PEAKS OF RANDOM PROCESSES

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INTRODUCTION

AT the present time the study of random processes is not only of profound physical interest (Brownian motion, radioactive decay, accuracy limit of measuring instruments, propagation of electromagnetic waves through the turbulent atmosphere, etc.), but also of appreciable practical significance. This is explained by the fact that further improvement in many physical devices by perfection of their construction and manufacturing technology has its own limit, determined by purely physical factors—fluctuations, so that it becomes necessary to search for principally new solutions. This state has been reached, for example, in many branches of modern radiophysics (radio astronomy, radar, radio communication, radio spectroscopy, etc.).

It is therefore natural that since the first papers by A. Einstein and M. Smoluchowski [1,2] on the theory of Brownian motion, more and more attention has been paid to random processes, particularly during the last 20 years, owing to the large technical progress in the field of radiophysics.

In the present article we consider among the large group of problems in the theory of random processes only a few special problems, which can be grouped under the heading "Peaks of Random Processes." Although not all of these problems have been completely solved analytically, the available literature and the experimental work performed do allow us to cite some general results, which can be useful to persons who are not specially engaged in problems of peaks.

Before we proceed to an exposition of the factual material, we shall present the fundamental definitions and indicate the practical significance of the individual problems and the main content of the article.

Figure 1 shows the realization of a stationary random process $\xi(t)$ of duration T. All physical real random processes are continuous functions of the time. Such a function has over a finite interval T a finite number of maxima and minima with different values

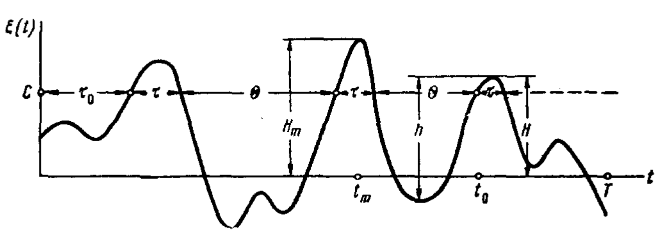


FIG. 1. Realization of a stationary random process.

of H, and at the instant t_m the realization has a largest maximum H_m. We denote the differences in height between a minimum and a neighboring maximum by h.

The realization $\xi(t)$ crosses N times (in Fig. 1, three times) a certain fixed level C in an upward direction (with positive derivative) and the first such crossing occurs at the instant τ_0.

We agree to call a positive peak an event in which a random process $\xi(t)$ crosses the level C in the upward direction. We can then say that the realization $\xi(t)$ has N peaks, and the values of τ and Θ indicated in the figure can be called the durations of the positive peaks and the intervals between peaks, respectively.

Within a single realization, the values of τ, Θ, H, and h can assume several values (depending on the level C and the interval T), and they vary, together with the quantities N, τ_0, and H_m, in random fashion from one realization to another.

In what follows, we shall consider the following problems for the specific forms of stationary random processes $\xi(t)$ most frequently encountered in radiophysics:

1) The quantities characterizing the distribution of the number of peaks N in an ensemble of realizations of different durations are determined for several values of the level C;

2) The distributions of the random quantities τ_0, τ, Θ, H_m, H, and h are given for an ensemble of realizations at several values of the duration of the interval T.

In addition to the fact that the quantities N, τ_0, τ, Θ, H_m, H, and h are of independent interest, being detailed characteristics of the random process $\xi(t)$, knowledge of these quantities is also essential in the solution of many practical problems. We cite several specific examples from the field of radiophysics, mechanics, reliability theory, and biophysics, although these examples do not cover the entire region of applicability of the results.

In various radio devices frequent use is made of electronic relays and flipflops. They are used in digital computers and in radar and radio communication apparatus where the information is displayed, in dosimetric instruments for the measurement of the intensity of radioactivity in instruments for precision measurement of small time intervals and frequencies of periodic oscillations, in various systems for information coding and decoding, in pulsed synchronization systems, and in other automation devices.

If a useful signal is accompanied by fluctuating noise, it becomes necessary to analyze the effect produced on the relay by the useful pulse signals together with the fluctuations. The influence of the latter on the operation of the relay depends on the ratio of the "threshold" operating voltage of the relay to the noise intensity.

If the noise level is low compared with the "threshold" voltage, the low-probability false operations of the relay can be neglected. The noise will cause fluctuations in both the instant of operation of the relay and the instant when operation terminates; the duration of the pulse generated by the electronic relay will be subject to certain fluctuations [3].

When the noise level is comparable with or exceeds the threshold voltage, false operations will occur, causing errors in the operation of the corresponding apparatus [4-7].

If it is permissible to regard the relay as a practically inertialess device, then the number of false operations will be determined by the number N of the peaks of fluctuation noise that exceed the operating threshold of the relay. To determine the number of false operations when the inertial properties of the relay are taken into account, it is necessary to know not only N but also the probability densities for τ and Θ [8]. Besides, the operation of an inertial relay is itself a problem highly typical of the theory of queuing.[9]

The quantities N, Θ, and H are under certain conditions important characteristics of fading of radio transmission [10-12]. For short-wave radio communication lines shorter than 200 km the electromagnetic field at the point of reception is made up by the atmospheric wave, reflected principally from the ionized F_2 layer. The ionization density of this layer is inhomogeneous and varies randomly with time. A harmonic wave incident on such an inhomogeneous medium splits after reflection into a series of elementary beams, which arrive at the point of reception with different intensities and phases. The received oscillation represents a narrow-band random process. In such a process N characterizes the frequency of the fading, Θ characterizes the length of the fadings below a definite threshold, and H represents the depth of the fading.

The probability density for the largest values of H_m in the realization of a random process of specified duration must be known in order to determine the region of applicability of one of the most important statistical estimating criteria, namely the maximum likelihood method [13].

In estimating the unknown parameters of a signal received together with fluctuation noise, the maximum likelihood method is frequently used [7]. The true value is assumed to be that value of the parameter, at which the likelihood function has the largest maximum over a specified time interval. However, at small signal/noise ratios this largest maximum can be due to peaks of noise and may be located a considerable distance away from the true value of the parameter. For such signal/noise ratios the maximum likelihood method results in unacceptable errors and becomes of no value.

Thus, a determination of the region of applicability of the maximum likelihood method is connected with a determination of the value of that signal/noise ratio at which a relatively small fraction of a sufficiently large number of realizations of fluctuation noise of fixed duration T contain peaks comparable in height with the peaks due to the signal. A complete solution of this problem includes as an essential stage the calculation of the probability density for the quantity H_m.

Knowledge of the distributions of H and H_m is necessary in radar detection of marine targets against the background of reflections from the billowing surface of the sea [14], in the analysis of the interference immunity of extremal regulation systems, [15] and in other problems. We note incidentally that when an extremal regulation system operates in the presence of fluctuation noise, it is necessary to know the probability density of the quantity h in order to determine the optimal step for the trial motions. [15,16]

A point of view recently adopted in mechanics is that if some material is under the influence of random loads, its rated strength must be based on the average number of times that the load exceeds a specified value (number of peaks) per unit time. It was established at the same time that a random load with the same average number of peaks as a harmonic one is the more dangerous one. Therefore, for example in calculating the strength of the wings of an airplane subject to random atmospheric turbulence, it is necessary to know the number of times that the bending and torsion moments exceed a specified level per unit time [17]. An analogous remark can be made with respect to the design of dams subject to random wave loads.

The peak parameters indicated above can be used in practice for a quantitative estimate of the microroughness of a finished surface. Individual parameters characterize the micro-unevennesses of the profile of the surface on different sides. In practice, of course, the tendency is to obtain as much information as possible with the aid of a minimum number of parameters.

It was established [18] that for certain types of finish (for example, grinding) it is possible to consider the curve of the surface profile, representing the dependence of the height of the irregularities on the abscissa of the profile, as a stationary normal random process. In this case a sufficiently complete description of the profile is obtained with the aid of three parameters: the average number that the profile curve crosses two levels C_1 and C_2, and the number of the maxima in the section under consideration.

We point out that most seismic instruments and medical instruments for the registration of biocurrents of the heart and the brain are based on the measurement of the height and duration of peaks and the intervals between them [19].

It is apparently possible to reduce to an investigation of peaks of random processes also certain problems in the theory of reliability of operation of complicated apparatus (in particular, radio apparatus) containing a large number of elements. In this case N can characterize the number of failures of the elements (apparatus) during the time T, τ_0 is the instant of occurrence of the first failure, τ is the time allotted to the elimination of the corresponding fault or the time allotted to the activation of a spare element which is turned on "cold," and Θ is the time of fault-free operation after the elimination of the last fault in the sequence. In practice one is usually interested in average values of these quantities [20]. The question of which random process $\xi(t)$ should be specified in this case must be solved on the basis of a statistical analysis of extensive experimental material pertaining to the specific apparatus.

From the examples given above we can gain an idea of the exhaustive scientific and applied significance of research on peaks of random processes. This research was initiated with a basic theoretical paper by S. O. Rice [21] in 1945, in which formulas were derived for certain types of random processes for the average number of peaks and the distribution of the maxima, and one approximate method of determining the probability density for the peak durations was also pointed out (see Sec. 3).

In subsequent years, peaks of random processes were considered in many theoretical and experimental investigations, the main contents of which will be indicated during the discussion of particular problems.

We note that some particular problems on peaks have received no analytic solution to this very day (for example, the distributions for the quantities τ_0 and h), while solutions of other problems suitable for practical use are quite approximate (the probability density for the quantities H_m, τ, and Θ). In view of this, we are paying proper attention in this paper to the experimental results.

The aim of the present article is to describe in an understandable form the main theoretical and experimental results on peaks of random processes. Inasmuch as the experimental results have been obtained for particular forms of noise, we present below their main characteristics.

1. MAIN CHARACTERISTICS OF THE INVESTIGATED FLUCTUATION NOISE

A very laborious experimental investigation, in which the statistical characteristics of the random quantities N, τ_0, τ, Θ, H_n, H, and h were determined, was carried out on several particular forms of stationary fluctuating processes, which must most frequently be taken into account because they interfere with the operation of radio systems and automation devices.*

1. Normal stationary fluctuation noise $\xi(t)$, whose one-dimensional probability density is determined by the formula

$$w(\xi)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{\xi^2}{2\sigma^2}}. \tag{1}$$

Three types of normal noise were used here, having approximately the following spectral density and the corresponding correlation functions

$$\left.\begin{aligned}S(f)=S_0\exp\left[-0.7\left(\frac{f}{\Delta f}\right)^2\right],\quad k(\tau)=S_0\Delta f\exp\left[-14.1\,(\tau\Delta f)^2\right],\\ \Delta f=14.6\ \text{kc},\end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned}S(f)=S_0\left[1+0.41\left(\frac{f}{\Delta f}\right)^2\right]^{-2},\quad k(\tau)=1{,}3S_0\Delta f\,(1+10\Delta f\,|\tau|)\,e^{-10\Delta f|\tau|},\\ \Delta f=21\ \text{kc},\end{aligned}\right\} \tag{3}$$

$$S(f)=\begin{cases}S_0 & 0\leqslant f\leqslant\Delta f,\\ 0 & f<0,\ f>\Delta f,\end{cases}\quad k(\tau)=S_0\Delta f\,\frac{\sin 2\pi\Delta f\tau}{2\pi\Delta f\tau},\ \Delta f=15\ \text{kc}. \tag{4}$$

Here S_0 is the spectral density at zero frequency and Δf is the width of the spectral density at the level $0.5S_0$.

Such noise was obtained from the outputs of low-frequency amplifiers with suitable amplitude-phase characteristics, with inputs in the form of broadband fluctuation noise from a thyratron placed in a magnetic field.

It must be noted that in practice it is impossible to obtain noise with spectral densities that can be sufficiently well approximated by curves (2)—(4). The real spectral densities, especially outside the band Δf, differed noticeably from the approximation curves. This circumstance is one of the essential reasons for the discrepancy between the theoretical and experimental results.

Normal random processes are encountered most frequently in practice and therefore occupy a special position among other random processes.

The majority of random processes encountered in practice, such as shot noise in vacuum tubes, thermal fluctuations, the internal noise of a typical radio re-

*We shall henceforth use as synonyms for the term "random process" also "fluctuation process," "fluctuation noise," or simply "noise."

ceiver ahead of the detector, atmospheric interference, atmospheric turbulence, noise of cosmic origin, and others are essentially a resultant effect (sum) of a large number of relatively small independent (or weakly dependent) elementary pulses which arise at random instants of time.

In accordance with the central limit theorem of probability theory, the probability density of a sum approaches without limit the normal density with increasing number of components, regardless of what the probability densities of the individual components may be.

Normal processes have the property of "stability" with respect to linear transformations, i.e., if a normal random process acts at the input of a linear system, a normal process is obtained also at the output of the system. Moreover, if a non-normal broadband random process acts on an inertial (narrow-band) system, then the process of the output of such a system approaches a normal one.

Let us indicate briefly the procedure for obtaining other types of fluctuation noise. It is known[21-24] that if a normal stationary noise has a spectral density S(f) which is symmetrical with respect to a certain frequency f_0, and the width of the spectral density Δf (say, at the 0.5 level relative to the value at the frequency f_0) is much less than f_0, then the noise recalls a quasi-harmonic oscillation in form. Corresponding to this, such a quasi-harmonic noise $\xi(t)$ can be represented in the form of a harmonic signal which is randomly modulated in amplitude and in phase

$$\xi(t) = A(t)\cos[2\pi f_0 t + \varphi(t)], \tag{5}$$

where A(t) and p(t) are slowly varying functions compared with $\cos\omega_0 t$. The random function A(t) can be called the envelope (amplitude) of the fluctuations, and the function $\varphi(t)$ the random phase of the fluctuations.

The correlation function of the quasi-harmonic noise (5) has the form

$$k(\tau) = \sigma^2\varrho(\tau)\cos\omega_0\tau,$$

where σ^2—the variance of the noise $\xi(t)$ and $\rho(\tau)$ is a slowly varying function compared with $\cos\omega_0\tau$.

The sum of the noise (5) and a harmonic signal $s(t) = A_m\cos[2\pi f_0 t + \varphi_0]$

$$\eta(t) = \xi(t) + s(t)$$

can be represented in analogous form

$$\eta(t) = E(t)\cos[2\pi f_0 t + \psi(t)]. \tag{6}$$

If certain conditions are satisfied, we can separate the envelopes A(t) and E(t) with a linear amplitude detector, the random frequencies $\dot{\varphi}(t)$ and $\dot{\psi}(t)$ with a frequency detector, and $\cos\varphi(t)$ and $\cos\psi(t)$ with a phase detector.

Several practical examples can be mentioned, in which the random processes indicated here are encountered. Thus, in radio communication equipment intended for the reception of amplitude, frequency, and phase modulated signals the random processes separated at the output, in the absence of a signal but with account of the internal noise of the receiver and the external fluctuation noise, are A(t), $\dot{\varphi}(t)$ and $\cos\varphi(t)$ respectively, while in the presence of a signal the outputs are the random processes E(t), $\dot{\psi}(t)$, and $\cos\psi(t)$.

In pulsed radar, owing to the choppy nature of the reflection pattern of the target, the reflected signal is frequently approximated in the form (5), while in the case of long-range radio communication, owing to the turbulent character of the ionization of the reflecting layer, it is approximated in the form (6).

2. The fluctuation processes A(t), and E(t), having the following respective probability densities

$$W(A) = \frac{A}{\sigma^2}\exp\left(-\frac{A^2}{2\sigma^2}\right), \qquad A \geqslant 0, \tag{7}$$

$$W(E) = \frac{E}{\sigma^2}\exp\left(-\frac{E^2+A_m^2}{2\sigma^2}\right)I_0\left(\frac{EA_m}{\sigma^2}\right), \qquad E \geqslant 0, \tag{8}$$

were investigated. Here $I_0(z)$ is the Bessel function of zero order of imaginary argument. In the absence of a signal ($A_m = 0$) formula (8) goes over into (7).

In both cases, the spectral density of the noise $\xi(t)$ had the form of a Gaussian curve

$$S(f) = S_0\exp\left[-2.8\left(\frac{f-f_0}{\Delta f}\right)^2\right], \tag{9}$$

with $f_0 = 12$ Mc and $\Delta f = 100$ kc. According to Khinchin's formula, the correlation function of such a noise is

$$k(\tau) = S_0\Delta f\exp[-3.5(\tau\Delta f)^2]\cos 2\pi f_0\tau. \tag{10}$$

For the fluctuation process E(t), several signal/noise ratios were chosen, namely:

$$a = \frac{A_m}{\sigma} = 1.5;\ 3.0;\ 5.0.$$

3. The statistical characteristics of the values of the peaks indicated above, for several values of signal/noise ratio and of other parameters, were also determined for the random processes $\dot{\varphi}(t)$, $\dot{\psi}(t)$, $\cos\varphi(t)$, and $\cos\psi(t)$. However, for lack of space these results are not presented here. One-dimensional probability densities of these processes and other characteristics are given in [22].

In the investigation of the random frequencies $\dot{\varphi}(t)$ and $\dot{\psi}(t)$, the initial normal fluctuation process $\xi(t)$ had a spectral density of the type (9) with parameters $f_0 = 140$ kc and $\Delta f = 1.5$ kc.

For the random processes $\cos\varphi(t)$ and $\cos\psi(t)$, the spectral density of the normal noise $\xi(t)$ was determined by the square of the resonance curve of a single resonant circuit with resonant frequency f_0

= 50 kc and a bandwidth $\Delta f = 3.8$ kc (at the 0.5 power level).

An experimental determination of the statistical characteristics of the random quantities N, τ_0, τ, Θ, H_m, H, and h was made by statistical processing of a large number of photographs (realizations) of the corresponding random processes. The correspondence of the fluctuation processes with the characteristics indicated above was verified essentially by measuring the spectral density and comparing the theoretical and experimental probability densities, the latter being determined photometrically.[25,26]

2. NUMBER OF PEAKS

We obtain formulas for the average number $\overline{N}(T)$ of the peaks of a stationary random process $\xi(t)$ on the interval T, in excess of a certain level C, and also for the variance $\sigma_N^2(t)$ of the number of peaks.

Formulas for the average number of peaks of normal random processes, and also for the envelopes A(t) and E(t), were first obtained by S. O. Rice[21,22] and were then discussed in greater detail in [27,23]. The relations for the variance of the number of peaks were first obtained in [28,29] and then by S. O. Rice[30].

We shall assume the random function $\xi(t)$ and its derivative $\dot{\xi}(t)$ to be continuous. We also assume that we know the joint probability density $W_2(\xi(t), \dot{\xi}(t))$.

It follows from the continuity that on a small interval Δt, i.e., within the interval $t \le t' \le t + \Delta t$, the function $\xi(t)$ is close to a straight line

$$\xi(t') = \xi(t) + \dot{\xi}(t)(t' - t).$$

Therefore the level C can be crossed not more than once within a sufficiently small Δt.

Thus, there are two possibilities: there will be either no peak or only one peak in the interval Δt. We denote by P_1 the probability of the occurrence of one peak, and by P_0 the probability that there will be not even one peak. Obviously, the average number of peaks in the interval Δt is

$$\overline{N}(\Delta t) = 1 \cdot P_1 + 0 \cdot P_0 = P_1,$$

i.e., it coincides with the probability P_1.

To calculate P_1 we note that the expression

$$dp = W_2(\xi(t), \dot{\xi}(t))\, \Delta\xi \Delta\dot{\xi}, \qquad \xi(t) = C,$$

determines the probability that the function $\xi(t)$, which is close to a straight line, crosses the vertical segment $AB = \Delta\xi$ (Fig. 2), and that at the same time the derivative is contained in the interval between $\dot{\xi}(t)$ and $\dot{\xi}(t) + \Delta\dot{\xi}$.

Let us consider the probability of crossing not the vertical segment $AB = \Delta\xi$, but the horizontal segment $AC = \Delta t$, assuming the derivative $\dot{\xi}(t)$ to be fixed. Obviously, when the derivative $\dot{\xi}(t)$ is fixed the crossing of a horizontal segment of length Δt is equivalent to crossing a vertical segment of length $\Delta\xi = \dot{\xi}(t)\,\Delta t$.

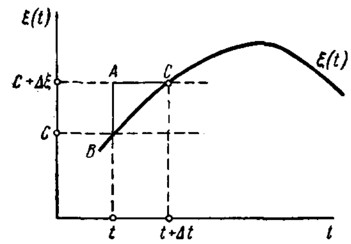


FIG. 2. Illustrating the calculation of the average number of peaks.

Therefore the probability of crossing the segment $AC = \Delta t$ with a derivative ranging between $\dot{\xi}(t)$ and $\dot{\xi}(t) + \Delta\dot{\xi}$ is

$$dp = W_2(\xi(t), \dot{\xi}(t))\, \dot{\xi}(t)\, \Delta\dot{\xi} \Delta t, \qquad \xi(t) = C.$$

The peaks of interest to us (crossings of the level C in the upward direction) will occur for all possible values of the derivative, i.e., when $0 \le \dot{\xi}(t) < \infty$. Therefore the total probability P_1 of crossing the level C in the interval $[t, t+\Delta t]$ is

$$P_1 = \Delta t \int_0^\infty \dot{\xi} W_2(C, \dot{\xi})\, d\dot{\xi}. \tag{11}$$

But the probability P_1 coincides with the average number of peaks occurring over the entire interval $[t, t+\Delta t]$. Dividing both parts of this equation by Δt, we obtain the average number of peaks per unit time within this interval

$$\overline{N}_1 = \int_0^\infty \dot{\xi} W_2(C, \dot{\xi})\, d\dot{\xi}. \tag{12}$$

The average number of peaks on the interval $[0, T]$ is obtained by integrating the right half of formula (11)

$$\overline{N}(T) = \int_0^T dt \int_0^\infty \dot{\xi} W_2(C, \dot{\xi})\, d\dot{\xi}. \tag{13}$$

For stationary processes, the integrand is independent of the time and consequently

$$\overline{N}(T) = \overline{N}_1 T. \tag{14}$$

The formula for the variance of the number of peaks can be obtained in the following manner[30]. We break up the total time interval $[0, T]$ into m equal elementary subintervals of small duration $\Delta t_1 = \Delta t_2 = \ldots = \Delta t_i = \ldots = \Delta t_m = T/m$. By virtue of the proposed continuity of the random function $\xi(t)$ and of its derivative $\dot{\xi}(t)$ the function $\xi(t)$ can have not more than one peak in each elementary subinterval. Let us relate to the i-th subinterval a random quantity δ_i, which assumes two values: $\delta_i = 1$ if the realization $\xi(t)$ has one peak and $\delta_i = 0$ if there is no peak.

Then the number of peaks of the specific realization $\xi(t)$ on the interval $[0, T]$ is

$$N_\xi(T) = \sum_{i=1}^m \delta_i.$$

The average number of peaks for the ensemble of realizations $\xi(t)$ of fixed duration T is obtained by statistical averaging

$$\overline{N}(T) = \overline{N_\xi(T)} = \sum_{i=1}^{m} \overline{\delta_i}.$$

But for each fixed subinterval Δt_i we have

$$\overline{\delta_i} = 1 \cdot P_1 + 0 \cdot P_0 = \overline{N}_1 \Delta t_i.$$

Consequently

$$\overline{N}(T) = \sum_{i=1}^{m} \overline{N}_1 \Delta t_i.$$

Going over now to the limit as $\Delta t_i \to 0$, we obtain formula (13).

The square of the number of peaks, averaged over the ensemble, is

$$\overline{N^2}(T) = \overline{N_\xi^2}(T) = \overline{\left[\sum_{i=1}^{m} \delta_i\right]^2} = \sum_{i=1}^{m} \overline{\delta_i^2} + \sum_{\substack{i,\, j=1 \\ (i \neq j)}}^{m} \overline{\delta_i \delta_j}.$$

As before, for each fixed subinterval Δt_i we have

$$\overline{\delta_i^2} = 0 \cdot 0 \cdot P_0 + 1 \cdot 1 \cdot P_1 = \overline{N}_1 \Delta t_i.$$

Analogously, we can write for the cross terms at fixed i and j, $i \neq j$,

$$\begin{aligned}\overline{\delta_i \delta_j} &= 0 \cdot 0 \cdot P(\delta_i = 0,\ \delta_j = 0) + 0 \cdot 1 \cdot P(\delta_i = 0,\ \delta_j = 1) \\ &+ 1 \cdot 0 \cdot P(\delta_i = 1,\ \delta_j = 0) + 1 \cdot 1 \cdot P(\delta_i = 1,\ \delta_j = 1) \\ &= P(\delta_i = 1,\ \delta_j = 1),\end{aligned}$$

where P denotes the probability of the corresponding event. For example, $P(\delta_i = 1, \delta_j = 1)$ denotes the probability that each of the two different subintervals Δt_i and Δt_j has one peak. By analogy with formula (11), this probability is equal to

$$\begin{aligned}P(\delta_i = 1,\ \delta_j = 1) &= \Delta t_i \Delta t_j \int_0^\infty \int_0^\infty \dot{\xi}(t_i) \dot{\xi}(t_j) W_4 \\ &\times (\xi(t_i), \dot{\xi}(t_i), \xi(t_j), \dot{\xi}(t_j))\, d\dot{\xi}(t_i)\, d\dot{\xi}(t_j)\end{aligned}$$

where we must put $\xi(t_i) = \xi(t_j) = C$.

We can thus write

$$\overline{N^2}(T) = \sum_{i=1}^{m} \overline{N}_1 \Delta t_i + \sum_{\substack{i,\, j=1 \\ (i \neq j)}}^{m} P(\delta_i = 1,\ \delta_j = 1).$$

Going over to the limit as $\Delta t \to 0$, we obtain

$$\begin{aligned}\overline{N^2}(T) &= \overline{N}(T) + \int_0^T \int_0^T dt_1\, dt_2 \int_0^\infty \int_0^\infty \dot{\xi}(t_1) \dot{\xi}(t_2) W_4 \\ &\times (C, \dot{\xi}(t_1), C, \dot{\xi}(t_2))\, d\dot{\xi}(t_1)\, d\dot{\xi}(t_2). \qquad (14')\end{aligned}$$

Using the well known relation

$$\sigma_N^2(T) = \overline{N^2}(T) - \{\overline{N}(T)\}^2,$$

we obtain the final formula for the variance of the number of peaks at the level C:

$$\begin{aligned}\sigma_N^2(T) &= \overline{N}(T) - \{\overline{N}(T)\}^2 + \int_0^T \int_0^T \int_0^\infty \int_0^\infty \dot{\xi}(t_1) \dot{\xi}(t_2) \\ &\times W_4(C, \dot{\xi}(t_1), C, \dot{\xi}(t_2))\, dt_1\, dt_2\, d\dot{\xi}(t_1)\, d\dot{\xi}(t_2). \qquad (15)\end{aligned}$$

As applied to processes that are stationary in the narrow sense, this formula can be simplified somewhat. We introduce the notation

$$F(t_1, t_2) = \int_0^\infty \int_0^\infty \dot{\xi}(t_1) \dot{\xi}(t_2) W_4(C, \dot{\xi}(t_1), C, \dot{\xi}(t_2))\, d\dot{\xi}(t_1)\, d\dot{\xi}(t_2). \qquad (16)$$

For processes that are stationary in the narrow sense, usually the following condition is satisfied

$$F(t_1, t_2) = F(\tau) = F(-\tau), \quad \tau = t_2 - t_1. \qquad (17)$$

Under this condition formula (15), with account of (14), is reduced to the form

$$\sigma_N^2(T) = \overline{N}_1 T - (\overline{N}_1 T)^2 + 2 \int_0^T (T - \tau) F(\tau)\, d\tau. \qquad (18)$$

Whereas the average number of peaks $\overline{N}(T)$ can be relatively easily calculated for many random processes, $\sigma_N^2(P)$ cannot be calculated analytically as a rule [28,30], and it becomes necessary to resort to numerical integration or to use the experimental results. In particular, it is shown in [28] with the aid of numerical integration that for a normal quasi-harmonic noise with Gaussian spectral density (9) at $\Delta f = 0.18 f_0$, the variance of the number of positive peaks at the zero level ($C = 0$) is determined by the formula

$$\sigma_N^2(T) = 0.067 f_0 T,$$

while the variance of the total number of zeroes (upward and downward crossings of the zero level) is

$$\sigma_{2N}^2(T) = 4\sigma_N^2(T) = 0.268 f_0 T.$$

We present without proof the final formulas for the average number of peaks per unit time $\overline{N}_1$ of certain stationary random processes.

The average number of positive peaks at the level $\gamma = C/\sigma$ of a normal stationary process is determined by the formula

$$\overline{N}_1 = \frac{1}{2\pi} \sqrt{-R''(0)}\, e^{-\frac{1}{2}\gamma^2}, \qquad (19)$$

where $R''(0)$ is a certain derivative of the correlation function at zero, connected with the spectral density by the relation

$$R''(0) = -\left(\frac{2\pi}{\sigma}\right)^2 \int_0^\infty f^2 S(f)\, df.$$

Let the random process $\eta(t)$ represent the sum

$$\eta(t) = \xi(t) + s(t),$$

where $\xi(t)$ is a normal stationary noise and

$$s(t) = A_m \cos(\omega_s t + \varphi_0)$$

is a determinate harmonic signal. For the average number of peaks $\overline{N}_1$ of such a process $\eta(t)$ the following formulas are obtained:

$$\begin{aligned}\overline{N}_1 &= \frac{1}{2\pi} \sqrt{-R''(0)} \Big\{ e^{-\alpha} I_0(\beta) \\ &+ 2b \int_0^\pi \varphi(\gamma - a\cos\theta) \sin\theta \Big[\int_0^{b \sin\theta} \varphi(x)\, dx \Big] d\theta \Big\}, \qquad (19')\end{aligned}$$

or

$$\overline{N}_1 = \sqrt{-\frac{R''(0)}{2\pi}} \times \sum_{n=0}^{\infty} \frac{\varphi^{(2n)}(\gamma)}{n!n!} \left(\frac{a}{2}\right)^{2n} {}_1F_1\left(-\frac{1}{2};\, n+1;\, -\frac{b^2}{2}\right). \quad (19'')$$

Here

$$b = \frac{a\omega_s}{\sqrt{-R''(0)}}, \quad \alpha = \frac{1}{4}(a^2+b^2), \quad \beta = \frac{1}{4}(a^2-b^2),$$

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \quad \varphi^{(2n)}(z) = \frac{d^{2n}}{dz^{2n}}\varphi(z),$$

${}_1F_1$ is the confluent hypergeometric function.

From formula (19′) with $\gamma = 0$ we can obtain the simpler relation

$$\overline{N}_1 = \frac{1}{2\pi}\sqrt{-R''(0)}\left[e^{-\alpha}I_0(\beta) + \frac{b^2}{2\alpha} I_e\left(\frac{\beta}{\alpha}, \alpha\right)\right],$$

where

$$I_e(k, x) = \int_0^x e^{-u} I_0(ku)\, du$$

is a tabulated integral [22].

For the envelope $A(t)$ of the quasi-harmonic noise (5) we have

$$\overline{N}_1 = \sqrt{\frac{-\varrho''(0)}{2\pi}}\, \gamma e^{-\frac{1}{2}\gamma^2}. \quad (20)$$

For the process $E(t)$, which represents the envelope of the sum of a quasi-harmonic noise and a harmonic signal, we obtain from (4)

$$\overline{N}_1 = \sqrt{\frac{-\varrho''(0)}{2\pi}}\, \gamma \exp\left[-\frac{1}{2}\gamma^2 - \frac{A_m^2}{2\sigma^2}\right] I_0\left(\gamma \frac{A_m}{\sigma}\right). \quad (21)$$

When $A_m = 0$ this formula goes over into (20). We can obtain a formula for $\overline{N}_1$ also in the case of the η-process [52].

It can be shown [53] that the average number per unit time of peaks exceeding a level $C = \cos\psi_0$ in a random process $\cos\psi(t)$, is determined by the formula

$$\overline{N}_1 = \frac{1}{\pi}\sqrt{-\varrho''(0)} \exp\left[-\frac{A_m^2}{2\sigma^2}(1-C^2)\right]\Phi\left(\frac{CA_m}{\sigma}\right), \quad (22)$$

where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}x^2}\, dx$$

is the tabulated probability integral. Putting in formula (22) $A_m = 0$, we obtain the average number of peaks of the random process $\cos\varphi(t)$

$$\overline{N}_1 = \frac{1}{2\pi}\sqrt{-\varrho''(0)}. \quad (23)$$

The formula for the average number of peaks of the process $\dot{\psi}(t)$ is in general very cumbersome. We shall therefore point out only two particular cases. If there is no signal $(A_m = 0)$ the average number of peaks at the level $C = \dot{\varphi}_0$ is

$$\overline{N}_1 = \frac{1}{2\pi[\dot{\varphi}_0^2 - \varrho''(0)]}\sqrt{\varrho''(0)[4\varrho''(0)\dot{\varphi}_0^2 + \varrho''^2(0) - \varrho^{(4)}(0)]}. \quad (24)$$

At large signal/noise ratios $(A_m/\sigma \geq 3)$ the average number of peaks at the level $C = \dot{\psi}_0$ can be approximately calculated from the formula

$$\overline{N}_1 = \frac{1}{2\pi}\sqrt{-\frac{\varrho^{(4)}(0)}{\varrho''(0)}} \exp\left[\frac{A_m^2}{2\sigma^2}\frac{\dot{\psi}_0^2}{\varrho''(0)}\right]. \quad (25)$$

We note, incidentally, that the character of variation of the average number of peaks (19)—(21) with varying level duplicates, apart from a constant factor, the probability densities (1), (7), and (8) of the fluctuating processes themselves. This result is normally used in an experimental determination of the one-dimensional probability densities of stationary random processes with the aid of computing circuits that determine the average number of peaks.

As can be seen from the fundamental formula (12), such a result is the consequence of the fact that for stationary random processes (1), (7), and (8) there is no statistical connection between the process itself and its derivative at coinciding instants of time (and the derivative of such a process has a normal probability density). It must be borne in mind, however that this property is possessed by not all the stationary random processes. One can cite examples of stationary processes which do not have this property. Nor are a process and its derivative independent at coinciding instants of time in the case of nonlinear inertialess transformations of the process.

As applied to normal processes with spectral densities (2), (3), and (4), formula (19) assumes respectively the form

$$\overline{N}_1 = n_1 = \frac{\Delta f}{\sqrt{1.4}} e^{-\frac{1}{2}\gamma^2}, \quad (19a)$$

$$\overline{N}_1 = n_2 = \frac{\Delta f}{0.5413} e^{-\frac{1}{2}\gamma^2}, \quad (19b)$$

$$\overline{N}_1 = n_3 = \frac{\Delta f}{\sqrt{3}} e^{-\frac{1}{2}\gamma^2}. \quad (19c)$$

For a normal noise, the spectral density of which is determined by the resonant curve of a single oscillating circuit with resonant frequency f_0 and bandwidth Δf (at the 0.5 power level), formula (19) yields

$$\overline{N}_1 = n_4 = \sqrt{f_0^2 + \frac{1}{4}\Delta f^2}\, e^{-\frac{1}{2}\gamma^2}. \quad (19d)$$

A thorough experimental investigation of the peaks of such a noise at different values of $f_0/\Delta f$ [31] gives good agreement between the experimental results and the results of calculations based on formula (19d). For a normal narrow-band noise with Gaussian spectral density (9) we obtain

$$\overline{N}_1 = n_5 = \sqrt{f_0^2 + \frac{1}{5.6}\Delta f^2}\, e^{-\frac{1}{2}\gamma^2}. \quad (19e)$$

From a comparison of formulas (19a)—(19c), which pertain to low frequency noise we see that for identical Δf and γ the average number of peaks is larger for noises that have a slower reduction in the spectral density with increase in frequency beyond the limits

Table I.
Peaks of normal noise

γ	$n_1/\Delta f$	$\sigma_N(T)/\overline{N}(T)$
0,0	0,845154	0.20
0,5	0,745846	—
1,0	0,512612	0,30
1,5	0,274381	—
2,0	0,114379	0,67
2,5	0,037133	—
3,0	0,009389	2,93
3,5	0,001849	—
4,0	0,000283	—
5,0	0,0000003	—

of the band Δf. In other words, the average number of peaks is greatly influenced by the behavior of the spectral density at high frequencies. Such a result is physically due to the fact that the presence of more sharply pronounced high-frequency spectral components causes large fluctuations in the random process, and consequently causes a large number of peaks.

The average number of peaks of a harmonic oscillation with amplitude A_m and frequency f_0 at any level $C < A_m$ is obviously equal to $\overline{N}_1 = f_0$. Formulas (19d) and (19e) show that the average number of peaks of a quasi-harmonic noise at zero level ($\gamma = 0$) is always larger than the average frequency f_0, but does not exceed the value of the upward frequency of the bandwidth $f_0 + 0.5\Delta f$. This result can be qualitatively attributed to the presence in the spectrum of the random process of high frequency components which have relatively low intensity.

The question of the variation in the average number of peaks when the normal noise $\xi(t)$ is subjected to integration or differentiation, was considered in [32]. A qualitative idea of the average number of peaks of normal noise is given in Table I, which lists results of calculations based on (19a).

The third column of Table I gives the ratio of the mean square value of the number of peaks to the average number of peaks for a normal noise (2). The data were obtained as a result of processing 800 oscillograms, each with duration $T = 10/\Delta f$. We see that the ratio $\sigma_N(t)/\overline{N}(t)$ increases with increasing γ. It turns out that such a result is general; it is valid not only for normal fluctuations, but also for fluctuating processes of other types.

If the spectral density of a quasi-harmonic noise $\xi(t)$ is constant and differs from zero only in the band Δf with central frequency $f_0 \gg \Delta f$, i.e.,

$$S(f) = \begin{cases} S_0 & |f - f_0| \leqslant \frac{1}{2}\Delta f, \\ 0 & |f - f_0| > \frac{1}{2}\Delta f, \end{cases} \tag{26}$$

then we obtain from (20)

$$\overline{N}_1 = n_6 = \Delta f \sqrt{\frac{\pi}{6}}\gamma e^{-\frac{1}{2}\gamma^2}. \tag{20a}$$

If the spectral density has the Gaussian form (9), formula (20) yields

$$\overline{N}_1 = n_7 = \Delta f \sqrt{\frac{\pi}{2,8}}\gamma e^{-\frac{1}{2}\gamma^2}. \tag{20b}$$

The results of calculations based on formulas (20a) and (20b) are listed in Table II. Here, too, the regularity noted above is observed: the longer the "skirts" of the spectral density, the larger the average number of peaks.

Table II lists also the experimental values of the ratio $\sigma_N(T)/\overline{N}(T)$ at different levels for different durations of the realizations P. The results pertain to a fluctuating process $A(t)$, when the spectral density of the quasi-harmonic noise $\xi(t)$ has the Gaussian form (9). The table indicates the number N of the photographs (realizations) the processing of which yields the corresponding data.

Analyzing the experimental results, we can conclude that the ratio $\sigma_N(T)/\overline{N}(T)$ decreases with increasing duration of the realizations T. At all durations T it has a minimum value at a level $C = \gamma\sigma$, which is approximately equal to the most probable value, and as the level deviates from this value, the ratio $\sigma_N(T)/\overline{N}(T)$ increases.

According to formula (21) the average number of peaks of the envelope $E(t)$ of the sum of the harmonic signal and the quasi-harmonic noise, with spectral density (26) or (9), is respectively equal to

Table II. Peaks of random process $A(t)$

γ	$n_6/\Delta f$	$n_7/\Delta f$	γ	$\sigma_N(T)/\overline{N}(T)$				
				$T\Delta f=2$, $M=1750$	$T\Delta f=3$, $M=503$	$T\Delta f=5$, $M=534$	$T\Delta f=10$, $M=588$	$T\Delta f=20$, $M=656$
0,0	0	0	0,66	1,40	1,12	1.08	1,17	0.80
0,5	0,319288	0.467390	1,25	0,94	0,79	0.57	0.41	0,50
1,0	0,438886	0,642464	2,0	1,54	1,09	0,88	0.73	0.45
1,5	0,352378	0.515829	2,6	2,36	1,98	1.48	1,47	0,71
2,0	0,195858	0,286706	3,3	3,43	3,60	2.76	2.76	1.21
2,5	0,079482	0,116350	4,0	5,46	—	6.26	5,30	2.14
3,0	0,024115	0,035301						
3,5	0,005540	0,008110						
4,0	0,000971	0,001421						
5,0	0,000013	0,000020						
6,0	0	0						

$$\overline{N}_1 = n_8 = \Delta f \sqrt{\frac{\pi}{6}} \gamma \exp\left[-\frac{1}{2}(\gamma^2 + a^2)\right] I_0(a\gamma), \quad (21a)$$

$$\overline{N}_1 = n_9 = \Delta f \sqrt{\frac{\pi}{2,8}} \gamma \exp\left[-\frac{1}{2}(\gamma^2 + a^2)\right] I_0(a\gamma). \quad (21b)$$

The results of the calculations based on formula (21b) are listed in Table IV.

We present a formula for the average number of peaks of one nonstationary normal process $\zeta(t)$, [33] representing the sum of a linearly increasing voltage and a normal stationary noise $\xi(t)$ with zero mean value, variance σ^2, and correlation coefficient $R(\tau)$

$$\zeta(t) = (C_0 + \beta t) + \xi(t). \quad (27)$$

If we use the well known expression for the joint probability density of the independent normal quantities $\xi(t)$ and $\dot{\xi}(t)$, and then change over in this expression from ξ and $\dot{\xi}$ to ζ and $\dot{\zeta}$, then from formula (13), written down for the nonstationary process

$$\overline{N}(T) = \int_0^T dt \int_0^\infty \dot{\zeta} W_2(C, \dot{\zeta})\, d\dot{\zeta},$$

we obtain after some transformations

$$\overline{N}(T) = \left[\frac{\sigma}{\beta} \sqrt{\frac{-R''(0)}{2\pi}} \exp\left(\frac{\beta^2}{2\sigma^2 R''(0)}\right) + \Phi\left(\frac{\beta}{\sigma\sqrt{-R''(0)}}\right)\right] \left[\Phi\left(\frac{C - C_0}{\sigma}\right) - \Phi\left(\frac{C - C_0 - \beta T}{\sigma}\right)\right]. \quad (28)$$

This formula is frequently used in an analysis of the accuracy of the operation of pulsed synchronization devices in the presence of an interfering noise.

3. DISTRIBUTION OF PEAK DURATIONS AND OF THE INTERVALS BETWEEN PEAKS

The problem of calculating the probability densities for the duration of peaks of fluctuating processes was posed in the basic paper of Rice [21] in 1945. The same paper indicates one approximate method of solving the problem, a method considered in greater detail in [34]. In later years this problem was considered in many theoretical [9,35-40] and experimental [31,41-44] papers. A brief summary of the basic papers is indicated below.*

The rigorous theoretical solution obtained in [35] gives an important particular result, namely that the distribution of the peaks for large durations should have an exponential form. However, at not very large durations the solution leads to very complicated final formulas very cumbersome to use in calculations. It is therefore of interest to consider approximate methods for calculating the probability densities for the durations of the peaks.

Three approximate methods can be mentioned: the Rice method, the uncorrelated pulse method, and the quadratic approximation. The nature of these methods and the results obtained with them are explained below.

From the physical premises on which all three methods are based it follows that they yield results that are valid for peaks of short duration at high levels. However, it is difficult to indicate the range of applicability of the results, since this is connected as a rule with a solution of the more complicated problem. Later on we shall present in this connection new experimental results.

In view of the limited extent of the article, we shall mention here from among the extensive experimental material only a small fraction of the results pertaining to the fluctuation processes $A(t)$ and $E(t)$. We note, incidentally, that for these processes calculations even on the basis of the approximate formulas turn out to be also very complicated.

The Rice method [21,34]. We assume that the peak starts at a certain instant of time t_0, i.e., the random function $\xi(t)$ crosses the level C in the upward direction (see Fig. 1) at $t = t_0$; then, obviously

$$\xi_0 = \xi(t_0) = C, \quad \dot{\xi}_0 = \dot{\xi}(t_0) > 0.$$

Let the duration of the peak be τ. Then at the end of the peak, i.e., at $t = t_0 + \tau$, the function $\xi(t)$ and its derivative should satisfy the condition

$$\xi_\tau = \xi(t_0 + \tau) = C, \quad \dot{\xi}_\tau = \dot{\xi}(t_0 + \tau) \leqslant 0.$$

By geometrical reasoning it is easy to show that if we know the joint probability density $W_4(\xi_0, \xi_T, \dot{\xi}_0, \dot{\xi}_T)$ for the values of the random function and its derivative at two different instants of time, then the probability density for the duration of the peaks is determined by the formula

$$P(\tau) = \frac{1}{\overline{N}_1} \int_0^\infty \int_0^{-\infty} \dot{\xi}_0 \dot{\xi}_\tau W_4(C, C, \dot{\xi}_0, \dot{\xi}_\tau)\, d\dot{\xi}_0\, d\dot{\xi}_\tau. \quad (29a)$$

The approximate character of this formula follows from the arguments that lead to (29a). In fact, several peaks can occur in the interval τ. Obviously, if τ is small compared with the correlation time of $\xi(t)$, then the probability of multiple crossings is small and formula (29a) will yield more correct results.

The method of uncorrelated pulses [37,36]. In place of the random function $\xi(t)$ let us consider the random sequence of rectangular pulses of unit height $\eta(t)$, obtained from $\xi(t)$ by means of the nonlinear transformation (Fig. 3):

$$\eta(t) = \begin{cases} 0 & \xi(t) < C, \\ 1 & \xi(t) \geqslant C. \end{cases}$$

It is clear that the distribution of the pulses by duration coincides with the distribution $P(\tau)$ of the peaks of $\xi(t)$ at the level C, and when $C \geq \sigma$ the peaks of the normal stationary fluctuations can be assumed approximately to be uncorrelated [42].

As applied to normal stationary fluctuations, such

*Note added in proof. An analogous problem was considered by Slepian [BSTJ, 41 (2), 463 (1962)].

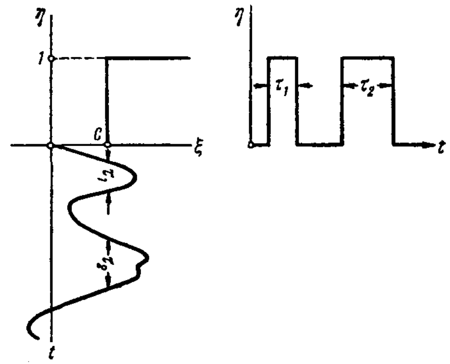


FIG. 3. Nonlinear transformation.

an approach yields for the probability density the Rayleigh law

$$P(\tau) = -\frac{1}{4}\gamma^2 R''(0)\,\tau \exp\left[\frac{1}{8}\gamma^2 R''(0)\tau^2\right]. \qquad (29b)$$

This formula enables us to calculate the mean value and the variance of the pulse durations.

Quadratic approximation[10,40]. The quadratic approximation is based on the assumption that at sufficiently high levels the overwhelming majority of the peaks have short durations, and that the shape of the peak is nearly parabolic.

Let t_0 be the instant of the start of the peak, i.e., $\xi_0 = C$, $\dot{\xi}_0 > 0$. For a smoothly varying fluctuating process we can expand $\xi(t)$ in a Taylor series in the vicinity of the point t_0 and retain only the quadratic term

$$\xi(t) = \xi(t_0) + \dot{\xi}(t_0)(t - t_0) + \frac{1}{2}\ddot{\xi}(t_0)(t - t_0)^2.$$

If we put $t = t_0 + \tau$, where τ is the duration of the peak, then $\xi(t_0 + \tau) = \xi(t_0) = C$, and we obtain from the preceding relation

$$\dot{\xi}_0 + \frac{1}{2}\ddot{\xi}_0\tau^2 = 0, \qquad \tau = -2\frac{\dot{\xi}_0}{\ddot{\xi}_0}. \qquad (30)$$

Inasmuch as the start of the peak corresponds to a positive derivative $\dot{\xi}_0 > 0$, Eq. (30) is valid only for negative values of the second derivative $\ddot{\xi}_0 < 0$.

If we now use the known formula for the normal three-dimensional probability density of the quantities ξ_0, $\dot{\xi}_0$, and $\ddot{\xi}_0$, we obtain the following formula for the probability density of the peak duration:

$$P(\tau) = -\frac{\nu R''(0)\,\tau}{4\varphi^2(\tau)}\left\{\frac{\gamma}{\sqrt{2\pi\gamma}}e^{-\frac{\gamma^2}{2\nu}} + \frac{\varphi(\tau) + \frac{\gamma^2}{\nu}}{2\sqrt{\varphi(\tau)}}\exp\left[\frac{R''(0)\gamma^2\tau^2}{8\varphi(\tau)}\right]\operatorname{erfc}\left(-\frac{\gamma}{\sqrt{2\nu\varphi(\tau)}}\right)\right\}, \qquad (29c)$$

where

$$\nu = \frac{R^{(4)}(0) - R''^2(0)}{R''^2(0)},$$

$$\varphi(\tau) = 1 - \frac{\nu}{4}R''^2(0)\,\tau^2, \quad \operatorname{erfc}(z) = 1 - \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt.$$

The quadratic approximation makes it also possible in principle to calculate analytically the peak distribution by areas[40], which is of interest in the analysis of the effect of noise on electronic relays and flipflops.

Comparison of the results of the calculations by means of formulas (29a)—(29c) with the experimental data for normal stationary fluctuations with spectral densities (2) and (26) given in [37], shows that in the case of smoothly varying and oscillating correlation functions a sufficiently good approximation to the experimental data is given by formula (29b) at not very long durations τ, starting with the level $\gamma \geq 1.5$.

For the fluctuating processes $A(t)$ and $E(t)$, the performance of the analytic calculations is difficult. Figures 4 and 5 show the experimental probability densities for the dimensionless quantities $\tau\Delta f$ and $\theta\Delta f$ of the random process $E(t)$ at two levels $\gamma_0 = 0$ and $\gamma_0 = 2$, for different signal/noise ratios, for the case when the spectral density of the initial quasi-harmonic noise $\xi(t)$ has the form (9).

By relative level γ_0 is meant in this case the quantity

$$\gamma_0 = \frac{C}{\sigma_E}, \qquad (31)$$

where C is the absolute level, reckoned from the average value $\bar{E}$, and σ_E is the mean-square value of the fluctuating process $E(t)$.

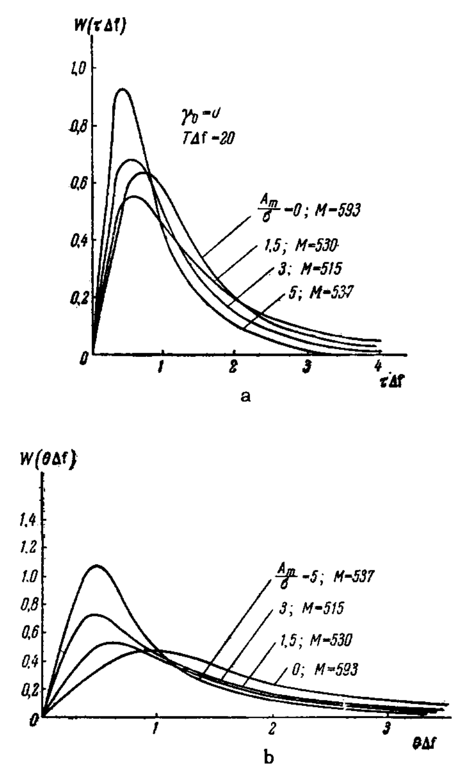


FIG. 4. Probability densities for the peak durations and intervals of the envelope of E(t) at the level $\gamma_0 = 0$.

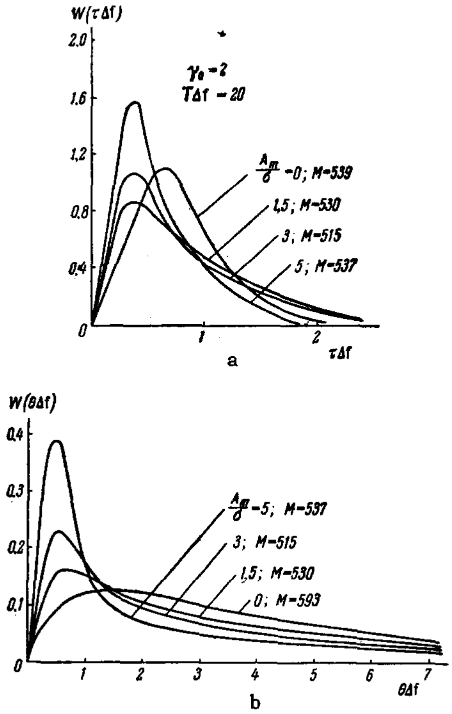


FIG. 5. Probability densities for the peak durations and intervals of the envelope E(t) at the level $\gamma_0 = 2$.

The individual curves were obtained as a result of processing of M realizations (photographs), each realization with duration $T = 20/\Delta f$.

From the analysis of the results we can draw the following qualitative conclusions.

1. The probability densities for both the peak durations and the intervals between peaks increase rapidly from zero to a certain maximum value, and then decrease slowly to zero, the probability density for the intervals dropping more slowly than the probability density for the peak durations.

2. Both the start of the signal and the increase in the level are accompanied by a shift in the most probable value of τ towards smaller τ. The maximum of the probability density for the peak durations at a fixed signal increases with increasing level γ_0.

3. A change in the magnitude of the signal hardly changes the most probable value of τ. However, when the signal is increased the maximum value of the probability density first decreases and then increases.

We shall not cite other conclusions, for if necessary this can be made by the reader himself, using the previous results.

Table III indicates the values of the average durations of the peaks $\bar{\tau}$ and of the intervals $\bar{\Theta}$, and also the mean square values σ_τ and σ_θ at a definite level γ_0. The data were obtained by statistical processing of 530 realizations for $A_m/\sigma = 1.5$. Each realization had a duration $T = 20/\Delta f$.

We see that with increasing level γ_0 the average peak duration $\bar{\tau}$ decreases, and the average interval

Table III. Main characteristics of the peaks of a fluctuating process E(t)

γ_0	-1	0	1	2	3
$\bar{\tau}\Delta f$	2,60	1,415	1,0	0,802	0,642
$\sigma_\tau\Delta f$	2,832	1,059	0,776	0,530	0,458
$\bar{\Theta}\Delta f$	0,802	1,482	2,89	4,870	5,510
$\sigma_\Theta\Delta f$	0,777	1,493	2,855	4,375	5,025

between peaks $\bar{\Theta}$ increases. The same holds true for the mean-square values: with increasing level the mean-square value of the pulse duration σ_τ decreases, and the mean square value of the interval duration σ_Θ increases.

We note that for average values of the peak duration $\bar{\tau}$ and for the duration of the interval between peaks $\bar{\Theta}$ we can obtain relatively simple formulas. Let us consider the realization of the stationary ergodic process $\xi(t)$ of sufficiently long duration T. Let the realization have a sufficiently large number of peaks N. We shall assume that for a stationary ergodic process the quantity $W(\xi)\Delta\xi$ is proportional to the relative time of stay of the random function $\xi(t)$ in the interval $(\xi, \xi+\Delta\xi)$. We can therefore write the relations

$$\frac{1}{T}\sum_{i=1}^{N}\tau_i = \int_C^\infty W(\xi)\,d\xi, \quad \frac{1}{T}\sum_{i=1}^{N}\Theta_i = \int_{-\infty}^{C} W(\xi)\,d\xi,$$

where τ_i and Θ_i are the durations of the i-th peak and the i-th interval at the level C.

The mean duration of the peaks and the mean duration of the intervals are obviously

$$\tau = \frac{1}{\bar{N}_1 T}\sum_{i=1}^{N}\tau_i, \quad \bar{\Theta} = \frac{1}{\bar{N}_1 T}\sum_{i=1}^{N}\Theta_i.$$

From this we obtain final formulas

$$\bar{\tau} = \frac{1}{\bar{N}_1}\int_C^\infty W(\xi)\,d\xi, \quad \bar{\Theta} = \frac{1}{\bar{N}_1}\int_{-\infty}^{C} W(\xi)\,d\xi. \tag{32}$$

Using (19), we find that for a normal fluctuating process formulas (32) assume the form

$$\bar{\tau} = \frac{2\pi}{\sqrt{-R''(0)}}[1-\Phi(\gamma)]e^{\frac{1}{2}\gamma^2}, \quad \bar{\Theta} = \frac{2\pi}{\sqrt{-R''(0)}}\Phi(\gamma)\,e^{\frac{1}{2}\gamma^2}. \tag{32a}$$

From formulas (7) and (20) we obtain for the fluctuating process A(t) the relations

$$\bar{\tau} = \frac{1}{\gamma}\sqrt{\frac{2\pi}{-\varrho''(0)}}, \quad \bar{\Theta} = \frac{1}{\gamma}\sqrt{\frac{2\pi}{-\varrho''(0)}}\left(e^{\frac{1}{2}\gamma^2}-1\right). \tag{32b}$$

If we substitute in (32a), (32b) the values of $R''(0)$ and $\rho''(0)$ expressed in terms of the spectral density of the fluctuating process $\xi(t)$, then we find that in both cases the average duration of the peaks and of the intervals is inversely proportional to the width of the spectral density.

For the envelope E(t), the following formulas are found to hold:

$$\bar{\tau}=\frac{1}{\gamma}\sqrt{\frac{2\pi}{-\varrho''(0)}}\exp\left(\frac{a^2+\gamma^2}{2}\right)\frac{J}{I_0(a\gamma)},$$

$$\bar{\Theta}=\frac{1}{\gamma}\sqrt{\frac{2\pi}{-\varrho''(0)}}\exp\left(\frac{a^2+\gamma^2}{2}\right)\frac{1-J}{I_0(a\gamma)}, \qquad (32c)$$

where

$$J=\int_0^{\gamma} z\exp\left(-\frac{a^2+z^2}{2}\right)I_0(az)\,dz.$$

The results of the calculations by formulas (32c) for four signal/noise ratios a = 0, 1.5, 3, and 5 and for several values of γ are listed in Table IV. It is assumed here that the spectral density of the initial quasi-harmonic noise $\xi(t)$ has a Gaussian form (9).

4. TIME OF FIRST ATTAINMENT OF THE LEVEL

In some problems it is interesting to know the distribution of the time τ_0 (see Fig. 1). By τ_0 is meant here the time interval between a certain fixed instant of time (say, the start of the realization) and the instant when the random process crosses for the first time the constant level C in an upward direction.

Such a problem is solved analytically only for the case of Markov processes [6,40,45,46]. For real smooth processes, on the other hand, there have been few theoretical results to date [47].

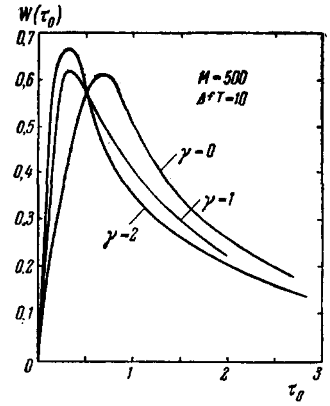


FIG. 6. Probability densities of the time of first attainment of the level for a normal noise.

We present below the experimental data. Figure 6 shows the probability densities $W(\tau_0)$ for a normal stationary noise $\xi(t)$ with spectral density (2) for three values of relative level $\gamma = C/\sigma$. The plots are the results of the processing of 500 realizations, each of duration $T = 10/\Delta f$.

In Table V are indicated the most important practical statistical characteristics of the random quantity τ_0—the mean value $\bar{\tau}_0$ and the mean-square value σ_0—for the indicated normal noise and also for the envelope A(t) of quasi-harmonic fluctuations with spectral density (9).

It follows from these experimental results that in both cases the mean value of the time of first attainment of the level and the mean square value have minima at the level C, a level equal to the mean value of the initial fluctuation processes. The mean and the

Table IV. Characteristics of peaks of the process E(t)

γ	$\bar{N}_1/\Delta f$			$\bar{\tau}\Delta f$				$\bar{\Theta}\Delta f$			
	$a=1,5$	3	5	$a=0$	1,5	3	5	$a=0$	1,5	3	5
0.5	0,173840	0,008550	0.000006	1.88814	5.51639	116.756	—	0,25140	0,23602	0.20233	—
1.0	0.343469	0,034835	0.000065	0.94407	2.43462	28.3889	—	0.61244	0,47683	0.31371	—
1.5	0,456691	0,100244	0,000515	0,62937	1.40953	9.56923	1960.47	1.30925	0,78013	0,40700	0,31371
2.0	0.454304	0,214143	0.003008	0.47201	0,93259	4,14084	331,870	3.01583	1.26858	0.52899	0.32226
2,5	0.344452	0.346606	0,013266	0.39484	0.67391	2,17506	74.9841	8,21710	2.22926	0.71001	0,35117
3.0	0.200346	0,428865	0,044683	0.31473	0,51824	1.32299	21.9796	28.0139	4,47302	1,00853	0.39973
3.5	0.089729	0,407885	0.115630	0.27003	0,41691	0,89653	8.17434	123,034	10,7276	1.55516	0,47393
4.0	0,031025	0.299199	0,230722	0.23944	0,34676	0,65678	3.75801	703.986	31.8801	2.68546	0,57623
5.0	0.001718	0.074472	0,424738		0.25581	0,41197	1.27160		581.140	13,0162	1.08278

Table V. Statistical characteristics of τ_0

	$T\Delta f=10$, $M=500$						$T\Delta f=20$, $M=593$				
$\xi(t)$	$\gamma=-2$	−1	0	1	2	$A(t)$	$\gamma=0.66$	1.25	2.0	2.6	3.3
	$\bar{\tau}_0\Delta f=3.46$	1.24	0,86	1,40	3.46		$\bar{\tau}_0\Delta f=3.63$	1.48	4.05	6.80	7.51
	$\sigma_0\Delta f=2.53$	0.96	0,65	1,03	2.5		$\sigma_0\Delta f=4.40$	1.64	3,97	5	4.79

mean square values increase with decreasing or increasing level. The incomplete data presented do not lead to more general conclusions and if necessary can be used only for tentative estimates.

5. DISTRIBUTION OF MAXIMA

Let us find the probability density for the maxima of the random function $\xi(t)$, i.e., let us find the law of distribution of the maxima as a function of their height [21]. In this case we assume a certain three-dimensional probability density $W_3(\xi(t), \dot{\xi}(t), \ddot{\xi}(t))$.

Let us consider some realization of a stationary random function $\xi(t)$ (Fig. 7), which we assume to be continuous together with its derivative $\dot{\xi}(t)$. At certain instants of time t_n, the realization has maxima. As is well known, at the points t_n the following conditions are satisfied

$$\dot{\xi}(t_n) = 0, \quad \ddot{\xi}(t_n) < 0, \qquad n = 0, 1, 2, \ldots$$

From the continuity of the function $\xi(t)$ it follows that in a sufficiently small interval Δt, i.e., within the interval $t < t' \leq t + \Delta t$, there can be not more than one maximum (or minimum). Thus, in the interval Δt there can be one maximum or no maximum at all. We denote by $P_1(H, t)$ the probability that there will be one maximum in the interval Δt, with a value within the limts $(H - \Delta H)$ and H, and by $P_0(H, t)$ the probability that there will be no such maximum. It is obvious that in the interval Δt the average number of maxima lies between $H - \Delta H$ and H and is

$$\bar{n}_m(H, t) = 1 \cdot P_1(H, t) + 0 \cdot P_0(H, t) = P_1(H, t), \tag{33}$$

i.e., it coincides with P_1.

Let a certain time instant t_0 correspond to the maximum of the random function $\xi(t)$. Then the expression

$$dp = W_3(\xi(t_0), \dot{\xi}(t_0), \ddot{\xi}(t_0))\, \Delta H\, (-\Delta\dot{\xi})\, \Delta\ddot{\xi}$$

at $\dot{\xi}(t_0) = 0$ and $\ddot{\xi}(t_0) < 0$ determines the probability that on the interval $t_0 \leq t' < t_0 + \Delta t$ there will be a maximum lying between $H - \Delta H$ and H, and at the same time the first derivative lies between zero and $-\Delta\dot{\xi}$, while the second derivative lies in the interval $(\ddot{\xi} - \Delta\ddot{\xi}, \ddot{\xi})$.

From the continuity of the first derivative $\dot{\xi}(t)$ it follows that on a small interval Δt, i.e., within the interval $t_0 \leq t' < t_0 + \Delta t$, it is close to the straight line

$$\dot{\xi}(t') = \dot{\xi}(t_0) + \ddot{\xi}(t_0)(t' - t_0)$$

or

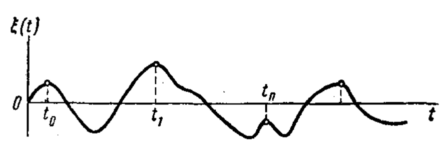


FIG. 7. Realization of stationary random process.

$$\Delta\dot{\xi} = \dot{\xi}(t + \Delta t) - \dot{\xi}(t) = \ddot{\xi}(t_0)\, \Delta t.$$

Substituting this value of $\Delta\dot{\xi}$ into the preceding expression, we obtain

$$dp = -\Delta t W_3(H, 0, \ddot{\xi})\, \ddot{\xi} \Delta H \Delta\ddot{\xi}.$$

Integrating the right half over all negative values of the second derivative (from $-\infty$ to 0) we obtain the probability $P_1(H, t)$ of finding the maximum in the elementary rectangle $\Delta H \Delta t$:

$$P_1(H, t) = -\Delta t \Delta H \int_{-\infty}^{0} \ddot{\xi} W_3(H, 0, \ddot{\xi})\, d\ddot{\xi}. \tag{34}$$

As shown by formula (33), the probability P_1 coincides with the average number of the corresponding maxima

$$\bar{n}_m(H, t) = -\Delta t \Delta H \int_{-\infty}^{0} \ddot{\xi} W_3(H, 0, \ddot{\xi})\, d\ddot{\xi}.$$

Dividing both halves of this equation by Δt, we obtain the average number of the maxima per unit time with values lying in the interval $(H - \Delta H, H)$:

$$\bar{n}_{1m}(H, t) = -\Delta H \int_{-\infty}^{0} \ddot{\xi} W_3(H, 0, \ddot{\xi})\, d\ddot{\xi}. \tag{35}$$

For the stationary random process $\xi(t)$ formula (35) is independent of the time and determines the average number of maxima per unit time between $H - \Delta H$ and H:

$$\bar{n}_{1m}(H) = -\Delta H \int_{-\infty}^{0} \ddot{\xi} W_3(H, 0, \ddot{\xi})\, d\ddot{\xi}. \tag{36}$$

The total average number of maxima per unit time, $\bar{N}_{1m}$, independently of their magnitude, is obtained from this by integrating the right half over all possible values of H:

$$\bar{N}_{1m} = -\int_{-\infty}^{\infty} dH \int_{-\infty}^{0} \ddot{\xi} W_3(H, 0, \ddot{\xi})\, d\ddot{\xi}. \tag{37}$$

The probability density for the maxima is obviously determined from the relation

$$P(H)\, \Delta H = \frac{\bar{n}_1(H)}{\bar{N}_{1m}},$$

i.e.,

$$P(H) = -\frac{1}{\bar{N}_{1m}} \int_{-\infty}^{0} \ddot{\xi} W_3(H, 0, \ddot{\xi})\, d\ddot{\xi}. \tag{38}$$

The average number of maxima per unit time exceeding a certain value C is obtained by integrating the right half of (36) over all values $H \geq C$, i.e., we can write

$$\bar{n}_{1m}(H \geqslant C) = -\int_{C}^{\infty} dH \int_{-\infty}^{0} \ddot{\xi} W_3(H, 0, \ddot{\xi})\, d\ddot{\xi}. \tag{39}$$

Let us assume that the random function $\xi(t)$ is a normal stationary process with zero mean value and with correlation function

$$k(\tau) = \sigma^2 R(\tau), \tag{40}$$

where σ^2 is the variance and $R(\tau)$ is the correlation coefficient.

In view of the fact that the autocorrelation functions of stationary processes are even, the correlation between the random function itself and its derivative, and also between the first and second derivatives at one and the same instants of time are zero. Recognizing that when normal processes are not correlated they are independent, we can write

$$W_3(\xi(t), \dot{\xi}(t), \ddot{\xi}(t)) = w(\dot{\xi}) w_2(\xi, \ddot{\xi}).$$

Using the well known expression for the normal probability densities, we obtain

$$W_3(\xi, 0, \ddot{\xi}) = \frac{1}{(2\pi)^{3/2}\sigma_1 \sqrt{\varepsilon}} \exp\left\{-\frac{1}{2\varepsilon}[\sigma_2^2\xi^2 + \sigma^2\ddot{\xi}^2 + 2\sigma_1^2\xi\ddot{\xi}]\right\}, \tag{41}$$

where

$$\sigma_1^2 = -\sigma^2 R''(0), \quad \sigma_2^2 = \sigma^2 R^{(4)}(0), \quad \varepsilon = \sigma^2\sigma_2^2 - \sigma_1^4. \tag{42}$$

Substituting this probability density in (32) and integrating first with respect to H and then with respect to $\ddot{\xi}$, we obtain

$$\overline{N}_{1m} = \frac{1}{2\pi}\frac{\sigma_2^2}{\sigma_1}. \tag{43}$$

Carrying out the calculations in (38) we obtain

$$P(H) = \frac{1}{\sqrt{2\pi}\sigma}\left[\frac{\sqrt{\varepsilon}}{\sigma\sigma_2}\exp\left(-\frac{\sigma_2^2 H^2}{2\varepsilon}\right) + \sqrt{2\pi}\frac{\sigma_1^2 H}{\sigma^2\sigma_2}\exp\left(-\frac{H^2}{2\sigma^2}\right)\Phi\left(\frac{\sigma_1^2}{\sigma\sqrt{\varepsilon}}H\right)\right]. \tag{44}$$

Formula (44) can be made more compact. For this purpose we consider in lieu of $\xi(t)$ the normalized random function

$$\zeta(t) = \frac{1}{\sigma}\xi(t),$$

which has a zero mean value and a unity variance.

From the obvious relation

$$P(\xi)\,d\xi = W(\zeta)\,d\zeta$$

we have

$$W(\zeta) = \sigma P(\sigma\xi).$$

We introduce the new quantity

$$\nu^2 = \frac{\varepsilon}{\sigma^2\sigma_2^2} = 1 - \frac{\sigma_1^4}{\sigma^2\sigma_2^2}. \tag{45}$$

Carrying out the necessary transformations we obtain a final formula for the probability density of the maxima of a random function

$$W(H) = \frac{1}{\sqrt{2\pi}}\left[\nu e^{-\frac{H^2}{2\nu^2}} + \sqrt{2\pi}(1-\nu^2)^{\frac{1}{2}} H e^{-\frac{1}{2}H^2}\Phi\left(\frac{\sqrt{1-\nu^2}}{\nu}H\right)\right]. \tag{46}$$

This formula enables us to conclude that the probability density of the maxima of a normal stationary function is determined uniquely by the single parameter ν. It can be shown that ν ranges from zero to one, where small values $\nu \ll 1$ correspond to a narrow band process of the type (5), while values $\nu \sim 1$ correspond to broadband processes. Putting $\nu = 0$ in (46) we obtain a Rayleigh probability density of the type (7):

$$W(H) = He^{-\frac{1}{2}H^2}, \qquad H \geqslant 0. \tag{46a}$$

When $\nu = 1$ we obtain from (46) the normal probability density

$$W(H) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}H^2}. \tag{46b}$$

These results can be explained physically. For quasi-harmonic fluctuations (5) we have

$$\dot{\xi}(t) = \dot{A}(t)\cos[\omega_0 t + \varphi(t)] - (\omega_0 + \dot{\varphi})\sin[\omega_0 t + \varphi(t)].$$

Near the points t_n, for which $\omega_0 t_n + \varphi(t_n) = 2\pi n$, $n = 0, 1, 2, \ldots$, the random function $\xi(t)$ has maxima. At these points the following equations hold true

$$\xi(t_n) = A(t_n), \quad \dot{\xi}(t_n) = \dot{A}(t_n),$$

i.e., near the maxima of $\xi(t)$ the envelope $A(t)$ and the random function $\xi(t)$ have common tangents. Inasmuch as the envelope $A(t)$ has a probability density (7), it is natural to expect the maxima of the random function $\xi(t)$, which are discrete values of the envelope, to have the same distribution.

The maxima of a normal broadband process, which are discrete values of a normal random function, also have a normal distribution. Thus, the probability density for the maximum values of a normal narrow-band process is in the limit (when $\nu = 0$) a Rayleigh density, and the probability density for the maximum values of a normal broadband process is in the limit (when $\nu = 1$) normal. Figure 8 shows the probability densities calculated by formula (46) for several values of ν.[48] The curves show the transition from the normal to the Rayleigh distribution.

For the fluctuation process $A(t)$ calculations with the aid of formula (38) can be carried through to conclusion only in particular cases, for example, for a

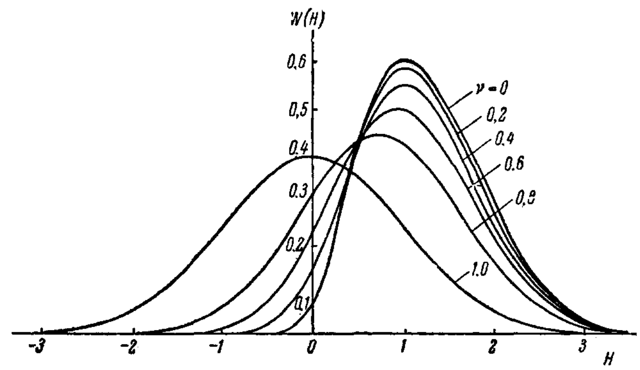


FIG. 8. Probability densities for the maxima of normal noise at different values of the parameter ν.

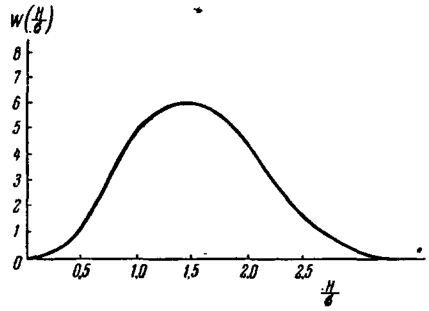


FIG. 9. Probability densities for the maxima of the envelope of quasi-harmonic fluctuations.

rectangular spectral density (26) [21]. (See note added in proof at the end of the article).

Figure 9 shows the non-normalized probability density for the maxima of the envelope A(t), obtained experimentally in [49], when the quasi-harmonic noise $\xi(t)$ has a Gaussian spectral density. Although it recalls the normal probability density in appearance, it does have a certain positive asymmetry.

Let us calculate the average number of maxima of the normal process per unit time, exceeding a certain level C. For this purpose it is necessary to substitute in (39) the expression (41) for the probability density. Carrying out the integration first with respect to $\ddot{\xi}$ and then with respect to H, we obtain

$$\bar{n}_{1m}(H \geqslant C) = \frac{1}{2\pi}\frac{\sigma_1}{\sigma}\left[e^{-\frac{1}{2}\gamma^2}\Phi\left(\frac{\sigma_1^2}{\sqrt{\varepsilon}}\gamma\right)+\frac{\sigma\sigma_2}{\sigma_1^2}\Phi\left(-\frac{\sigma\sigma_2}{\sqrt{\varepsilon}}\gamma\right)\right],$$

where $\gamma = C/\sigma$.

If we introduce the parameter ν (45), we obtain as a final formula

$$\bar{n}_{1m}(H \geqslant C) = \bar{N}_{1m}\left[\Phi\left(-\frac{\gamma}{\nu}\right)+\sqrt{1-\nu^2}\,e^{-\frac{1}{2}\gamma^2}\Phi\left(\frac{\sqrt{1-\nu^2}}{\nu}\gamma\right)\right], \quad (47)$$

where $\bar{N}_{1m}$ is the average number of all the maxima of the random function per unit time, calculated by formula (37).

We note that at sufficiently large γ we can put in (47) $\Phi(z \geq 3) \approx 1$ and $\Phi(z \leq -3) \approx 0$. In this case formula (47) goes over into formula (19) for the average number of peaks per unit time exceeding a level γ. In practice $\bar{N}_1$ and $\bar{n}_{1m}$ coincide when $\gamma \geq 3$.

From the equality of $\bar{N}_1$ and $\bar{n}_{1m}$ when $\gamma \geq 3$ we can draw the following conclusion. In Sec. 3 we pointed out an approximate method for calculating the peak-duration probability density, based on approximating the random function in the vicinity of the peak by a parabola. Obviously, such an approximation excludes the possibility of existence of several maxima in the vicinity of the peak. Consequently the quadratic approximation can be regarded as applicable for large levels ($\gamma \geq 3$).

6. DISTRIBUTION OF MAXIMUM VALUES

A rigorous mathematical calculation of the probability density for the largest values H_m (see Fig. 1) in the realizations of a random process $\xi(t)$ of finite duration T is apparently a very complicated problem [50]. It is necessary to calculate for this purpose the maxima, to determine the largest maximum, and to compare its value with the values of the random function assumed at the ends of the interval $[0, T]$.

We can indicate the following method of solving this problem. We break up the time interval T into m different segments of duration $\Delta = T/m$ each. We denote the values of the random function at the $(m+1)$ selected reference points respectively by

$$\xi_0 = \xi(0), \quad \xi_1 = \xi(\Delta), \quad \xi_2 = \xi(2\Delta), \quad \ldots, \quad \xi_m = \xi(m\Delta) = \xi(T).$$

The probability density $W_{m+1}(\xi_0, \xi_1, \ldots, \xi_m)$ for the values of the random function at these points is assumed known.

The probability $p_i(H_n)$ that the random function $\xi(t)$ assumes at $t = i\Delta$ $(i = 0, 1, 2, \ldots, m)$ a value $\xi(i\Delta) = H_m$, and that it assumes at all other reference points values smaller than H_m, is

$$p_i(H_m) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} W_{m+1}(\xi_0, \ldots, \xi_{i-1}, H_m, \xi_{i+1}, \ldots, \xi_m) \times d\xi_0 \ldots d\xi_{i-1}\, d\xi_{i+1} \ldots d\xi_m. \quad (48)$$

Inasmuch as for stationary fluctuations all the reference points are of equal weight, the probability $P(H_m)$ that the random function assumes a value H_m at any arbitrary reference point and assumes values smaller than H_m in the remaining points is

$$P(H_m) = N \sum_{i=0}^{m} p_i(H_m), \quad (49)$$

where N is a normalization factor, defined by

$$\int_{-\infty}^{\infty} P(H_m)\, dH_m = 1.$$

Unfortunately, the multiple integral (48) can be calculated only in particular cases, for example when the values of the random function $\xi(t)$ at the reference points are independent.

If the values of the random function at the reference points are independent and have a normal distribution

$$w_\xi(\xi_i) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{\xi_i^2}{2\sigma^2}\right)$$

or have a Rayleigh distribution

$$W_A(\xi_i) = \frac{\xi_i}{\sigma^2}\exp\left(-\frac{\xi_i^2}{2\sigma^2}\right),$$

then the probability densities (49) assume respectively the form

$$P_\xi(H_m) = (m+1)\, w_\xi(H_m)\, \Phi^m\left(\frac{H_m}{\sigma}\right), \quad (49a)$$

$$P_A(H_m) = (m+1)\, W_A(H_m)\left[1 - \exp\left(-\frac{H_m^2}{2\sigma^2}\right)\right]^m. \quad (49b)$$

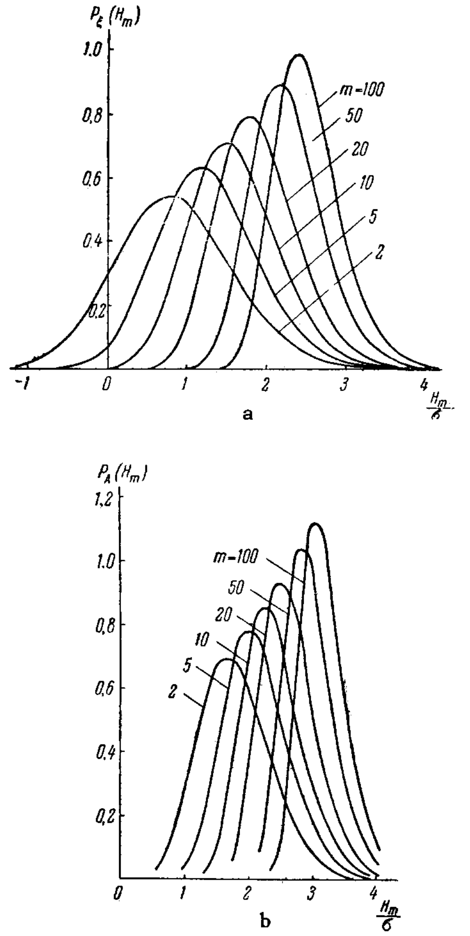


FIG. 10. Probability densities for the largest values.

These probability densities are plotted in Fig. 10.

It must be noted that in order for the values of the random function to be independent at the reference points it is necessary to choose $\Delta \gg \tau_k$, where τ_k is the correlation time of $\xi(t)$. But at such large values of Δ large errors are obtained, since the position of the largest maximum may not coincide with the reference points t_i and may be "left out."

Figure 11 shows the probability densities for the largest values in realizations of normal noise (2), (3), and (4) for three values of the durations of the realizations. Each curve is plotted from results of processing of M realizations of corresponding duration.

Figure 12 shows analogous curves for the fluctuation process $A(t)$, when the spectral density of the

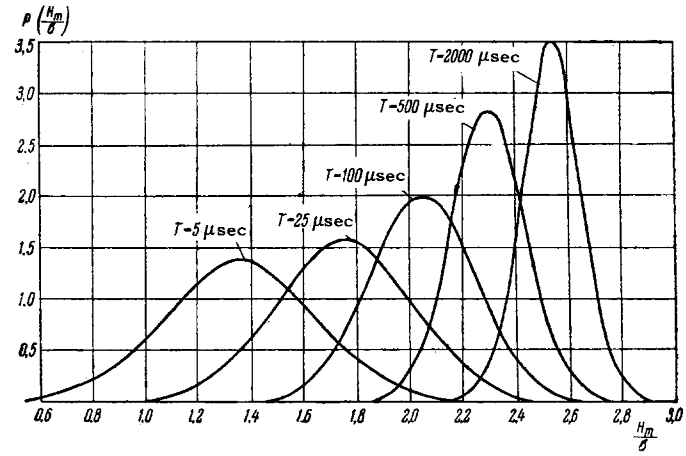


FIG. 12. Probability densities for the largest maxima of realizations of the envelope of a quasi-harmonic noise of different duration.

initial quasi-harmonic fluctuations $\xi(t)$ has the form (9), with $f_0 = 30$ Mc, $\Delta f = 0.92$ Mc. Similar curves for the $E(t)$ process can be found in [51].

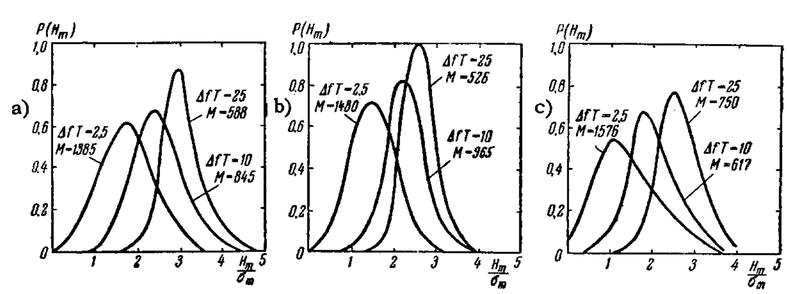
It follows from the foregoing results that with increasing durations of the realizations, the probability densities become narrower and shift towards the larger values of H_m. This is explained by the fact that with increase in duration T first, the value of H_m increases in each individual realization, and second, the scatter in the values of H_m decreases.

7. DISTRIBUTION OF DISTANCE BETWEEN A MINIMUM AND A NEIGHBORING MAXIMUM

Until recently no attempts were made to obtain theoretically the probability densities for the distances h between the minima and the neighboring maxima (see Fig. 1). We therefore confine ourselves to an indication of several specific experimental results [54].

Figure 13 shows the probability densities of the random quantity h for normal noise. Each curve is plotted from the results of the processing of M realization of duration $T = 10/\Delta f$ each, with curve 1 pertaining to a normal noise with spectral density (2), curve 2—with spectral density (3), and curve 3—with spectral density (4). The abscissa axis shows the relative value h/σ_h, where σ_h is the mean square value of h.

FIG. 11. Probability densities for the largest values and realizations of normal noise of duration T (a – spectral density is determined by formula (2), b – by formula (3) and c – by formula (4)).



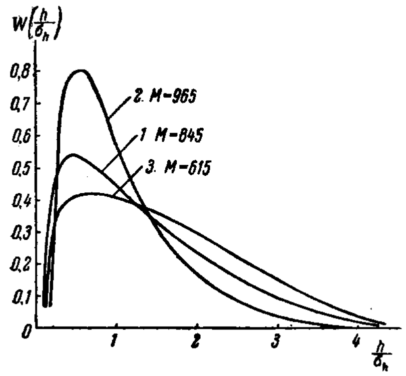


FIG. 13. Probability densities of the difference between a minimum and a neighboring maximum in normal noise.

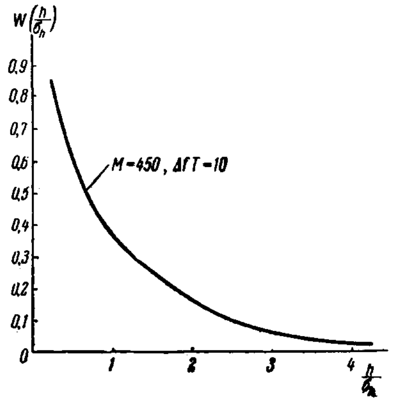


FIG. 14. Probability densities of the difference in the heights between a minimum and a neighboring maximum in the envelope of a quasi-harmonic noise.

From a comparison of the curves we can conclude that all the curves are similar in shape to the Rayleigh probability density. All three probability densities have a most probable value at approximately $h/\sigma_h \approx 0.5$. The slower the noise spectrum decreases with increasing frequency, the larger the most probable value and the faster the decrease of the probability density to the right of this value.

Figure 14 shows a plot of the probability density of the relative h/σ_h for the envelope $A(t)$ of a quasi-harmonic noise with spectral density (9). The curve is based on the results of the processing of 450 realizations, each of duration $T = 10/\Delta f$. In this case the probability density has the form of a hyperbola.

We note that the probability density for the distances between the maximum and neighboring minimum coincides with the probability density for the distances between the minima and the neighboring maxima.

In conclusion we note, that owing to the space limitation we did not give here the statistical characteristics of the peaks of the fluctuation processes $\dot\varphi(t)$, $\dot\psi(t)$, $\cos\varphi(t)$, and $\cos\psi(t)$, which are of great interest in the case of radio circuits with frequency and phase modulation. These results will be published elsewhere.

Note added in proof. For the fluctuating process $A(t)$, formulas (37) and (48) assume the form

$$\bar N_{1m} = \frac{(a^2-1)^2}{(2a)^{5/2}} \sqrt{\frac{-\varrho''(0)}{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n}{2}+\frac{5}{4}\right)}{\Gamma\left(\frac{n}{2}+\frac{7}{4}\right)} \frac{A_n}{a^n}, \qquad (37')$$

$$P(z) = \frac{1}{4\bar N_{1m}} \sqrt{\frac{-\varrho''(0)}{\pi}} (a^2-1)^{\frac{3}{2}} z^{\frac{3}{2}} e^{-a^2z^2} \sum_{n=0}^{\infty} \frac{z^n A_n}{\Gamma\left(\frac{n}{2}+\frac{7}{4}\right)}. \qquad (38')$$

We used here the following notation

$$z = \frac{-\varrho''(0)}{\sqrt{2}\sqrt{\varrho^{(4)}(0)-\varrho''^2(0)}} \frac{H}{\sigma}, \quad a^2 = \frac{\varrho^{(4)}(0)}{\varrho''^2(0)}, \quad b = \frac{1}{2}(3-a^2),$$

$$A_n = \sum_{m=0}^{n} \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\dots\left(m-\frac{1}{2}\right)}{m!} (n-m+1)\, b^m,$$

$$A_n \approx (n+1)(1-b)^{-\frac{1}{2}} - \frac{b}{2}(1-b)^{-\frac{3}{2}},$$

if n is large.

Calculations based on formula (37') show that the average number of the maxima of the envelope of a quasi-harmonic noise with rectangular spectral density of width Δf is

$$\bar N_{1m} = 0.6411\,\Delta f, \qquad (37'a)$$

while for a noise with a Gaussian spectral density (of width Δf) we get

$$\bar N_{1m} = 1.06\,\Delta f, \qquad (37'b)$$

i.e., it is approximately twice as large for the same values of Δf.

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