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INDUCED RADIATIVE PROCESSES IN QUANTUM AND CLASSICAL THEORIES

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1. INTRODUCTION

THE existence of induced emission of radiation was first postulated by Einstein[1] on the basis of thermodynamic arguments, before the development of the quantum theory of radiative processes. The quantum theory, however, has given the simplest and most natural explanation of this effect and of its main features. According to this theory the probability of transition of an atom from state a with energy E_a to state b with energy E_b, accompanied by the emission of a photon, of frequency $\omega = (E_a - E_b)/\hbar$ and polarized along the direction $\mathbf{e}_\rho$, into the element of solid angle dO, is given by the expression

$$dW_{ab} = \frac{\omega^3}{2\pi\hbar c^3} |\mathbf{e}_\rho \mathbf{d}_{ab}|^2 \left(1 + \frac{8\pi^3 c^2}{\hbar\omega^3} I_{\rho\mathbf{k}}\right) dO. \qquad (1.1)$$

The probability of the inverse transition b → a, in which absorption occurs, is

$$dW_{ba} = \frac{\omega^3}{2\pi\hbar c^2} |\mathbf{e}_\rho \mathbf{d}_{ab}|^2 \frac{8\pi^3 c^2}{\hbar\omega^3} I_{\rho\mathbf{k}}\, dO. \qquad (1.2)$$

In Eqs. (1.1) and (1.2) $\mathbf{d}_{ab}$ is the matrix element of the dipole moment of the atom*; $I_{\rho\mathbf{k}}$ is the spectral intensity of radiation with polarization $\mathbf{e}_\rho$ and wave vector $\mathbf{k}$. This last quantity is defined so that $I_{\rho\mathbf{k}}\, d\omega\, dO$ is the energy arriving from the solid angle dO through 1 cm^2 in 1 sec. Consequently the units of $I_{\rho\mathbf{k}}$ are erg cm^{-2} sr^{-1}.

According to Eq. (1.1) the emission probability is made up of two parts. The first does not depend on the intensity of the radiation incident on the atom, and is due to spontaneous emission. If $I_{\rho\mathbf{k}} \neq 0$, we have in addition to the spontaneous emission a term proportional to $I_{\rho\mathbf{k}}$, which corresponds to the induced emission. Equation (1.1) shows an important feature of the induced emission: it occurs with the same frequency, in the same direction, and with the same polarization as the incident radiation.

In the case of isotropic naturally-polarized radiation, for which $I_{1\mathbf{k}} = I_{2\mathbf{k}} = \frac{1}{2} I(\omega)$, we can carry out a summation over polarization directions and an integration over angles in Eqs. (1.1) and (1.2). We thus obtain as the formulas for the spontaneous emission probability W^{sp}_{ab}, the induced emission probability W^{ind}_{ab}, and the absorption probability W^{abs}_{ab}:

$$\left.\begin{aligned} W^{ind}_{ab} &= W^{abs}_{ba} = W^{sp}_{ab} \frac{4\pi^3 c^2}{\hbar\omega^3} I(\omega), \\ W^{sp}_{ab} &= \frac{4\omega^3}{3\hbar c^3} |\mathbf{d}_{ab}|^2. \end{aligned}\right\} \qquad (1.3)$$

*Here and in what follows we confine ourselves for simplicity to the consideration of electric dipole radiation.

For transitions between levels γ, γ' which are degenerate with multiplicities g, g' we have

$$W^{ind}_{\gamma\gamma'} = \frac{g'}{g} W^{abs}_{\gamma'\gamma} = W^{sp}_{\gamma\gamma'} \frac{4\pi^3 c^2}{\hbar\omega^3} I(\omega), \quad W^{sp}_{\gamma\gamma'} = \frac{1}{g} \sum_{a,b} W^{sp}_{ab}, \qquad (1.4)$$

where the indices a, b are used to number the states belonging to the levels γ, γ'. If the concentrations of atoms in the levels γ, γ' are N_γ, $N_{\gamma'}$, then the numbers of transitions $\gamma \to \gamma'$ and $\gamma' \to \gamma$ per cubic centimeter and second are given by the respective formulas

$$N_\gamma (W^{sp}_{\gamma\gamma'} + W^{ind}_{\gamma\gamma'}) = N_\gamma \left(W^{sp}_{\gamma\gamma'} + \frac{4\pi^3 c^2}{\hbar\omega^3} I(\omega) W^{sp}_{\gamma\gamma'}\right), \qquad (1.5)$$

$$N_{\gamma'} W^{abs}_{\gamma'\gamma} = N_{\gamma'} \frac{g}{g'} \frac{4\pi^3 c^2}{\hbar\omega^3} I(\omega) W^{sp}_{\gamma\gamma'}. \qquad (1.6)$$

Under conditions of thermodynamic equilibrium the number of transitions $\gamma \to \gamma'$ is equal to the number of transitions $\gamma' \to \gamma$, and furthermore

$$\frac{N_\gamma}{N_{\gamma'}} = \frac{g}{g'} e^{-\frac{E_\gamma - E_{\gamma'}}{kT}} = \frac{g}{g'} e^{-\frac{\hbar\omega}{kT}}. \qquad (1.7)$$

Equating the expressions (1.5), (1.6) and using Eq. (1.7), we get

$$I(\omega) = \frac{\hbar\omega^3}{4\pi^3 c^2 (e^{\hbar\omega/kT} - 1)}, \quad U(\omega) = \frac{4\pi}{c} I(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3 (e^{\hbar\omega/kT} - 1)}, \qquad (1.8)$$

which is Planck's formula for the spectral density of black-body radiation.

It is obvious that Planck's formula follows from the equality of the numbers of $\gamma \to \gamma'$ and $\gamma' \to \gamma$ transitions in thermal equilibrium only if we postulate the existence of induced emission. It is from just these considerations that Einstein postulated the relations (1.4) between $W^{sp}_{\gamma\gamma'}$, $W^{ind}_{\gamma\gamma'}$, and $W^{abs}_{\gamma'\gamma}$. For low frequencies, $\hbar\omega \ll kT$, Eq. (1.8) goes over into the classical Rayleigh-Jeans formula, which does not contain Planck's constant $\hbar$:

$$I(\omega) = \frac{\omega^2 kT}{4\pi^3 c^2}, \quad U(\omega) \frac{\omega^2 kT}{\pi^2 c^3}. \qquad (1.9)$$

It is not hard to see that this formula also could not be obtained from Eqs. (1.5)—(1.7) if the induced emission were absent. (As is well known, when the induced emission is omitted Planck's formula goes over into Wien's formula.) It follows from this fact that the induced emission cannot be a purely quantum effect, and must be present in the classical theory of radiative processes. As we shall show, however, the passage from the quantum formulas to the classical limit occurs in a very peculiar way. In particular, the quantities

$$S^{ind} = \hbar\omega N_\gamma W^{ind}_{\gamma\gamma'}, \quad S^{abs} = \hbar\omega N_{\gamma'} W^{abs}_{\gamma'\gamma}, \qquad (1.10)$$

which give the power of the induced emission and the absorbed power, have no direct classical analogs.

It must be pointed out that the classical theory of radiative processes is usually constructed for the example of the harmonic-oscillator model. This is the simplest model, but at the same time a very limited one. A number of effects in the interaction of oscillators with the field disappear when we go over to this model. In particular, the harmonic-oscillator model is quite useless for the treatment of radiative processes induced by the field. The reason for this is a specific feature of the harmonic oscillator, the fact that the energy levels are equidistant. Meanwhile there have been almost no general treatments in classical theory of the interaction between the radiation field and nonlinear oscillators (whose frequency depends on the energy). For this reason, and also because of the recent interest in the problem of quantum generators based on the use of induced emission (masers and lasers), a special treatment of the question of the role of induced emission in the classical theory is justified.

In Secs. 2—6 we develop the classical theory of radiative processes and bring out some specific features of the behavior of classical systems as compared with quantum systems. A detailed comparison of the classical and quantum theories is made in Sec. 7. In this section, and also in Sec. 8, it is shown that under certain conditions classical systems can strengthen incident radiation instead of absorbing it.

2. RADIATIVE PROCESSES IN CLASSICAL THEORY

Let us consider the interaction of a classical oscillator with the electromagnetic field. We shall here in general take an oscillator to mean any system capable of emitting and absorbing electromagnetic waves: for example, a harmonic oscillator, an anharmonic oscillator, a rotator, and so on. For definiteness let the oscillator be a particle of charge e and mass m in a potential well $U(\mathbf{r})$. The interaction of such an oscillator with the electromagnetic field is described by the equation of motion of the charge in the field and the Maxwell field equations, with the current in the right members of these equations being that given by the motion of the charge. For the derivation of this system of equations, and also for all of the further treatment, it is convenient to start from Hamilton's equations.

The Hamiltonian function for the oscillator and the electromagnetic field treated as interacting can be written in the form

$$H = H_{\text{osc}} + H_{\text{f}} + H', \tag{2.1}$$

where H_{osc} and H_{f} are the Hamiltonians for the free oscillator and the field, and H' is that for the interaction between them. We confine ourselves to the nonrelativistic approximation. The Hamiltonian of the free oscillator is

$$H_{\text{osc}} = \frac{\mathbf{p}^2}{2m} + U(\mathbf{r}). \tag{2.2}$$

Let the field be concentrated in some finite volume V, while the oscillator is at the point $\mathbf{R} = 0$ and its dimensions are small in comparison with the wavelength of the radiation. Then (cf. e.g., [2,4])

$$H_{\text{f}} = \sum_{\mu} H_{\mu} = \frac{1}{2}\sum_{\mu}(P_{\mu}^2 + \omega_{\mu}^2 Q_{\mu}^2) = \frac{1}{2}\sum_{\mathbf{k},\varrho=1,2}(P_{\mathbf{k},\varrho}^2 + \omega_{\mathbf{k}}^2 Q_{\mathbf{k},\varrho}^2), \tag{2.3}$$

$$H' = -2\sqrt{\frac{\pi}{V}}\frac{e}{m}\sum_{\mu} Q_{\mu}\mathbf{e}_{\mu}\mathbf{p} = -2\sqrt{\frac{\pi}{V}}\frac{e}{m}\sum_{\mathbf{k},\varrho=1,2} Q_{\mathbf{k},\varrho}\mathbf{e}_{\mathbf{k},\varrho}\mathbf{p}, \tag{2.4}$$

where Q_μ, P_μ are "canonically conjugate" field variables; the index μ consists of the wave vector $\mathbf{k}$ and the polarization direction $\mathbf{e}_{\mathbf{k},\rho}$; $\mathbf{e}_{\mathbf{k},\rho}$ is the unit vector of the polarization. The expressions for the vector potential $\mathbf{A}$ and the electric field strength $\mathbf{E}$ in terms of Q_μ and P_μ are as follows:

$$\mathbf{A}(\mathbf{r}, t) = 2\sqrt{\frac{\pi}{V}}\sum_{\mathbf{k},\varrho=1,2}\mathbf{e}_{\mathbf{k},\varrho}\left(cQ_{\mathbf{k},\varrho}\cos\mathbf{kR} - \frac{1}{k}P_{\mathbf{k},\varrho}\sin\mathbf{kR}\right), \tag{2.5}$$

$$\mathbf{E}(\mathbf{r}, t) = -2\sqrt{\frac{\pi}{V}}\sum_{\mathbf{k},\varrho=1,2}\mathbf{e}_{\mathbf{k},\varrho}(ckQ_{\mathbf{k},\varrho}\sin\mathbf{kR} + P_{\mathbf{k},\varrho}\cos\mathbf{kR}). \tag{2.6}$$

Using Hamilton's equations

$$\left.\begin{aligned}\dot{\mathbf{r}} &= \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p}}{m} - 2\sqrt{\frac{\pi}{V}}\frac{e}{m}\sum_{\mu} Q_{\mu}\mathbf{e}_{\mu},\\ \dot{\mathbf{p}} &= -\frac{\partial H}{\partial \mathbf{r}} = -\frac{\partial U}{\partial \mathbf{r}},\end{aligned}\right\} \tag{2.7}$$

$$\left.\begin{aligned}\dot{Q}_{\mu} &= \frac{\partial H}{\partial P_{\mu}} = P_{\mu},\\ \dot{P}_{\mu} &= -\frac{\partial H}{\partial Q_{\mu}} = -\omega_{\mu}^2 Q_{\mu} + 2\sqrt{\frac{\pi}{V}}\frac{e}{m}\mathbf{e}_{\mu}\mathbf{p},\end{aligned}\right\} \tag{2.8}$$

we get

$$\ddot{\mathbf{r}} + \frac{1}{m}\frac{\partial u}{\partial \mathbf{r}} = -2\sqrt{\frac{\pi}{V}}\frac{e}{m}\sum_{\mu}\dot{Q}_{\mu}\mathbf{e}_{\mu}, \tag{2.9}$$

$$\ddot{Q}_{\mu} + \omega_{\mu}^2 Q_{\mu} = 2e\sqrt{\frac{\pi}{V}}\mathbf{e}_{\mu}\dot{\mathbf{r}}. \tag{2.10}$$

[In accordance with the approximation used hereafter we have omitted from the right member of Eq. (2.10) a term $2(\pi/V)^{1/2}(e^2/m)\sum_{\mu} Q_{\mu}\mathbf{e}_{\mu}$ proportional to e^2.]

The equations of motion (2.9), (2.10) together with the first of the equations (2.8) allow us to find the field in the volume V. Let us also find the increase per unit time $\dot{H}_\mu$ of the energy of the μ-component of the field owing to the interaction with the oscillator. The quantity $\dot{H}_\mu$ is expressed in terms of a Poisson bracket:

$$\dot{H}_{\mu} = \frac{\partial H_{\mu}}{\partial Q_{\mu}}\frac{\partial H}{\partial P_{\mu}} - \frac{\partial H_{\mu}}{\partial P_{\mu}}\frac{\partial H}{\partial Q_{\mu}} = 2\sqrt{\frac{\pi}{V}}\frac{e}{m}P_{\mu}\mathbf{e}_{\mu}\mathbf{p}. \tag{2.11}$$

In an analogous way we easily get for the increase per unit time of the oscillator energy H_{osc} owing to interaction with the field:

$$\dot{H}_{\text{osc}} = -2\sqrt{\frac{\pi}{V}}\frac{e}{m}\sum_{\mu} P_\mu \mathbf{e}_\mu \mathbf{p} = -\sum_{\mu}\dot{H}_\mu. \qquad (2.12)$$

The quantities (2.11), (2.12) can be expressed in terms of the electric field strength $\mathbf{E}$ or the vector potential $\mathbf{A}$ at the point $\mathbf{R} = 0$. For the average increase $\overline{\dot{H}}_{\text{osc}}$ of the oscillator energy over a period of the field we get

$$\overline{\dot{H}}_{\text{osc}} = -\sum_{\mu}\overline{\dot{H}}_\mu = e\sum_{\mu}\overline{\dot{\mathbf{r}}\dot{\mathbf{E}}}_\mu = -\frac{e}{c}\sum_{\mu}\overline{\dot{\mathbf{r}}\dot{\mathbf{A}}}_\mu. \qquad (2.13)$$

The equations of motion (2.9), (2.10) can also be written in terms of the quantity A_μ that appears in Eq. (2.13):

$$\ddot{\mathbf{r}} + \frac{1}{m}\frac{\partial u}{\partial \mathbf{r}} = -\frac{e}{mc}\sum_{\mu}\dot{\mathbf{A}}_\mu, \qquad (2.14)$$

$$\ddot{\mathbf{A}}_\mu + \omega_\mu^2 \mathbf{A}_\mu = \frac{4\pi}{V} ec\,(\mathbf{e}_\mu \dot{\mathbf{r}})\,\mathbf{e}_\mu. \qquad (2.15)$$

This is the required system of equations, which enable us to calculate $\dot{\mathbf{r}}$, $\dot{\mathbf{A}}_\mu$, and consequently also $\overline{\dot{H}}_{\text{osc}}$.

Let the interaction between the oscillator and the field be turned on at the time $t = 0$. Up to this time there were only the free vibrations $\mathbf{r}^{(0)}$ of the oscillator and $\mathbf{A}_\mu^{(0)}$ of the field. We shall solve the equations (2.14), (2.15) by a method of successive approximations, i.e., in terms of series $\mathbf{r} = \mathbf{r}^{(0)} + \mathbf{r}^{(1)} + \ldots$, $\mathbf{A}_\mu = \mathbf{A}_\mu^{(0)} + \mathbf{A}_\mu^{(1)} + \ldots$, in which we confine ourselves to the terms proportional to e, while in Eq. (2.13) we keep the terms proportional to e^2. This is equivalent to the first-order perturbation theory used in quantum theory to derive the formulas (1.1), (1.2). It is not hard to show that Eqs. (1.1) and (1.2) can also be obtained by averaging the operator $(e/c)\sum_{\mu}\dot{\mathbf{r}}\dot{\mathbf{A}}_\mu$ over the wave functions of the system (atom + radiation field) calculated in first-order perturbation theory. Thus we have a right to expect that the approximation we have chosen will lead to the same results as the taking of the quasi-classical limit in quantum theory. We accordingly write, by Eq. (2.13),

$$\overline{\dot{H}}_\mu = \frac{e}{c}\overline{\dot{\mathbf{r}}^{(0)}\dot{\mathbf{A}}_\mu^{(0)}} + \frac{e}{c}\overline{\dot{\mathbf{r}}^{(1)}\dot{\mathbf{A}}_\mu^{(0)}} + \frac{e}{c}\overline{\dot{\mathbf{r}}^{(0)}\dot{\mathbf{A}}_\mu^{(1)}}, \quad \overline{\dot{H}}_{\text{osc}} = -\sum_{\mu}\overline{\dot{H}}_\mu. \qquad (2.16)$$

In the general case the free vibrations $\mathbf{r}^{(0)}$ of the oscillator can be written in the form of a Fourier series

$$\mathbf{r}^{(0)} = \sum_{l}\mathbf{b}_l \cos(\omega_l t + \delta_l), \qquad (2.17)$$

where $\omega_l = \omega_0 l = (2\pi/T)l$, where T is the period and $l = 1, 2, \ldots$

The free field vibrations $\mathbf{A}_\mu^{(0)}$ are of the form

$$\mathbf{A}_\mu^{(0)} = \mathbf{a}_\mu \cos(\omega_\mu t + \varphi_\mu), \qquad (2.18)$$

where $\mathbf{a}_\mu$ is directed along $\mathbf{e}_\mu$: $\mathbf{a}_\mu = a_\mu \mathbf{e}_\mu$.

When we substitute Eqs. (2.17) and (2.18) in the right members of Eqs. (2.14) and (2.15) we can find $\mathbf{r}^{(1)}$ and $\mathbf{A}_\mu^{(1)}$, and then calculate $\overline{\dot{H}}_{\text{osc}}$.

3. SPONTANEOUS EMISSION

Let us begin with an examination of the third term in Eq. (2.16), the only one that is not zero when there are no free vibrations of the field, i.e., when $\mathbf{A}_\mu = 0$.

Substituting Eq. (2.17) in the right member of Eq. (2.15), we get

$$\mathbf{A}_\mu^{(1)} = -\frac{4\pi}{V} ec \sum_{l}(\mathbf{e}_\mu \mathbf{b}_l)\,\mathbf{e}_\mu \frac{\omega_l}{\omega_\mu^2 - \omega_l^2} \times \left[\sin(\omega_l t + \delta_l) - \sin\delta_l \cos\omega_\mu t - \frac{\omega_l}{\omega_\mu}\cos\delta_l \sin\omega_\mu t\right]. \qquad (3.1)$$

Since Eq. (3.1) contains the resonance factor $\omega_l/(\omega_\mu^2 - \omega_l^2)$, we can set

$$\frac{\omega_l}{\omega_\mu} \approx 1$$

and

$$\frac{\omega_l}{\omega_\mu^2 - \omega_l^2} = \frac{1}{2}\frac{1}{\omega_\mu - \omega_l}$$

in the square brackets in Eq. (3.1). When we do this we get

$$\frac{e}{c}\overline{\dot{\mathbf{r}}^{(0)}\dot{\mathbf{A}}_\mu^{(1)}} = \frac{1}{\tau}\int_0^\tau \frac{e}{c}\dot{\mathbf{r}}^{(0)}\dot{\mathbf{A}}_\mu^{(1)}\,dt$$

$$= \frac{\pi e^2}{V}\sum_{l}\omega_l^2 |\,\mathbf{e}_\mu \mathbf{b}_l\,|^2 \frac{1-\cos(\omega_\mu - \omega_l)\tau}{\tau(\omega_\mu - \omega_l)^2}. \qquad (3.2)$$

Let us sum Eq. (3.2) over μ, i.e., over k and the polarization directions $\mathbf{e}_{\mathbf{k},\rho}$. We can replace the summation over k by an integration (for $V \to \infty$), setting

$$\sum_{\mathbf{k}} \to V\int \frac{d\mathbf{k}}{(2\pi)^3}.$$

For $\tau \to \infty$ the factor

$$\frac{1}{\pi}\frac{1-\cos(\omega_\mu - \omega_l)\tau}{\tau(\omega_\mu - \omega_l)^2} \qquad (3.3)$$

has the property of the δ function $\delta(\omega_\mu - \omega_l)$. Therefore

$$\frac{e}{c}\sum_{\mu}\overline{\dot{\mathbf{r}}^{(0)}\dot{\mathbf{A}}_\mu^{(1)}} = \sum_{l}\pi^2 e^2 \omega_l \sum_{\rho=1,2}\int |\,\mathbf{e}_{\mathbf{k},\rho}\mathbf{b}_l\,|^2\, d\Omega_{\mathbf{k}} \int \delta(\omega_{\mathbf{k}} - \omega_l)\frac{\omega_{\mathbf{k}}^2 d\omega_{\mathbf{k}}}{(2\pi c)^3}$$

Since

$$\sum_{\rho=1,2}\int |\,\mathbf{e}_{\mathbf{k},\rho}\mathbf{b}_l\,|^2\, d\Omega_{\mathbf{k}} = \mathbf{b}_l^2\frac{8\pi}{3},$$

we finally get

$$\frac{e}{c}\sum_{\mu}\overline{\dot{\mathbf{r}}^{(0)}\dot{\mathbf{A}}_\mu^{(1)}} = \sum_{l}\frac{e^2\omega_l^4}{3c^3}\mathbf{b}_l^2 = \sum_{l}\frac{\omega_l^4}{3c^3}\mathbf{d}_l^2. \qquad (3.4)$$

As is well known, in the classical limit the square of the absolute value of the matrix element, $|\,f_{ab}\,|^2$, of an operator f corresponding to a physical quantity

$$f = \sum_{\omega} f_\omega \cos(\omega t + \alpha_\omega),$$

goes over into $\frac{1}{4}f_\omega^2$, where $\omega = (E_a - E_b)/\hbar$. This is the content of the so-called correspondence principle.

Therefore from a comparison of Eq. (3.4) and (1.3) it follows that each term of the sum (3.4) is the classical limit of an expression $S^{sp} = \hbar\omega_{ab} W^{sp}_{ab}$, where $\omega_{ab} = \omega_l$.

4. INDUCED RADIATIVE PROCESSES

If for some values of μ $A^{(0)}_\mu \neq 0$, then in addition to the spontaneous emission there are radiative processes induced by the field. These processes are described by the first and second terms in Eq. (2.16). It must be pointed out that already at this stage there comes to light some difference between the classical and quantum-mechanical descriptions of absorption and induced emission. In quantum theory we can distinguish these two effects in a quite natural way and consider them independently, since it follows from general intuitive ideas that the downward transition ($\gamma \to \gamma''$, $E_\gamma > E_{\gamma''}$) is responsible for induced emission and the upward transition ($\gamma \to \gamma'$, $E_\gamma < E_{\gamma'}$), for absorption. The situation is different in classical theory, since there is no basis for separating out from the first two terms of the expression (2.16) any parts responsible only for absorption or only for induced emission. Thus the absorption and the induced emission have to be treated together. For this reason we shall call processes described by the first and second terms of the expression (2.16) induced radiative processes. It is easy to see that the same properties of coherence with the incident field hold for these processes as in the quantum theory [for those components $\mu(\mathbf{k}, \mathbf{e}_{\mathbf{k},\rho})$ of the unperturbed field that have $A^{(0)}_\mu = 0$, the first two terms in Eq. (2.16) are zero].

Let us begin by considering the first term in Eq. (2.16). Using the fact that for $V \to \infty$

$$\sum_\mu a_\mu \cos(\omega_\mu t + \varphi_\mu) \to \int A(\omega)\cos(\omega t + \varphi_\omega)\, d\omega,$$

we write this term in the form

$$\begin{aligned}
&\frac{e}{c}\sum_\mu \overline{\dot{\mathbf{r}}^{(\omega)}\dot{\mathbf{A}}^{(\omega)}_\mu} \\
&= \frac{1}{\tau}\int_0^\tau dt \frac{e}{c}\sum_l \mathbf{b}_l\omega_l \sin(\omega_l t + \delta_l)\int d\omega\omega\mathbf{A}(\omega)\sin(\omega t + \varphi_\omega) \\
&= \frac{1}{\tau}\frac{e}{c}\sum_l \omega_l \mathbf{b}_l \int d\omega\omega\mathbf{A}(\omega)\frac{1}{2}\left[\frac{\sin(\omega_l-\omega)\tau}{\omega_l-\omega}\cos(\delta_l-\varphi_\omega)\right. \\
&\left. - \frac{1-\cos(\omega_l-\omega)\tau}{\omega_l-\omega}\sin(\delta_l-\varphi_\omega)\right]. \qquad (4.1)
\end{aligned}$$

For $\tau \to \infty$ we can replace $\tau^{-1}(\omega_l-\omega)^{-1}\sin(\omega_l-\omega)\tau$ by $(\pi/\tau)\,\delta(\omega_l-\omega)$ and $\tau^{-1}(\omega_l-\omega)^{-1}[1-\cos(\omega_l-\omega)\tau]$ by $\pi(\omega_l-\omega)\,\delta(\omega_l-\omega)$. When after this we integrate over $d\omega$, we get

$$\frac{e}{c}\sum_\mu \overline{\dot{\mathbf{r}}^{(\omega)}\dot{\mathbf{A}}^{(\omega)}_\mu} = \frac{1}{\tau}\frac{\pi}{2}\frac{e}{c}\sum_l \omega_l^2 \mathbf{b}_l \mathbf{A}(\omega_l)\cos(\delta_l - \varphi_{\omega_l}). \qquad (4.2)$$

Thus the total increase of the field energy in a large time τ owing to the first term in Eq. (2.16) is

$$\int_0^\tau dt \frac{e}{c}\sum_\mu \dot{\mathbf{r}}^{(\omega)}\dot{\mathbf{A}}^{(\omega)}_\mu = \frac{\pi}{2}\frac{e}{c}\sum_l \omega_l^2 \mathbf{b}_l \mathbf{A}(\omega_l)\cos(\delta_l - \varphi_{\omega_l}) \qquad (4.3)$$

and is finite and independent of τ. It can be seen from Eq. (4.3) that this quantity can have either sign, depending on the phase relation between the free vibrations of the oscillator and the field, i.e., it is possible for energy to be transferred from the field to the oscillator, or, on the other hand, from the oscillator to the field.

The average rate of energy transfer is zero, however, when we average over a sufficiently large time interval τ. The expressions (4.2), (4.3) also give zero when averaged over the phases δ_l of the oscillators. This shows that the role of the first term in Eq. (2.16) in induced radiative processes depends essentially on the physical statement of the problem. When we consider the interaction with the field of oscillators with fixed energy E and arbitrary phases, the first term in Eq. (2.16) is equal to zero and all the induced radiative processes are described by the term

$$\frac{e}{c}\sum_\mu \overline{\dot{\mathbf{r}}^{(1)}\dot{\mathbf{A}}^{(\omega)}_\mu}. \qquad (4.4)$$

We note that it is just this statement of the problem that corresponds to the quantum-mechanical formulas for S^{ind} and S^{abs} given above, since S^{ind} is the average of the induced radiation (per unit time) over an ensemble of oscillators with the energy E_γ, and S^{abs} is the average of the absorption over an ensemble of oscillators with the energy $E_{\gamma'}$.

We therefore proceed to the calculation of the expression (4.4). In Sec. 8 we shall look into the role of the first term in Eq. (2.16) in the interaction of oscillators with the field, and the conditions under which it becomes important.

5. HARMONIC OSCILLATOR

Let us begin with the treatment of the simplest case, that of the harmonic oscillator.

Setting $m^{-1}\partial U/\partial \mathbf{r} = \omega_0^2\mathbf{r}$ in Eq. (2.14), we get

$$\ddot{\mathbf{r}} + \omega_0^2\mathbf{r} = \frac{e\omega_\mu}{mc}\mathbf{a}_\mu \sin(\omega_\mu t + \varphi_\mu). \qquad (5.1)$$

In the case of interest, $\omega_\mu \approx \omega_0$ [compare with Eqs. (3.1) and (3.2)],

$$\mathbf{r}^{(1)} = \frac{e}{2mc}\frac{\mathbf{a}_\mu}{\omega_0-\omega_\mu}[\sin(\omega_\mu t + \varphi_\mu) - \sin(\omega_0 t + \varphi_\mu)] \qquad (5.2)$$

and

$$\frac{e}{c}\overline{\dot{\mathbf{r}}^{(1)}\dot{\mathbf{A}}^{(\omega)}_\mu} = \frac{e^2\omega_0^2}{2mc^2}\frac{|\mathbf{a}_\mu|^2}{\omega_0-\omega_\mu}\frac{1}{\tau}\int_0^\tau \sin(\omega_\mu t + \varphi_\mu)\cos(\omega_0 t + \varphi_\mu)\, dt. \qquad (5.3)$$

Let a light beam $I_{\mathbf{k}}\Delta\omega\Delta O = \sum_{\rho=1,2} I_{\mathbf{k},\rho}\Delta\omega\Delta O$ fall on the oscillator. Let us sum Eq. (5.2) over the values of $\mu(\mathbf{k},\rho)$ which correspond to this beam. Replacing the summation over $\mathbf{k}$ by an integration and repeating

the arguments used in the derivation of (3.4), we get

$$c\sum_{\varrho=1,2}\int\frac{Vd\mathbf{k}}{(2\pi)^3}\frac{\omega^2}{c^2}\frac{|a_{\mathbf{k}\varrho}|^2}{8\pi}=\sum_{\varrho=1,2}I_{\mathbf{k},\varrho}\Delta\omega\,\Delta O, \tag{5.4}$$

$$\frac{e}{c}\sum_{\mu}\overline{\dot{\mathbf{r}}^{(1)}\dot{\mathbf{A}}_{\mu}^{(0)}}=-\frac{2\pi^2e^2}{mc}\sum_{\varrho=1,2}I_{\mathbf{k}_0,\varrho}\Delta O=-\frac{2\pi^2e^2}{mc}I_{\mathbf{k}_0}\Delta O. \tag{5.5}$$

For isotropic radiation

$$\frac{e}{c}\sum_{\mu}\overline{\dot{\mathbf{r}}^{(1)}\dot{\mathbf{A}}_{\mu}^{(0)}}=-\frac{8\pi^3e^2}{mc}I(\omega). \tag{5.6}$$

The sign of this expression, unlike that of the quantity (4.2), does not depend on the phase relation between the vibrations of the oscillator and of the field. It follows from Eq. (2.13) that the time average of the increase of the oscillator energy described by the expression (5.6) is positive. In other words, the harmonic oscillator always absorbs radiation.

To bring out the reason for this, let us see what quantum theory gives for the harmonic oscillator.

The energy levels of the harmonic oscillator, given by the formula

$$E_n=\hbar\omega_0\left(n+\frac{1}{2}\right), \tag{5.7}$$

are equidistant, i.e., separated by equal intervals $\hbar\omega_0$. If we place an oscillator which is in its level n in the light beam $I_{\mathbf{k}}\Delta\omega\,\Delta O$, two types of induced radiative processes are possible: the transition $n\to n+1$, accompanied by absorption, and the transition $n\to n-1$, accompanied by induced emission.

In accordance with Eqs. (1.1), (1.2) the probabilities of these transitions are

$$\left.\begin{aligned}\Delta W_{n,\,n+1}&=\frac{4\pi^2e^2}{\hbar^2c}|x_{n,\,n+1}|^2I_{\mathbf{k}_0}\Delta O,\\ \Delta W_{n,\,n-1}&=\frac{4\pi^2e^2}{\hbar^2c}|x_{n,\,n-1}|^2I_{\mathbf{k}_0}\Delta O,\end{aligned}\right\} \tag{5.8}$$

where

$$|x_{n,\,n+1}|^2=\frac{(n+1)\hbar}{2m\omega_0},\quad |x_{n,\,n-1}|^2=\frac{n\hbar}{2m\omega_0}. \tag{5.9}$$

Obviously the only actually observable quantity is the total energy transferred to the oscillator (per second) as the result of both processes. This quantity is given by

$$\hbar\omega_0(\Delta W_{n,\,n+1}-\Delta W_{n,\,n-1})=\frac{2\pi^2e^2}{mc}I_{\mathbf{k}_0}\Delta O. \tag{5.10}$$

Thus quantum theory leads to just the same sort of result as classical theory [compare Eqs. (5.10) and (5.6)]; a harmonic oscillator always absorbs radiation, independently of what energy level it is in. This fact is due to a specific feature of the system of energy levels of the harmonic oscillator—the fact that they are equidistant and that the probability for transitions $n\to n+1$ is larger than that for transitions $n\to n-1$. Therefore it is natural that for the harmonic oscillator the classical theory also gives only absorption.

6. ANHARMONIC OSCILLATOR

For a system with nonequidistant levels, in the general case the transitions $\gamma\to\gamma'$ (upward) and $\gamma\to\gamma''$ (downward) will correspond to different frequencies $\omega'=(E_{\gamma'}-E_{\gamma})/\hbar$ and $\omega''=(E_{\gamma}-E_{\gamma''})/\hbar$.

Therefore in quantum theory it follows that for such a system there will be absorption at the frequencies ω' and induced emission at the frequencies ω''. Therefore it is interesting to find out what classical theory gives for the anharmonic oscillator.

For an arbitrary anharmonic potential U(r) one cannot find the explicit form of $\mathbf{r}^{(1)}$ by integrating Eq. (2.14). Therefore calculations of the same type as those we have made for the harmonic oscillator are possible only in particular special cases. Nevertheless, by using a somewhat different way of calculating the quantity (4.4) we can get a general formula valid for any nonlinear system (cf. e.g., [6,8]).

To make the point of this method clear it is helpful to refer to the example of quantum mechanics. The general formulas (1.1), (1.2) for the probabilities of radiative transitions are valid in all cases to which perturbation theory can be applied, independently of whether or not the Schrödinger equation can be integrated for the unperturbed system. Everything specific to the unperturbed system is included in the matrix elements d_{ab}. In classical mechanics also there exists a method of small perturbations of the very same kind, which allows us to make all the calculations in general form without choosing concrete properties for the unperturbed system, and all the results are expressed in terms of characteristics of the unperturbed motion. This method was widely used in atomic physics in the years preceding the development of quantum mechanics.

It is essentially a prototype of the quantum-mechanical perturbation theory (a detailed exposition of the method can be found in Born [6]).

The method consists of the use of successive approximations to integrate the Hamilton equations written in terms of action and angle variables I, w. These canonically conjugate variables have a very remarkable property: expressed in terms of I and w the Hamiltonian depends only on the I's, while the angular variables are linear functions of the time, and any single-valued function of the coordinates and momenta is periodic in the w's.

Precisely these properties of the variables I and w are very convenient in the solution of a number of problems by the method of successive approximations.

Let us proceed to the proof of our formula (on this cf. also [8]). For simplicity we shall confine ourselves to the treatment of an oscillator with one degree of freedom.

In the Hamiltonian H_{osc} of the unperturbed oscillator we change from the variables p, r to new canonic-

ally conjugate variables, the action variable I and the angle variable w. These variables have the property that H_{osc} depends only on I:

$$\dot{I} = -\frac{\partial H_{osc}}{\partial w} = 0, \quad I = \text{const} \tag{6.1}$$

and

$$\dot{w} = \frac{\partial H_{osc}}{\partial I} = \frac{\omega_0}{2\pi}, \quad w = \frac{1}{2\pi}(\omega_0 t + \delta), \tag{6.2}$$

where $2\pi/\omega_0 = T$ is the period of the motion. Suppose that beginning at time t = 0 the oscillator is acted on by the perturbation

$$H' = -\frac{e}{mc}\mathbf{pA} = -\frac{e}{mc}\sum_{\mu} a_\mu (\mathbf{e}_\mu \mathbf{p}) \cos(\omega_\mu t + \varphi_\mu); \tag{6.3}$$

for the perturbed system the variables I and w satisfy the equations

$$\dot{I} = -\frac{\partial H'}{\partial w}, \quad \dot{w} = \frac{\partial H_{osc}}{\partial I} + \frac{\partial H'}{\partial I}, \tag{6.4}$$

whereas for $t \leq 0$ they satisfy Eqs. (6.1), (6.2). We denote the unperturbed values satisfying Eqs. (6.1), (6.2) by $I^{(0)}$, $w^{(0)}$. In solving Eq. (6.4) we shall confine ourselves to the first approximation in the perturbation. In this case we can substitute in Eq. (6.3) the unperturbed value of p/m

$$\frac{\mathbf{p}^{(0)}}{m} = \dot{\mathbf{r}}^{(0)} = -\sum_l \omega_l \mathbf{b}_l \sin 2\pi l w^{(0)} \tag{6.5}$$

and write H′ in the form

$$\begin{aligned} H' &= \frac{e}{c}\sum_{\mu, l} \omega_l (\mathbf{e}_\mu \mathbf{b}_l) a_\mu \sin 2\pi l w^{(0)} \cos(\omega_\mu t + \varphi_\mu) \\ &= \frac{e}{2c}\sum_{\mu, l} \omega_l (\mathbf{e}_\mu \mathbf{b}_l) a_\mu \sin[2\pi l w^{(0)} - \omega_\mu t - \varphi_\mu]. \end{aligned} \tag{6.6}$$

In Eq. (6.6) we have dropped terms corresponding to combination frequencies, which do not give any contribution to the final expression for the energy transferred. Substituting Eq. (6.6) in the first of the equations (6.4), we get

$$\left.\begin{aligned} \dot{I} &= -\frac{\pi e}{c}\sum_{\mu, l} l\omega_l (\mathbf{e}_\mu \mathbf{b}_l) a_\mu \cos[(\omega_l - \omega_\mu)t + \delta_l - \varphi_\mu], \\ \Delta I = I - I^{(0)} &= \frac{\pi e}{c}\sum_{\mu, l} l\omega_l (\mathbf{e}_\mu \mathbf{b}_l) a_\mu \frac{\sin(\delta_l - \varphi_\mu) - \sin[(\omega_l - \omega_\mu)t + \delta_l - \varphi_\mu]}{\omega_l - \omega_\mu}. \end{aligned}\right\} \tag{6.7}$$

In the second of the equations (6.4)

$$\frac{\partial H_{osc}}{\partial I} = \frac{\omega}{2\pi} \approx \frac{\omega_0}{2\pi} + \frac{1}{2\pi}\frac{\partial \omega_0}{\partial I^{(0)}}\Delta I. \tag{6.8}$$

Substituting Eqs. (6.6), (6.8), and (6.7) in this equation, we get

$$\begin{aligned} \Delta w = w - w^{(0)} = \frac{e}{2c}\sum_{\mu, l}\Bigg\{ &\frac{\partial \omega_l}{\partial I^{(0)}} \omega_l (\mathbf{e}_\mu \mathbf{b}_l) a_\mu \Bigg[\frac{\sin(\delta_l - \varphi_\mu)}{\omega_l - \omega_\mu} t \\ &- \frac{\cos(\delta_l - \varphi_\mu) - \cos[(\omega_l - \omega_\mu)t + \delta_l - \varphi_\mu]}{(\omega_l - \omega_\mu)^2}\Bigg] \\ &+ \frac{\partial}{\partial I^{(0)}}[\omega_l (\mathbf{e}_\mu \mathbf{b}_l)] a_\mu \frac{\cos(\delta_l - \varphi_\mu) - \cos[(\omega_l - \omega_\mu)t + \delta_l - \varphi_\mu]}{\omega_l - \omega_\mu}\Bigg\}. \end{aligned} \tag{6.9}$$

We can now calculate $\dot{\mathbf{r}}^{(1)}$

$$\begin{aligned} \dot{\mathbf{r}}^{(1)} &= \frac{\partial \dot{\mathbf{r}}^{(0)}}{\partial I^{(0)}}\Delta I + \frac{\partial \dot{\mathbf{r}}^{(0)}}{\partial w^{(0)}}\Delta w = -\Delta I \sum_l \frac{\partial}{\partial I^{(0)}}(\omega_l \mathbf{b}_l) \sin(\omega_l t + \delta_l) \\ &- \Delta w 2\pi \sum_l l\omega_l \mathbf{b}_l \cos(\omega_l t + \delta_l). \end{aligned} \tag{6.10}$$

Substituting Eqs. (6.10) and (6.9) in Eq. (4.4) and averaging over the phases δl, φl, we obtain without difficulty the result

$$\begin{aligned} \frac{e}{c}\sum_{\mu} \overline{\dot{\mathbf{r}}^{(1)} \dot{\mathbf{A}}_\mu^{(0)}} = &-\frac{\pi e^2}{4c^2}\sum_{\mu, l}\frac{\partial}{\partial I^{(0)}}[\omega_l (\mathbf{e}_\mu \mathbf{b}_l)]\, \omega_\mu a_\mu^2 l\omega_l (\mathbf{e}_\mu \mathbf{b}_l)\frac{\sin(\omega_l - \omega_\mu)t}{\omega_l - \omega_\mu} \\ &- \frac{\pi e^2}{4c^2}\sum_{\mu, l}\frac{\partial \omega_l}{\partial I^{(0)}}\omega_\mu a_\mu^2 l\omega_l^2 |\mathbf{e}_\mu \mathbf{b}_l|^2 \\ &\times\left(\frac{t\cos(\omega_l - \omega_\mu)t}{\omega_l - \omega_\mu} - \frac{\sin(\omega_l - \omega_\mu)t}{(\omega_l - \omega_\mu)^2}\right). \end{aligned} \tag{6.11}$$

Using the obvious relation

$$\frac{t\cos(\omega_l - \omega_\mu)t}{\omega_l - \omega_\mu} - \frac{\sin(\omega_l - \omega_\mu)t}{(\omega_l - \omega_\mu)^2} = -\frac{d}{d\omega_\mu}\frac{\sin(\omega_l - \omega_\mu)t}{\omega_l - \omega_\mu}$$

and then averaging Eq. (6.11) over the time, we get

$$\begin{aligned} \frac{e}{c}\sum_{\mu} \overline{\dot{\mathbf{r}}^{(1)} \dot{\mathbf{A}}_\mu^{(0)}} = &-\frac{\pi e^2}{4c^2}\sum_{\mu, l}\frac{\partial}{\partial I^{(0)}}[\omega_l^2 |\mathbf{e}_\mu \mathbf{b}_l|^2]\, l\omega_\mu a_\mu^2 \frac{1 - \cos(\omega_l - \omega_\mu)\tau}{\tau(\omega_l - \omega_\mu)^2} \\ &+ \frac{\pi e^2}{4c^2}\sum_{\mu, l}\frac{\partial \omega_l}{\partial I^{(0)}} l\omega_l^2 |\mathbf{e}_\mu \mathbf{b}_l|^2 \omega_\mu a_\mu^2 \frac{d}{d\omega_\mu}\frac{1 - \cos(\omega_l - \omega_\mu)\tau}{\tau(\omega_l - \omega_\mu)^2}. \end{aligned} \tag{6.12}$$

Replacing the summation over k by an integration and using Eqs. (3.3), (5.4), we get finally

$$\begin{aligned} \frac{e}{c}\sum_{\mu} \overline{\dot{\mathbf{r}}^{(1)} \dot{\mathbf{A}}_\mu^{(0)}} &= -\frac{8\pi^4 e^2}{3c}\sum_l l\frac{\partial}{\partial I^{(0)}}[\omega_l \mathbf{b}_l^2 I(\omega_l)] \\ &= -\frac{4\pi^3 e^2}{3c}\sum_l \omega_l \frac{d}{dE}[\omega_l \mathbf{b}_l^2 I(\omega_l)]. \end{aligned} \tag{6.13}$$

We must take the derivative of the intensity I with respect to the energy as given by

$$\frac{dI}{dE} = \frac{dI}{d\omega}\frac{d\omega}{dE}.$$

In the case of the harmonic oscillator [frequency independent of E, and $\mathbf{b}^2 = (2/m\omega^2)E$] the formula (6.13) takes the form

$$\frac{e}{c}\sum_{\mu} \overline{\dot{\mathbf{r}}^{(1)} \dot{\mathbf{A}}_\mu^{(0)}} = -\frac{8\pi^3 e^2}{3mc} I(\omega). \tag{6.14}$$

Apart from a factor $\frac{1}{3}$ this formula is the same as Eq. (5.6). The factor $\frac{1}{3}$ is due to the fact that Eq. (6.14) has been derived for an oscillator with one

degree of freedom, whereas Eq. (5.6) corresponds to an oscillator with three degrees of freedom.*

If the intensity does not depend on the frequency, dI/dE = 0, the formula (6.13) takes the form

$$\frac{e}{c}\sum_{\mu}\overline{\dot{\mathbf{r}}^{(1)}\dot{\mathbf{A}}_{\mu}^{(0)}}=-\frac{4\pi^3e^2}{3c}\sum_{l}I(\omega_l)\,\omega_l\,\frac{d}{dE}(\omega_l\mathbf{b}_l^2). \qquad (6.15)$$

Obviously the sign of the quantity (6.15) depends on the sign of the derivative, i.e., on whether the quantity $\omega_l\mathbf{b}_l^2$ increases or decreases as the energy of the system increases. It is not hard to show that for quite a number of nonlinear systems Eq. (6.15) gives absorption, and not induced emission, at all frequencies ω_l.

We shall show this for the examples of a particle in a field $U(x) = \alpha|x|^q$, where $q > 0$, and of a rigid rotator.

From the definition of $\mathbf{b}_l$ it follows that

$$\omega_l\mathbf{b}_l^2=\frac{16}{l^2\pi^2}\omega_0\left[\int_0^a\cos l\omega_0 t(x)\,d(x)\right]^2, \qquad (6.16)$$

where

$$t(x)=\int_0^x\frac{dy}{\sqrt{\frac{2}{m}[E-U(y)]}},\quad \omega_0=\frac{\pi}{2}\left\{\int_0^a\frac{dy}{\sqrt{\frac{2}{m}[E-U(y)]}}\right\}^{-1}; \qquad (6.17)$$

a is the turning point given by the condition[3] $E - U(y) = 0$. Substituting Eq. (6.17) in Eq. (6.16) and making the change of variables

$$z=y\left(\frac{a}{E}\right)^{\frac{1}{q}},\quad u=x\left(\frac{a}{E}\right)^{\frac{1}{q}},$$

we get

$$\omega_l\mathbf{b}_l^2=\frac{8\sqrt{2}}{l^2\pi\sqrt{m}}\left(\frac{1}{a}\right)^{\frac{1}{q}}E^{\frac{2+q}{q}}\left[\int_0^1\frac{dz}{\sqrt{1-z^q}}\right]^{-1}$$

$$\times\left[\int_0^1\cos\left(2\pi l\frac{\int_0^u\frac{dz}{\sqrt{1-z^q}}}{\int_0^1\frac{dz}{\sqrt{1-z^q}}}\right)du\right]^2,$$

$$\omega_l\frac{d}{dE}(\omega_l\mathbf{b}_l^2)=\frac{2+q}{mq}\left[\frac{2}{\int_0^1\frac{dz}{\sqrt{1-z^q}}}\int_0^1\cos\left(2\pi l\frac{\int_0^u\frac{dz}{\sqrt{1-z^q}}}{\int_0^1\frac{dz}{\sqrt{1-z^q}}}\right)du\right]^2. \qquad (6.18)$$

For all values of l the expression (6.18) is positive. Consequently, again according to Eq. (6.15) the field energy decreases at all frequencies ω_l, i.e., only absorption by the oscillator is possible, and not induced emission.

A second and extremely characteristic example is the rigid rotator. We assume for simplicity that the rotator consists of a particle with charge e and mass m revolving around a fixed axis at the distance R. In this case the rotation frequency is connected with the energy by the relation

$$E=\frac{1}{2}I\omega^2=\frac{mR^2}{2}\omega^2, \qquad (6.19)$$

where I is the moment of inertia of the rotator. Therefore the sum over l in Eq. (6.15) reduces to a single term, for which

$$\omega\frac{d}{dE}(\omega\mathbf{b}^2)=2\omega\frac{d}{dE}(\omega R^2)=R^2\frac{d}{dE}(\omega^2)=\frac{2}{m},$$

and Eq. (6.15) goes over into the expression obtained earlier for the absorption by a harmonic oscillator.

At the same time, in quantum theory the energy levels of such a rotator are given by the formula

$$E=\frac{\hbar^2k^2}{2I},\quad k=0,\ \pm1,\ \pm2,\ \ldots, \qquad (6.20)$$

and thus are not equidistant. The frequency that corresponds to the transition $|k| \to |k| - 1$ is $\omega'' = (\hbar/2I)(2|k| - 1)$, and that for the transition $|k| \to |k| + 1$ is $\omega' = (\hbar/2I)(2|k| + 1)$, which is not equal to ω''. Therefore there must be induced emission at the frequency ω'' and absorption at ω', with

$$\left.\begin{aligned}S^{\text{ind}}&=\frac{4\pi^3e^2}{3mc}(2|k|-1)I(\omega''),\\ S^{\text{abs}}&=\frac{4\pi^3e^2}{3mc}(2|k|+1)I(\omega').\end{aligned}\right\} \qquad (6.21)$$

Many examples can be given of other systems for which quantum theory gives absorption at some frequencies and induced emission at others, whereas in classical theory, in the framework of our formulation of the problem, only absorption is possible. The cause of this fact, which is due to general properties of classical motions, and also the conditions under which a classical system can amplify radiation, will be brought out in the following section.

7. THE PASSAGE FROM QUANTUM THEORY TO THE LIMIT OF CLASSICAL THEORY

As is well known (cf. e.g., [5]), in the quasi-classical approximation in quantum mechanics what corresponds to motion along a trajectory is a wave packet formed by the superposition of a large number of closely spaced stationary states. We denote by n the average value of the quantum number which labels these stationary states. Then the average value of an operator F over such a wave packet can be written in the form (cf. [5])

$$\langle F\rangle=\sum_{l>0}(F_{n+l,\,n}e^{i\omega_{n+l,\,n^t}}+F_{n-l,\,n}e^{-i\omega_{n,\,n-l^t}}). \qquad (7.1)$$

In the limit the expression (7.1) must coincide with the classical quantity F(t). Since F(t) is a real quantity, it follows from Eq. (7.1) that in the quasi-classical limit

$$F_{n-l,\,n}\to F^*_{n+l,\,n},\quad \omega_{n,\,n-l}\to\omega_{n+l,\,n}\to\omega_l=\omega_0 l, \qquad (7.2)$$

*For simplicity we confine ourselves in what follows to the treatment of the oscillator with one degree of freedom.

where ω_0 is the frequency of the classical motion, and $l = 1, 2, \ldots$ is the number of the harmonic. From this, however, it can be seen that in the classical limit the energy levels of any system are equidistant.

We can also arrive at this same result in a different way. In the quasi-classical approximation the energy levels are determined by the Bohr quantization rule

$$\oint p\,dx = 2\pi\hbar\left(n+\frac{1}{2}\right). \tag{7.3}$$

Calculating the difference between the energy levels $n + \Delta n$ and n by means of Eq. (7.3), we easily get

$$\omega = \frac{\Delta E}{\hbar} = \frac{2\pi}{T}\Delta n\left[1+\frac{1}{2}\hbar\frac{\partial}{\partial E}\left(\frac{2\pi}{T}\right)\Delta n\right], \tag{7.4}$$

where

$$T = \oint \frac{\partial p}{\partial E}\,dx = \oint \frac{dx}{\sqrt{\frac{2}{m}[E-U(x)]}} \quad \text{and} \quad \omega_0 = \frac{2\pi}{T}$$

are the period and frequency of the classical motion. For $\hbar \to 0$

$$\omega \to \frac{2\pi}{T}\Delta n = \omega_0 \Delta n. \tag{7.5}$$

Consequently, $\omega_{n,n-l} \to \omega_0 l$, $\omega_{n+l,n} \to \omega_0 l$, and the difference $\omega_{n+l,n} - \omega_{n,n-l}$, being proportional to $\hbar$, goes to zero. It can be seen from this that in the quasi-classical limit the energy levels of any system are equidistant. According to Eq. (7.4), in the general case ω_0 is a function of the energy. Therefore the distances $\hbar\omega_0$ between adjacent levels are different for different values of E. But in any arbitrarily small, but finite, energy range ΔE there is a very large (infinite for $\hbar \to 0$) number of levels, at distances which are equal up to terms proportional to $\hbar^2$.

Thus the situation that arises is the same as in the case of the harmonic oscillator. At any natural frequency ω_l there will always be transitions $n \to n + l$ (absorption) and $n \to n - l$ (emission), and consequently the only observable quantity is the difference of the energies absorbed and emitted, not each amount separately.

Let us now see how the formulas of quantum theory for radiative processes behave on passage to the limit of classical theory. For definiteness we assume that the incident radiation is isotropic and of natural polarization. In this case the formulas (1.4) lead to the following expressions for the power spontaneously emitted S^{sp}, the induced emission S^{ind}, and the absorption S^{abs}:

$$S^{sp} = \frac{4e^2\omega_{n,n-l}^4}{3c^3}|r_{n,n-l}|^2, \tag{7.6}$$

$$S^{ind} = \frac{16\pi^3e^2\omega_{n,n-l}}{3c\hbar}|r_{n,n-l}|^2 I(\omega_{n,n-l}). \tag{7.7}$$

$$S^{abs} = \frac{16\pi^3e^2\omega_{n+l,n}}{3c\hbar}|r_{n,n+l}|^2 I(\omega_{n+l,n}). \tag{7.8}$$

In these equations $|r_{n,n'}|^2$ is to be understood as the value averaged over all transitions between the degenerate states a and a′ which belong to the levels n and n′: $|r_{n,n'}|^2 = (1/g_n)\sum_{a,a'}|r_{a,a'}|^2$.

To go to the classical limit in these formulas we must let $\hbar \to 0$ and replace $4|r_{n,n\pm l}|^2$ by $|b_l|^2$ [see discussion of Eq. (3.4)].

As a result of this passage to the limit the formula (7.6) for S^{sp}, which does not contain $\hbar$, goes over into the previously obtained formula (3.4) of classical theory. On the other hand the quantities S^{abs} and S^{ind} increase without limit as $\hbar \to 0$. This shows that each of these quantities has no physical meaning by itself in the classical theory. At the same time the difference of these quantities

$$\tilde{S}_l = S_l^{abs} - S_l^{ind} = \frac{16\pi^3e^2}{3c}\frac{1}{\hbar}[\omega_{n+l,n}|r_{n,n+l}|^2 I(\omega_{n+l,n}) - \omega_{n,n-l}|r_{n,n-l}|^2 I(\omega_{n,n-l})] \tag{7.9}$$

remains finite, in accordance with Eq. (7.2). Thus the only quantities which can appear in the classical theory and actually be observed are S^{sp} and $\tilde{S} = S^{abs} - S^{ind}$.

Multiplying and dividing the right member of Eq. (7.9) by ω_l and using the fact that for $\hbar \to 0$

$$\frac{\omega_{n+l,n}|r_{n,n+l}|^2 I(\omega_{n+l,n}-\omega_{n,n-l})|r_{n,n-l}|^2 I(\omega_{n,n-l})}{\hbar\omega_l} \to \frac{1}{4}\frac{d}{dE}[\omega_l b_l^2 I(\omega_l)],$$

we get

$$\tilde{S}_l = \frac{4\pi^3e^2}{3c}\omega_l\frac{d}{dE}[\omega_l b_l^2 I(\omega_l)]. \tag{7.10}$$

Thus in the quasi-classical limit the difference $\tilde{S}_l = S^{abs} - S^{ind}$ goes over into the formula (6.13) of the classical theory. If the intensity is independent of the frequency, Eq. (7.10) is the same as Eq. (6.15). It is convenient to write the last formula in the form

$$\tilde{S}_l = \frac{8\pi^3e^2}{3mc}I(\omega_l)f_l, \tag{7.11}$$

where

$$f_l = \frac{m\omega_l}{2}\frac{d}{dE}[\omega_l b_l^2]. \tag{7.12}$$

This quantity is none other than the classical limit of the difference of the absolute values of the oscillator strengths $f_{n,n+l}$ and $f_{n,n-l}$ of the transitions. Since

$$f_{n,n'} = \frac{2m}{\hbar}\omega_{n,n'}|r_{nn'}|^2, \tag{7.13}$$

in the classical limit

$$|f_{n,n+l}| - |f_{n,n-l}| \to f_l. \tag{7.14}$$

The oscillator strengths $f_{nn'}$ of the transitions obey the sum rule

$$\sum_{n'} f_{nn'} = \sum_{l>0}(|f_{n,n+l}| - |f_{n,n-l}|) = 1. \tag{7.15}$$

Consequently

$$\sum_l f_l = 1. \tag{7.16}$$

According to Eq. (7.16) the sum of the $\widetilde{S}_l$ over all frequencies at which radiative processes are possible (i.e., over all l) is greater than zero. Thus as the overall result there is always absorption. If the system is such that radiative processes are possible at only one frequency, then again $\widetilde{S} > 0$. $\widetilde{S}$ is then determined by the same formula as in the case of the harmonic oscillator. An example of such a system is the rigid rotator considered above. As for the individual terms of the sums (7.15) and (7.16), the problem of their signs is a difficult one in the general case. The condition (7.16) does not impose any restrictions on the signs of the individual terms in the sum. Still, as was already stated in the preceding section, consideration of a number of concrete examples with very different properties shows that for a wide class of systems $\widetilde{S}_l > 0$, i.e., absorption occurs at all frequencies ωl.

Thus at least for oscillators of this general type the quantum and classical theories lead to essentially different results. A quantum-mechanical oscillator in the level E_γ absorbs radiation at frequencies $\omega_{\gamma'\gamma}$ ($E_{\gamma'} > E_\gamma$) and gives induced emission, i.e., intensifies the radiation, at frequencies $\omega_{\gamma\gamma''}$ ($E_{\gamma''} < E_\gamma$).

In the classical theory there is just one harmonic corresponding to each pair of frequencies $\omega_{\gamma'\gamma}$ and $\omega_{\gamma\gamma''}$, and an oscillator with the energy $E = E_\gamma$ (energy of the free vibrations) absorbs radiation at all the harmonics ωl.

It does not, however, follow from what has been said that induced emission, and thus also amplification of radiation, are altogether impossible in the classical theory. The point is that the results obtained above for oscillators with a fixed energy cannot be carried over to the case of an ensemble of oscillators distributed over a range of energies. In particular, in the general case the absorption at the l-th harmonic by oscillators with energies in the range E, $E+\Delta E$ is not given by

$$\int_E^{E+\Delta E} \widetilde{S}_l(E)\, N(E)\, dE. \qquad (7.17)$$

As will be shown below, this is again due to specific features of the quasi-classical motion.

Let us consider an assembly of oscillators with a quasi-classical (i.e., almost equidistant) energy spectrum, with the distribution over the energy levels given by occupation numbers N_m. We then find the absorption power ΔQ^{abs} corresponding to transitions $m \to m+1$ and the induced-emission power ΔQ^{ind} corresponding to transitions $m \to m-1$, in the frequency range ω, $\omega+\Delta\omega$. The quantities ΔQ^{abs} and ΔQ^{ind} can be found by summing $N_m S^{abs}_{m,m+1}$ and $N_m S^{ind}_{m,m-1}$ over all levels m for which $\omega_{m+1,m}$ and $\omega_{m,m-1}$ are contained in the narrow range of frequencies. We denote by the indices k and k′ the lowest and the highest levels for which the frequencies $\omega_{k+1,k}$ and $\omega_{k',k'-1}$ still fall in the interval ω, $\omega+\Delta\omega$ (Fig. 1).

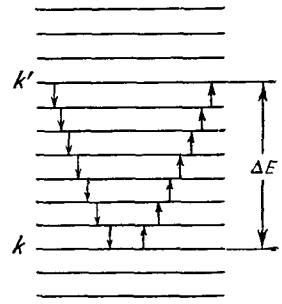

FIG. 1. Scheme of transitions inside the interval ΔE.

Then the total increase of the energy of the oscillators per unit time resulting from the two processes will be given by

$$\Delta Q = \Delta Q^{abs} - \Delta Q^{ind} = \sum_{m=k}^{k'-1} N_m S^{abs}_{m,m+1} - = \sum_{m=k+1}^{k'} N_m S^{ind}_{m,m-1}$$
$$= -\sum_{m=k+1}^{k'} (N_m - N_{m-1})\, S^{ind}_{m,m-1}.$$

Substituting in this formula the expression (7.7) and replacing the occupation numbers N_m by $N(E_m)$ $\hbar\omega_0(E_m)$, where N(E) is the energy distribution function of the oscillators, we get in the quasi-classical limit

$$\frac{N_m - N_{m-1}}{E_m - E_{m-1}} \to \hbar \frac{\partial}{\partial E}(\omega_0 N)$$

and

$$\Delta Q = -\int_E^{E+\Delta E} dE \cdot \frac{4\pi^3 e^2}{3c} \omega_0 b^2 \frac{\partial}{\partial E}(\omega_0 N)\, I(\omega_0), \qquad (7.18)$$

where

$$\omega_0(E) = \omega, \quad \omega_0(E+\Delta E) = \omega + \Delta\omega.$$

By repeating these same arguments in the general case of transitions $m \to m+l$, $m \to m-l$, we can get an analogous expression for the total increase of energy of the oscillators per unit time corresponding to the l-th harmonic:

$$\Delta Q_l = -\frac{4\pi^3 e^2}{3c} \int_E^{E+\Delta E} dE \cdot \omega_l b_l^2 \frac{\partial}{\partial E}(\omega_l N)\, I(\omega_l), \qquad (7.19)$$

where

$$\omega_l(E) = \omega, \quad \omega_l(E+\Delta E) = \omega + \Delta\omega.$$

Let us compare this expression with Eq. (7.18). We integrate Eq. (7.19) by parts. Using Eqs. (7.11), (7.13), we get

$$\Delta Q_l = -\frac{4\pi^3 e^2}{3c} I(\omega_l)\, \omega_l^2 b_l^2 N \Big|_E^{E+\Delta E} + \int_E^{E+\Delta E} \widetilde{S}_l(E)\, N(E)\, dE. \qquad (7.20)$$

This expression agrees with Eq. (7.17) only when the first term in Eq. (7.20) is zero at the end points of the interval E, $E+\Delta E$. In the general case ΔQ_l is not equal to the expression (7.17), and in particular can be of the opposite sign. As can be seen from Eq. (7.19), the sign of ΔQ_l is determined by the sign of the deriv-

ative $(\partial/\partial E)(\omega_l N)$. If $(\partial/\partial E)(\omega_l N) < 0$, then $\Delta Q_l > 0$, and consequently there is absorption by the oscillators. If, on the other hand, $(\partial/\partial E)(\omega_l N) > 0$, then $\Delta Q_l < 0$. This means that the oscillators lose energy and amplify the incident radiation. Thus a system of classical oscillators can either absorb or amplify incident radiation, depending on the sign of $(\partial/\partial E)(\omega_l N)$. The condition $(\partial/\partial E)(\omega_l N) > 0$ is completely equivalent to the condition for inverted populations $N_m > N_{m-l}$ (or $N_{m+l} > N_m$) in quantum theory, since the quantities which correspond to the occupation numbers N_m in the quasi-classical limit are

$$N(E_m)\hbar\omega_0(E_m) = \frac{\hbar}{l} N(E_m)\omega_l(E_m).$$

If N(E) varies with E much faster than does ω_l then we can replace $(\partial/\partial E)(\omega_l N)$ by $\omega_l(\partial N/\partial E)$. In this case the sign of ΔQ_l is entirely determined by the form of the distribution function N(E). For $(\partial N/\partial E) < 0$ there is absorption, and for $(\partial N/\partial E) > 0$ there is induced emission.

What we have just said is graphically illustrated in Fig. 2. In the frequency range $\Delta\omega_1$ which corresponds to the energy range ΔE_1 we have $(\partial N/\partial E) > 0$, $\Delta Q_l < 0$, and consequently the incident radiation is amplified. In the frequency range $\Delta\omega_2$ (the energy range ΔE_2) we have $(\partial N/\partial E) < 0$, $Q_l > 0$, i.e., there is absorption, Finally, in the range $\Delta\omega$ (the energy range ΔE) there are frequencies at which radiation is absorbed, and also frequencies at which the radiation is amplified. The expression (7.19), which in this case agrees with (7.17), describes the total effect. The smaller the width of the distribution N(E), the smaller the energy ranges for which $(\partial N/\partial E)$ is larger and smaller than zero, and the more difficult it is to distinguish the frequency range in which induced emission occurs. In the case of a δ-function distribution it is obviously impossible to distinguish frequency ranges corresponding to absorption and to induced emission. When in Eq. (7.19) we set $N = N_0\delta(E - E_0)$ and integrate over an arbitrarily small range ΔE containing the point E_0 we get

$$\Delta Q_l = N_0\tilde{S}_l(E_0) > 0.$$

It is for this reason that in our treatment of an ensemble of oscillators all having the same energy we did not get any induced emission.

We also note that since $\Delta\omega_1 \sim (\partial\omega_l/\partial E)\Delta E_1$, when we have a given distribution function N(E) the range of frequencies $\Delta\omega_1$ in which induced emission occurs will be smaller if the anharmonicity is smaller, i.e., if the energy dependence of the frequency ω_l is weaker. Therefore the smaller the anharmonicity the harder it is to distinguish a frequency range in which induced emission can be observed.

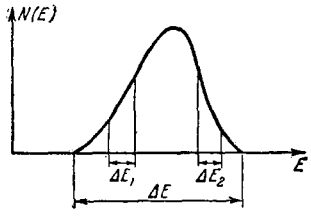

FIG. 2. Example of a distribution function N(E) with particular portions corresponding to induced emission and to absorption.

For the harmonic oscillator the only quantity which can be observed or calculated in the theory is the integrated effect, since in this case the frequency does not depend on the energy and in Eqs. (7.19) and (7.20) the integration must be extended over the interval $0, \infty$. Then the first term in Eq. (7.20) is zero [b(0) = 0, $N(\infty) = 0$], and since $\tilde{S}(E)$ does not depend on E the formula (7.20) gives

$$\Delta Q = \tilde{S}\int_0^\infty N(E)\,dE = \tilde{S}N > 0,$$

where N is the total number of oscillators.

Thus under specific conditions both quantum and classical systems have to equal degrees the capacity to amplify incident radiation instead of absorbing it. The differences existing between the effects for quantum and classical systems (for example, for a δ-function distribution) are entirely due to specific features of their energy spectra.

8. CLASSICAL THEORY FOR AN ENSEMBLE OF OSCILLATORS DISTRIBUTED OVER ENERGY

It now remains for us to obtain the general formula (7.19) for the increase of energy ΔQ_l of an ensemble of oscillators not by making a passage to a limit from quantum theory, as we did in the preceding section, but entirely within the framework of a classical treatment. Let us consider the l-th-harmonic absorption of radiation in the frequency range ω, $\omega+\Delta\omega$ by an ensemble of oscillators. It is obvious that the absorption will be due to oscillators with energies in the range E, $E+\Delta E$, where $\omega_l(E) = \omega$ and $\omega_l(E+\Delta E) = \omega+\Delta\omega$. The change of the energy of the oscillators per unit time owing to the absorption consists of two parts. First, there is a change of the distribution function in the range ΔE, and second, a certain number of oscillators leave this range, and consequently cease to take part in the absorption at frequencies in the range ω, $\omega+\Delta\omega$. The energy of an oscillator which leaves the interval ΔE through the upper boundary will thereafter remain equal to $E+\Delta E$, and that of one leaving through the lower boundary, to E. Therefore

$$\Delta Q_l = \int_{\Delta E} \dot{N}(E)\,E\,dE + [Ej(E)]\Big|_E^{E+\Delta E}, \qquad (8.1)$$

where j(E) is the flux in the energy space. The distribution function N(E) and the flux j(E) in the energy space can be expressed in terms of the corresponding quantities in phase space, N(g) and j(g) (g is a phase volume):

$$N(E)\,dE = N(g)\,dg, \quad j(E) = j(g). \qquad (8.2)$$

The distribution function in this problem satisfies the Fokker-Planck equation*

$$\dot{N}+\frac{\partial}{\partial g}\left\{\alpha N-\frac{1}{2}\frac{\partial}{\partial g}(\beta N)\right\}=0. \tag{8.3}$$

Since N(g) and j(g) are connected by the equation of continuity

$$\dot{N}+\frac{\partial}{\partial g}j=0, \tag{8.4}$$

it follows from Eq. (8.3) that

$$j=\alpha N-\frac{1}{2}\frac{\partial}{\partial g}(\beta N). \tag{8.5}$$

The quantities α and β in Eqs. (8.3) and (8.5) can be expressed in terms of the mean increment of g and the mean square increment of g per unit time:

$$\alpha=\langle\delta g\rangle,\quad \beta=\langle\delta g^2\rangle. \tag{8.6}$$

Since

$$\delta g=g(E+\delta E)-g(E)=\frac{\partial g}{\partial E}\delta E+\frac{1}{2}\frac{\partial^2 g}{\partial E^2}\delta E^2+\ldots,$$

the calculation of the quantities (8.6) reduces to the calculation of averages of the energy transfer δE, the squared energy transfer δE^2, and so on.

As is well known

$$\frac{\partial g}{\partial E}=T(E)=\frac{2\pi}{\omega_0}, \tag{8.7}$$

where T is the period of the motion. The quantity $\langle\delta E\rangle$ has been calculated earlier—see Eq. (6.13).

In the calculation of $\langle\delta E^2\rangle$ it suffices to use only the first term in Eq. (2.16), since it is the only one that gives a contribution $\sim e^2$.

Accordingly,

$$\langle\delta E^2\rangle=\frac{1}{\tau}\left\langle\left(\int_0^\tau\frac{e}{c}\sum_\mu\dot{\mathbf{r}}^{(0)}\dot{\mathbf{A}}_\mu^{(0)}\,dt\right)^2\right\rangle. \tag{8.8}$$

Performing the integration over dt and averaging over the phases of the free vibrations of the oscillator and the field, we get

$$\frac{1}{\tau}\frac{e^2\omega_l^2}{4c^2}\sum_\mu(\mathbf{e}_\mu\mathbf{b}_l)^2a_\mu^2\omega_\mu^2\frac{1-\cos(\omega_l-\omega_\mu)\tau}{(\omega_l-\omega_\mu)^2}. \tag{8.9}$$

Repeating the arguments used for the derivation of Eq. (3.4) and using Eq. (5.4), we get for the case of isotropic radiation

$$\langle\delta E^2\rangle=\frac{8\pi^3e^2}{3c}\omega_l^2\mathbf{b}_l^2I(\omega_l). \tag{8.10}$$

Thus we have

$$\alpha=\frac{\partial g}{\partial E}\langle\delta E\rangle+\frac{1}{2}\frac{\partial^2 g}{\partial E^2}\langle\delta E^2\rangle=\frac{8\pi^4e^2}{3c}\omega_0l^2\frac{d}{dE}[\mathbf{b}_l^2I(\omega_l)]$$
$$=\frac{16\pi^3e^2}{3c}l^2\frac{\partial}{\partial g}[\mathbf{b}_l^2I(\omega_l)], \tag{8.11}$$

*We recall that the Fokker-Planck equation in the general case must be written in precisely the form (8.3). It is only in certain special cases that this equation goes over into a corresponding equation for N(E),

$$\dot{N}(E)+\frac{\partial}{\partial E}\left\{\alpha N(E)-\frac{1}{2}\frac{\partial}{\partial E}[\beta N(E)]\right\}=0.$$

$$\beta=\left(\frac{\partial g}{\partial E}\right)^2\langle\delta E^2\rangle=\frac{32\pi^5e^2}{3c}l^2\mathbf{b}_l^2I(\omega_l). \tag{8.12}$$

It can be seen from Eqs. (8.11) and (8.12) that α and β satisfy the relation

$$\alpha-\frac{1}{2}\frac{\partial\beta}{\partial g}=0. \tag{8.13}$$

Therefore we have by Eqs. (8.5) and (8.13)

$$j=-\frac{1}{2}\beta\frac{\partial N}{\partial g}. \tag{8.14}$$

Let us return to the expression (8.1). Using Eqs. (8.4) and (8.14) and integrating Eq. (8.1) by parts, we get

$$\Delta Q_l=-(Ej)\int_E^{E+\Delta E}+\int_{\Delta g}\frac{\partial E}{\partial g}j\,dg+(Ej)\Big|_E^{E+\Delta E}$$
$$=-\frac{1}{2}\int_{\Delta g}\beta\frac{\partial E}{\partial g}\frac{\partial N}{\partial g}dg. \tag{8.15}$$

Since according to Eqs. (8.2) and (8.7) we have

$$N(g)=\frac{\omega_0}{2\pi}N(E),$$

Equation (8.15) finally gives

$$\Delta Q_l=-\frac{4\pi^3e^2}{3c}\int_E^{E+\Delta E}\omega_l\mathbf{b}_l^2I(\omega_l)\frac{\partial}{\partial E}(\omega_lN)\,dE. \tag{8.16}$$

This is the general formula for the absorption in the frequency range $\Delta\omega$, and is exactly Eq. (7.19). A detailed discussion of this formula is given in the preceding section. We note once more that according to this formula a system of classical oscillators can either absorb or amplify incident radiation, depending on the sign of the derivative $(\partial/\partial E)(\omega_lN)$.

It can be seen from the derivation of Eq. (8.16) that in the general case the induced processes are determined both by the mean energy $\langle\delta E\rangle$ transferred to an oscillator and by the mean square fluctuation $\langle\delta E^2\rangle$. It is essential that contributions to these processes come from both the first term and the second term in Eq. (2.16).

We note that in the quantum theory of radiative processes there are no expressions analogous to the first term in Eq. (2.16). This is due to the fact that the average value of the operator $\dot{\mathbf{r}}$ is equal to zero for any stationary state. As has already been pointed out, however, in the quasi-classical approximation of quantum mechanics a stationary state of a system by no means gives the motion of the particle along a definite trajectory (cf. e.g., [5]). What corresponds to motion along a trajectory in the quasi-classical approximation is a superposition of a large number of closely spaced stationary states, i.e., a wave packet $\psi=\sum_n C_n\psi_n$. For such a wave packet the average value of $\dot{\mathbf{r}}$ is indeed different from zero, and is of just the form necessary to get the expression $(e/c)\sum_\mu\dot{\mathbf{r}}^{(0)}\dot{\mathbf{A}}_\mu^{(0)}$.

In conclusion we shall show that the formula (8.16) allows us to obtain the Rayleigh-Jeans distribution directly from the equality of the absorbed energy ΔQ_l and the quantity

$$\Delta Q_l^{sp} = \int_E^{E+\Delta E} S_l^{sp}(E) N\, dE$$

of energy emitted by an ensemble of arbitrary nonlinear oscillators in a state of thermodynamic equilibrium.

Since in this case

$$N(g) \sim e^{-\frac{E}{kT}}$$

and consequently

$$\omega_0(E) N(E) \sim e^{-\frac{E}{kT}}, \quad \frac{\partial}{\partial E}[\omega_0(E) N(E)] \sim -\frac{1}{kT} e^{-\frac{E}{kT}},$$

it follows from Eqs. (3.4) and (8.16) that

$$\int_E^{E+\Delta E} dE \frac{e^2}{3c^3} \omega_l^2 l b_l^2 e^{-\frac{E}{kT}} = \int_E^{E+\Delta E} dE \cdot \frac{4\pi^3 e^2}{3c} \omega_l l b_l^2 I(\omega_l) \frac{1}{kT} e^{-\frac{E}{kT}}.$$

From this we get

$$I(\omega) = \frac{\omega^2 kT}{4\pi^3 c^2}. \tag{8.17}$$

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