SPECTRAL DISTRIBUTION OF RADIANT ENERGY

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1. About 10 years ago A.A. Gershun [1] directed the attention of physicists and illumination engineers to the fact that in discussing the laws of thermal radiation one inevitably encounters certain difficulties associated with the energy distribution in the spectrum of an ideal black body, especially regarding the location of the spectral maximum.

It is sometimes forgotten that the spectral density of radiation measured according to some particular spectral scale is not itself associated with a value of energy (or power), but is only an auxiliary quantity enabling us to determine the energy radiated in a selected spectral range. This auxiliary quantity depends, of course, on the scale used for the energy distribution.

Three scales merit discussion at the present time. Most physicists and illumination engineers customarily use a linear scale of wavelength λ in which the radiant energy (i.e., power) element dP in the interval λ, $\lambda + d\lambda$ is given by

$$dP = p(\lambda)\,d\lambda \quad \text{watt/cm}^2 \tag{1}$$

Secondly, many theorists prefer a scale of frequencies or the practically equivalent scale of wave numbers n, in which the energy element dP in the interval n, n + dn is

$$dP = p(n)\,dn \quad \text{watt/cm}^2 \tag{2}$$

Thirdly, we have the infrequently used logarithmic scale, which is worthy of attention for several reasons. Here the logarithms of wavelengths ($\ln\lambda$) or of wave numbers ($\ln n$) are given linearly and the energy element is

$$dP = p(\ln\lambda)\,d(\ln\lambda) = p(\ln\lambda)\,\frac{d\lambda}{\lambda} \quad \text{watt/cm}^2 \tag{3}$$

With each scale the width of the spectral interval ($d\lambda$, dn, or $d\lambda/\lambda$) is a multiplicative factor of the energy element. It can therefore not be indicated at which of two wavelengths λ_1, and λ_2 in the continuous spectrum more energy is radiated. This depends on intervals $d\lambda_1$, and $d\lambda_2$ that are determined entirely independently. The same comments apply to all of the spectral scales.

Each scale leads to a different form of the Planck function with a corresponding position of the peak given by the displacement law

$$\lambda_m T = \text{const.} \tag{4}$$

The respective values of the constants are [2]

$$\text{for} \quad p(\lambda) \quad C(\lambda) = 2896.2\ \mu\text{-deg}, \tag{4a}$$

$$\text{for} \quad p(n) \quad C(n) = 5096.8\ \mu\text{-deg}, \tag{4b}$$

$$\text{for} \quad p(\ln\lambda) \quad C(\ln\lambda) = 3667.7\ \mu\text{-deg}. \tag{4c}$$

The linear scale of wavelengths squeezes an enormous region of short-wave radiation (ultraviolet, x rays, and γ rays) into a small region close to its zero point. The scale of wave numbers squeezes all long-wave radiation (infrared and radio waves) close to the zero point. Since all graphs of the spectral distribution are plotted to indicated the proportionality between the energy radiated in a given interval and the area lying under the corresponding portion of the curve, (between the curve, the abscissa axis, and the two limiting ordinates), this one-sided compression of the scale must lead to a corresponding increase of the ordinates. In other words, from general considerations one can expect a shift of the peak towards short waves on the wavelength scale and towards long waves on the wave-number (or frequency) scale.

The logarithmic scale possesses no zero point and represents all electromagnetic radiation by equal segments for equal relative changes of wavelength (or wave number) in any spectral range.

2. All the aforementioned scales are undoubtedly correct from a theoretical point of view, but are not of equal merit in practice. For example, let us consider the problem of determining the temperature at which a black body exhibits its maximum radiation efficiency. An opticist or illumination engineer accustomed to using the spectral density curve $p(\lambda)$ expressed in the linear scale of wavelengths can easily arrive at an incorrect result if he uses the customary value of the displacement-law constant (4a) to calculate the desired temperature as

$$T_m = 2896.2/0.555 \approx 5200^\circ\,\text{K}. \tag{4d}$$

This computation is based on the assumption that the most favorable conditions for the light yield exist when the spectral density peak is found at the wavelength to which the eye is most sensitive. This seems incontestable at first glance.

It is known, however, that the maximum luminous efficiency of a black body occurs at about 6600° K, which is obtained by using the displacement-law constant on the logarithmic scale (4c):

$$T_m = 3667.7/0.555 \approx 6600^\circ\,\text{K}. \tag{4e}$$

A correct result can, of course, be obtained on the

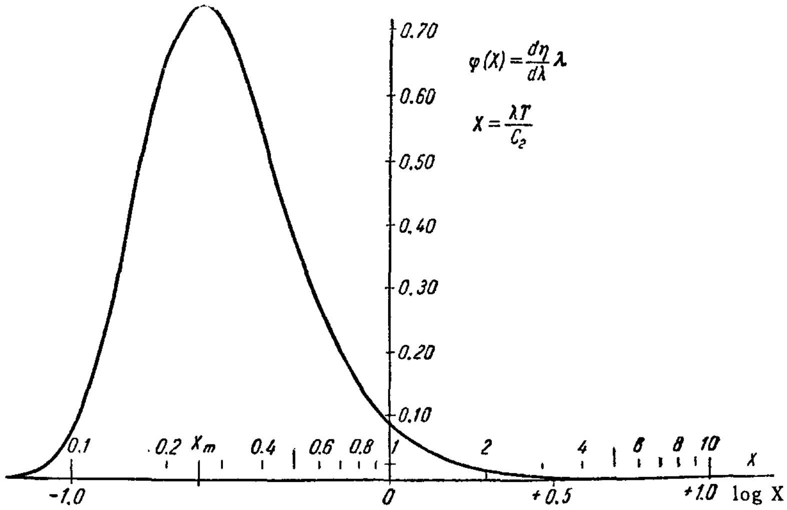

FIG. 1. Radiation efficiency of an ideal black body calculated for an octave as a function of $\lambda T/C_2$.

basis of the function $p(\lambda)$, keeping in mind that if the maximum of this function at a temperature T pertains to a wavelength λ_m, the largest fraction of the radiation will be found in an interval around the same wavelength but for a different temperature T_m, which must be obtained by an additional calculation.

3. Let us consider once more the problem of determining the radiation efficiency of a black body in a narrow spectral interval.* By this we mean the ratio of the energy ΔP in a given interval λ, $\lambda + \Delta\lambda$ to the total radiant energy P. We denote this efficiency by $\Delta\eta$, which from the definition of a black body is given by

$$\Delta\eta = \frac{\Delta P}{P} = \frac{C_1\lambda^{-5}\Delta\lambda}{(e^{C_2/\lambda T}-1)\sigma T^4} = \frac{C_1}{C_2^4\sigma}\,\frac{X^{-4}}{e^{1/X}-1}\,\frac{\Delta\lambda}{\lambda}, \tag{5}$$

where $X = \lambda T/C_2$.

The efficiency $\Delta\eta$ is a function of the black body temperature, the wavelength λ, and the given interval $\Delta\lambda$. We now consider the derived equation in detail. The first of the three factors is a constant dimensionless quantity equal to 1/6.4939. The second factor, $f(X) = X^{-4}/(e^{1/X} - 1)$, is also dimensionless and depends only on the product λT. It is easily seen that for $X = X_m = 0.25505$ this factor reaches its maximum $f(X_m) = 4.7799$, and that it approaches zero both for larger and smaller values of X.

The third, and also dimensionless, factor $\Delta\lambda/\lambda$ gives the relative spectral width of the selected interval. The unit of this relative spectral interval can be called an "octave" since it represents a change $\Delta\lambda$ of wavelength equal to the wavelength itself. The radiation efficiency of a black body in a single octave is represented by

$$\varphi(X) = \frac{\Delta\eta}{\Delta\lambda/\lambda} = \frac{f(X)}{6.4939} \tag{6}$$

which is plotted on a logarithmic scale in Fig. 1.

*This problem has recently been studied by R. A. Sapozhnikov,[3] who has given an extensive bibliography.

Considering $\Delta\eta$ as a function of temperature, we find that it reaches its maximum value at a temperature T_m along with the function $\varphi(X)$. We thus have

$$\Delta\eta_m = \frac{4.7799}{6.4939}\,\frac{\Delta\lambda}{\lambda} = 0.73605\,\frac{\Delta\lambda}{\lambda} \tag{7}$$

and

$$\lambda T_m = 3667.7 \quad \mu\text{-deg}^\circ\,\text{K}.$$

Thus, in order to determine the temperature at which an interval λ, $\lambda + \Delta\lambda$ contains the maximum fraction of the total radiant energy of a black body, we must be guided by (7); this also clarifies the aforementioned case of the radiation efficiency of a black body. The latter expression does not depend on the scale used to represent the spectrum, whether it be wavelengths, wave numbers, or logarithms.

The function $\varphi(X)$ in Fig. 1 shows how the efficiency depends on temperature; the form of this dependence is identical for all wavelengths. The only difference lies in the fact that, according to (7), with a change of wavelength the peak will be found at a different temperature.

4. Equation (5) defining the efficiency $\Delta\eta$ can be regarded as a function of wavelength representing the spectral dependence of the energy element ΔP relative to the total energy P radiated by a black body at a constant temperature. It follows from (6) that the wavelength dependence computed for an octave of the efficiency coincides with the temperature dependence and can also be represented by the curve in Fig. 1.

In order to achieve a more thorough understanding of the derived relations, we plot in Fig. 2 the logarithm of the wavelength ($\ln\lambda$) as the abscissa and the logarithm of temperature ($\ln T$) as the ordinate. Each point in the plane corresponds to a pair of values (λ, T) determining $\varphi(X)$, which can be represented by a perpendicular to the plane on an arbitrary scale. The shape of the surface $\varphi(\lambda T/C_2)$ obtained in this way can easily be determined. A family of parallel

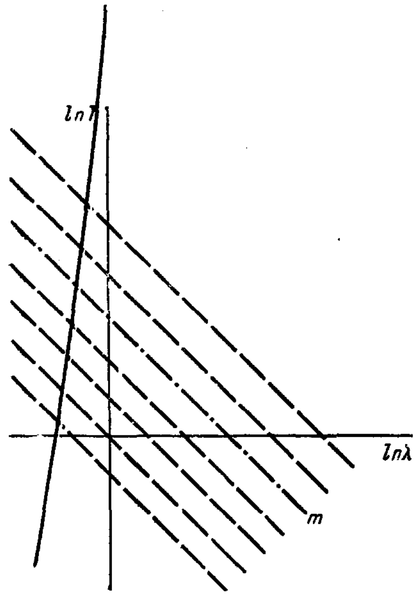

FIG. 2. Logarithmic plot of independent variables in Planck's function.

straight lines represented by the equation $\ln \lambda + \ln T = \text{const}$ (the dashed lines in Fig. 2) forms the geometric locus of constant values of λT and thus of $\varphi(X)$.

The plane U perpendicular to the plane of Fig. 2 and parallel to the ordinate axis ($\lambda = \text{const}$) intersects the φ surface along a line duplicating the curve represented in Fig. 1. By shifting the plane U parallel to itself we obtain identical curves at intersections with the φ surface. The maximum height of these curves is obtained above the dashed line corresponding to (7), which is designated by the letter m in Fig. 2.

The intersection of the surface $\varphi(\lambda T/C_2)$ with the plane V perpendicular to the plane of Fig. 2 and parallel to the abscissa axis $T = \text{const}$ will duplicate its intersection with the plane U. The maximum value of $\varphi(X)$ in this intersection will be found above the same dashed line designated by the letter m. When the plane V is shifted parallel to itself the curve representing its intersection with the surface $\varphi(\lambda T/C_2)$ will also retain its shape.

Figure 3 shows the shape of the surface $\varphi(\lambda T/C_2)$ in logarithmic coordinates. The surface is a kind of "wave" with a straight horizontal crest lying above the dashed line m. The surface approaches zero at both large and small values of λT.

5. It should be noted that the foregoing simple dependences of the efficiency of monochromatic blackbody emission on wavelength and on temperature occur only on logarithmic scales. The relations are enormously more complicated in the analogous picture formed with linear scales of λ and T. The differentials $d\lambda$ and dT are dimensional quantities on the linear scales. We plot the surface

$$\psi(\lambda, T) = \frac{d\eta}{d\lambda} = \frac{1}{\lambda}\varphi\left(\frac{\lambda T}{C_2}\right), \tag{8}$$

where the values of $\psi(\lambda, T)$ are segments of per-

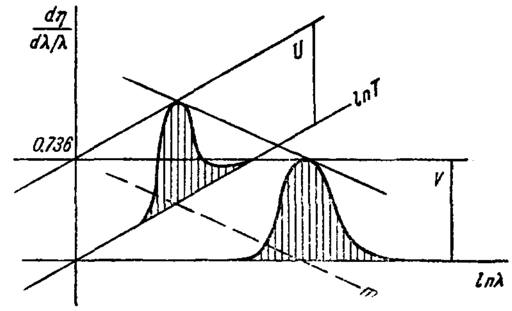

FIG. 3. The surface $\varphi(\lambda T/C_2)$ and its intersections with the planes $U(\lambda = \text{const})$ and $V(T = \text{const})$.

pendiculars erected at points λ, T of the coordinate plane (Fig. 4); the values of $\psi(\lambda, T)$ have the dimension of a reciprocal length.

The lines $\lambda T = \text{const}$ along which $\varphi(X)$ is constant are equilateral hyperbolas (the dashed lines in Fig. 4). The hyperbola $\lambda T = 3667.7$ μ-deg has been designated by the letter m_1. The function $\psi(\lambda, T) = (d\eta)/(d\lambda)$ will not be constant along the hyperbolas $\lambda T = \text{const}$ because, as Eq. (8) shows, the constant value of $\varphi(X)$ must be divided by the varying wavelength λ. As λ decreases, $\psi(\lambda, T)$ will increase hyperbolically.

Let us consider the intersection of the surface $\psi(\lambda, T)$ with the plane $T = \text{const}$. Equation (8) shows that for constant λ the shape of this intersection will not differ from that of the curve $\varphi(X)$ shown in Fig. 1. $\psi(\lambda, T)$ reaches its highest value in the plane $\lambda = \text{const}$ at the point T_m lying on the hyperbola m_1.

When we consider the intersection of $\psi(\lambda, T)$ with the planes $T = \text{const}$ we find that the maxima of these curves also lie above an equilateral hyperbola, but with a different value of the constant ($\lambda T = 2896.2$ μ-deg), which is designated in Fig. 4 by the letter m_2.

The form of the surface $\psi(\lambda, T)$ will be found to differ greatly from that of the surface $\varphi[(\lambda T)/(C_2)]$

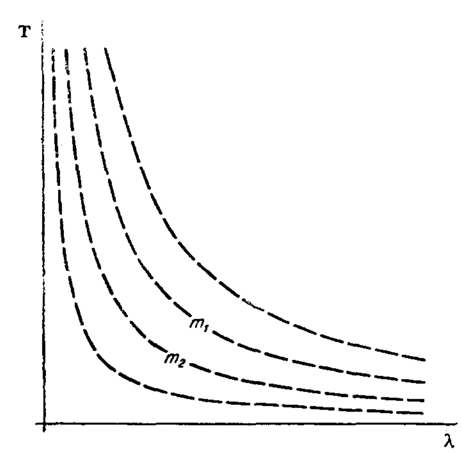

FIG. 4. Plane of independent variables in Planck's function on linear scales.

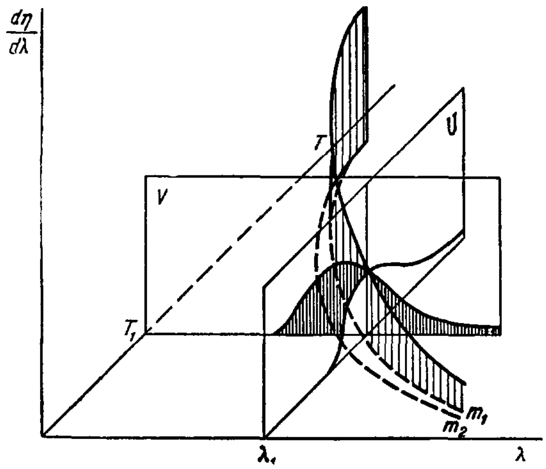

FIG. 5. The surface $\psi(\lambda, T)$ and its intersections with the planes $U(\lambda = \text{const})$ and $V(T = \text{const})$.

shown in Fig. 3. The straight "wave" having a horizontal crest becomes a kind of hill that is bent in the horizontal λ, T plane along the hyperbola m_1, above which its crest lies while rising steeply as λ decreases. Figure 5 shows an example of this hill in the form of the curves along which it intersects the planes $U(\lambda = \text{const} = \lambda_1)$ and $V(T = \text{const} = T_1)$. It is seen from the figure that the shift of the curve maximum in the plane V relative to its intersection with the plane U where $T_1 = T_m$, results from the general rise of the entire hill in the short-wave direction.

The shapes of the efficiency surfaces could be discussed in greater detail for the logarithmic and linear scales of wave numbers n and temperatures T. It would again be found that the shape of a surface is incomparably simpler on a logarithmic than on a linear scale. However, this would be largely a repetition of the preceding discussion, and will therefore be omitted.

6. We use a procedure discussed by Fabry [4]. Retaining the scale of $\varphi(X)$, we take as the unit of the independent variable its value $X_m = 0.25505$ at the maximum of $\varphi(X)$. Denoting the new function by y and the new independent variable by x, we obtain

$$y = 36.387 \frac{x^{-4}}{e^{3.9207/x} - 1} = 36.387 \frac{x^{-4}}{10^{1.70273/x} - 1}. \quad (9)$$

It is easily seen that for x = 1 we have $y = 0.7360_5$, which corresponds to the highest possible black-body radiation efficiency in a single octave. From all of the foregoing it follows that this function, which is plotted in Fig. 6, can be used conveniently to calculate black-body emission.

Let us suppose that we wish to know the energy radiated by a black body at T = 2000°K in the spectral interval $0.50-0.60\mu$. We define the mean wavelength in this interval as the half-sum of its extremes (0.55μ). We now obtain the product $\lambda T = 0.55 \times 2000 = 1100\ \mu$-deg and its ratio to the product $(\lambda T)_m$ corresponding to the maximum of $\varphi(X)$: $1100/3667.7 \approx 0.3$. It is easily seen that this latter ratio is an independent variable in (9).

Figure (6) shows that $y(0.3) = 0.01$; therefore the efficiency of black-body radiation in the interval of interest equals $y(0.3)\,\Delta\lambda/\lambda = 0.01(0.1)/(0.55) = 0.00182$. In order to determine the energy ΔP radiated by a black body under the given conditions, this result must be multiplied by $P = \sigma T^4$, i.e., in our case by $5.68 \times 16 = 90.8$ watt/cm². Thus in the interval $0.5-0.6\mu$ at T = 2000°K each square centimeter of a black body radiates $0.00182 \times 90.8 \approx 0.165$ watt.

The foregoing shows that the radiation efficiency of a black body in a single octave is determined by the ratio of the given λT product to $(\lambda T)_m = 3667.7\ \mu$-deg.

The numerals strung out along the curve in Fig. 6 show how rapidly the efficiency varies with changing temperature at constant wavelength. For example, the value 9 at y(0.3) denotes that a 1% temperature

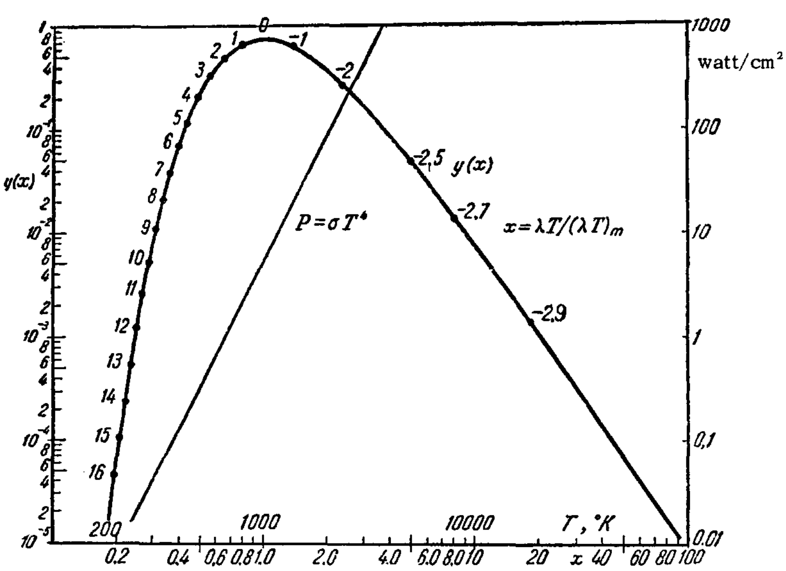

FIG. 6. Graph for the computation of black-body efficiency and radiant energy in narrow spectral intervals.

increase is accompanied by a 9% efficiency increase. The rate of radiation change will be 4% greater, i.e., for $\lambda T/(\lambda T)_m = 0.3$ a 1% temperature increase leads to a 13% increase of radiant energy. The figure also shows that for $\lambda T/(\lambda T)_m = 3$ a 1% temperature increase leads to a 2.2% decrease of efficiency for radiation that increases by $4 - 2.2 = 1.8\%$.

7. The foregoing furnishes sufficient evidence that the logarithmic scale of wavelengths (wave numbers, or frequencies) is very suitable for solving many problems associated with the black-body spectral energy distribution. The linear scale, while entirely usable and resulting in no fundamental errors, does lead to extremely more complicated relations.

The advantages of the logarithmic scale of wavelengths (wave numbers, or frequencies) are as follows:

a) The spectral density has the same dimension as the integral radiation density (watt/cm^2) [2].

b) All regions of the electromagnetic spectrum are represented uniformly in a graph; changes of the spectral coordinate (λ, n, or ν) by a single octave in any part of the spectrum are represented by equal ranges on this scale.

c) The black-body radiation efficiencies calculated for a single octave have identical dependences on wavelength and on temperature.

d) The constant of Wien's displacement law determining the position (λ_m) of the highest spectral density of black-body radiation at a given temperature T coincides with the constant determining the temperature T_m at which the efficiency of black-body radiation in a narrow spectral interval around a wavelength λ reaches its maximum; for the logarithmic scale we thus have

$$\lambda_m T = \lambda T_m = 3667.7 \ \mu\text{-deg.} \tag{10}$$

[1] A. A. Gershun, On the Spectral Density of Radiation, UFN **46**, 366 (1952).

[2] M. M. Gurevich, On the Spectral Distribution of Radiant Energy, UFN **56**, 417 (1955).

[3] R. A. Sapozhnikov, On the Spectral Distribution of Radiant Energy, UFN **70**, 387 (1960), Soviet Phys. Uspekhi **3**, 172 (1960).

[4] Ch. Fabry, Les principes de la photométrie, Paris, 1934. (Russ. transl., ONTI, M., 1934).

Translated by I. Emin