plasma. From (1.36), (1.37), and (1.10) we obtain in this case the following eikonal equation for the longitudinal oscillations:

$$1 - \frac{\omega_{Le}^2}{\omega^2}\left(1 + 3\frac{k^2 v_{Te}^2}{\omega^2}\right) + i\sqrt{\frac{\pi}{2}}\frac{\omega\omega_{Le}^2}{k^3 v_{Te}^3} e^{-\frac{\omega^2}{2k^2 v_{Te}^2}} = 0. \qquad (2.6)$$

On the other hand, using (1.37) and (1.27) and taking into account the conditions for the applicability of geometrical optics (2.3), we can obtain from Maxwell's equations the following equation for the high-frequency longitudinal field in a plasma:

$$\left(1 - \frac{\omega_{Le}^2}{\omega^2}\right) E^l + 3\frac{\omega_{Le}^2 v_{Te}^2}{\omega^2}\Delta E^l = -i\int dx' \hat{\varepsilon}^{l''}(\omega, x - x', x) E^l(x'), \qquad (2.7)$$

where

$$\hat{\varepsilon}^{l''}(\omega, x - x', x) = \frac{1}{2\pi}\int dk_x e^{ik_x(x-x')}\varepsilon^{l''}(\omega, k, x) = \frac{1}{2\pi}\int dk_x e^{ik_x(x-x')}\sqrt{\frac{\pi}{2}}\frac{\omega\omega_{Le}^2}{k^3 v_{Te}^3} e^{-\frac{\omega^2}{2k^2 v_{Te}^2}}. \qquad (2.8)$$

Equation (2.6) is the eikonal equation for the integral equation (2.7) in the zeroth approximation of geometical optics. The quasiclassical spectrum of the eigenvalues of Eq. (2.7) can be determined by two different methods. The first was used in [9] and consists in the following. Recognizing that the imaginary part of the dielectric constant in the frequency region under consideration is small, we note that the integral term in (2.7) is small. In the zeroth approximation, neglecting this term, we have

$$\left(1 - \frac{\omega_{Le}^2}{\omega^2}\right) E^l + 3\frac{\omega_{Le}^2 v_{Te}^2}{\omega^4}\Delta E^l = 0. \qquad (2.9)$$

This equation no longer contains the integral and is similar to Eq. (2.2), that is, to the Schrödinger equation with a real potential. The eigenvalue spectrum of this equation is therefore determined in the geometrical optics approximation by the dispersion equation

$$\int dx k_x(\omega, x) = \int dx \left\{\frac{\omega^4}{3\omega_{Le}^2 v_{Te}^2}\left[1-\frac{\omega_{Le}^2}{\omega^2}-3\frac{\omega_{Le}^2 v_{Te}^2}{\omega^4}(k_y^2+k_z^2)\right]\right\}^{1/2} = \pi n. \tag{2.10}$$

Integration in this formula, as before, is carried out over the region of transparency of the plasma, with the eigenfunctions of (2.9) inside and outside the transparency region respectively (that is, between the transition points and beyond these points) having the following forms [9]

$$\left.\begin{aligned}\Psi_{0n} &= \alpha E_{\nu n}^l = \frac{1}{\sqrt{k_x}}\cos\left[\int_{x_\nu}^{x} dx k_x(\omega, x) - \frac{\pi}{4}\right] \quad \text{if } x < x_\nu,\\ \Psi_{0n} &= \alpha E_{\nu n}^l\\ &= \frac{1}{2\sqrt{k_x}}\exp\left[-\int_{x_\nu}^{x} dx\,|k_x(\omega, x)| - i\frac{\pi}{4}\right] \quad \text{if } x > x_\nu.\end{aligned}\right\} \tag{2.11}$$

where x_ν is the right-hand transition point (x_μ—left-hand transition point), with $x_\mu < x \le x_\nu$ corresponding to the transparency region of the plasma and $\alpha = \omega_{Le}^2 v_{Te}^2/\omega^4$. Analogous formulas hold also in the vicinity of the transition point x_μ.

Formulas (2.10) and (2.11) have been obtained by completely neglecting the dissipative processes, and determine therefore only the real part of the longitudinal plasma oscillation frequency. To determine the small imaginary part of the oscillation frequency ($\omega \to \omega + i\gamma$) we write Eq. (2.7) in the following approximate form:

$$\frac{d^2\Psi}{dx^2} + k_x^2(\omega, x)\Psi + i\gamma\left(\frac{\partial}{\partial\omega}k_x^2(\omega, x)\right)\Psi = -i\int dx' \frac{1}{3\alpha(\omega, x')}\varepsilon^{l''}(\omega, x-x', x)\Psi(x'), \tag{2.12}$$

where $\Psi(x) = \alpha E^l$. From this, owing to the orthogonality of the eigenfunctions Ψ_{0n} (see formulas (2.11) with different n), we obtain in first perturbation-theory approximation the correction to the eigenvalue for the n-th level [9]

$$\gamma = -\frac{\int dx \int dx' \Psi_{0n}(x)\Psi_{0n}(x')\varepsilon^{l''}(\omega, x-x', x)/3\alpha(\omega, x')}{\int dx \Psi_{0n}^2(x)\frac{\partial k_x^2(\omega, x)}{\partial\omega}}. \tag{2.13}$$

The main contribution to these integrals is made by regions of plasma transparency, and the integration is therefore carried out only over these regions. Owing to the rapid oscillations of the function Ψ_{0n}, these integrals can be simplified by the stationary-phase method (for details see [9]). Taking in addition account of (2.8), we obtain ultimately

$$\gamma = -\frac{1}{6}\frac{\int \frac{dx}{k_x(\omega, x)}\frac{\varepsilon^{l''}(\omega, k, x)}{\alpha(\omega, x)}}{\int dx \frac{\partial k_x(\omega, x)}{\partial\omega}} = -\frac{1}{6}\sqrt{\frac{\pi}{2}}\frac{\int \frac{dx}{k_x(\omega, x)}\frac{\omega^5}{k^3 v_{Te}^5} e^{-\frac{\omega^2}{2k^2 v_{Te}^2}}}{\int dx \frac{\partial k_x(\omega, x)}{\partial\omega}}. \tag{2.14}$$

Formulas (2.10) and (2.14) determine the spectrum of the high-frequency longitudinal oscillations of a weakly inhomogeneous isotropic plasma.

We now derive these equations by a different method (which is simpler albeit less direct), described in [3] and based on the concept of the approximately-equivalent equation. Using the smallness of the imaginary part of the longitudinal dielectric constant, we easily determine the complex quantity $k_x^2(\omega, x)$ from the eikonal equation (2.6). We have

$$\left.\begin{aligned}\operatorname{Re} k_x^2(\omega, x) &= \frac{1}{3\alpha(\omega, x)}\left[1-\frac{\omega_{Le}^2}{\omega^2} - 3\alpha(\omega, x)(k_y^2+k_z^2)\right],\\ \operatorname{Im} k_x^2(\omega, x) &= \frac{\varepsilon^{l''}(\omega, k, x)}{3\alpha(\omega, x)} = \frac{1}{3}\sqrt{\frac{\pi}{2}}\frac{\omega\omega_{Le}^2}{\alpha k^3 v_{Te}^3} e^{-\frac{\omega^2}{2k^2 v_{Te}^2}}.\end{aligned}\right\} \tag{2.15}$$

We note that in the expression for the imaginary part $\operatorname{Im} k_x^2(\omega, x)$ we must imply $k^2 = k_y^2 + k_z^2 + (\operatorname{Re} k_x)^2$. Formulas (2.15) can now be considered independently of the connection with the initial integro-differential equation (2.7), the quasiclassical eigenvalue spectrum of which we are determining. Namely, it can be said that formulas (2.15), which determine the solutions of (2.7) in the zeroth approximation of geometrical optics, determine at the same time the solutions of the zeroth geometrical-optics approximation of the second-order differential equation

$$\frac{d^2\Phi}{dx^2} + k_x^2(\omega, x)\Phi = 0. \tag{2.16}$$

In this sense we can speak of the equivalence of Eqs. (2.7) and (2.16) in the zeroth approximation of the geometrical-optics method. We can now transfer the results of the theory of asymptotic solutions of the differential second-order equations, with the aid of the approximately-equivalent equation (2.16), to the theory of asymptotic solutions of integral equations.*

Equation (2.16) is analogous to the Schrödinger equation with complex potential. The theory of asymptotic solutions of equations of this type has been well developed. For the case of real potentials [real $k_x^2(\omega, x)$] such a theory is well known to physicists from quantum mechanics. Therefore we used above the quasiclassical quantization condition (2.4) to determine the eigenvalue spectrum of (2.2) without further explanation. For the case of complex potentials, the theory of asymptotic solutions of equations of the type (2.16) is far less known to physicists. We therefore restate here the main premises of this theory, which concerns the Stokes phenomenon and the eigenvalue spectrum of Eq. (2.16).

*The hydrodynamic theory of stability, which originated with the classical researches of Rayleigh, a splendid exposition of which the reader can find in the reviews[10], is also customarily connected with the use of asymptotic solution methods. It must be emphasized, however, that in the most interesting cases of hydrodynamic instability we can not confine ourselves to the geometrical-optics approximation.

For a detailed exposition we refer the reader to the specialized literature [11] (see also [12-14]).

We confine ourselves to the case when the imaginary part of $k_x^2(\omega, x)$ is small. In practice this is apparently the most interesting case, and we shall deal with only such cases. We assume also that the frequencies of the natural oscillations are almost real, that is, $\omega \to \omega + i\gamma$, with $\gamma \ll \omega$. Finally, we assume that the transition points, determined by relation (2.5) and lying in general in the complex region of x, are near the real axis, that is, they likewise have small imaginary parts.

The asymptotic solution of Eq. (2.16) is of the form

$$\Phi(x) = C_+ e^{i\int^x k_x dx} + C_- e^{-i\int^x k_x dx} \quad (2.17)$$

In the general case the complex x plane can be broken up into regions in which the coefficients $C_\pm$ have definite values. The transition from one region to the other is accompanied by a jumplike change in these coefficients. This is the Stokes phenomenon, and the lines separating the indicated regions from one another are called Stokes lines ($\mathrm{Im}\, k_x = 0$ on such lines). It is obvious that the transition points lie on Stokes lines. In the vicinity of the transition points the solution of (2.16) is expressed in terms of Bessel functions. The asymptotic behavior and the Stokes phenomenon for such functions have been well investigated. By stipulating that such an asymptotic solution coincide with (2.17), we obtain the coefficients $C_\pm$. This on the other hand leads to the quantization rule which determines the eigenvalue spectrum of (2.16). Without discussing this question in detail (as already pointed out, this question is adequately treated in the literature), we present the final result for the case when there are two transition points. The dispersion equation for the eigenvalue spectrum is of the form [12-15]

$$\int_{x_\mu}^{x_\nu} k_x(\omega, x)\, dx = \pi\left(n + \frac{1}{2}\right). \quad (2.18)$$

The integration in this formula is over the transparency region, where $\mathrm{Re}\, x_x^2(\omega, x) \geq 0$, between the complex transition points. In view of the small imaginary part of the frequency ω, of the function $k_x(\omega, x)$, and of the points x_μ and x_ν (the latter enables us to neglect the small contribution due to the integration along the imaginary axis), we can obtain from (2.18) two relations which determine the frequency ω and the oscillation damping decrement γ [15]:

$$\left.\begin{aligned} &\int dx\, \mathrm{Re}\, k_x(\omega, x) = \pi n, \\ &\gamma = -\frac{\int dx\, \mathrm{Im}\, k_x(\omega, x)}{\int dx \frac{\partial}{\partial\omega} \mathrm{Re}\, k_x(\omega, x)}, \end{aligned}\right\} \quad (2.19)$$

in which the integration is over the transparency region between the projections of the complex points of transition on the real axis. (In (2.19) and thereafter we neglect the ½ compared with the large quantity n.) Formulas (2.19) will be used later to determine the oscillation spectra of an inhomogeneous plasma. In particular, if we substitute (2.15) in these relations, we obtain expressions which coincide exactly with (2.10) and (2.14).

It is important to note that the second relation of (2.19), for the damping decrement (or the growth increment) of the oscillations, should be used only in those cases when the frequencies determined from the first relation (that is, neglecting completely the dissipative processes) are real or, as is customarily stated, the plasma oscillations are hydrodynamically stable. The sign of the quantity γ shows in this case whether these plasma oscillations are kinetically stable or not [16].

We now employ relations (2.19) to determine the spectrum of the low-frequency longitudinal oscillations of a non-isothermal inhomogeneous plasma in the absence of a magnetic field (ionic sound in an inhomogeneous plasma*). Such requirements exist in the frequency region $kv_{Ti} \ll \omega \ll kv_{Te}$, and, in accordance with (1.26), the eikonal equation for these oscillations takes the form†

$$1 + \frac{\omega_{Le}^2}{k^2 v_{Te}^2} - \frac{\omega_{Li}^2}{\omega^2} + i\sqrt{\frac{\pi}{2}} \frac{\omega\omega_{Le}^2}{k^3 v_{Te}^3} = 0. \quad (2.20)$$

Taking into account the smallness of the imaginary part of this equation and using (2.19), we obtain the following dispersion equation for the spectrum of the low-frequency oscillations of an inhomogeneous non-isothermal plasma in which $T_e \gg T_i$:

$$\int dx\, \mathrm{Re}\, k_x(\omega, x) = \int dx \left(-k_y^2 - k_z^2 + \frac{\omega^2}{\omega_{Li}^2 - \omega^2} \frac{\omega_{Le}^2}{v_{Te}^2}\right)^{1/2} = \pi n, \quad (2.21)$$

$$\gamma = -\sqrt{\frac{\pi}{8}}\, \omega^2 \frac{\int \frac{dx}{\mathrm{Re}\, k_x} \frac{\omega_{Le}^2}{\omega_{Li}^2 - \omega^2} \frac{1}{v_{Te}^3 \sqrt{k_y^2 + k_z^2 + (\mathrm{Re}\, k_x)^2}}}{\int \frac{dx}{\mathrm{Re}\, k_x} \frac{\omega_{Le}^2}{v_{Te}^2} \frac{\omega_{Li}^2}{(\omega_{Li}^2 - \omega^2)^2}}. \quad (2.22)$$

From this it follows, in particular, that in an inhomogeneous plasma, like in a homogeneous one, the low-frequency longitudinal oscillations are possible only if $\omega^2 < \omega_{Li}^2$.

We note that the presence of plasma inhomogeneity

*For a comparison see[9], where a perturbation theory method is used to investigate the spectrum of ionic sound in an inhomogeneous plasma.

†We neglect here the ion absorption of the waves in the plasma. This is legitimate if

$$1 \gg \left(\frac{T_e}{T_i}\right)^{3/2} \sqrt{\frac{M}{m}}\, e^{-\frac{\omega^2}{2k^2 v_{Ti}^2}}.$$

In the presence of a magnetic field it is necessary to write in this k_z in place of k.

only in the x direction makes the plasma anisotropic. This is particularly manifest in the anisotropy of the velocity of sound, which follows from the relation for the sound spectrum obtained from (2.21) in the limit $\omega^2 \ll \omega_{Li}^2$:

$$\int dx \sqrt{\frac{\omega^2}{v_s^2(x)} - k_y^2 - k_z^2} = \pi n, \qquad (2.23)$$

where $v_s(x) = (T_e/M)^{1/2}$ is the local velocity of sound. Formula (2.23) becomes particularly simple for the propagation of sound along the inhomogeneity direction. Indeed, in this case we have

$$\omega^2 = (\pi n)^2 \left[\int \frac{dx}{v_s(x)} \right]^{-2}. \qquad (2.24)$$

The damping of the sound waves is determined by the relation

$$\frac{\gamma}{\omega} = -\sqrt{\frac{\pi}{8}\frac{m}{M}}. \qquad (2.25)$$

It must be noted that the right half of (2.25) does not depend on the inhomogeneity of the plasma and coincides with the corresponding expression of the theory of the homogeneous plasma.

To conclude this section we present simple formulas for the spectrum and the damping of high-frequency oscillations in a Maxwellian plasma with homogeneous temperature and with a density that varies in a finite region near x = 0 like

$$N(x) = \frac{N_0}{1-(x/d)^2} \quad (x < d). \qquad (2.26)$$

We confine ourselves here to wave propagation along the inhomogeneity. Then there exists near x = 0 a transparency region, bounded by the points

$$\pm d \left[1 - \frac{\omega_{Le}^2(0)}{\omega^2} \right]^{1/2}.$$

The oscillation frequency ω is close to $\omega_{Le}(0)$ and is equal to [9]

$$\omega^2 = \omega_{Le}^2(0) \left[1 + \sqrt{3}\frac{r_{De}}{d}(2n+1) \right], \qquad (2.27)$$

where $r_{De} = \sqrt{T_e/4\pi e^2 N_0}$. The damping is given by

$$\frac{\gamma}{\omega} = -\frac{3\sqrt{3}d}{2r_{De}(2n+1)} \exp\left[-\frac{\sqrt{3}d}{2r_{De}(2n+1)} - \frac{3}{2} \right]. \qquad (2.28)$$

As in a homogeneous plasma, the damping of the oscillations in question, with a wavelength much larger than the Debye radius, is exponentially small.

3. SPECTRUM OF OSCILLATIONS OF AN INHOMOGENEOUS MAGNETOACTIVE PLASMA

We now proceed to investigate the oscillation spectrum of a weakly inhomogeneous magnetoactive plasma. In view of the large variety of oscillation modes of an inhomogeneous magnetoactive plasma, we consider separately the region of frequencies that are large compared with the drift frequencies of the particles, and the region of frequencies that are comparable with them. The present section is devoted to investigation of oscillations in the first of these regions. Drift oscillations of an inhomogeneous magnetoactive plasma will be investigated in the next two sections.

As already noted, in the region of frequencies that are large compared with the drift frequencies, $\omega \gg \omega_d$, the gradient terms can be neglected in the expression for the dielectric tensor of the inhomogeneous plasma. The eikonal equation (1.10) then coincides in form with the dispersion equation of the homogeneous plasma, but differs essentially from it in the fact that the density and the temperature of the particles (and in this frequency region also the magnetic field) are assumed dependent on the coordinates. Because of this circumstance, the natural frequencies of the oscillations of the inhomogeneous plasma are determined not by local but by integral relations. We confine ourselves here to an examination of only several branches of the oscillations of a magnetoactive plasma, namely high-frequency oscillations in the frequency region $\omega \gg \sqrt{M/m}\Omega_i$, when the motion of the ions can be neglected, and low-frequency oscillations lying in the region $\omega \ll \Omega_i$, when the motion in the plasma plays a decisive role. The cyclotron oscillations of the inhomogeneous plasma will not be considered. This question is dealt with in [17]. It can be shown that in the frequency region under consideration, when the gradient terms can be neglected, only long-wave oscillations are possible in the inhomogeneous plasma, like in the homogeneous one (an exception are cyclotron oscillations, which are not considered here). We shall therefore assume throughout this section that the condition $z = k_\perp^2 v_T^2/\Omega^2 \ll 1$ is satisfied.

We start the analysis with the simplest case of longitudinal (potential) oscillations of an inhomogeneous plasma. The eikonal equation (1.6) takes the form

$$\varepsilon_{33} k_z^2 + \varepsilon_{11} k_\perp^2 = 0. \qquad (3.1)$$

To obtain the dispersion equations of the oscillations of an inhomogeneous plasma we need to determine from this equation the complex function $k_x(\omega, x)$ and substitute it in (2.19). We consider the high-frequency region $\omega \gg \sqrt{M/m}\Omega_i$. If in addition, $\omega \pm \Omega_e$ and ω are much larger than $k_z v_{Te}$, then we get from (1.32) and (1.34)

$$\left.\begin{aligned} \varepsilon_{11} &= 1 - \frac{\omega_{Le}^2}{\omega^2 - \Omega_e^2} + i\sqrt{\frac{\pi}{8}} \frac{\omega_{Le}^2}{\omega |k_z| v_{Te}} \left[e^{-\frac{(\omega-\Omega_e)^2}{2k_z^2 v_{Te}^2}} + e^{-\frac{(\omega+\Omega_e)^2}{2k_z^2 v_{Te}^2}} \right], \\ \varepsilon_{33} &= 1 - \frac{\omega_{Le}^2}{\omega^2} + i\sqrt{\frac{\pi}{2}} \frac{\omega\omega_{Le}^2}{|k_z|^3 v_{Te}^3} e^{-\frac{\omega^2}{2k_z^2 v_{Te}^2}} \end{aligned}\right\} \qquad (3.2)$$

Substituting these expressions into (3.1) and neglecting in first approximation the exponentially small dissipative terms, we obtain the following dispersion equation for the determination of the oscillation frequency

$$\int dx \operatorname{Re} k_x(\omega, x) = \int dx \left[-k_y^2 - \frac{k_z^2(1-\omega_{Le}^2/\omega^2)}{1-\dfrac{\omega_{Le}^2}{\omega^2-\Omega_e^2}} \right]^{1/2} = \pi n. \quad (3.3)$$

We see therefore that these oscillations are hydrodynamically stable, $\omega^2 > 0$. Account of the small dissipative terms leads to damping of the oscillations in question. The damping decrement γ is exponentially small and is determined by the formula

$$\gamma = -\left(\int \frac{dx}{\operatorname{Re} k_x} \frac{\partial}{\partial\omega} \operatorname{Re} \frac{\varepsilon_{33}}{\varepsilon_{11}}\right)^{-1} \int \frac{dx}{\operatorname{Re} k_x} \operatorname{Im} \frac{\varepsilon_{33}}{\varepsilon_{11}}$$

$$= -\frac{\displaystyle\int \frac{dx}{\operatorname{Re} k_x} \frac{1}{(\operatorname{Re}\varepsilon_{11})^2} (\operatorname{Re}\varepsilon_{11} \operatorname{Im}\varepsilon_{33} - \operatorname{Re}\varepsilon_{33} \operatorname{Im}\varepsilon_{11})}{\displaystyle\int \frac{dx}{\operatorname{Re} k_x} \frac{1}{(\operatorname{Re}\varepsilon_{11})^2} \left(\operatorname{Re}\varepsilon_{11} \frac{\partial}{\partial\omega} \operatorname{Re}\varepsilon_{33} - \operatorname{Re}\varepsilon_{33} \frac{\partial}{\partial\omega} \operatorname{Re}\varepsilon_{11}\right)}. \quad (3.4)$$

In the region of low-frequency oscillations, when $\omega \ll \Omega_i$, the plasma oscillation spectrum depends essentially on the motion of the ions. The quantities ε_{11} and ε_{33} in this frequency region, if $\omega \gg k_z v_{Te}$, are of the form

$$\varepsilon_{11} = 1 + \frac{\omega_{Li}^2}{\Omega_i^2}, \quad \varepsilon_{33} = 1 - \frac{\omega_{Le}^2}{\omega^2} + i\sqrt{\frac{\pi}{2}} \frac{\omega\omega_{Le}^2}{|k_z|^3 v_{Te}^3} e^{-\frac{\omega^2}{2k_z^2 v_{Te}^2}}. \quad (3.5)$$

Substituting these expressions in (3.1) and taking into account the smallness of the dissipative terms, we obtain from (2.19) dispersion equations for the determination of the spectrum of the low frequency potential oscillations of the inhomogeneous plasma:

$$\int dx \operatorname{Re} k_x = \int dx \left[-k_y^2 - \frac{k_z^2(1-\omega_{Le}^2/\omega^2)}{1+c^2/v_A^2} \right]^{1/2} = \pi n, \quad (3.6)$$

$$\gamma = -\sqrt{\frac{\pi}{2}}\, \omega^2 \left[\int \frac{dx}{\operatorname{Re} k_x} \frac{\omega_{Le}^2 k_z^2}{\omega^2(1+c^2/v_A^2)} \right]^{-1} \times \int \frac{dx}{\operatorname{Re} k_x} \frac{\omega_{Le}^2}{|k_z| v_{Te}^3} \frac{\exp\left(-\dfrac{\omega^2}{2k_z^2 v_{Te}^2}\right)}{1+c^2/v_A^2},$$

where $v_A = \sqrt{B_0^2/4\pi NM}$ is the Alfven velocity. We see therefore that the oscillations under consideration are damped, although the damping is exponentially small.

Low frequency oscillations are damped much more intensely under conditions when $\omega \ll k_z v_{Te}$ in the entire transparency region. It is necessary to assume here that $\omega \gg k_z v_{Ti}$, for the opposite case there is no transparency region and oscillations become impossible. The last condition enables us to neglect the ion absorption of the waves in the plasma. The components ε_{11} and ε_{33} have under these conditions the form

$$\left.\begin{aligned} \varepsilon_{11} &= 1 + \frac{\omega_{Li}^2}{\Omega_i^2} = 1 + \frac{c^2}{v_A^2}, \\ \varepsilon_{33} &= 1 - \frac{\omega_{Li}^2}{\omega^2} + \frac{\omega_{Le}^2}{k_z^2 v_{Te}^2} + i\sqrt{\frac{\pi}{2}} \frac{\omega\omega_{Le}^2}{|k_z|^3 v_{Te}^3}. \end{aligned}\right\} \quad (3.7)$$

The dispersion equations for the spectrum of the oscillations under consideration are written, on the other hand, in the form

$$\left.\begin{aligned} &\int dx \operatorname{Re} k_x = \int dx \left[-k_y^2 - \frac{\omega_{Le}^2}{v_{Te}^2} + \frac{k_z^2(1-\omega_{Li}^2/\omega^2)}{1+c^2/v_A^2} \right]^{1/2} = \pi n, \\ &\gamma = -\sqrt{\frac{\pi}{2}}\, \omega^2 \left[\int \frac{dx}{\operatorname{Re} k_x} \frac{k_z^2 \omega_{Li}^2}{\omega^2(1+c^2/v_A^2)} \right]^{-1} \int \frac{dx}{\operatorname{Re} k_x} \frac{\omega_{Le}^2}{|k_z| v_{Te}^3} \frac{1}{1+c^2/v_A^2} \end{aligned}\right\} \quad (3.8)$$

It is easy to see that these oscillations are also damped, and the damping occurs in this case much more strongly than in the preceding case, since the damping decrement γ is no longer exponentially small. In the limit of a homogeneous plasma the formulas (3.3), (3.4), (3.6), and (3.8) obtained above go over into the known formulas for the spectrum of the potential oscillations of a homogeneous magnetoactive plasma [1,18].

A magnetoactive plasma is an anisotropic medium. The potential oscillations of the electromagnetic field in such a plasma are in general not natural oscillations. We therefore consider here general nonpotential oscillations of an inhomogeneous magnetoactive plasma under conditions corresponding to the potential oscillations discussed above. In the region of high frequencies $\omega \gg \sqrt{M/m}\,\Omega_i$ and under the condition $\omega, \omega \pm \Omega_e \gg k_z v_{Te}$, the total eikonal equation (1.5) assumes the form

$$k_\perp^4 \varepsilon_{11} + \left[\left(k_z^2 - \frac{\omega^2}{c^2}\varepsilon_{11}\right)(\varepsilon_{11}+\varepsilon_{33}) - \frac{\omega^2}{c^2}\varepsilon_{12}^2 \right] k_\perp^2 + \varepsilon_{33}\left[\left(k_z^2 - \frac{\omega^2}{c^2}\varepsilon_{11}\right)^2 + \frac{\omega^4}{c^4}\varepsilon_{12}^2 \right] = 0, \quad (3.9)$$

where ε_{11} and ε_{33} are given by (3.2), and

$$\varepsilon_{12} = i\frac{\omega_{Le}^2 \Omega_e}{\omega(\omega^2-\Omega_e^2)} + \sqrt{\frac{\pi}{2}} \frac{\omega_{Le}^2}{\omega |k_z| v_{Te}} \left[e^{-\frac{(\omega-\Omega_e)^2}{2k_z^2 v_{Te}^2}} - e^{-\frac{(\omega+\Omega_e)^2}{2k_z^2 v_{Te}^2}} \right]. \quad (3.10)$$

Equation (3.9) determines two functions $k_x^2(\omega, x)$, which corresponds to two branches of the high frequency electron oscillations of a magnetoactive plasma—ordinary and extraordinary waves. Neglecting the exponentially small dissipative terms in (3.2) and (3.10), we obtain for the determination of the spectrum of the oscillations under consideration the following dispersion equations:

$$\int dx \operatorname{Re} k_x = \int dx \left(-k_y^2 - P \mp \sqrt{P^2 - q}\right)^{1/2} = \pi n, \quad (3.11)$$

describing the ordinary and extraordinary waves in an inhomogeneous magnetoactive plasma. We use here the notation

$$P = \frac{1}{2\varepsilon_{11}} \left[\left(k_z^2 - \frac{\omega^2}{c^2}\varepsilon_{11}\right)(\varepsilon_{11}+\varepsilon_{33}) - \frac{\omega^2}{c^2}\varepsilon_{12}^2 \right],$$

$$q = \frac{\varepsilon_{33}}{\varepsilon_{11}} \left[\left(k_z^2 - \frac{\omega^2}{c^2}\varepsilon_{11}\right)^2 + \frac{\omega^4}{c^4}\varepsilon_{12}^2 \right].$$

It is easy to see that in the limit, when $c^2 k_z^2/\omega^2 \gg \varepsilon_{11}$ or ε_{12}, Eq. (3.9) goes over into the eikonal equation

(3.1) for the potential oscillations. One equation of (3.11) goes over here into Eq. (3.3), which describes the high frequency potential oscillations of a magnetoactive plasma. In this limit, oscillations described by the second equation of (3.11) are impossible. Thus, the high-frequency potential oscillations of a magnetoactive inhomogeneous plasma can propagate only at a sufficiently acute angle to the magnetic field.

In the opposite limiting case $c^2k_z^2/\omega^2 \ll \varepsilon_{11}$ or ε_{12}, that is, when the oscillations propagate almost transversely to the magnetic field, Eqs. (3.11) assume the simple form

$$\left.\begin{aligned}\int dx\,\mathrm{Re}\,k_x &= \int dx\left[-k_y^2+\frac{\omega^2}{c^2}\left(1-\frac{\omega_{Le}^2}{\omega^2}\right)\right]^{1/2}=\pi n,\\ \int dx\,\mathrm{Re}\,k_x &= \int dx\left[-k_y^2+\frac{1}{c^2}\frac{(\omega^2-\Omega_e^2)^2-\omega^2\Omega_e^2}{\omega^2-\Omega_e^2-\omega_{Le}^2}\right]^{1/2}=\pi n.\end{aligned}\right\} \tag{3.12}$$

In strictly transverse propagation ($k_z = 0$), the oscillations of a collisionless plasma are undamped, since the dissipative terms of the dielectric tensor (antihermitian part) are in this case exactly equal to zero. However, when $k_z \neq 0$ the oscillations described by Eqs. (3.11) and (3.12) attenuate only exponentially, but attentuate all the same. Thus, in the limit of $c^2k_z^2/\omega^2 \ll \varepsilon_{11}$ or ε_{12}, when relations (3.12) are valid, the oscillations determined by the first of these relations are undamped, since the phase velocity of such oscillations exceeds the velocity of light. On the other hand, the oscillations determined by the second equation of (3.12) are damped, with a damping decrement*

$$\gamma = -\frac{\int \frac{dx}{\mathrm{Re}\,k_x}\mathrm{Im}\left(\varepsilon_{11}+\frac{\varepsilon_{12}^2}{\varepsilon_{11}}\right)}{\int \frac{dx}{\mathrm{Re}\,k_x}\frac{\partial}{\partial\omega}\mathrm{Re}\left(\varepsilon_{11}+\frac{\varepsilon_{12}^2}{\varepsilon_{11}}\right)} = -\frac{\int \frac{dx}{\mathrm{Re}\,k_x}\left[\varepsilon_{11}''+\frac{\varepsilon_{12}''^2}{\varepsilon_{11}'}\left(-\frac{\varepsilon_{11}''}{\varepsilon_{11}'}+2\frac{\varepsilon_{12}'}{\varepsilon_{12}''}\right)\right]}{\int \frac{dx}{\mathrm{Re}\,k_x}\left[\frac{\partial\varepsilon_{11}'}{\partial\omega}+\frac{\varepsilon_{12}''^2}{\varepsilon_{11}'}\left(-\frac{\partial\varepsilon_{11}'/\partial\omega}{\varepsilon_{11}'}+2\frac{\partial\varepsilon_{12}''/\partial\omega}{\varepsilon_{12}''}\right)\right]}. \tag{3.13}$$

We now proceed to consider low-frequency oscillations of an inhomogeneous plasma, when $\omega \ll \Omega_i$. In the frequency region $\omega \gg k_z v_{Te}$, corresponding to the limit of two-fluid hydrodynamics of a collisionless plasma [1], the eikonal equation breaks up into the following two equations

$$k^2-\frac{\omega^2}{c^2}\varepsilon_{22}=0,\qquad \varepsilon_{33}\left(k_z^2-\frac{\omega^2}{c^2}\varepsilon_{11}\right)+\varepsilon_{11}k_\perp^2=0, \tag{3.14}$$

where ε_{11} and ε_{33} are determined by expressions (3.4), and $\varepsilon_{22} = \varepsilon_{11}$. The first of these equations leads, taking (2.19) into account, to the dispersion equation

$$\int dx\,\mathrm{Re}\,k_x=\int dx\left[-k_y^2-k_z^2+\frac{\omega^2}{c^2}\left(1+\frac{c^2}{v_A^2}\right)\right]^{1/2}=\pi n, \tag{3.15}$$

*To avoid misunderstanding we note that the hermitian part of the component ε_{12} is imaginary and the antihermitian part is real [see (3.10)].

which describes undamped low-frequency oscillations of an inhomogeneous plasma. From the second equation of (3.14) we obtain the following dispersion relations:

$$\left.\begin{aligned}\int dx\,\mathrm{Re}\,k_x &= \int dx\left[-k_y^2-\frac{k_z^2-\frac{\omega^2}{c^2}\left(1+\frac{c^2}{v_A^2}\right)}{1+c^2/v_A^2}\left(1-\frac{\omega_{Le}^2}{\omega^2}\right)\right]^{1/2} = \pi n,\\ \gamma &= -\sqrt{\frac{\pi}{8}}\frac{\int\frac{dx}{\mathrm{Re}\,k_x}\left(\omega^2-\frac{k_z^2c^2}{1+c^2/v_A^2}\right)\frac{\omega_{Le}^2}{|k_z|^3v_{Te}^3}e^{-\frac{\omega^2}{2k_z^2v_{Te}^2}}}{\int\frac{dx}{\mathrm{Re}\,k_x}\left[1-\frac{\omega_{Le}^2k_z^2c^2}{\omega^4(1+c^2/v_A^2)}\right]}.\end{aligned}\right\} \tag{3.16}$$

We see therefore that the low-frequency oscillations of an inhomogeneous magnetoactive plasma, which we are considering, are damped, although the damping is exponentially small.

In the frequency region $\omega \ll k_z v_A$, Eqs. (3.15) have no solutions, that is, such oscillations cannot occur in a plasma. On the other hand, Eqs. (3.16) go over in this limit into Eqs. (3.6), which describe low-frequency potential oscillations of a magnetoactive plasma. Thus, the condition $\omega \ll k_z v_A$ determines the region of applicability of formulas (3.6) or, what is the same, the region where the oscillations are potential. In the opposite limit, when $\omega \gg k_z v_E$, the oscillations described by Eqs. (3.16) become impossible if we assume $\Omega_i \ll \omega_{Le}$, which is practically always satisfied in a real plasma. To the contrary, Eq. (3.15) has in this frequency region solutions which correspond to transverse Alfven oscillations of an inhomogeneous plasma.

To conclude this section let us consider oscillations of an inhomogeneous plasma in the frequency region $k_z v_{Ti} \ll \omega \ll k_z v_{Te}$, corresponding to the limit of the one-fluid hydrodynamics of a non-isothermal ($T_e \gg T_i$) collisionless plasma [19]. The eikonal equation (1.5) in this frequency region breaks up into the following two equations [16,20]:

$$k_z^2-\frac{\omega^2}{c^2}\varepsilon_{11}=0,\qquad \left(k^2-\frac{\omega^2}{c^2}\varepsilon_{22}\right)\varepsilon_{33}+\frac{\omega^2}{c^2}\varepsilon_{23}^2=0, \tag{3.17}$$

where (we confine ourselves here to a plasma of relatively high density, in which $c^2 \gg v_A^2$, for strictly speaking the equations of one-fluid hydrodynamics are applicable only to such a plasma)

$$\left.\begin{aligned}\varepsilon_{11} &= \frac{c^2}{v_A^2},\qquad \varepsilon_{22}=\frac{c^2}{v_A^2}+i\sqrt{2\pi}\,\frac{\omega_{Le}^2k_\perp^2v_{Te}}{\omega\Omega_e^2|k_z|},\\ \varepsilon_{23} &= i\frac{k_\perp}{k_z}\frac{\omega_{Le}^2}{\omega\Omega_e}\left(1+i\sqrt{\frac{\pi}{2}}\frac{\omega}{|k_z|v_{Te}}\right),\\ \varepsilon_{33} &= -\frac{\omega_{Li}^2}{\omega^2}+\frac{\omega_{Le}^2}{k_z^2v_{Te}^2}\left(1+i\sqrt{\frac{\pi}{2}}\frac{\omega}{|k_z|v_{Te}}\right).\end{aligned}\right\} \tag{3.18}$$

From the first equation of (3.15) follows directly the relation $\omega^2 = k_z^2v_A^2$, which describes the spectrum of undamped Alfven oscillations of an inhomogeneous plasma, in which the mass velocity vector lies in the plane perpendicular to the plane of the vectors **k** and

B_0 (that is, is parallel to the y axis). The dependence of the frequency of such oscillations on the coordinates should not cause any misunderstanding, since the eigenvalue problem does not arise for these field oscillations (an analogous situation occurs also in ordinary magnetohydrodynamics of an ideally conducting liquid [21]).

From the second equation of (3.17), taking into account relations (2.19), we obtain the following dispersion equations for the determination of the spectrum of the magnetic-sound oscillations of an inhomogeneous non-isothermal plasma:

$$\left.\begin{aligned} &\int dx \,\mathrm{Re}\, k_x = \int dx \left[-k_y^2 + \frac{(\omega^2 - k_z^2 v_A^2)(\omega^2 - k_z^2 v_s^2)}{\omega^2 (v_A^2 + v_s^2) - k_z^2 v_A^2 v_s^2} \right]^{1/2} = \pi n, \\ &\gamma = -\sqrt{\frac{\pi}{8}} \left[\int \frac{dx}{\mathrm{Re}\, k_x} \frac{2[\omega^2 (v_A^2 + v_s^2) - 2k_z^2 v_A^2 v_s^2]}{\omega^2 (v_A^2 + v_s^2) - k_z^2 v_A^2 v_s^2} \right]^{-1} \\ &\quad \times \int \frac{dx}{\mathrm{Re}\, k_x} \frac{\omega^2 - k_z^2 v_z^2}{|k_z| v_{Te} [\omega^2 (v_A^2 + v_s^2) - k_z^2 v_A^2 v_s^2]} \\ &\quad \times \left[1 - \frac{v_A^2 (\omega^2 - k_z^2 v_s^2) \left(1 + \frac{k_z^2 v_s^4}{\omega^2 v_A^2} \right)}{\omega^2 (v_A^2 + v_s^2) - k_z^2 v_A^2 v_s^2} \right], \end{aligned}\right\} \tag{3.19}$$

where $v_s = \sqrt{T_e/M}$ is the local velocity of sound in the inhomogeneous plasma. Formulas (3.19) generalize the spectrum (2.21) to the case when a magnetic field is present. In the case of a homogeneous plasma these formulas go over into the well-known formulas for the spectrum of the low-frequency oscillations of a non-isothermal magnetoactive plasma [1,19,20].

4. SPECTRUM OF LOW FREQUENCY DRIFT POTENTIAL OSCILLATIONS OF AN INHOMOGENEOUS PLASMA

In the preceding two sections we considered, using the method of geometrical optics, the oscillations of an inhomogeneous plasma under conditions when the particle drift could be neglected. We now proceed to an investigation of low-frequency drift oscillations of an inhomogeneous plasma. In an inhomogeneous plasma confined by a strong magnetic field, particle drifts are produced transverse to the magnetic field. The electrons and ions of the plasma drift in opposite directions, as a result of which a relative motion of particles in the plasma takes place. The relative motion of the charged particles can in turn lead to a buildup of plasma oscillations, that is, to instability of an inhomogeneous plasma. Such an instability is analogous to two-stream instability in the sense that it occurs under conditions when there are in the plasma slow waves, whose phase velocity is close to the velocity of the relative particle drift. In the present review we confine ourselves essentially to an account of only Larmor particle drift, that is, drift due to inhomogeneity of the density and temperature of the particles. Diamagnetic drift, which is due either to inhomogeneity or curvature of the magnetic-field force lines, will not be considered (see [22-24] concerning oscillations of an inhomogeneous plasma with account of diamagnetic drift*). Moreover, we confine ourselves here to an investigation of drift oscillations of an inhomogeneous low-pressure plasma, when $\beta = 8\pi P/B_0^2 \ll 1$. It is precisely under this assumption that we have obtained an expression for the dielectric tensor of an inhomogeneous plasma. We emphasize once more that this limitation is not essential in the region of frequencies that are large compared with the drift frequencies, so that the results of the preceding two sections are valid for a plasma of arbitrary pressure under conditions when particle collisions can be neglected. On the other hand, in the region of frequencies that are comparable with drift frequencies, $\omega \lesssim \omega_d \sim k_y v_T^2/\Omega L$, this limitation is important to us. All the results discussed in Secs. 4-6 are valid only for low-pressure plasma.

It will be shown below that in the region of drift frequencies $\omega \lesssim \omega_d$ new branches appear in the spectrum of the oscillations of the inhomogeneous plasma, connected with the drift of the particles and missing in the case of a homogeneous plasma. Under real conditions the drift frequencies are quite low (the observed drift frequencies are of the order of several dozen kcs [25]). Taking this into account, we confine ourselves only to an investigation of low-frequency drift oscillations of an inhomogeneous plasma, when $\omega \ll \Omega_i$.†

The present section is devoted to potential drift oscillations of an inhomogeneous plasma. The drift oscillations of the field can be regarded as potential only under certain conditions. These conditions will be derived in the next section, where we investigate general oscillations, in the main non-potential, of an inhomogeneous plasma and analyze the total eikonal equation (1.10) in the region of drift frequencies.

From (1.35) we obtain the following eikonal equation for the potential drift oscillations of an inhomogeneous plasma in the frequency region $\omega \ll \Omega_i$:

$$1 + \sum \frac{1}{k^2 r_D^2} \left\{ 1 - \left[1 - \frac{k_y v_T^2}{\omega\Omega} \left(\frac{\partial \ln N}{\partial x} + \frac{\partial T}{\partial x} \frac{\partial}{\partial T} \right) \right] \times A_0(z) J_+ \left(\frac{\omega}{|k_z| v_T} \right) \right\} = 0, \tag{4.1}$$

*The article[22] is among the first devoted to the investigation of the stability of an inhomogeneous plasma on the basis of the kinetic equation. A great influence on the development of the theory of oscillations of inhomogeneous plasma was exerted by the paper[23a], in which the finite Larmor radius of the ions was first taken into account.

†Drift oscillations of an inhomogeneous plasma in the region of the ion and electron cyclotron frequencies are discussed in[26,27]. The interaction between a beam of charged particles and the drift oscillations of an inhomogeneous plasma is considered in[28].

where $r_D = \sqrt{T/4\pi e^2 N}$—Debye radius of the particles. In writing out this equation we have also assumed that $k_z v_T \ll \Omega$. Only under this condition can we confine ourselves to terms with $n = 0$ in (1.35). To obtain the dispersion equations of the oscillations we must determine from (4.1) the complex function $k_x(\omega, x)$ and substitute it in (2.19). We then distinguish between three limiting cases: a) $z_e \ll z_i$, corresponding to longwave oscillations of the plasma; b) $z_e \ll 1$ and $z_i \gg 1$, corresponding to short-wave oscillations, with wavelength shorter than the Larmor radius of the ions, but longer than the Larmor radius of the electrons; c) $z_i \gg z_e \gg 1$, corresponding to very short wavelengths, shorter than the Larmor radii of both ions and electrons. In all these cases we are considering oscillations in the frequency region $\omega \gg k_z v_{Ti}$, for in the opposite limit, as can be readily seen from (4.1), the potential oscillations of an inhomogeneous plasma are impossible if the inhomogeneity of the magnetic field is neglected (there occurs the usual Debye screening of the field in the plasma in this frequency region).

a) In the region of long-wave oscillations $z_i \ll 1$, and under the condition $\omega \gg k_z v_{Ti}$, the eikonal equation (4.1) assumes the form

$$k_\perp^2 \left[1 + \frac{c^2}{v_A^2} \left(1 - \frac{k_y v_{Ti}^2}{\omega \Omega_i} \frac{\partial}{\partial x} \ln NT_i \right) \right] + k_z^2 + \sum r_D^{-2} \left\{ 1 - \left[1 - \frac{k_y v_T^2}{\omega \Omega} \left(\frac{\partial \ln N}{\partial x} + \frac{\partial T}{\partial x} \frac{\partial}{\partial T} \right) \right] \times J_+\left(\frac{\omega}{|k_z| v_T} \right) \right\} = 0, \qquad (4.2)$$

where $v_A = \sqrt{B_0^2/4\pi NM}$ is the Alfven velocity. By determining from this equation the complex function $k_x(\omega, x)$ and using (2.19), we obtain the following dispersion equations for the spectrum of the long-wave oscillations of the inhomogeneous plasma:

$$\left.\begin{aligned} \int dx \operatorname{Re} k_x &= \int dx \left\{ -k_y^2 - \frac{k_z^2 + \sum r_D^{-2} \left\{ 1 - \left[1 - \frac{k_y v_T^2}{\omega \Omega} \left(\frac{\partial \ln N}{\partial x} + \frac{\partial T}{\partial x} \frac{1}{\partial T} \right) \right] \right\} \operatorname{Re} J_+ \left(\frac{\omega}{|k_z| v_T} \right)^{1/2}}{1 + \frac{c^2}{v_A^2} \left(1 - \frac{k_y v_{Ti}^2}{\omega \Omega_i} \frac{\partial}{\partial x} \ln NT_i \right)} \right. = \pi n, \\ \gamma &= \sqrt{\frac{\pi}{8}} \left(\int dx \frac{\partial \operatorname{Re} k_x}{\partial \omega} \right)^{-1} \int \frac{dx}{\operatorname{Re} k_x} \frac{\omega \omega_{Le}^2}{|k_z| v_{Te}^3} \frac{\left[1 - \frac{k_y}{\omega} \frac{v_{Te}^2}{\Omega_e} \left(\frac{\partial}{\partial x} \ln \frac{N}{\sqrt{T_e}} + \frac{\omega^2}{2k_z^2 v_{Te}^2} \frac{\partial \ln T_e}{\partial x} \right) \right] e^{-\frac{\omega^2}{2k_z^2 v_{Te}^2}}}{1 + \frac{c^2}{v_A^2} \left(1 - \frac{k_y v_{Ti}^2}{\omega \Omega_i} \frac{\partial}{\partial x} \ln NT_i \right)}. \end{aligned}\right\} \qquad (4.3)$$

It is important to note that Eqs. (4.3) are valid under the assumption that in the entire region of plasma transparency the integrands have no singularities, that is,

$$1 + \frac{c^2}{v_A^2} \left(1 - \frac{k_y v_{Ti}^2}{\omega \Omega_i} \frac{\partial}{\partial x} \ln NT_i \right) \neq 0. \qquad (4.4)$$

In the region of "high" frequencies, considerably larger than the drift frequencies of the electrons and the ions, the gradient terms in (4.3) can be neglected. These equations go over into Eqs. (3.6), which were considered in the preceding section. The plasma oscillations, as shown above, are always damped in this frequency region.

The situation is different in the region of "low" frequencies, comparable with the drift frequencies of the particles in an inhomogeneous plasma. The plasma oscillations in this frequency region can become unstable under certain conditions. In order to verify this, we consider long-wave hydrodynamic oscillations ($\omega \gg k_z v_{Te}$) of an inhomogeneous plasma in the region of frequencies smaller than the drift frequencies of both the electrons and the ions. In a plasma of relatively high frequency, in which $c^2 \gg v_A^2$ (such a case is encountered in practice quite frequently), we then obtain from the first equation of (4.3) [29]

$$\int dx \operatorname{Re} k_x = \int dx \left(-k_y^2 - \frac{M}{m} \frac{\Omega_i^2}{\omega^2} \frac{T_e}{T_i} k_z^2 \frac{\partial \ln NT_e}{\partial \ln NT_i} \right)^{1/2} = \pi n. \qquad (4.5)$$

We see therefore that when the local inequality

$$\frac{\partial \ln NT_e}{\partial \ln NT_i} > 0 \qquad (4.6)$$

is satisfied in the transparency region of the plasma, which in this case is determined by the condition $\omega \gg k_z v_{Te}$, the oscillation frequencies are pure imaginary and correspond to aperiodic instability of the inhomogeneous plasma relative to such oscillations. This instability of the plasma is not connected with the dissipative processes and in this sense is purely hydrodynamic. The growth increment of such oscillations can reach sufficiently large values, namely the maximum increment is $\gamma_{max} \sim v_{Ti}/L \ll \Omega_i$. However, from the condition $\omega \gg k_z v_{Te}$ it follows that such an instability is possible in a plasma whose longitudinal dimensions (along the magnetic field) exceed the transverse ones by more than $\sqrt{M/m} \sim 40$ times. We note also that from the condition that the long wave oscillations be potential, namely $\omega \ll k_z v_A$ (see formula (5.4) of the next section), it follows that the oscillations in question can exist only in a plasma of very low pressure, when $\beta = 8\pi P/B_0^2 \ll m/M$.

We now consider long-wave drift oscillations in the frequency region $k_z v_{Ti} \ll \omega \ll k_z v_{Te}$. From the dispersion equations (4.3) in this frequency region we obtain

$$\int dx \, \mathrm{Re}\, k_x = \int dx \times \left\{ -k_y^2 - \frac{k_z^2\left[1-\frac{\omega_{Li}^2}{\omega^2}\left(1-\frac{k_y v_{Ti}^2}{\omega\Omega_i}\frac{\partial}{\partial x}\ln NT_i\right)\right]+\frac{\omega_{Le}^2}{v_{Te}^2}\left(1-\frac{k_y v_{Te}^2}{\omega\Omega_e}\frac{\partial \ln N}{\partial x}\right)}{1+\frac{c^2}{v_A^2}\left(1-\frac{k_y v_{Ti}^2}{\omega\Omega_i}\frac{\partial}{\partial x}\ln NT_i\right)} \right\}^{1/2} = \pi n \tag{4.7a}$$

and

$$\gamma = \sqrt{\frac{\pi}{8}}\left(\int dx \frac{\partial \,\mathrm{Re}\, k_x}{\partial \omega}\right)^{-1} \int \frac{dx}{\mathrm{Re}\, k_x} \frac{\omega \omega_{Le}^2}{|k_z| v_{Te}^3} \frac{1-\frac{k_y v_{Te}^2}{\omega\Omega_e}\frac{\partial}{\partial x}\ln\frac{N}{\sqrt{T_e}}}{1+\frac{c^2}{v_A^2}\left(1-\frac{k_y v_{Ti}^2}{\omega\Omega_i}\frac{\partial}{\partial x}\ln NT_e\right)}. \tag{4.7b}$$

If we neglect the gradient terms in the region of frequencies larger than drift frequencies, these equations go over into Eqs. (3.7). The frequencies determined by the first equation of (4.7) are always real, thus evidencing hydrodynamic stability of the inhomogeneous plasma relative to the oscillations under consideration. The damping decrement γ, however, can in this case become positive. This corresponds to kinetic instability of the plasma (with γ becoming the growth increment of the oscillations). A simple analysis of (4.7) shows that for both negative and positive values of (4.4) the following local condition must be satisfied in the transparency region if the plasma is to be unstable relative to the oscillations in question:

$$1-\frac{k_y v_{Te}^2}{\omega\Omega_e}\frac{\partial \ln N}{\partial x}\left(1-\frac{1}{2}\frac{\partial \ln T_e}{\partial \ln N}\right) < 0. \tag{4.8}$$

In the case when (4.4) is positive and $\omega > k_z v_s$, this condition can be replaced by the stronger local condition for plasma instability [3]

$$1-\frac{1}{1+(k_y^2+k_z^2)r_{De}^2} > \frac{1}{2}\frac{\partial \ln T_e}{\partial \ln N}. \tag{4.9}$$

It follows from this, in particular, that oscillations with wavelength longer than the Debye radius of the electrons are certainly unstable, provided $\partial \ln T_e/\partial \ln N \le 0$ in the entire region of plasma transparency. On the other hand, oscillations with wavelength shorter than the Debye radius are unstable if $\partial \ln T_e/\partial \ln N \le 2$. Such instability conditions for an inhomogeneous plasma are indicated in [3,15] (the first of these conditions, with an inequality sign was derived earlier in [30-32]*). We must point out also the instability condition which holds when (4.4) is positive. Namely, in the region of very low frequencies $\omega < k_z v_s$, which are possible only in a strongly nonisothermal plasma in which $T_e \gg T_i$, it follows from the local condition (4.8) that the inhomogeneous plasma can be stable relative to oscillations with wavelength larger than the Debye radius of the electrons if $\partial \ln T_e/\partial \ln N \le 2$ in the entire transparency region.

In the case of negative values of (4.4) the local instability condition (4.8) can be replaced by the stronger condition

$$1+\left(1+\frac{v_A^2}{c^2}\right)\frac{T_e}{T_i}\frac{\partial \ln N/\sqrt{T_e}}{\partial \ln NT_i} < 0. \tag{4.10}$$

In a nonisothermal plasma in which $T_e \gg T_i$, the quantity (4.4) is positive and such an instability is therefore impossible. As regards an isothermal plasma, condition (4.10) can be satisfied in it (for $c^2 \gg v_A^2$) if $-4 < \partial \ln T/\partial \ln N < -1$.

The growth increments of the oscillations under consideration can reach values comparable with the oscillation frequency, which is of the order of the drift frequency of the electrons, that is,

$$\gamma_{\max} < \omega \ll \frac{T_e}{T_i}\frac{v_{T_i}}{L}.$$

It must be noted that the local instability conditions given above are necessary but not sufficient conditions for the instability of an inhomogeneous plasma. On the other hand, satisfaction of these conditions in the entire plasma transparency region is certainly sufficient for the instability. The necessary and sufficient condition of instability can be written in the form of the integral relation $\gamma > 0$. The foregoing pertains to all the kinetic plasma instabilities which will be considered below. This will therefore be implied without special stipulation.

b) We now proceed to consider short-wave drift oscillations of an inhomogeneous plasma under conditions when $z_i \gg 1$ and $z_e \ll 1$. In a homogeneous plasma there are no low frequency oscillations in this region of wavelengths ($\omega \ll \Omega_i$). We shall therefore consider here only the region of frequencies that are comparable with the drift frequencies of the particles, in which the plasma inhomogeneity becomes essentially manifest. Recognizing that $k_\perp \gg k_z$ in the wavelength region under consideration, we obtain from (4.1) in the limit $\omega \gg k_z v_{Te}$ the following eikonal equation:

$$k_\perp^2\left(1+\frac{\omega_{Le}^2}{\Omega_e^2}\right)+\frac{\omega_{Li}^2}{v_{Ti}^2}+\frac{\omega_{Le}^2}{v_{Te}^2}\left\{\frac{k_y v_{Te}^2}{\omega\Omega_e}\frac{\partial \ln N}{\partial x}\right.$$
$$+ i\sqrt{\frac{\pi}{2}}\frac{\omega}{|k_z| v_{Te}} e^{-\frac{\omega^2}{2k_z^2 v_{Te}^2}}$$
$$\left.\times\left[1-\frac{k_y v_{Te}^2}{\omega\Omega_e}\left(\frac{\partial}{\partial x}\ln\frac{N}{\sqrt{T_e}}+\frac{\omega^2}{2k_z^2 v_{Te}^2}\frac{\partial \ln T_e}{\partial x}\right)\right]\right\} = 0. \tag{4.11}$$

From this, taking into account (2.19), we obtain the following dispersion equations for the spectrum of the shortwave oscillations of the inhomogeneous plasma:

*See also[33], which contains the statement that the inhomogeneous plasma is unstable against long-wave oscillations at constant temperature, that is, when $\partial \ln T/\partial \ln N = 0$.

$$\left.\begin{aligned}&\int dx\,\mathrm{Re}\,k_x=\int dx\left\{-k_y^2-\frac{\omega_{Li}^2}{v_{Ti}^2}\frac{1-\frac{k_y v_{Ti}^2}{\omega\Omega_i}\frac{\partial\ln N}{\partial x}}{1+\omega_{Le}^2/\Omega_e^2}\right\}^{1/2}=\pi n,\\ &\gamma=-\sqrt{\frac{\pi}{2}}\,\omega^2\left(\int\frac{dx}{\mathrm{Re}\,k_x}\frac{\frac{k_y\omega_{Li}^2}{\omega\Omega_i}\frac{\partial\ln N}{\partial x}}{1+\omega_{Le}^2/\Omega_e^2}\right)^{-1}\int\frac{dx}{\mathrm{Re}\,k_x}\frac{\omega_{Le}^2}{|k_z|\,v_{Te}^3}\frac{1}{1+\omega_{Le}^2/\Omega_e^2}\,e^{-\frac{\omega^2}{2k_z^2v_{Te}^2}}\left[1+\frac{T_e}{T_i}\frac{k_y v_{Ti}^2}{\omega\Omega_i}\frac{\partial\ln N}{\partial x}\left(1+\frac{\omega^2}{2k_z^2v_{Te}^2}\frac{\partial\ln T_e}{\partial\ln N}\right)\right].\end{aligned}\right\}\quad(4.12)$$

It is easy to see that the spectrum determined by these equations corresponds to hydrodynamically stable oscillations. If in addition we recognize that in the entire plasma transparency region we have

$$\frac{k_y}{\omega\Omega_i}\frac{\partial}{\partial x}\ln N>0,$$

then we get from the expression for γ that if $\partial\ln T_e/\partial\ln N\geq 0$ (when it is satisfied in the entire transparency region) the oscillations in question are also kinetically stable. However, since $\omega^2\gg k_z^2v_{Te}^2$, these oscillations can become unstable for any finite negative value of the quantity $\partial\ln T_e/\partial\ln N<0$ in the transparency region of the plasma. The growth increment of such oscillations is exponentially small, and is therefore of little interest from the point of view of plasma instability. We note that in accordance with the condition $\omega\gg k_z v_{Te}$, such oscillations can exist only in a plasma whose longitudinal dimensions exceed the transverse dimensions by at least a factor $\sqrt{M/m}\sim 40$. From the condition that the oscillations be potential [see (5.4)] it follows that such oscillations are possible when

$$\beta\ll\frac{m}{M}z_i\sim z_e\ll 1.$$

From the point of view of plasma stability, of greater interest are the short-wave drift oscillations in the frequency region $k_z v_{Ti}\ll\omega\ll k_z v_{Te}$. As will be shown below, an inhomogeneous plasma is practically always unstable against such oscillations, with the growth increment of the oscillations being far from small [see (4.15) and (4.16)]. The eikonal equation for the potential oscillations of the inhomogeneous plasma (4.1) assumes in this region of wavelengths and oscillation frequencies the form*

$$\left(1+\frac{T_e}{T_i}\right)+\frac{T_e}{T_i}\frac{|\Omega_i|}{\sqrt{2\pi}k_\perp v_{Ti}}\frac{k_y v_{Ti}^2}{\omega\Omega_i}\frac{\partial}{\partial x}\ln\frac{N}{\sqrt{T_i}}$$
$$+i\sqrt{\frac{\pi}{2}}\frac{\omega}{|k_z|v_{Te}}\frac{T_e}{T_i}\frac{k_y v_{Ti}^2}{\omega\Omega_i}\frac{\partial}{\partial x}\ln\frac{N}{\sqrt{T_e}}=0.\quad(4.13)$$

*For the sake of simplicity we confine ourselves here to an investigation of oscillations with wavelength larger than the Debye radius of the particles, which is legitimate under these conditions only in a plasma of relatively high density, with $c^2\gg v_A^2$. An account of the finite Debye radius of the particles does not change the local instability condition (4.18) derived below for an inhomogeneous plasma. On the other hand, the oscillation frequencies decrease in this case by an amount $1/(1+k_\perp^2 r_{Di}^2)$.

To determine the spectrum of the oscillations of the inhomogeneous plasma we obtain in this case the following dispersion equation [3]

$$\left.\begin{aligned}&\int dx\,\mathrm{Re}.k_x=\int dx\,|k_y|\left[-1+\frac{v_{Ti}^2}{2\pi\omega^2}\frac{\left(\frac{\partial}{\partial x}\ln\frac{N}{\sqrt{T_i}}\right)^2}{(1+T_i/T_e)^2}\right]^{1/2}=\pi n,\\ &\gamma=-\sqrt{\frac{\pi}{2}}\\ &\times\frac{\int\frac{dx}{\mathrm{Re}\,k_x}\frac{\omega}{|k_z|\,v_{Te}}\frac{k_y v_{Ti}^2}{\Omega_i\left(1+\frac{T_i}{T_e}\right)^2}\left(\frac{\partial}{\partial x}\ln\frac{N}{\sqrt{T_i}}\right)^2\left(1+\frac{1}{2}\frac{\partial\ln\frac{T_i}{T_e}}{\partial\ln N/\sqrt{T_i}}\right)}{\int\frac{dx}{\mathrm{Re}\,k_x}\frac{v_{Ti}^2}{\left(1+\frac{T_i}{T_e}\right)^2}\left(\frac{\partial}{\partial x}\ln\frac{N}{\sqrt{T_i}}\right)^2}.\end{aligned}\right\}\quad(4.14)$$

The spectrum obtained for the short-wave oscillations in question offers evidence of hydrodynamic stability of the inhomogeneous plasma. However, kinetic instability takes place for practically any inhomogeneity in this case. Indeed, if we recognize that in the entire transparency region of the plasma we have

$$\frac{k_y}{\omega\Omega_i}\frac{\partial}{\partial x}\ln\frac{N}{\sqrt{T_i}}<0,$$

we get from the second relation in (4.14) the local condition for the instability of the inhomogeneous plasma

$$1+\frac{1}{2}\frac{\partial\ln T_i/T_e}{\partial\ln N/\sqrt{T_i}}>0,\quad(4.15)$$

which apparently is satisfied in all practical real cases of inhomogeneous plasma. This instability condition was derived in such a form in [3]. On the other hand, in the case of an isothermal plasma the deduction of the so-called "universal" instability of the inhomogeneous plasma relative to the oscillations under consideration was first made in [31-33].

The growth increments of these oscillations can reach values comparable with the frequency $\gamma_{max}<\omega<v_{Ti}/L\ll\Omega_i$ (that is, they are smaller than the drift frequencies of the particles).

c) Finally, it remains to consider the potential oscillations of an inhomogeneous plasma in the region of the shortest wavelengths, shorter than the electron Larmor radius. It can be shown that such oscillations are possible in an inhomogeneous plasma only in the frequency region $\omega\gg k_z v_{Te}$. (In the frequency region $\omega\lesssim k_z v_{Te}$ such oscillations are impossible.) The eikonal equation (4.1) assumes under these conditions the form

$$k_{\perp}\left(1+\frac{T_e}{T_i}\right)-\frac{k_y v_{Te}}{\sqrt{2\pi}\omega}\frac{\partial}{\partial x}\ln\frac{N}{\sqrt{T_e}}$$
$$+i\sqrt{\frac{\pi}{2}}\frac{k_y}{\sqrt{2\pi}|k_z|}e^{-\frac{\omega^2}{2k_z^2v_{Te}^2}}\left(\frac{\partial}{\partial x}\ln\frac{N}{\sqrt{T_e}}+\frac{\omega^2}{2k_z^2v_{Te}^2}\frac{\partial\ln T_e}{\partial x}\right)=0. \quad (4.16)$$

Taking into account the last footnote, which remains in force also in this case, we confine ourselves here to oscillations with wavelength longer than the electron Debye radius (this means that the formulas (4.17) and (4.18) derived below are strictly speaking valid only for a high-density plasma, in which $c^2 \gg Mv_A^2/m$). From (4.16) and (2.19) we obtain the following dispersion equations for the determination of the spectrum of the short wave plasma oscillations:

$$\left.\begin{aligned}&\int dx\,\mathrm{Re}\,k_x=\int dx\,|k_y|\left[-1+\frac{M}{m}\frac{T_i}{T_e}\frac{v_{Ti}^2}{2\pi\omega^2}\frac{\left(\frac{\partial}{\partial x}\ln\frac{N}{\sqrt{T_e}}\right)^2}{(1+T_i/T_e)^2}\right]^{1/2}=\pi n,\\&\gamma=-\sqrt{\frac{\pi}{2}}\omega^2\frac{\displaystyle\int\frac{dx}{\mathrm{Re}\,k_x}\frac{T_iv_{Ti}^2}{T_e|k_z|v_{Te}}\frac{\left(\frac{\partial}{\partial x}\ln\frac{N}{\sqrt{T_e}}\right)^2}{(1+T_i/T_e)^2}e^{-\frac{\omega^2}{2k_z^2v_{Te}^2}}\left(1+\frac{\omega^2}{2k_z^2v_{Te}^2}\frac{\partial\ln T_e}{\partial\ln N/\sqrt{T_e}}\right)}{\displaystyle\int\frac{dx}{\mathrm{Re}\,k_x}\frac{T_iv_{Ti}^2}{T_e\left(1+\frac{T_i}{T_e}\right)^2}\left(\frac{\partial}{\partial x}\ln\frac{N}{\sqrt{T_e}}\right)^2}.\end{aligned}\right\} \quad (4.17)$$

We see therefore that these oscillations are hydrodynamically stable. The kinetic instability is possible if the following local condition is satisfied in the transparency region of the plasma:

$$1+\frac{\omega^2}{2k_z^2v_{Te}^2}\frac{\partial\ln T_e}{\partial\ln N/\sqrt{T_e}}<0. \quad (4.18)$$

Thus, for an inhomogeneous plasma to be unstable against such short-wave potential oscillations it is necessary to satisfy one of the conditions:

$$\text{either } \frac{\partial\ln T_e}{\partial\ln N}<0, \quad \text{or} \quad \frac{\partial\ln T_e}{\partial\ln N}>2.$$

The growth increment of the oscillations is exponentially small, but at the border of the frequency region under consideration, $\omega \sim k_z v_{Te}$, the increment can reach values comparable with the oscillation frequency, which is of the order of $\omega \sim v_{Te}/L$ (larger than all the drift-oscillation frequencies considered above). From the condition that the oscillations be potential [see (5.4)] it follows that such oscillations are possible when $\beta \ll mz_i/M \sim z_e$. For $\beta \ll 1$ this condition is known to be satisfied.

In conclusion let us dwell briefly on the question of the role of ion absorption of waves in an inhomogeneous plasma. We have neglected above everywhere ion absorption compared with electron absorption, as being an exponentially small effect. The validity of such a neglect is due to the inequality

$$1\gg\left(\frac{T_e}{T_i}\right)^{3/2}\sqrt{\frac{M}{m}}e^{-\frac{\omega^2}{2k_z^2v_{Ti}^2}}. \quad (4.19)$$

However, the presence of a large factor in front of the small exponential (particularly in the case of a non-isothermal plasma in which $T_e \gg T_i$) may violate this inequality. It is no longer possible to neglect ion absorption of waves in this case. It is obvious that such a situation can occur only in the frequency region $\omega \ll k_z v_{Te}$. On the other hand, in the frequency region $\omega \gg k_z v_{Te}$ ion absorption of waves is always negligibly small. Without presenting the corresponding derivations, we merely point out that in the case of a homogeneous ion temperature, that is, when $\nabla T_i = 0$, the terms that account for the Cerenkov effect on the ions in the expression for γ always lead to absorption of waves. Therefore in the case of unstable oscillations of plasma these terms assume a stabilizing role.* On the other hand, if the plasma ion temperature is not uniform, then the Cerenkov effect on the ions can lead under certain conditions to a buildup of oscillations, that is, it may cause instability. It can be shown that in the limit of long-wave oscillations, $z_i \ll 1$, this is possible if the following local condition is satisfied in the transparency region of the plasma:

$$\frac{\partial\ln T_i}{\partial\ln N}<0. \quad (4.20)$$

In the limit of short-wave oscillations, when $z_i \gg 1$ and $z_e \ll 1$, the Cerenkov effect on the ions corresponds to buildup of oscillations under the condition

$$\frac{\partial\ln T_i}{\partial\ln N/\sqrt{T_i}}<0, \quad (4.21)$$

That is, either when $\partial\ln T_i/\partial\ln N<0$, or else when $\partial\ln T_i/\partial\ln N>2$. We note that conditions (4.20) and (4.21) can be obtained directly from the analogy between the ion buildup of the oscillations with the electronic buildup in the frequency region $\omega \gg k_z v_{Te}$ (see (4.12), (4.17), and (4.18) with their corollaries).

5. NON-POTENTIAL DRIFT OSCILLATIONS OF AN INHOMOGENEOUS PLASMA

Proceeding to the investigation of low-frequency ($\omega \ll \Omega_i$) non-potential drift oscillations of an in-

*It must be noted that the Cerenkov effect for drift oscillations on the plasma ions increases with increasing plasma pressure (with increasing $\beta = 8\pi P/B_0^2$). As shown in[24], the drift oscillations become stabilized in the frequency region $\omega > k_z v_{Ti}$ at $\beta \gtrsim 0.13$, owing to ion absorption of waves in a plasma with constant temperature.

homogeneous plasma, we first note that the condition $v_A^2 \gg (T_e + T_i)/M$ (or, what is equivalent, $\beta = 8\pi P/B_0^2 \ll 1$), leads to the inequality

$$\omega \sim \frac{k_y v_T^2}{\Omega L} \sim \frac{k_y v_s^2}{\Omega} \frac{1}{L} \ll k_y v_s \ll k_y v_A. \quad (5.1)$$

As in the case of potential oscillations, we also assume that the conditions $\omega \gg k_z v_{Ti}$ and $k_z v_T \ll \Omega$ are satisfied. From the inequality (5.1) we then get $k_y \gg k_z$. Taking all this into consideration, we can write the eikonal equation (1.10) for the drift non-potential oscillations in the form

$$\varepsilon_{33}\left(k_z^2 - \frac{\omega^2}{c^2}\varepsilon_{11}\right) + k_\perp^2 \varepsilon_{11} = 0, \quad (5.2)$$

where

$$\left.\begin{aligned} \varepsilon_{11} &= 1 + \sum \frac{1}{T}\left[1 - \frac{k_y v_T^2}{\omega\Omega}\left(\frac{\partial N}{\partial x}\frac{\partial}{\partial N} + \frac{\partial T}{\partial x}\frac{\partial}{\partial T}\right)\right] \\ &\times T\frac{\omega_L^2}{\Omega^2}\frac{1-A_0(z)}{z}, \\ \varepsilon_{33} &= 1 + \sum \frac{\omega_L^2}{k_z^2 v_T^2}\left[1 - \frac{k_y v_T^2}{\omega\Omega}\left(\frac{\partial \ln N}{\partial x} + \frac{\partial T}{\partial x}\frac{\partial}{\partial T}\right)\right] \\ &\times A_0(z)\left[1 - J_+\left(\frac{\omega}{|k_z| v_T}\right)\right]. \end{aligned}\right\} \quad (5.3)$$

Equation (5.2) goes over into (6.14) and (6.13) of [8] in the frequency regions $\omega \gg k_z v_{Te}$ (neglecting exponentially small terms) and $k_z v_{Ti} \ll \omega \ll k_z v_{Te}$, and $z_e \ll 1$ (wavelength larger than the Larmor radius of electrons).

It can be seen that if $\omega^2 \varepsilon_{11} \ll c_2 k_z^2$ equation (5.2) goes over into the eikonal equation for the potential field oscillations (4.1), investigated in the preceding section. Thus, the condition for the low-frequency drift oscillation of an inhomogeneous plasma to be potential can be written in the form

$$\omega^2, \omega\omega_d \ll \frac{k_z^2 v_A^2 z_i}{1 - A_0(z_i) + \frac{T_i}{T_e}(1 - A_0(z_e))} \equiv k_z^2 v_A^{*2}. \quad (5.4)$$

All the results of the preceding section are valid if this condition is satisfied. Consequently, in particular, it follows that the long-wave potential oscillations in the frequency region $\omega \gg k_z v_{Te}$ are possible in a plasma of very low pressure, when $\beta \ll m/M$. On the other hand, shortwave potential oscillations in this frequency region can exist in a lower pressure plasma, in which $\beta \ll m z_i/M$.

For the opposite limit, when $\omega^2 \varepsilon_{11} \gg k_z^2 c^2$, Eq. (5.2) breaks up into the following two equations:

$$\varepsilon_{11} = 0, \quad (5.5)$$

$$k_\perp^2 - \frac{\omega^2}{c^2}\varepsilon_{33} = 0, \quad (5.6)$$

the first of which describes potential oscillations of an inhomogeneous plasma propagating transversely to the magnetic field, and the second describes non-potential oscillations.

We restrict the investigation of non-potential drift oscillations of an inhomogeneous plasma to an analysis of Eq. (5.6). The reason is that, on the one hand, the analysis of (5.2) in the general case is quite cumbersome, and on the other hand, Eq. (5.6) is in itself of interest, since it describes the oscillations of an inhomogeneous plasma in a limit opposite the limit of potential oscillations.

Equation (5.5) leads in all cases to a spectrum of stable plasma oscillations with frequency on the order of the drift frequencies $\omega \sim \omega_d$. This can be readily verified from Eq. (5.5) itself, which is linear relative to ω and contains no imaginary terms. We shall therefore not analyze this equation and concentrate our attention on Eq. (5.6). In the long-wave region, when $z_i \ll 1$, expression (5.3) for ε_{33} takes the form

$$\begin{aligned} \varepsilon_{33} &= 1 + \frac{\omega_{Le}^2}{k_z^2 v_{Te}^2}\left[1 - \frac{k_y v_{Te}^2}{\omega\Omega_e}\left(\frac{\partial \ln N}{\partial x} + \frac{\partial T_e}{\partial x}\frac{\partial}{\partial T_e}\right)\right] \\ &\times \left[1 - \operatorname{Re} J_+\left(\frac{\omega}{|k_z| v_{Te}}\right)\right] + i\sqrt{\frac{\pi}{2}}\frac{\omega\omega_{Le}^2}{|k_z|^3 v_{Te}^3} e^{-\frac{\omega^2}{2k_z^2 v_{Te}^2}} \\ &\times \left[1 - \frac{k_y v_{Te}^2}{\omega\Omega_e}\left(\frac{\partial}{\partial x}\ln\frac{N}{\sqrt{T_e}} + \frac{\omega^2}{2k_z^2 v_{Te}^2}\frac{\partial \ln T_e}{\partial x}\right)\right]. \end{aligned} \quad (5.7)$$

Substituting this expression in (5.6) and taking (2.19) into account we obtain the following dispersion equations for the spectrum of the non-potential oscillations of an inhomogeneous plasma in the frequency region $\omega \gg k_z v_{Te}$:

$$\left.\begin{aligned} &\int dx \operatorname{Re} k_x = \int dx \left[-k_y^2 - \frac{\omega_{Le}^2}{c^2}\left(1 - \frac{k_y v_{Te}^2}{\omega\Omega_e}\frac{\partial}{\partial x}\ln NT_e\right)\right]^{1/2} = \pi n, \\ &\gamma = \sqrt{\frac{\pi}{2}}\frac{\int \frac{dx}{\operatorname{Re} k_x}\frac{\omega^4\omega_{Le}^2}{|k_z|^3 v_{Te}^3} e^{-\frac{\omega^2}{2k_z^2 v_{Te}^2}}\left[1 - \frac{k_y v_{Te}^2}{\omega\Omega_e}\left(\frac{\partial}{\partial x}\ln\frac{N}{\sqrt{T_e}} + \frac{\omega^2}{2k_z^2 v_{Te}^2}\frac{\partial \ln T_e}{\partial x}\right)\right]}{\int \frac{dx}{\operatorname{Re} k_x}\frac{\omega_{Le}^2 k_y v_{Te}^2}{\omega\Omega_e}\frac{\partial}{\partial x}\ln NT_e} \end{aligned}\right\}. \quad (5.8)$$

It follows from the first equation of (5.8) that the oscillations under consideration are hydrodynamically stable, with the oscillation frequencies being on the order of the electron drift frequencies

$$\omega \sim \omega_d \sim \frac{k_y v_{Te}^2}{\omega\Omega_e}.$$

Recognizing that in the plasma transparency region we have in accordance with the first equation in (5.8)

$$1 - \frac{k_y v_{Te}^2}{\omega\Omega_e}\frac{\partial}{\partial x}\ln NT_e < 0,$$

we obtain from the expression for γ the necessary local condition for the kinetic instability of the inhomogeneous plasma relative to such oscillations:

$$1 - \frac{k_y v_{Te}^2}{\omega\Omega_e}\frac{\partial}{\partial x}\ln NT_e\left(1 + \frac{\omega^2}{2k_z^2 v_{Te}^2}\frac{\partial \ln T_e}{\partial \ln NT_e}\right) > 0. \quad (5.9)$$

It follows therefore that the inhomogeneous plasma can be unstable for any finite negative value of the quantity $\partial \ln T_e/\partial \ln NT_e$ in the transparency region, that is, under the condition

$$-1 < \frac{\partial \ln T_e}{\partial \ln N} < 0.$$

The growth increment of such oscillations is generally speaking exponentially small, but at the border of the frequency region considered, that is, when $\omega \sim k_z v_{Te}$, it can reach values comparable with the oscillation frequency $\gamma_{max} < \omega \sim \omega_d$.

In the frequency region $k_z v_A \ll \omega \ll k_z v_{Te}$, which is possible in a plasma when

$$\beta = \frac{8\pi P}{B_0^2} \gg \frac{m}{M},$$

we obtain from (5.6), (5.7), and (2.19) the following dispersion equations for the spectrum of the long-wave nonpotential oscillations of the inhomogeneous plasma:

$$\left.\begin{aligned} &\int dx\, \mathrm{Re}\, k_x = \int dx \left[-k_y^2 + \frac{\omega^2 \omega_{Le}^2}{k_z^2 c^2 v_{Te}^2} \left(1 - \frac{k_y v_{Te}^2}{\omega \Omega_e} \frac{\partial \ln N}{\partial x} \right) \right]^{1/2} = \pi n, \\ &\gamma = -\sqrt{\frac{\pi}{2}}\, \omega^2 \frac{\displaystyle\int \frac{dx}{\mathrm{Re}\, k_x} \frac{\omega_{Le}^2}{|k_z|^3 v_{Te}^2} \left[1 - \frac{k_y v_{Te}^2}{\omega \Omega_e} \frac{\partial \ln N}{\partial x} \left(1 - \frac{1}{2} \frac{\partial \ln T_e}{\partial \ln N} \right) \right]}{\displaystyle\int \frac{dx}{\mathrm{Re}\, k_x} \frac{\omega_{Le}^2}{k_z^2 v_{Te}^2} \left(2 - \frac{k_y v_{Te}^2}{\omega \Omega_e} \frac{\partial \ln N}{\partial x} \right)}. \end{aligned}\right\} \tag{5.10}$$

It is easy to see that these oscillations are hydrodynamically stable, and the oscillation frequencies, like those considered above, are of the order of the electron drift frequency. The local condition for the kinetic instability of the inhomogeneous plasma relative to such oscillations is of the form (recognizing that according to the first equation of (5.10) we have $1 - k_y v_{Te}^2/\omega\Omega_e \partial \ln N/\partial x > 0$)

$$1 - \frac{k_y v_{Te}^2}{\omega \Omega_e} \frac{\partial \ln N}{\partial x} \left(1 - \frac{1}{2} \frac{\partial \ln T_e}{\partial \ln N} \right) < 0. \tag{5.11}$$

From this we find that in the case

$$\frac{k_y v_{Te}^2}{\omega \Omega_e} \frac{\partial \ln N}{\partial x} > 0$$

the plasma can be unstable if $\partial \ln T_e/\partial \ln N < 0$, while in the case

$$\frac{k_y v_{Te}^2}{\omega \Omega_e} \frac{\partial \ln N}{\partial x} < 0 \text{ it is unstable [31] when } \frac{\partial \ln T_e}{\partial \ln N} > 2.$$

The growth increments of the oscillations on the border of the frequency region $\omega \sim k_z v_{Te}$ under consideration can likewise reach in this case values comparable with the oscillation frequency, that is, $\gamma_{max} < \omega \sim \omega_d$.

Formulas (5.7)—(5.11), obtained in the limit of longwave oscillations, remain valid also for the case of shortwave oscillations, when $z_i \gg 1$ and $z_e \ll 1$. Of course, the local instability conditions (5.9) and (5.11) also remain in force. All that change are the conditions under which such short-wave oscillations are possible. Namely, the oscillations described by formulas (5.10) and (5.11) are possible in the case of short waves in a plasma in which

$$\beta = \frac{8\pi P}{B_0^2} \gg \frac{m}{M} Z_i,$$

whereas in the limit of long-wave oscillations they are possible in a plasma in which $\beta \gg m/M$.

Finally, it remains to analyze Eq. (5.6) in the region of the shortest wavelengths, when $z_e \gg 1$. Here, as in the case of potential oscillations, oscillations are impossible in the frequency region $k_z v_A^* \ll \omega \ll k_z v_{Te}$. On the other hand, in the frequency region $\omega \gg k_z v_{Te}$ we have from (5.3)

$$\begin{aligned} \varepsilon_{33} = 1 &- \frac{\omega_{Le}^2}{\omega^2} \frac{|\Omega_e|}{\sqrt{2\pi} k_\perp v_{Te}} \left(1 - \frac{k_y v_{Te}^2}{\omega \Omega_e} \frac{\partial}{\partial x} \ln N \sqrt{T_e} \right) \\ &+ i \sqrt{\frac{\pi}{2}} \frac{\omega \omega_{Le}^2}{|k_z|^3 v_{Te}^2} \frac{|\Omega_e|}{\sqrt{2\pi} k_\perp v_{Te}} e^{-\frac{\omega^2}{2k_z^2 v_{Te}^2}} \\ &\times \left[1 - \frac{k_y v_{Te}^2}{\omega \Omega_e} \left(\frac{\partial \ln N}{\partial x} + \frac{\omega^2}{2k_z^2 v_{Te}^2} \frac{\partial \ln T_e}{\partial x} \right) \right]. \end{aligned} \tag{5.12}$$

From (5.6) and (2.19) we obtain in this case the following dispersion relations for the spectrum of the short-wave oscillations of an inhomogeneous plasma

$$\begin{aligned} &\int dx\, \mathrm{Re}\, k_x = \int dx \left\{ -k_y^2 + \left[-\frac{\omega_{Le}^2 |\Omega_e|}{\sqrt{2\pi} c^2 v_{Te}} \left(1 - \frac{k_y v_{Te}^2}{\omega \Omega_e} \frac{\partial}{\partial x} \ln N \sqrt{T_e} \right) \right]^{2/3} \right\}^{1/2} = \pi n, \\ &\gamma = \sqrt{\frac{\pi}{2}} \left\{ \int \frac{dx}{\mathrm{Re}\, k_x} \frac{\dfrac{\omega_{Le}^2 |\Omega_e|}{\sqrt{2\pi} c^2 v_{Te}} \dfrac{k_y v_{Te}^2}{\omega \Omega_e} \dfrac{\partial}{\partial x} \ln N \sqrt{T_e}}{\left[-\dfrac{\omega_{Le}^2 |\Omega_e|}{\sqrt{2\pi} c^2 v_{Te}} \left(1 - \dfrac{k_y v_{Te}^2}{\omega \Omega_e} \dfrac{\partial}{\partial x} \ln N \sqrt{T_e} \right) \right]^{1/3}} \right\}^{-1} \\ &\times \int \frac{dx}{(\mathrm{Re}\, k_x)^2} \frac{\omega^4 \omega_{Le}^2 |\Omega_e|}{\sqrt{2\pi} |k_z|^3 c^2 v_{Te}^4} e^{-\frac{\omega^2}{2k_z^2 v_{Te}^2}} \\ &\times \left\{ 1 - \frac{k_y v_{Te}^2}{\omega \Omega_e} \frac{\partial}{\partial x} \ln N \sqrt{T_e} \left(1 + \frac{\omega^2}{2k_z^2 v_{Te}^2} \frac{\partial \ln T_e}{\partial \ln N \sqrt{T_e}} \right) \right\}. \end{aligned} \tag{5.13}$$

We see from these formulas that the oscillations under consideration are hydrodynamically stable, and are possible in the frequency region in which

$$1 - \frac{k_y v_{Te}^2}{\omega \Omega_e} \frac{\partial}{\partial x} \ln N \sqrt{T_e} < 0.$$

From the expression for γ we obtain in this case the following local condition for the kinetic instability of an inhomogeneous plasma relative to such oscillations:

$$1 - \frac{k_y v_{Te}^2}{\omega \Omega_e} \frac{\partial}{\partial x} \ln N \sqrt{T_e} \left(1 + \frac{\omega^2}{2k_z^2 v_{Te}^2} \frac{\partial \ln T_e}{\partial \ln N \sqrt{T_e}} \right) > 0. \tag{5.14}$$

Thus, for the plasma to be stable it is necessary to satisfy the condition $\partial \ln T_e/\partial \ln N\sqrt{T_e} < 0$ in the transparency region or, what is the same, $-2 < \partial \ln T_e/\partial \ln N < 0$. Although the growth increment of the oscillations does contain an exponentially small factor, it can reach at the border of the frequency region $\omega \sim k_z v_{Te}$ values that are comparable with the oscillation frequency, which has the same order of magnitude as the drift frequency of the electrons, that is, $\gamma_{max} < \omega \lesssim \omega_d$.

It must be noted that all the non-potential oscillations considered above and described by (5.6) are stable if the electron temperature is uniform, $T_e = \text{const}$, in that in all cases the oscillation frequency is of the order of the drift frequency of the electrons. The instabilities have a kinetic character, and therefore the oscillation growth increment is smaller than the frequency ω_d. We note also that unlike potential oscillations ($\omega_d \ll k_z v_A^*$), which are unstable for all density and particle-temperature gradients, non-potential oscillations which are described by Eq. (5.6) ($\omega_d \gg k_z v_A^*$) are unstable only outside the region determined by the inequality $0 < \partial \ln T_e / \partial \ln N < 2$.

It was pointed our earlier, in the analysis of potential oscillations of an inhomogeneous plasma, that the condition (4.19) under which the ion absorption of waves can be neglected may become violated in the low-frequency region, when $\omega \ll k_z v_{Te}$. It then becomes necessary to take into account the Cerenkov effect on the ions, which under certain conditions can lead to stabilization of the unstable drift oscillations even in a low pressure plasma, in which

$$\beta = \frac{8\pi P}{B_0^2} \ll 1.$$

It is easy to see that in the region of non-potential oscillations, investigated in this section, ion absorption of waves can always be neglected. Indeed, from the condition that the oscillations are potential we have $\omega \sim \omega_d \gg k_z v_A^* \gg k_z v_{Ti}$. Thus, in this region of frequencies the phase velocity of the oscillations should exceed the thermal velocity of the ions by at least an order of magnitude, causing the condition (4.19) to be satisfied without doubt.

6. EFFECT OF NONPARALLEL MAGNETIC FORCE LINES. STABILIZATION OF DRIFT OSCILLATIONS

So far our analysis pertains, strictly speaking, to the case of a magnetic field with parallel force lines. In the formulas derived above we used wave numbers k_y and k_z which in the case of planar geometry should be taken literally, while in the case of cylindrical geometry k_y stands for l/r [see (1.39)]. However, all the results can be directly extended to the case when the force lines of the magnetic field are not parallel, namely when along with a longitudinal magnetic field B_z there is also a small transverse field $B_\perp \ll B_z$, perpendicular to the direction of inhomogeneity of the plasma.* In the case of planar geometry the field $B_\perp$ is directed along the y axis, while in the case of cylindrical geometry it has an azimuthal direction. All the formulas obtained above retain the same form, if k_y and k_z are taken to mean the quantities defined by (1.40). In the case when the transverse magnetic field $B_\perp$ varies slightly in the region of plasma transparency relative to some oscillations (compared with the variation of the density or temperature), it naturally exerts no influence on these plasma oscillations, since the force lines of the total magnetic field remain in this case parallel in the transparency region, and the entire matter reduces to the usual rotation of the coordinate frame. The situation is different if the transverse field $B_\perp$ varies strongly in the transparency region (as indicated above, the field $B_\perp$ can be arbitrarily inhomogeneous, provided $B_\perp \ll B_z$). In this case the transparency region itself is determined essentially by the transverse magnetic field $B_\perp$, and under certain conditions it can be generally missing. In this sense the inhomogeneous transverse field exerts an appreciable influence on the spectrum of the plasma oscillations.

All the inhomogeneous plasma oscillations considered above lie without exception in the frequency region $\omega \gg k_z v_{Ti}$. This is just the condition that determines the region of transparency of the plasma relative to the oscillations in question. The drift oscillations fall in this frequency region if $\omega_d \gg k_z v_{Ti}$. With increasing transverse magnetic field, the effective k_z increases and this condition may become violated. Drift oscillations then become impossible. Naturally, in this case there are likewise none of the drift instabilities of the inhomogeneous plasma considered above. In this sense we can say that drift oscillations of inhomogeneous plasma become forbidden. Taking formulas (1.40) into account, the condition for the forbiddenness of drift oscillations can be written in the form [35]

$$\frac{k_z}{k_y} \sim \frac{B_\perp}{B_z} \gtrsim \frac{v_T^2}{\Omega L} \sim \frac{T_m}{T_i} \frac{v_{Ti}}{L\Omega_i}, \tag{6.1}$$

where $T_m = \max(T_e, T_i)$. We see from this condition that even sufficiently small inhomogeneous transverse magnetic fields stabilize the drift instability of the inhomogeneous plasma.

It is even easier to stabilize drift instabilities relative to oscillations in the frequency region $\omega \gg k_z v_{Te}$ and $\omega \gg k_z v_A^*$ [the latter, in particular, include the nonpotential oscillations of the inhomogeneous plasma considered above, see (5.4)]. Thus, the condition for forbiddenness of drift oscillations in the frequency region $\omega \gg k_z v_{Te}$ can be written in the form

$$\frac{k_z}{k_y} \sim \frac{B_\perp}{B_z} \gtrsim \frac{v_T^2}{v_{Te} L\Omega} \sim \sqrt{\frac{m}{M}} \sqrt{\frac{T_m^2}{T_e T_i}} \frac{v_{Ti}}{L\Omega_i}. \tag{6.2}$$

On the other hand, the condition that forbids the nonpotential oscillations which we have considered, in the long-wave region when $z_i \ll 1$, is of the form

$$\frac{k_z}{k_y} \sim \frac{B_\perp}{B_z} \gtrsim \frac{v_T^2}{v_A L\Omega} \sim \sqrt{\frac{8\pi P}{B_0^2}} \frac{v_{Ti}}{L\Omega_i} \sim \sqrt{\beta} \frac{v_{Ti}}{L\Omega_i}. \tag{6.3}$$

Finally, in the case of short-wave nonpotential oscillations of the plasma, the forbiddenness condition is written in the form

*We do not consider here the effect of gravitational (convective) plasma instability, connected with the presence of a magnetic field component along the inhomogeneity.

Frequency region, type of oscillations	Local instability condition	Maximum possible increments	Stabilization condition
1. $\omega \gg k_z v_{Te}$, $z_i \ll 1$, potential oscillation	$\frac{\partial \ln NT_e}{\partial \ln NT_i} > 0$	$\gamma_{max} \lesssim \frac{v_{Ti}}{L} \frac{T_{min}}{T_i} \frac{k_y v_{Ti}}{\Omega_i}$	(6,2)
2. $k_z v_{Ti} \ll \omega \ll k_z v_{Te}$, $z_i \ll 1$, potential oscillation	$\frac{\partial \ln T_e}{\partial \ln N} \leqslant \frac{2k^2 r_{De}^2}{1+k^2 r_{De}^2}$	$\gamma_{max} < \omega \approx \frac{v_{Ti}}{L} \frac{T_e}{T_i} \frac{k_y v_{Ti}}{\Omega_i}$	(6.1)
3. $\omega \gg k_z v_{Te}$, $z_i \gg 1$, $z_e \ll 1$, potential oscillation	$\frac{\partial \ln T_e}{\partial \ln N} < 0$	$\gamma_{max} < \omega \approx \frac{v_{Ti}}{L} \frac{k_y v_{Ti}}{\Omega} \frac{1}{1+k_\perp^2 r_{Di}^2}$	(6.2)
4. $k_z v_{Ti} \ll \omega \ll k_z v_{Te}$, $z_i \gg 1$, $z_e \ll 1$, potential oscillation	$1+\frac{1}{2} \frac{\partial \ln T_i/T_e}{\partial \ln N/\sqrt{T_i}} > 0$	$\gamma_{max} < \omega \approx \frac{v_{Ti}}{L} \frac{T_e}{T_e+T_i} \frac{1}{1+k_\perp^2 r_{Di}^2}$	(6.1)
5. $\omega \gg k_z v_{Te}$, $z_e \gg 1$, potential oscillation	$\frac{\partial \ln T_e}{\partial \ln N} < 0$, $\frac{\partial \ln T_e}{\partial \ln N} > 2$	$\gamma_{max} < \omega \approx \frac{v_{Te}}{L} \frac{T_i}{T_e+T_i} \frac{1}{1+k_\perp^2 r_{De}^2}$	(6.2)
6. $\omega \gg k_z v_{Te}$, $z_i \ll 1$, nonpotential oscillations	$-1 < \frac{\partial \ln T_e}{\partial \ln N} < 0$	$\gamma_{max} > \omega \approx \frac{T_e}{T_i} \frac{v_{Ti}}{L} \frac{k_y v_{Ti}}{\Omega_i} \frac{1}{1+(k_\perp^2 c^2/\omega_{Le}^2)}$	(6.2), (6.3)
7. $k_z v_A \ll \omega \ll k_z v_{Te}$, $z_i \ll 1$, nonpotential oscillations	$0 > \frac{\partial \ln T_e}{\partial \ln N}$, $\frac{\partial \ln T_e}{\partial \ln N} > 2$	$\gamma_{max} < \omega \approx \frac{T_e}{T_i} \frac{v_{Ti}}{L} \frac{k_y v_{Ti}}{\Omega_i}$	(6.3)
8. $\omega \gg k_z v_{Te}$, $z_i \gg 1$, $z_e \ll 1$, nonpotential oscillations	$-1 < \frac{\partial \ln T_e}{\partial \ln N} < 0$	$\gamma_{max} < \omega \approx \frac{v_{Ti}}{L} \frac{T_e}{T_i} \frac{k_y v_{Ti}}{\Omega_i} \frac{1}{1+(k_\perp^2 c^2/\omega_{Le}^2)}$	(6.2), (6.4)
9. $k_z v_A^* \ll \omega \ll k_z v_{Te}$, $z_i \gg 1$, $z_e \ll 1$, nonpotential oscillations	$0 > \frac{\partial \ln T_e}{\partial \ln N}$, $\frac{\partial \ln T_e}{\partial \ln N} > 2$	$\gamma_{max} < \omega \approx \frac{v_{Ti}}{L} \frac{T_e}{T_i} \frac{k_y v_{Ti}}{\Omega_i}$	(6.4)
10. $\omega \gg k_z v_{Te}$, $z_e \gg 1$ nonpotential oscillations	$-2 \lessdot \frac{\partial \ln T_e}{\partial \ln N} < 0$	$\gamma_{max} < \omega \approx \frac{k_y v_{Te}}{\Omega_e} \frac{v_{Te}}{L} < \frac{v_{Te}}{L}$	(6.2), (6.4)

$$\frac{k_z}{k_y} \sim \frac{B_\perp}{B_z} \geqslant \frac{v_T^2}{v_A^* L\Omega} \sim \sqrt{\frac{8\pi P}{B_0^2}} \sqrt{\frac{1}{z_i}} \frac{v_{Ti}}{L\Omega_i} \sim \sqrt{\frac{\beta}{z_i}} \frac{v_{Ti}}{L\Omega_i}. \quad (6.4)$$

From conditions (6.1)—(6.4) it follows that the most difficult to eliminate is the instability of an inhomogeneous plasma relative to potential oscillations in the frequency region $k_z v_{Ti} \ll \omega \ll k_z v_{Te}$ or $k_z v_A^*$, especially if the plasma is furthermore non-isothermal, $T_e \gg T_i$. These conditions that forbid drift oscillations were derived in [34-36]. In the same papers we can find the conditions for the stabilization of the inhomogeneous plasma against some specific form of instability of drift oscillations. Such conditions are obtained without difficulty by substituting the frequencies of the drift oscillations, determined in the preceding sections, in the conditions $\omega \gg k_z v_{Ti}$, $\omega \gg k_z v_{Te}$, and $\omega \gg k_z v_A^*$ (depending on which of these conditions determines the particular region of frequencies of the drift oscillations). The table lists the local conditions for instability of an inhomogeneous plasma relative to the drift oscillations considered above, and indicates the conditions under which they are forbidden.

We have restricted the present review to an exposition of the applications of the methods of geometrical optics in the theory of oscillations of one-dimensionally inhomogeneous plasma. This is due primarily to the fact that only such a case is treated in the literature. On the other hand, it is by now already clear that practical demands will call for the development of a theory of plasma oscillations for both the case of two-dimensionally inhomogeneous distributions and for the case of three-dimensional inhomogeneities. Only the first steps are being made at present in this direction [37,38]. Furthermore, if the eikonal equation can be set in correspondence with an approximately equivalent partial differential equation of the elliptic type, then we can determine the oscillation spectrum by using the Bohr and Sommerfeld multidimensional phase integrals

$$(2\pi)^m n = \int dx_1 \ldots dx_m \int dk_1 \ldots dk_m,$$

where n is an integer much larger than unity, m the number of dimensions of the space, and **k** the wave vector. It is important to indicate the region of integration. First of all, integration with respect to **k** is carried out over a volume bounded by a surface described by the eikonal equation. We note that the corresponding volume is finite for equations of the elliptic type, and also for certain oscillations described by an equation of the parabolic type [38]. Integration over the space variables is carried out either up to the boundaries on which the nondissipative boundary conditions are specified, or over the region bounded by the surface (or line) $k^2 \equiv \Sigma k_i^2 = 0$. In the latter case of locked rays, the ray trajectory practically fills the transparency region of the plasma. Here obviously, we cannot speak of separation of the variables in the field equations. To the contrary, when such a separation is possible it is obviously possible to reduce the three-dimensional problem to a two-dimensional one and the two-dimensional problem to a one-dimensional one.

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