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## *ELECTRON THEORY OF TRANSITION METALS, II.\**

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THE first part of the review contained an analysis of the main experimental facts pertaining to the properties of the electron system of transition metals, and it was shown that the entire specific character of their physical and chemical properties is connected with the presence of unfilled electron shells (d or f) with uncompensated spin in these atoms. Experiment has shown that in spite of certain changes that can occur in the closed electron shells of the atoms that combine in the crystal, the individuality of these shells is retained in the metal to a considerable degree, meaning a large degree of localization of the spin density near the lattice sites. This pertains in particular to the deep lying 4f-shells of the rare-earth metals.

The exchange interaction between the electrons of the unfilled shells in transition metals frequently leads to the establishment of a ferromagnetic or antiferromagnetic state, while their metallic properties are connected primarily with the former valence electrons, which form in the crystal a collectivized system of conduction electrons.

To study the connection between the electric and magnetic properties of the transition metals, the so-called s-d exchange model was proposed<sup>[1]</sup>, an elementary and phenomenological treatment of which was given in Sec. 5. The second part of the review is devoted to a more rigorous exposition of both the principles of the s-d model and its various applications to the theoretical treatment of ferromagnetic and antiferromagnetic metals, with Chapter III devoted to the properties of the spin system of a metal from the point of view of the influence exerted on it by the conduction electrons, while Chapter IV is a

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study of the reaction of the spin system on the conduction electrons.

### III. PROPERTIES OF THE SPIN SYSTEM OF A TRANSITION METAL

#### 6. Hamiltonian of the Electron System of a Transition Metal

In accordance with the model assumed (see Sec. 5) we segregate in the transition metal two groups of electrons, those of the unfilled shells (d or f)\* and the conduction electrons (s), the former being described by the atomic localized functions (2.11) and the latter by the Bloch functions

$$\psi_{k\sigma}(\mathbf{r}, s) = \frac{1}{\sqrt{N}} e^{i\mathbf{k}\mathbf{r}} U_{\mathbf{k}}(\mathbf{r}) c_{\sigma}(s) = \psi_{\mathbf{k}}(\mathbf{r}) c_{\sigma}(s), \quad (6.1)$$

where  $c_{\sigma}(s)$  is the so-called spin function of the electron (see [73], Sec. 60).

According to the Dirac vector model, we can write for the s-d exchange interaction operator, accurate to a constant factor,

$$H_{sd} = -2 \sum_{i, n} J(\mathbf{r}_i - \mathbf{R}_n) (\mathbf{s}_i \mathbf{S}_n), \quad (6.2)$$

where  $\mathbf{s}_i$ ,  $\mathbf{r}_i$  and  $\mathbf{S}_n$ ,  $\mathbf{R}_n$  are respectively the spin operators and radius vectors of the conduction electron and of the summary spin of the ferromagnetic ion in the n-th site. The spin  $\mathbf{S}_n$  is made up of the spins  $z$  of the unpaired electrons of the unfilled shell, so that the maximum projection of the site spin is  $s = z/2$ , and  $J(\mathbf{r}_i - \mathbf{R}_n)$  is the exchange integral.

In the second-quantization representation for the conduction electrons, the Hamiltonian (6.2) has the form

$$H_{sd} = -2 \sum_{\mathbf{k}} \sum_{\mathbf{k}'\sigma\sigma'} S_n (\mathbf{k}\sigma | J(\mathbf{r} - \mathbf{R}_n) \mathbf{s} | \mathbf{k}'\sigma') a_{\mathbf{k}\sigma}^{\dagger} a_{\mathbf{k}'\sigma'}. \quad (6.3)$$

In formula (6.3),  $a_{\mathbf{k}\sigma}$  and  $a_{\mathbf{k}\sigma}^{\dagger}$  are the Fermi operators for the creation and annihilation of electrons in the state  $\mathbf{k}\sigma$ . Using the multiplicative spin and coordinate parts of the function (6.1), and also translation invariance  $U_{\mathbf{k}}(\mathbf{r}) = U_{\mathbf{k}}(\mathbf{r} - \mathbf{R}_n)$ , (where  $\mathbf{R}_n$  is any vector of the crystal lattice), the matrix element in (6.3) can be written in the form

$$(\mathbf{k}\sigma | J(\mathbf{r} - \mathbf{R}_n) \mathbf{s} | \mathbf{k}'\sigma') = \frac{1}{N} e^{i(\mathbf{k}' - \mathbf{k}) \mathbf{R}_n} J(\mathbf{k}\mathbf{k}') (\sigma | \mathbf{s} | \sigma'), \quad (6.4)$$

where  $J(\mathbf{k}\mathbf{k}')$  is the s-d exchange integral [1]. The quantities  $(\sigma | \mathbf{s}^{\alpha} | \sigma')$  are Pauli matrices (see [73], Secs. 59, 60). Thus, the Hamiltonian (6.3) can be represented in the form [2]

$$H_{sd} = -\frac{1}{N} \sum_{\mathbf{k}\mathbf{k}'n} J(\mathbf{k}\mathbf{k}') e^{i(\mathbf{k}' - \mathbf{k}) \mathbf{R}_n} \{ a_{\mathbf{k}(-)}^{\dagger} a_{\mathbf{k}'(+)} S_n^- + a_{\mathbf{k}(+)}^{\dagger} a_{\mathbf{k}'(-)} S_n^+ + (a_{\mathbf{k}(-)}^{\dagger} a_{\mathbf{k}'(-)} - a_{\mathbf{k}(+)}^{\dagger} a_{\mathbf{k}'(+)} S_n^z) \}, \quad (6.5)$$

where  $S_n^{\pm} = S_n^x \pm iS_n^y$ . In (6.5), as well as from now on, we shall write merely + and - for the indices  $\sigma = 1/2$  and  $-1/2$ .

The operator (6.5) describes the exchange interaction between the collectivized s-electrons and the localized "magnetic" electrons. The total Hamiltonian of the electron system of a transition metal consists of three parts:  $H = H_s + H_d + H_{sd}$ , where  $H_s$  is the Hamiltonian of the conduction electrons, which in the second-quantization representation can be written in the form

$$H_s = \sum_{\mathbf{k}\sigma} [E_{\mathbf{k}} - sJ(\mathbf{k}\mathbf{k})] a_{\mathbf{k}\sigma}^{\dagger} a_{\mathbf{k}\sigma}. \quad (6.6)$$

$E_{\mathbf{k}}$  is the energy of the conduction electron in the metal without allowance for the s-d interaction. The additional term  $-sJ(\mathbf{k}\mathbf{k})$  is due to the spin-independent term of the exchange Hamiltonian, which leads to an equal energy shift of the conduction electrons with both spin projections.

We shall henceforth choose for the Hamiltonian  $H_d$  of the "magnetic" electrons the exchange operator of the interaction between the d electrons of neighboring ions.

A Hamiltonian in the form (6.5) was first obtained by Vonsovskiĭ and Turov [2] and then, independently, in [3-6]. In the derivation of (6.5) from the overall Hamiltonian of the system of interacting s and d electrons, the homopolarity condition, i.e., the constancy of the number of d electrons near the sites, was used in addition. This condition corresponds in the scheme adopted here to writing the exchange energy in the form (6.2).

The following expression was obtained in [1] for the s-d exchange integral:

$$J(\mathbf{k}\mathbf{k}') = \iint \psi_{\mathbf{k}}^*(\mathbf{r}) \varphi^*(\mathbf{r}') \Phi(\mathbf{r} - \mathbf{r}') \psi_{\mathbf{k}'}(\mathbf{r}') \varphi(\mathbf{r}) d\mathbf{r} d\mathbf{r}', \quad (6.7)$$

where  $\varphi(\mathbf{r})$  is the atomic function of the d electron, and  $\Phi(\mathbf{r} - \mathbf{r}')$  is the Coulomb interaction potential of the s and d electrons. It is difficult to conclude from such a general formulation anything specific regarding the dependence of  $J(\mathbf{k}\mathbf{k}')$  on the quasi-momenta  $\mathbf{k}$  and  $\mathbf{k}'$ , but this is not very important. It will be shown below that in the expressions for the different physical quantities,  $J(\mathbf{k}\mathbf{k}')$  enters for values of  $\mathbf{k}$  and  $\mathbf{k}'$  that are close to the limiting Fermi momentum  $k_0$ , inasmuch as the electrons in the metal are strongly degenerate. If we assume that  $J(\mathbf{k}\mathbf{k}')$  varies slowly in the metal when  $\mathbf{k} \sim \mathbf{k}'$ , we can put everywhere  $J(\mathbf{k}\mathbf{k}') \approx J(k_0, k_0) = J_0$ . According to various estimates [7], the absolute value of  $J_0$  is  $\sim 10^{-14} - 10^{-13}$  erg, which is one or two orders of magnitude below

\*Unless there is danger of misunderstanding, we shall use the subscript d throughout to denote both unfilled shells, 3d and 4f. In this case the words "s-d exchange interaction" denote the interaction between the conduction electrons and the electrons of either the 3d or the 4f unfilled shell.

the Fermi energy  $\xi \sim 10^{-12}$  erg. Under these conditions there appears a small parameter  $J_0/\xi \ll 1$ , in which the thermodynamic and kinetic quantities are expanded if the s-d interaction is regarded as a perturbation. The smallness of this parameter enables us to confine ourselves in many cases to the first two perturbation-theory approximations. Within the framework of perturbation theory, we shall consider in Secs. 7 and 8 the distribution of the spin density of the s electrons in a transition metal and the indirect interaction of the d electrons via the conduction electrons.

### 7. Spin Polarization of the Conduction Electrons

We consider the crystal of a transition metal or of the alloy of some simple metal with diamagnetic ionic cores and a transition metal, say an alloy such as Cu-Mn. Assume that a ferromagnetic or antiferromagnetic spin order has been established in the transition metal atoms of such crystals as a result of the exchange coupling. Let us determine the influence of this order on the conduction electrons with account of the s-d exchange coupling.

The Hamiltonian of the system of s electrons in a magnetic field H parallel to the quantization axis z has the form

$$H_s = \sum_{\mathbf{k}} [E_{\mathbf{k}} - sJ(\mathbf{k}\mathbf{k}) - \mu_0 H] a_{\mathbf{k}(-)}^{\dagger} a_{\mathbf{k}(-)} + \sum_{\mathbf{k}} [E_{\mathbf{k}} - sJ(\mathbf{k}\mathbf{k}) + \mu_0 H] a_{\mathbf{k}(+)}^{\dagger} a_{\mathbf{k}(+)} + H_{sd}, \quad (7.1)$$

where  $H_{sd}$  is determined by expression (6.5), and the index n numbers in this case the sites occupied by the ferromagnetic ions.

We regard the s-d interaction as a small perturbation; we must then take into account in the first order of perturbation theory only the diagonal part of  $H_{sd}$ , equal to

$$- \sum_{\mathbf{k}} J(\mathbf{k}\mathbf{k}) (a_{\mathbf{k}(-)}^{\dagger} a_{\mathbf{k}(-)} - a_{\mathbf{k}(+)}^{\dagger} a_{\mathbf{k}(+)}) \frac{1}{N} \sum_n S_n^z, \quad (7.2)$$

and we then obtain in lieu of the exact Hamiltonian (7.1)

$$H = \sum_{\mathbf{k}} (\varepsilon_{\mathbf{k}}^{\pm} a_{\mathbf{k}(+)}^{\dagger} a_{\mathbf{k}(+)} + \varepsilon_{\mathbf{k}}^{-} a_{\mathbf{k}(-)}^{\dagger} a_{\mathbf{k}(-)}), \quad (7.3)$$

where

$$\varepsilon_{\mathbf{k}}^{\pm} = \varepsilon_{\mathbf{k}} - sJ(\mathbf{k}\mathbf{k}) \pm \mu_0 H \pm J(\mathbf{k}\mathbf{k}) \frac{1}{N} \sum_n S_n^z. \quad (7.4)$$

We see that if the summary spin of the crystal differs from zero, a term due to the s-d exchange appears in the expression for the conduction-electron energy, having the meaning of the internal molecular field and leading to spontaneous magnetization of these electrons. In fact, let us calculate the average values of the numbers  $n_-$  and  $n_+$  of the s electrons with left-hand and right-hand spins:

$$n_{\pm} = \sum_{\mathbf{k}} n_{\mathbf{k}}^{\pm}, \quad (7.5)$$

where

$$n_{\mathbf{k}}^{\pm} = \langle a_{\mathbf{k}(\pm)}^{\dagger} a_{\mathbf{k}(\pm)} \rangle = \frac{1}{e^{\beta(\varepsilon_{\mathbf{k}}^{\pm} - \xi)} + 1}, \quad \beta = \frac{1}{kT}, \quad (7.6)$$

are the distribution functions for electrons with different spin orientations. Expressions (7.5) are calculated by means of the usual transition from summation over  $\mathbf{k}$  to integration. Assuming a quadratic dispersion law for the s electrons and using the approximation  $J(\mathbf{k}\mathbf{k}')$  for  $J_0$ , we obtain

$$n_{\pm} = \frac{N_s}{2} \mp \frac{3}{4} \frac{N_s}{\xi} \left( \mu_0 H \pm J_0 \frac{1}{N} \sum_n S_n^z \right), \quad (7.7)$$

$$\xi = \xi_0 \left\{ 1 - \frac{1}{4} \left( \frac{\mu_0 H \pm J_0 \frac{1}{N} \sum_n S_n^z}{\xi_0} \right)^2 \right\},$$

$$\xi_0 = \frac{\hbar^2}{2m} \left( 3\pi^2 \frac{N_s}{V} \right)^{2/3}, \quad (7.8)$$

where V is the volume of the crystal and  $N_s$  is the total number of s electrons in it.

The magnetic moment of the conduction-electron gas, neglecting its weak diamagnetism, is equal to

$$M_s = \mu_0 \frac{1}{2} (n_- - n_+) = \frac{3}{2} N_s \mu_0 \frac{\mu_0 H}{\xi} + \frac{3}{2} N_s \mu_0 \frac{J_0}{\xi} \frac{1}{N} \sum_n S_n^z. \quad (7.9)$$

The first term in (7.9) gives the Pauli paramagnetic magnetization [see also (4.8) and below], while the second, which does not depend on the magnetic field, gives the spontaneous magnetic moment of the s electrons. The latter is due to the s-d exchange coupling and has an order of magnitude of  $J_0/\xi$  times the magnetic moment of the d electrons. In an antiferromagnetic metal, where  $\sum_n S_n^z = 0$ , there is no

spontaneous magnetization of the conduction electrons. In ferromagnets it reaches a maximum value at 0°K, and vanishes above the Curie point.\*

The foregoing deduction concerning the s-d exchange polarization of the conduction electrons has a statistical character. Let us ascertain now, following the paper of Yosida [6], how the density of electrons with different spin projections varies from point to point.

The density of s electrons with spin of given orientation  $\rho_{\pm}(\mathbf{r})$  can be expressed in term of their wave functions (6.1):

$$\rho_{\pm}(\mathbf{r}) = \sum_{\mathbf{k}} \sum_n \varphi_{\mathbf{k}(\pm)}^* \varphi_{\mathbf{k}(\pm)}, \quad (7.10)$$

where  $k_0^{\pm}$  are the limiting Fermi momenta of the right- and left-hand spin projections, respectively, while  $\sum$  denotes summation over the spin variables.

The wave functions  $\varphi_{\mathbf{k}(\pm)}$  can be determined from perturbation theory. We assume as the zeroth approximation for these functions the plane waves

\*These results were first obtained by Vonsovskii in 1946.

$$\varphi_{\mathbf{k}(\pm)}^0 = \frac{1}{\sqrt{V}} e^{i\mathbf{k}\mathbf{r}} C_{\pm}(s). \quad (7.11)$$

Under the influence of the s-d exchange we have in first-order perturbation theory

$$\varphi_{\mathbf{k}\sigma} = \varphi_{\mathbf{k}\sigma}^0 + \sum_{\mathbf{k}'\sigma'} \frac{(\mathbf{k}\sigma | H_{sd} | \mathbf{k}'\sigma')}{\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}'}} \varphi_{\mathbf{k}'\sigma'}. \quad (7.12)$$

After calculating the matrix elements  $(\mathbf{k}\sigma | H_{sd} | \mathbf{k}'\sigma')$  with the aid of the functions (7.11) and substituting them in (7.12) we obtain

$$\varphi_{\mathbf{k}(\pm)} = \varphi_{\mathbf{k}(\pm)}^0 - \frac{1}{N} \sum_{\mathbf{k}'} \frac{J(\mathbf{k}\mathbf{k}')}{\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}'}} \sum_n e^{i(\mathbf{k}-\mathbf{k}')\mathbf{R}_n} \times \{ \mp S_n^z \varphi_{\mathbf{k}'(\pm)}^0 + S_n^{\pm} \varphi_{\mathbf{k}'(\mp)}^0 \} \quad (7.13)$$

The prime on the summation sign denotes that the term with  $\mathbf{k}' = \mathbf{k}$  is excluded. Substituting (7.13) in (7.10) and summing over  $s$ , we obtain

$$\varrho_{\pm}(\mathbf{r}) = \sum_{\mathbf{k}} \frac{h_0^{\pm}}{V} \mp \frac{1}{VN} \sum_{\mathbf{k}} \sum_{\mathbf{k}'} \frac{J(\mathbf{k}\mathbf{k}')}{\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}'}} \times \sum_k \{ e^{i(\mathbf{k}-\mathbf{k}')(\mathbf{r}-\mathbf{R}_n)} + e^{-i(\mathbf{k}-\mathbf{k}')(\mathbf{r}-\mathbf{R}_n)} \} S_n^z. \quad (7.14)$$

It is obvious that at ordinary temperatures

$$\sum_{\mathbf{k}} \frac{h_0^{\pm}}{V} = \frac{1}{V} n_{\pm},$$

so that we can use expression (7.7) for the case  $H = 0$ . In the second term of (7.14) we change the summation variables to  $\mathbf{k} - \mathbf{k}' = \mathbf{q}$ . As a result we can rewrite (7.14) in the form

$$\varrho_{\pm}(\mathbf{r}) = \frac{1}{2} \left( \frac{N_s}{V} \right) \pm \frac{3}{4} \left( \frac{N_s}{V} \right) \frac{J_0}{\xi} \frac{1}{N} \sum_n S_n^z \pm \frac{1}{VN} \sum_{\mathbf{q}} \sum_{\mathbf{k}} \frac{J(\mathbf{k}, \mathbf{k}-\mathbf{q})}{\varepsilon_{\mathbf{k}-\mathbf{q}} - \varepsilon_{\mathbf{k}}} \sum_n \{ e^{i\mathbf{q}(\mathbf{r}-\mathbf{R}_n)} + e^{-i\mathbf{q}(\mathbf{r}-\mathbf{R}_n)} \} S_n^z. \quad (7.15)$$

Further calculation of (7.15) calls for knowledge of the dependence of  $J(\mathbf{k}\mathbf{k}')$  on the momenta. In the approximation  $J(\mathbf{k}\mathbf{k}') = J_0$ , the summation over  $\mathbf{k}$  in (7.15) can be carried through to conclusion. Assuming a quadratic dispersion law, we obtain

$$\sum_{\mathbf{k}} \frac{1}{\varepsilon_{\mathbf{k}-\mathbf{q}} - \varepsilon_{\mathbf{k}}} = \frac{3}{16} \frac{N_s}{\xi} f(q), \quad (7.16)$$

$$f(q) = 1 + \frac{4k_0^2 - q^2}{4k_0q} \ln \left| \frac{2k_0 + q}{2k_0 - q} \right|. \quad (7.17)$$

Taking these relations into account, (7.15) assumes the form

$$\varrho_{\pm}(\mathbf{r}) = \frac{1}{2} \left( \frac{N_s}{V} \right) \pm \frac{3}{4} \left( \frac{N_s}{V} \right) \frac{J_0}{\xi} \frac{1}{N} \sum_n S_n^z \pm \frac{3}{16} \left( \frac{N_s}{V} \right) \frac{J_0}{\xi} \frac{1}{N} \sum_{\mathbf{q}} f(q) \sum_n \{ e^{i\mathbf{q}(\mathbf{r}-\mathbf{R}_n)} + e^{-i\mathbf{q}(\mathbf{r}-\mathbf{R}_n)} \} S_n^z. \quad (7.18)$$

Inasmuch as  $f(0) = 2$ , the second term in the right half of (7.18) is exactly equal to the missing term in the sum over  $\mathbf{q}$  when  $\mathbf{q} = 0$ , so that it can be combined with this sum, after which (7.18) is written in the form

$$\varrho_{\pm}(\mathbf{r}) = \frac{1}{2} \left( \frac{N_s}{V} \right) \pm \frac{3}{16} \left( \frac{N_s}{V} \right) \frac{J_0}{\xi} \frac{1}{N} \sum_{\mathbf{q}} f(\mathbf{q}) \times \sum_n \{ e^{i\mathbf{q}(\mathbf{r}-\mathbf{R}_n)} + e^{-i\mathbf{q}(\mathbf{r}-\mathbf{R}_n)} \} S_n^z. \quad (7.19)$$

It is easy to show that [8]

$$\frac{1}{N} \sum_{\mathbf{q}} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} = -12\pi \frac{N_s}{N} F(2k_0R). \quad (7.20)$$

where

$$F(x) = \frac{x \cos x - \sin x}{x^4}. \quad (7.21)$$

Using (7.20), we can rewrite (7.19) in final form:

$$\varrho_{\pm}(\mathbf{r}) = \frac{1}{2} \left( \frac{N_s}{V} \right) \mp \frac{9}{2} \pi \left( \frac{N_s}{V} \right) \frac{J_0}{\xi} \frac{N_s}{N} \sum_n F(2k_0|\mathbf{r}-\mathbf{R}_n|) S_n^z. \quad (7.22)$$

The first term in the right half of (7.22) gives the homogeneous spin density of the conduction electrons without account of the s-d interaction. The second term describes the inhomogeneous density variation, due to the s-d exchange coupling, the absolute value of which at the given point depends essentially on the distance from it to the paramagnetic sites. It has a maximum near the paramagnetic site and decreases as the cube of the distance from it, and oscillates at the same time. Near the site itself each term of the sum over  $n$  in (7.22) becomes infinite. This is apparently connected with the fact that in calculating the wave functions only the first perturbation-theory approximation was taken into account. It can be assumed that the spin density changes from point to point with amplitude proportional to  $(J_0/\xi)(N_s/V)$ .

It is seen from expression (7.22) that the inhomogeneous component of the density has a different sign for electrons with different spin projections. Therefore in the general case the result is inhomogeneous polarization of the conduction electrons,  $\Delta\rho = \rho_- - \rho_+$ . In a dilute solution of a transition element in a diamagnetic metal, the spin polarization of the electrons will increase appreciably near the paramagnetic sites. In some sense this is equivalent to a Friedel screening of the charged impurities in the metal. [9]

The result obtained is also fully equivalent to the result obtained in the theory of nuclear resonance in metals by Ruderman and Kittel [6], inasmuch as the hyperfine interaction between the nucleus and the conduction electrons is described by the same Hamiltonian as the s-d exchange coupling.

### 8. Indirect Exchange Interaction of the Electrons of Spin-unsaturated Shells Via the Conduction Electrons

If there exists a system of particles which do not interact directly with one another, but are weakly coupled to another system, then it is sometimes possible within the framework of perturbation theory to eliminate this interaction from the Hamiltonian and to replace it by a certain effective coupling between the particles of the first system. Thus, the coupling between the particle system and the other system leads to an indirect interaction between the particles. Let us ascertain the character possessed by the indirect interaction between the spins of unfilled shells under the influence of their exchange coupling with the *s*-electrons.

To derive the effective spin Hamiltonian we average the Hamiltonian of the electron system of the transition metal (or alloy) over the statistical states of the conduction electrons. The expression obtained in this manner will contain spin operators of paramagnetic ions and should describe the effective interaction between them.

Thus, let us calculate the quantity

$$H_{\text{eff}} = \frac{\text{Sp}_e (e^{-\beta H} H)}{\text{Sp}_e e^{-\beta H}}, \quad (8.1)$$

where the trace is taken only over the states of the conduction electrons. The operator  $H = H_0 + H_{\text{sd}}$  consists of the Hamiltonian of the system of conduction electrons (6.6) and the Hamiltonian of the *s*-*d* exchange (6.5). With the aid of the well known expansion of the exponential operator

$$e^{-\beta(H_0+H')} = e^{-\beta H_0} \left\{ 1 - \int_0^\beta d\lambda_1 H'(\lambda_1) + \int_0^\beta d\lambda_1 \int_0^{\lambda_1} d\lambda_2 H'(\lambda_1) H'(\lambda_2) + \dots \right\}, \quad (8.2)$$

we can represent (8.1), accurate to within terms of second order of smallness in  $H'$ , in the form

$$H_{\text{eff}} = \langle H_0 \rangle_e - \int_0^\beta d\lambda_1 \langle H'(\lambda_1) H' \rangle_e + \int_0^\beta d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \{ \langle H'(\lambda_1) H'(\lambda_2) H_0 \rangle_e - \langle H'(\lambda_1) H'(\lambda_2) \rangle_e \langle H_0 \rangle_e \}, \quad (8.3)$$

where  $\langle \dots \rangle_e = \text{Sp}_e (\exp[-\beta H_0] \dots) / \text{Sp} \exp[-\beta H_0]$  denotes averaging over the conduction-electron state only. In such an averaging the terms linear in  $H'$  drop out of  $H_{\text{eff}}$ . In expression (8.3) we have  $H'(\lambda) = \exp[\lambda H_0] H' \exp[-\lambda H_0]$ . These quantities can be readily calculated if account is taken of the following relations that hold for a Hamiltonian of the type (6.6)

$$a_{\mathbf{k}\sigma}(\lambda) = e^{-\lambda e_{\mathbf{k}\sigma}^\sigma} a_{\mathbf{k}\sigma}, \quad a_{\mathbf{k}\sigma}^+(\lambda) = e^{\lambda e_{\mathbf{k}\sigma}^\sigma} a_{\mathbf{k}\sigma}^+. \quad (8.4)$$

After averaging over the variables of the conduction electrons in (8.3) we obtain in lieu of (8.3), in explicit form,

$$H_{\text{eff}} = \frac{1}{N^2} \sum_{\mathbf{k}\mathbf{k}'} \sum_{nm} \frac{J^2(\mathbf{k}\mathbf{k}')}{\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}'}} e^{i(\mathbf{k}-\mathbf{k}')\mathbf{R}_{nm}} \{ n_{\mathbf{k}}^+(1-n_{\mathbf{k}'}) S_n^- S_m^+ + n_{\mathbf{k}}^-(1-n_{\mathbf{k}'}) S_n^+ S_m^- + [n_{\mathbf{k}}^+(1-n_{\mathbf{k}'}) + n_{\mathbf{k}}^-(1-n_{\mathbf{k}'})] S_n^z S_m^z \}, \quad (8.5)$$

where  $n_{\mathbf{k}}^\pm$  is the electron distribution function (7.6) with quasi-momentum  $\mathbf{k}$  and spin  $\sigma = \pm 1/2$ . In the absence of an external magnetic field  $n_{\mathbf{k}}^\pm = n_{\mathbf{k}}^- = n_{\mathbf{k}}$ , so that (8.5) can be written in the form

$$H_{\text{eff}} = - \sum_{nm} I(\mathbf{R}_{nm}) (S_n S_m), \quad (8.6)$$

where

$$I(\mathbf{R}_{nm}) = \frac{3}{8} \frac{N_s}{N^2} \frac{J_0^2}{\zeta} \sum_{\mathbf{q}} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{R}_{nm}}, \quad (8.7)$$

and the function  $f(\mathbf{q})$  is determined by expression (7.17). In deriving (8.6) and (8.7) we assumed the approximation  $J(\mathbf{k}\mathbf{k}') = J_0$ .

Expression (8.6) means that the exchange interaction between the *d* and *s* electrons causes an indirect exchange between the former. The value of the corresponding effective exchange integral is  $\sim J_0^2/\zeta$ . An essential feature of this interaction is its long-range character.\* Indeed, taking (7.20) into account, we can rewrite (8.7) in the form

$$I(\mathbf{R}_{nm}) = -\frac{9}{2} \pi \frac{J_0^2}{\zeta} \left( \frac{N_s}{N} \right)^2 F(2k_0 R_{nm}), \quad (8.8)$$

from which it is clear that  $I(\mathbf{R}_{nm})$  decreases as the third power of the distance between the paramagnetic ions. We note that the spatial distribution of the spin polarization of the conduction electrons near the paramagnetic sites (7.22) is described by the same distance function as the indirect exchange between the sites.

In view of the long-range character of the indirect coupling (8.7), we cannot confine ourselves in (8.6) to the nearest-neighbor approximation, as is done for the direct short-range exchange, since the possibility of ferromagnetic or antiferromagnetic ordering under the influence of the indirect exchange only calls for an additional investigation which we shall carry out in Sec. 9. However, the principal possibility of such an ordering is already clear. It is evident that the long-range character of the indirect exchange is particularly important for an explanation of the ferromag-

\*The first to advance the idea of indirect exchange between unclosed shells via their exchange interaction with the conduction electrons was Zener,<sup>[10]</sup> to whom the phenomenological theory of exchange interaction in metals is due (see Sec. 5). The calculation presented here, which is a variant of Kasuya's calculation,<sup>[2]</sup> is a microscopic development of Zener's idea, but it is more meaningful than Zener's theory, since it leads to a clear-cut proof of the long-range character of the indirect exchange.

netic and antiferromagnetic ordering in dilute alloys of transition elements, and also for rare-earth metals (see Sec. 3).

### 9. Spin-wave Theory of a Ferromagnetic Metal

The theory developed in Secs. 7–8 for a ferromagnetic metal has several shortcomings. First, it does not take into account the possibility of direct exchange between the “magnetic” electrons, something particularly important for d-metals; in addition, it is not fully consistent from the mathematical point of view: a perturbation theory is constructed only for the conduction-electron states, the operators of the d-electron spin are regarded as c-numbers. These shortcomings can be eliminated in a more rigorous theory, which, however, is suitable only for low temperatures near the magnetic-saturation states.

We consider first a ferromagnetic metal described by a Hamiltonian in which is included, in addition to the s-d exchange operator, also an operator that describes the direct exchange between the d-electrons, and we shall attempt to determine the spectrum of the elementary excitations. Inasmuch as we are unable to obtain the eigenvalues of the Hamiltonians expressed in terms of the spin operator, we change over, following Holstein and Primakoff [11], to other dynamic variables  $b_n$  and  $b_n^+$ :

$$\begin{aligned} S_n^z &= (2s)^{1/2} f_s(\hat{n}_n) b_n, & S_n^- &= (2s)^{1/2} b_n^+ f_s(\hat{n}_n), & S_n^+ &= s - \hat{n}_n, \\ \hat{n}_n &= b_n^+ b_n, & f_s(\hat{n}) &= \left(1 - \frac{\hat{n}}{2s}\right)^{1/2}. \end{aligned} \quad (9.1)$$

The operators  $b_n$  and  $b_n^+$  obey the commutation relations [12]

$$\left. \begin{aligned} b_n b_{n'} - b_{n'} b_n &= 0, \\ b_n b_{n'}^+ - b_{n'}^+ b_n &= 0, \quad n \neq n', \end{aligned} \right\} \quad (9.2)$$

$$b_n b_n^+ - b_n^+ b_n = 1 - \frac{2s+1}{(2s)!} b_n^{+2s} b_n^{2s}. \quad (9.2')$$

Relations (9.2) have a Bose character, while (9.2') is more complicated and depends essentially on the value of the spin  $s$ . In the particular case  $s = 1/2$ , (9.2) has the character of the Fermi commutation relation

$$b_n b_n^+ + b_n^+ b_n = 1 \quad (s = 1/2). \quad (9.2'')$$

Relations (9.2) and (9.2'') describe the Pauli operators. In the second limiting case  $s \rightarrow \infty$ , the last term of (9.2') can be neglected, so that (9.2') acquires a Bose character

$$b_n b_n^+ - b_n^+ b_n = 1 \quad (s = \infty). \quad (9.2''')$$

In the intermediate case the commutation relations are complicated, and it can be shown [12] that the eigenvalues of the operator  $\hat{n}_n = b_n^+ b_n$  are the integers  $0, 1, 2, \dots, 2s$ . The operator  $\hat{n}_n$  describes the deviation from the maximum spin projection at the

site  $n$ . In the ground state of the ferromagnet, all the spins of the d-shells have the same direction, that is, they have a maximum projection  $s$  on the quantization axis  $Oz$ . At small deviations of the system from the ground state, the number of sites in which the spin deviates from its maximum value is not large, so that the mean value of the operator  $n_n$  in the case of weak excitations is small:

$$\langle \hat{n}_n \rangle_{av} \ll 2s. \quad (9.3)$$

In this case  $f_s(\hat{n}) \approx 1$  and (9.2) becomes

$$S_n^+ = (2s)^{1/2} b_n, \quad S_n^- = (2s)^{1/2} b_n^+, \quad S_n^z = s - b_n^+ b_n. \quad (9.4)$$

With the same degree of accuracy we can neglect the last term in (9.2'). Thus, the operators  $b_n$  and  $b_n^+$  are approximately Bose operators if condition (9.3) is satisfied. This approximation is called the spin-wave approximation.

Let us consider the total Hamiltonian of the ferromagnet in this approximation. If we take the Fourier expansion of the operators  $b_n$  and  $b_n^+$ ,

$$b_n = \frac{1}{\sqrt{N}} \sum_{\mathbf{q}} e^{i\mathbf{q}\mathbf{R}_n} b_{\mathbf{q}}, \quad b_n^+ = \frac{1}{\sqrt{N}} \sum_{\mathbf{q}} e^{-i\mathbf{q}\mathbf{R}_n} b_{\mathbf{q}}^+,$$

then the Hamiltonian can be represented in the form

$$H = H_0 + H', \quad (9.5)$$

$$H_0 = \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}}^{\sigma} a_{\mathbf{k}\sigma}^{\dagger} a_{\mathbf{k}\sigma} + \sum_{\mathbf{q}} \varepsilon_{\mathbf{q}} b_{\mathbf{q}}^{\dagger} b_{\mathbf{q}}, \quad (9.6)$$

$$\begin{aligned} H' = & - \left(\frac{2s}{N}\right)^{1/2} \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}} J(\mathbf{k}\mathbf{k}') \delta(\mathbf{k}' - \mathbf{k} + \mathbf{q}) (a_{\mathbf{k}'}^{\dagger} a_{\mathbf{k}}^{\dagger} b_{\mathbf{q}}) \\ & + a_{\mathbf{k}'(-)}^{\dagger} a_{\mathbf{k}(+)} b_{\mathbf{q}}^{\dagger} + \frac{1}{N} \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}\mathbf{q}'} J(\mathbf{k}\mathbf{k}') \delta(\mathbf{k}' - \mathbf{k} + \mathbf{q}' - \mathbf{q}) \\ & \times (a_{\mathbf{k}'(-)}^{\dagger} a_{\mathbf{k}'(-)} - a_{\mathbf{k}(+)}^{\dagger} a_{\mathbf{k}(+)}) b_{\mathbf{q}}^{\dagger} b_{\mathbf{q}'}, \end{aligned} \quad (9.7)$$

where

$$\varepsilon_{\mathbf{k}}^{\sigma} = E_{\mathbf{k}} - 2sJ(\mathbf{k}\mathbf{k}) \delta_{\sigma, -1/2} + \sigma \hbar \omega_s, \quad (9.8)$$

$$\varepsilon_{\mathbf{q}} = Iq^2 + \hbar \omega_d \quad (9.9)$$

are respectively the energies of the s-electrons with spin  $\sigma = \pm 1/2$  and of the spin waves with quasi momentum  $\mathbf{q}$ , arising in the d-electron system as a result of the direct exchange;  $I$  is the integral of the direct exchange for the nearest-neighbor sites,  $\hbar \omega_s \equiv g_s \mu_0 H$ ,  $\hbar \omega_d \equiv g_d \mu_0 H$ , and  $g_s$ ,  $g_d$  are respectively the Zeeman energies and the Lande factors of the s and d electrons.

The part of the Hamiltonian  $H$  which is diagonal in the occupation numbers represents the energy of the elementary excitations of the spin-wave system and the conduction electrons, the latter turning out to be magnetized by the exchange interaction with the d electrons, which are magnetized to saturation. Thus, the spectrum of the elementary excitations of the electron system of a ferromagnetic metal contains two branches, a Fermi branch (conduction electrons) and a Bose branch (spin waves).

The operator  $H'$  in (9.7) describes the interaction of these two types of excitations. Consequently, the coupling between the s and d electrons leads in this approximation to two effects: 1) change in the energy of the s electrons [the term with  $J(\mathbf{k}\mathbf{k})$  in (9.8)], and 2) their interaction with the spin waves, causing transitions between the zero energy levels of the system. The operator  $H'$  contains two types of terms: "ternary" and "quaternary," depending on the number of creation and annihilation operators. The quaternary terms describe the elastic scattering of the spin waves by the conduction electrons, while the ternary ones describe inelastic scattering processes, in which spin waves can be created and annihilated. In these processes, however, the total spin of the colliding particles is conserved, so that the operator  $H'$  does not change the summary spin of the electron system of the metal. The delta-functions under the summation sign in (9.7) take into account the conservation of the quasi momentum upon collision of the elementary excitations.

Using ordinary perturbation theory it would be possible to find the corrections to the elementary-excitation energies, brought about by the interaction described by the operator  $H'$ . On the other hand, as already mentioned, the perturbation (9.7) gives rise to transitions between the zero energy levels, that is, leads to the damping of the elementary excitations, the lifetimes of which can be determined within the framework of perturbation theory with the aid of the kinetic-equation technique<sup>[13]</sup>. However, we shall not solve these two problems separately, and approach the damping problem from more general positions.

For different applications it is not essential to know the energy spectrum of the system, and it is sufficient to be able to calculate some correlation functions and distribution functions, which are statistical means of the dynamic variables of the system. We therefore consider the statistical properties of elementary excitations of the electron system of a ferromagnetic metal.

We first calculate the equilibrium distribution functions for the conduction electrons and the spin waves, defined as the statistical means of the corresponding occupation numbers:

$$n_{\mathbf{k}}^{\sigma} = \langle a_{\mathbf{k}\sigma}^{\dagger} a_{\mathbf{k}\sigma} \rangle, \quad n_{\mathbf{q}} = \langle b_{\mathbf{q}}^{\dagger} b_{\mathbf{q}} \rangle. \quad (9.10)$$

Here and throughout the symbol  $\langle \dots \rangle$  denotes averaging over the Gibbs ensemble with the total Hamiltonian  $H'$ , namely

$$\langle \dots \rangle = \text{Sp}(e^{-\beta H} \dots) / \text{Sp} e^{-\beta H}.$$

To calculate the functions (9.10) it is convenient to use the statistical Green's function method\* developed

\*A detailed exposition of the Green's function method in quantum statistics can be found in the reviews.<sup>[15]</sup>

by Bogolyubov and Tyablikov<sup>[14]</sup>. We note first that the distribution function (9.10) represents especially the correlation functions when the arguments coincide,  $t = t'$ . In the general case, to calculate such functions

$$\langle A(t) B(t') \rangle, \quad \langle B(t') A(t) \rangle, \quad (9.11)$$

where

$$A(t) = e^{\frac{i}{\hbar} H t} A e^{-\frac{i}{\hbar} H t} \quad (9.12)$$

are arbitrary operators, taken in the Heisenberg approximation, Bogolyubov and Tyablikov proposed to consider retarded and advanced Green's function

$$\langle\langle A(t) | B(t') \rangle\rangle_{\text{ret}} \text{ and } \langle\langle A(t) | B(t') \rangle\rangle_{\text{adv}},$$

defined by the relations

$$\begin{aligned} \langle\langle A(t) | B(t') \rangle\rangle_{\text{ret}} &\equiv \theta(t-t') \langle [A(t), B(t')]_{\eta} \rangle, \\ \langle\langle A(t) | B(t') \rangle\rangle_{\text{adv}} &\equiv -\theta(t-t') \langle [A(t), B(t')]_{\eta} \rangle, \end{aligned} \quad (9.13)$$

in which the following equation is introduced

$$\theta(t-t') = \begin{cases} 1, & t > t', \\ 0, & t < t', \end{cases} \quad [A, B]_{\eta} = AB - \eta BA \quad (\eta = \pm 1). \quad (9.14)$$

Both functions satisfy one and the same equation

$$\begin{aligned} i\hbar \frac{d}{dt} \langle\langle A(t) | B(t') \rangle\rangle &= i\hbar \delta(t-t') \langle [A(t), B(t')]_{\eta} \rangle \\ &+ \langle\langle i\hbar \frac{d}{dt} A(t) | B(t') \rangle\rangle, \end{aligned} \quad (9.15)$$

which contains the more complicated Green's functions  $\langle\langle i\hbar \frac{d}{dt} A(t) | B(t') \rangle\rangle$ . Setting up an equation for these functions, we can verify that it gives rise to even more complicated Green's functions. The chain of equations obtained in this manner should be terminated with the aid of some sort of approximation.

The Green's function (9.13) enables us to find the correlation functions (9.11) with the aid of the spectral representation

$$\langle B(t') A(t) \rangle = \int_{-\infty}^{+\infty} J(E) e^{-\frac{i}{\hbar} E(t-t')} dE, \quad (9.16)$$

$$\langle A(t) B(t') \rangle = \int_{-\infty}^{+\infty} J(E) e^{\beta E} e^{-\frac{i}{\hbar} E(t-t')} dE. \quad (9.17)$$

The spectral density  $J(E)$  is determined by the Fourier components  $\langle\langle A | B \rangle\rangle_E$  of the functions (9.13) on the basis of the limiting relation (spectral theorem)

$$\langle\langle A | B \rangle\rangle_{E+i\epsilon} - \langle\langle A | B \rangle\rangle_{E-i\epsilon} = (e^{\beta E} - \eta) J(E), \quad \epsilon \rightarrow 0. \quad (9.18)$$

With an aim toward calculating the distribution functions (9.10) for the quasi particles of a ferromagnetic metal, we introduce into consideration the boson and fermion Green's functions corresponding to them

$$\langle\langle b_{\mathbf{q}}(t) | b_{\mathbf{q}}^{\dagger}(t') \rangle\rangle, \quad \langle\langle a_{\mathbf{k}\sigma}(t) | a_{\mathbf{k}\sigma}^{\dagger}(t') \rangle\rangle, \quad (9.19)$$

with the former preferably used when  $\eta = 1$  and the latter when  $\eta = -1$ . It is easy to write for these functions the equations (9.15), using the Hamiltonian (9.5).

These equations will include, in addition to (9.19), also mixed Green's functions such as

$$\langle\langle a_{\mathbf{k}(+)a_{\mathbf{k}-\mathbf{q}}(-)}^{\pm} | b_{\mathbf{q}}^{\pm} \rangle\rangle, \quad (9.20)$$

$$\langle\langle b_{\mathbf{q}}^{\pm} a_{\mathbf{k}(+)a_{\mathbf{k}-\mathbf{q}}(-)}^{\pm} | b_{\mathbf{q}}^{\pm} \rangle\rangle, \quad \langle\langle a_{\mathbf{k}\sigma}^{\pm} a_{\mathbf{k}\sigma} b_{\mathbf{q}} | b_{\mathbf{q}}^{\pm} \rangle\rangle. \quad (9.21)$$

Let us terminate the resultant chain of equations, by expressing the complex Green's functions (9.21) in terms of the simpler ones (9.19) and (9.20) with the aid of the approximations

$$\langle\langle b_{\mathbf{q}}^{\pm} a_{\mathbf{k}(+)a_{\mathbf{k}-\mathbf{q}}(-)}^{\pm} | b_{\mathbf{q}}^{\pm} \rangle\rangle \cong n_{\mathbf{q}} \langle\langle a_{\mathbf{k}(+)a_{\mathbf{k}-\mathbf{q}}(-)}^{\pm} | b_{\mathbf{q}}^{\pm} \rangle\rangle,$$

$$\langle\langle a_{\mathbf{k}\sigma}^{\pm} a_{\mathbf{k}\sigma} b_{\mathbf{q}} | b_{\mathbf{q}}^{\pm} \rangle\rangle \cong n_{\mathbf{q}}^{\sigma} \langle\langle b_{\mathbf{q}} | b_{\mathbf{q}}^{\pm} \rangle\rangle. \quad (9.22)$$

The quantities  $n_{\mathbf{q}}$  and  $n_{\mathbf{k}}^{\sigma}$ , defined in accord with (9.10), must be regarded for the time being as unknown distribution functions. The meaning of the approximation (9.22) becomes clear if the definition (9.13) is used for the Green's functions. As will be shown below, it corresponds to second order perturbation theory.

With the aid of (9.22), the chain of equations for the Green's functions turns into a closed system of three equations\*:

$$i\hbar \frac{d}{dt} \langle\langle b_{\mathbf{q}} | b_{\mathbf{q}}^{\pm} \rangle\rangle = i\hbar \delta(t-t') + \left\{ \varepsilon_{\mathbf{q}} + \frac{1}{N} \sum_{\mathbf{k}} J(\mathbf{k}\mathbf{k}) (n_{\mathbf{k}}^{-} - n_{\mathbf{k}}^{+}) \right\} \langle\langle b_{\mathbf{q}} | b_{\mathbf{q}}^{\pm} \rangle\rangle + \left( \frac{2s}{N} \right)^{1/2} \sum_{\mathbf{k}} J(\mathbf{k}, \mathbf{k}-\mathbf{q}) \langle\langle a_{\mathbf{k}(+)a_{\mathbf{k}-\mathbf{q}}(-)}^{\pm} | b_{\mathbf{q}}^{\pm} \rangle\rangle, \quad (9.23)$$

$$i\hbar \frac{d}{dt} \langle\langle a_{\mathbf{k}(+)a_{\mathbf{k}-\mathbf{q}}(-)}^{\pm} | b_{\mathbf{q}}^{\pm} \rangle\rangle = (\tilde{\varepsilon}_{\mathbf{k}}^{+} - \tilde{\varepsilon}_{\mathbf{k}-\mathbf{q}}^{-}) \langle\langle a_{\mathbf{k}(+)a_{\mathbf{k}-\mathbf{q}}(-)}^{\pm} | b_{\mathbf{q}}^{\pm} \rangle\rangle + \left( \frac{2s}{N} \right)^{1/2} J(\mathbf{k}, \mathbf{k}-\mathbf{q}) (n_{\mathbf{k}-\mathbf{q}}^{-} - n_{\mathbf{k}}^{+}) \langle\langle b_{\mathbf{q}} | b_{\mathbf{q}}^{\pm} \rangle\rangle. \quad (9.24)$$

$$i\hbar \frac{d}{dt} \langle\langle a_{\mathbf{k}(\pm)} | a_{\mathbf{k}(\pm)}^{\pm} \rangle\rangle = i\hbar \delta(t-t') + (\tilde{\varepsilon}_{\mathbf{k}}^{\pm} - \zeta) \langle\langle a_{\mathbf{k}(\pm)} | a_{\mathbf{k}(\pm)}^{\pm} \rangle\rangle, \quad (9.25)$$

where

$$\tilde{\varepsilon}_{\mathbf{k}}^{\pm} = \varepsilon_{\mathbf{k}} - s(1 \mp \mu) J(\mathbf{k}\mathbf{k}), \quad (9.26)$$

and  $\mu$  is the relative magnetization of the d electrons:

$$\mu = 1 - \frac{1}{SN} \sum_{\mathbf{q}} n_{\mathbf{q}}. \quad (9.27)$$

We obtain the solution of the system of equations with the aid of the Fourier transformation. For the Fourier transforms of the Green's functions we obtain

$$\langle\langle a_{\mathbf{k}(\pm)} | a_{\mathbf{k}(\pm)}^{\pm} \rangle\rangle_E = \frac{i\hbar}{2\pi} \frac{1}{E - \tilde{\varepsilon}_{\mathbf{k}}^{\pm} + \zeta}, \quad (9.28)$$

$$\langle\langle b_{\mathbf{q}} | b_{\mathbf{q}}^{\pm} \rangle\rangle_E = \frac{i\hbar}{2\pi} \frac{1}{E - \varepsilon_{\mathbf{q}} - \Delta - P_{\mathbf{q}}(E)}, \quad (9.29)$$

where

$$\Delta = \frac{1}{N} \sum_{\mathbf{k}} J(\mathbf{k}\mathbf{k}) (n_{\mathbf{k}}^{-} - n_{\mathbf{k}}^{+}), \quad (9.30)$$

$$P_{\mathbf{q}}(E) = \frac{2s}{N} \sum_{\mathbf{k}} \frac{J^2(\mathbf{k}, \mathbf{k}-\mathbf{q}) (n_{\mathbf{k}-\mathbf{q}}^{-} - n_{\mathbf{k}}^{+})}{E - \tilde{\varepsilon}_{\mathbf{k}}^{+} + \tilde{\varepsilon}_{\mathbf{k}-\mathbf{q}}^{-}}, \quad (9.31)$$

With the aid of the symbolic identity

$$\frac{1}{x \pm i\varepsilon} = \frac{\mathcal{P}}{x} \mp i\lambda\delta(x), \quad (9.32)$$

where  $\mathcal{P}$  is the principal-value symbol, we readily can show that

$$P_{\mathbf{q}}(E \pm i\varepsilon) = P_{\mathbf{q}}(E) \mp i\gamma_{\mathbf{q}}(E), \quad (9.33)$$

with

$$\gamma_{\mathbf{q}}(E) = \pi \frac{2s}{N} \sum_{\mathbf{k}} J^2(\mathbf{k}, \mathbf{k}-\mathbf{q}) (n_{\mathbf{k}-\mathbf{q}}^{-} - n_{\mathbf{k}}^{+}) \delta(E - \tilde{\varepsilon}_{\mathbf{k}}^{+} + \tilde{\varepsilon}_{\mathbf{k}-\mathbf{q}}^{-}). \quad (9.34)$$

On the basis of (9.16) with  $t = t'$ , using the spectral theorem (9.18) and also relations (9.32)–(9.34), we obtain the distribution functions (9.10):

$$n_{\mathbf{k}}^{\pm} = \frac{1}{e^{\beta(\tilde{\varepsilon}_{\mathbf{k}}^{\pm} - \zeta)} + 1}, \quad (9.35)$$

$$n_{\mathbf{q}} = \int_{-\infty}^{+\infty} \frac{dE}{e^{\beta E} - 1} \frac{1}{\pi} \frac{\gamma_{\mathbf{q}}(E)}{(E - \varepsilon_{\mathbf{q}} - \Delta - P_{\mathbf{q}}(E))^2 + \gamma_{\mathbf{q}}^2(E)}. \quad (9.36)$$

In the approximation considered, (9.35) is the Fermi distribution function for the s electrons, the energy of which, (9.26), is modified by the s-d interaction. The latter corresponds fully to formula (7.4), which is obtained with the aid of perturbation theory.

Expression (9.36) is a superposition of elementary Bose excitations with resonant intensity and damping  $\gamma_{\mathbf{q}}(E)$ . In the case of small damping, the resonant function under the integral sign in (9.36) has a sharp peak at the point  $E = \tilde{\varepsilon}_{\mathbf{q}}$ , determined from the equation

$$\tilde{\varepsilon}_{\mathbf{q}} - \varepsilon_{\mathbf{q}} - P_{\mathbf{q}}(\tilde{\varepsilon}_{\mathbf{q}}) - \Delta = 0, \quad (9.37)$$

and can be replaced approximately by the function  $\delta(E - \tilde{\varepsilon}_{\mathbf{q}})$ . Thus, neglecting damping we obtain in place of (9.36) the ordinary Bose distribution function

$$n_{\mathbf{q}} = \frac{1}{e^{\beta \varepsilon_{\mathbf{q}}} - 1}. \quad (9.38)$$

Consequently, the quantities  $\tilde{\varepsilon}_{\mathbf{q}}$  and  $\tilde{\varepsilon}_{\mathbf{k}}^{\pm}$  have the meanings of elementary-excitation energies at a specified temperature. In the approximation employed here for the Fermi Green's function, the Fermi excitations are not damped, and damping appears when higher-order Green's functions are taken into consideration. The energy and the damping of the Bose excitations will be considered below.

The distribution functions obtained enable us to calculate the equilibrium thermodynamic quantities of the metal, for example the magnetization. The operator of the magnetic moment in the direction of the Oz axis can be written in the form

\*The parameter  $\zeta$  can be formally introduced by replacing in the zeroth approximation Hamiltonian (9.6)  $\varepsilon_{\mathbf{k}}^{\pm}$  by  $\varepsilon_{\mathbf{k}}^{\pm} - \zeta$  or by means of an equivalent averaging in (9.11) over the Gibbs grand ensemble.



$$\begin{aligned} \hat{M}_z = g\mu_0 s N \left( 1 - \frac{1}{sN} \sum_{\mathbf{q}} b_{\mathbf{q}}^+ b_{\mathbf{q}} \right) \\ + \frac{1}{2} g\mu_0 \sum_{\mathbf{k}} (a_{\mathbf{k}(-)}^+ a_{\mathbf{k}(-)} - a_{\mathbf{k}(+)}^+ a_{\mathbf{k}(+)}). \end{aligned} \quad (9.39)$$

Averaging (9.39) over the statistical ensemble, we express the magnetization in terms of the distribution functions

$$M_z = \langle \hat{M}_z \rangle = g\mu_0 s N \left\{ 1 - \frac{1}{N_s} \sum_{\mathbf{q}} n_{\mathbf{q}} \right\} + \frac{1}{2} g\mu_0 \sum_{\mathbf{k}} (n_{\mathbf{k}}^- - n_{\mathbf{k}}^+). \quad (9.40)$$

The first term yields the magnetization of the d-electrons of a ferromagnet of the Heisenberg type in the spin-wave approximation. The second term yields the contribution to the magnetization due to the conduction electrons. It coincides with the previously obtained expression (7.9). Calculating the last sum in (9.40) and using (9.27), we can write the magnetization in the form

$$M_z = \left( 1 + \frac{3}{4} \frac{N_s}{N} \frac{J_0}{\zeta} \right) M_d, \quad (9.41)$$

where  $M_d = g\mu_0 s N \mu$  is the d-electron magnetization. We shall show below that, with account of s-d exchange, the dispersion law for the spin waves remains quadratic, so that we have the ordinary  $T^{3/2}$  temperature law for the magnetization of a ferromagnetic metal.

Let us determine now the explicit form of  $\tilde{\epsilon}_{\mathbf{q}}$ ; for this purpose we write (9.37) in the expanded form

$$\begin{aligned} \tilde{\epsilon}_{\mathbf{q}} = \epsilon_{\mathbf{q}} + \frac{1}{N} \sum_{\mathbf{k}} J(\mathbf{k}\mathbf{k}) (n_{\mathbf{k}}^- - n_{\mathbf{k}}^+) \\ + \frac{2s}{N} \sum_{\mathbf{k}} \frac{J^2(\mathbf{k}, \mathbf{k}-\mathbf{q}) (n_{\mathbf{k}-\mathbf{q}}^- - n_{\mathbf{k}}^+)}{\epsilon_{\mathbf{q}} - \tilde{\epsilon}_{\mathbf{k}}^+ + \tilde{\epsilon}_{\mathbf{k}-\mathbf{q}}^-}. \end{aligned} \quad (9.42)$$

We first find the solution of (9.42) for  $\mathbf{q} = 0$ . Taking it into account that according to (9.9) and (9.26) we have

$$\epsilon_0 = \hbar\omega_d, \quad \tilde{\epsilon}_{\mathbf{k}}^+ - \tilde{\epsilon}_{\mathbf{k}}^- = \hbar\omega_s + 2sJ(\mathbf{k}\mathbf{k})\mu,$$

and taking also the approximation  $J(\mathbf{k}\mathbf{k}) = J_0$  into account, we can reduce (9.42) to a quadratic equation, with two solutions

$$\begin{aligned} \tilde{\epsilon}_0 = \frac{1}{2} \{ \hbar(\omega_s + \omega_d) + 2sJ_0(1 + \mu_s) \pm [ \hbar(\omega_s + \omega_d) + 2sJ_0(1 + \mu_s) ]^2 \\ - 4[\hbar^2\omega_s\omega_d + 2sJ_0\hbar(\omega_s + \omega_d)]^{1/2} \}, \end{aligned} \quad (9.43)$$

where  $\mu_s$  is the relative magnetization of the conduction electrons

$$\mu_s = \frac{1}{2sN} \sum_{\mathbf{k}} (n_{\mathbf{k}}^- - n_{\mathbf{k}}^+). \quad (9.44)$$

In the particular case when  $g_s = g_d$  and  $\omega_s = \omega_d = \omega_0$ , we therefore obtain from (9.42)

$$\epsilon_0^{(1)} = \hbar\omega_0, \quad \tilde{\epsilon}_0^{(2)} = \hbar\omega_0 + 2sJ_0(1 + \mu_s). \quad (9.45)$$

In the absence of a magnetic field,  $\omega_0 = 0$  and  $\tilde{\epsilon}_0^{(1)}$

= 0. Thus, the ferromagnetic metal has one branch of spin waves which are not separated from the ground state by a gap. From the mathematical point of view this result is quite interesting. Expression (9.42) is the first-approximation correction due to the quaternary terms of the perturbation operator (9.7). Its ternary terms yield zero in first approximation, since they do not contain diagonal matrix elements. The third term in (9.42) is the second-approximation correction due to the ternary terms. For the case  $\mathbf{q} = 0$  it is equal to the first-approximation correction with the sign reversed. Thus, the first-approximation correction in the s-d interaction, which leads to the appearance of a gap in the spin-wave spectrum, is compensated by the second-approximation correction. This is the consequence of the special structure of the perturbation operator as an exchange operator. This circumstance was noted already in [16].

The occurrence of a second spin-wave branch is connected with the magnetization of the conduction electrons by the s-d exchange interaction, as a result of which the latter have their own spontaneous magnetic moments, so that the metal as a whole becomes equivalent to a ferrite with two magnetic sublattices. A ferrite is characterized by the presence in the excitation spectrum of a high-energy branch with energy on the order of the exchange interaction of the different sublattices.

Let us find now the solutions of (9.42) for  $\mathbf{q} \neq 0$ . Inasmuch as  $\epsilon_{\mathbf{q}}$  is an even function of  $\mathbf{q}$ ,  $\tilde{\epsilon}_{\mathbf{q}}$  is also an even function of  $\mathbf{q}$ , so that the solution of (9.42), which branches away from  $\tilde{\epsilon}_0^{(1)} = \hbar\omega_0$ , will be sought in the form

$$\tilde{\epsilon}_{\mathbf{q}} = \hbar\omega_0 + I^*q^2, \quad (9.46)$$

where  $I^*$  is the effective exchange-interaction parameter.

In the case of a quadratic dispersion law for the s electrons ( $\epsilon_{\mathbf{k}} = Ak^2$ ), neglecting the difference in the g-factors, we obtain for the sums in (9.42), after changing over in them from summation to integration,

$$\Delta = \frac{3sJ_0^2}{2\zeta} \frac{N_s}{N} + \frac{3\mu_0 J_0 H}{2\zeta} \frac{N_s}{N}, \quad (9.47)$$

$$P_{\mathbf{q}}(E) = \frac{sJ_0^2}{4\pi^2 A q} \{ \psi_-(C_-, q) - \psi_+(C_+, q) \}, \quad (9.48)$$

where

$$\psi_{\pm}(C_{\pm}, q) = \frac{C_{\pm}}{2Ag} + \frac{1}{2} \left( k_{\pm}^2 - \frac{C_{\pm}^2}{4A^2q^2} \right) \ln \left| \frac{C_{\pm} + 2Ak_{\pm}}{C_{\pm} - 2Ak_{\pm}} \right|, \quad (9.49)$$

with

$$C_{\pm} = E - \tilde{\epsilon}_{\mathbf{k}}^+ + \tilde{\epsilon}_{\mathbf{k}}^- \pm Aq^2. \quad (9.50)$$

Expanding (9.48) in powers of  $q^2$ , using (7.8) for  $\zeta_0$ , and putting  $I \ll \zeta$ , we obtain

$$P_{\mathbf{q}}(\tilde{\epsilon}_{\mathbf{q}}) = -\Delta + \frac{s \left( 3\pi^2 \frac{N_s}{N} \right)^{1/3}}{24\pi^2} \frac{J_0^2}{\zeta} q^2 + \dots \quad (9.51)$$

Substituting (9.51) in (9.42), we see that the quantities  $\Delta$  cancel each other and we obtain for the effective exchange integral  $I^*$  in (9.46)

$$I^* = I + \frac{s \left( 3\pi^3 \frac{N_s}{N} \right)^{1/3}}{24\pi^3} \frac{J_0^2}{\zeta}. \quad (9.52)$$

Such an expression was obtained in [16-18]. The second term in (9.52) represents the indirect exchange integral.\* Inasmuch as the indirect exchange integral is always positive, it is not necessary for the existence of ferromagnetism that the direct-exchange integral be of necessity positive. Ferromagnetism can exist also when  $I = 0$ , which apparently occurs in the case of crystals such as of dilute alloys of manganese and copper, and also when  $I < 0$ . A criterion of ferromagnetism is now the more general requirement  $I^* > 0$ .

Let us calculate now the values of the damping  $\gamma_{\mathbf{q}}$  for spin waves with momentum  $\mathbf{q}$ . Changing over in (9.34) from summation to integration, we obtain

$$\gamma_{\mathbf{q}}(\tilde{\epsilon}_{\mathbf{q}}) \approx \tilde{\epsilon}_{\mathbf{q}} \frac{3\pi s}{8} \frac{N_s}{V} \left( \frac{J_0}{\zeta} \right)^2 \frac{k_0}{q} \begin{cases} 1, & q > q_0, \\ 0, & q < q_0, \end{cases} \quad (9.54)$$

where

$$q_0 = \frac{1}{2} \frac{|\epsilon_{\mathbf{q}} - 2sJ_0|}{\zeta} k_0, \quad (9.55)$$

$k_0$  is a certain "average" ( $\approx k_0^- \approx k_0^+$ ) Fermi momentum. For spin waves with energy  $\epsilon_{\mathbf{q}} \ll 2sJ_0$  we have  $q_0 \approx sI_0 k_0 / \zeta$ .

\*It can also be obtained from the expression (8.6), if we go over from the spin operator to the Bose operator and diagonalize the resultant quadratic form. We then obtain for the energy of the spin wave the expression

$$\epsilon_{\mathbf{q}} = 2s \sum_n I(\mathbf{R}_n) (1 - e^{i\mathbf{q}\mathbf{R}_n}), \quad (9.53)$$

where the summation is over all the signs. Taking formula (8.7) for  $I(\mathbf{R}_n)$  into account, we obtain

$$\epsilon_{\mathbf{q}} = \frac{3}{4} \frac{N_s s J_0^2}{N \zeta} [f(0) - f(q)], \quad (9.53')$$

where  $f(\mathbf{q})$  is determined by (7.17). Inasmuch as we have  $f(\mathbf{q}) \approx 2 - q^2/6k_0^2$  when  $q$  is small, we get  $\epsilon_{\mathbf{q}} = I^*(aq)^2$ , where  $I^*$  is exactly equal to the second term in (9.52).

The character of the dispersion law of the spin waves is determined by the behavior of the function  $f(\mathbf{q})$  in (9.53'). In particular, Woll and Nettel[84] noticed recently that the derivative of  $f(\mathbf{q})$  has a logarithmic divergence at the point  $q = 2k_0$ . If we recognize that in place of the expression (9.53') it is more correct to write

$$\epsilon_{\mathbf{q}} = \frac{3}{4} \frac{N_s s J_0^2}{N \zeta} \sum_{\boldsymbol{\tau}} [f(\boldsymbol{\tau}) - f(\boldsymbol{\tau} + \mathbf{q})], \quad (9.53'')$$

where  $\boldsymbol{\tau}$  is the reciprocal lattice vector, then this fact causes the function  $\epsilon_{\mathbf{q}}$  to become sharply non-monotonic at values of  $q$  satisfying the condition  $|\boldsymbol{\tau} + \mathbf{q}| = 2k_0$ . Numerical calculations carried out in [84] have shown that the magnitude of such "breaks" on the  $\epsilon_{\mathbf{q}}$  curve amounts to several per cent. Although this effect is difficult to observe, it is very interesting since it demonstrates the influence of the form of the Fermi surface (which is spherical in this case) on the spectrum of the spin waves.

It follows thus, from (9.54) that spin waves with small quasi-momenta are not damped by the s-d interaction. This is connected with the fact that at small quasi-momenta the energy of the spin wave cannot satisfy the energy conservation law in collisions between the wave and the conduction electron. In such a collision the electron spin should flip, and for this purpose it is necessary to change its energy by an amount equal to the magnetization energy, that is, by an amount  $\sim 2sJ$ . As soon as the energy of the spin wave becomes sufficient for this purpose, it will become damped. The damping energy has an order of magnitude  $\sim J_0 \tilde{\epsilon}_{\mathbf{q}} / \zeta$ , that is, it is small compared with the self-energy. This justifies the applicability of perturbation theory.\*

In conclusion we note that the Green's function method makes it possible to separate more consistently† the elementary excitations in the electron system of a ferromagnetic metal, to determine their damping, and to determine the influence of the spin system on the conduction electrons (the magnetization effect) and the reaction of the conduction electrons on the spin system (the effective indirect exchange).‡

\*It must be noted that expression (9.34) for  $\gamma_{\mathbf{q}}$ , obtained with the aid of the Green's function method, coincides with that previously obtained by Turov,[19] and also by Mitchell[5] and Abrahams[3] with the aid of the kinetic-equation technique. However, the last two authors did not take into account the magnetization of the conduction electrons, so that their result coincides formally with (9.54), where one must put  $q_0 = 0$ . For spin waves with  $q_0$  their results are incorrect.

†In particular, it is easy to see that no g-factor shift due to the s-d exchange appears for the d electrons, in spite of the deduction of Yosida[6] and of [20,21] according to which for a ferromagnetic metal  $\tilde{g} = g \left( 1 + \frac{3}{4} \frac{N_s J_0}{N \zeta} \right)$ . This occurs because the corrections in (9.42), which depend on the magnetic field, cancel each other. If we neglect the dependence of  $P_{\mathbf{q}}(E)$  on the magnetic field, then the term remaining in (9.47) leads precisely to the g-factor shift indicated above. Thus, the fictitiousness of this result is due to the fact that in the second-approximation correction to the spin-wave energy the term that depends on the magnetic field was neglected without justification. No "shift" was likewise observed in experiment.

We note that the qualitative agreement obtained in [85] between the experimental values of the g-factors for several rare-earth metals and those calculated from the above formula for  $\tilde{g}$  is illusory, inasmuch as it is assumed in this case that the sign of the s-d exchange integral for rare-earth metals (in particular for Gd) is opposite the sign that is obtained from other experiments (see, for example, [86]), whereas the values are close to each other in absolute magnitude, with  $J_0 \sim 2 \times 10^{-13}$  erg for Gd[86,78]. The deviation of the values of the g-factor for these metals from two should apparently be attributed to other factors.

‡The method of retarded and advanced Green's function was recently used in the theory of ferromagnetic metals also by Potapkov and Tyablikov,[22] who obtained results which essentially coincided with those given above. However, for the spin-wave spectrum they obtained a gap proportional to the third power of the s-d exchange integral. This result seems inaccurate to us. To obtain the corrections to the energy of the spin wave in the

On the basis of the results obtained here, we consider in Secs. 11 and 12 ferromagnetic resonance and the scattering of slow neutrons in metals.

### 10. Spin-wave Theory of an Antiferromagnetic Metal

Let us consider now a metal with antiferromagnetic spin ordering. We assume that it is possible to separate in it two collinear magnetic sublattices. This very simple model describes satisfactorily the properties of an antiferromagnet. An important role is played for the antiferromagnetic state by magnetic anisotropy<sup>[23]</sup>. It is therefore necessary to take into account along with the exchange also the anisotropic interaction between the atomic spins, which in the simplest case of an anisotropic crystal which is uniaxial and has a symmetry axis Oz leads to an additional term in the spin Hamiltonian of the type (9.5)

$$-K \left\{ \sum_n (s_n^z)^2 + \sum_m (s_m^z)^2 \right\},$$

where K is the anisotropy constant, while n and m are the numbers of the sites of the different sublattices. Then, in the spin-wave approximation, the Hamiltonian of the antiferromagnetic metal has the form<sup>[24-26]</sup>

$$\begin{aligned} H = & \sum_{\mathbf{k}\sigma} [E_{\mathbf{k}} + sJ(\mathbf{k}\mathbf{k}) + \sigma h\omega_s] a_{\mathbf{k}\sigma}^{\dagger} a_{\mathbf{k}\sigma} + \sum_{\mathbf{q}} [\varepsilon_{1\mathbf{q}} \xi_{1\mathbf{q}}^{\dagger} \xi_{1\mathbf{q}} + \varepsilon_{2\mathbf{q}} \xi_{2\mathbf{q}}^{\dagger} \xi_{2\mathbf{q}}] \\ & + \left( \frac{2s}{N} \right)^{1/2} \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}} J(\mathbf{k}\mathbf{k}') \delta(\mathbf{k} - \mathbf{k}' + \mathbf{q}) g(\mathbf{q}) \\ & \times \{ a_{\mathbf{k}^{(+)}\sigma}^{\dagger} a_{\mathbf{k}'^{(-)}\sigma} (\xi_{1\mathbf{q}} + \xi_{2\mathbf{q}}^{\dagger}) + a_{\mathbf{k}^{(-)}\sigma}^{\dagger} a_{\mathbf{k}'^{(+)}\sigma} (\xi_{1\mathbf{q}}^{\dagger} + \xi_{2\mathbf{q}}) \} \\ & + \frac{1}{N} \sum_{\mathbf{k}\mathbf{q}} J(\mathbf{k}\mathbf{k}) (a_{\mathbf{k}^{(-)}\sigma}^{\dagger} a_{\mathbf{k}^{(-)}\sigma} - a_{\mathbf{k}^{(+)}\sigma}^{\dagger} a_{\mathbf{k}^{(+)}\sigma}) \\ & \times (\xi_{1\mathbf{q}}^{\dagger} \xi_{1\mathbf{q}} - \xi_{2\mathbf{q}}^{\dagger} \xi_{2\mathbf{q}}), \end{aligned} \quad (10.1)$$

where

$$\begin{aligned} \varepsilon_{1\mathbf{q}} = & szI \sqrt{(1+\alpha)^2 - \Gamma_{\mathbf{q}}^2} + h\omega_s, \\ \varepsilon_{2\mathbf{q}} = & szI \sqrt{(1+\alpha)^2 - \Gamma_{\mathbf{q}}^2} - h\omega_s, \end{aligned} \quad (10.2)$$

$$\alpha = \frac{2K}{zI}, \quad \Gamma_{\mathbf{q}} = \frac{1}{z} \sum_{(n)} e^{i\mathbf{q}\mathbf{R}_n}, \quad g^2(\mathbf{q}) = \left( \frac{1+\alpha - \Gamma_{\mathbf{q}}}{1+\alpha + \Gamma_{\mathbf{q}}} \right)^{1/2}. \quad (10.3)$$

Here  $\varepsilon_{1\mathbf{q}}$ ,  $\varepsilon_{1\mathbf{q}}^{\dagger}$  and  $\varepsilon_{2\mathbf{q}}$ ,  $\varepsilon_{2\mathbf{q}}^{\dagger}$  are the creation and annihilation operators of the antiferromagnet spin waves with energies  $\varepsilon_{1\mathbf{q}}$  and  $\varepsilon_{2\mathbf{q}}$ . The summation in the expression for  $\Gamma_{\mathbf{q}}$  is carried out only over the values of z of the nearest lattice sites.

Thus, two spin-wave systems with energies (10.2) exist in the antiferromagnet and split in a magnetic field. For small values of the quasi-momentum q we get  $\Gamma_{\mathbf{q}} \approx 1 - q^2/z$ . In this case we can write two

limiting forms for the spin-wave dispersion law ( $\gamma = 1, 2$ ):

$$\begin{aligned} \varepsilon_{\nu\mathbf{q}} = & s \sqrt{2z} I q \pm h\omega_d \left( \frac{1}{z} q^2 \gg \alpha \right), \quad (10.4) \\ \varepsilon_{\nu\mathbf{q}} = & szI \sqrt{\alpha(\alpha+2)} + \frac{sI}{\sqrt{\alpha(\alpha+2)}} q^2 \pm h\omega_d \left( \frac{1}{z} q^2 \ll \alpha \right). \end{aligned} \quad (10.5)$$

In the case of small anisotropy with respect to the exchange interaction, the dispersion law for the spin waves is approximately linear, while in the case of large anisotropy it is quadratic, with a gap  $\sim \sqrt{KI}$ . Thus, the dispersion law is quadratic near the energy gap and linear away from it.

The influence of the s-d exchange on the spectrum of the spin waves (10.2) can be investigated with the aid of the Green's function method<sup>[26]</sup>, as in the case of a ferromagnet. It turns out here that the spin-wave distribution functions (neglecting damping) have the usual Bose character, and we have for the energies of the spin waves in lieu of (10.2)

$$\begin{aligned} \tilde{\varepsilon}_{1\mathbf{q}} = & \left( szI - \frac{3s}{8(2+\alpha)} \frac{N_s}{N} \frac{J_0^2}{\zeta} \right) \sqrt{(1+\alpha)^2 - \Gamma_{\mathbf{q}}^2} + \tilde{g}\mu_0 H, \\ \tilde{\varepsilon}_{2\mathbf{q}} = & \left( szI - \frac{3s}{8(2+\alpha)} \frac{N_s}{N} \frac{J_0^2}{\zeta} \right) \sqrt{(1+\alpha)^2 - \Gamma_{\mathbf{q}}^2} - \tilde{g}\mu_0 H, \end{aligned} \quad (10.6)$$

where

$$\tilde{g} = g \left\{ 1 + \sqrt{\frac{z}{2}} \frac{J_0(kT)}{(szI)^2} \right\}. \quad (10.7)$$

We see that the effective exchange integral\*, as well as (9.52) for the ferromagnetic metal, consists of two terms corresponding to the direct and indirect exchanges, only in this case the indirect exchange reduces the antiferromagnetic coupling.<sup>†</sup>

For the damping energy of the spin waves we have the expressions

$$\begin{aligned} \gamma_{\nu\mathbf{q}} = & \pm g^2(q) \frac{\pi s}{N} \sum_{\mathbf{k}} J^2(\mathbf{k}, \mathbf{k} + \mathbf{q}) \\ & \times (n_{\mathbf{k}+\mathbf{q}} - n_{\mathbf{k}}) \delta(\tilde{\varepsilon}_{\nu\mathbf{q}} \mp \tilde{\varepsilon}_{\mathbf{k}}^{\pm} \pm \tilde{\varepsilon}_{\mathbf{k}+\mathbf{q}}), \end{aligned} \quad (10.8)$$

which are analogous to (9.34), except for the factor  $g^2(\mathbf{q})$  which depends on the anisotropy.

\*In the absence of anisotropy ( $\alpha = 0$ ) and for small q formulas (10.6) go over into the corresponding formulas of Berdyshev and Karpenko<sup>[24]</sup>, where they were first derived with the aid of ordinary perturbation theory. Kasuya,<sup>[4]</sup> however, arrived at the conclusion that the indirect interaction in terms of the conduction electrons contributes in an antiferromagnetic metal to the establishment of antiferromagnetic order. This conclusion of his is in error.

†In this connection, a difficulty arises in the explanation of the antiferromagnetism of certain dilute alloys of transition elements in diamagnetic metals.<sup>[27]</sup> Indirect exchange via the conduction electrons cannot lead, as was indicated here, to antiferromagnetism. Several authors have advanced the idea<sup>[28,29]</sup> of an indirect exchange coupling in such alloys via excited and localized electron states.

third approximation of the s-d exchange it would be necessary to take into account Green's functions of higher order than those used by these authors, and then the terms that appear in the energy should cancel out the "gap."

Calculation of the sum in (10.8) leads to the approximate expression

$$\gamma_{vq} = \tilde{\epsilon}_{vq} \frac{3\pi s}{16} \frac{N_s}{N} \left( \frac{J_0}{\xi} \right)^2 \frac{k_0}{q} g^2(q) \left( \frac{1}{0} \right); \quad (10.9)$$

The "1" is used if  $q$  satisfies the inequality

$$\frac{\tilde{\epsilon}_{vq} k_0}{2\xi q} < 1, \quad (10.10)$$

and in the opposite case  $\gamma_{vq}$  is equal to zero. In the case of small anisotropy [see (10.4)] condition (10.10) is always satisfied, inasmuch as  $(ak_0) \sim 1$ ,  $J_0/\xi \ll 1$ , and expression (10.9) reduces in this case to

$$\gamma_{vq} = \tilde{\epsilon}_{vq} \frac{3\pi s}{8\sqrt{2z}} \frac{N_s}{N} \left( \frac{J_0}{\xi} \right)^2 k_0. \quad (10.11)$$

Thus, at sufficiently large  $q$  (in the region of the linear dispersion law) the damping is proportional to the spin-wave energy, their ratio being equal to  $(J_0/\xi)^2$ .

In the case (10.5), that is, near the gap, the spin waves with quasi-momentum

$$q < \frac{\tilde{\epsilon}_{vq_0}}{2\xi} k_0 \equiv q_0 \quad (10.12)$$

are not damped at all. For a spin wave with  $q = q_0$ , for which damping begins, we have

$$\gamma_{vq_0} = \frac{3\pi s}{8} \frac{N_s}{N} \left( \frac{\alpha}{2+\alpha} \right)^{1/2} \frac{J_0^2}{\xi}. \quad (10.13)$$

This quantity depends appreciably on the anisotropy constant. If we assume the usual estimates  $K \sim 10^{-17}$  erg,  $I \sim 10^{-14}$  erg,  $J_0 \sim 10^{-14}$  erg, and  $\xi \sim 10^{-12}$  erg then we get  $\gamma_{vq_0} \sim 10^{-16}$  erg. Thus, the damping  $\gamma_{vq}$  is a complicated function of the quasi-momentum  $q$ . When  $q < q_0$  the spin waves do not attenuate. When  $q = q_0$  the attenuation is given by formula (10.13). When  $q^2 \gg \alpha$  (in the region of the linear dispersion law), the damping is determined by formula (10.11) and begins to increase with increasing  $q$ . In this region it does not depend on the anisotropy constants and amounts to  $\sim 10^{-4} \tilde{\epsilon}_{vq}$  for the estimates made. In a ferromagnetic metal (see Sec. 9) the value of the damping varies qualitatively with increasing  $q$  in the same manner, but  $q_0$  is much larger there ( $q_0 \sim J_0 k_0 / \xi$ ), owing to the exchange magnetization of the conduction electrons.

The results obtained here will be used in Sec. 11 in a discussion of magnetic resonance in transition metals.

## 11. Magnetic Relaxation and Resonance in Ferromagnetic and Antiferromagnetic Metals

1. The s-d exchange leads to a dynamic interaction between the spin waves of the ferromagnet and the conduction electrons, so that this interaction, being the strongest of all interactions between these two systems of quasi-particles, determines the relaxation processes between them.

With the aid of the kinetic-equation technique<sup>[13]</sup> it is possible to determine the relaxation time  $\tau_{\mathbf{q}}$  of a spin wave with quasi-momentum  $\mathbf{q}$  resulting from its interaction with the s electrons. The value of  $\tau_{\mathbf{q}}$  is determined from the relation

$$n_{\mathbf{q}}(t) - n_{\mathbf{q}} = \Delta n_{\mathbf{q}} e^{-\frac{t}{\tau_{\mathbf{q}}}}, \quad (11.1)$$

where  $n(t)$  is the number of spin waves with momentum  $q$  at the instant of time  $t$ ,  $n_{\mathbf{q}}$  is their equilibrium number [see (9.38)], and  $\Delta n_{\mathbf{q}}$  is the deviation of the number of spin waves at  $t = 0$  from the equilibrium value.

For the mechanism under consideration (s-d exchange) the relaxation time  $\tau_{\mathbf{q}}$  was calculated in<sup>[5,18,19]</sup>. It turned out here that  $\tau_{\mathbf{q}}$  is connected with the damping  $\gamma_{\mathbf{q}}$  of the spin wave [see (9.34)] by the simple relation

$$\frac{1}{\tau_{\mathbf{q}}} = 2\gamma_{\mathbf{q}}, \quad (11.2)$$

which is characteristic not only of s-d exchange, but of any other spin-wave interaction which can be described with the aid of perturbation theory.

To describe the s-d exchange relaxation process it is useful to calculate the average relaxation time  $\tau_{sd}$  (the average probability of disappearance of the spin wave), defined by the expression

$$\frac{1}{\tau_{sd}} = \frac{\sum_{\mathbf{q}} \frac{1}{\tau_{\mathbf{q}}} n_{\mathbf{q}}}{\sum_{\mathbf{q}} n_{\mathbf{q}}}. \quad (11.3)$$

This quantity was calculated by several authors<sup>[19,30]</sup>; in particular, Bar'yakhtar and Peletminskiĭ<sup>[30]</sup> have found

$$\frac{1}{\tau_{sd}} = \frac{(ak_0)^2}{\pi^{3/2} \xi} \frac{T_0}{\hbar} \left( \frac{T}{\theta_K} \right)^{1/2} \Psi \left( e^{-\frac{T}{T_0}} \right), \quad (11.4)$$

where  $\theta_K$  is the Curie temperature,  $T_0 = \theta_K (ak_0)^2 \times (J_0/\xi)^2$ ,  $\xi^{3/2}$  are the Riemann  $\xi$  functions, and

$$\Psi(x) = \frac{1}{2} \ln(1-x) \ln \frac{x^2}{1-x} + \int_0^{x^{1-x}} \frac{\ln(1+t)}{t} dt. \quad (11.5)$$

In the customary estimates  $T_0$  amounts to several degrees Kelvin. Neglecting magnetic interactions, we can consider the region of not too low temperatures, so that (11.4) is actually meaningful when  $T \gg T_0$ . For this case we obtain<sup>[30]</sup>

$$\frac{1}{\tau_{sd}} = \frac{\sqrt{\pi}}{6\xi} \frac{(ak_0)^2}{\hbar} \left( \frac{T}{\theta_K} \right)^{1/2}. \quad (11.6)$$

for  $T \sim 10^3$  K we obtain from (11.6)  $\tau_{sd} \sim 10^{-10}$  sec.

Investigations have shown<sup>[31]</sup> that the s-d exchange relaxation time is much shorter than all the other times due to other interactions between the spin waves and s electrons, particularly the interaction between

the latter and the magnetic field produced by the spin waves ( $\tau_m \sim 10^{-8}$  sec).

At the same time it turns out<sup>[30]</sup> that this magnetic interaction is responsible for the establishment of the equilibrium value of the magnetic-moment projection on the easiest magnetization axis with a relaxation time  $\tau \sim 10^{-8}-10^{-9}$  sec, and the latter is independent of the temperature. In view of the fact that  $\tau_{sd} \ll \tau_m$  and  $\tau$ , we can draw the following conclusion: in a ferromagnetic metal, owing to the s-d exchange, the first to be established is a quasi-equilibrium distribution of the spin waves and of the conduction electrons with specified nonequilibrium value of the magnetic-moment projection on the easiest magnetization axis. The equilibrium value of the magnetic-moment projections is then established by the magnetic interaction between these quasi-particles.

2. The smallness of the relaxation time  $\tau_{sd}$  makes it probable that the s-d exchange plays an important role in ferromagnetic resonance in a metal. As is well known, absorption of microwave radiation in ferromagnets has a resonant character, and during the absorption definite spin waves are excited. The presence of conduction electrons in a ferromagnetic metal does not change the character of the resonant absorption, but greatly modifies the conditions for its observation. This is connected with the fact that the presence of conduction electrons in the metal gives rise to skin effect, by virtue of which the magnetization component due to the alternating field is not homogeneous over the depth of the metal. This changes the selection rules for transitions under the influence of the interaction between the magnetic material and the field, in other words, the electromagnetic field excites in the metal other spin waves as compared, for example, with a ferromagnetic semiconductor, where the inhomogeneity of the alternating field can be neglected.

The exact solution of ferromagnetic resonance in metals presupposes a simultaneous solution of Maxwell's equations and the equation for the spin-system density matrix. There is still no consistent solution of this problem, so that we shall therefore confine ourselves to crude estimates only. However, in the case when the dimensions of the specimen are much smaller than the depth of the skin layer, the inhomogeneity of the alternating field can be neglected.

In weak homogeneous magnetic fields of frequency  $\omega$ , the behavior of the magnet is described by a magnetic susceptibility tensor  $\chi_{\alpha\beta}(\omega)$ , defined with the aid of the relation

$$M_{\alpha}(t) = M_{\alpha}^0 + \sum_{\beta} \chi_{\alpha\beta}(\omega) h_{\beta}(t) \quad (\alpha\beta = x, y, z), \quad (11.7)$$

where  $M_{\alpha}(t)$  is the magnetization of the instant of time  $t$ ,  $M_{\alpha}^0$  is the equilibrium magnetization of the ferromagnet, and  $h_{\beta}(t)$  is the component of the

alternating magnetic field vector. The tensor of the magnetic susceptibility, after Kubo and Tomita<sup>[32]</sup>, can be expressed in terms of the magnetic-moment operator and the Hamiltonian  $H$ :

$$\chi_{\alpha\beta}(\omega) = \int_0^{\infty} \frac{i}{h} \langle [\hat{M}_{\alpha}(\tau), \hat{M}_{\beta}] \rangle e^{-i\omega\tau + \epsilon\tau} d\tau. \quad (11.8)$$

Here  $\hat{M}_{\alpha}(t)$  is the operator  $\hat{M}_{\alpha}$  in the Heisenberg representation. It is seen from (11.8)<sup>[33]</sup> that  $\chi_{\alpha\beta}(\omega)$  is the Fourier component of the retarded branch function  $\ll \hat{M}_{\alpha}(t) \hat{M}_{\beta}(t') \gg$  [see (9.13)]

$$\chi_{\alpha\beta}(\omega) = \frac{2\pi i}{h} \langle \langle \hat{M}_{\alpha} | \hat{M}_{\beta} \rangle \rangle_{E=\hbar\omega + i\epsilon}. \quad (11.9)$$

In order to use relation (11.9) it is necessary to express the operator  $M_{\alpha}$  in terms of the same dynamic variables as the Hamiltonian (9.5). It is obvious that

$$\hat{M} = g_d \mu_0 \hat{S}_d + g_s \mu_0 \hat{S}_s, \quad (11.10)$$

where  $\hat{S}_d$  and  $\hat{S}_s$  are the summary-spin operators of the d and s electrons.

Let resonance be observed under the following conditions: the constant magnetic field  $H$  is directed along the  $z$  axis, and the alternating field perpendicular to it is linearly polarized along the  $x$  axis. It is not sufficient to calculate only one component of the tensor  $\chi_{xx}(\omega)$ . In the spin-wave approximation, using (9.4), we obtain

$$S_d^x = \sum_j S_j^x = \frac{1}{2} (2sN)^{1/2} (b_0 + b_0^{\dagger}). \quad (11.11)$$

This means that in homogeneous resonance the electromagnetic field excites spin waves only with  $q = 0$ . The summary spin projections of the s electrons are expressed on the other hand in terms of the Fermi operators<sup>[34]</sup>:

$$\begin{aligned} S_s^x &= \frac{1}{2} \sum_{\mathbf{k}} [a_{\mathbf{k}(+)}^{\dagger} a_{\mathbf{k}(+)} + a_{\mathbf{k}(+)}^{\dagger} a_{\mathbf{k}(+)}], \\ S_s^y &= \frac{i}{2} \sum_{\mathbf{k}} [a_{\mathbf{k}(+)}^{\dagger} a_{\mathbf{k}(-)} - a_{\mathbf{k}(-)}^{\dagger} a_{\mathbf{k}(+)}], \\ S_s^z &= \frac{1}{2} \sum_{\mathbf{k}} [a_{\mathbf{k}(-)}^{\dagger} a_{\mathbf{k}(-)} - a_{\mathbf{k}(+)}^{\dagger} a_{\mathbf{k}(+)}]. \end{aligned} \quad (11.12)$$

The total magnetic moment operator (11.10) does not commute with the Hamiltonian (9.5) in view of the presence in it of non-diagonal terms with  $\mathbf{k} \neq \mathbf{k}'$ , describing the s-d exchange processes which can change the momentum of the conduction electron. This gives rise to the possibility of energy transfer from the spin system of the d electrons to increase the kinetic energy of the s electrons. In view of this we can expect the s-d exchange to broaden the ferromagnetic resonance absorption line.

By virtue of (11.10) we have

$$\begin{aligned} \langle \langle M_x | M_x \rangle \rangle &= \mu_0^2 \{ g_d^2 \langle \langle S_d^x | S_d^x \rangle \rangle + g_s g_d \langle \langle S_s^x | S_d^x \rangle \rangle \\ &+ g_s g_d \langle \langle S_d^x | S_s^x \rangle \rangle + g_s^2 \langle \langle S_s^x | S_s^x \rangle \rangle \}. \end{aligned} \quad (11.13)$$

Each of the four Green's functions in (11.13) is expressed by virtue of (11.11) and (11.12) in terms of the second-quantization operators

$$\left. \begin{aligned} \langle\langle S_d^x | S_d^x \rangle\rangle &\approx \frac{1}{2} sN \langle\langle b_0 | b_0^+ \rangle\rangle + \text{c. c.}, \\ \langle\langle S_s^x | S_d^x \rangle\rangle &\approx \frac{1}{4} (2sN)^{1/2} \langle\langle \sum_{\mathbf{k}} a_{\mathbf{k}(-)}^+ a_{\mathbf{k}(+)} | b_0^+ \rangle\rangle + \text{c. c.}, \\ \langle\langle S_d^x | S_s^x \rangle\rangle &\approx \frac{1}{4} (2sN)^{1/2} \langle\langle b_0 | \sum_{\mathbf{k}} a_{\mathbf{k}(+)}^+ a_{\mathbf{k}(-)} \rangle\rangle + \text{c. c.}, \\ \langle\langle S_s^x | S_s^x \rangle\rangle &\approx \frac{1}{4} \langle\langle \sum_{\mathbf{k}} a_{\mathbf{k}(-)}^+ a_{\mathbf{k}(+)} | \sum_{\mathbf{k}} a_{\mathbf{k}(+)}^+ a_{\mathbf{k}(-)} \rangle\rangle + \text{c. c.} \end{aligned} \right\} \quad (11.14)$$

The symbol c.c. denotes the addition to the corresponding expression of the Green's function made up of the conjugate operators, for example  $\langle\langle b_0^+ | b_0 \rangle\rangle$  is added to  $\langle\langle b_0 | b_0^+ \rangle\rangle$ . We note that expressions

(11.14) are approximate, since we discard in them the Green's functions of the form  $\langle\langle b_0 | b_0 \rangle\rangle$  etc, which turn out to be of higher order, as can be readily verified by setting up the corresponding equations of motion.

Thus, to calculate  $\chi_{xx}$  it is necessary to find four Green's functions, and the four others (c.c.) do not have to be found, since a relation exists between their Fourier components:

$$\langle\langle A^+ | B^+ \rangle\rangle_E = -\langle\langle A | B \rangle\rangle_{-E}^*.$$

The first two functions in (11.14) are readily obtained from (9.29) and (9.24), in which one must put  $q = 0$ . For the two other functions it is necessary to set up the equations of motion. Approximating the higher Green's functions in analogy with (9.22), we can readily write down the system of equations

$$\left. \begin{aligned} i\hbar \frac{d}{dt} \langle\langle \sum_{\mathbf{k}} a_{\mathbf{k}(-)}^+ a_{\mathbf{k}(+)} | \sum_{\mathbf{k}'} a_{\mathbf{k}'(+)}^+ a_{\mathbf{k}'(-)} \rangle\rangle &= i\hbar \delta(t-t') \sum_{\mathbf{k}} (n_{\mathbf{k}}^- - n_{\mathbf{k}}^+) \\ &+ \sum_{\mathbf{k}} (\tilde{\epsilon}_{\mathbf{k}} - \tilde{\epsilon}_{\mathbf{k}}) \langle\langle a_{\mathbf{k}(-)}^+ a_{\mathbf{k}(+)} | \sum_{\mathbf{k}'} a_{\mathbf{k}'(+)}^+ a_{\mathbf{k}'(-)} \rangle\rangle \\ &- \left(\frac{2s}{N}\right)^{1/2} \sum_{\mathbf{k}} J(\mathbf{k}\mathbf{k}) (n_{\mathbf{k}}^- - n_{\mathbf{k}}^+) \langle\langle b_0 | \sum_{\mathbf{k}'} a_{\mathbf{k}'(+)}^+ a_{\mathbf{k}'(-)} \rangle\rangle, \\ i\hbar \frac{d}{dt} \langle\langle b_0 | \sum_{\mathbf{k}'} a_{\mathbf{k}'(+)}^+ a_{\mathbf{k}'(-)} \rangle\rangle &= \left[ \epsilon_0 + \frac{1}{N} \sum_{\mathbf{k}} J(\mathbf{k}\mathbf{k}) (n_{\mathbf{k}}^- - n_{\mathbf{k}}^+) \right] \langle\langle b_0 | \sum_{\mathbf{k}'} a_{\mathbf{k}'(+)}^+ a_{\mathbf{k}'(-)} \rangle\rangle \\ &- \left(\frac{2s}{N}\right)^{1/2} \sum_{\mathbf{k}} J(\mathbf{k}\mathbf{k}) \langle\langle a_{\mathbf{k}(-)}^+ a_{\mathbf{k}(+)} | \sum_{\mathbf{k}'} a_{\mathbf{k}'(+)}^+ a_{\mathbf{k}'(-)} \rangle\rangle. \end{aligned} \right\} \quad (11.15)$$

In the approximation employed, this chain turns into a closed system which can be readily solved. The Green's functions determined by (11.15) have only real poles. In view of the fact that, according to (9.54),  $\gamma_{\mathbf{q}} = 0$  when  $q = 0$ , the Green's functions  $\langle\langle b_0 | b_0^+ \rangle\rangle$  and  $\langle\langle \sum_{\mathbf{k}} a_{\mathbf{k}(-)}^+ a_{\mathbf{k}(+)} | b_0^+ \rangle\rangle$  have also only real poles.

It is easy to see that all the functions from (11.14) have two real poles each, which are the same for all of these functions and are defined by (9.45)—the energies of the spin waves with zero quasi-momentum.

Substituting the obtained values of the Green's functions in (11.9), we obtain the value of the real part of the magnetic susceptibility

$$\begin{aligned} \chi_{xx}(\omega) &= \frac{\mu_0^2 s N}{(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)} \{ -\omega^2 [g_d^2 \omega_d + g_s^2 \omega_s \mu_s + 2sJ_0 \mu_s (g_s - g_d)^2] \\ &+ [g_d^2 \omega_s + g_s^2 \omega_d \mu_s + 2sJ_0 (g_d + g_s \mu_s)^2] \\ &\times [\omega_s \omega_d + 2sJ_0 (\omega_d + \omega_s \mu_s)] \}. \end{aligned} \quad (11.16)$$

Here  $\omega_1 = \epsilon_0^{(1)}/\hbar$  and  $\omega_2 = \epsilon_0^{(2)}/\hbar$  are the resonant frequencies. Inasmuch as there is no damping in this case, the imaginary part of the magnetic susceptibility can be formally expressed in terms of a  $\delta(\omega \pm \epsilon_0^{(1,2)})$  function. Expression (11.16) coincides exactly with

the corresponding expression for the susceptibility of a ferrite with two sublattices and magnetizations  $\mu_s$  and 1. Thus, there should exist in a ferromagnetic metal two resonant frequencies. As can be seen from (9.45), one of them is close to the Larmor frequency  $\omega_0$ , and the other is shifted over from it into the infrared region by an amount  $\sim 2sJ_0/\hbar$ . When the g-factors are equal, expression (11.16) becomes much simpler\*:

$$\chi_{xx}(\omega) = \chi_0 \frac{\omega^2}{\omega_0^2 - \omega^2}, \quad (11.17)$$

where  $\chi_0$  is the static susceptibility of the metal

$$\chi_0 = \frac{g\mu_0 s N (1 + \mu_s)}{H} \equiv \frac{M_d + M_s}{H}. \quad (11.18)$$

The second frequency

$$\omega_2 = \omega_0 + \frac{2sJ_0}{\hbar} (1 + \mu_s) \quad (11.19)$$

drops out in this case entirely from the expression for the susceptibility, as is the situation in ferrites,

\*A microscopic derivation of formula (11.17) was given by Izyumov and Polyak<sup>[35]</sup> using a variant of the Kubo and Tomita method, based on perturbation theory.

so that resonance can be observed in the infrared region of the spectrum (s-d exchange resonance) only when the d and s electrons have different g-factors.

It is useful to note that the part of the susceptibility proportional to the magnetization  $\mu_s$  of the s electrons is due to a Green's function of the mixed type, and also to functions which pertain only to conduction electrons [see the last three lines in (11.14)]. The other part, proportional to the magnetization of the d electrons, is due to the functions  $\ll b_0 | b_0^+ \gg$ . Therefore, in neglecting the first three Green's functions, we neglect the magnetization of the s electrons compared with the magnetization of the d electrons. The latter is always appreciably larger, as can be seen from (9.41). At small values of the ratio  $J_0/\zeta$ , the magnetization of the s electrons can be neglected. Thus, for this case the resonant properties of the system are described essentially by the Green's function  $\ll b_0 | b_0^+ \gg$ . By leaving only this function in expression (11.13), we take into account only the contribution made by the d electrons to the magnetic susceptibility of the metal.

After substituting (9.29) with  $q = 0$  in (11.9) we obtain

$$\chi_{xx}^d(\omega) = \frac{\chi_0 \omega_0}{2} \left\{ \frac{\frac{\tilde{\epsilon}_0}{\hbar} - \omega}{(\tilde{\epsilon}_0/\hbar - \omega)^2 + \gamma_0^2/\hbar^2} + \frac{\frac{\tilde{\epsilon}_0}{\hbar} + \omega}{(\tilde{\epsilon}_0/\hbar + \omega)^2 + \gamma_0^2/\hbar^2} \right\} + i \frac{\chi_0 \omega_0}{2} \left\{ \frac{\gamma_0/\hbar}{(\tilde{\epsilon}_0/\hbar - \omega)^2 + \gamma_0^2/\hbar^2} - \frac{\gamma_0/\hbar}{(\tilde{\epsilon}_0/\hbar + \omega)^2 + \gamma_0^2/\hbar^2} \right\}. \quad (11.20)$$

Thus, the magnetic susceptibility would be described by the classical Lorentz function were the damping to be  $\gamma_0 \neq 0$ . The line width in this case is  $\Delta\omega = 2\gamma_0/\hbar$ . Although for the s-d interaction we have in accord with (9.54)  $\gamma_0 = 0$  and therefore the line does not broaden in homogeneous resonance, we use expression (11.20) to estimate the line width when taking the skin effect into account.

When the specimen dimensions are smaller than the depth of the skin layer, the alternating field excites spin waves with quasi-momenta that lie near the value of the reciprocal of the skin depth  $\delta$ , that is,

$$q \sim \frac{1}{\delta} = \frac{1}{c} \sqrt{4\pi\sigma\omega\mu_2(\omega_0)}, \quad (11.21)$$

where  $\sigma$  is the electric conductivity of the metal, and  $\mu_2(\omega_0)$  is the imaginary part of the magnetic susceptibility at the resonant frequency. Using (9.54) we can calculate the value of the damping corresponding to this spin wave, and, in analogy with formula (11.20) for homogeneous resonance, assume that the line width due to the s-d exchange is

$$\Delta\omega = \frac{2\gamma_2}{\hbar}. \quad (11.22)$$

It must be borne in mind here that  $\mu_2$  is itself dependent on the line width, and therefore (11.12) is essentially a certain equation that defines implicitly the

line width  $\Delta\omega$ . This was first pointed out by Akhiezer, Kaganov, and Bar'yakhtar [36].

We now substitute the expression for  $q$  in formula (9.44), assuming that  $q > q_0$ . This assumption sets the upper limit of the s-d exchange integral, at which the s-d exchange interaction can be called the line broadening. In the opposite case the line width vanishes. Thus, solving (11.12), we obtain [31]

$$\Delta\omega = \left( \frac{3\pi}{4} s \frac{N_s}{N} \frac{J_0^2}{\zeta^2 \hbar} \omega k_0 \right)^2 \frac{c^2}{4\pi\sigma\Omega_0^2}. \quad (11.23)$$

Here  $\Omega_0$  is the resonant frequency. An exact quantitative estimate of  $\Delta\omega$  in accordance with (11.23) is quite difficult, since the order of magnitude of  $J_0$ , on which  $\Delta\omega$  depends very strongly ( $\Delta\omega \sim J_0^4$ ), is not known. However, from condition (9.55), which is the only one for which formula (11.23) is valid, we find that  $J_0$  must not exceed  $10^{-14}$  erg. This means that the maximum value of  $\Delta\omega$  due to the relaxation mechanism under consideration at  $\sigma \sim 10^{17}$  sec $^{-1}$  (room temperatures) and  $\omega = \Omega_0 \sim 10^4$  sec $^{-1}$  amounts to  $10^9$  sec $^{-1}$ .

According to (11.23),  $\Delta\omega$  should decrease with decreasing temperature in direct proportion to the electric resistivity. However, at low temperatures the skin effect becomes anomalous and formula (11.23) no longer applies. The condition of applicability of the theory of normal skin effect in a ferromagnetic metal at resonant frequency has the form

$$\delta_0^2 \gg l^2 \mu_2, \quad (11.24)$$

where  $\delta_0$  is the skin depth at  $\mu = 1$ , and  $l$  is the mean free path of the electrons. Inasmuch as  $\mu_2 \gg 1$  at resonance, the condition (11.24) can be easily violated. For the extremely anomalous skin effect ( $\delta_0^2 \ll l^2 \mu_2$ ), as is well known, we can use formally the expressions which are valid in the case of normal skin effect, replacing in them the electric conductivity  $\sigma$  by  $\sigma_{\text{eff}}$ , which is determined from the condition

$$\sigma_{\text{eff}} = \sigma \frac{\delta(\sigma_{\text{eff}})}{l}. \quad (11.25)$$

Correcting in this manner the expression (11.23), we obtain the line width in the region of anomalous skin effect [31]

$$\Delta\omega_a = \left( \frac{3\pi}{4} s \frac{N_s}{N} \frac{J_0^2}{\zeta^2 \hbar} \omega k_0 \right)^{3/2} \left( \frac{c^2 l}{4\pi\sigma\Omega_0^2} \right)^{1/2}. \quad (11.26)$$

$\Delta\omega_a$  is independent of the temperature, since  $\sigma/l$  is independent of the temperature. Assuming that  $\sigma/l \sim 10^{22}$  sec $^{-1}$  cm $^{-1}$ , and taking all other estimates as before, we obtain again  $\Delta\omega_a \sim 10^9$  sec $^{-1}$ , whereas at lower temperatures ( $\sigma \sim 10^{18}$  sec $^{-1}$ ) formula (11.23) yields  $\Delta\omega \sim 10^8$  sec $^{-1}$ .

Turov [31] considered the contribution made to the resonant line width by the interaction of the s-electrons with the magnetic fields due to the spin waves. He showed that these processes make a much smaller contribution to the line width. Thus, s-d exchange

relaxation predominates among all the spin-electron relaxation processes in a ferromagnetic metal.

3. Let us consider now resonance in an antiferromagnetic metal. The summary spin operators of the *s* electrons are expressed by the same formulas (11.12). It is easy to show that for the summary spin of the *d* electrons in the spin-wave approximation the following relations hold true (see Sec. 10):

$$\left. \begin{aligned} S_d^x &= \frac{1}{2} (sN)^{1/2} g(0) (\xi_{10} + \xi_{20} + \xi_{10}^\dagger + \xi_{20}^\dagger), \\ S_d^y &= \frac{1}{2i} (sN)^{1/2} g(0) (\xi_{10} - \xi_{20} + \xi_{20}^\dagger - \xi_{10}^\dagger), \\ S_d^z &= \sum_q (\xi_{2q}^\dagger \xi_{2q} - \xi_{1q}^\dagger \xi_{1q}). \end{aligned} \right\} \quad (11.27)$$

With the aid of (11.27) it is possible, as in the case of a ferromagnetic metal, to calculate the magnetic susceptibility as a function of the frequency. In the particular case when the *s*-*d* interaction is missing, we obtain for the real part of the susceptibility the expression

$$\begin{aligned} \operatorname{Re} \chi_{xx}(\omega) &= \chi_s^0 \frac{\omega_0^2}{\omega_0^2 - \omega^2} \\ &+ \frac{g^2 \mu_B^2 s N}{2\hbar} \left( \frac{\alpha}{\alpha + 2} \right)^{1/2} \left\{ \frac{\frac{\epsilon_0}{\hbar} + \omega_0}{(\epsilon_0/\hbar + \omega_0)^2 - \omega^2} + \frac{\frac{\epsilon_0}{\hbar} - \omega_0}{(\epsilon_0/\hbar - \omega_0)^2 - \omega^2} \right\}; \end{aligned} \quad (11.28)$$

in this case there is no absorption, and therefore  $\operatorname{Im} \chi_{xx}(\omega) = 0$ .

The first term in (11.28) corresponds to spin resonance on the *s*-electrons, and  $\chi_s^0$  is the static susceptibility of the electron gas. The second term describes resonance in the *d*-electron system. It breaks up into two terms, each of which describes resonance due to excitation of spin waves with energies  $\epsilon_0 \pm \hbar\omega_0$ . When  $\omega = 0$ ,  $H \rightarrow 0$ , and  $\alpha \rightarrow 0$  this term goes over into the expression for the static susceptibility of an antiferromagnet

$$\chi_{\perp} = \frac{g^2 \mu_B^2 N}{2zI}. \quad (11.29)$$

Thus, in the weak *s*-*d* exchange coupling one should observe in the antiferromagnet three resonant frequencies,  $\omega_0$  and  $\epsilon_0/\hbar \pm \omega_0$ .

Near the frequencies of spin electron resonance ( $\omega \sim \omega_0$ ) the contribution from the *s*-electrons to  $\chi_{xx}(\omega)$  predominates; near the antiferromagnetic frequencies ( $\omega \sim \epsilon_0/\hbar \pm \omega_0$ ), the contribution from the *d* electrons predominates. However, in view of the fact that

$$\frac{\chi_s^0}{\chi_{\perp}} = \frac{3z}{4} \frac{N_s}{N} \frac{I}{\epsilon} \ll 1, \quad (11.30)$$

the intensity of the spin electron resonance is smaller than that of the antiferromagnetic resonance.

The *s*-*d* exchange does not change essentially the intensity of the resonance lines, but influences their form [26]. As in the case of the ferromagnet, in an

antiferromagnetic metal the *s*-*d* exchange coupling does not lead to a broadening of the resonance lines in a homogeneous field. However, in a ferromagnet the reason for it was the exchange magnetization of the *s* electrons, while in an antiferromagnet it is the presence of the gap in the spectrum of the spin waves, resulting from the action of the anisotropic field [see (10.9)].

When account is taken of the skin effect, the *s*-*d* exchange leads to a line broadening of the type (11.22). In the case of anti-ferromagnetic resonance the *s*-*d* exchange is closely intertwined with the action of the anisotropy. Thus, the intensity and the width of the antiferromagnetic resonance lines turn out to be proportional to certain powers of the anisotropic constant. However, the anisotropy enters into these quantities by virtue of different causes. The intensity depends on the anisotropy because the expressions for the transverse magnetization components of the *d* electrons are determined in terms of the function  $g(0)$  [see (11.27)], containing the anisotropic constant. On the other hand, the expression for the attenuation of the spin waves, meaning also for the line width, is connected with the anisotropy principally through the *s*-*d* exchange mechanism [see (10.1) and (10.9)].

## 12. Magnetic Scattering of Slow Neutrons in a Ferromagnetic Metal

As was already noted in Sec. 2, important information concerning the state of the electron system of a ferromagnetic metal can be obtained by a study of the magnetic scattering of slow neutrons. In those cases when the neutron wavelength is of the order of the interatomic distances, coherent scattering of the neutrons arises in the ferromagnet, owing to their interaction with the spontaneously ordered atomic magnetic moments. The conduction electrons will also make a contribution to the scattering, both directly and by changing the scattering from the inner shells. We consider here the influence of the conduction electrons on the scattering of neutrons by a ferromagnet.

The effective scattering cross section  $d^2\sigma/d\Omega dE_{p'}$  per unit solid angle and per unit energy interval of the unpolarized neutron beam can be calculated with the aid of the temporal formalism technique in the scattering theory, which leads to the following general formula [37]:

$$\frac{d^2\sigma}{d\Omega dE_{p'}} = \frac{M^2}{(2\pi)^3 \hbar^5} \frac{p'}{p} \int_{-\infty}^{+\infty} dt e^{i(E_{p'} - E_p)t} \langle V_{p'p}^+ V_{p'p}(t) \rangle. \quad (12.1)$$

Here  $M$  is the neutron mass,  $(p, E_p)$  and  $(p', E_{p'})$  are the initial and final momenta and energies of the neutron,  $V_{p'p}$  is the matrix element of the operator of interaction between the neutron and the scatterer, taken over the states of the neutron in the initial and final beams, and  $V_{p'p}(t)$  is the same operator in the



Heisenberg representation with the Hamiltonian of the scatterer. The bar above the operator product denotes averaging over the spin states in the neutron beam.

The interaction between a slow neutron and the electron system of the crystal must be regarded as an interaction between the electron current  $\mathbf{j}_l$  and the field generated by the magnetic moment of the neutron, so that the interaction operator can be written in the form

$$V = \sum_l \frac{2\gamma\mu_{\text{nuc}}\mathbf{s}_n \times (\mathbf{r}_l - \mathbf{r})}{|\mathbf{r}_l - \mathbf{r}|^3} \frac{1}{c} \mathbf{j}_l, \quad (12.2)$$

where  $\mathbf{s}_n$  is the neutron spin operator,  $\mu_{\text{nuc}}$  is the Bohr nuclear magneton,  $\gamma = 1.93$  is the gyromagnetic ratio for the neutron, and  $\mathbf{r}_l$  and  $\mathbf{r}$  are the coordinates of the  $l$ -th electron and neutron. Summation in (12.2) is over all unpaired electrons of the crystal.

The current  $\mathbf{j}_l$  has in the general case orbital and spin parts, and the matrix element between the electron states  $\psi_n$  and  $\psi_m$  has the form [38]

$$\frac{1}{c} \psi_n^* \mathbf{j} \psi_m = i\mu_0 (\psi_m \nabla \psi_n^* - \psi_n^* \nabla \psi_m) + 2\mu_0 \text{rot} (\psi_n^* \mathbf{s} \psi_m). \quad (12.3)$$

Here  $\mathbf{s}$  is the electron spin operator.

In the case of a transition metal, the neutrons will be scattered by the unfilled atomic shells and by the conduction electrons. Halpern and Johnson [39] have shown that the matrix element  $V_{\mathbf{p}'\mathbf{p}}^d$  of the interaction between a neutron and the spins  $\mathbf{s}_j$  of the unfilled shells, situated at the lattice sites, is

$$V_{\mathbf{p}'\mathbf{p}}^d = -\frac{4\pi\hbar^2}{M} r_0 \gamma F(\mathbf{q}) \sum_j e^{i\mathbf{q}\cdot\mathbf{R}_j} (S_j, (\mathbf{s}_n \mathbf{e}) \mathbf{e} - \mathbf{s}_n). \quad (12.4)$$

Here  $\mathbf{q} = \mathbf{p} - \mathbf{p}'$  is the scattering vector,  $\mathbf{e} = \mathbf{q}/q$ ,  $\mathbf{R}_j$  is the coordinate of the  $j$ -th atom,  $r_0 = e^2/mc^2$  is the possible radius of the electron, and

$$F(\mathbf{q}) = \int d\tau \psi^* \sum_{\nu=1}^z \frac{e^{i\mathbf{q}\cdot\mathbf{r}_\nu} (s_\nu S)}{s(s+1)} \psi \quad (12.5)$$

is the magnetic atomic form factor. Here  $\psi$  is the wave function of the atom, and  $\mathbf{s}_\nu$  and  $\mathbf{r}_\nu$  are the spin and coordinate of the  $\nu$ -th electron of the atom. The summation is over all  $z$  electrons which form the uncompensated shell of the atom with spin  $s = z/2$ .

The matrix element  $V_{\mathbf{p}'\mathbf{p}}^s$  of the operator of interaction between the neutron and the conduction electrons is [40]

$$V_{\mathbf{p}'\mathbf{p}}^s = \sum_{\mathbf{k}\mathbf{k}'\sigma\sigma'} (\mathbf{k}'\sigma' | v_{\mathbf{p}'\mathbf{p}} | \mathbf{k}\sigma) a_{\mathbf{k}'\sigma'}^* a_{\mathbf{k}\sigma}, \quad (12.6)$$

where

$$(\mathbf{k}'\sigma' | v_{\mathbf{p}'\mathbf{p}} | \mathbf{k}\sigma) = -\frac{4\pi\hbar^2}{M} r_0 \gamma \left( \int_V e^{i\mathbf{q}\cdot\mathbf{r}} \varphi_{\mathbf{k}'\sigma'}^* \varphi_{\mathbf{k}\sigma} d\tau, (\mathbf{s}_n \mathbf{e}) \mathbf{e} - \mathbf{s}_n \right), \quad (12.7)$$

$\varphi_{\mathbf{k}\sigma}$  are the wave functions of the  $s$  electron.

With the aid of (12.4), (12.6), and (12.7) it is now easy to write the correlator

$$\begin{aligned} \langle V_{\mathbf{p}'\mathbf{p}}^d V_{\mathbf{p}'\mathbf{p}}^d(t) \rangle &= \left( \frac{4\pi\hbar^2}{M} r_0 \gamma \right)^2 \frac{1}{4} (\delta_{\alpha\beta} - e_\alpha e_\beta) \\ &\times \left\{ F^2(\mathbf{q}) \sum_{jj'} e^{i\mathbf{q}(\mathbf{R}_j - \mathbf{R}_{j'})} \langle S_j^z S_{j'}^z(t) \rangle \right. \\ &+ F(\mathbf{q}) \sum_j e^{i\mathbf{q}\cdot\mathbf{R}_j} \sum_{\mathbf{k}\mathbf{k}'\sigma\sigma'} \left( \int_V e^{i\mathbf{q}\cdot\mathbf{r}} \varphi_{\mathbf{k}'\sigma'}^* \varphi_{\mathbf{k}\sigma} d\tau \right)^* \langle a_{\mathbf{k}'\sigma'}^* a_{\mathbf{k}\sigma} S_j^z(t) \rangle \\ &+ F(\mathbf{q}) \sum_j e^{-i\mathbf{q}\cdot\mathbf{R}_j} \sum_{\mathbf{k}\mathbf{k}'\sigma\sigma'} \int_V e^{i\mathbf{q}\cdot\mathbf{r}} \varphi_{\mathbf{k}'\sigma'}^* \varphi_{\mathbf{k}\sigma} d\tau \langle S_j^z a_{\mathbf{k}'\sigma'}(t) a_{\mathbf{k}\sigma}(t) \rangle \\ &+ \sum_{\substack{\mathbf{k}\mathbf{k}'\mathbf{k}_1\mathbf{k}_1' \\ \sigma\sigma'\sigma_1\sigma_1'}} \left( \int_V e^{i\mathbf{q}\cdot\mathbf{r}} \varphi_{\mathbf{k}'\sigma'}^* \varphi_{\mathbf{k}\sigma} d\tau \right)^* \int_V e^{i\mathbf{q}\cdot\mathbf{r}} \varphi_{\mathbf{k}_1\sigma_1}^* \varphi_{\mathbf{k}_1\sigma_1} d\tau \\ &\times \langle a_{\mathbf{k}\sigma}^* a_{\mathbf{k}'\sigma'} a_{\mathbf{k}_1\sigma_1}^* a_{\mathbf{k}_1\sigma_1}(t) a_{\mathbf{k}_1\sigma_1}(t) \rangle \left. \right\}. \quad (12.8) \end{aligned}$$

It is therefore clear that the scattering cross section (12.1) is expressed in terms of the correlators of the dynamic variables of the system: the spin operators of the unfilled shells of the atoms and the second-quantization operators for the conduction electrons. At low temperatures we use the spin-wave approximation and change over from the spin operators to the Bose operators for the creation and annihilation of the spin waves, by means of formulas (9.4). Of all the correlation functions arising in (12.8), we shall consider only  $\langle \mathbf{b}_{\mathbf{g}}^+ \mathbf{b}_{\mathbf{g}}(t) \rangle$ ,  $\langle \mathbf{b}_{\mathbf{g}} \mathbf{b}_{\mathbf{g}}^+(t) \rangle$ ,  $\langle a_{\mathbf{k}\sigma}^+ a_{\mathbf{k}\sigma}(t) \rangle$ , and  $\langle a_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}^+(t) \rangle$ . Functions for example of the form  $\langle \mathbf{b}_{\mathbf{g}} \mathbf{b}_{\mathbf{g}}(t) \rangle$  or  $\langle \mathbf{b}_{\mathbf{g}} a_{\mathbf{k}\sigma}^+(t) a_{\mathbf{k}\sigma}(t) \rangle$  make a small contribution to the scattering either by virtue of the small number of spin waves or because of the assumed smallness of the  $s$ - $d$  interaction.

The correlation functions pertaining to the conduction electrons can be readily calculated with the aid of the single-fermion Green's functions (9.28) on the basis of the spectral theorem (9.18). We have

$$\begin{aligned} \langle a_{\mathbf{k}\sigma}^+ a_{\mathbf{k}\sigma}(t) \rangle &= n_{\mathbf{k}}^\sigma e^{-\frac{i}{\hbar} (\epsilon_{\mathbf{k}}^\sigma - \epsilon) t}, \\ \langle a_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}^+(t) \rangle &= (1 - n_{\mathbf{k}}^\sigma) e^{\frac{i}{\hbar} (\epsilon_{\mathbf{k}}^\sigma - \epsilon) t}. \quad (12.9) \end{aligned}$$

As the wave functions  $\varphi_{\mathbf{k}\sigma}$  entering into expression (12.7), we should take the Bloch functions (6.1), corrected for the  $s$ - $d$  exchange interaction. They are calculated in (7.13). After these remarks we can write the expression for the scattering cross section in the form

$$\frac{d^2\sigma}{d\Omega dE_{\mathbf{p}'}} = \frac{d^2\sigma_{\text{el}}}{d\Omega dE_{\mathbf{p}'}} + \frac{d^2\sigma_{\text{inel}}^d}{d\Omega dE_{\mathbf{p}'}} + \frac{d^2\sigma_{\text{inel}}^s}{d\Omega dE_{\mathbf{p}'}} \quad (12.10)$$

where

$$\begin{aligned} \frac{d^2\sigma_{\text{el}}}{d\Omega dE_{\mathbf{p}'}} &= (r_0 \gamma)^2 (1 - e_z^2) |F(\mathbf{q}) \sum_j e^{i\mathbf{q}\cdot\mathbf{R}_j} \langle S_j^z \rangle| \\ &+ \frac{1}{2} \int_V e^{i\mathbf{q}\cdot\mathbf{r}} \{ \varrho_-(\mathbf{r}) - \varrho_+(\mathbf{r}) \} d\mathbf{r} |\delta(E_{\mathbf{p}} - E_{\mathbf{p}'})|, \quad (12.11) \end{aligned}$$

$$\frac{d^2\sigma_{\text{inel}}^d}{d\Omega dE_{p'}} = (r_0\gamma)^2 \frac{p'}{p} F^2(q) (1 + \varepsilon_z) \times \frac{(2\pi)^3}{V_0} \sum_{\tau} \sum_{\mathbf{g}} \{ \delta(\mathbf{g} - \mathbf{q} - 2\pi\tau) Q'_g(E_{p'} - E_p) + \delta(-\mathbf{g} - \mathbf{q} - 2\pi\tau) Q''_g(E_{p'} - E_p) \}, \quad (12.12)$$

$$\frac{d^2\sigma_{\text{inel}}^s}{d\Omega dE_{p'}} = (r_0\gamma)^2 \frac{p'}{p} \frac{1}{4} \frac{(2\pi)^3}{NV_0} \sum_{\tau} |\varphi(2\pi\tau)|^2 \times \sum_{\mathbf{k}\mathbf{k}'\sigma\sigma'} \delta(\mathbf{k} - \mathbf{k}' + \mathbf{q} - 2\pi\tau) \delta \varepsilon_{\sigma\sigma'} \times (E_{p'} - E_p + \tilde{\varepsilon}_{\mathbf{k}'}^{\sigma'} - \tilde{\varepsilon}_{\mathbf{k}}^{\sigma}) \times n_{\mathbf{k}}^{\sigma} (1 - n_{\mathbf{k}'}^{\sigma'}) (1 - \varepsilon_{\sigma\sigma'} e_z^2), \quad (12.13)$$

$$\varepsilon_{\sigma\sigma'} = \begin{cases} 1, & \sigma = \sigma', \\ -1, & \sigma \neq \sigma'. \end{cases}$$

In expression (12.10) we put for brevity

$$\left. \begin{aligned} Q'_g &= \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} e^{i\hbar Et} \langle b_{\mathbf{g}} b_{\mathbf{g}}^{\dagger}(t) \rangle dt, \\ Q''_g &= \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} e^{i\hbar Et} \langle b_{\mathbf{g}}^{\dagger} b_{\mathbf{g}}(t) \rangle dt, \end{aligned} \right\} \quad (12.14)$$

$$\varrho_-(\mathbf{r}) - \varrho_+(\mathbf{r}) = \frac{3}{8} \frac{N_s}{V} \frac{J_0}{\zeta} \frac{1}{N} \times \sum_{\mathbf{g}} f(\mathbf{g}) \sum_j \{ e^{i\mathbf{g}(\mathbf{r}-\mathbf{R}_j)} + e^{-i\mathbf{g}(\mathbf{r}-\mathbf{R}_j)} \} S_j^z u^2(\mathbf{r}), \quad (12.15)$$

$u^2(\mathbf{r})$  is a certain value (averaged over the quasi-momenta near the Fermi surface)  $|\mathbf{u}_{\mathbf{k}}(\mathbf{r})|^2$ , and  $f(\mathbf{g})$  is given by (7.17).

In the derivation of (12.10) we used the following relation [4]:

$$\frac{1}{N} \left| \sum_j e^{i\mathbf{q}\mathbf{R}_j} \right|^2 = \frac{(2\pi)^3}{V_0} \sum_{\tau} \delta(\mathbf{q} - 2\pi\tau), \quad (12.16)$$

where  $\tau$  is the vector of the reciprocal lattice.

Expression (12.11) describes the elastic scattering of neutrons on the spins of the d and s electrons of the metal; (12.13) describes the incoherent scattering of the neutrons on the conduction electrons, and (12.12) yields the inelastic scattering of the neutrons with emission and absorption of one spin wave.

Let us consider first the elastic scattering. Substituting (12.15) in (12.11) and integrating the latter over the energies of the scattered neutrons, we obtain the scattering cross section per unit solid angle

$$\frac{d\sigma_{\text{el}}}{d\Omega} = (r_0\gamma)^2 (1 - e_z^2) |F(\mathbf{q}) \sum_j e^{i\mathbf{q}\mathbf{R}_j} \langle S_j^z \rangle + \frac{3}{8} \frac{N_s}{N} \frac{J_0}{\zeta} \frac{1}{N} \sum_{\mathbf{g}} f(\mathbf{g}) \varphi(\mathbf{q} - \mathbf{g}) \times \sum_s e^{i(\mathbf{q}-\mathbf{g})\mathbf{R}_s} \sum_j e^{i\mathbf{g}\mathbf{R}_j} \langle S_j^z \rangle|^2, \quad (12.17)$$

where

$$\varphi(\mathbf{q}) = \int_{V_0} e^{i\mathbf{q}\mathbf{r}} u^2(\mathbf{r}) d\mathbf{r} \quad (12.18)$$

is the electron form factor of the elementary cell, and  $V_0$  is its volume. Summation over  $s$  is carried out over all elementary cells.

In the particular case of weak coupling between the  $s$  electron and the lattice, when the electron density in the metal can be assumed constant, that is, when  $u^2(\mathbf{r}) \approx 1$ , we have  $\varphi(\mathbf{q}) \sim \delta(\mathbf{q})$ , so that in place of (12.17) we have

$$\frac{d\sigma_{\text{yel}}}{d\Omega} = (r_0\gamma)^2 (1 - e_z^2) \langle S_j^z \rangle^2 \times \left[ F(\mathbf{q}) + \frac{3}{8} \frac{N_s}{N} \frac{J_0}{\zeta} f(\mathbf{q}) \right]^2 \left| \sum_j e^{i\mathbf{q}\mathbf{R}_j} \right|^2. \quad (12.19)$$

The quantity in the square bracket describes the effective magnetic form factor of the ferromagnetic ion clothed by the polarized conduction electrons. The angular dependence of the scattering of the neutrons by the polarized cloud itself is determined by the function  $f(\mathbf{q})$ . The more general expression (12.17) takes into account the influence of the periodic structure of the electron density of the crystal on the form factor of the polarization cloud. The phase factor in (12.19) yields in the case of a pure metal, on the basis of (12.16), the conditions for the regular coherent reflections. Expressions (12.19) is suitable also for a description of scattering in an alloy of a transition metal with a simple metal, and in this case the summation over  $j$  denotes summation over all sites occupied by the atoms of the transition metal. In the case of random distribution of these sites in the alloy, the phase sum can be readily calculated:

$$\left| \sum_j e^{i\mathbf{q}\mathbf{R}_j} \right|^2 = C^2 \left| \sum_s e^{i\mathbf{q}\mathbf{R}_s} \right|^2 + NC(1 - C), \quad (12.20)$$

where  $C$  is the concentration of the impurity element. The latter expression yields in the case of an alloy the resolution of the elastic scattering into coherent and incoherent components. We see thus that a study of the (elastic) scattering in ferromagnetic metals or alloys makes it possible to determine the distribution of the spin density of the conduction electrons near the paramagnetic ions.

Let us discuss now the scattering of neutrons with emission or absorption of one spin wave. In the absence of interaction between the spin waves and the conduction electrons, the correlation functions in (12.14) have the same structure as in (12.9), so that after integration with respect to time delta functions arise in (12.14) and account for the conservation of energy in the emission or absorption of a spin wave by the neutron. When account is taken of the  $s$ - $d$  interaction, the values of (12.14) can be readily calculated with the aid of the boson Green's function (9.29). As a result we have

$$Q'_g = N(E) \frac{1}{\pi} \frac{\gamma_{\mathbf{g}}(E)}{[E - \varepsilon_{\mathbf{g}} - \Delta - P_{\mathbf{g}}(E)]^2 + \gamma_{\mathbf{g}}^2(E)},$$

$$Q_{\mathbf{g}}'' = [1 + N(-E)] \frac{1}{\pi} \frac{\gamma_{\mathbf{g}}(-E)}{[-E - \varepsilon_{\mathbf{g}} - \Delta - P_{\mathbf{g}}(-E)]^2 + \gamma_{\mathbf{g}}^2(E)}, \quad (12.21)$$

where  $\Delta$ ,  $P_{\mathbf{g}}(E)$  and  $\gamma_{\mathbf{g}}(E)$  are determined by expressions (9.30), (9.31), and (9.34), while  $N(E)$  is the Bose distribution function. As was shown above, at small  $\gamma_{\mathbf{g}}$  the quantities (12.21) have sharp maxima at the points  $E = \pm \tilde{\varepsilon}_{\mathbf{g}}$ . Inasmuch as  $N(E)$  are smoothly varying functions of  $E$  in  $E$  intervals on the order of  $\gamma_{\mathbf{g}}$ , it is possible to make the following approximate substitutions in (12.21): in the first expression  $N(E)$  can be replaced by  $N(\tilde{\varepsilon}_{\mathbf{g}}) \equiv n_{\mathbf{g}}$  and in the second  $N(-E)$  can be replaced by  $N(\tilde{\varepsilon}_{\mathbf{g}})$ . The expression (12.12) then assumes the form ( $\mathbf{q}_1 = \mathbf{q} + 2\pi\tau$ ) [42]

$$\begin{aligned} \frac{d^2\sigma_{\text{inel}}^d}{d\Omega dE_{\mathbf{p}'}} &= (r_0\gamma)^2 \frac{p'}{p} F^2(\mathbf{q}) \frac{s}{2} (1 + e^{\varepsilon}) \\ &\times \sum_{\tau} \left\{ (1 + n_{\mathbf{q}_1}) \frac{1}{\pi} \frac{\gamma_{\mathbf{q}_1}}{(E_{\mathbf{p}} - E_{\mathbf{p}'} - \tilde{\varepsilon}_{\mathbf{q}_1})^2 + \gamma_{\mathbf{q}_1}^2} \right. \\ &\left. + n_{\mathbf{q}_1} \frac{1}{\pi} \frac{\gamma_{-\mathbf{q}_1}}{(E_{\mathbf{p}} - E_{\mathbf{p}'} + \tilde{\varepsilon}_{-\mathbf{q}_1})^2 + \gamma_{-\mathbf{q}_1}^2} \right\}. \end{aligned} \quad (12.22)$$

Thus, owing to the finite lifetime of the spin waves in the metal, due to their interaction with the conduction electrons, the energy distribution of the neutrons scattered in a given direction has a finite width proportional to the damping energy of the corresponding spin wave. It must be borne in mind that actually there are many causes in a ferromagnet leading to a finite lifetime of the spin waves. The most important of these are the spin-spin and spin-phonon interactions, and therefore the real width of the neutron line (just as in ferromagnetic resonance) is due to a whole series of interactions. However, the s-d exchange interaction considered here is specific only for metals, and moreover it is apparently the decisive factor in the overall line broadening in the case of metals.

Experimentally this broadening can be readily separated from others by virtue of the specific dependence of the damping  $\gamma_{\mathbf{q}}$  on the quasi-momentum of the spin wave. Inasmuch as the spin waves are not damped by the s-d interaction in the case of momenta that are smaller than a certain value of  $q_0$  [see (9.55)], the line width in the metal should change sufficiently sharply near the corresponding scattering vector. An investigation of inelastic scattering of neutrons in ferromagnets aimed at a study of their energy spectrum has many advantages compared with the magnetic-resonance method, inasmuch as the neutrons can excite the entire spin-wave spectrum, whereas resonance makes it possible to investigate only a limited number of points of the spectrum.

#### IV. CONDUCTION-ELECTRON SYSTEM IN A TRANSITION METAL

##### 13. Energy of Conduction Electrons in a Ferromagnetic Transition Metal

In Sec. 5 and in Chapter III it was shown that the exchange interaction between the conduction electrons and the uncompensated spin momenta of the metal ions leads to a lifting of the degeneracy in the conduction-electron energy with respect to the spin quantum number  $\sigma$ . The energies of electrons with different spin orientations but with the same quasi-momenta  $\mathbf{k}$  differ by an amount of the order of the s-d exchange integral.

In Sec. 9 we calculated with the aid of the statistical Green's functions the equilibrium distribution functions for the conduction electrons in the approximation equivalent to the first order of perturbation theory in the s-d interaction. We now use the same method to calculate the next approximation. Using the decoupling of the complicated Green's functions of the type (9.22), we can readily obtain a closed system of equations for the Green's functions  $\ll a_{\mathbf{k}\sigma} | a_{\mathbf{k}\sigma}^{\dagger} \gg$ :

$$\begin{aligned} ih \frac{d}{dt} \ll a_{\mathbf{k}(-)} | a_{\mathbf{k}(-)}^{\dagger} \gg &= ih\delta(t-t') + (\tilde{\varepsilon}_{\mathbf{k}} - \zeta) \ll a_{\mathbf{k}(-)} | a_{\mathbf{k}(-)}^{\dagger} \gg \\ &- \left(\frac{2s}{N}\right)^{1/2} \sum_{\mathbf{q}} J(\mathbf{k} + \mathbf{q}, \mathbf{k}) \ll a_{\mathbf{k} + \mathbf{q}(+) } b_{\mathbf{q}}^{\dagger} | a_{\mathbf{k}(-)}^{\dagger} \gg, \\ ih \frac{d}{dt} \ll a_{\mathbf{k}(+)} | a_{\mathbf{k}(+)}^{\dagger} \gg &= ih\delta(t-t') + (\tilde{\varepsilon}_{\mathbf{k}}^{\dagger} - \zeta) \ll a_{\mathbf{k}(+)} | a_{\mathbf{k}(+)}^{\dagger} \gg \\ &- \left(\frac{2s}{N}\right)^{1/2} \sum_{\mathbf{q}} J(\mathbf{k}, \mathbf{k} - \mathbf{q}) \ll a_{\mathbf{k} - \mathbf{q}(-)} b_{\mathbf{q}} | a_{\mathbf{k}(+)}^{\dagger} \gg, \\ ih \frac{d}{dt} \ll a_{\mathbf{k} - \mathbf{q}(-)} b_{\mathbf{q}} | a_{\mathbf{k}(+)}^{\dagger} \gg &= (\tilde{\varepsilon}_{\mathbf{k} - \mathbf{q}} - \tilde{\varepsilon}_{\mathbf{q}}) \ll a_{\mathbf{k} - \mathbf{q}(-)} b_{\mathbf{q}} | a_{\mathbf{k}(+)}^{\dagger} \gg \\ &- \left(\frac{2s}{N}\right)^{1/2} J(\mathbf{k}, \mathbf{k} - \mathbf{q}) (1 + n_{\mathbf{q}} - n_{\mathbf{k} - \mathbf{q}}) \ll a_{\mathbf{k}(+)} | a_{\mathbf{k}(+)}^{\dagger} \gg, \\ ih \frac{d}{dt} \ll a_{\mathbf{k} + \mathbf{q}(+) } b_{\mathbf{q}}^{\dagger} | a_{\mathbf{k}(-)}^{\dagger} \gg &= (\tilde{\varepsilon}_{\mathbf{k} + \mathbf{q}}^{\dagger} - \tilde{\varepsilon}_{\mathbf{q}}) \ll a_{\mathbf{k} + \mathbf{q}(+) } b_{\mathbf{q}}^{\dagger} | a_{\mathbf{k}(-)}^{\dagger} \gg \\ &- \left(\frac{2s}{N}\right)^{1/2} J(\mathbf{k} + \mathbf{q}, \mathbf{k}) (n_{\mathbf{q}} + n_{\mathbf{k} + \mathbf{q}}) \ll a_{\mathbf{k}(-)} | a_{\mathbf{k}(-)}^{\dagger} \gg, \end{aligned} \quad (13.1)$$

where  $\tilde{\varepsilon}_{\mathbf{k}}^{\pm}$  and  $\tilde{\varepsilon}_{\mathbf{q}}$  are given by formulas (9.26) and (9.42). For the Fourier components of the Green's functions we obtain from (13.1) the following expression:

$$\ll a_{\mathbf{k}(\pm)} | a_{\mathbf{k}(\pm)}^{\dagger} \gg_E = \frac{ih}{2\pi} \frac{1}{E - \tilde{\varepsilon}_{\mathbf{k}}^{\pm} - \mathfrak{M}_{\mathbf{k}}^{\pm}(E)}, \quad (13.2)$$

where

$$\left. \begin{aligned} \mathfrak{M}_{\mathbf{k}}^{-} &= \frac{2s}{N} \sum_{\mathbf{q}} \frac{J^2(\mathbf{k} + \mathbf{q}, \mathbf{k}) (n_{\mathbf{q}} + n_{\mathbf{k} + \mathbf{q}}^{\dagger})}{E - \tilde{\varepsilon}_{\mathbf{k} + \mathbf{q}}^{\dagger} + \tilde{\varepsilon}_{\mathbf{q}}}, \\ \mathfrak{M}_{\mathbf{k}}^{+} &= \frac{2s}{N} \sum_{\mathbf{q}} \frac{J^2(\mathbf{k}, \mathbf{k} - \mathbf{q}) (1 + n_{\mathbf{q}} - n_{\mathbf{k} - \mathbf{q}})}{E - \tilde{\varepsilon}_{\mathbf{k} - \mathbf{q}} - \tilde{\varepsilon}_{\mathbf{q}}} \end{aligned} \right\} \quad (13.3)$$

are the mass operators.

To obtain the distribution functions it is necessary

to use the spectral theorem (9.18), after first separating the real and imaginary parts in the expression

$$\begin{aligned} \mathfrak{M}_{\mathbf{k}}^{\pm}(E \pm i\epsilon) &= \mathfrak{M}_{\mathbf{k}}^{\pm}(E) \mp i\Gamma_{\mathbf{k}}^{\pm}(E), \\ \Gamma_{\mathbf{k}}^{-} &= \pi \frac{2s}{N} \sum_{\mathbf{q}} J^2(\mathbf{k} + \mathbf{q}, \mathbf{k}) (n_{\mathbf{q}} + n_{\mathbf{k}+\mathbf{q}}^+) \delta(E - \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}}^+ + \tilde{\epsilon}_{\mathbf{q}}), \\ \Gamma_{\mathbf{k}}^{+} &= -\pi \frac{2s}{N} \sum_{\mathbf{q}} J^2(\mathbf{k}, \mathbf{k} - \mathbf{q}) (1 + n_{\mathbf{q}} - n_{\mathbf{k}-\mathbf{q}}^-) \delta(E - \tilde{\epsilon}_{\mathbf{k}-\mathbf{q}}^- - \tilde{\epsilon}_{\mathbf{q}}). \end{aligned} \quad (13.4)$$

Then on the basis of (9.16) we obtain the distribution functions

$$n_{\mathbf{k}}^{\pm} = \int_{-\infty}^{+\infty} \frac{dE}{e^{\beta E} + 1} \frac{1}{\pi} \frac{\Gamma_{\mathbf{k}}^{\pm}(E)}{[E - \tilde{\epsilon}_{\mathbf{k}}^{\pm} - \mathfrak{M}_{\mathbf{k}}^{\pm}(E) + \zeta]^2 + \Gamma_{\mathbf{k}}^{\pm 2}(E)}. \quad (13.5)$$

$\Gamma_{\mathbf{k}}^{\pm}$  describes the damping of the  $s$  electrons under the influence of their collisions with the spin waves. If we neglect attenuation and the values of  $\mathfrak{M}_{\mathbf{k}}^{\pm}(E)$ , expression (13.5) goes over into the Fermi distribution function (9.35).

Thus, the  $s$ - $d$  interaction leads not only to magnetization of the  $s$  electrons but also to their damping. The first effect appears in first-order perturbation theory, and the second in the next order. In other words, the energy shift of the conduction electrons is proportional to  $K_0$ , while the damping is proportional to  $J_0^2/\xi$ . On the other hand, the conduction electrons act on the spin waves in such a way that their actual energy shift (in the presence of a magnetic field) and their damping are proportional to  $J_0^2/\xi$ .

#### 14. Effective Interaction of Conduction Electrons Via Spin Waves and Its Influence on the Superconducting State

As was shown in Sec. 8, the interaction between the electrons of the unfilled shells with the conduction electrons in a ferromagnetic metal leads to an indirect exchange coupling between the former. On the other hand the  $s$ - $d$  exchange should lead to an effective interaction between the  $s$  electrons. Let us determine the effective Hamiltonian of this interaction within the framework of the spin-wave approximation. We start from the fundamental Hamiltonian (9.5) and subject it to the unitary transformation

$$H_T = e^{-i\hat{S}} H e^{i\hat{S}} = H + i[H, \hat{S}] + \frac{i^2}{2!} [[H, \hat{S}], \hat{S}] + \dots, \quad (14.1)$$

where the Hermitian operation  $\hat{S}$  is chosen such as to eliminate the triple terms in  $H$ , namely:

$$\begin{aligned} \hat{S} &= \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}} \delta(\mathbf{k}' - \mathbf{k} + \mathbf{q}) \{ A_{\mathbf{k}\mathbf{k}'\mathbf{q}} a_{\mathbf{k}'}^+ a_{\mathbf{k}}^+ a_{\mathbf{k}'}^- a_{\mathbf{k}}^- b_{\mathbf{q}} \\ &+ A_{\mathbf{k}\mathbf{k}'\mathbf{q}}^* a_{\mathbf{k}'}^- a_{\mathbf{k}}^- a_{\mathbf{k}'}^+ a_{\mathbf{k}}^+ b_{\mathbf{q}}^+ \}; \end{aligned} \quad (14.2)$$

The unknown coefficients  $A_{\mathbf{k}\mathbf{k}'\mathbf{q}}$  are chosen from the condition

$$H_1' + i[H_0, \hat{S}] = 0, \quad (14.3)$$

where  $H_1'$  is the part of the operator (9.7) with the triple terms. From (14.3) we get

$$A_{\mathbf{k}\mathbf{k}'\mathbf{q}} = -i \left( \frac{2s}{N} \right)^{1/2} \frac{J(\mathbf{k}\mathbf{k}')}{\epsilon_{\mathbf{k}}^+ - \epsilon_{\mathbf{k}'}^- - \epsilon_{\mathbf{q}}}.$$

Accurate to the second approximation in the interaction constant  $J_0$  we have

$$\begin{aligned} H_T &= \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}}^{\sigma} a_{\mathbf{k}\sigma}^+ a_{\mathbf{k}\sigma} + \sum_{\mathbf{q}} \epsilon_{\mathbf{q}} b_{\mathbf{q}}^+ b_{\mathbf{q}} - \frac{1}{2} \frac{2s}{N} \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}} J(\mathbf{k}, \mathbf{k} - \mathbf{q}) J(\mathbf{k}', \mathbf{k}' - \mathbf{q}) \\ &\times \left\{ \frac{1}{\epsilon_{\mathbf{k}'}^+ - \epsilon_{\mathbf{k}}^- - \epsilon_{\mathbf{q}}} + \frac{1}{\epsilon_{\mathbf{k}}^+ - \epsilon_{\mathbf{k}'}^- - \epsilon_{\mathbf{q}}} \right\} \\ &\times a_{\mathbf{k}-\mathbf{q}}^+ a_{\mathbf{k}'-\mathbf{q}}^- a_{\mathbf{k}'}^+ a_{\mathbf{k}}^+ \\ &+ \frac{1}{N} \sum_{\mathbf{k}\mathbf{q}} J(\mathbf{k}\mathbf{k}') (a_{\mathbf{k}}^+ a_{\mathbf{k}'}^- - a_{\mathbf{k}}^- a_{\mathbf{k}'}^+) b_{\mathbf{q}}^+ b_{\mathbf{q}} \\ &+ \frac{2s}{N} \sum_{\mathbf{k}\mathbf{q}} \frac{J^2(\mathbf{k}, \mathbf{k} - \mathbf{q})}{\epsilon_{\mathbf{k}}^+ - \epsilon_{\mathbf{k}-\mathbf{q}}^- - \epsilon_{\mathbf{q}}} \\ &\times \left\{ a_{\mathbf{k}+\mathbf{q}}^+ a_{\mathbf{k}}^+ (1 + b_{\mathbf{q}}^+ b_{\mathbf{q}}) - a_{\mathbf{k}-\mathbf{q}}^- a_{\mathbf{k}}^- b_{\mathbf{q}}^+ b_{\mathbf{q}} \right\}. \end{aligned} \quad (14.4)$$

We see that in the new representation there appear in the Hamiltonian terms that describe the effective interaction of the  $s$  electrons.

We shall show first that it is possible to obtain from (14.4) the energies of the elementary excitations of the spin waves and  $s$  electrons, modified with account of the  $s$ - $d$  interaction. For this purpose we average the Hamiltonian (14.4) over the conduction-electron states, using (7.6). Then (14.4) assumes the form of a diagonal quadratic Bose-operator form:

$$\begin{aligned} H_T' &= U_1 + \sum_{\mathbf{q}} \left\{ \epsilon_{\mathbf{q}} + \frac{1}{N} \sum_{\mathbf{k}} J(\mathbf{k}\mathbf{k}') (n_{\mathbf{k}}^- - n_{\mathbf{k}}^+) \right. \\ &\left. + \frac{2s}{N} \sum_{\mathbf{k}} \frac{J^2(\mathbf{k}, \mathbf{k} - \mathbf{q}) (n_{\mathbf{k}-\mathbf{q}}^- - n_{\mathbf{k}}^+)}{\epsilon_{\mathbf{k}}^+ - \epsilon_{\mathbf{k}-\mathbf{q}}^- - \epsilon_{\mathbf{q}}} \right\} b_{\mathbf{q}}^+ b_{\mathbf{q}}. \end{aligned} \quad (14.5)$$

Further, averaging over the spin-wave states, using (9.10), and averaging over the  $s$  electrons with any one spin projection, we obtain the diagonal quadratic forms of the Fermi operators:

$$\begin{aligned} H_T'' &= U_2 + \sum_{\mathbf{k}} \left\{ \epsilon_{\mathbf{k}}^+ - \frac{1}{N} J(\mathbf{k}\mathbf{k}') \sum_{\mathbf{q}} n_{\mathbf{q}} \right. \\ &\left. + \frac{2s}{N} \sum_{\mathbf{q}} \frac{J^2(\mathbf{k}, \mathbf{k} - \mathbf{q}) (1 + n_{\mathbf{q}} - n_{\mathbf{k}-\mathbf{q}}^-)}{\epsilon_{\mathbf{k}}^+ - \epsilon_{\mathbf{k}-\mathbf{q}}^- - \epsilon_{\mathbf{q}}} \right\} a_{\mathbf{k}}^+ a_{\mathbf{k}}^+, \end{aligned} \quad (14.6)$$

$$\begin{aligned} H_T''' &= U_3 + \sum_{\mathbf{k}} \left\{ \epsilon_{\mathbf{k}}^- + \frac{1}{N} J(\mathbf{k}\mathbf{k}') \sum_{\mathbf{q}} n_{\mathbf{q}} \right. \\ &\left. + \frac{2s}{N} \sum_{\mathbf{q}} \frac{J^2(\mathbf{k}, \mathbf{k} - \mathbf{q}) (n_{\mathbf{q}} + n_{\mathbf{k}+\mathbf{q}}^+)}{\epsilon_{\mathbf{k}}^- - \epsilon_{\mathbf{k}+\mathbf{q}}^+ + \epsilon_{\mathbf{q}}} \right\} a_{\mathbf{k}}^- a_{\mathbf{k}}^-. \end{aligned} \quad (14.7)$$

In formulas (14.5)–(14.7)  $U_1$ ,  $U_2$ , and  $U_3$  are different constants which depend on the distribution functions of the spin waves and the conduction electrons. The expressions in the curly brackets of (14.5)–(14.7) have the meanings of elementary-excitation energies of the spin waves and of the conduc-

tion electrons—with account of the s-d interaction. These expressions coincide with the corresponding expressions (9.42) and (9.26) [see (13.2)], which determine the poles of the boson and fermion single-particle Green's functions. This demonstrates the equivalence of the perturbation-theory approximations made in decoupling the Green chains. This result explains the physical meaning of the poles of the statistical Green's function (9.19). Each of these functions gives a description of elementary excitations of spin waves or conduction-electrons, according to which all the other excitations, except the one considered, are averaged and make a constant contribution to the Hamiltonian. In such an analysis, the system energy cannot be equated to the energy of the sum of independent quasi particles. The Green's-function method, however, is more convenient not only because it makes it possible to take into account the damping of the excitations, but because it also shows how the thermodynamic functions of the system are expressed in terms of the elementary-excitation energies.

Let us turn now to (14.4). The third term in this expression describes the effective interaction between the conduction electrons via the spin waves. An analogous expression can be obtained also for an antiferromagnetic metal, described by Hamiltonian (10.1), after eliminating from it the triple terms. In this case

$$H_{\text{int}} = -\frac{s}{2N} \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}} g^2(\mathbf{q}) J(\mathbf{k}, \mathbf{k}-\mathbf{q}) J(\mathbf{k}', \mathbf{k}'-\mathbf{q}) \left\{ \frac{1}{\varepsilon_{\mathbf{k}'}^+ - \varepsilon_{\mathbf{k}'-\mathbf{q}}^- - \varepsilon_{\mathbf{q}}} - \frac{1}{\varepsilon_{\mathbf{k}'}^+ - \varepsilon_{\mathbf{k}'-\mathbf{q}}^- + \varepsilon_{2\mathbf{q}}} + \frac{1}{\varepsilon_{\mathbf{k}'}^+ - \varepsilon_{\mathbf{k}'-\mathbf{q}}^- - \varepsilon_{\mathbf{q}}} - \frac{1}{\varepsilon_{\mathbf{k}'}^+ - \varepsilon_{\mathbf{k}'-\mathbf{q}}^- + \varepsilon_{2\mathbf{q}}} \right\} \times a_{\mathbf{k}-\mathbf{q}}^+ (-) a_{\mathbf{k}'-\mathbf{q}} (-) a_{\mathbf{k}'}^+ (+) a_{\mathbf{k}} (+). \quad (14.8)$$

It is easy to see that in both cases the energy of the effective interaction of the electrons near the Fermi surface is proportional to  $J_0^2/\xi$ . The expressions obtained for the effective interaction of the electrons enable us to investigate the role of the scattering of the conduction electrons by spin waves and the appearance of the superconducting state.

It is known<sup>[43,44]</sup> that superconductivity arises in a metal because of the mutual attraction between the conduction electrons which have opposite momenta and spins near the Fermi surface. This attraction is brought about by the electron-phonon interaction. The operator for the effective interaction (via the phonons) of electrons with opposite momenta and spins has the form

$$H_{\text{red}}^{\text{ph}} = \frac{1}{N} \sum_{\substack{\mathbf{k}\mathbf{k}' \\ (\mathbf{q}=\mathbf{k}-\mathbf{k}')}} \frac{g'^2 \omega_{\mathbf{q}}^2}{(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}'})^2 - \omega_{\mathbf{q}}^2} a_{\mathbf{k}'}^+ (-) a_{-\mathbf{k}} (-) a_{-\mathbf{k}'}^+ (+) a_{\mathbf{k}} (+), \quad (14.9)$$

where  $\omega_{\mathbf{q}}$  is the energy of a phonon with quasi-momentum  $\mathbf{q}$ , and  $g'$  is the electron-phonon coupling constant. It is known that the interaction between the

electrons near the Fermi surface, described by such an operator, has the character of an attraction. We now separate from (14.4) and (14.8) the parts corresponding to the interaction of electrons with opposite momenta, for which purpose we put  $\mathbf{q} = \mathbf{k} + \mathbf{k}'$ . In addition, we take into account the fact that in the absence of a magnetic field in an antiferromagnet we have  $\varepsilon_{1\mathbf{q}} = \varepsilon_{2\mathbf{q}} \equiv \varepsilon_{\mathbf{q}}$  and  $\varepsilon_{\mathbf{k}}^{\pm} \equiv \varepsilon_{\mathbf{k}}$ . In the case of a ferromagnet we put approximately  $\varepsilon_{\mathbf{k}}^+ \approx \varepsilon_{\mathbf{k}}^- \equiv \varepsilon_{\mathbf{k}}$ , i.e., we neglect the shift of the Fermi spheres for electrons with different spin projections, inasmuch as an account of this difference in the formula for  $H_{\text{int}}$  exceeds the accuracy of the approximation, and the effect of the shift of the Fermi surface will be investigated separately in Sec. 15. Taking all the foregoing into account, we obtain in lieu of the corresponding parts of (14.4) and (14.8)

$$H_{\text{red}}^{\text{f}} = -\frac{2s}{N} \sum_{\substack{\mathbf{k}\mathbf{k}' \\ (\mathbf{q}=\mathbf{k}-\mathbf{k}')}} \frac{J^2(\mathbf{k}\mathbf{k}') \varepsilon_{\mathbf{q}}}{(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}'})^2 - \varepsilon_{\mathbf{q}}^2} a_{\mathbf{k}'}^+ (-) a_{-\mathbf{k}} (-) a_{-\mathbf{k}'}^+ (+) a_{\mathbf{k}} (+), \quad (14.10)$$

$$H_{\text{red}}^{\text{af}} = -\frac{2s}{N} \sum_{\substack{\mathbf{k}\mathbf{k}' \\ (\mathbf{q}=\mathbf{k}-\mathbf{k}')}} \frac{J^2(\mathbf{k}\mathbf{k}') g^2(\mathbf{q}) \varepsilon_{\mathbf{q}}}{(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}'})^2 - \varepsilon_{\mathbf{q}}^2} a_{\mathbf{k}'}^+ (-) a_{-\mathbf{k}} (-) a_{-\mathbf{k}'}^+ (+) a_{\mathbf{k}} (+) \quad (14.11)$$

for ferromagnetic and antiferromagnetic metals respectively.

The structure of the last expressions is completely identical with (14.9), the main difference being in the signs. Inasmuch as the electron-phonon interaction leads to the attraction of electrons with opposite momenta and spins and to the formation of stable Cooper pairs, the interaction between the conduction electrons and the spin waves in ferromagnetic and antiferromagnetic metals leads to the inverse process. It contributes to the destruction of the pairs, i.e., to the destruction of the superconducting state. Comparison of expressions (14.10) and (14.11) for the effective interaction in ferromagnetic and antiferromagnetic metals shows that there is no principal difference in the character of the influence of the electron-magnon interaction on the superconductivity of these metals, although the dispersion laws for spin waves are different in ferromagnetic and antiferromagnetic substances.

It is clear that such a qualitative equivalence of the results is due to the identical dynamics of the electron-magnon scattering: in either case the interaction is such that emission or absorption of a spin wave causes the electron to change not only its momentum but of necessity also the spin direction. In this lies the specific nature of the scattering of electrons by spin waves, as compared with their scattering by phonons, which does not lead to spin flip.

To explain why the spin flip of the electron in electron-magnon interaction hinders the occurrence of the superconductive state in ferromagnetic and antiferromagnetic metals, let us consider in succes-

sion the interaction of electrons with phonons and spin waves. Electron-phonon interaction leads to the formation of a bound pair of electrons with opposite momenta and spins. If in addition there is also electron-magnon interaction, the following can occur: one of the electrons of the pair can absorb a spin wave and thereby reverse its spin direction. We then have in place of a pair two electrons with like spins, but such a system is not stable. Thus, electron-magnon interaction can lead to destruction of the Cooper pairs. These arguments, however, hold true only in the analysis of processes of second order in perturbation theory, to which we confined ourselves above. It is obvious that other processes are also possible, in which both electrons forming the bound pair simultaneously absorb (or emit) spin waves, and then the pair is not destroyed. Processes are also possible which occur without any change in spin whatever. A detailed analysis shows [46], however, that processes of this type are of higher order than the scattering of only one of the pair electrons by a spin wave. Second-order perturbation theory must lead to a spin flip of one of the pair electrons.

The thermodynamics of superconducting ferromagnets and antiferromagnets was considered in [46]. In particular, an expression was derived for the gap of the single-fermion excitations at zero temperature (under the assumption that the regions of  $k$ -space are near the Fermi surface, where the effective spin-electron and electron-phonon interactions are of the same order of magnitude)

$$\Delta = 2\hbar\omega_e^{-\frac{1}{2}}, \quad (14.12)$$

where  $\rho = \rho_{ph} - \rho_m$  is the difference between the corresponding quantities for the electron-phonon and electron-magnon interactions.

Thus, the electron-magnon interaction in ferromagnetic and antiferromagnetic metals counteracts the formation of the superconducting state.\* For antiferromagnetic metals this is apparently the main reason why they do not include superconductors. On the other hand, in the case of ferromagnetic metals this cause is not the only one. It turns out that the magnetization of the conduction electrons by the  $s$ - $d$  interaction also creates conditions that hinder pair production as a result of electron-phonon interaction. This effect will be considered in Sec. 15 below.

\*The question of the influence of the electron-magnon interaction on the superconducting state in ferromagnetic metals was considered first by Kasuya [47] who established the repulsion character of the interaction between electrons with opposite momenta and spins. This deduction agrees with the subsequent calculations of Karpenko, [45] and also of Vonsovskiĭ and Svirskiĭ, [48] who used the Bogolyubov-method scheme. In [49], which is devoted to the same problem, no account was taken of the sign of the matrix element of the electron-electron interaction, and this led to the wrong deduction that this coupling is an attraction.

### 15. Condition for the Existence of a Superconducting State in a Ferromagnetic Metal

Until recently there was not a single ferromagnet among the many superconducting metals and alloys. Only in 1958 was the existence of a superconducting ferromagnet established apparently by Matthias et al [50] in the (Ce, Ge)  $Ru_2$  system. This means that conditions that do not favor superconductivity, but do not exclude it in principle exist in ferromagnetic metals.

Vonsovskiĭ and Svirskiĭ [51] advanced the hypothesis that such an unfavorable condition may be the exchange magnetization of the conduction electrons. Indeed, by virtue of such a magnetization, the electrons lying on the Fermi sphere with different spin orientation have energies that differ by an amount  $2sJ_0\mu$ , meaning that they have also different momenta  $k_0^-$  and  $k_0^+$ . Inasmuch as the superconducting state is connected with the formation (under the influence of the electron-phonon interaction) of electron pairs with opposite spins and momenta near the Fermi surface, it is clear that the magnetization of the electrons acts unfavorably on the superconducting state, because it decreases the number of electrons which have opposite momenta and spins near the Fermi surface. In the case when the "magnetic" gap  $2sJ_0\mu$  exceeds twice the average phonon energy  $\hbar\bar{\omega}$ , at zero temperature, near the Fermi sphere, there will be no electron pairs having opposite momenta with different spin orientations. In this case the superconducting state is impossible. It actually turns out impossible even when the magnetization is lower.

Let us investigate in detail now the role of the exchange magnetization on the condition for the existence of the superconducting state. The electron system of the superconductor in the Bardeen-Cooper-Schrieffer [43] (BCS) model is described by a Hamiltonian

$$H = \sum_{k\sigma} T_k^\sigma a_{k\sigma}^+ a_{k\sigma} - \frac{1}{V} \sum_{kk'} A(kk') a_{k(+)}^+ a_{-k(-)}^+ a_{-k(-)} a_{k(+)} \quad (15.1)$$

The second term describes here the effective interaction of the Cooper pairs due to the electron-phonon interaction:  $T_k^\sigma = \epsilon_k^\sigma - \zeta$ . The BCS theory corresponds to the case when  $\epsilon_k^\sigma$  is independent of the electron spin  $\sigma$ . In a ferromagnetic metal  $\epsilon_k^\sigma$  is given by expression (9.26). The role that the dynamic part of the  $s$ - $d$  interaction (interaction of the  $s$ -electrons with the spin waves) plays in the superconducting state has already been investigated in Sec. 14.

Following the Bogolyubov-Tyablikov method [14], we set up an equation for the Green's functions

$$G_k^\sigma(t-t') = \langle\langle a_{k\sigma} | a_{k\sigma}^+ \rangle\rangle, \quad (15.2)$$

with the aid of which we can calculate the distribution function for the  $s$  electrons with account of the pair interaction. It is easy to see that this equation contains a Green's function made up of four Fermi oper-

ators, resulting from an account of the second term in the Hamiltonian (15.1), which we uncouple with the aid of the following approximation:

$$\langle (a_{\mathbf{k}}^{\pm}, -\sigma a_{-\mathbf{k}'\sigma} a_{\mathbf{k}'\sigma}^{\pm}) | a_{\mathbf{k}\sigma}^{\pm} \rangle = \langle a_{-\mathbf{k}'\sigma} a_{\mathbf{k}'\sigma}^{\pm} \rangle \langle (a_{\mathbf{k}}^{\pm}, -\sigma | a_{\mathbf{k}\sigma}^{\pm} \rangle) + \delta_{-\mathbf{k}, \mathbf{k}'} \langle a_{\mathbf{k}}^{\pm}, -\sigma a_{-\mathbf{k}, -\sigma} \rangle \langle (a_{\mathbf{k}\sigma} | a_{\mathbf{k}\sigma}^{\pm} \rangle). \quad (15.3)$$

This approximation corresponds to an account of the correlation of the electron pairs with opposite momenta and spins. It is easy to see that the second term in the right half of (15.3) leads in the equation for  $G_{\mathbf{k}}^{\sigma}$  to an asymptotically small term, which tends to zero as  $V \rightarrow \infty$ , so that it can be left out. Thus, the decoupling consists of replacing the complicated Green's function in (15.3) by

$$\Gamma_{\mathbf{k}}^{\sigma}(t-t') = \langle (a_{\mathbf{k}}^{\pm}, -\sigma | a_{\mathbf{k}\sigma}^{\pm} \rangle). \quad (15.4)$$

The equation for this Green's function includes in the analogous approximation only the functions  $G_{\mathbf{k}}^{\sigma}$ . Thus, we obtain two pairs of closed equations

$$\left. \begin{aligned} i\hbar \frac{d}{dt} G_{\mathbf{k}}^{-} &= i\hbar \delta(t-t') + T_{\mathbf{k}}^{-} G_{\mathbf{k}}^{-} + L_{\mathbf{k}} \Gamma_{\mathbf{k}}^{-}, \\ i\hbar \frac{d}{dt} \Gamma_{\mathbf{k}}^{-} &= -T_{\mathbf{k}}^{+} \Gamma_{\mathbf{k}}^{-} + L_{\mathbf{k}} G_{\mathbf{k}}^{-}, \end{aligned} \right\} \quad (15.5)$$

$$\left. \begin{aligned} i\hbar \frac{d}{dt} G_{\mathbf{k}}^{+} &= i\hbar \delta(t-t') + T_{\mathbf{k}}^{+} G_{\mathbf{k}}^{+} - L_{\mathbf{k}} \Gamma_{\mathbf{k}}^{+}, \\ i\hbar \frac{d}{dt} \Gamma_{\mathbf{k}}^{+} &= -T_{\mathbf{k}}^{-} \Gamma_{\mathbf{k}}^{+} - L_{\mathbf{k}} G_{\mathbf{k}}^{+}, \end{aligned} \right\} \quad (15.6)$$

where we introduce for brevity the notation

$$L_{\mathbf{k}} = \frac{1}{V} \sum_{\mathbf{k}'} A(\mathbf{k}\mathbf{k}') \langle a_{-\mathbf{k}'(-)} a_{\mathbf{k}'(+)} \rangle. \quad (15.7)$$

Solving Eqs. (15.5) for the Fourier components  $G_{\mathbf{k}}^{-}(E)$  and  $\Gamma_{\mathbf{k}}^{-}(E)$  of the Green's functions  $G_{\mathbf{k}}^{-}(t-t')$  and  $\Gamma_{\mathbf{k}}^{-}(t-t')$  we obtain the expressions

$$G_{\mathbf{k}}^{-}(E) = \frac{i\hbar}{2\pi} \frac{E + T_{\mathbf{k}}^{+}}{(E - T_{\mathbf{k}}^{-})(E + T_{\mathbf{k}}^{+}) - L_{\mathbf{k}}^2}, \quad (15.8)$$

$$\Gamma_{\mathbf{k}}^{-}(E) = \frac{i\hbar}{2\pi} \frac{L_{\mathbf{k}}}{(E - T_{\mathbf{k}}^{-})(E + T_{\mathbf{k}}^{+}) - L_{\mathbf{k}}^2}. \quad (15.9)$$

Analogous expressions can also be obtained for the other pair of functions. On the basis of the spectral theorem (9.16)–(9.18) we can readily obtain now an expression for the average number of electrons in the state  $(\mathbf{k}\sigma)$ :

$$\langle a_{\mathbf{k}}^{\pm}(\pm) a_{\mathbf{k}}(\pm) \rangle = \frac{1}{2} \left\{ 1 + f_{\mathbf{k}}^{\pm} - f_{\mathbf{k}}^{\mp} - \frac{T_{\mathbf{k}}}{\omega_{\mathbf{k}}} (1 - f_{\mathbf{k}}^{\pm} - f_{\mathbf{k}}^{\mp}) \right\}, \quad (15.10)$$

where

$$f_{\mathbf{k}}^{\pm} = \frac{1}{e^{\beta(\omega_{\mathbf{k}} \pm \Delta_{\mathbf{k}})} + 1}, \quad (15.11)$$

with

$$\omega_{\mathbf{k}} = \sqrt{\left(\frac{T_{\mathbf{k}}^{+} + T_{\mathbf{k}}^{-}}{2}\right)^2 + L_{\mathbf{k}}^2}, \quad \Delta_{\mathbf{k}} = \frac{T_{\mathbf{k}}^{+} - T_{\mathbf{k}}^{-}}{2}. \quad (15.12)$$

The quantities  $\omega_{\mathbf{k}} \pm \Delta_{\mathbf{k}}$ , which are poles of the Green's functions (15.2) and (15.4), represent the energies of the elementary excitations, separated by a

gap  $L_{\mathbf{k}}$  from the Fermi level. When  $L_{\mathbf{k}} = 0$  they go over into the expressions for  $T_{\mathbf{k}}^{+}$  and  $T_{\mathbf{k}}^{-}$ , while the formulas (15.10) and (15.11) go over into the ordinary Fermi distribution functions for the conduction electrons of a ferromagnetic metal in the normal state.

The size of the gap  $L_{\mathbf{k}}$ , as can be seen from (15.7), is due to the correlation of the electrons with opposite momenta and spins. So far the quantity  $\langle a_{-\mathbf{k}(-)} a_{\mathbf{k}(+)} \rangle$  remains indeterminate. Inasmuch as

$$\langle a_{-\mathbf{k}(-)} a_{\mathbf{k}(+)} \rangle = \langle a_{\mathbf{k}(+)} a_{-\mathbf{k}(-)}^{\dagger} \rangle,$$

the sought quantity can be expressed in terms of the Green's function  $\Gamma_{\mathbf{k}}^{-}(E)$  on the basis of the spectral representation

$$\langle a_{-\mathbf{k}(-)} a_{\mathbf{k}(+)} \rangle = \int_{-\infty}^{+\infty} \frac{\Gamma_{\mathbf{k}}^{-}(E + i\epsilon) - \Gamma_{\mathbf{k}}^{-}(E - i\epsilon)}{e^{\beta E} + 1} dE. \quad (15.13)$$

Calculating the integral in (15.13) and substituting the result in (15.7) we obtain an integral equation for  $L_{\mathbf{k}}$ :

$$L_{\mathbf{k}} = \frac{1}{2V} \sum_{\mathbf{k}'} \frac{A(\mathbf{k}, \mathbf{k}')}{\omega_{\mathbf{k}'}} (1 - f_{\mathbf{k}'} - f_{\mathbf{k}'}) L_{\mathbf{k}'}. \quad (15.14)$$

An analysis of the solution of Eq. (15.14) determines the conditions for the existence of the superconducting state of a ferromagnetic metal. Let the size of the gap in the energy of the elementary Fermi excitations of the superconductor, due to the electron-phonon interaction, be

$$\Delta_{\text{ph}} = \hbar \tilde{\omega} e^{-1/\epsilon}, \quad (15.15)$$

where  $\tilde{\omega}$  is the average phonon energy. Berdyshev and Alievskii<sup>[74]</sup> in an analysis of Eq. (15.14), have reached the conclusion that so long as the energy gap of the exchange magnetization

$$\Delta_{\text{ex}} = s\mu J(\mathbf{k}\mathbf{k}) \quad (15.16)$$

is smaller than the size of the gap (15.15), the s-d exchange magnetization does not influence altogether the size of the "superconducting" gap (15.15). However, if  $\Delta_{\text{ex}} \geq \Delta_{\text{ph}}$ , the "superconducting" gap becomes immediately equal to zero and the superconductivity thus disappears.

This conclusion of the authors of<sup>[74]</sup> is in some contradiction with the earlier work of Vonsovskii and Svirskii<sup>[52]</sup>, according to which a decrease in the superconducting gap (15.15) sets in at the occurrence of as small an s-d exchange interaction as desired. However, even in that paper the authors reached the conclusion that when  $\Delta_{\text{ex}} \geq \Delta_{\text{ph}}$  the superconductivity disappears. The incomplete agreement in the results of<sup>[52, 74]</sup> (see also the note<sup>[75]</sup>) indicates that there is not yet full clarity in this question. This is connected with the great mathematical difficulties of the problem, on which we shall not dwell here, referring the reader to the original literature.

By way of a summary of the foregoing analysis of the influence of the s-d exchange interaction on the

superconducting state, we can state the following.

In ferromagnets the s-d exchange interaction prevents the establishment of a superconducting state, and this influence is effected by two mechanisms. One is connected with the fact that the electron-ferromagnon interaction leads to an effective repulsion of the electrons in the Cooper pairs. The other mechanism is due to the exchange magnetization of the conduction electrons. In antiferromagnetic metals the second effect is missing, so that from this point of view they are under more favorable conditions for simultaneous realization of the superconducting state than in ferromagnetic metals. It is possible that this is precisely what is realized in vanadium (see Sec. 2, item a) of the first part of the review, footnote on p. 553).

### 16. Anomalous Electric Resistivity of Ferromagnetic Metals

As was already indicated in the first part of the review (see Secs. 2 and 5), the electric resistivity of ferromagnetic metals is characterized by a larger value compared with the non-transition metals. In addition, they display an anomalous temperature variation both at low temperatures and near the Curie point. The latter is connected also with the singularities of the law governing the dispersion of the carriers in ferromagnetic transition metals and also with the manifestation in these crystals of "magnetic" (spin-electronic) mechanisms for the scattering of these carriers. We shall consider here a more rigorous calculation of the electric resistivity of ferromagnets, due to the mechanism of the s-d exchange\*.

Let us divide the Hamiltonian of a ferromagnetic metal into two parts

$$H = H_0 + H',$$

$$H_0 = \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}\sigma}^{\sigma} a_{\mathbf{k}\sigma}^{\dagger} a_{\mathbf{k}\sigma} + H_e, \quad H_e = - \sum_{j,j'} J(jj') (S_j S_{j'}), \quad (16.1)$$

$$H' = - \frac{1}{N} \sum_{\mathbf{k}\mathbf{k}'j} J(\mathbf{k}\mathbf{k}') e^{i(\mathbf{k}'-\mathbf{k})\mathbf{R}_j} \{ a_{\mathbf{k}'(-)}^{\dagger} a_{\mathbf{k}'(+)} S_j^- + a_{\mathbf{k}'(+)}^{\dagger} a_{\mathbf{k}'(-)} S_j^+ \}. \quad (16.2)$$

In this representation we average the part of the s-d exchange operator that includes the z component of the spin over the states of the spin system, ascribing the addition obtained in this manner to the conduction electrons, so that in (16.1) the energy of the s-electron is

$$\varepsilon_{\mathbf{k}}^{\pm} = E_{\mathbf{k}} - (s \pm \langle S_j^z \rangle) J(\mathbf{k}\mathbf{k}). \quad (16.3)$$

Formula (16.3) reflects the known fact of exchange magnetization of the conduction electrons of a ferromagnet. In what follows we shall regard  $H'$  as a perturbation. This means that we choose as the zeroth approximation the state of the magnetized con-

duction electrons and the electrons of the unfilled shells, with allowance for their exchange interaction.

To calculate the electric conductivity we use the method of Kubo<sup>[53]</sup>, according to which the electric conductivity  $\sigma$  is connected with the current correlation function by

$$\sigma = \beta \int_0^{\infty} \langle j_{\alpha} j_{\alpha}(\tau) \rangle d\tau \quad (\alpha = x, y, z). \quad (16.4)$$

Here  $j_{\alpha}$  is the current operator in the metal and  $j_{\alpha}(\tau)$  is the current operator in the Heisenberg representation with total Hamiltonian  $H$ . The current correlator can be expanded in powers of the perturbation  $H'$ . In the second perturbation-theory approximation we obtain the following expression for the electric resistivity<sup>[54]</sup>:

$$\varrho = \frac{1}{\sigma} = \frac{kT}{\hbar^2 \langle j_{\alpha}^2 \rangle} \operatorname{Re} \int_0^{\infty} \langle [j_{\alpha}, H'(\tau)] [H', j_{\alpha}] \rangle d\tau. \quad (16.5)$$

Here

$$H'(\tau) = e^{\frac{i}{\hbar} H_0 \tau} H' e^{-\frac{i}{\hbar} H_0 \tau}. \quad (16.6)$$

The Heisenberg representation of the operator  $H'$  with Hamiltonian  $H_0$  and the averaging in formula (16.5) are also carried out over the statistical ensemble with zero-order Hamiltonian. It is easy to see that in the case of Hamiltonian (16.1)–(16.2)

$$H'(\tau) = - \frac{1}{N} \sum_{\mathbf{k}\mathbf{k}'j} J(\mathbf{k}\mathbf{k}') e^{i(\mathbf{k}'-\mathbf{k})\mathbf{R}_j} e^{\frac{i}{\hbar}(\varepsilon_{\mathbf{k}}^- - \varepsilon_{\mathbf{k}'}^+) \tau} a_{\mathbf{k}'(-)}^{\dagger} a_{\mathbf{k}'(+)} \times S_j^-(\tau) + \text{c.c.}, \quad (16.7)$$

where

$$S_j^{\pm}(\tau) = e^{\frac{i}{\hbar} H_0 \tau} S_j^{\pm} e^{-\frac{i}{\hbar} H_0 \tau}. \quad (16.8)$$

The current operator has in the second-quantization representation the form

$$j_{\alpha} = \frac{e}{\hbar} \sum_{\mathbf{k}\sigma} \frac{\partial \varepsilon_{\mathbf{k}}^{\sigma}}{\partial k_{\alpha}} a_{\mathbf{k}\sigma}^{\dagger} a_{\mathbf{k}\sigma}. \quad (16.9)$$

With the aid of (16.7) and (16.9) we readily obtain

$$\begin{aligned} \langle [j_{\alpha}, H'(\tau)] [H', j_{\alpha}] \rangle &= \frac{1}{N^2} \left( \frac{e}{\hbar} \right)^2 \sum_{j_1 \mathbf{k}_1 \mathbf{k}_2} \sum_{j_2 \mathbf{k}'_1 \mathbf{k}'_2} J(\mathbf{k}_1 \mathbf{k}_2) J(\mathbf{k}'_1 \mathbf{k}'_2) \\ &\times e^{i(\mathbf{k}_2 - \mathbf{k}_1)(\mathbf{R}_{j_1} - \mathbf{R}_{j_2})} \left( \frac{\partial \varepsilon_{\mathbf{k}_1}}{\partial k_{1\alpha}} - \frac{\partial \varepsilon_{\mathbf{k}_2}}{\partial k_{2\alpha}} \right)^2 \\ &\times \{ n_{\mathbf{k}_1}^- (1 - n_{\mathbf{k}_2}^+) \langle S_{j_1}^-(\tau) S_{j_2}^+(\tau) \rangle e^{\frac{i}{\hbar}(\varepsilon_{\mathbf{k}_1}^- - \varepsilon_{\mathbf{k}_2}^+) \tau} \\ &+ n_{\mathbf{k}_1}^+ (1 - n_{\mathbf{k}_2}^-) \langle S_{j_1}^+(\tau) S_{j_2}^-(\tau) \rangle e^{-\frac{i}{\hbar}(\varepsilon_{\mathbf{k}_2}^- - \varepsilon_{\mathbf{k}_1}^+) \tau} \}, \quad (16.10) \end{aligned}$$

and also

$$\langle j_{\alpha}^2 \rangle = \left( \frac{e}{\hbar} \right)^2 \sum_{\mathbf{k}\sigma} \left( \frac{\partial \varepsilon_{\mathbf{k}}^{\sigma}}{\partial k_{\alpha}} \right)^2 n_{\mathbf{k}}^{\sigma} (1 - n_{\mathbf{k}}^{\sigma}), \quad (16.11)$$

where  $n_{\mathbf{k}}^{\sigma}$  are the Fermi distribution functions.

The main difficulty in the further calculations is the

\*In connection with this question, we call the reader's attention to the review<sup>[76]</sup> and also to the original papers.<sup>[59,77,78]</sup>



determination of the quantities  $S_j^\pm(\tau)$  and the averaging of their products. This has not been possible to do so far in general form, but two limiting cases can be considered.

a) Low temperatures. In this region a spin-wave approximation, by which it is possible to calculate these quantities, is feasible and calculation of the averages of the spin operators leads to a Bose distribution function for the spin waves. Recognizing that the spin operators can be expressed in terms of Bose operators for the creation and annihilation of spin waves  $b_{\mathbf{q}}$  and  $b_{\mathbf{q}}^\dagger$  by means of formulas (9.4), and that in the Heisenberg representation with the Hamiltonian (16.1) we have

$$b_{\mathbf{q}}(t) = e^{-\frac{i}{\hbar} \epsilon_{\mathbf{q}} t} b_{\mathbf{q}}, \quad b_{\mathbf{q}}^\dagger(t) = e^{\frac{i}{\hbar} \epsilon_{\mathbf{q}} t} b_{\mathbf{q}}^\dagger,$$

where  $\epsilon_{\mathbf{q}}$  is the spin-wave energy, we obtain in place of (16.10)

$$\begin{aligned} \langle [j_\alpha, H'(\tau)] [H', j_\alpha] \rangle &= \left( \frac{e J_0}{\hbar} \right)^2 \frac{2s}{N} \sum_{\mathbf{k}\mathbf{q}} \left[ \frac{\partial \epsilon_{\mathbf{k}}^-}{\partial k_\alpha} - \frac{\partial \epsilon_{\mathbf{k}-\mathbf{q}}^+}{\partial (k-q)_\alpha} \right]^2 \\ &\times \{ n_{\mathbf{q}} n_{\mathbf{k}-\mathbf{q}}^- (1 - n_{\mathbf{k}}^-) e^{i\Omega_{\mathbf{k}\mathbf{q}} \tau} + (1 + n_{\mathbf{q}}) n_{\mathbf{k}}^- (1 - n_{\mathbf{k}-\mathbf{q}}^+) e^{-i\Omega_{\mathbf{k}\mathbf{q}} \tau} \}, \end{aligned} \quad (16.12)$$

where

$$\Omega_{\mathbf{k}\mathbf{q}} = \frac{1}{\hbar} (\epsilon_{\mathbf{k}-\mathbf{q}}^+ - \epsilon_{\mathbf{k}}^- + \epsilon_{\mathbf{q}}).$$

We now substitute (16.12) into (16.5) and use the identity

$$\int_0^\infty e^{i\Omega \tau} d\tau = \pi \delta(\Omega) + \frac{i}{\Omega}, \quad (16.13)$$

obtaining

$$\begin{aligned} \rho &= \frac{\pi k T}{\hbar (j_\alpha^2)^2} \left( \frac{e}{\hbar} J_0 \right)^2 \frac{2s}{N} \sum_{\mathbf{k}\mathbf{q}} \left[ \frac{\partial \epsilon_{\mathbf{k}}^-}{\partial k_\alpha} - \frac{\partial \epsilon_{\mathbf{k}-\mathbf{q}}^+}{\partial (k-q)_\alpha} \right]^2 \{ n_{\mathbf{q}} n_{\mathbf{k}-\mathbf{q}}^+ (1 - n_{\mathbf{k}}^-) \\ &+ (1 + n_{\mathbf{q}}) n_{\mathbf{k}}^- (1 - n_{\mathbf{k}-\mathbf{q}}^+) \} \delta(\epsilon_{\mathbf{k}-\mathbf{q}}^+ - \epsilon_{\mathbf{k}}^- + \epsilon_{\mathbf{q}}). \end{aligned} \quad (16.14)$$

In the isotropic case  $\rho$  will not depend on  $\alpha$ . The sums over the quasi-momenta contained in these equations can be replaced by integrals. After all the calculations, which follow the standard procedure, we arrive at the following temperature dependence of the electric resistivity<sup>[55]</sup>:

$$\rho = C_1 \left( T_0 \ln \operatorname{cth} \frac{T_0}{2T} \right) T + C_2 \left( \int_{T_0/T}^\infty \frac{x dx}{\operatorname{sh} x} \right) T^2. \quad (16.15)^*$$

The temperature-independent coefficients  $C_1$  and  $C_2$  represent in the general case very cumbersome expressions, which depend on the functions  $\epsilon_{\mathbf{k}}^\sigma$  and  $J(\mathbf{k}\mathbf{k}')$  and their derivatives on the Fermi boundary. We shall not write out these expressions, but note only a characteristic feature of the coefficient  $C_1$ . The value of  $C_1$  differs from zero if the dispersion law for  $\epsilon_{\mathbf{k}}^\sigma$  is not quadratic. In the case of a quadratic

dispersion law we have  $C_1 \neq 0$  only if  $[dJ(\mathbf{k}\mathbf{k}')/dk]_{k_0} = 0$ .

The parameter  $T_0$  [see (11.4)] is some critical temperature, characterized by the fact that scattering processes of this type become ineffective below this temperature (as a result of the impossibility of satisfying simultaneously the conservation and momentum laws in collisions between electrons and spin waves) and the corresponding part of the electric conductivity vanishes exponentially when  $T \ll T_0$ .

For temperatures  $T \gg T_0$  the integral in (16.15) can be readily expanded in powers of the small parameter  $T_0/T$ :

$$\int_{T_0/T}^\infty \frac{x dx}{\operatorname{sh} x} = \frac{\pi^2}{4} - \frac{T_0}{T} + \dots$$

Expanding  $\operatorname{coth}(T_0/2T)$ , too, we obtain for the temperature dependence of the electric resistivity

$$\rho = C_1 \left( T_0 \ln \frac{2T}{T_0} \right) T - C_2 T_0 T + \frac{\pi^2}{4} C_2 T^2. \quad (16.16)$$

$\rho$  decreases exponentially with the temperature when  $T \gg T_0$  so that  $\rho \sim \exp[-T_0/T]$ . In the intermediate temperature region it is necessary to use the general formula (16.15).

The case of greatest interest is  $T \gg T_0$ . If the coefficients  $C_1$  and  $C_2$  have the same order of magnitude, then the s-d exchange interaction when  $T \gg T_0$  will yield essentially a quadratic dependence of  $\rho$  on  $T$ . The linear term is comparable with the quadratic only when the ratio  $C_1/C_2$  is sufficiently large. The latter can occur only when the dispersion law for  $\epsilon_{\mathbf{k}}$  deviates quite strongly from a quadratic one. In the case of a quadratic dispersion law  $C_1$  should not exceed  $C_2$  in value.

Thus, the principal term in the electric resistivity of ferromagnets at low temperature is  $\sim T^2$ . This term was first derived in the 1954 paper by Turov<sup>[56]</sup>. Later on Turov's calculation was confirmed in different versions in<sup>[57,58]</sup>.

As is well known, scattering of conduction electrons by phonons leads to a temperature dependence  $\rho \sim T^5$  at low temperatures. We see therefore that in this temperature region the contribution to the resistivity from the s-d exchange should predominate for ferromagnetic metals, and therefore such metals should obey, in accordance with the theory developed, a temperature law close to  $\sim T^2$ .

Experimental investigations of the temperature dependence for the region of helium temperatures in ferromagnetic metals, carried out in<sup>[59]</sup>, are in qualitative agreement with the theory. It must be noted, however, that it is impossible for the time being to separate the temperature term from the total resistivity corresponding to the exchange scattering of the electrons by spin waves on the basis of the experimental data, inasmuch as there exist other mechanisms of scattering in transition metals, which also

\* $\operatorname{cth} = \operatorname{coth}$ ,  $\operatorname{sh} = \operatorname{sinh}$ .

lead to a quadratic temperature term in the resistivity (see Secs. 2 and 5). For a more detailed analysis of the phenomena considered it is necessary to have additional experimental research on the temperature dependence of the electric resistivity, and also on other kinetic coefficients. Therefore, naturally, it cannot be regarded that the aforementioned comparison with the theory is a final confirmation of the correctness of the obtained temperature variation of the contribution made by the electron-ferromagnon interaction to the electric resistivity of ferromagnets.

b) **High temperatures.** Let us consider the range of temperatures close to the Curie temperature. As is well known (see Sec. 5) in this region one can employ for a description of the spin-system states the method of molecular field, which consists in replacing the exchange interaction in the spin system by an effective molecular field  $\lambda$  acting on the given spin, so that  $H_I \rightarrow -\lambda S^Z$  (the constant  $\lambda$  can be readily expressed in terms of the Curie temperature). In this approximation

$$\begin{aligned} S_j^\pm(\tau) &= e^{-\frac{i}{\hbar} \lambda S^Z \tau} S_j^\pm e^{\frac{i}{\hbar} \lambda S^Z \tau} = e^{\mp \frac{i}{\hbar} \lambda \tau} S^\pm, \\ \langle S_j^\pm(\tau) S_j^\mp \rangle &= e^{\mp \frac{i}{\hbar} \lambda \tau} \langle S^\pm S^\mp \rangle \delta_{jj'}. \end{aligned} \quad (16.17)$$

Here, for example,

$$\begin{aligned} \langle S^- S^+ \rangle &= \langle (s - S^Z)(s + S^Z + 1) \rangle = \sum_{m=-s}^{+s} W_m (s - m)(s + m + 1), \\ W_m &= \frac{e^{\beta \lambda m}}{\sum_{m=-s}^{+s} e^{\beta \lambda m}}. \end{aligned} \quad (16.18)$$

$W_m$  is the probability that a given spin has a projection  $m$  on the direction of the molecular field.

Substituting the approximate relations (16.17) in (16.10) and integrating with respect to  $\tau$  from zero to infinity we obtain

$$\begin{aligned} \text{Re} \int_0^\infty \langle [j_\alpha, H'(\tau)] [H', j_\alpha] \rangle d\tau \\ = \frac{\pi \hbar}{N} \left( \frac{e}{\hbar} \right)^2 \sum_{\mathbf{k}\mathbf{k}'} J^2(\mathbf{k}\mathbf{k}') \left[ \frac{\partial e_{\mathbf{k}}}{\partial k_\alpha} - \frac{\partial e_{\mathbf{k}'}}{\partial k'_\alpha} \right]^2 \{ n_{\mathbf{k}} (1 - n_{\mathbf{k}'}) \langle S^- S^+ \rangle \\ + n_{\mathbf{k}'} (1 - n_{\mathbf{k}}) \langle S^+ S^- \rangle \} \delta(e_{\mathbf{k}} - e_{\mathbf{k}'} + \lambda). \end{aligned} \quad (16.19)$$

It is easy to show that

$$\langle S^+ S^- \rangle = \langle S^- S^+ \rangle e^{\beta \lambda}. \quad (16.20)$$

With the aid of this relation and with account of the  $\delta$ -function under the summation sign, (16.19) simplifies to

$$\begin{aligned} \text{Re} \int_0^\infty \langle [j_\alpha, H'(\tau)] [H', j_\alpha] \rangle d\tau = \frac{2\pi e^2}{N\hbar} \sum_{\mathbf{k}\mathbf{k}'} J^2(\mathbf{k}\mathbf{k}') \left[ \frac{\partial e_{\mathbf{k}}}{\partial k_\alpha} - \frac{\partial e_{\mathbf{k}'}}{\partial k'_\alpha} \right]^2 \\ \times n_{\mathbf{k}} (1 - n_{\mathbf{k}'}) \delta(e_{\mathbf{k}} - e_{\mathbf{k}'} + \lambda) \langle (s - S^Z)(s + S^Z + 1) \rangle. \end{aligned} \quad (16.21)$$

Calculating the sums over the quasi-momenta in the approximation of quadratic dispersion for the conduc-

tion electrons, we obtain on the basis of (16.21) and (16.5) the following formula for the electric resistivity:

$$\rho = \frac{3\pi}{4} \frac{J_0^2 m V}{e^2 \hbar \zeta N} \langle (s - S^Z)(s + S^Z + 1) \rangle \frac{\beta \lambda}{e^{\beta \lambda} - 1}. \quad (16.22)$$

Neglecting the dispersion of the spin variable, i.e., assuming that  $\langle (S^Z)^2 \rangle \approx \langle S^Z \rangle^2$ , we can replace the average of the spin operators in the last formula by

$$(s - \langle S^Z \rangle)(s + \langle S^Z \rangle + 1). \quad (16.23)$$

Inasmuch as  $\lambda \rightarrow 0$  near the Curie point, the last factor in (16.22) is practically equal to unity, and therefore the temperature of the resistivity is determined for this temperature interval by the factor (16.23). It is clear therefore that near the Curie point the resistance due to the s-d exchange increases sharply, whereas above the Curie point, when  $\langle S^Z \rangle = 0$ , it is constant and equal to

$$\rho = \frac{3\pi}{4} \frac{J_0^2 m V}{e^2 \hbar \zeta N} s(s + 1). \quad (16.24)$$

In the ferromagnetic region, away from the Curie point, it is small because  $\langle S^Z \rangle \approx s$  and vanishes at the temperature of absolute zero, although generally speaking in this temperature region formula (16.22) is not applicable. Thus, qualitatively formula (16.22) describes correctly the anomalous course of the additional electric resistivity in ferromagnets. The numerical value of the resistivity is also in satisfactory agreement with the experimental data. Under ordinary estimates:  $m \sim 10^{-27}$  g,  $\zeta \sim 10^{-12}$  erg,  $N/V \sim 10^{22}$  cm<sup>-3</sup>, and  $J_0 \sim 10^{-14}$  erg we have near the Curie point and in the paramagnetic region  $\rho_p \sim 10^{-18}$  sec.

We note that formula (16.24) reflects the tendency observed in transition metals towards an increase in the additional electric resistivity of the metal with increasing value of the spin  $s$  of the unfilled shell.

The question of additional electric resistivity of ferromagnetic and antiferromagnetic metals is considered, both from the theoretical and from the experimental point of view, in a review by Coles<sup>[76]</sup>.

## V. CONCLUSION

The foregoing review of the present status of the experimental and theoretical study of the properties of the transition metals enables us to state that this branch of solid state physics is a wide field for scientific research, since we are far from having complete knowledge of the main features of the electronic structure of these substances. Our principal attention was paid to the exposition of the principles and applications of the s-d (f) exchange model of transition metals (see Secs. 5-16), based on the assumption that it is possible to divide in the crystals

the electron system of the outer atomic shell into two subshells: the conduction electrons and the electrons of the former unfilled d or f shells. With this we emphasize not only the very existence of these two subshells, but indicate the presence of two branches in the spectrum of the single system of electrons of the transition-metal crystal. A favorable aspect of this model is that it makes it possible to describe in the most consistent and general form several most important specific features of the electron system of the transition-metal crystal: 1) establish the singularities in the structure of the Fermi surface of the conduction-electron system, connected with the lifting of the spin degeneracy in ferromagnetic crystals or with the appearance of an additional energy gap in antiferromagnetic metals; these singularities of the energy spectrum made it possible to understand in principle the entire specific nature of the equilibrium statistical properties and kinetic coefficients of these substances; 2) take into account the active participation of the conduction electrons in the indirect exchange phenomenon, particularly in the case of f metals and dilute solid solutions of transition metals in diamagnetic solvents.

It must be noted that with the aid of new quantum-statistical methods (the theory of the Fermi liquid<sup>[61]</sup>, temperature Green's functions<sup>[14,17,18,22,60]</sup>, diagram techniques<sup>[63]</sup> etc., and also the phenomenological treatment of the s-d(f) exchange, see Sec. 5), it was possible to demonstrate the correctness not only of its general physical premises (the presence of collective motions of both types), but the most important quantitative deductions (the displacement of the Fermi surfaces for conduction electrons with different spin projections, the expression for the indirect-exchange parameters and others, see Chapter III and IV).

The weakest feature of the s-d(f) model is that it does not take practical account of the collectivization of the electrons of the former inner unfilled shells and their participation in the transport of electron charge in the crystal. This model is therefore more applicable to the case of the f metals and their alloys, and also to the case of ferromagnetic and antiferromagnetic semiconductors, in which the effect of collectivization of the f or d electrons is minimal.\*

The treatment considered likewise does not take account at all of the real distribution of the electron charge and spin densities,  $\rho(\mathbf{r})$  and  $\sigma(\mathbf{r})$ , in the crystal (see Sec. 5). Therefore it is impossible to take into account within the framework of the s-d(f) model the actual form of the Fermi surface and consequently impossible to treat all the effects that are typical of Fermi systems, or to determine the x-ray and neutron (magnetic) form factors. The model is perfectly

incapable of determining with any degree of accuracy the value of the average atomic magnetic moments (including estimates of the contribution made by the effect of magnetization of the conduction electrons), particularly in the case of d metals and alloys. In exactly the same way, no account is being taken at all of the role of the magnitude and symmetry of the internal crystalline (electric and magnetic) field in the distribution of the electron and spin density of the d and f electrons, and also of the inner electrons of the ionic cores. In this connection, the question that remains completely open in the theory is the relation between the spin and orbital momenta of the d and f shells.\* At the same time, as shown by the latest experimental researches on the internal effects of fields in ferromagnetic and antiferromagnetic crystals (with the aid of the Mossbauer effect, nuclear magnetic resonance, see Sec. 5), they are subject to a considerable influence of the state of the outer shells of the electronic system on the internal electronic shell of the ionic cores of the crystal.

In connection with the foregoing principal shortcomings of the s-d(f) model, we can formulate the following principal problems in the physics of transition metals, their alloys, and components, which at the present time need to be solved experimentally and theoretically.

a) It is necessary to determine with very high precision the charge and spin electron densities  $\rho(\mathbf{r})$  and  $\sigma(\mathbf{r})$  in the crystals of the investigated substances. For this purpose it is necessary first to improve the experimental methods for the determination of the x-ray and neutron form factors, and also develop further more accurate theoretical computation methods for their theoretical analysis (development of the Hartree-Fock method, calculation of scattering cross sections of polarized neutrons, etc.). It is simultaneously necessary to take into consideration the influence of the symmetry and magnitude of the crystalline field, to take into account effects of crystalline splitting (Stark and Zeeman) of the internal electronic levels, etc. It must be noted that although recently certain papers were published on this subject (see, for example, research on the calculation of the form factor<sup>[64]</sup>), questions involving the accuracy of the experimental measurements, particularly the theoretical treatment of the experimental data, are far from being complete (it is sufficient to mention the "discussion" in connection with the treatment of the results of the experiments in the paper by Weiss and De Marco<sup>[65]</sup>). At the same time, the determination of the correct picture of the electron density in a crystal is the most essential premise for the con-

\*In connection with the problem of ferromagnetic and antiferromagnetic as well as paramagnetic alloys of transition metals, we call the reader's attention to the interesting article by Lomer<sup>[63]</sup>.

\*As follows from certain theoretical papers<sup>[62]</sup>, even crude model representations of the form of the function  $\rho(\mathbf{r})$  make it possible to obtain a qualitative explanation of some physical properties of transition metals and alloys.

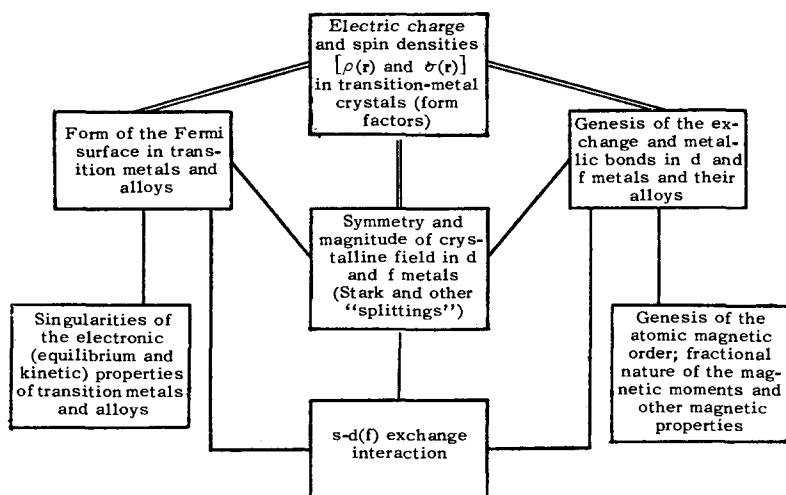


Diagram of principal "problems" in the theory of transition metals.

struction of a consistent theory of transition metals and their alloys, the only one that can disclose the still unknown potential promises of their practical utilization.

b) The solution of the problem of the "geometrical" distribution of the electrons in the investigated crystals must be closely related with the question of their distribution in the quasi momentum space, i.e., with problems in the determination of the shape of the Fermi surface in crystals of transition metals and alloys for a complicated dispersion law for the conduction electrons, determination of the anomalously high values of the electron density near the Fermi surface, etc.<sup>[66,67]</sup>. All these problems have so far not been touched upon at all and not treated in the s-d(f) exchange model\*. At the same time, a study of the conduction-electron subsystem in a metal and alloy without solving these fundamental problems cannot be regarded as anywhere near complete. This problem (of the Fermi surface) is of particular interest for d metals which have no atomic magnetic ordering (primarily for the start of the series of 3d metals and the series of 4d and 5d metals).

c) The third most important problem is the solution of the problem of the nature of the exchange and metallic bonds in transition metals, alloys, and components. Here, likewise, very little is known as yet. Essentially, we know only in principle what is the general (electrostatic) nature of the exchange interaction and what can be said concerning the direct and indirect exchange interactions. So far there is no rigorous quantitative theory of the exchange bond in crystals. It is likewise not clear, for example, what the genesis of this bond is in d metals (see, for example, the calculation of Marshall and Stuart<sup>[68]</sup> and

also the work of Freeman and Watson<sup>[81]</sup>). It is precisely for this reason that we still cannot answer the question why the greater part of the d metals are paramagnetic, and a smaller part is ferromagnetic or antiferromagnetic, while the f metals, to the contrary, are all ferromagnetic or antiferromagnetic. Naturally, this third fundamental problem concerning the genesis and magnitude of the exchange interaction is in close relationship with the first two, for only when we know exactly the distribution of the charge and spin densities can we raise the question of the calculation of the energetic exchange parameters. In particular, knowledge of the Fermi surfaces will resolve the question of the competition between the exchange interaction and the kinetic energy in the system of collectivized electrons<sup>[69-72,87]</sup>. Closely connected with the problem of the genesis of the exchange bond is the question of the nature of the bonding forces in transition metals and alloys, which determine their predominant place among many strong structural materials. Here again the solution of all three foregoing questions must be a necessary stage in the development of a consistent theory.

Naturally, a solution of these fundamental problems and all the particular specific problems that follow from them involving the explanation of the known and predicted properties (see the diagram) should follow parallel paths both in experiment and in theory. In the experimental field the leading role should be assumed by purposeful model experiments, and not merely accumulation of facts. One such important method will be the method for studying highly dilute solid solutions of transition elements, when the lattice of the diamagnetic solvent contains a small number of practically non-interacting paramagnetic d or f ions. Detailed studies of the electron density (charge and spin), of the average magnetic moment, of the exchange bond, of the effective fields acting on the electron shell and on the nucleus, etc.—all these are essential elements without which one cannot solve the main theoretical problems raised above. An important

\*Vonsovskii and Kobelev<sup>[82]</sup> made the first attempt to solve with the aid of the Green's function method the more general problem concerning the magnetic properties of two subsystems of interacting electrons, without making any assumptions beforehand concerning their spatial localization or delocalization.

element in such model experiments is also the investigation of ordered alloys with participation of transition elements, and also the study of the influence of various imperfections (where possible isolated ones) in the structure (vacancies, interstitial and substitutional atoms, dislocations, etc.) on various properties of transition metals and alloys.

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