# ON THE "SPECIAL ROLE" OF THE ELECTROMAGNETIC POTENTIALS IN QUANTUM MECHANICS

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## 1. INTRODUCTION

SOME years ago the assertion was made in a paper by Aharonov and Bohm [1] that the potentials of the electromagnetic field play a special role in quantum mechanics, which they do not have in classical mechanics; that, unlike the potentials in classical electrodynamics, they must here be regarded as "primary" physical quantities, and the field strengths must be regarded as "secondary" quantities, in the sense of derived concepts; that owing to this "special" role of the potentials, "contrary to the conclusions of classical mechanics, there exist effects of potentials on charged particles, even in the region where all the fields (and therefore the forces on the particles) vanish" (abstract of [1], page 485); that owing to this "some further development of the theory is needed. Two possible directions are clear," write the authors. "First, we may try to formulate a nonlocal theory in which, for example, the electron could interact with a field that was at a finite distance away. . . . Secondly, we may retain the present local theory" if we "regard $A_\mu(x)$ as a physical variable. This means that we must be able to define the physical difference between two quantum states which differ only by gauge transformation" ( [1], pp. 490—491). It is true that the authors are evidently not inclined to insist on this latter extreme possibility (the statements on this point in their second paper [2] are more cautious). Nevertheless it is clear that fundamental propositions of quantum mechanics are at stake. If the question as to which is more important, field or potential, can still be regarded as a matter of taste, there are other assertions which have a concrete meaning and are based on the analysis of two possible experiments proposed by the authors. (The second of these experiments had actually been indicated long before by Ehrenberg and Siday [3] in connection with an analysis of problems of electron microscopy; but although these authors indeed came to the conclusion that a source influences an electron even when it is in a part of space where the field strength vanishes and only the potential is different from zero, they did not draw such far-reaching conclusions about the foundations of quantum electrodynamics.) In addition, in their second paper [2] Aharonov and Bohm analyze a third physical example—the stationary states of an electron in the field of a solenoid.

These assertions gave rise to a theoretical discussion [4,5,14,16] (partly in private letters, cf. [2]). New experiments were arranged and old ones were reexamined, [6-9] and experimental authors believe that their data confirm the conclusions of Aharonov and Bohm (although Aharonov and Bohm themselves admit that none of the experiments is as yet a completely clean case [2]). A sympathetic reference to this point of view can be found in a paper devoted to the analysis of other problems. [10]

Thus the question already "has a literature." It is worth examination, since after all there are no finally decisive statements in the theoretical papers either for or against the point of view of Aharonov and Bohm.*

We shall analyze all three physical examples and try to obtain the answers to two questions.

1. Is there really an experimental possibility of finding a physical effect when an electron (its wave function) is entirely in a region where the field strength of the source vanishes but the potential is different from zero?

2. Is there a special effect of the potential in quantum mechanics, different from its effect in classical physics, which would allow one to regard the potential as a more fundamental quantity than the field strength, and would require a reformulation of the foundations of the theory?

We shall see that these two questions do not reduce to a single one, as it might seem at first glance.

*After this article had been written, there appeared a paper by De Witt[14] and an answer to it by Aharonov and Bohm.[15] De Witt did not agree with the idea that it is necessary to reexamine the concept of the potential in quantum mechanics, and pointed out that the potential can be replaced in the Schrödinger equation and in all other uses by a line integral of the field strength. Thus the theory can be formulated with the field strengths alone, but at the expense of introducing nonlocality: the effect of the field at a given point is determined by its values at other points, and in general at other times. This criticism essentially fails to refute the thesis that in some sense of the word there is a special role of the potentials in quantum mechanics. In their reply Aharonov and Bohm regard the replacement as a trivial step. A formulation of quantum electrodynamics without potentials has been given V. I. Ogievetskiĭ and I. V. Polubarinov (Joint Institute for Nuclear Research Preprint E-975, 1962).

## 2. THE EFFECT OF THE SCALAR POTENTIAL

The first suggested experiment is as follows. [1,4] A plane electron wave, which in the direction of motion is a packet of length L, is separated in a transverse direction into two parts (for example, as the result of passing through a screen $S_1$ with two slits, Fig. 1,a). Each part passes through its own cylindrical metal tube (Faraday cage) of length $l \gg L$. When the packets have completely entered the tubes, a potential difference $\varphi$ is applied to the tubes and kept constant for the time t (Fig. 1,b). For simplicity we can suppose that one of the tubes is grounded and is at potential zero. Then, before the packets begin to emerge from the tubes, the potential is removed. After they come out the packets are deflected by the prisms P and interfere, giving bands on a screen $S_2$ (Fig. 1,c).

The essential point of the experiment is that the packet which has been subjected to the action of the potential $\varphi$ acquires an additional phase $(e/\hbar)\varphi t$ (where e is the charge of the electron), which must produce a shift of the interference pattern which increases with increase of $\varphi$. At the same time, inside the cylinder the potential $\varphi$ does not depend on the coordinates and therefore the field strength is zero. Thus there must be an observable effect although the electron has not been acted on by a field strength. This is the basis for the fundamental assertions which have been mentioned above.

The argument as given does not seem open to any doubt. In fact, [1] we can write for the function $\psi$ the equation

$$i\hbar \frac{\partial \psi}{\partial t} = (\mathscr{H}_0 + e\varphi)\psi,$$

where $\mathscr{H}_0$ is the unperturbed Hamiltonian. In the absence of the potential the solution $\psi = \psi_0$ is of the form $\psi_0 = \psi_1^0 + \psi_2^0$, where $\psi_1^0$ is different from zero in the first tube, and $\psi_2^0$, in the second.

When there is a potential $\varphi = \varphi_1 + \varphi_2$, with $\varphi_1$ and $\varphi_2$ different from zero only in the first or in the second tube, respectively, then as is easily verified by substitution, the solution is

$$\psi = \psi_1^0 e^{-\frac{iS_1}{\hbar}} + \psi_2^0 e^{-\frac{iS_2}{\hbar}},$$

where

$$S_1 = \int^t e\varphi_1(t)\,dt,$$

$$S_2 = \int^t e\varphi_2(t)\,dt.$$

The presence of the phase difference $S_1 - S_2$ causes the effect. The quantum character of the effect is shown by the fact that it depends on the quantity $\hbar$.

One might doubt the effect because at the moments when the field is switched on and off a redistribution of charges occurs and therefore the field might temporarily penetrate into the tubes. It is clear, however, that this effect is limited to the on and off switching

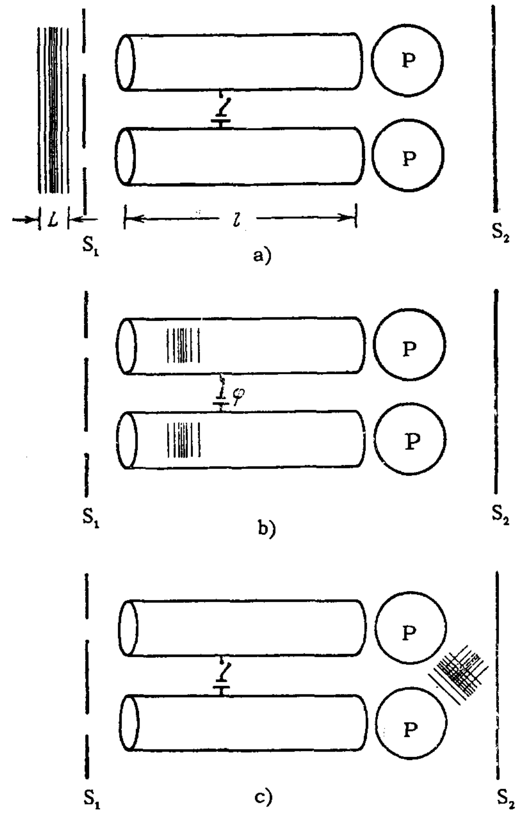


FIG. 1

time intervals $\Delta t = \Delta t_1 + \Delta t_2$. Meanwhile the phase shift $S_1 - S_2$ is proportional to the time t and can be made arbitrarily large, so that the phase shift in the time $\Delta t$ can be neglected.

It also follows from this that the experiment could be done under less restrictive conditions: one could impose a constant field over all space from the beginning, so that before going into the tubes and after coming out of them the electron would be acted on by the field. But the scheme described in [1] and [4] relieves us of any need to discuss this point further.

Furry and Ramsey showed [4] that if we take into account the reaction of the electron's charge on the charge distribution on the tube it is possible (from the change of potential of the tube) to determine through which of the two tubes the electron has gone. But if we arrange the experiment so that such a determination can be made the interference pattern disappears. This indeed must be the case: if it is known that the electron is in one tube, then its $\psi$ function in the other tube must be zero and there is nothing to interfere with. Therefore Furry and Ramsey come to the convincing conclusion that the predicted result of the experiment must without doubt occur according to the fundamental propositions of quantum mechanics.

We shall try, however, to make sure whether we really have here a manifestation of a property of the potential unknown in classical physics.

The basis of the argument is the assertion that the electron in the tube has an additional energy $e\varphi$.

There is nothing specifically quantum-mechanical about this fact. If we were working with a classical electron, its energy would also be increased by the amount $e\varphi$ when the field was turned on. The increase of energy is quite real. If we want to get the electron out of the tube before the potential is switched off, we have to collect this energy from it. Consequently there is a quite perceptible "effect of the potential on the charged particle" in a region where "all the fields (and therefore the forces on the particle) vanish" also in classical electrodynamics. There is nothing nonclassical in the fact that the electron acquires an additional energy. We can even show where this energy comes from: in giving the tube the constant potential $\varphi$ we have placed an additional charge on it. In flowing onto the tube it had, depending on the sign, either to overcome the repulsive force of the electron in the tube, or else to be attracted by it (see the calculation in [4]). In fine, $e\varphi$ is the purely classical energy of the interaction of the electron with the charges of the source.

Whence therefore an effect which has no classical analog? The answer is obvious: the electron wave function has a particular frequency which depends on the total energy of the electron. Furthermore, whereas the only quantities important for the motion of a classical particle are the derivatives of the action function at a given point and a given time, the energy $E = -\partial S/\partial t$ and the momentum $p = \partial S/\partial q$, in quantum mechanics the frequency and the resulting phase determine the absolute magnitude of the action S. If we know that the electron goes through only one tube, then the change of its frequency causes the constant phase shift $S_1/\hbar = (e/h)\int \varphi_1(t)\,dt$. This shift has no effect at all (just as changing the action by a constant has no effect for the classical electron). If, however, the electron is described by a wave function which has different parts in which there are different integrated phase shifts, so that the difference of the shifts is $(S_1 - S_2)/\hbar$, then there is an interference effect.

Thus the quantum peculiarity of the effect has its hiding place in two circumstances. First, in the fact that the energy of the particle (in the present case the energy of interaction with the charges on the tube) has any effect on the frequency of the wave function (in classical electrodynamics, although this energy is present and real, it does not affect the character of the motion, as long as there is no dependence of the potential energy on the coordinates); second, in the fact that the position of the electron is in principle undetermined, it "is in both tubes simultaneously" and is under the influence of different potentials in different parts of its packet. Both of these facts make the process very different from the corresponding classical case. In both of them, however, it is difficult to perceive any new and special role of the potential (in particular, even in the simple passage of a particle through a screen with two slits a difference of an external influence on two parts of the same wave packet will produce an effect).

It must be noted that various actions of the potential which are not inherent in Newtonian physics can be encountered elsewhere than in quantum mechanics. For example, in the general theory of relativity the rate of a clock depends just on the (gravitational) potential at the given point, and not on the potential gradient. An atom in the field of a constant potential has an altered frequency of radiation. This, however, gives us no reason to suppose that in the theory of gravitation the potential has special features which could, say, require us to look for a nonlocal formulation of the theory.

We note that a stationary sphere in a gravitational field, held in equilibrium by two equal forces acting in opposite directions, has potential energy, and the frequency of its wave function (which it has in principle, like the electron in the experiment described above) also depends on this energy.

Thus from the example of the first of the experiments suggested in [1] we can already see what answers there may be to the questions posed at the end of Section 1: a positive answer to the first question and a negative answer to the second. We still have to verify that the same conclusion follows from the other two physical examples.

## 3. SCATTERING OF AN ELECTRON IN THE FIELD OF A VECTOR POTENTIAL

The second experiment which has been proposed (and actually carried out [6]), and which was first analyzed in detail in [3], is as follows.

Let a plane electron wave, which as before has been separated into two packets by a screen with two slits, as in Fig. 1, or else by a biprism BP and two deflecting prisms, as in Fig. 2, pass on both sides of an infinitely long solenoid or magnetized rod S (Fig. 2) perpendicular to the plane of the drawing. Then the two packets are brought together by other prisms and give interference bands on a screen $S_2$.

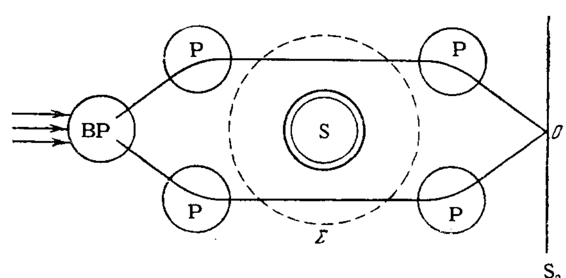


FIG. 2

Outside the solenoid the magnetic field is zero, and therefore no forces act on the electron. The vector potential, however, cannot be zero here: according to Stokes' theorem, the integral along a curve Σ passing around the solenoid must give the flux of induction through the curve

$$\oint_{\Sigma} A_s\,ds = \int \mathrm{rot}_n \mathbf{A}\,dS = \int H_n\,dS = \pi a^2 H = \Phi, \quad (1)$$

where a is the radius of the solenoid, H is the magnetic field strength in it, and $\Phi$ is the flux of induction in the solenoid. In particular, we can pick the gauge for the potentials so that $A = A_\theta = \Phi/2\pi r$, where $\theta$ is the polar angle measured in the plane of the drawing and r is the distance from the axis of the solenoid (Fig. 3). This vector potential appears in the Schrödinger equation for the electron and can cause an effect on the electron even in cases in which the wave function of the electron does not penetrate into the solenoid anywhere.

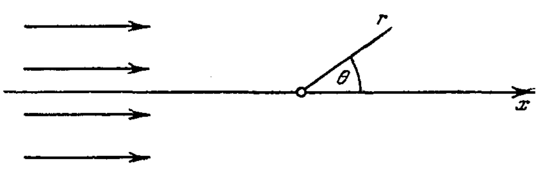


FIG. 3

To prove this, Aharonov and Bohm treat the scattering of a plane wave by a solenoid whose radius a goes to zero while the field strength H increases so that the flux is finite. One must deal with the Schrödinger equation separately outside and inside the solenoid and join the results at the surface r = a. Outside we have

$$\frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2 \psi = E\psi, \quad (2)$$

or in cylindrical coordinates r, $\theta$, z, on the assumption that under the conditions of the experiment nothing depends on z,

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\left(\frac{\partial}{\partial\theta} + i\alpha\right)^2\psi + k^2\psi = 0, \quad k^2 = \frac{2mE}{\hbar^2}, \quad (3)$$

where the parameter $\alpha$, which is of fundamental importance for what follows, is given by the relations

$$\alpha = -\frac{e\Phi}{ch}, \quad \Phi = \pi a^2 H. \quad (4)$$

We shall verify later that we do not need to consider the interior region of the solenoid, and can simply take Eq. (3) to hold for all space.

This equation can be solved by separation of variables; the equation for the radial function has solutions in terms of Bessel functions, for example

$$\psi_n \sim e^{\pm in\theta} J_{n+\alpha}(kr). \quad (5)$$

From the periodicity condition when $\theta$ changes by $2\pi$ it follows that n must be an integer. A plane electron wave incident from the left (it gives $\psi$ for $r \to \infty$ and $\theta \to \pi$) can be expanded in terms of Bessel functions ($\delta_{p0}$ is the Kronecker symbol)

$$\psi_0 = e^{ikx} = \sum_{p=0}^{\infty} (2 - \delta_{p0})\, i^p \cos p\theta\, J_p(kr). \quad (6)$$

If $\alpha$ is an integer, $n + \alpha = p$, it is seen that this function is made up of exact solutions (5) even in the presence of the potential $A_\theta$. Consequently, it is the solution of the Schrödinger equation and there is no scattering. It is only in rare cases, however, that the magnetic flux $\Phi$ is quantized. This occurs, for example, in superconductors [11] (here the "quantum of flux" is the quantity $ch/2e = 2 \cdot 10^{-7}$ Mx). If indeed $\alpha$ is not an integer, than $\psi$ must be the sum of $\psi_0$ and a scattered wave $\psi_1$. In [1] $\psi_1$ is found as an expansion in terms of the particular solutions $\psi_n$. It allows the authors to determine the scattering cross section [1]:

$$d\sigma = \frac{\sin^2 \pi\alpha}{2\pi k}\,\frac{d\theta}{\sin^2\frac{\theta}{2}}. \quad (7)$$

We can also use a simpler approach, if we confine ourselves to the case of a value of $\alpha$ close to an integer. We can then regard the difference from an integer as a perturbation,

$$\alpha_n = \alpha - n, \quad (8)$$

where n is the nearest integer, and apply the Born approximation of perturbation theory. It is clear that this is enough to settle the question in principle as to whether or not a scattering exists. Then, setting $\psi = \psi_0 + \psi_1$ and keeping only terms of first order in $\alpha_n$, we have in the three-dimensional formulation

$$(\nabla^2 + k^2)\psi_1 = -\frac{2i\alpha_n}{r^2}\frac{\partial\psi_0}{\partial\theta}, \quad (9)$$

$$\psi_1 = \frac{1}{4\pi}\int \frac{2i\alpha_n}{r'^2}\,\frac{e^{ik|\mathbf{R}-\mathbf{R}'|}}{|\mathbf{R}-\mathbf{R}'|}\,\frac{\partial}{\partial\theta'}\,e^{ikx'}\,d^3\mathbf{R}'. \quad (9a)$$

By the usual method, integrating over z from $-\infty$ to $+\infty$, we get for $r \to \infty$

$$\psi_1 = \frac{2i\alpha_n}{4\pi}\int i\pi H_0^{(1)}(k|\mathbf{r}-\mathbf{r}'|)\frac{\partial\psi_0(\mathbf{r}')}{\partial\theta}\,\frac{dr'\,d\theta'}{r'}$$

$$\to \frac{ik\alpha_n}{\sqrt{2\pi kr}}\,e^{ikr - i\frac{\pi}{4}}\int e^{i(\mathbf{k}_0-\mathbf{k},\,\mathbf{r}')}\sin\theta'\,d\theta'\,dr'. \quad (10)$$

The angle $\chi$ between $\mathbf{q} = \mathbf{k}_0 - \mathbf{k}$ and $\mathbf{r}'$ (Fig. 4) can be expressed in terms of $\theta'$ and the scattering angle $\theta$, $\chi = \pi/2 + \theta' - \theta/2$, so that after integrating over $\theta'$ from 0 to $2\pi$ and over r′ from a to $\infty$ we get

$$\psi_{1\,\mathrm{scat}} = A(\theta)\, i\pi H_0^{(1)}(kr), \quad A(\theta) = \frac{i\alpha_n}{2\,\mathrm{tg}\frac{\theta}{2}}\,J_0(ka), \quad (11)^*$$

and the scattering cross section (calculated as the ratio of the flux of the scattered wave $\psi_1$ to the incident flux) is, when we set $ka \to 0$

$$d\sigma = \frac{2\pi}{k}|A(\theta)|^2\,d\theta = \frac{\pi\alpha_n^2}{2k\,\mathrm{tg}^2\frac{\theta}{2}}\,d\theta. \quad (12)$$

This expression is somewhat different from the re-

*tg = tan.

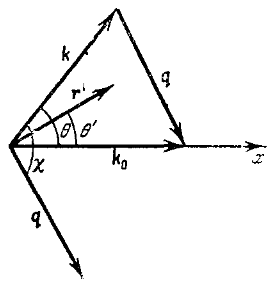


FIG. 4

sult of taking $\alpha \ll 1$ in the result (7) which was given in [1] (there is an extra factor $\cos^2 \theta/2$). In fact, the solution found in [1] is, strictly speaking, incorrect: the total function changes when $\theta$ is changed by $2\pi$ (this can be seen particularly clearly from Eqs. (21) and (23) of [1]). This does not change the essential point, however: there is scattering when $\alpha$ is not an integer. Moreover, the total cross section is infinite (on account of the small values of $\theta$, i.e., distant passages).

In our treatment here we have supposed that a = 0. This is actually permissible, since the probability of finding the particle inside the solenoid can be made vanishingly small. As was pointed out in [1], we could surround the solenoid with an impenetrable barrier and this would not change the result. In fact, for $ka \ll 1$ such a barrier would mean that as the wave unperturbed by the potential **A** we would have to take instead of $e^{ikz}$ the expression [12]

$$\psi_0 = e^{ikx} - i\sum_{p=0}^{\infty} (2-\delta_{p0})\, e^{\frac{1}{2}p\pi i - \delta_p(ka)} \sin\delta_p(ka)\, H_p^{(1)}(kr) \times \cos p\theta \approx e^{ikx} - \frac{i\pi H_0^{(1)}(kr)}{2\ln ka} \qquad (13)$$

(where $\delta_p(ka)$ is the scattering phase shift), and make corresponding changes in the application of perturbation theory. In effect we would have to proceed as before and integrate only over the region $r > a$, which is what we have done.

Thus there must be a scattering, although the field strength is zero in the region where the electron is. We note that this is also a quantum effect, since $\sigma$ is proportional to $1/k = \hbar/p$.

This conclusion has been subjected to experimental test. Chambers [6] has observed the scattering in the field of a magnetized iron "whisker" C (of diameter about 1μ) for electrons emitted by a source S (Fig. 5, not drawn to scale). The electrons passed through the biprism efe and gave an interference pattern at 0. The magnetic whisker was in the shadow of the aluminized quartz fiber f of diameter about 1.5μ. The magnetic flux was about 400 hc/e and varied somewhat along the length of the whisker, by about one unit hc/e (one "fluxon") per micron. It was found that along the coordinate z perpendicular to the drawing the interference bands have a slope which is in good agreement with that to be expected from the theory of Aharonov and Bohm: the shift amounted to one band for a displacement along the z axis of 1μ—that is, for a change of $\alpha$ by unity.

Unfortunately, as was shown by Pryce (see [2]), the nonuniformity of the magnetization produces a stray magnetic field which by itself can cause a similar displacement of the bands. Besides this, the diameter of the magnet is comparable with the width of a band at 0. Therefore ka cannot be regarded as small. Consequently the experiment cannot be accepted as conclusive. It is hard to doubt, however, that the result must be positive.

In the other experiments [7-9] it has only been shown that the direct effect of the potential must also be taken into account in cases in which the magnetic field is different from zero; otherwise one cannot give an exact explanation of the experimental facts. This in itself only confirms the correctness of the Schrödinger equation.

Again, as in the case of the experiment with the electric potential, the answer to the first of the questions formulated at the end of Section 1 must be positive.

Is it correct, however, to suppose that here there is no physical action of the magnetic field in the classical sense of the word? The field strength does indeed vanish at every point where the probability for the presence of the electron is different from zero. We must, however take account of the fact that the interaction of the electron with the source of the field does not vanish. The electric current of the moving electron produces a magnetic field which acts on the solenoid and produces an interaction energy represented by the term $-e/c\ \mathbf{A}$ in the expression for the total energy $1/2m\,(\hat{p} - e/c\ \mathbf{A})^2$. This expression itself is purely classical (it does not contain $\hbar$), and therefore in this case also the action of the potential has a classical basis.

In fact, when we make a perturbation calculation in first order in $\alpha_n$ we have for the matrix element

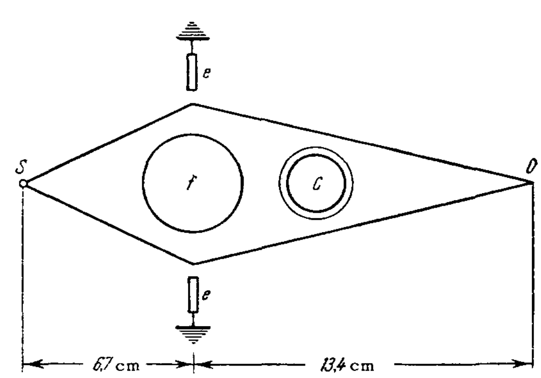


FIG. 5

for the transition of the electron in being scattered from state $\psi_0$ to state $\psi_f$

$$M \sim \int \psi_f^*(\mathbf{r}') \frac{e}{mc} (\hat{\mathbf{p}}\mathbf{A}(\mathbf{r}')) \psi_0(\mathbf{r}')\, d\mathbf{r}'. \quad (14)$$

But **A** is produced by the current **j** in the solenoid,

$$\mathbf{A}(\mathbf{r}') = \frac{1}{c} \int \frac{\mathbf{j}(\mathbf{r}'')}{|\mathbf{r}' - \mathbf{r}''|} d\mathbf{r}''.$$

On the other hand, $\hat{\mathbf{p}}/m = \mathbf{v}$ is the velocity operator, so that the expression that appears in Eq. (14), $\psi_f^*(\mathbf{r}')\, e/m\, \hat{\mathbf{p}}\psi_0(\mathbf{r}') = \mathbf{j}_{0f}(\mathbf{r}')$, is the transition current density of the electron. Therefore M takes the symmetrical form of an interaction between currents:

$$M \sim \frac{1}{c^2} \int \frac{\mathbf{j}_{0f}(\mathbf{r}')\, \mathbf{j}(\mathbf{r}'')}{|\mathbf{r}' - \mathbf{r}''|} d\mathbf{r}'\, d\mathbf{r}''. \quad (15)$$

Here, strictly speaking, the change of the state of the source under the influence of the interaction has not been taken into account. In order to do so, we must write instead of $\mathbf{j}(\mathbf{r}'')$ the transition current density of the source $J_{\text{of.source}}(\mathbf{r}'')$. The expression (15) explicitly is of the classical form of the energy of interaction of currents. It is, by the way, clear even without this that **A** in the Schrödinger equation appears after variation of the expression for the energy of the system, and therefore represents the interaction of the electron with the solenoid or magnet. This energy, for example, is expended in the form of the work of the additional electromotive force which must be applied to the solenoid to keep the current in it constant when the approaching electron produces an induced current in the solenoid; it is essential that we are always speaking here about the energetic changes with a constant current in the source (cf. e.g., [13], Section 52). The complex result of these interactions can be expressed in the simple fact that the action for the electron, according to classical electrodynamics, acquires the added term $\int e/c\, A_s\, ds$, where the integral is taken along the path of the electron. Since in a passage around the solenoid this would give an increase of the action $\Delta S = e/c\, \Phi$, then in passing by (without going around) from $x = -\infty$ to $x = +\infty$, independent of the path, the classical electron receives a constant increase of action $\frac{1}{2}\, e/c\, \Phi$ if it passes on one side of the solenoid or $-\frac{1}{2}\, e/c\, \Phi$ if it passes on the other side. This has no effect on its motion (in particular, the total work done by the electron on the solenoid and the source of current is zero). In quantum mechanics there is an increase of the phase by $\pm\, e/2c\hbar\, \Phi$, which is the same at all points of the packet if the entire wave packet is on one side of the solenoid. If, however, the packet envelops the solenoid, then there are different phase shifts in different parts of it, and there is a disturbance of the interference pattern which increases with increase of $\Phi$.

The complete similarity of this picture with the case of the scalar potential (Section 2) is quite obvious. Again the interaction with the source is basically a classical one. The quantum character of the process is due to the facts that a) the existence of an energy of interaction with the source of the field is important for the frequency of the wave function, and b) the position of the electron is indefinite, so that there are different phase shifts in different parts of the packet.

The similarity with Section 2 can be made even more graphic if we consider the process in a system in which the electron is at rest and the solenoid moves. In this reference system there is an electric potential $\varphi'$ caused by the energy $e\varphi'$ of the interaction of the electron with the electric polarization $\mathbf{P} = \mathbf{v}/c \times \mathbf{M}$, where **M** is the magnetic moment per unit volume of the solenoid (the field $\mathbf{E}' = -\text{grad}\, \varphi' - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$ which acts on the electron is zero as before).

The question of the effect on the source has been discussed in detail by Aharonov and Bohm. [2] The whole purpose of their treatment, however, was to prove that the interaction actually leads to the equation (2) for the electron. This scarcely needs proving, however.

## 4. THE STATIONARY STATES IN THE FIELD OF A SOLENOID

An additional example analyzed in [2] is that of the stationary states of an electron in the field of the same infinitely long solenoid that was considered in Section 3. Here also, if we assume for example that there is available to the electron only the space in a ring (bounded by infinitely high impenetrable potential walls) surrounding the solenoid, the electron's energy depends on **A**, although the field strength outside the solenoid is zero. If we treat the electron as a plane rotator of radius r, we have as the equation for its wave function

$$-\frac{\hbar^2}{2mr^2}\left(\frac{\partial}{\partial \theta} + i\alpha\right)^2 \psi = E\psi. \quad (16)$$

A solution of this equation is the function

$$\psi \sim e^{in\theta}, \quad (17)$$

where it follows from the requirement of single-valuedness that n is an integer. Therefore

$$E = \frac{\hbar^2}{2mr^2}(n + \alpha)^2. \quad (18)$$

To prove that this result is also of classical origin, let us consider a current of strength I flowing in a ring of radius r around the same solenoid. On the z axis it produces the magnetic field (cf., e.g., [13], Section 42)

$$H_1(z) = \frac{2\pi I}{c} \frac{r^2}{(r^2 + z^2)^{3/2}}, \quad (19)$$

which is added to the magnetic field $H_0$ of the solenoid. The energy of the system is

$$W = \frac{1}{8\pi}\int H^2\,dV = \frac{1}{8\pi}\int (H_0^2 + H_1^2 + 2H_0H_1)\,dV. \quad (20)$$

Here the first term gives the proper energy of the solenoid, the second the proper energy of the ring current, and the last the interaction energy. If the radius of the solenoid is infinitely small, then substituting Eq. (19) in Eq. (20) and integrating over the volume of the solenoid, where $H_0$ is different from zero, we get

$$W_{12} = \frac{I\Phi}{c}. \quad (21)$$

This result is of course the same as we would have found by integrating the expression $1/c \int (\mathbf{A}\cdot\mathbf{j})\,dV$ over the volume of the ring current.

In the quantum case of a plane rotator, according to the usual rules of quantum mechanics and Eq. (17), the current density of the electron is given by

$$j = j_\theta = \frac{e\hbar}{2mi}(\psi^*\nabla_\theta\psi - \psi\nabla_\theta\psi^*) - \frac{e^2}{mc}A_\theta\psi^*\psi$$
$$= \left(\frac{e\hbar}{m}\frac{n}{r} - \frac{e^2}{mc}A_\theta\right)\psi^*\psi = \frac{e\hbar}{mr}(n+\alpha)\,\psi^*\psi. \quad (22)$$

Here again the vector potential appears, although we are dealing with a region where by hypothesis there is no magnetic field strength. In this too, however, it is hard to perceive any "new" or "unclassical" properties of the potential: in classical electrodynamics also, if we express the current density $\mathbf{j} = \rho\mathbf{v}$ ($\rho$ is the charge density and $\mathbf{v}$ the velocity) in terms of the generalized momentum $\mathbf{p}$, we must replace $\mathbf{v}$ by $\mathbf{p} - e/c\,\mathbf{A}$.

If dS is an element of the cross section of the ring and $dV = 2\pi r dS$ is an element of its volume, the total current is

$$I = \int j_\theta\,dS = \frac{e\hbar}{m}\frac{n+\alpha}{2\pi r^2}\int |\psi|^2\,dV = \frac{e\hbar}{2\pi m r^2}(n+\alpha), \quad (23)$$

Substituting this expression in Eq. (21), we get

$$W_{12} = -\frac{\hbar^2}{mr^2}(n\alpha + \alpha^2). \quad (24)$$

During the process, however, of increase of $\Phi$ by $d\Phi$ and the accumulation of this energy, an emf $\mathcal{E} = -\frac{1}{c}\frac{d\Phi}{dt}$ is induced in the ring contour and does work on the current, so that its energy is changed by $-I/c\,d\Phi$. Expressing I and $\Phi$ in terms of $\alpha$ by means of Eqs. (4) and (23) and integrating over $\alpha$ from 0 to $\alpha$ (with n = const, since $p_\theta$ commutes with $A_\theta$), we get for the change of the energy of the ring current

$$\frac{\hbar^2}{2mr^2}(2n\alpha + \alpha^2),$$

i.e., the terms which must be added to the energy of the electron in the absence of the field, $\hbar^2 n^2/2mr^2$, in order to get the total energy (18).

Thus there is no change in the interaction energy, and consequently no change from the classical meaning of the potential. The quantum feature (proportionality to $\hbar$) appears only because the position of the electron in the ring is in principle indeterminate, so that it can be treated like a current filling the entire ring at each instant of time; the strength of the current is proportional to $\hbar$ [Eq. (23)].

## 5. SUMMARY

We have examined three physical examples which display the effects of potentials which are constant in space and time on an electrically charged particle within the framework of nonrelativistic quantum mechanics. In all cases the basis of the effects is the ordinary classical energy of interaction of particle and source, which is different from zero, in spite of the fact that the field strength at the position of the particle is zero (a charge at points where the electrical potential is constant in space and time; an electric current surrounding a solenoid; and so on). This actual energy of the system has been accumulated in the process of setting up the system.

The quantum peculiarity of the behavior of the particle under the action of such a potential arises only because the energy of the system is directly related to the frequency of the wave function, and if the change of the frequency is different in different parts of the packet there can be an interference effect. There do not appear here any new, "nonlocal" properties of the electromagnetic potential itself which would not be present in classical electrodynamics. The feature in which one can perceive an element of new "nonlocality" is the diffuseness of the wave function of the particle, which requires us to calculate the action of the external field as if the electron itself were diffuse and existed simultaneously at all points of space with a probability density proportional to the square of the absolute value of the wave function. This, however, is due to the indeterminacy in principle (in the framework of the uncertainty relation) of the position of the electron, and has no direct relation to the properties of the electromagnetic potential. A reexamination or "further development" of the theory, as spoken of in [1], (see above, Section 1) can essentially be directed only against the concept of the potential in classical electrodynamics or against the uncertainty relation. There are, however, no grounds for doing this.

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