THE MÖSSBAUER EFFECT AND THE THEORY OF RELATIVITY

Ya. A. SMORODINSKIĬ

Uspekhi Fiz. Nauk **79**, 589-594 (April, 1963)

1. INTRODUCTION

IMMEDIATELY after the discovery of the Mössbauer effect (cf. [1,2]), there were discussions in the literature of the possibility of testing the general theory of relativity by using this effect. The experiments of Pound and others [2-4] showed that under laboratory conditions one can observe the influence of the gravitational field on the frequency of a photon, and that the magnitude of the effect is in agreement with the predictions of the theory (cf. the paper of Sherwin [5]). The question arises: just what is being tested in the Pound experiments? In order to test a theory we must first agree on what features we regard as already established and which ones we question. The situation is simplest when the experiment must distinguish between two theories which give different predictions. In that case we look for an answer to a "yes-no" question, and there are usually no ambiguities in the interpretation of the experiments.

The problem is more complicated in the case of the general theory of relativity. There is no other theory which can combine the special theory of relativity with the gravitational field. The general theory of relativity is connected by a such a strong logical chain to other branches of physics that any test of it reduces in the last analysis to a test of the conclusions of the special theory or even simply to a test of the law of conservation of energy. Since the actual experiments are not very accurate, it is obvious that a rigorous test of the equivalence (locally, at a given point and a given instant of time) of the gravitational field and an acceleration is not possible. For this reason, within the group of crude experiments now available, one can (as has been done in the literature) give an explanation on the basis of other theories. In this sense the general theory of relativity is distinguished from the unlimited number of other theories by its internal structure and theoretical completeness and by its better agreement with experiment.

A good example of a test of the general theory of relativity which has led to a great deal of discussion is the experiment of Pound. These experiments could be interpreted as a test of one of the equations of the theory of gravitation—the Schwarzschild central field equation. Let us discuss these experiments in somewhat more detail than has been given previously, and see what direct conclusions can be drawn from the results without using other considerations which enter in constructing a theory of gravitation. We first recall the fundamental property of the Schwarzschild solution, which describes the nature of the gravitational field of a material body (for more detail, see [6,7]).

2. THE SCHWARZSCHILD SOLUTION

The metric of a spherically symmetric field, which at large distances gives the Newtonian attraction, is usually written in the form

$$ds^2 = (c^2 + 2\varphi)\, dt^2 - \frac{dr^2}{1+\frac{2\varphi}{c^2}} - r^2\, d\Omega^2. \tag{2.1}$$

Here $\varphi(r)$ is the Newtonian potential $\kappa M/r$, $r^2 d\Omega$ is the surface element on the sphere. a) The metric in the form of (2.1) chooses the time coordinate so that the coefficients are independent of time; b) the scale is chosen so that the area of the sphere is always equal to $4\pi r^2$, while the radius of the sphere is always less than r. We point out that dr differs from the Euclidean line element by an amount of order $1/c^2$. Setting $\varphi \sim gh$, where h is the height above the Earth's surface, we get $\varphi/c^2 \sim gh/c^2$. Under laboratory conditions, $h \sim 10\,\mathrm{m}$, $\varphi/c^2 \sim 10^{-15}$. A quantity of this same order also gives the difference in rate of clocks at different points.*

We note that in the metric (2.1) the coordinate r is still not given a definite meaning, since there is no indication of how to measure it or to compare it with the usual Euclidean coordinate.

The coordinate r appears in the argument of the potential $\varphi(r)$. If we stop with terms of order $1/c^2$, it is irrelevant how we define r, and we can take the Euclidean value for r. But if we are interested in effects of higher order, the question of the determination of distances requires special treatment.

The spatial metric can be changed by a coordinate transformation. It is convenient to use the so-called isotropic metric, obtained from (2.1) by the substitution

$$r = r_1\left(1 + \frac{r_0}{4r_1}\right), \tag{2.2}$$

where $r_0 = \kappa M/c^2$ is the gravitational radius of the source, so that $\varphi/c^2 = r_0/r$. With the variable r_1, the metric becomes

$$ds^2 = \frac{\left(1-\frac{r_0}{4r_1}\right)^2}{\left(1+\frac{r_0}{4r_1}\right)^2} c^2\, dt^2 - \left(1 + \frac{r_0}{4r_1}\right)^2 (dr_1^2 + r_1^2\, d\Omega^2). \tag{2.3}$$

*On the surface of the sun, $\varphi \approx 2 \times 10^{-6}$. This number also gives the accuracy of the astronomical experiments (cf. [8]).

In this metric the magnitude of the line element does not depend on direction, in accordance with the usual measurement of length using a rigid measuring rod. We note that the velocity of light in both metrics depends on the coordinates:

$$v_l^2 = c^2\left(1-\frac{2r_0}{r}\right) = \frac{\left(1-\frac{r_0}{r_1}\right)^2}{\left(1+\frac{r_0}{r_1}\right)^4}. \tag{2.4}$$

An important point is that the difference in the metric occurs only starting with terms $\sim c^{-4}$. It then follows that in experiments whose accuracy does not permit determination of terms of order c^{-4} one can get no information about the spatial curvature of the space, and all the effects are described phenomenologically as a change in the light velocity. It also follows that more precise experiments should also include the measurement of geometrical lengths and angles. Experiments of this type are the measurement of the deflection of light in the field of the sun and the motion of the perihelion of Mercury. (In both experiments one measures angles.) We note that in measuring the deflection of a light ray one measures a null interval, and in the experiment one determines only one quantity (the light velocity). For a planet the two effects—curvature of space and the change in mass with velocity, give different contributions (this is related to the fact that $ds^2 \neq 0$), and the experiments on the perihelion motion together with others give information on the geometry of space in the neighborhood of the sun.

3. THE FREQUENCY OF A QUANTUM IN A GRAVITATIONAL FIELD

The usual derivation of the change in frequency in a gravitational field reduces to the following. We start from the fact that: 1) the number of vibrations of a quantum between two events is independent of the observer; 2) the frequency of radiation or the internal properties of a radiating system which is at rest relative to the observer and located at the same place as the observer are independent of the location. This means that we are assuming that the effect of a gravitational field on the properties of nuclei and atoms is negligibly small. More precisely, we are assuming that the Planck constant does not depend on the gravitational field and that the whole dependence comes from the dependence of the mass-energy of the system on its coordinates.

Under these assumptions the product $\omega\Delta t$ will be an invariant. Setting $\varphi = 0$ and $\omega = \omega_0$ on the surface of the Earth, we can write the formula for the frequency corresponding to the potential φ:

$$\omega = \omega_0\left(1-\frac{2r_0}{r}\right)^{1/2} \tag{3.1}$$

in the metric (2.1), or in the form

$$\omega = \omega_0\left(\frac{1-\frac{r_0}{4r_1}}{1+\frac{r_0}{4r_1}}\right)^2 \tag{3.2}$$

in the metric (2.2). The difference between formulas (3.1) and (3.2) is caused by the different choices of the coordinate r. It is convenient to write both formulas in a common form:

$$\omega = \omega_0\left(1+\frac{2\varphi}{c^2}\right)^{1/2} \tag{3.3}$$

In the metric (2.1), φ coincides with the Newtonian potential $\varphi = -r_0/r$, while in the isotropic metric

$$\varphi = -\frac{r_0/r}{\left(1+\frac{r_0}{4r_1}\right)^2}, \tag{3.3'}$$

which differs from the Newtonian potential by terms of order c^{-4}.

The form (3.3) is convenient because in it the potential serves as a natural coordinate, almost equal to the reciprocal of the distance (in units of r_0). For discussion of experiments it is convenient to change the definition slightly, writing (3.3) in the form

$$\omega = \omega_0\left(1+\frac{\psi}{c^2}\right), \tag{3.4}$$

so that

$$\psi = -1+\sqrt{1+2\varphi}. \tag{3.5}$$

Using formula (3.4), one can in principle associate a potential ψ with each point in space. If we introduce in place of ψ the reciprocal ψ^{-1}, we can use it as a coordinate differing from r only by terms of order c^{-4}.

From these remarks it is clear that an experiment which measures the shift in frequency can give information about the space metric only if space measurements have already been done. To order c^{-2} these distance measurements can be made in the Euclidean approximation; to order c^{-4}, they must include terms up to c^{-2}. The determination of corrections of higher order already requires the inclusion of the gravitational radiation (of order c^{-5}), and the whole problem becomes very complicated, even theoretically.

Now let us go on to experiments using the Mössbauer effect. The experiments of Pound and Rebka, and Cranshaw, Schiffer, and Whitehead consisted in comparing the frequency of the photon radiated by an excited Fe^{57} nucleus when it is located at a height of about 10 m above the Earth's surface with the resonance frequency which can be absorbed by an unexcited Fe^{57} nucleus at the surface of the Earth. The comparison was done by absorbing the photon in a target moving at such a velocity that the Doppler shift just compensated the change in frequency due to the gravitational field.

Let us try to treat the observed effect starting only with the conservation laws, and see what must be added to these laws for a complete description of the experiment.

First we must use a system which has at least two quantum states (iron nucleus), which serve as clocks whose frequency, for an observer at the same location in space, is independent of the position of the system. It is obvious that experiments of this type could not have been done with water clocks of the kind used by Galileo in the first experiments on gravitation; the rate of such clocks decreases as they are raised above the Earth's surface. One cannot see how the experiment could be done if instead of a quantum source one had a classical dipole such as a radio antenna. In such a case it would be difficult to determine what is meant by the statement that the final state of the receiver is the same as the initial state of the radiator.

We emphasize this point because it is not a trivial matter that one must use quantum clocks in the general theory of relativity, a fact which indicates a deep connection between geometry and quanta (cf. Wigner[9]).

The experiment of Pound and Rebka is described schematically as follows.

Before the experiment. The radiator system, having mass M_0, is at rest at a height corresponding to the potential φ. At the Earth's surface there is an absorber system, with mass m_1, moving with velocity v. By a system we mean the nucleus together with all the equipment (the answer will of course not depend on M_0 and m_1).

After the experiment. The system M_0 has gone over into a state with mass M_1 and (after radiating a quantum) has gotten a velocity u; the system m_1 has changed to a state m_0 and changed its velocity by an amount Δv.

Since these are quantum systems, the energy difference Δm is the same for an observer located with the system. The energy difference for an observer on the Earth is given by the Newtonian law of gravitation and is equal to Δm on the Earth and $\Delta m(1+\varphi)$ at the upper level.

If we assume that the masses are large and the velocity changes small, conservation of energy and momentum gives two equations relating the states of the system before and after the experiment:

$$\Delta m\cdot(1+gh)-M_1\frac{u^2}{2}=\Delta m\cdot\frac{v^2}{2}+v\Delta v\cdot m_1, \tag{3.6a}$$

$$M_1 v=\Delta m\cdot v+m_1\Delta v. \tag{3.6b}$$

In order to write a third equation we must use one of the properties of the electromagnetic field. Namely, in the radiation of an electromagnetic wave there is an energy transfer equal to the momentum transfer. This is a consequence of the special theory of relativity and is verified, for example, in the experiments of Lebedev on light pressure. It contains no assumptions about the action of the gravitational field on quanta. Thus the left sides of (3.6a) and (3.6b) are equal. Neglecting u^2 on the left side of (3.6b) we get a third equation

$$\Delta m\cdot(1+gh)=M_1 v. \tag{3.6c}$$

From (3.6b) and (3.6c) we immediately get the desired result:

$$v=\frac{\varphi}{c},$$

i.e., the target must move with a velocity gh/c in order to absorb the light.

We see that the conservation laws in the form of the special theory of relativity and the quantum nature of the target (the fact that the universal constants are independent of the gravitational field) are sufficient for deriving the formula for the effect.

For comparison we note that if the energy were transmitted vertically not by an electromagnetic field but by a nonrelativistic body such as a ball, the answer would be different. If we ask at what velocity we must fire a rocket (model of the absorber) in order that the ball drop and not go up (model of the absorption), the answer on the basis of the conservation laws will be $\Delta v = 0$; this replaces (3.6c), and $v = \sqrt{2gh}$. This is precisely the result found by Galileo, who did not know that only bodies whose velocities are small compared to the velocity of light fall with the same speed. In experiments with quanta, the velocity does not change with height, but the frequency changes.

Thus, contrary to the common assertion (especially in popular articles) the experiments on the Mössbauer effect test nothing but the law of conservation of energy. In order for experiments of this type to give more information about the geometry, it would be necessary at least to measure the distance between the source and detector. To do this one must, for example, measure the time for light to go from the detector to the source and back. From such an experiment we would get information about the spatial part of the metric.

These remarks can be illustrated by a geometrical argument. The measurement of the frequency shift is simply the establishing of a scale for space axes or for the time axis, but not the establishing of a correspondence between the scales for the two. It is obvious that from measurements along the coordinate axes one cannot determine the curvature of space-time. The measurements of the time of propagation of light can be described as measurements of the base of the isosceles triangle Δt in the (x, t) plane. In this triangle we also know the height (the distance to the source) and the base angles which are determined by the velocity of light. Knowing four elements of the triangle (two angles, the altitude and the base), we can find how much it differs from a triangle in Euclidean geometry and calculate the curvature.

It is clear that the frequently discussed experiments (cf. [9]) on the frequency shift using artificial satellites give no more information than the laboratory experiments, since in this case also we must have a very precise method for measuring the time of propagation of the signal.

It remains to emphasize that the question of the effect of a gravitational field on light was already treated completely by Einstein in his 1911 paper[10] several years before the appearance of the general theory of relativity. It is curious that the computation of the deflection of the light in the field of the sun given in this paper gave a result (0.83″) which is half the correct value, since this effect is related to the curvature of the space.

[1] Barit, Podgoretskiĭ, and Shapiro, JETP 38, 301 (1960), Soviet Phys. JETP 11, 218 (1960).

[2] R. V. Pound and G. A. Rebka Jr., Phys. Rev. Letters 3, 439 (1959); 4, 337 (1960).

[3] Cranshaw, Schiffer, and Whitehead, Phys. Rev. Letters 4, 163 (1960).

[4] Hay, Schiffer, Cranshaw, and Egelstaff, Phys. Rev. Letters 4, 165 (1960).

[5] C. S. Sherwin, Phys. Rev. 120, 17 (1960).

[6] A. S. Eddington, The Mathematical Theory of Relativity, Cambridge, 1924.

[7] L. Landau and E. Lifshitz, The Classical Theory of Fields, Pergamon, 1962.

[8] V. L. Ginzburg, in the collection "Einstein and the Development of Physico-Mathematical Thought," Moscow, AN SSSR, 1962, p. 117.

[9] E. Wigner, Revs. Modern Phys. 29, 255 (1957).

[10] A. Einstein, Ann. Physik 35, 898 (1911).

Translated by M. Hamermesh