THEORY OF CERENKOV RADIATION (III)

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INTRODUCTION

CERENKOV radiation in a medium limited by boundary surfaces exhibits a number of characteristic features that do not appear in unbounded media. The first work in which dielectric boundary effects were considered in Cerenkov radiation was published in 1947. At that time the theory of Cerenkov radiation in infinite media had already been well developed and the application of this phenomenon for various physical purposes was being considered. The earliest proposed application was the use of the Cerenkov effect for the generation of ultrashort radio waves (V. L. Ginzburg,[182] 1947). This proposal has, in recent years, stimulated the investigation of a large number of boundary-value problems: Cerenkov radiation in a channel cut into a refractive medium,[189] the radiation produced by a charge moving over a plane interface between two media,[116,272,329,307,364,171,260] etc. Several years ago V. I. Veksler proposed the so-called coherent technique for acceleration of charged particles.[407] Certain versions of this method are based on the entrainment of a charged particle by electron plasma bunches, a process which is essentially an "inverse" Cerenkov effect. Estimates of the acceleration efficiency require the solution of various boundary-value problems.[64,58-63] Boundary-value problems in Cerenkov theory have also been treated in connection with controlled thermonuclear reactions (stabilization of the current in high-current discharges[310] and Cerenkov radiation of magnetoacoustic waves[309]). Boundary-value problems in Cerenkov theory are also important in connection with the theory of linear accelerators and waveguide systems.[1,8-14,317,353,205,206]

A number of workers have been concerned with the design of Cerenkov generators for millimeter and submillimeter radio waves.[279-281,52,314] The large number of experiments carried out in recent years[52,279-281,114,115,274,313-315,324] indicate that Cerenkov generation of radio waves is engaging the attention of research workers to an ever increasing degree. It is obvious that the design of Cerenkov counters is also intimately involved with boundary-value problems in Cerenkov radiation.

It will be evident from this brief survey that many boundary-value problems in Cerenkov radiation theory have already been solved.

The present review is devoted to an analysis of the characteristic features of the radiation field of a charge moving in a bounded or semi-bounded medium. This paper is a continuation of a review article[65] in which Cerenkov radiation in infinite media was considered. The same numbering system is used for the equations in both papers.

The problems that arise when boundary conditions are taken into account can be classified as stationary and nonstationary problems. Stationary problems are those in which the field produced by the charge moves as a whole with the velocity of the charge, that is to

*The Cerenkov Effect in Infinite Media and in Crystals (Parts I and II) appeared in Usp. Fiz. Nauk 62, 201 (1957).

say, all fields depend solely on the argument $\mathbf{x}-\mathbf{v}t$. A typical example is the motion of a charge along the axis of a cylindrical channel in a dielectric.

In nonstationary problems the field does not move with the same velocity as the charge. An example of a nonstationary problem is the motion of a charge from one medium into another, with the resultant production of transition radiation. We shall be interested in stationary problems only.

III. CERENKOV RADIATION IN THE PRESENCE OF BOUNDARIES

III. 1. Boundary Conditions

The field produced by the motion of a charge in a refractive medium is given by the usual system of Maxwell equations

$$\operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad \operatorname{div}\mathbf{B} = 0,$$
$$\operatorname{rot}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{j}, \quad \operatorname{div}\mathbf{D} = 4\pi\varrho,$$
$$\mathbf{D} = \varepsilon\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H}. \qquad \text{(III.1)*}$$

Here, as usual, we regard in the coordinate representation the quantities ϵ and μ as operators that reduce to functions of frequency ω and wave vector $\mathbf{k}$ in the Fourier representation. In other words, if

$$(\mathbf{E};\ \mathbf{D};\ \mathbf{H};\ \mathbf{B}) = \int (\mathbf{E}_{\mathbf{k}\omega};\ \mathbf{D}_{\mathbf{k},\omega};\ \mathbf{H}_{\mathbf{k},\omega};\ \mathbf{B}_{\mathbf{k},\omega})\, e^{i\mathbf{k}\mathbf{x}-i\omega t}\, d\mathbf{k}\, d\omega,$$

then

$$\mathbf{D}_{\mathbf{k},\omega} = \varepsilon(\mathbf{k},\ \omega)\,\mathbf{E}_{\mathbf{k},\omega} \qquad \text{(III.2)}$$

and

$$\mathbf{B}_{\mathbf{k},\omega} = \mu(\mathbf{k},\ \omega)\,\mathbf{H}_{\mathbf{k},\omega}. \qquad \text{(III.3)}$$

The quantities ϵ and μ can be different functions of frequency ω (and wave vector $\mathbf{k}$) on the two sides of a boundary separating two media. Unless otherwise specified we assume below that ϵ and μ are functions of frequency ω only.

The fields must satisfy the usual boundary conditions at each boundary.

Consider a boundary separating two media. We denote one medium by the subscript "1" and the second by "2". The normal to the boundary surface $\mathbf{n}$ is directed from medium 1 into medium 2. The electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ satisfy the following conditions at the boundary:

$$[\mathbf{n},\ \mathbf{E}_2 - \mathbf{E}_1] = 0, \quad [\mathbf{n},\ \mathbf{H}_2 - \mathbf{H}_1] = 0 \qquad \text{(III.4)}\dagger$$

(it is assumed that there are no surface currents at the boundary). These conditions express the continuity of the tangential components of $\mathbf{E}$ and $\mathbf{H}$ across the boundary surface. In addition to these conditions there are the continuity conditions on the normal components of the induction vectors:

$$(\mathbf{n},\ \mathbf{D}_2 - \mathbf{D}_1) = 0, \quad (\mathbf{n},\ \mathbf{B}_2 - \mathbf{B}_1) = 0. \qquad \text{(III.5)}$$

It will be shown below that all these conditions are not necessary when the boundary is a geometrically simple one. However, all six boundary conditions must be used for complicated curved surfaces. In many cases it is convenient to use the approximate boundary conditions given by M. A. Leontovich. The use of the Leontovich boundary conditions in Cerenkov radiation theory was first proposed by A. I. Morozov, who also estimated the errors that arise when these conditions are used to determine the field of a moving charge.[308]

The Leontovich conditions* apply for media in which the absolute magnitude of the complex refractive index $n = \sqrt{\epsilon\mu}$ is large (by virtue of a large complex dielectric constant ϵ or a large complex permeability μ). The following condition is satisfied approximately at the interface between such a medium and a vacuum:

$$\sqrt{\varepsilon}\,E_t = \sqrt{\mu}\,H_t, \qquad \text{(III.6)}$$

where ϵ and μ are the dielectric constant and magnetic permeability of the medium while E_t and H_t are the tangential components of the electric and magnetic fields.

In addition to being simple these boundary conditions are useful in that we can proceed without determining the fields inside a medium with high values of n; on the other hand, the presence of the medium is taken into account by the boundary conditions at the interface.

The condition (III.6) is an exact one for waves normally incident on the interface, but is approximate for waves that are incident upon the interface at an angle φ; the correction is of order $\frac{\sin^2\varphi}{n^2}$ where $n = \sqrt{\epsilon\mu}$ is the refractive index of the medium. It is easy to show that $\frac{\sin^2\varphi}{n^2} = \frac{1}{n^2\beta^2}$ for Cerenkov radiation. Consider a point charge moving uniformly in vacuum parallel to the boundary of a medium with a high value of refractive index n, which we assume to be real. Suppose that Cerenkov radiation is excited in the medium. The radiation wave vector is directed inward to the medium and forms an angle θ with the normal to the surface; this angle is given by

$$\sin\theta = \frac{1}{n\beta}.$$

Knowing θ we can determine the angle φ (the angle of incidence of the radiation at the interface). It is evident that $\sin\varphi = 1/\beta$. The fact that $\sin\varphi$ is

*rot = curl.

†$[\mathbf{n},\ \mathbf{E}_2 - \mathbf{E}_1] = \mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1)$ etc.

*L. A. Vainshtein, Электромагнитные волны (Electromagnetic Waves) Soviet Radio Press, 1957.

greater than unity indicates that there is no radiation in the vacuum. This value of $\sin\varphi$ gives the relative magnitude of the correction term when the approximate boundary conditions (III.6) are used:

$$\frac{\sin^2\varphi}{n^2}=\frac{1}{\beta^2 n^2}.$$

Thus, the validity of the Leontovich conditions in Cerenkov radiation problems is limited by the condition

$$\frac{1}{\beta^2|n|^2}=\frac{1}{|\varepsilon\mu|\beta^2}\ll 1. \tag{III.7}$$

For motion near a curved surface the error arising from the use of (III.6) is of order

$$h\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{\beta\sqrt{\varepsilon\mu}}, \tag{III.8}$$

where h is the distance of the field source from the surface while R_1 and R_2 are the principal radii of curvature.

It follows from (III.7) and (III.8) that the Leontovich boundary conditions apply for charge velocities appreciably greater than the phase velocity of light in the medium through which the charge moves [if the medium is not transparent (III.6) applies if the absorption is strong].

In the nonrelativistic case the problem of finding the field of a moving source can be simplified still further by expanding the field in powers of $\beta = v/c$:

$$\mathbf{E}=\mathbf{E}^{(0)}+\beta\mathbf{E}^{(1)}+\beta^2\mathbf{E}^{(2)}+\ldots,$$
$$\mathbf{H}=\mathbf{H}^{(0)}+\beta\mathbf{H}^{(1)}+\beta^2\mathbf{H}^{(2)}+\ldots \tag{III.9}$$

To be specific we consider the motion of a charge near a surface at which (III.6) is satisfied. It can be shown that at this surface

$$\frac{E_t^{(0)}}{E_n^{(0)}}\approx\beta\sqrt{\frac{\mu}{\varepsilon}}, \tag{III.10}$$

where $E_n^{(0)}$ and $E_t^{(0)}$ are the normal and tangential components of the vector $\mathbf{E}$.

It is evident from (III.10) that the electric field "bulges" out of the medium when $\mu \gg \epsilon$ and enters the medium almost normally when $\mu \ll \epsilon$. Thus, Laplace's equation can be solved with the boundary conditions $E_n^{(0)} = 0$ or $E_t^{(0)} = 0$; then, $\mathbf{E}^{(1)}$, $\mathbf{H}^{(1)}$ and the radiated power can be found from (III.6).

III.2. Radiation of a Charge Moving Along the Axis of a Cylindrical Dielectric-Filled Channel

Suppose that an isotropic medium of dielectric constant ϵ_2 and permeability μ_2 contains a circular cylindrical channel filled with a medium of dielectric constant ϵ_1 and permeability μ_1 (Fig. 1). The motion of a charge along the axis of such a channel was analyzed in 1947 by V. L. Ginzburg and I. M. Frank.[189] Some seven years earlier, in 1940, the problem of a charge moving along the axis of a channel in a dielectric was posed by L. I. Mandel'shtam in his remarks at the defense of the doctoral dissertation of P. A. Cerenkov. Mandel'shtam noted that a charge moving along the axis of an empty channel in a dense medium (small channel radius, i.e., smaller than the wavelength of the Cerenkov radiation in the medium) can lose energy by Cerenkov radiation. This remark was extremely important because it indicated that the production of Cerenkov radiation of wavelength λ is due to that region of the medium which is at least a distance λ from the particle path; the region of the medium in direct proximity to the path of the charge does not make an important contribution.

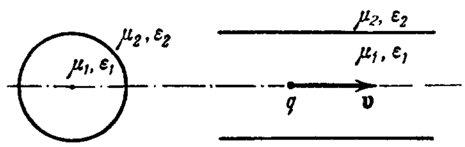

FIG. 1

Suppose that a charge moves through a transparent medium and that the Cerenkov radiation conditions are satisfied. We remove the medium near the path of the charge so that the charge now moves in a narrow channel in a vacuum. The fact that a charge moving in vacuum close to a medium radiates as though it were moving through the medium is of extreme importance from the practical point of view because it offers the possibility of generating Cerenkov radiation without polarization (Bohr) radiation; the latter is responsible for most of the energy loss of a charge in a continuous medium.

a) Let us analyze the problem quantitatively. Suppose that a point charge q moves with velocity v along the axis of a cylinder of radius a. The cylinder is filled with a medium ϵ_1 and μ_1 while the external region is a medium ϵ_2 and μ_2. The electromagnetic field produced by the charge is determined from Maxwell's equations (III.1). We start with the equation for the potentials $\mathbf{A}$ and φ (1.6). This approach simplifies the calculations since the symmetry of the problem means that $\mathbf{A} = \frac{\epsilon\mu}{c}\mathbf{v}\varphi$ (1.9). We could use the potential equations (1.8), in which the longitudinal field is separated from the transverse field, but the calculations would be more complicated in this case.[64] We introduce a cylindrical coordinate system r, φ, and z, with the z-axis along the axis of the cylinder and write φ (rz − vt) in the form (I.33):

$$\varphi=\int_{-\infty}^{\infty}e^{i\frac{\omega}{v}(z-vt)}\Phi(\omega,r)\,d\omega,\quad \mathbf{A}=\frac{\mathbf{v}}{c}\varepsilon\mu\varphi. \tag{III.11}$$

The following equation is then obtained for $\Phi(\omega, r)$:

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{\omega^2}{v^2}(1-\varepsilon\mu\beta^2)\right]\Phi(\omega,r)=-\frac{q}{\pi\varepsilon v r}\delta(r). \tag{III.12}$$

The values of ϵ and μ inside the channel (r < a) are ϵ_1 and μ_1 respectively; the values outside the channel are ϵ_2 and μ_2. The solution of (III.12) is

$$\Phi_{\omega>0}(\omega, r) = \begin{cases} \frac{q}{\pi\varepsilon_1 v}\left[K_0\left(\frac{\omega}{v}\sqrt{1-\varepsilon_1\mu_1\beta^2}\,r\right)+\alpha I_0\left(\frac{\omega}{v}\sqrt{1-\varepsilon_1\mu_1\beta^2}\,r\right)\right] & \text{for } r<a, \\ \frac{q}{\pi\varepsilon_2 v}\left[\eta K_0\left(\frac{\omega}{v}\sqrt{1-\varepsilon_2\mu_2\beta^2}\,r\right)+\gamma I_0\left(\frac{\omega}{v}\sqrt{1-\varepsilon_2\mu_2\beta^2}\,r\right)\right] & \text{for } r>a. \end{cases} \quad \text{(III.13)}$$

Here, K_0 and I_0 are Bessel functions of imaginary argument. The complex conjugates of all quantities are taken when the sign of ω is reversed. When $r < a$ the coefficient of K_0 is determined by the singularity in the right side of (III.12). The coefficients α, η and γ must be determined from the boundary conditions at $r = a$. It is immediately clear that γ must vanish because I_0 increases exponentially with r. The two remaining coefficients are determined from two boundary conditions; these boundary conditions may be conveniently taken as the continuity conditions on E_z and H_φ at the interface. When these conditions are satisfied all the remaining boundary conditions are automatically satisfied. The continuity conditions on the components E_z and H_φ reduce to continuity conditions on the two quantities

$$(1-\varepsilon\mu\beta^2)\,\Phi(\omega, r) \text{ and } \varepsilon\frac{\partial\Phi(\omega, r)}{\partial r}$$

at $r = a$. We introduce the notation

$$k_1=\frac{\omega}{v}\sqrt{1-\varepsilon_1\mu_1\beta^2}, \quad k_2=\frac{\omega}{v}\sqrt{1-\varepsilon_2\mu_2\beta^2}. \quad \text{(III.14)}$$

Matching the fields then gives the following formula for $\Phi(\omega, r)$:

$$\Phi(\omega, r)=\begin{cases} \frac{q}{\pi v\varepsilon_1}[K_0(k_1 r)+\alpha I_0(k_1 r)], \\ \frac{q}{\pi v\varepsilon_2}\eta K_0(k_2 r), \end{cases} \quad \text{(III.15)}$$

where

$$\alpha=\frac{Bk_1K_1(k_1a)K_0(k_2a)-k_2K_0(k_1a)K_1(k_2a)}{Bk_1I_1(k_1a)K_0(k_2a)+k_2I_0(k_1a)K_1(k_2a)},$$

$$\eta=\frac{1}{a}\,\frac{1}{Bk_1I_1(k_1a)K_0(k_2a)+k_2I_0(k_1a)K_1(k_2a)},$$

$$B=\frac{\varepsilon_1}{\varepsilon_2}\,\frac{1-\varepsilon_2\mu_2\beta^2}{1-\varepsilon_1\mu_1\beta^2}=\frac{k_2^2\varepsilon_1}{k_1^2\varepsilon_2}. \quad \text{(III.16)}$$

Equations (III.11, 14–16) together with (I.9) completely determine the field produced in the medium by the moving charge. The equations were first given in this form by A. G. Sitenko.[360]

The field in this problem can also be determined approximately through the use of the Leontovich boundary conditions (III.6). When this procedure is used we avoid the necessity of computing the coefficient η, which gives the field outside the channel. The boundary condition (III.6) gives an equation for α immediately and we have

$$\alpha=\frac{\sqrt{\mu}\beta K_1(k_1a)+i\sqrt{\varepsilon(1-\beta^2)}\,K_0(k_1a)}{\sqrt{\mu}\beta I_1(k_1a)-i\sqrt{\varepsilon(1-\beta^2)}\,I_0(k_1a)}.$$

For convenience, we write $\epsilon_1 = 1$, $\mu_1 = 1$, $\epsilon_2 = \epsilon$, and $\mu_2 = \mu$. When $\epsilon\mu\beta^2 \to \infty$ the approximation obtained by means of the Leontovich boundary conditions coincides with the exact expression for α (III.16). Thus the Leontovich boundary conditions can be used far beyond the Cerenkov threshold.

Equation (III.15) indicates that the nature of the field is determined by the quantity $k_{1,2} = \frac{\omega}{v}\sqrt{1-\epsilon\mu\beta^2}$. The Cerenkov condition is not satisfied in a medium in which this quantity is real and the field is essentially made up of monotonic nonradiating modes. However, if the Cerenkov condition is satisfied inside or outside the channel, the field in the medium in which the condition is satisfied is given by a superposition of cylindrical waves; the charge then loses energy by radiation of these waves.

We now determine the energy loss of the charge. The energy loss of the charge per unit length of path is determined by the retardation force exerted on the charge by the field produced by the charge

$$\frac{dW}{dz}=qE_z\Big|_{\substack{z\to vt\\ r\to 0}}=\frac{2q^2}{\pi c^2}\operatorname{Re}\int_0^\infty \mu_1(\omega)\left(1-\frac{1}{\varepsilon_1\mu_1\beta^2}\right)[K_0(k_1 r_{\min}) + \alpha I_0(k_1 r_{\min})]\,i\omega\,d\omega, \quad \text{(III.17)}$$

where α is given by (III.15). The quantity r_{min} is the minimum average distance to the field source for which classical electrodynamics still holds. We assume for simplicity that both media (inside and outside the channel) are transparent.

b) Analysis of (III.17) indicates that the charge loses energy in several frequency ranges.

Energy is lost at the discrete frequencies at which the dielectric constant of the medium in the channel vanishes. This is the Bohr polarization energy lost in the production of a longitudinal field. This polarization loss is given by the expression

$$\frac{dW}{dz_{\text{Bohr}}}=-\frac{q^2}{v^2}\times\sum_{\omega_s}\frac{2\omega_s}{\left|\frac{d\varepsilon(\omega)}{d\omega}\right|_{\omega=\omega_s}}\left[K_0\left(\frac{\omega_s r_{\min}}{v}\right)-I_0\left(\frac{\omega_s r_{\min}}{v}\right)\frac{K_0\left(\frac{\omega_s a}{v}\right)}{I_0\left(\frac{\omega_s a}{v}\right)}\right]. \quad \text{(III.18)}$$

The summation is taken over all frequencies ω_s for which $\epsilon_1(\omega_s) = 0$. The second term in the square brackets is due to the presence of the boundary. As the channel radius a increases, this term approaches zero as $\exp[-2\omega_s a/v]$ and (III.18) becomes the expression for the Bohr loss of a particle in an infinite medium. As far as the Bohr losses are concerned, the medium inside the channel may be assumed to be infinite when the channel radius is of the order of or greater than the maximum value of v/ω_s. The medium then shields the longitudinal field and the charge does not "feel" the boundary.

c) Energy losses due to Cerenkov radiation also occur. In this case the field and loss spectrum depend primarily on which of the two media satisfies the Cerenkov radiation condition. We consider various possible cases.

We first consider the case where the Cerenkov condition is not satisfied inside the channel but is satisfied outside the channel. This case corresponds to the inequality

$$k_1^2 > 0; \quad k_2^2 < 0 \quad (\varepsilon_1\mu_1\beta^2 < 1;\ \varepsilon_2\mu_2\beta^2 > 1). \qquad \text{(III.19)}$$

This problem is the one suggested by L. I. Mandel'shtam and was analyzed by Ginzburg and Frank. In moving along the axis of the channel the charge radiates into the outer medium. The energy losses are given by the expression

$$\frac{dW}{dz} = \frac{2q^2}{\pi c^2}\,\mathrm{Re}\int_0^\infty \mu_1\left(1-\frac{1}{\varepsilon_1\mu_1\beta^2}\right)\alpha(\omega)\,i\omega\,d\omega. \qquad \text{(III.20)}$$

The integral is taken over the frequency interval for which the inequalities in (III.19) are satisfied. This formula is obtained from (III.17) as follows. When (III.19) is satisfied the first term in the rectangular brackets under the integral in (III.17) can be neglected since it does not make a contribution to the real part of the integral. We can write $r_{min} = 0$ in the second term. This procedure yields (III.20).

The integral in (III.20) can be given in another form. Only the imaginary part of the coefficient $\alpha(\omega)$ makes a contribution to the real part of the integral. It can be shown that Im $\alpha(\omega)$ is related to the factor $\eta(\omega)$, which determines the field outside the channel [cf. (III.15, 16)]:

$$\mathrm{Im}\,\alpha(\omega) = -\frac{B\pi}{2}|\eta(\omega)|^2.$$

Thus, we have

$$\frac{dW}{dx} = -\frac{q^2}{c^2}\int \mu_2\left(1-\frac{1}{\varepsilon_2\mu_2\beta^2}\right)|\eta(\omega)|^2\,\omega\,d\omega,$$
$$\varepsilon_1\mu_1\beta^2 < 1, \quad \varepsilon_2\mu_2\beta^2 > 1. \qquad \text{(III.20')}$$

Equation (III.20') differs from the expression for Cerenkov losses in an infinite medium characterized by ϵ_2 and μ_2 by the factor $|\eta(\omega)|^2$ under the integral. In the spectral interval under consideration, where (III.19) is satisfied, the coefficient $\eta(\omega)$ (III.16) is complex because k_2 is imaginary. Hence, the functions $K_0(k_2a)$ and $K_1(k_2a)$ become $H_0^{(2)}(s_2a)$ and $H_1^{(2)}(s_2a)$ where $s_2 = \frac{\omega}{v}\sqrt{\epsilon_2\mu_2\beta^2-1}$ and $H_0^{(2)}$ and $H_1^{(2)}$ are Hankel functions.

The factor η approaches unity as the channel radius approaches zero. Hence, the Cerenkov radiation of a charge moving in a narrow channel is essentially the same as that in an infinite medium characterized by ϵ_2 and μ_2. If the presence of the channel is not to affect the radiation spectrum the following inequalities must be satisfied:

$$k_1a \ll 1, \quad s_2a \ll 1 \quad (s_2 = ik_2). \qquad \text{(III.21)}$$

The case being considered ($k_1^2 > 0$ and $k_2^2 < 0$) is very important from the point of view of theory. When a charge moves in a medium in which the Cerenkov radiation is not satisfied it radiates into the medium where the condition is satisfied and the radiation intensity falls off sharply with increasing distance between the particle trajectory and the boundary between the two media.

The quantities $s_{1,2}$ are the radial components of the Cerenkov radiation wave vectors in the first and second media respectively. If the Cerenkov condition is not satisfied then s is imaginary and the field falls off exponentially with a decay factor $-is = k = \frac{\omega}{v}\sqrt{1-\epsilon\mu\beta^2}$.

As a increases the functions $I_0(k_1a)$ and $I_1(k_1a)$ grow exponentially (as e^{k_1a}). Hence $|\eta(\omega)|^2$ diminishes as e^{-2k_1a} and the radiation more or less vanishes when the channel radius a is appreciably greater than $1/2k_1$ (Fig. 2). This result was used by Schonberg to check the theory of Cerenkov losses in a transparent medium against experiment.[211] Schonberg and Huybrechts were unacquainted with the work of Ginzburg and Frank, which had been carried out five years earlier, and repeated the calculations of Ginzburg and Frank (although for another purpose). We may note that even in 1958, eleven years after publication of the work by Ginzburg and Frank, another paper appeared[350] in which the work reported in reference 189 was repeated.

If $k_1 = 0$, as when a charge moves in vacuum with the velocity of light, the factor $|\eta(\omega)|^2$ does not fall

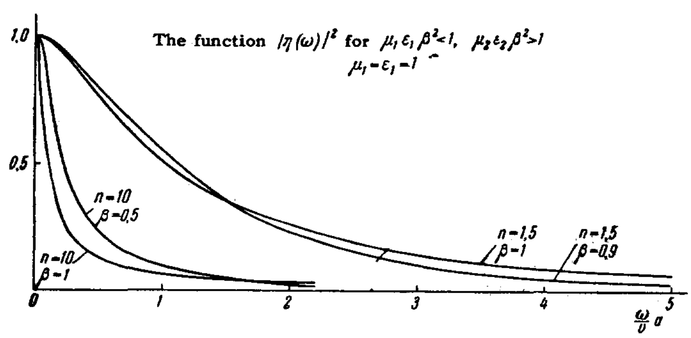

FIG. 2

off exponentially as the channel radius increases, but is proportional to

$$\frac{1}{(s_2 a)^3} = \left(\frac{\omega}{v} a \sqrt{\varepsilon_2 \mu_2 \beta^2 - 1}\right)^{-3}.$$

Here we may note an interesting analogy. If an extended charged beam moves in a continuous medium the Cerenkov radiation losses are described by (III.20′) except that $|\eta(\omega)|^2$ is replaced by the square of the modulus of the form factor describing the charge distribution in a bunch. Thus, in a sense the introduction of a channel is equivalent to the smearing out of a point charge radiating in a continuous medium.

d) We now consider the case where the Cerenkov condition is satisfied both inside and outside the channel. In this case

$$k_1^2 < 0, \quad k_2^2 < 1 \quad (\varepsilon_1 \mu_1 \beta^2 > 1, \ \varepsilon_2 \mu_2 \beta^2 > 1)$$

[k_1 and k_2 are defined in (III.14)].

The Cerenkov loss integral (III.17) now becomes

$$\frac{dW}{dx} = \frac{2q^2}{\pi c^2} \operatorname{Re} \int\limits_{\substack{\varepsilon_1\mu_1\beta^2 > 1 \\ \varepsilon_2\mu_2\beta^2 > 1}} \mu_1(\omega)\left(1 - \frac{1}{\varepsilon_1 \mu_1 \beta^2}\right)\left[\frac{\pi i}{2} + \alpha(\omega)\right] i\omega \, d\omega, \tag{III.22}$$

where the integration extends over all frequencies for which the Cerenkov condition is satisfied both inside and outside the channel.

The integral in (III.22) is obtained from (III.17) as follows. Since $k_1 = -is_1$,

$$K_0(k_1 r) = \frac{\pi i}{2} H_0^{(1)}(s_1 r) = \frac{\pi i}{2}[J_0(s_1 r) + iN_0(s_1 r)], \tag{III.23}$$

where $H_0^{(1)}$ is the Hankel function, J_0 is the Bessel function, N_0 is the Neumann function. Since we are interested in the real part of the integral (III.17) we can neglect the term $N_0(s, r)$; however, it is then formally possible to take the limiting case $r_{\min} \to 0$, which gives (III.22) directly.

Equation (III.22) can be simplified by means of the relation

$$\operatorname{Im}\left[\frac{\pi i}{2} + \alpha(\omega)\right] = -\frac{\pi}{2} B |\eta(\omega)|^2, \tag{III.24}$$

which applies in the frequency region considered. The loss integral then assumes the form

$$\frac{dW}{dz} = -\frac{q^2}{c^2} \int \mu_2(\omega)\left(1 - \frac{1}{\varepsilon_2 \mu_2 \beta^2}\right)|\eta(\omega)|^2 \omega \, d\omega. \tag{III.25}$$

This expression is superficially similar to (III.21), the expression for losses in the other frequency region, where $\epsilon_1\mu_1\beta^2 < 1$ and $\epsilon_2\mu_2\beta^2 > 1$. However, this similarity is not a true one because the dependence of the factor $|\eta(\omega)|^2$ on the channel radius a is different in the frequency region for which (III.19) applies. Specifically, $|\eta(\omega)|^2$ does not vanish as $a \to \infty$.

The physical picture of the effect is as follows (Fig. 3). Cerenkov radiation is excited in the first medium. It propagates at an angle ϑ_1 with the velocity, where $\cos \vartheta_1 = 1/n_1\beta$. This radiation is incident upon

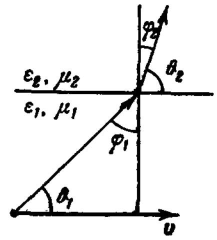

FIG. 3

the channel boundary at an angle $\varphi_1 = \frac{\pi}{2} - \vartheta_1$ so that

$$\sin \varphi_1 = \frac{1}{n_1 \beta}. \tag{III.26}$$

In passing through the boundary the Cerenkov wave is refracted in accordance with the familiar relation

$$\frac{\sin \varphi_1}{\sin \varphi_2} = \frac{n_2}{n_1}, \tag{III.27}$$

whence

$$\sin \varphi_2 = \frac{1}{n_2 \beta}. \tag{III.28}$$

Consequently, the refracted radiation propagates at an angle ϑ_2 to the velocity, where $\cos \vartheta_2 = 1/n_2\beta$. It will be evident that if the radiation propagates at the Cerenkov angle in the first medium it also propagates at the Cerenkov angle in the second medium, after being refracted.

We now give the expression for the factor $|\beta(\omega)|^2$ when the channel radius is large ($s_1 a \gg 1$, $s_2 a \gg 1$):

$$|\eta(\omega)|^2 = \frac{s_1}{s_2} \frac{1}{\frac{s_2^2 \varepsilon_1^2}{s_1^2 \varepsilon_2^2} \sin^2\left(s_1 a - \frac{\pi}{4}\right) + \cos^2\left(s_1 a - \frac{\pi}{4}\right)}. \tag{III.29}$$

This factor oscillates about the value

$$\frac{2s_1}{s_2} \frac{1}{\frac{s_2^2 \varepsilon_1^2}{s_1^2 \varepsilon_2^2} + 1}.$$

By choosing the channel radius or the parameters of both media we can enhance the radiation intensity in a narrow frequency range.

e) We now consider the case where the Cerenkov condition is satisfied inside the channel ($k_1^2 < 0$; $\epsilon_1\mu_1\beta^2 > 1$) but is not satisfied outside the channel ($k_2^2 > 0$, $\epsilon_2\mu_2\beta^2 < 1$). In this case the energy loss (III.17) becomes

$$\frac{dW}{dz} = \frac{2q^2}{\pi c^2} \operatorname{Re} \int \mu_1 \left(1 - \frac{1}{\varepsilon_1 \mu_1 \beta^2}\right)\left(\frac{\pi i}{2} + \alpha\right) i\omega \, d\omega. \tag{III.30}$$

In the frequency range under consideration, where

$$k_1^2 < 0, \quad k_2^2 > 0,$$

we can write $\alpha(\omega)$, (III.16), in the form

$$\alpha(\omega) = -\frac{\pi i}{2}\left[1 - i\frac{Bs_1 N_1(s_1 a) K_0(k_2 a) - k_2 N_0(s_1 a) K_1(k_2 a)}{Bs_1 J_1(s_1 a) K_0(k_2 a) + k_2 J_0(s_1 a) K_1(k_2 a)}\right]. \tag{III.31}$$

Substitution of (III.31) in (III.30) yields the following expression for the loss:

$$\frac{dW}{dz} = -\frac{q^2}{c^2}\,\mathrm{Re} \times \int \mu_1\left(1-\frac{1}{\varepsilon_1\mu_1\beta^2}\right)\frac{Bs_1N_1(s_1a)K_0(k_2a)-k_2N_0(s_1a)K_1(k_2a)}{Bs_1J_1(s_1a)K_0(k_2a)+k_2J_0(s_1a)K_1(k_2a)}\,i\omega\,d\omega. \quad \text{(III.32)}$$

It is evident from the last formula that the only contributions to the loss come from the poles of the integrand. The integrand is purely imaginary and the contribution to the real part can come only from the residues at the poles.

The poles of the integrand are given by

$$D(\omega) = Bs_1J_1(s_1a)K_0(k_2a) + k_2J_0(s_1a)K_1(k_2a) = 0. \quad \text{(III.33)}$$

The frequencies for which this equation is satisfied form a discrete Cerenkov spectrum.

The origin of the discrete Cerenkov spectrum can be understood on the basis of certain observations made by Frank[149] (cf. also reference 64).

If the Cerenkov condition is satisfied both inside and outside the channel, then it follows from (III.27) and (III.28) that the Cerenkov wave emitted by the charge satisfies the radiation condition in the second medium after refraction at the channel boundary. Suppose now that the Cerenkov condition is not satisfied in the second medium. Equation (III.28) then shows that the sine of the angle of refraction is greater than unity. This corresponds to the case of total reflection, which is well-known in optics. Since the Cerenkov condition is not satisfied in the outside medium, the Cerenkov wave emitted by the charge is reflected back into the inside medium when it reaches the boundary; it then travels to the other boundary, where it is again reflected, and so on; thus, the wave "travels" inside the channel by multiple reflection. Under these conditions waves emitted earlier interfere with waves emitted at later times. Suppose that a charge at point A on the channel axis radiates a Cerenkov wave (Fig. 4). The

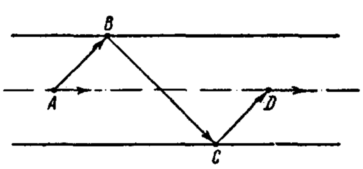

FIG. 4

radiated light travels to point B, is reflected, travels to point C, is again reflected, and then intersects the path of the charge at point D, at the same angle at which it was emitted. During this time the charge travels to point D and emits another Cerenkov wave at this point. These two waves will reinforce each other if the difference in path length is an integral number of wavelengths. If this condition is not satisfied the waves cancel each other. Hence, the radiation spectrum must be a discrete one.

If the channel radius is large ($s_1a \gg 1$, $k_2a \gg 1$), Eq. (III.33) for the radiation frequency assumes the relatively simple form

$$\mathrm{tg}\left(\frac{\omega}{v}a\sqrt{\varepsilon_1\mu_1\beta^2-1}-\frac{\pi}{4}\right) = -\frac{\varepsilon_2}{\varepsilon_1}\sqrt{\frac{\varepsilon_1\mu_1\beta^2-1}{1-\varepsilon_2\mu_2\beta^2}}, \quad \text{(III.34)*}$$

or

$$\frac{\omega}{v}a\sqrt{\varepsilon_1\mu_1\beta^2-1}-\frac{\pi}{4}+n\pi = -\mathrm{arctg}\,\frac{\varepsilon_2}{\varepsilon_1}\sqrt{\frac{\varepsilon_1\mu_1\beta^2-1}{1-\varepsilon_2\mu_2\beta^2}}. \quad \text{(III.35)*}$$

The first term on the left is proportional to the phase change of the Cerenkov ray along the path ABCD while the right-hand side gives the phase jump due to total reflection at the channel boundary.

The dispersion equation (III.33) indicates an interesting feature of Cerenkov radiation in a channel. Consider two cases of particle radiation in a channel. In both cases let the particle velocity be close to the velocity of light ($\beta = 1$) and let the Cerenkov condition be satisfied inside the channel. In one case we assume that the outside medium is a vacuum ($\epsilon_2 = 1$, $\mu_2 = 1$); in the second we assume that it is an ideally conducting metal ($\epsilon_2 = i\infty$). It is evident from the dispersion equation (III.33) that the radiation spectrum is determined by the same condition in both cases

$$J_0\left(\frac{\omega}{v}a\sqrt{\varepsilon_1\mu_1\beta^2-1}\right) = 0. \quad \text{(III.36)}$$

This result indicates that it is not necessary to silver cylindrical Cerenkov counters used for detecting relativistic particles ($\beta = 1$): the emitted light is essentially totally reflected without silvering. It will be obvious that this remark applies when the trajectory of the fast radiating particle in the cylindrical Cerenkov counter is close to the counter axis.

The particle energy losses due to Cerenkov radiation in the case under consideration are due to the residues at the poles of the integrand (III.32). We denote the numerator of the fraction in the integrand in (III.32) by $A(\omega)$:

$$A(\omega) = Bs_1N_1(s_1a)K_0(k_2a) - k_2N_0(s_1a)K_1(k_2a). \quad \text{(III.37)}$$

The Cerenkov loss of a particle in the frequency range of interest is then given by

$$\frac{dW}{dz} = -\pi\frac{q^2}{c^2}\sum_s \mu_{1s}\left(1-\frac{1}{\varepsilon_{1s}\mu_{1s}\beta^2}\right)\left|\frac{A(\omega_s)}{D'(\omega_s)}\right|\omega_s, \quad \text{(III.38)}$$

where D' is the derivative of the function $D(\omega)$ defined by (III.33), ω_s is the s-th root of Eq. (III.33), and μ_{1s} and ϵ_{1s} are the values of $\mu_1(\omega)$ and $\epsilon_1(\omega)$ at $\omega = \omega_s$. The summation is taken over all frequencies for which radiation is excited inside the channel but not excited outside the channel.

The wave outside the channel decays exponentially because the Cerenkov condition is not satisfied in the outside medium [cf. (III.15)]. Thus, the Poynting flux through a cylindrical surface coaxial with the channel in the external medium is zero. This does not mean that the charge loses no energy by radiation. It simply means that the radiation does not penetrate into the outside medium. For this reason, the presence

*tg = tan; arctg = $\tan^{-1}$.

of a boundary makes the determination of the energy loss by the radiation flux an extremely complicated problem compared with the case of a uniform medium. One must take account of the possible reflection and refraction at the boundary. Neglect of these effects can lead to erroneous conclusions. This would appear to be the reason for the incorrect statement given in reference 360 that the particle energy losses due to radiation can be determined from the radiation flux outside the channel. This erroneous approach to the determination of the loss led the authors of reference 189 to the incorrect conclusion that there is no radiation loss when $\epsilon_1\beta^2 > 1$ and $\epsilon_2\beta^2 < 1$.

The cases of Cerenkov radiation in a channel analyzed here are characteristic of all stationary boundary-value problems of this kind.

III.3. Motion of a Point Charge Parallel to the Axis of a Channel in a Dielectric

The case in which the charge moves parallel to the axis of a channel and at a distance r_0 from it has been considered in reference 63. In this case the field produced by the particle is not axially symmetric. The radiation conditions are exactly the same as in the preceding case, where the charge moves along the channel axis. Specifically, if the radiation condition is satisfied both inside and outside the channel the radiation is excited in the first medium and then enters the second medium after being refracted at the boundary. The radiation spectrum is continuous in this case. If the radiation condition is satisfied only inside the channel, the radiation is excited in the first medium and experiences total reflection at the channel boundary. Thus the radiation does not enter the second medium, but propagates along the channel axis. In this case the radiation spectrum is discrete. Finally, when the Cerenkov condition is not satisfied inside the channel but is satisfied outside the channel the radiation is excited only in the second medium and its intensity falls off exponentially with increasing distance between the particle and the boundary surface. (If the channel radius is a and the distance of the particle from the axis is r_0 the distance of the particle from the channel boundary is $a - r_0$.)

It is evident from the above considerations that the basic features of the radiation process are not changed if the particle trajectory is displaced from the channel axis. There is one new effect, however, when the particle trajectory is displaced from the channel axis. First, the angular distribution of the radiation is no longer isotropic. The particle field can be expanded in harmonics of the form

$$\varphi = \sum_{m=-\infty}^{\infty} e^{im\varphi} \int_{-\infty}^{\infty} e^{i\frac{\omega}{v}(z-vt)} \Phi_m(r, \omega)\, d\omega,$$
$$\mathbf{A} = \sum_{m=-\infty}^{\infty} e^{im\varphi} \int_{-\infty}^{\infty} e^{i\frac{\omega}{v}(z-vt)} \mathbf{a}_m(r, \omega)\, d\omega, \qquad \text{(III.39)}$$

where m is a whole number and φ is the angle between the radius vector to the particle and the radius vector to the point of observation (Fig. 5). We do not give the expressions for Φ_m and $\mathbf{a}_m$ here, but refer the reader to the original work.[63] It follows from the geometry of the problem that the field is symmetric with respect to the plane passing through the channel axis and the line of motion of the particle. For this reason only $\cos m\varphi$ terms appear in (III.39).

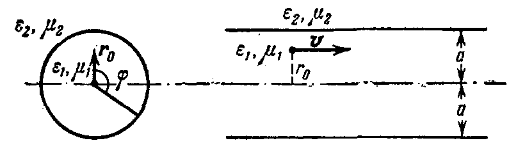

FIG. 5

The lack of axial symmetry means that the charge is subject to a force that tends to deflect it from its linear path in addition to a retarding force. It is clear from the symmetry of the field that this force can only have a radial component and that the φ component must vanish.

The radial force that arises when the charge is displaced from the channel axis can act to focus the charged particle, i.e., it can act as a restoring force tending to return the particle to the axis.

Equation (III.39) can be analyzed to find the frequency regions in which a charged particle moving near the channel axis experiences a focusing force. It is shown in reference 63 that the radial force acting on the charge is ($\mu_1 = 0$, $\mu_2 = 1$)

$$F_r = \frac{q^2}{\pi v^2} \sum_{m=0}^{\infty} a_m \int_{-\infty}^{\infty} \frac{(1-\varepsilon_1\beta^2)^{\frac{3}{2}}}{\varepsilon_1} \lambda_{1m} I_m\left(\frac{|\omega|}{v} \sqrt{1-\varepsilon_1\beta^2}\, r_0\right) \times I'_m\left(\frac{|\omega|}{v} \sqrt{1-\varepsilon_1\beta^2}\, r_0\right) \omega\, d\omega, \qquad \text{(III.40)}$$

where $a_m = 1$ when $m = 0$, $a_m = 2$ when $m = 1, 2, \ldots$ and λ_{1m} is a complicated function that depends on the properties of the inside and outside media and the channel radius a. The radial force can be a focusing force in certain regions and a defocusing force in others. The resultant sign of the radial force depends on the integral over all frequencies; to determine this sign we must know the dispersion properties of both media.

We consider in greater detail the case in which the Cerenkov condition is satisfied inside the channel but not outside. The radiation is discrete in this case, as we have already indicated. The frequencies at which radiation is produced are harmonics characterized by the number m (the angular dependence of this harmonic is given by the factor $\exp[im\varphi]$) and are determined by the dispersion equation:

$$(\varepsilon_1\beta^2-1)(1-\varepsilon_2\beta^2)\Big[\varepsilon_1\sqrt{1-\varepsilon_2\beta^2}\times K_m\Big(\frac{\omega}{v}\sqrt{1-\varepsilon_2\beta^2}\,a\Big)J'_m\Big(\frac{\omega}{v}\sqrt{\varepsilon_1\beta^2-1}\,a\Big)+\varepsilon_2\sqrt{\varepsilon_1\beta^2-1}\,J_m\Big(\frac{\omega}{v}\sqrt{\varepsilon_1\beta^2-1}\,a\Big)K'_m\Big(\frac{\omega}{v}\sqrt{1-\varepsilon_2\beta^2}\,a\Big)\Big]\times\Big[\sqrt{\varepsilon_1\beta^2-1}\,J_m\Big(\frac{\omega}{v}\sqrt{\varepsilon_1\beta^2-1}\,a\Big)K'_m\Big(\frac{\omega}{v}\sqrt{1-\varepsilon_2\beta^2}\,a\Big)+\sqrt{1-\varepsilon_2\beta^2}\,J'_m\Big(\frac{\omega}{v}\sqrt{\varepsilon_1\beta^2-1}\,a\Big)K_m\Big(\frac{\omega}{v}\sqrt{1-\varepsilon_2\beta^2}\,a\Big)\Big]-\Big(\frac{mv}{\omega a}\Big)^2 J_m^2\Big(\frac{\omega}{v}\sqrt{\varepsilon_1\beta^2-1}\,a\Big)\times K_m^2\Big(\frac{\omega}{v}\sqrt{1-\varepsilon_2\beta^2}\,a\Big)(\varepsilon_1-\varepsilon_2)^2\beta^2=0. \quad \text{(III.41)}$$

When $m = 0$ this dispersion equation coincides with (III.33) for the case of motion of a particle along the axis of the channel. Thus, when the particle trajectory is displaced from the channel axis, in addition to the frequencies corresponding to the case of motion along the axis, there are additional frequencies, at which field harmonics characterized by $m \neq 0$ are radiated.

The radiation spectrum of a charged particle moving parallel to the axis of a channel is also characterized by the same interesting feature as the radiation spectrum for motion along the axis. Specifically, if the velocity of the particle is close to the velocity of light ($\beta = 1$) the same radiation spectrum is obtained if the outside medium is vacuum ($\epsilon_2 = 1$, $\mu_2 = 1$) or a metal ($\epsilon_2 = i\infty$). Hence, the radiation spectrum of relativistic particles is the same in silvered and unsilvered Cerenkov counters. It is assumed that the counter is cylindrical in shape and that the particle moves parallel to the axis.

Displacement of the particle trajectory from the channel axis affects the magnitude of the loss. This effect is of particular importance when the radiation wavelength becomes comparable with the channel radius. If the radiation wavelength is much smaller than the channel radius, as is usually the case in Cerenkov counters, the dependence of radiation loss on particle displacement is important only when the distance from the particle trajectory to the edge of the channel is of the same order of magnitude as the radiated wavelength.

III.4. Radiation of a Dipole Moving Along the Axis of a Cylindrical Channel

The radiation of a point charge moving along the axis of an empty channel in a dielectric possesses an important feature: as the channel radius becomes vanishingly small the Cerenkov radiation becomes the same as that in a continuous medium.

On the other hand, the radiation of more complex systems such as electric or magnetic dipoles is not characterized by this property. It has been pointed out by V. L. Ginzburg and V. Ya. Éĭdman[191] and by L. S. Bogdankevich[60] that the dipole radiation in a channel exhibits important differences from the radiation in a continuous medium, even when the channel radius is arbitrarily small. The reason can be understood on the basis of the following considerations.

A charge moving along the axis of a cylindrical channel is subject to a force qE_z ($z = vt$, $r = 0$). The work done by the charge against this force then gives the energy loss due to Cerenkov radiation. Since E_z is continuous at the interface then, when the channel radius is small, E_z at the axis does not differ greatly from E_z at points lying close to the channel boundary in the outside medium. Thus, at small values of the channel radius the radiation of a charge moving along the channel axis is determined completely by the properties of the outside medium.

Now suppose that a point electric dipole rather than a point charge moves along the channel axis (magnitude and direction of the dipole moment given by the vector $\mathbf{p}$); the radiation losses of the dipole are determined, as in the case of the charge, by the force exerted on the dipole by the field produced by the dipole. This force is given by $(\mathbf{p}\nabla)E_z = p_r\frac{\partial E_z}{\partial r} + p_z\frac{\partial E_z}{\partial z}$, where E_z is the component of the dipole field along the channel axis while p_z and p_r are the projections of the dipole moment on the z axis and in the plane perdicular to the channel axis. It is evident that E_z and $\partial E_z/\partial z$ are continuous at the edge of the channel. On the other hand $\partial E_z/\partial r$ goes through a discontinuity at the edge of the channel. Thus, if the dipole moment is oriented along the channel axis the dipole radiation losses in a narrow channel are the same as in a continuous medium characterized by ϵ_2. In contrast, if the dipole moment has a radial component, when the channel radius becomes vanishingly small the radiation losses do not approach the value which would obtain in a continuous medium characterized by ϵ_2.

Let us consider this effect in greater detail. Suppose that a point object having an electric dipole moment $\mathbf{p}$ and a magnetic moment $\mathbf{m}$ moves in a medium characterized by a dielectric constant $\epsilon(\omega)$. We first consider the case in which the medium is continuous, then the motion of dipoles in a narrow channel, and then compare the two cases. We assume below that the magnetic moment $\mathbf{m}$ and the electric moment $\mathbf{p}$ are perpendicular to the dipole velocity, since this is the case in which a vanishing channel radius does not give the field of a moving dipole in a continuous medium. To be specific we also assume that the vectors $\mathbf{p}$ and $\mathbf{m}$ are mutually perpendicular.

To compute the field in the medium produced by a source with electric moment $\mathbf{P} = \mathbf{p}\delta(\mathbf{x}-\mathbf{v}t)$ and magnetic moment $\mathbf{M} = \mathbf{m}\delta(\mathbf{x}-\mathbf{v}t)$ we must solve Maxwell's equations with the external current j and the current density ρ; these quantities are given by the well-known relations

$$\mathbf{j} = c\,\mathrm{rot}\,\mathbf{M} + \frac{\partial\mathbf{P}}{\partial t}, \quad \rho = -\mathrm{div}\,\mathbf{P}. \quad \text{(III.42)}$$

The solution of Maxwell's equations with sources of this kind is obtained in the usual fashion. Having determined the field we can find the dipole energy loss due to Cerenkov radiation. We compute the Poynting flux through a cylindrical surface surrounding the dipole trajectory, thereby obtaining the following expression for the energy loss per unit length:

$$\frac{dW}{dz}=\frac{1}{2v^2c^2}\int\limits_{\varepsilon\beta^2>1}\left[\frac{1}{\varepsilon\beta^2}(m-\beta p)^2+\left(m-\frac{p}{\varepsilon\beta}\right)^2\right]\varepsilon\omega^3\,d\omega. \quad \text{(III.43)}$$

Suppose that the radiation source is a dipole having a magnetic moment m_0 in the system in which it is at rest. In the laboratory system the moving dipole will have an electric moment in addition to its magnetic moment. If the effect of the medium is neglected the electric dipole moment produced by the motion of the magnetic dipole m_0 can be written in the form

$$\mathbf{p}=\frac{1}{c}[\mathbf{v},\,\mathbf{m}_0]. \quad \text{(III.44)}$$

Substitution of this relation in (III.43) gives

$$\frac{dW}{dz}=\frac{m_0^2}{2v^2c^2}\int\limits_{\varepsilon\beta^2>1}\left[2\left(1-\frac{1}{\varepsilon}\right)^2-\left(1-\frac{\beta^2}{\varepsilon}\right)\left(1-\frac{1}{\varepsilon\beta^2}\right)\right]\varepsilon\omega^3\,d\omega. \quad \text{(III.45)}$$

This expression was obtained by I. M. Frank in 1942;[144] *it is noteworthy that the integrand, which gives the radiation intensity at frequency* ω, does not vanish at the radiation threshold. However, the total energy loss and the range of radiated frequencies do vanish as the radiation threshold is approached: $\beta^2 \to 1/\epsilon$.

If the effect of the medium is now considered, i.e., the fact that the magnetic moment is "permeated" by the medium, we find that the moving magnetic dipole now has an electric moment.

$$\mathbf{p}=\frac{\varepsilon}{c}[\mathbf{v},\,\mathbf{m}]. \quad \text{(III.46)}$$

Substitution of this relation in (III.43) gives an expression for the Cerenkov radiation loss that differs from (III.45):

$$\frac{dW}{dz}=\frac{m_0^2}{2c^4}\int\omega^3\varepsilon^2\left(1-\frac{1}{\varepsilon\beta^2}\right)^2d\omega. \quad \text{(III.47)}$$

The two formulas for the radiation loss of a magnetic dipole (III.45) and (III.47) are perfectly compatible because, as we have indicated above, they apply to different situations.

The radiation of point dipole moments moving along the axis of a circular channel of finite radius has been considered by L. S. Bogdankevich.[60] We limit ourselves here to the limiting case of small channel radius. If $k_{r1}a \ll 1$ and $k_{r2}a \ll 1$, where a is the channel radius, while k_{r1} and k_{r2} are the radial components of the radiation wave vector inside and outside the channel, the radiation field in the outside medium is obtained from the dipole field in a continuous medium by the substitution

$$p\to p^*=\frac{2\varepsilon_2}{\varepsilon_1+\varepsilon_2}\,p. \quad \text{(III.48)}$$

Correspondingly, the radiation energy loss is now written in the form:

$$\frac{dW}{dz}=\frac{1}{2v^2c^2}\int\limits_{\varepsilon\beta^2>1}\left[\frac{1}{\varepsilon_2\beta^2}\left(m-\beta p\frac{2\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right)^2+\left(m-\frac{p}{\varepsilon_2\beta}\cdot\frac{2\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right)^2\right]\varepsilon_2\omega^3\,d\omega. \quad \text{(III.49)}$$

It is evident from this formula that the radiation of an electric dipole ($m = 0$) in a narrow channel is $4\epsilon_2^2/(\epsilon_1+\epsilon_2)^2$ *times greater than that of a dipole in a continuous medium.* We again emphasize that all of these considerations refer to the case in which the dipole is perpendicular to the channel axis. When the dipole is oriented in the longitudinal direction the radiation in a narrow channel is the same as in a continuous medium.

The magnitude of the radiation loss for a dipole moving in a narrow channel can be found easily by means of the reciprocity theorem (Ginzburg and Éĭdman[191]).

Suppose that a point electric dipole $\mathbf{p}$ moves along the axis of a channel in a dielectric. The field produced by this dipole outside the channel is denoted by $\mathbf{E}$. We now place an additional dipole moment $p_1\exp[-i\omega t]$ outside the channel at the point $\mathbf{x}_0$. The field produced by this dipole inside the channel is denoted by $\mathbf{E}_1$. The reciprocity theorem then states that

$$\int \mathbf{p}\mathbf{E}_{1\omega}(0,0,z)\,e^{-i\frac{\omega}{v}z}\,dz=\mathbf{p}_1\mathbf{E}_\omega(\mathbf{x}_0), \quad \text{(III.50)}$$

where $\mathbf{p}\exp[i\omega z/v]$ is the Fourier component of the moving dipole moment density $\mathbf{p}\delta(x)\delta(y)\delta(z-vt)$ and $\mathbf{E}_{1\omega}$ and $\mathbf{E}_\omega$ are the Fourier components of the fields $\mathbf{E}_1$ and $\mathbf{E}$. If the dipole $\mathbf{p}$ is oriented along the channel axis the factor $\mathbf{E}_{1z}$ appears alone in the integrand. In a narrow channel this factor has the same value as in a continuous medium because the tangential components of $\mathbf{E}$ are continuous across the boundary. Consequently, the value of the total integral is the same in a channel and in a continuous medium. It follows that the field $\mathbf{E}_\omega$ on the right side of the equation is the same in a continuous medium and in a narrow channel.

Suppose now that the dipole $\mathbf{p}$ is perpendicular to the channel axis. In this case, the field component $\mathbf{E}_r$, perpendicular to the channel axis, appears in the integrand in (III.50). In a narrow channel this value differs by a factor of $\frac{2\epsilon_2}{\epsilon_1+\epsilon_2}$ from the field in the outside medium* or, what is the same thing, from the field in the absence of a channel. Hence, the presence of a channel makes the left side of the equation $\frac{2\epsilon_2}{\epsilon_1+\epsilon_2}$ times larger, where ϵ_1 and ϵ_2 are the dielectric constants of the media inside and outside the channel. The

*W. R. Smythe, Static and Dynamic Electricity, McGraw-Hill, N. Y., 1950.

right-hand side of the equation is increased by the same factor, that is to say, the field of the moving dipole $\mathbf{E}_\omega$ is increased by the presence of the channel. The magnetic field in the wave zone is proportional to the electric field so that the radiated energy is $\frac{4\epsilon_2^2}{(\epsilon_1+\epsilon_2)^2}$ times greater.

III.5 Cerenkov Effect in Periodic Linear Structures

a) General theory. In order for a uniformly moving particle to radiate under stationary conditions its velocity must be greater than the phase velocity of the radiated wave. This condition can only be satisfied when the phase velocity of the electromagnetic wave is smaller than the velocity of light in vacuum. Electromagnetic waves of this kind are called slow waves. As an example we consider a simple waveguide of circular cross section. The phase velocity of electromagnetic waves propagating along the axis of an empty waveguide is greater than the velocity of light in vacuum. For this reason it is impossible to excite Cerenkov radiation in an empty waveguide. On the other hand, if the waveguide is filled with a dielectric the phase velocity is smaller than the velocity of light in vacuum when

$$\omega > \frac{\omega_0}{\sqrt{\varepsilon-1}} \tag{III.51}$$

(ω_0 is the cutoff frequency of the empty waveguide and ϵ is the dielectric constant of the material that fills the waveguide). Cerenkov radiation can be excited in this case. However, it is not essential that the waveguide be filled completely with the dielectric. An empty cylindrical channel can be left close to the axis of the waveguide, giving a waveguide partially filled with dielectric. Slow waves can also propagate in a waveguide of this kind and radiation is produced if the velocity of the charged particle moving through the waveguide is greater than the velocity of the slow waves.

The phase velocity in a waveguide can be made smaller than the velocity of light in vacuum without the use of dielectrics; for instance, this condition can be achieved if suitably located metal irises are inserted in the waveguide. A charged particle moving with uniform motion in a system of this kind with velocity greater than the phase velocity will radiate in the same way as a charged particle moving in a dielectric.

This radiation can be determined simply in the case of a linear periodic structure; such a structure is essentially a periodic array of identical unit cells coupled to each other by the apertures through which the charged particle moves. The field of a charge moving through a linear periodic structure was first analyzed in a paper by A. I. Akhiezer, G. L. Lyubarskii and Ya. B. Faĭnberg.[8,9] We follow their treatment.

Consider a linear periodic structure of period l. This can be an iris-loaded waveguide, a corrugated waveguide, or a smooth waveguide (in the last case $l \to \infty$).

In the absence of the charge the field equations in the structure have solutions which can be written in the form $\mathbf{A}_\lambda(\mathbf{r})e^{i\omega_\lambda t}$ ($\mathbf{A}$ is the vector potential). The index λ denotes all the parameters that characterize the various electromagnetic waves that can propagate in the periodic structure with no charge present.

Suppose that the structure is periodic in the z direction (along which the cells are arranged) and that the period is l. The equations for the $\mathbf{A}_\lambda$ are then equations with periodic coefficients (along the z axis). It is convenient to write λ in terms of two parameters, κ and s; κ is a continuous parameter associated with the propagation of a wave along the structure and determined from the solution of equations with periodic coefficients

$$\mathbf{A}_\lambda(\mathbf{r}) = e^{i\varkappa z}\mathbf{a}_\lambda(\mathbf{r}), \quad (\lambda = \varkappa,\ s); \tag{III.52}$$

$\mathbf{a}_\lambda$ is a periodic function of the z coordinate (along which the cells are arranged) with period l, while the quantity κ lies between $-\pi/l$ and $+\pi/l$. The index s denotes the remaining discrete parameters that characterize the wave. The function $\mathbf{a}_\lambda$ is normalized so that

$$\int\limits_{V_1} |\mathbf{a}_\lambda|^2\, dV = 4\pi c^2, \tag{III.53}$$

where V_1 is the volume of a single cell. The functions characterized by different values of the parameters $\lambda \equiv (\kappa, s)$ are orthogonal to each other and form a complete set.

The field produced by a charge moving along the structure can conveniently be written as an expansion

$$\mathbf{A} = \sum_\lambda q_\lambda(t)\,\mathbf{A}_\lambda(\mathbf{r}), \tag{III.54}$$

where $q_\lambda(t)$ is a function of time still to be determined. Since $\mathbf{A}_\lambda(\mathbf{r})$ is a solution of the field in the structure in the absence of charges the expansion can only describe the transverse part of the vector potential. However this is sufficient for our purposes since we are interested in the radiation field only.

The coefficients $q_\lambda(t)$ obey the equation

$$\ddot{q}_\lambda + \omega_\lambda^2 q_\lambda = \frac{1}{N_c}\int\limits_{V_N} \mathbf{j}\mathbf{A}_\lambda^*\, dV = f_\lambda, \tag{III.55}$$

where N_c is the total number of cells and V_N is their volume. If a point charge q moves with uniform motion along the z axis with velocity v, the "force" term f_λ on the right side of (III.55) is simplified and assumes the form

$$f_\lambda = \frac{qv}{c}A_{\lambda z}^*(0,\ 0,\ vt), \tag{III.56}$$

where we have written $x = y = 0$ and $z = vt$.

Using (III.52) we can write (III.56) in the form

$$f_\lambda = \frac{qv}{c}e^{-i\varkappa vt}a_{\lambda z}^*(0,\ 0,\ vt), \tag{III.57}$$

where the function $a_{\lambda z}(0, 0, vt)$ is periodic with period l in the only nonvanishing term in the argument. We expand $a_{\lambda z}$ in a Fourier series:

$$a_{\lambda z}(z) = \sum_{n=-\infty}^{\infty} b_{\lambda n} e^{\frac{2\pi i n}{l} z}, \quad b_{\lambda n} = \frac{1}{l} \int_0^l a_{\lambda z}(z) e^{\frac{2\pi i n}{l} z} dz. \quad \text{(III.58)}$$

Finally, substituting the expression for f_λ in (III.55) we obtain the following equation for the field component $q_\lambda(t)$ due to the uniformly moving point charge:

$$\ddot{q}_\lambda + \omega_\lambda^2 q_\lambda = \frac{qv}{c} \sum_{n=-\infty}^{\infty} b^*_{n\lambda} e^{-i\left(\varkappa + \frac{2\pi n}{l}\right)vt}. \quad \text{(III.59)}$$

What we have derived is an equation for a linear oscillator driven by an external force. The frequency spectrum $\Omega_{\kappa n}$ of the external force is given by

$$\Omega_{\varkappa n} = \left(\varkappa + \frac{2\pi n}{l}\right) v. \quad \text{(III.60)}$$

A given field component $q_\lambda(t)$ is excited only at resonance, that is to say, if the spectrum of the exciting force f_λ contains frequencies ω_λ that coincide with natural frequencies of the oscillator. Growth of q_λ indicates that a given harmonic of the field $\mathbf{A}_\lambda$ is radiated. Thus, the radiation condition can be written

$$\omega_\lambda^2 \equiv \omega_{\varkappa s}^2 = \Omega_{\varkappa n}^2 = \left(\varkappa + \frac{2\pi n}{l}\right)^2 v^2. \quad \text{(III.61)}$$

The condition (III.61) can be satisfied by various combinations of the quantities s, n, and κ. The significance of this condition is clear. Equation (III.58) gives the electromagnetic field (III.52) in terms of a sum of waves with wave vectors $k_{\kappa n} = \kappa + \frac{2\pi n}{l}$. The condition in (III.61) gives the relation between the wave vector $k_{\kappa n}$ and the frequency of the radiated wave ω_λ. It follows that the projection of the phase velocity of the radiated wave on the axis of the system, given by the ratio $\omega_\lambda / k_{\kappa n}$, must coincide with the particle velocity.

The condition in (III.61) assumes a simple form when a charge radiates in a medium composed of periodic layers if the alternating layers have approximately the same dielectric constant.[392-394] In this case, as an approximation we can write $\kappa = \frac{\omega}{c}\sqrt{\epsilon}\cos\vartheta$, where ϵ is the value of the dielectric constant averaged over the period; in this way we obtain

$$\frac{\omega}{v}(1 - \sqrt{\varepsilon}\beta\cos\vartheta) = \frac{2\pi n}{l}.$$

The case n = 0 corresponds to Cerenkov radiation while $n \neq 0$ corresponds to higher order radiation.

The energy of the electromagnetic field is

$$H(t) = \frac{N}{2} \sum_\lambda (\dot{q}_\lambda^2 + \omega_\lambda^2 q_\lambda^2).$$

If radiation is produced then H(t) is proportional to t when $t \to \infty$. The general formula for radiation intensity is of the form (particle radiation energy loss per unit time)

$$I = \lim_{t\to\infty} \frac{H(t)}{t} = \frac{q^2 v^2 l}{4c^2} \left\{ \sum_{n,\lambda'} \frac{|b_{\lambda n}|^2}{\left|\frac{d\omega_\lambda}{d\varkappa} - v\right|_{\lambda=\lambda'}} + \sum_{n,\lambda''} \frac{|b_{\lambda n}|^2}{\left|\frac{d\omega_\lambda}{d\varkappa} + v\right|_{\lambda=\lambda''}} \right\}, \quad \text{(III.62)}$$

where λ' represents the set of quantities κ' and s' that satisfy the equation $\omega_\lambda - \Omega_{\kappa n} = 0$ while λ'' represents the quantities satisfying the equation $\omega_\lambda + \Omega_{\kappa n} = 0$.

The radiation energy loss can be determined in another way. Having determined q_λ from (III.59) we can find the vector potential **A**, (III.54), of the field produced in the periodic structure by the moving charge. The radiation energy loss is determined from the force exerted on the charge by the radiated field. The radiation intensity is

$$I = \lim_{T\to\infty} \frac{1}{T} \int_0^T q\,(\mathbf{E}\mathbf{v})_{z=vt}\, dt,$$

where

$$\mathbf{E}(z = vt) = \sum_\lambda q_\lambda(t) \mathbf{A}_\lambda(\mathbf{v}t). \quad \text{(III.63)}$$

It is not necessary to know all the components of the vectors $\mathbf{A}_\lambda$ of the free field to determine the radiation loss. A knowledge of the projection of $\mathbf{A}_\lambda$ in the direction of the particle velocity is sufficient.

b) Radiation of a charge in an iris-loaded waveguide. As an example of the application of this theory we consider the Cerenkov radiation in an iris-loaded waveguide. Consider a waveguide of circular cross section with radius a (Fig. 6). Suppose that this waveguide contains thin irises with apertures of radius b separated by a distance l along the axis of the waveguide.

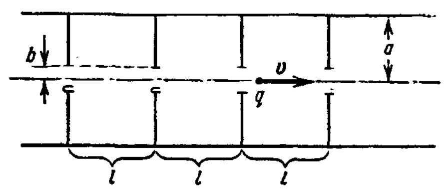

FIG. 6

This system may be regarded as a chain of cylindrical resonators of length l and radius a. The resonators are mechanically connected end to end and are electrically coupled to each other by apertures of radius b cut into the ends. Finding the waves that can propagate in such a linear periodic structure is a complicated problem. This problem was treated approximately by V. V. Vladimirskii* for the case of weak coupling between the resonators, i.e., aperture radius b much smaller than the resonator radius a (or, what is the same thing, iris aperture much smaller than the radius of the waveguide). Under these conditions it may be assumed as a rough approximation that each resonator oscillates at approximately the natural oscillation frequency it would have as a closed resonator. The oscillation of each successive resonator is shifted in phase with respect to that of the proceeding resonator by the same amount ψ. Thus, when the resonators are coupled (apertures between the resonators) an elec-

*V. V. Vladimirskii, J. Tech. Phys. (U.S.S.R.) **17**, 1269, 1277 (1947).

tromagnetic wave travels along the resonators chain; the wavelength of this wave can be expressed simply in terms of the phase shift between the oscillations in adjacent resonators:

$$\lambda = \frac{2\pi}{\varkappa} = \frac{2\pi}{\psi} l, \quad \psi = \varkappa l, \tag{III.64}$$

where l is the length of the resonator (distance between irises) and κ is the wave vector. The condition $\lambda \gg l$ must be satisfied if a wave is to be propagated.

Actually, it cannot be assumed that the natural oscillations are excited in each resonator of the chain. Because the resonators are coupled the allowed frequencies in such a system become frequency bands rather than narrow lines. The lower limit of each frequency band is the natural oscillation frequency of a closed resonator. To summarize, because the resonators are coupled each natural frequency becomes a frequency band. The width of each band increases with aperture radius and the bands can overlap.

The lowest band, which starts with the fundamental natural frequency of a closed cylindrical resonator $\omega_0 = c\mu_1/a$ [μ_1 is the first root of the Bessel function $J_0(x)$] is determined approximately by the relation

$$\omega = \omega_0 [1 + \alpha (1 - \cos \varkappa l)], \tag{III.65}$$

where the quantity

$$\alpha = \frac{2}{3\pi J_1^2(\mu_1)} \cdot \frac{b^3}{a^2 l} \tag{III.66}$$

is the degree of coupling between the resonators. The quantity α is assumed to be small so that the lowest band does not overlap the next band $\alpha \ll 1$.

The dispersion equation (III.65) for a wave propagating along a chain of coupled resonators (i.e., a wave traveling along an iris-loaded waveguide) is an expansion in powers of b/a and is accurate to higher orders of this parameter. The phase velocity and group velocity of the wave described by (III.65) are given with this same accuracy. The phase velocity of the wave is

$$v_{\mathrm{ph}} = \frac{\omega}{\varkappa} = \frac{\omega_0}{\varkappa} [1 + \alpha (1 - \cos \varkappa l)]. \tag{III.67}$$

Since the absolute magnitude of κ is smaller than π/l the phase velocity of a wave in an iris-loaded waveguide satisfies the inequality

$$\frac{\omega_0 l}{\pi} (1 + 2\alpha) < v_{\mathrm{ph}} < \infty; \tag{III.68}$$

the wavelength increases with phase velocity. The group velocity of the wave is given by

$$v_g = \frac{d\omega}{d\varkappa} = \omega_0 \alpha l \sin \varkappa l. \tag{III.69}$$

The group velocity first increases with κ and then diminishes when $\kappa = \pi/2l$.

Now assume that a charged particle moves along the axis of the iris-loaded waveguide. The energy lost in excitation of a wave in the lowest frequency band can be determined by using the general theory given above. The radiation condition (III.61) is written as follows:

$$\omega_\lambda^2 = \omega_0^2 [1 + \alpha (1 - \cos \varkappa l)]^2 = \Omega_{\varkappa n}^2 = \varkappa^2 v^2 \tag{III.70}$$

(in the band n = 0). Assuming that α is small we find immediately

$$\varkappa = \frac{\omega_0}{v}. \tag{III.71}$$

Since $|\kappa| < \pi/l$, to satisfy the radiation condition it is necessary that

$$v > \frac{l\omega_0}{\pi}. \tag{III.72}$$

To accuracy of order α, this inequality means that the particle velocity v must lie within an interval of possible wave phase velocities; this interval is given by (III.68). The normalized function for the lowest frequency band $a_{\lambda z}$ can be written

$$a_{\lambda z} = \frac{2c}{a\sqrt{l}} \frac{J_0\left(\mu_1 \frac{r}{a}\right)}{J_1(\mu_1)} e^{-i\varkappa z}. \tag{III.73}$$

(This is the eigenfunction for a closed resonator of radius a and length l, corresponding to the lowest frequency ω.) When $\kappa l \ll 1$ the radiation is determined by $b_{\lambda 0}$:

$$b_{\lambda 0} = \frac{2c}{a\sqrt{l}} \frac{1}{J_1(\mu_1)} \frac{2}{\varkappa l} \sin \frac{\varkappa l}{2} e^{-i\frac{\varkappa l}{2}}. \tag{III.74}$$

This quantity is to be substituted in (III.62), which gives the radiation intensity for a charged particle in an iris-loaded waveguide. To use this expression we must still compute $\frac{d\omega}{d\kappa} - v$. Using (III.65) we easily show that

$$\left|\frac{d\omega}{d\varkappa} - v\right| = v\left(1 - \frac{\alpha\omega_0 l}{v} \sin \varkappa l\right). \tag{III.75}$$

In substituting all these expressions in the final formula we must use the value of κ that satisfies (III.70), i.e., $\kappa = v/\omega_0$. In this way we obtain the following expression[9] for the radiation intensity (we have corrected an error in reference 9 and use the numerical factor 8 rather than 16):

$$I = \frac{8q^2 v^3}{a^2 l^2 \omega_0^2 J_1^2(\mu_1)} \frac{\sin^2 \frac{\omega_0 l}{2v}}{1 - \frac{\alpha\omega_0 l}{v} \sin \frac{\omega_0 l}{v}}. \tag{III.76}$$

When $\alpha = 0$ (chain of uncoupled closed cylindrical resonators) this expression gives the energy loss due to radiation of frequency ω_0 in a single resonator multiplied by the number of resonators traversed per unit time v/l.

III.6. Cerenkov Effect in Waveguides

As we have indicated above, a charged particle moving uniformly in a waveguide can only interact with slow electromagnetic waves, i.e., waves with phase velocity smaller than the velocity of light. The simplest "slow" waveguide is one filled with dielectric. We start with an analysis of this case.

a) Waveguide filled with an isotropic dielectric. First a preliminary remark. The field produced by a charge in a cylindrical waveguide can be determined by the method of the preceding section because a smooth waveguide is actually a particular case of a linear periodic system (infinite period). However, it will be more condenient to use the results of Sec. III.2, where we have considered the motion of a charge along the axis of a channel filled with a medium ϵ_1 cut into a medium ϵ_2. When a charge moves in a waveguide the outside medium is a metal, i.e., $\epsilon_2 = 4\pi i\sigma/\omega \to i\infty$ (assuming that the metal is ideally conducting). We use the limiting transition $\epsilon_2 \to i\infty$ in (III.11, 13, 15, 16), obtaining

$$\varphi = \int_{-\infty}^{\infty} e^{i\frac{\omega}{v}(z-vt)}\Phi(\omega, r)\,d\omega, \quad \mathbf{A} = \varepsilon\mu\frac{\mathbf{v}}{c}\varphi, \tag{III.77}$$

where

$$\Phi(\omega, r) = \begin{cases} \frac{q}{\pi\varepsilon_1 v}\left[K_0(kr) + \alpha I_0(kr)\right], & r < a, \\ 0, & r > a, \end{cases} \tag{III.78}$$

while

$$\alpha = -\frac{K_0(ka)}{I_0(ka)}, \quad k = \frac{\omega}{v}\sqrt{1 - \varepsilon\mu\beta^2}. \tag{III.79}$$

The expressions given above indicate that the Cerenkov condition for a waveguide filled with dielectric is the usual one, $\epsilon\mu\beta^2 > 1$. The radiation frequency spectrum is given by the dispersion equation

$$J_0\left(\frac{\omega}{v}\sqrt{\varepsilon\mu\beta^2 - 1}\,a\right) = 0, \tag{III.80}$$

hence

$$\frac{\omega}{v}\sqrt{\varepsilon\mu\beta^2 - 1}\,a = \mu_{0s}, \tag{III.81}$$

$$\omega_s = \frac{v\mu_{0s}}{a\sqrt{\varepsilon\mu\beta^2 - 1}}, \tag{III.82}$$

where μ_{0s} is the s-th root of the Bessel function J_0. The Cerenkov loss integral is of the form

$$\frac{dW}{dx} = -\frac{q^2}{c^2}\,\mathrm{Re}\int_0^{\infty}\mu\left(1 - \frac{1}{\varepsilon\mu\beta^2}\right)\frac{N_0\left(\frac{\omega}{v}\sqrt{\varepsilon\mu\beta^2 - 1}\,a\right)}{J_0\left(\frac{\omega}{v}\sqrt{\varepsilon\mu\beta^2 - 1}\,a\right)}\,i\omega\,d\omega. \tag{III.83}$$

The integrand has poles at frequencies for which the radiation condition (III.81) is satisfied. Taking the residues at these poles we obtain the energy loss in the form of a summation over all radiated harmonics[64]:

$$\frac{dW}{dx} = -\frac{2q^2}{a^2}\sum_s \frac{\mu_{0s}}{\varepsilon(\omega_s)\,\omega_s\,[J_0'(\mu_{0s})]^2}\,\frac{1}{\left|\frac{d}{d\omega}\frac{\omega}{v}\sqrt{\varepsilon\mu\beta^2 - 1}\,a\right|_{\omega=\omega_s}}. \tag{III.84}$$

The summation is taken over all harmonics for which the radiation condition $\epsilon\mu\beta^2 > 1$ is satisfied. The energy loss expression is simplified if the medium is dispersionless, in which case ϵ and μ are independent of frequency:

$$\frac{dW}{dx} = -\frac{2q^2}{a^2\varepsilon}\sum_{s=1}^{\infty}\frac{1}{[J_0'(\mu_{0s})]^2}. \tag{III.85}$$

This expression was first obtained by Akhiezer, Lyubarskii, and Faĭnberg.[8] If the waveguide radius is much greater than the radiated wavelength (III.85) becomes the well-known formula of Frank and Tamm.

It is also easy to obtain the field in the case of noncentral motion of a charge in a waveguide filled with dielectric. Suppose that a charge moves parallel to and at a distance r_0 from the axis of a cylindrical waveguide. The field of the charge is of the same form as in non-central motion in a channel (III.39). The excitation condition for a harmonic denoted by the number m is

$$J_m\left(\frac{\omega}{v}\sqrt{\varepsilon\mu\beta^2 - 1}\,a\right) = 0, \tag{III.86}$$

or

$$\frac{\omega}{v}\sqrt{\varepsilon\mu\beta^2 - 1}\,a = \mu_{ms}, \tag{III.87}$$

where μ_{ms} is the s-th root of the Bessel function J_m. The Cerenkov energy loss in the waveguide[63] is:

$$\frac{dW}{dx} = -\frac{2q^2}{a^2} \times \sum_{m,s} a_m\frac{\mu_{ms}}{\omega_{ms}}\frac{1}{\varepsilon(\omega_{ms})}\frac{J_m^2\left(\mu_{ms}\frac{r_0}{a}\right)}{|J_m'(\mu_{ms})|^2}\frac{1}{\left|\frac{d}{d\omega}\frac{\omega}{v}\sqrt{\varepsilon\mu\beta^2 - 1}\,a\right|_{\omega=\omega_{ms}}} \tag{III.88}$$

In this formula a = 1 when m = 0 and a = 2 when $m \neq 0$; ω_{ms} is the frequency that satisfies the radiation condition (III.87). When $r_0 \to 0$, (III.88) becomes (III.84), giving the energy loss for the case of motion along the waveguide axis. If the waveguide is filled with a dispersionless dielectric, (III.88) assumes the form

$$\frac{dW}{dx} = -\frac{4q^2}{a^2\varepsilon}\sum_{s=1}^{\infty}\left[\frac{1}{2}\frac{J_0^2\left(\mu_{0s}\frac{r_0}{a}\right)}{[J_0'(\mu_{0s})]^2} + \sum_{m=1}^{\infty}\frac{J_m^2\left(\mu_{ms}\frac{r_0}{a}\right)}{|J_m'(\mu_{ms})|^2}\right]. \tag{III.89}$$

This expression was given by Muzicar.[317] When $r_0 \to 0$ this expression goes over into (III.85).

It is evident from (III.88, 89) that the energy loss vanishes when $r_0 \to a$. This result can be explained simply. If the charge is close to the wall of the waveguide it may be assumed that the charge moves over an ideally conducting plane. In this case the field is essentially the sum of the field of the original charge and the field of its image, taken with opposite sign. As the true charge approaches the wall, so does its image. Since the sign of the image charge is the opposite of the sign of the real charge, the fields cancel and no energy is radiated. This situation arises when the distance of the charge from the wall is smaller than the radiated wavelength.

Equation (III.89) can be used to estimate the loss as a function of the displacement r_0 (the distance between the charge trajectory and the waveguide axis). It is found that the loss is a maximum when $r_0 = 0$, i.e., when the charge moves along the waveguide axis. This is easily checked by calculating the first and second derivatives of the loss [expression (III.89)] at $r_0 = 0$. The first derivative with respect to r_0 vanishes and the second is negative.

If the charge moves in a waveguide filled with a dispersionless dielectric and the charge velocity is greater than the velocity of light in the dielectric there will be no forces acting on the charge to deflect it from its rectilinear path. The charge does not "feel" the walls of the waveguide because the entire field produced by the charge trails behind it. Actually, however, any medium will always exhibit some dispersion. There will always be waves with phase velocity greater than the velocity of the charge and these waves result in the production of a radial force on the charge. This force can either be a focusing force, directed toward the center, or a defocusing force.

b) Waveguide partially filled with an isotropic dielectric. It is not advisable to fill a waveguide completely with a dielectric because a particle in such a waveguide will lose energy by polarization or Bohr radiation in addition to Cerenkov radiation. The polarization losses are actually responsible for the greater part of the energy loss and the particle is quickly brought to a stop. It is much better to investigate Cerenkov radiation in a waveguide partially filled with a dielectric. The appropriate theoretical analysis was first carried out by Abele.[1]

We consider a circular waveguide of radius a filled in such a way that there is an empty cylindrical channel of radius b along the axis (Fig. 7). It is assumed that the material partially filling the waveguide is isotropic (dielectric constant ϵ and permeability μ). The field of a charge moving with velocity v along the axis of the waveguide is given by (III.11), where

$$\Phi(\omega, r) = \begin{cases} \frac{q}{\pi v}\left[K_0\left(\frac{\omega}{v}\sqrt{1-\beta^2}r\right) + \alpha I_0\left(\frac{\omega}{v}\sqrt{1-\beta^2}r\right)\right], & r < b, \\ \frac{q}{\pi\varepsilon v}\left[\eta K_0\left(\frac{\omega}{v}\sqrt{1-\varepsilon\mu\beta^2}r\right) + \gamma I_0\left(\frac{\omega}{v}\sqrt{1-\varepsilon\mu\beta^2}r\right)\right], & r > b. \end{cases} \quad \text{(III.90)}$$

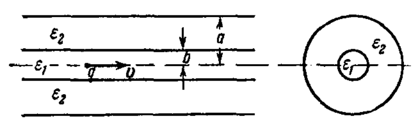

FIG. 7

The coefficients α, η, and γ are determined from three boundary conditions; it is convenient to take as these boundary conditions the continuity of E_z and H_φ at the edge of the empty channel ($r = b$), and the fact that E_z must vanish at the surface of the waveguide ($r = a$). We shall determine the coefficients α, η, and γ under more general assumptions than those above. In particular, we assume the waveguide is filled with a medium characterized by ϵ_1 and μ_1 for $r < b$ and ϵ_2 and μ_2 for $r > b$. Then the field in (III.90) depends on the argument $k_1 r = \frac{\omega}{v}\sqrt{1-\epsilon_1\mu_1\beta^2}\, r$ when $r < b$ and on the argument $k_2 r = \frac{\omega}{v}\sqrt{1-\epsilon_2\mu_2\beta^2}\, r$ when $r > b$. The coefficients α, η, and γ for this case are:

$$\alpha = \frac{Bk_1K_1(k_1b)K_0(k_2b)f_1(a, b, k_1, k_2) - k_2K_0(k_1b)K_1(k_2b)f_2(a, b, k_1, k_2)}{Bk_1I_1(k_1b)K_0(k_2b)f_1(a, b, k_1, k_2) + k_2I_0(k_1b)K_1(k_2b)f_2(a, b, k_1, k_2)},$$

$$\eta = \frac{1}{b}\,\frac{I_0(k_2a)}{Bk_1I_1(k_1b)K_0(k_2b)f_1(a, b, k_1, k_2) + k_2I_0(k_1b)K_1(k_2b)f_2(a, b, k_1, k_2)},$$

$$\gamma = -\eta\frac{K_0(k_2a)}{I_0(k_2a)}, \quad \text{(III.91)}$$

where

$$f_1 = I_0(k_2a) - K_0(k_2a)\frac{I_0(k_2b)}{K_0(k_2b)},$$

$$f_2 = I_0(k_2a) + K_0(k_2a)\frac{I_1(k_2b)}{K_1(k_2b)}; \quad \text{(III.92)}$$

as $a \to \infty$ the functions f_1 and f_2 both become $I_0(k_2a)$ while the coefficients α and η coincide with the coefficients (III.16), which determine the field in the channel with ϵ_1 and μ_1 in a medium with ϵ_2 and μ_2. This limiting case holds for an arbitrarily small absorption in the outside medium.

The radiation field produced by a particle in a waveguide partially filled with a dielectric is similar to that produced in a channel. The Cerenkov condition can be satisfied in the medium with ϵ_1 and μ_1 only, in the medium with ϵ_2 and μ_2 only, or in both media. In the first two cases the field falls off exponentially with increasing distance from the boundary in the medium in which the radiation condition is not satisfied.

The radiation spectrum is discrete in all three cases.

We now consider the original case in greater detail, i.e., the case of an empty waveguide partially filled with dielectric ($\epsilon_1 = 1$, $\mu_1 = 1$). This case is of greatest interest from the point of view of microwave generation. If the Cerenkov condition is satisfied in the medium with ϵ_2 and μ_2 the charge loses energy by wave excitation. Under these conditions the wave field is concentrated in the region $b < r < a$.

The energy loss of a particle caused by radiation in a waveguide partially filled with a dielectric is determined in the usual fashion; we find the reaction of the radiation field on the particle:

$$\frac{dW}{dx} = +qE_z\Big|_{\substack{r=vt\\ r\to 0}} = -\frac{2q^2}{\pi v^2}(1-\beta^2)\,\mathrm{Re}\int\limits_{\varepsilon_2\mu_2\beta^2>1} \alpha(\omega)\, i\omega\, d\omega. \quad \text{(III.93)}$$

When $\epsilon_1 = 1$, $\mu_1 = 1$ and $\epsilon_2\mu_2\beta^2 > 1$, the coefficient $\alpha(\omega)$ assumes the form

$$\alpha(\omega) = \frac{s_2K_1(k_1b)\psi_1 + k_1\varepsilon_2K_0(k_1b)\psi_0}{s_2I_1(k_1b)\psi_1 - k_1\varepsilon_2I_0(k_1b)\psi_0}, \quad \text{(III.94)}$$

where

$$s_2 = \frac{\omega}{v}\sqrt{\varepsilon_2\mu_2\beta^2 - 1}, \qquad k_1 = \frac{\omega}{v}\sqrt{1-\beta^2}, \qquad \text{(III.95)}$$

while ψ_1 and ψ_0 are the functions introduced by Abele:

$$\psi_0 = J_1(s_2b)\,N_0(s_2a) - J_0(s_2a)\,N_1(s_2b),$$
$$\psi_1 = J_0(s_2b)\,N_0(s_2a) - J_0(s_2a)\,N_0(s_2b). \qquad \text{(III.96)}$$

The integrand has poles where the denominator of $\alpha(\omega)$ (III.94) vanishes. The corresponding values of the frequency give the radiation spectrum in a waveguide partially filled with a dielectric. We write the equation that determines the radiation frequency when (III.95) holds:

$$\frac{s_2}{k_1\varepsilon_2}\frac{I_1(k_1b)}{I_0(k_1b)} = \frac{\psi_0}{\psi_1}; \qquad \text{(III.97)}$$

when $k_1b \ll 1$, (III.60) becomes

$$\psi_0 = 0. \qquad \text{(III.98)}$$

However, when $s_2b \ll 1$ the radiation spectrum is determined by the equation

$$J_0(s_2a) = 0, \qquad \text{(III.99)}$$

in the same way as for a waveguide completely filled by a dielectric with ϵ_2 and μ_2. This means that if the empty channel along the waveguide axis is narrow enough the radiation spectrum is the same as that of a waveguide completely filled with a dielectric. In many respects the situation is reminiscent of the radiation in an empty channel surrounded by an infinite medium (III.2). It was found in that case, too, that when the channel radius is arbitrarily small the radiation pattern is the same as when there is no channel. The difference between these two cases is the fact that the spectrum is discrete in the waveguide case. When $k_1b \gg 1$, (III.97) assumes the form

$$\frac{s_2}{k_1\varepsilon_2} = \frac{\psi_0}{\psi_1}. \qquad \text{(III.100)}$$

When k_1b, k_1a, s_2b, and s_2a are large compared with unity, (III.17) becomes

$$\frac{s_2}{k_1\varepsilon_2} = \operatorname{ctg} k_2(a-b). \qquad \text{(III.101)*}$$

If dispersion is neglected the frequency spectrum given by (III.97) is a sequence of increasing values of ω_λ with no limit; the difference between two neighboring values of ω_λ approaches a constant, as indicated by (III.101). The presence of dispersion introduces a cutoff at high values of ω.

Integrating the loss expression (III.93) we have

$$\frac{dW}{dx} = -\frac{2q^2}{v^2}(1-\beta^2)\sum_\lambda \left.\frac{[s_2K_1(k_1b)\,\psi_1 + k_1\varepsilon_2K_0(k_1b)\,\psi_0]\,\omega}{\frac{d}{d\omega}[s_2I_1(k_1b)\,\psi_1 - k_1\varepsilon_2I_0(k_1b)\,\psi_0]}\right|_{\omega=\omega_\lambda}. \qquad \text{(III.102)}$$

It follows from (III.102) that the dependence of the radiation intensity in a waveguide partially filled with a dielectric on the radius of the empty channel is qualitatively the same as for an empty channel in an infinite medium. In particular, if the radius of the empty channel b approaches zero, (III.102) becomes (III.85), which applies for a waveguide completely filled with a dielectric. Thus, in a waveguide or in an infinite medium, a narrow channel does not affect the intensity of radiation produced by a charged particle.

If the channel radius is so large that $k_1b \gg 1$, it is evident from (III.102) that the radiation intensity falls off exponentially (as e^{-2k_1b}), again in complete analogy with the case of an empty channel in an infinite medium. It should be recalled, however, that in contrast with an infinite medium, a channel in a waveguide yields a discrete radiation spectrum.

Several other features of the radiation in a waveguide partially filled with a dielectric are of interest. S. N. Stolyarov has shown that when a charge moves in an empty channel parallel to and at a distance $r_0 < b$ from the axis the energy loss due to radiation is always greater than for motion along the axis. The minimum energy loss obtains when the charge moves along the axis of a waveguide partially filled with a dielectric.

c) Waveguide filled with an anisotropic dielectric. Anisotropic dielectrics can also be used to retard electromagnetic waves in a waveguide. We consider the radiation in a circular waveguide filled with a uniaxial crystalline dielectric. Let the optical axis of the dielectric be along the waveguide axis. If the axis of the waveguide is taken as the z axis the dielectric tensor for the material filling the waveguide can be written in the form

$$\varepsilon = \begin{pmatrix} \varepsilon_r & 0 & 0 \\ 0 & \varepsilon_r & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix}. \qquad \text{(III.103)}$$

The equation for the field potentials in the waveguide can be obtained from (II.4), where we assume that the only nonvanishing components are A_z and φ, upon which we impose the added condition:

$$A_z = \varepsilon_r\beta\varphi. \qquad \text{(III.104)}$$

The following equation is obtained for A_z (the subscript z is omitted):

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} - \left(1 - \frac{1}{\varepsilon_r\beta^2}\right)\frac{\varepsilon_z}{c^2}\frac{\partial^2 A}{\partial t^2} = -\frac{4\pi}{c}\,j. \qquad \text{(III.105)}$$

In deriving this equation we have assumed that the field produced by the uniformly moving charge depends on z and t in the combination $z - vt$. The equation for φ can then easily be obtained from (III.104). The current density j in the right side of (III.105) is described in terms of a point charge moving with velocity v:

$$j = qv\delta(x)\,\delta(y)\,\delta(z-vt) = qv\frac{\delta(r)}{2\pi r}\,\delta(z-vt). \qquad \text{(III.106)}$$

It is convenient to write $\delta(r)$ in the representation

*ctg = cot.

$$\delta(r)=\frac{2r}{a^2}\sum_{n=1}^{\infty}\frac{J_0\left(\mu_n\frac{r}{a}\right)}{J_0'^2(\mu_n)}, \qquad \text{(III.107)}$$

where a is the waveguide radius. We write $\delta(z-vt)$ in the well-known Fourier integral expansion. The solution of (III.105) is

$$A(r, z-vt)=-\frac{2q}{\pi ca^2}\sum_{n=1}^{\infty}\frac{J_0\left(\mu_n\frac{r}{a}\right)}{J_0'^2(\mu_n)}\int_{-\infty}^{\infty}\frac{e^{i\frac{\omega}{v}(z-vt)}d\omega}{\frac{\mu_n^2}{a^2}-\frac{\omega^2}{v^2}\frac{\varepsilon_z}{\varepsilon_r}(\varepsilon_r\beta^2-1)}. \qquad \text{(III.108)}$$

Using this expression we determine the particle radiation loss

$$\frac{dW}{dz}=-\frac{4q^2}{\pi cva^2}\sum\frac{1}{J_0'^2(\mu_n)}\operatorname{Re}\int_0^{\infty}\frac{\left(\frac{1}{\beta\varepsilon_r}-\beta\right)i\omega\,d\omega}{\frac{\mu_n^2}{a^2}-\frac{\omega^2}{v^2}\frac{\varepsilon_z}{\varepsilon_r}(\varepsilon_r\beta^2-1)}. \qquad \text{(III.109)}$$

It is evident that the radiated frequencies can be obtained by setting the integrand in (III.109) equal to zero. This procedure yields (M. I. Kaganov[232])

$$\omega_n^2=\frac{\mu_n^2v^2\varepsilon_r}{a^2\varepsilon_z}\frac{1}{\varepsilon_r\beta^2-1}. \qquad \text{(III.110)}$$

The radiation condition becomes

$$\frac{\varepsilon_z}{\varepsilon_r}(\varepsilon_r\beta^2-1)>0. \qquad \text{(III.111)}$$

This condition is the same as the radiation condition in an infinite uniaxial crystal (II.23).

Since ϵ_r and ϵ_z are functions of frequency ω, one value of μ_n in (III.110), which determines the radiation frequency, can correspond to several values of ω. Integrating (III.109) we have[239]

$$\left(\frac{dW}{dz}\right)_{\text{Cer}}=\frac{2q^2}{a^2c^2}\sum_n\frac{1}{J_1^2(\mu_n)}\left\{\frac{1-\frac{1}{\beta^2\varepsilon_r}}{\frac{d}{d\omega^2}\left[\frac{\omega^2}{v^2}\frac{\varepsilon_z}{\varepsilon_r}(\varepsilon_r\beta^2-1)\right]}\right\}_{\omega=\omega_n} \qquad \text{(III.112)}$$

The expression is simplified if ϵ_r and ϵ_z are independent of frequency, and the energy loss per unit length of path becomes

$$\frac{dW}{dz}=-\frac{2q^2}{a^2\varepsilon_z}\sum_n\frac{1}{J_1^2(\mu_n)}. \qquad \text{(III.113)}$$

This quantity diverges in precisely the same way as the energy loss of a charge due to Cerenkov radiation in a dispersionless medium. However, it can be used to obtain a correct result if the summation is terminated at some frequency. We recall that each term in (III.113) gives the radiation intensity at a frequency ω_n determined by (III.110).

We now compare (III.113) with (III.85), the Cerenkov loss in a waveguide filled with an isotropic dielectric. A comparison shows that the use of an anisotropic dielectric makes it possible to change the radiation intensity by a factor ϵ/ϵ_z without changing the particle velocity or the waveguide radius.

It is of interest to investigate the way in which the energy loss in a waveguide filled with an anisotropic dielectric depends on the orientation of the optical axis of the dielectric with respect to the waveguide axis. This problem has been investigated by L. S. Bogdankevich for the case of a rectangular waveguide.[59]

Consider a waveguide of a rectangular cross section. The sides of the waveguide are 2a and 2b. Suppose that the waveguide is filled with an anisotropic dielectric. As in the preceding example we consider a particular anisotropic dielectric, a uniaxial crystal characterized by two parameters: the dielectric constant along the optical axis (ϵ_0) and the dielectric constant perpendicular to the axis $(\epsilon_\perp)$.

The equations for the field potentials due to the motion of a charge in a rectangular waveguide filled with an anisotropic dielectric are of the form given in (II.4) with the additional condition $\operatorname{div}\epsilon\mathbf{A}=0$. If the charged particle moves along the axis of the waveguide the charge density is

$$\varrho=q\delta(x)\,\delta(y)\,\delta(z-vt)$$
$$=\frac{q}{8\pi vab}\sum_{m,\,n=-\infty}^{\infty}\cos\frac{2n+1}{2a}\pi x\cdot\cos\frac{2m+1}{2b}\pi y\int e^{i\frac{\omega}{v}(z-vt)}\,d\omega. \qquad \text{(III.114)}$$

The field potentials can be determined in the same way as in part II. These are

$$\left.\begin{aligned}\varphi&=\frac{q}{2vab}\sum_{m,\,n}\cos\frac{2n+1}{2a}\pi x\cdot\cos\frac{2m+1}{2b}\pi y\int\frac{e^{i\frac{\omega}{v}(z-vt)}d\omega}{(\mathbf{k}\varepsilon\mathbf{k})},\\ \mathbf{A}&=-\frac{q}{2vcab}\sum_{m,\,n}\cos\frac{2n+1}{2a}\pi x\cdot\cos\frac{2m+1}{2b}\pi y\\ &\quad\times\int\Lambda^{-1}\left[\mathbf{S}-\frac{(\mathbf{k}\varepsilon\Lambda^{-1}\mathbf{s})}{(\mathbf{k}\varepsilon\Lambda^{-1}\mathbf{k})}\right]e^{i\frac{\omega}{v}(z-vt)}\,d\omega,\end{aligned}\right\} \qquad \text{(III.115)}$$

where ϵ is the dielectric constant operator, Λ is an operator connected with ϵ by the relation $\Lambda=\epsilon\frac{\omega^2}{c^2}-k^2$, and $\mathbf{k}$ is a vector with components

$$k_x=\frac{2n+1}{2a}\pi,\qquad k_y=\frac{2m+1}{2b}\pi,\qquad k_z=\frac{\omega}{v},$$
$$\mathbf{s}=\mathbf{v}-\varepsilon\mathbf{k}\frac{\omega}{(\mathbf{k}\varepsilon\mathbf{k})}. \qquad \text{(III.116)}$$

The operators Λ and ϵ are diagonal in the coordinate system corresponding to the principal axes of the crystal and

$$(\mathbf{k}\varepsilon\mathbf{k})=\varepsilon_xk_x^2+\varepsilon_yk_y^2+\varepsilon_zk_z^2,$$
$$\mathbf{k}\varepsilon\Lambda^{-1}\mathbf{s}=\frac{k_xs_x\varepsilon_x}{\varepsilon_x\frac{\omega^2}{c^2}-k^2}+\frac{k_ys_y\varepsilon_y}{\varepsilon_y\frac{\omega^2}{c^2}-k^2}+\frac{k_zs_z\varepsilon_z}{\varepsilon_z\frac{\omega^2}{c^2}-k^2} \qquad \text{(III.117)}$$

The expressions for the field potentials in the waveguide (III.115) and (III.116, 117) differ from the analogous expressions for an infinite crystal (II.16–19) only in that the components of the wave vector perpendicular to the walls, k_x and k_y, must assume discrete values.

Bogdankevich[59] has determined the energy loss of a charge due to radiation in a rectangular waveguide filled with an anisotropic uniaxial dielectric. Two cases have been considered. In the first case the optical axis of the dielectric is parallel to the waveguide axis and parallel to the velocity of the charge. In the second case the optical axis of the dielectric is perpendicular to the waveguide axis.

Crystal axis parallel to the waveguide axis. In this case radiation is produced when the following condition is satisfied:

$$\frac{\varepsilon_0}{\varepsilon_\perp}[\varepsilon_\perp\beta^2-1]>0, \qquad \text{(III.118)}$$

which is the same as the radiation condition in an infinite uniaxial crystal with a charge moving along the optical axis (II.22). Only extraordinary waves are radiated. The ordinary waves are not radiated even when their phase velocity is smaller than the velocity of the charge. The ordinary waves are not radiated because the electric vector of the ordinary wave is perpendicular to the optical axis of the uniaxial crystal and, thus, to the velocity of the charge (cf. Sec. 11.4). We note that precisely the same situation arises in an example we have considered earlier, i.e., radiation of a charge in a circular waveguide. Only the extraordinary waves are radiated and the radiation condition (III.111) for these waves is exactly the same as (III.118). This result follows because in both cases the axis of the crystal filling the waveguide is parallel to the charge velocity.

The energy loss due to radiation of the extraordinary waves can be found from the field potentials (III.115)

$$\frac{dW_e}{dz}=-\frac{\pi q^2}{ab}\sum_{m,n}\left|\frac{(\varepsilon_\perp\beta^2-1)\,\omega}{\varepsilon_\perp\frac{d}{d\omega}\left[\omega^2\frac{\varepsilon_0}{\varepsilon_\perp}(\varepsilon_\perp\beta^2-1)\right]}\right|_{\omega=\omega_{mn}}, \qquad \text{(III.119)}$$

where ω_{mn} is the radiation frequency, given by the relation

$$\frac{\omega^2}{v^2}\frac{\varepsilon_0(\omega)}{\varepsilon_\perp(\omega)}[\varepsilon_\perp(\omega)\beta^2-1]=\left(\frac{2n+1}{2a}\right)^2\pi^2+\left(\frac{2m+1}{2a}\right)^2\pi^2. \qquad \text{(III.120)}$$

The energy loss relation (III.119) becomes particularly simple if the material filling the waveguide is dispersionless, that is, if ϵ_0 and $\epsilon_\perp$ are independent of frequency. In this case

$$\frac{dW}{dz}=-\frac{\pi q^2}{2ab}\frac{1}{\varepsilon_0}N, \qquad \text{(III.121)}$$

where N is the number of harmonics excited in the waveguide. It is evident that a charge loses the same amount of energy in the excitation of each harmonic in a dispersionless rectangular waveguide.

Crystal axis perpendicular to waveguide axis. Both ordinary and extraordinary waves are radiated in this case. The radiation condition is the same as for motion in an infinite crystal [cf. (II.6b)] with the sole difference that in the waveguide case the components k_x and k_y can only assume discrete values.

The frequencies of the radiated ordinary waves are determined from the equation

$$\frac{\omega^2}{v^2}[\varepsilon_\perp(\omega)\beta^2-1]=\left(\frac{2n+1}{2a}\pi\right)^2+\left(\frac{2m+1}{2a}\pi\right)^2, \qquad \text{(III.122)}$$

while the frequencies of the extraordinary waves are determined from the equation

$$\frac{\omega^2}{v^2}\frac{\varepsilon_\perp(\omega)[\varepsilon_0(\omega)\beta^2-1]}{\varepsilon_0(\omega)k_x^2+\varepsilon_\perp(\omega)k_y^2}=1, \qquad \text{(III.123)}$$

where k_x and k_y are determined by (III.116).

The energy losses due to the radiation of ordinary and extraordinary waves are

$$\frac{dW_0}{dz}=-\frac{\pi q^2}{c^2ab}\sum_{m,n}\left|\frac{\omega k_y^2(\varepsilon_\perp\beta^2-1)}{(\varepsilon_\perp\beta^2k_y^2+k_x^2)\frac{d}{d\omega}\left[\frac{\omega^2}{v^2}(\varepsilon_\perp\beta^2-1)\right]}\right|_{\omega=\omega_{mn}>0},$$

$$\frac{dW_e}{dz}=-\frac{\pi q^2}{v^2ab}\sum_{m,n}\left|\frac{\omega k_x^2(\varepsilon_0\beta^2-1)}{(\varepsilon_\perp\beta^2k_y^2+k_x^2)\frac{d}{d\omega}\left[\frac{\omega^2}{c^2}\varepsilon_0\varepsilon_\perp-(\mathbf{k}\varepsilon\mathbf{k})\right]}\right|_{\omega=\omega_{mn}>0}. \qquad \text{(III.24)}$$

The values of ω_{mn} for the ordinary waves are determined from (III.122) and the values for the extraordinary waves are found from (III.123).

It is evident from (III.124) that when dispersion can be neglected the terms under the summation sign do not depend explicitly on ω_{mn}. This situation can be exploited to obtain a simple expression for the total radiation loss. Suppose that ϵ_0 and $\epsilon_\perp$ are independent of frequency; furthermore, suppose that for some choice of m and n the radiation conditions are satisfied for both the ordinary and extraordinary waves. In this case the total energy loss due to radiation of ordinary and extraordinary waves is written in the form

$$\frac{dW}{dz}=-\frac{\pi q^2}{2ab}\frac{1}{\varepsilon_\perp}N', \qquad \text{(III.125)}$$

where N′ is the number of pairs (m, n) for which the radiation conditions are satisfied for both kinds of waves. It is evident from the last relation that the total loss is the same for each harmonic (m, n). In this case the loss due to radiation of ordinary waves in each harmonic is proportional to

$$W_0^{(m,n)}\sim\frac{k_y^2\beta}{k_y^2\varepsilon_\perp\beta+k_x^2}, \qquad \text{(III.126)}$$

while the loss due to radiation of extraordinary waves is proportional to

$$W_e^{(m,n)}\sim\frac{k_x^2}{\varepsilon_\perp}\frac{1}{k_y^2\varepsilon_\perp\beta+k_x^2} \qquad \text{(III.127)}$$

with the same proportionality constant. The total radiation in a given harmonic is proportional to $1/\epsilon_\perp$, as follows from (III.125). It is evident from (III.126, 127) that only the extraordinary waves are radiated in the cross section of the waveguide in the xz ($k_y = 0$) plane while only the ordinary waves are radiated in the cross section of the waveguide in the yz ($k_x = 0$) plane. A similar situation obtains for an infinite medium (II.6b). This effect is due to the nature of the polarization of the electric vector.

III.7. Field of a Charged Particle Moving Parallel to the Boundary Between Two Media

We consider the field of a point charge that moves with constant velocity parallel to a plane boundary separating two media. The simplest case of this kind is the one in which the charge moves in vacuum parallel to and over a semi-infinite plane dielectric. This particular problem is also of great interest for various radiophysical purposes, primarily problems involving the generation of microwaves. The first qualitative estimates of the radiation produced in this case were carried out[182,183] for just this purpose; these analyses were then carried out in greater detail.

The problem of a charge moving over a plane boundary is, in many respects, similar to the problem of a charge moving in a channel, which we have considered earlier.

a) We consider the general case of motion of a charge moving along a boundary. The charge is denoted by q, the velocity by v, and the distance of the charge from the boundary by d. Suppose that the charge moves in medium 1 (ϵ_1, μ_1) at a distance d from a plane boundary with medium 2 (ϵ_2, μ_2) (Fig. 8).

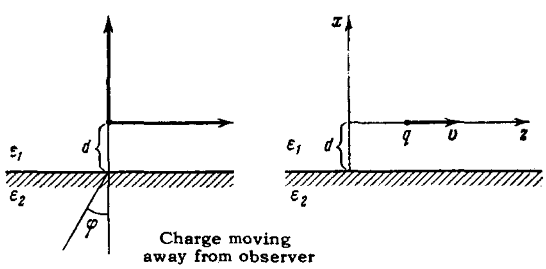

FIG. 8. In the figure at the left the x axis is vertical and the y axis is horizontal.

The solution of this problem was obtained by V. E. Pafomov,[329] who also investigated a number of particular cases in detail.* The case in which medium 1 is a vacuum ($\epsilon_1 = 1$, $\mu_1 = 1$) has been considered earlier by a number of authors: Danos,[116] Linhart,[272] and Motz. Investigations of various aspects of this problem have been published by Sitenko and Tkalich[364] and by Garibyan and Mergelyan.[171] The method used by Danos is the simplest and will be used below.

We introduce a Cartesian coordinate system with the z axis along the line of motion of the charge. The x axis is normal to the boundary so that the equation of the boundary plane is $x = -d$. The field in this problem is made up of the field of a charge in an infinite medium and the fields due to the presence of the boundary.

Maxwell's equations for the field potentials in the first medium are:

$$\left(\Delta - \frac{\varepsilon_1\mu_1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi_0 = -\frac{4\pi p}{\varepsilon_1}, \qquad A_{0z} = \varepsilon_1\mu_1\beta\varphi_0. \qquad \text{(III.128)}$$

In the second medium, on the other side of the boundary, we take $\Phi_0 = A_{0Z} = 0$.

The solution of (III.128) can be conveniently written in the form

$$A_{0z} = \frac{qi}{2\pi c}\int \frac{dk_y\, d\omega\, \mu_1(\omega)}{g_1(k_y, \omega)} e^{i\left\{g_1|x| + k_y y + \frac{\omega}{v}(z - vt)\right\}},$$

$$\varphi_0 = \frac{1}{\varepsilon_1\mu_1\beta} A_{0z}, \qquad \text{(III.129)}$$

where

$$g_1 = \sqrt{(\varepsilon_1\mu_1\beta^2 - 1)\frac{\omega^2}{v^2} - k_y^2} = k_x \qquad \text{(III.130)}$$

is the projection of the wave vector **k** along the x axis.

It is evident that the solution of (III.129) is the same as the expression for A_Z and φ given in the first part of this work [cf. (I.49–52)]. The difference in form is due to the fact that we must satisfy the boundary conditions in the plane $x = -d$, parallel to the yz plane; hence it is convenient to write all Fourier coefficients as functions of k_y and $k_Z = \omega/v$ alone.

The potential corresponding to the homogeneous Maxwell equations is

$$\mathbf{A} = \mathbf{A}_1 = \frac{iqc}{2\pi v}\iint dk_y\, d\omega\, \mathbf{a}_1(k_y, \omega)\, e^{i\left\{2dg_1 + g_1 x + k_y y + \frac{\omega}{v}(z - vt)\right\}} \qquad \text{(III.131)}$$

for $x > -d$ and

$$\mathbf{A} = \mathbf{A}_2 = \frac{iqc}{2\pi v}\iint dk_y \cdot d\omega\, \mathbf{a}_2(k_y, \omega)\, e^{i\left\{dg_1 + g_2(x+d) + k_y y + \frac{\omega}{v}(z - vt)\right\}} \qquad \text{(III.132)}$$

for $x < -d$.

Here $\mathbf{A}_1$ describes the field in the first medium due to the presence of the boundary while $\mathbf{A}_2$ describes the field in the second medium. We impose the following obvious requirements on $\mathbf{A}_1$ and $\mathbf{A}_2$:

$$\operatorname{div}\mathbf{A}_1 = 0, \qquad \operatorname{div}\mathbf{A}_2 = 0. \qquad \text{(III.133)}$$

Further, since $\mathbf{A}_1$ and $\mathbf{A}_2$ represent solutions of the homogeneous Maxwell equations we can write $\varphi_1 = \varphi_2 = 0$.

The boundary conditions then yield the following expressions for $\mathbf{a}_1(k_y, \omega)$ and $\mathbf{a}_2(k_y, \omega)$:

$$a_{1x} = -\frac{1}{\omega\varepsilon_1}\frac{\varepsilon_1 g_2 - \varepsilon_2 g_1}{\varepsilon_1 g_2 + \varepsilon_2 g_1},$$

$$a_{1y} = -\frac{k_y}{\omega\varepsilon_1}\left[\frac{2\varepsilon_1(\mu_1 g_1 + \mu_2 g_2)}{(\mu_1 g_2 + \mu_2 g_1)(\varepsilon_1 g_2 + \varepsilon_2 g_1)} - \frac{1}{g_1}\right],$$

$$a_{1z} = \frac{v}{\omega^2\varepsilon_1}\left[k_y^2\left(\frac{2\varepsilon_1(\mu_1 g_1 + \mu_2 g_2)}{(\mu_1 g_2 + \mu_2 g_1)(\varepsilon_1 g_2 + \varepsilon_2 g_1)} - \frac{1}{g_1}\right)\right.$$

$$\left. + g_1\frac{\varepsilon_1 g_2 - \varepsilon_2 g_1}{\varepsilon_1 g_2 + \varepsilon_2 g_1}\right], \qquad \text{(III.134)}$$

*The author is indebted to V. E. Pafomov for this private communication.

$$a_{2x} = -\frac{2g_1}{\omega(\varepsilon_1 g_2 + \varepsilon_2 g_1)},$$
$$a_{2y} = -\frac{2k_y}{\omega(\varepsilon_1 g_2 + \varepsilon_1 g_1)} \frac{\mu_1 g_1 + \mu_2 g_2}{\mu_1 g_2 + \mu_2 g_1},$$
$$a_{2z} = \frac{2v}{\omega^2(\varepsilon_1 g_2 + \varepsilon_2 g_1)} \left(g_1 g_2 + k_y^2 \frac{\mu_1 g_1 + \mu_2 g_2}{\mu_1 g_2 + \mu_2 g_1} \right). \qquad \text{(III.135)}$$

The fields were computed by other methods in references 364 and 171. The formulas which have been obtained completely determine the fields produced by the moving charge in both media. The field in the first medium is determined by the potentials φ_0 and $\mathbf{A}_0 + \mathbf{A}_1$ [cf. (III.129–134)] while the field in the second medium is determined by the vector potential $\mathbf{A}_2$.

The structure of the solution can be easily understood if we consider the analogous behavior of a light wave incident on a boundary between two media. The potentials $\mathbf{A}_0$ and φ describe the "incident" field, the potential $\mathbf{A}_1$ describes the "reflected" field, and the potential $\mathbf{A}_2$ describes the "refracted" field.

If the substitutions $\epsilon_1 = 1$, $\mu_1 = 1$, and $\mu_2 = 1$ are made we obtain the Danos solution for motion of a charge in vacuum.*

We now consider briefly certain particular cases of the motion of a charge over a boundary separating two media. Both media are assumed to be transparent.

b) First let us consider the frequency range for which the Cerenkov condition is satisfied in the second medium but is not satisfied in the first:

$$\varepsilon_1 \mu_1 \beta^2 < 1, \quad \varepsilon_2 \mu_2 \beta^2 > 1. \qquad \text{(III.136)}$$

This case is of most interest for Cerenkov generation of microwaves.

The expressions describing the field are a sum of plane waves of the form $\exp\{i[gx + k_y y + \frac{\omega}{v}(z - vt)]\}$ where $g = g_1 = \sqrt{(\epsilon_1\mu_1\beta^2 - 1)\frac{\omega^2}{v^2} - k_y^2}$ in the first medium while $g = g_2 = \sqrt{(\epsilon_2\mu_2\beta^2 - 1)\frac{\omega^2}{v^2} - k_y^2}$ in the second medium. The quantity g_1 is purely imaginary in the present case. This means that when the distance from the boundary is large the waves are damped exponentially in the first medium, i.e., there is no radiation in the first medium. Radiation is produced in the second medium at those values of ω and k_y for which the projection of g_2 (the wave vector) on the x axis is real. This condition gives

$$k_y^2 < \frac{\omega^2}{v^2}(\varepsilon_2\mu_2\beta^2 - 1). \qquad \text{(III.137)}$$

As in the case of a continuous medium, the wave vectors for a given frequency ω lie on the surface of a cone whose opening angle is determined by the Cerenkov condition. However, in contrast with the case of an infinite medium the electric vector of the radiated wave does not lie in the plane containing the particle velocity and the wave vector. Instead the projection of the wave vector on the xy plane forms an angle φ with the negative x axis:

$$\operatorname{tg}\varphi = -\frac{k_y}{k_x} = \frac{k_y}{g_2}. \qquad \text{(III.138)}$$

The projection of $\mathbf{E}_2(k_y, \omega)$ on the xy plane makes an angle $\varphi' \neq \varphi$ with the negative x axis:

$$\operatorname{tg}\varphi' = -\frac{E_y}{E_x} = -\frac{a_{2y}}{a_{2x}} = -\frac{k_y}{g_1}\frac{\mu_1 g_1 + \mu_2 g_2}{\mu_1 g_2 + \mu_2 g_1}. \qquad \text{(III.139)}$$

Since g_1 is pure imaginary in the case at hand (III.139) indicates that the wave is elliptically polarized. The direction of polarization is reversed when the sign of k_y is changed. This is to be expected because the field is symmetric under reflection in the xz plane. When k_y approaches zero the elliptically polarized wave degenerates into a linearly polarized wave. The wave whose wave vector lies in the xz plane ($k_y = 0$). is also polarized in the xz plane ($E_y = 0$). This is evident from (III.135).

We now find the energy loss of a charge moving parallel to the interface. The energy loss of the charged particle per unit path can be computed from the work done by the moving charge against the reaction force due to the field:

$$\frac{dW}{dz} = qE_z\Big|_{\substack{x=0\\y=0\\z=vt}} = -q\left[\frac{\partial\varphi_0}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\left(A_{0z} + A_{1z}\right)\right]\Big|_{\substack{x=0\\y=0\\z=vt}}$$
$$= -\frac{q^2}{2\pi v}\int\int_{-\infty}^{\infty}\frac{dk_y\,d\omega}{\varepsilon_1 g_1}\left\{(\varepsilon_1\mu_1\beta^2 - 1)\frac{\omega}{v}\right.$$
$$\left. + \frac{v}{\omega} g_1 e^{2ig_1 d}\left[\frac{2\varepsilon_1 k_y^2(\mu_1 g_1 + \mu_2 g_2)}{(\mu_1 g_2 + \mu_2 g_1)(\varepsilon_1 g_2 + \varepsilon_2 g_1)} - \frac{k_y^2}{g_1} + g_1\frac{\varepsilon_1 g_2 - \varepsilon_2 g_1}{\varepsilon_1 g_2 + \varepsilon_2 g_1}\right]\right\}. \qquad \text{(III.140)}$$

In the case considered here the integration over ω extends over the range for which the two inequalities $\epsilon_1\mu_1\beta^2 < 1$ and $\epsilon_2\mu_2\beta^2 > 1$ are satisfied: the region of integration over k_y is determined by (III.137).

The first term in the curly brackets under the integral can be neglected in the present case. This term describes the energy loss of a charge in an infinite medium ϵ_1 and μ_1 and does not make a contribution to the Cerenkov loss because the Cerenkov condition is not satisfied in the first medium (polarization losses are considered below). Hence, in the frequency range of interest the loss integral is

$$\frac{dW}{dz} = -\frac{q^2}{\pi}\operatorname{Re}\int\int\frac{dk_y\,d\omega}{\omega}e^{2ig_1 d}$$
$$\times\left[\frac{2\varepsilon_1 k_y^2(\mu_1 g_1 + \mu_2 g_2)}{(\mu_1 g_2 + \mu_2 g_1)(\varepsilon_1 g_2 + \varepsilon_2 g_1)} - \frac{k_y^2}{g_1} + g_1\frac{\varepsilon_1 g_2 - \varepsilon_2 g_1}{\varepsilon_1 g_2 + \varepsilon_2 g_1}\right]. \qquad \text{(III.141)}$$

The integration over ω is limited to positive values and is bounded by the inequalities $\epsilon_2\mu_2\beta^2 > 1$, $\epsilon_1\mu_1\beta^2 < 1$, and $k_y^2 < \frac{\omega^2}{v^2}(\epsilon_2\mu_2\beta^2 - 1)$. When these inequalities are

*There is an obvious error in the paper by Danos:[110] in Eq. (66), the factor l should not appear in the numerator of the expression for B_z.

satisfied radiation is generated in the second medium but not the first. In the usual way, the symbol Re denotes the real part of a quantity.

We consider the radiation energy loss in the particularly simple case $\epsilon_1 = 1$, $\mu_1 = 1$, $\mu_2 = 1$, and $\epsilon_2 = \epsilon$. When the real part of the integrand is taken (III.141) assumes the form

$$\frac{dW}{dz} = -\frac{2q^2}{\pi} \int \frac{d\omega}{\omega} dk_y e^{-2sd} \frac{g(k_y^2 + \varepsilon s^2)}{g^2 + \varepsilon^2 s^2}, \qquad \text{(III.142)}$$

where

$$g = g_2 = \sqrt{\frac{\omega^2}{v^2}(\varepsilon\beta^2 - 1) - k_y^2}, \quad s = -ig_1 = \sqrt{\frac{\omega^2}{v^2}(1-\beta^2) + k_y^2}. \qquad \text{(III.143)}$$

It is convenient to replace the variable k_y by another variable, the angle φ, formed by the negative x axis and the projection of the radiation wave vector in the xy plane. It is evident that

$$k_y = \frac{\omega}{v} \sqrt{\varepsilon\beta^2 - 1} \sin\varphi. \qquad \text{(III.144)}$$

Then (III.142) can be written in the form

$$\frac{dW}{dz} = \frac{2q^2}{\pi v^2} \int\limits_{\substack{\varepsilon\beta^2 > 1 \\ \omega > 0}} \omega\, d\omega \frac{\varepsilon\beta^2 - 1}{\varepsilon - 1} \times \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\varphi\, d\varphi \frac{(\varepsilon+1)(\varepsilon\beta^2-1)\sin^2\varphi + \varepsilon(1-\beta^2)}{(\varepsilon+1)(\varepsilon\beta^2-1)\cos^2\varphi - \varepsilon^2\beta^2} \times e^{-2d\frac{\omega}{v}\sqrt{(\varepsilon\beta^2-1)\sin^2\varphi + 1 - \beta^2}}. \qquad \text{(III.145)}$$

Equation (III.145) gives the Cerenkov radiation energy of a particle moving in vacuum above a dielectric. The integrand is proportional to the intensity of the radiation at frequency ω at an angle φ in the xy plane. The region of integration is given by the inequalities.

We now consider in greater detail the features of the radiation in this, the simplest case. As in a continuous medium the wave vectors of the radiated waves form a conical surface. The axis of this surface is parallel to the charge velocity. The angle between the axis and the generatrices of the cone is given by the familiar relation $\cos\vartheta = 1/\sqrt{\epsilon}\,\beta$. However, radiation is generated in the second medium only, i.e., the Cerenkov wave-vector surface is actually only half of a conical surface. Moreover, the radiation intensity is not uniform over different generatrices of this cone.

We write the radiation intensity at a frequency ω and azimuthal angle φ, where $-\pi/2 < \varphi < \pi/2$ (we recall that φ is the angle formed by the normal to the boundary and the projection of the wave vector on the xy plane perpendicular to the charge velocity):

$$I(\omega, \varphi) = -\frac{2q^2}{\pi v^2} \omega \frac{\varepsilon\beta^2 - 1}{\varepsilon - 1} \frac{(\varepsilon+1)(\varepsilon\beta^2-1)\sin^2\varphi + \varepsilon(1-\beta^2)}{(\varepsilon+1)(\varepsilon\beta^2-1)\cos^2\varphi - \varepsilon^2\beta^2} \times \cos^2\varphi \cdot e^{-2d\frac{\omega}{v}\sqrt{(\varepsilon\beta^2-1)\sin^2\varphi + 1 - \beta^2}} \qquad \text{(III.146)}$$

and compare this expression with the radiation intensity in an infinite medium, where the radiation intensity is uniform over azimuthal angles:

$$I_0(\omega, \varphi) = \frac{q^2}{2\pi v^2} \omega \frac{\varepsilon\beta^2 - 1}{\varepsilon}. \qquad \text{(III.147)}$$

We assume that the velocity of the charge is high so that $\beta = 1$. Then, the radiation intensity $I(\omega, \varphi = 0)$ vanishes at $\varphi = 0$. (More precisely, this quantity is proportional to $1 - \beta^2$, i.e., inversely proportional to the square of the particle energy.) Thus, as the particle velocity increases the radiation intensity in the plane perpendicular to the boundary approaches zero. When $\varphi = \pm\pi/2$ the radiation intensity is identically zero for any charge velocity. The radiation reaches a maximum value between $\varphi = 0$ and $\varphi = \pm\pi/2$. This maximum can be extremely high as compared with the radiation intensity in a continuous medium $I_0(\omega, \varphi)$. For purposes of illustration we consider $I(\omega, \varphi)$ when $\varphi = 45°$ (it is again assumed that $\beta = 1$):

$$I\left(\omega, \varphi = \frac{\pi}{4}\right) = \frac{2q^2}{\pi v^2} \omega \frac{\varepsilon^2 - 1}{\varepsilon^2 + 1} e^{-\sqrt{2}\, d\frac{\omega}{c}\sqrt{\varepsilon - 1}}. \qquad \text{(III.148)}$$

If d, the distance of the charge from the boundary, is small, so that the exponential factor can be set equal to unity, the quantity $I(\omega, \varphi = \pi/4)$ may be appreciably greater than $I_0(\omega, \varphi)$. When $\epsilon = 1.5$ the ratio I/I_0 is almost 4 when $\varphi = 45°$. According to (III.146) the function $I(\omega, \varphi)$ is independent of φ apart from a constant; this function is shown in Figs. 9a to 9d. (It is assumed that the charge moves toward the observer and radiates into the lower space, which is filled with dielectric.)

To obtain the total radiation intensity at frequency ω we integrate $I(\omega, \varphi)$ over all angles at which radiation is excited $(-\pi/2 < \varphi < \pi/2)$. The integration is difficult because of the presence of the exponential factor. If we consider only those wavelengths for which the exponential factor can be set equal to unity $(d\frac{\omega}{c}\sqrt{\epsilon - 1} \ll 1)$ the integration is carried out easily and yields

$$I(\omega) = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} I(\omega, \varphi)\, d\varphi = \frac{2q^2}{v^2} \omega \cdot \frac{\varepsilon\beta^2 - 1}{\varepsilon - 1} \left[\frac{1}{2} - \frac{1}{(\varepsilon+1)(\varepsilon\beta^2-1)}\left(\frac{1}{\sqrt{1+\varepsilon-\varepsilon\beta^2}} - 1\right)\right]. \qquad \text{(III.149)}$$

This quantity is to be compared with the radiation intensity at frequency ω in a continuous medium

$$I_0(\omega) = \frac{q^2}{v^2} \omega \frac{\varepsilon\beta^2 - 1}{\varepsilon}. \qquad \text{(III.150)}$$

When $\epsilon\beta^2 = 1$ the ratio I/I_0 is unity at the radiation threshold:

$$\frac{I}{I_0} = 1 \qquad (\varepsilon\beta^2 = 1). \qquad \text{(III.151)}$$

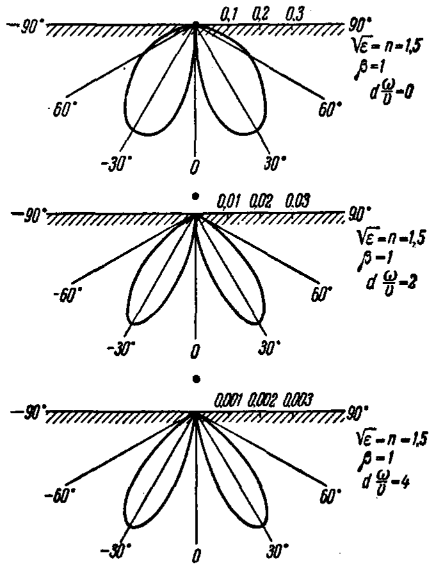

FIG. 9a

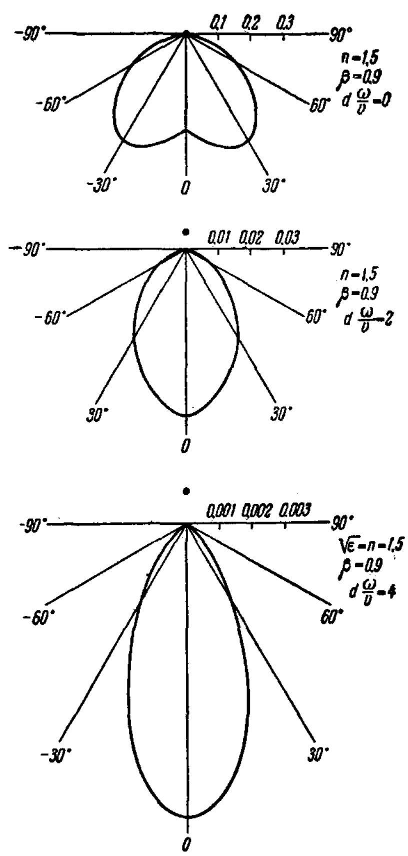

FIG. 9b

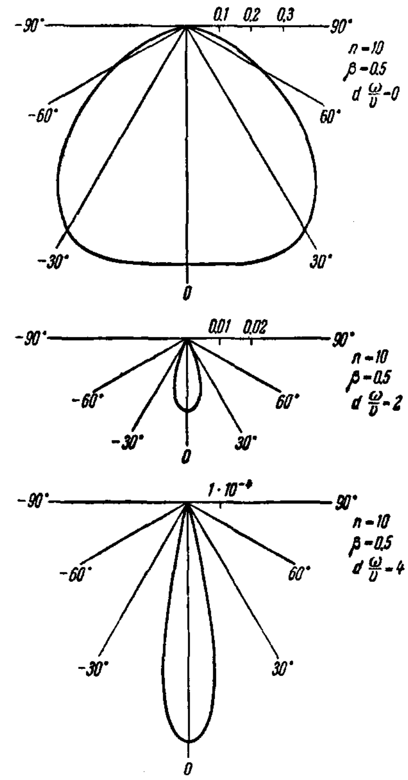

FIG. 9c

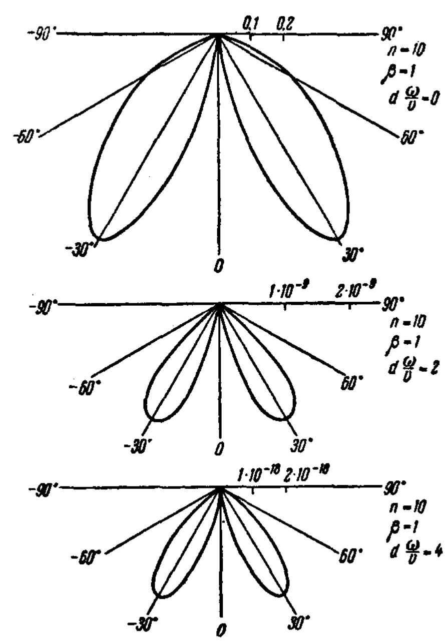

FIG. 9d

Thus, near the radiation threshold (for sufficiently small distances between the charge and the boundary) the radiation intensity is the same as in an infinite medium. Although $I = I_0$ at velocities close to the threshold velocity it should be noted that both of these quantities are proportional to $\epsilon\beta^2 - 1$ and vanish at the radiation threshold.

As the particle velocity increases still further the quantity $I(\omega)$ becomes smaller than $I_0(\omega)$; when $\beta = 1$, the ratio I/I_0 is

$$\frac{I}{I_0} = \frac{\varepsilon}{\varepsilon + 1} \qquad (\beta = 1). \qquad \text{(III.152)}$$

Although this quantity is smaller than unity it is evident that the radiation intensity in the presence of a boundary is comparable with that in an infinite medium.

In the present analysis we have assumed that the particle moves close to the boundary, that is,

$$d\,\frac{\omega}{c}\sqrt{\varepsilon-1}\ll 1.$$

If this condition is not satisfied the Cerenkov radiation spectrum of a particle moving in vacuum and radiating into a medium is cut off because of the factor

$$e^{-2d\frac{\omega}{v}\sqrt{(\varepsilon\beta^2-1)\sin^2\varphi+1-\beta^2}} \tag{III.152'}$$

in the integrand in (III.145). The radiation spectrum at a given azimuth is proportional to ω as long as the exponential factor (III.152) can be neglected and then cuts off sharply at this point. The frequency at which the radiation spectrum at angle φ starts to cut off is determined by the order of magnitude of the quantity

$$\omega=\frac{v}{2d\sqrt{(\varepsilon\beta^2-1)\sin^2\varphi+1-\beta^2}}\,.$$

In many respects this behavior is reminiscent of radiation in an empty channel, being a typical feature of boundary-value problems of this kind.

We recall that in the case of radiation in an empty channel the limiting frequency is determined by the analogous relation

$$\omega=\frac{v}{a\sqrt{\varepsilon\beta^2-1}}\,,$$

where a is the channel radius.

The total radiation energy loss can be estimated if it is assumed that the medium is dispersionless (ϵ independent of ω). Carrying out the integration over frequency and then angle in (III.145) we obtain the total energy loss per unit path[307] $\mu_2 = 1$

$$\frac{dW}{dz}=-\frac{q^2}{2\beta d^2}\frac{1}{\varepsilon-1}\left[\frac{\varepsilon}{\sqrt{1+\varepsilon-\varepsilon\beta^2}}-\sqrt{(\varepsilon-1)(1-\beta^2)}-\beta\right]. \tag{III.153}$$

It is interesting that (III.153) is simplified considerably in the ultrarelativistic case, ($\beta = 1$) and assumes the form

$$\frac{dW}{dz}=-\frac{q^2}{2d^2}\,. \tag{III.154}$$

Strictly speaking, this relation does not apply if there is dispersion. However (III.154) still applies with high accuracy if the exponent (III.152′) becomes large far from an absorption pole (i.e., if the exponent increases because of the factor ω rather than the factor $\sqrt{(\epsilon\beta^2-1)\sin^2\varphi+1-\beta^2}$); the radiation of a charge moving in a vacuum along an interface with a medium does not depend on the behavior of ϵ if the charge velocity is close to the velocity of light. It is evident that (III.154) becomes more accurate with increasing distance d between the particle trajectory and the interface.

We have already indicated that the waves radiated by the particle into the second medium are elliptically polarized. The polarization vector for waves with $k_y > 0$ rotates from the x axis to the y axis and in the opposite direction for waves with $k_y < 0$.

A charged particle moving near the surface of a dielectric experiences not only a retarding force, caused by the radiation reaction, but also a transverse force, caused by the reaction of the dielectric. This transverse force is given by the expression

$$F_x=-\frac{q}{c}\left\{\frac{\partial A_{1x}}{\partial t}-[\mathbf{v},\,\mathrm{rot}\,\mathbf{A}_1]_x\right\}\Big|_{x=y=0,\;z=vt},$$

where $\mathbf{A}_1$ is obtained from (III.131) and (III.134). Calculations carried out by A. I. Morozov, who neglected dispersion, indicate that the force F_x points toward the interface for any charge velocity and is of the same order of magnitude as the retarding force $F_x = dW/dz$.

c) We now briefly consider several other possibilities. The Cerenkov condition can be satisfied in both media, that is, $\epsilon_1\mu_1\beta^2 > 1$ and $\epsilon_2\mu_2\beta^2 > 1$. In this case radiation is excited on both sides of the boundary. The energy loss of the charge due to Cerenkov radiation is given by (III.140), where the integration extends over the range for which the inequalities $\epsilon_1\mu_1\beta^2 > 1$ and $\epsilon_2\mu_2\beta^2 > 1$ are satisfied. At certain values of k_y radiation can be produced in one medium while the field is damped exponentially in the other. To be specific, let us assume that $\epsilon_1\mu_1 < \epsilon_2\mu_2$, i.e., that the second medium is more dense optically than the first. Then, if k_y lies in the interval

$$\frac{\omega^2}{v^2}(\varepsilon_1\mu_1\beta^2-1)<k_y^2<\frac{\omega^2}{v^2}(\varepsilon_2\mu_2\beta^2-1),$$

the field in the first medium is damped $g_1^2 < 0$ but there is a radiation field in the second medium $g_2^2 > 0$. If $k_y^2 < \frac{\omega^2}{v^2}(\epsilon_1\mu_1\beta^2-1)$ the fields in both media are radiation fields and if $k_y^2 > \frac{\omega^2}{v^2}(\epsilon_2\mu_2\beta^2-1)$ the fields in both media are damped exponentially. We define the azimuthal angle φ by the relation

$$k_y=\frac{\omega}{v}\sqrt{\varepsilon_2\mu_2\beta^2-1}\sin\varphi.$$

Then, if

$$\sin^2\varphi<\frac{\varepsilon_1\mu_1\beta^2-1}{\varepsilon_2\mu_2\beta^2-1}\,,$$

there are radiation fields in both media. In this case the field is linearly polarized in the second medium. However, if

$$\sin^2\varphi>\frac{\varepsilon_1\mu_1\beta^2-1}{\varepsilon_2\mu_2\beta^2-1}\,,$$

the field in the second medium is a radiation field and the field in the first medium is damped. In this case the field in the second medium is elliptically polarized. The radiation at frequency ω is distributed over a cone in the second medium and the intensity is different on different generatrices.

d) The Cerenkov condition is satisfied in the first medium but not in the second ($\epsilon_1\mu_1\beta^2 > 1$, $\epsilon_2\mu_2\beta^2 < 1$,

$g_1^2 > 0$, $g_2^2 < 0$). In this case the Cerenkov radiation excited in the first medium experiences total internal reflection at the boundary and does not enter the second medium. The radiation is a superposition of linearly and elliptically polarized waves.

e) To conclude, we consider the polarization loss of a charge moving parallel to the interface between two dielectrics. Polarization radiation is produced at frequencies ω_s such that the dielectric constant ϵ_1 vanishes. The polarization energy loss is

$$\frac{dW}{dz}_{\text{pol}} = -\frac{q^2}{2v^2}\int dk_y \sum_s \left|\frac{\omega_s}{\varepsilon'(\omega_s)}\right| \frac{1-e^{-2d\sqrt{\frac{\omega_s^2}{v^2}+k_y^2}}}{\sqrt{\frac{\omega_s^2}{v^2}+k_y^2}}, \quad \text{(III.155)}$$

where $\epsilon'(\omega_s) = \left.\frac{d\epsilon}{d\omega}\right|_{\omega=\omega_s}$ and the summattion is taken over all roots ω_s. Because the polarization waves are damped in all directions from the particle trajectory, as d increases this formula becomes an expression for the polarization loss in an infinite medium. When $d\frac{\omega_s}{v} \gg 1$ the exponential factor in the integrand can be neglected and we obtain the usual formula for polarization energy loss in an infinite medium. In this case the integration must be cut off at some maximum value k_{max}. We may note that the integrand in the loss expression (III.140) can have poles where

$$\varepsilon_1 g_2 + \varepsilon_2 g_1 = 0 \text{ and } \mu_1 g_2 + \mu_2 g_1 = 0.$$

We neglect the loss corresponding to this case.

In various situations it is frequently necessary to consider the radiation due to an extended charge rather than a point charge, for example the radiation of a plane modulated electron beam moving in vacuum, parallel to a dielectric boundary. In this case the charge density is written in the form

$$\varrho = \varrho_0\,\delta(x)\,[1+\alpha\cos(\varkappa z-\omega t)], \quad \text{(III.156)}$$

where $2\pi/\kappa$ is the modulation wavelength, ω is the modulation frequency and α is the depth of modulation. The field produced by extended sources of this kind can be obtained by multiplying the Fourier components of the solution (III.129–135) for a point charge by the Fourier component of the density of the extended source. In the example being considered this factor is

$$\varrho_{k_x,\,k_y,\,k_z} = \frac{\varrho_0}{2\pi}\,[\delta(\omega)+\alpha\delta(k_z^2-\frac{\omega^2}{v^2})]\,\delta(k_y).$$

Because of this factor, only the waves for which $k_y = 0$ and $k_z = \pm\omega/v$ are radiated. The radiation propagates in the xz plane. The electric vector of the radiated wave also lies in this plane (because the component a_{2y} is proportional to k_y, we have $E_{2y} = (i\omega/c)a_{2y} = 0$.

We have assumed above that the media on both sides of the boundary are absolutely transparent, that is, we have assumed ϵ_1, μ_1, ϵ_2 and μ_2 to be real. The expressions for the field of the charge (III.129–135) still apply if one or both media exhibit absorption. Equation (III.140) for the particle energy loss will still be valid. In this case all calculations must be carried out with complex ϵ and μ. The appropriate calculations for a plane modulated electron beam have been carried out by Lashinsky.[260]

The literature contains several papers in which the radiation of charges and currents moving in plane slits in a dielectric have been considered.[304,305] We shall not discuss these papers here, but note that as the width of the slit approaches zero the radiation of a charge moving in the slit goes over to the radiation of a charge in a continuous medium. However, the radiation of a dipole in a narrow slit is the same as in a continuous medium only if the dipole is parallel to the plane of the slit. As the width of the slit approaches zero the radiation of an electric dipole perpendicular to the plane of the slit differs from the radiation of a dipole in a continuous medium by a factor $(\epsilon_2/\epsilon_1)^2$ where ϵ_1 and ϵ_2 are the dielectric constants of the media inside and outside the slit.[191]

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[1] M. Abele, Nuovo cimento, Suppl. **9**, p. 207 (1952).

[2] Agranovich, Pafomov, and Rukhadze, JETP **36**, 238 (1959), Soviet Phys. JETP **9**, 160 (1959).

[3] V. M. Agranovich and A. A. Rukhadze, JETP **35**, 1171 (1958), Soviet Phys. JETP **8**, 819 (1959).

[4] A. Ts. Amatuni, Izv. AN Arm.S.S.R., ser. fiz.-mat. nauk (Physico-Mathematical Science Series) **12**, 111 (1960).

[5] A. Ts. Amatuni and N. A. Korkhmazyan, ibid. **13**, 55 (1960).

[6] A. Ts. Amatuni and N. A. Korkhmazyan, JETP **39**, 1011 (1960), Soviet Phys. JETP **12**, 703 (1961).

[7] A. I. Akhiezer, Nuovo cimento Suppl. **3** Ser. X, No. 4, 591 (1956).

[8] A. I. Akhiezer, Doklady AN SSSR **73**, 55 (1950).

[9] Akhiezer, Lyubarskii, and Faĭnberg, J. Tech. Phys. (U.S.S.R.) **25**, 2526 (1955).

[10] Akhiezer, Lyubarskii, and Faĭnberg, CERN Symposium Vol. 1, p. 220 (1956).

[11] A. I. Akhiezer and A. G. Sitenko, JETP **23**, 161 (1952).

[12] A. I. Akhiezer and Ya. B. Faĭnberg, Usp. Fiz. Nauk **44**, 321 (1951).

[13] Akhiezer, Faĭnberg, and Lyubarskii, Тезисы докладов на Всесоюзной конференции по физике частиц высоких энергий 14–22 мая 1956 г. (Abstracts of Reports to All-Union Conference on Physics of High Energy Particles 14–22 May, 1956), Moscow, AN SSSR (1956).

[14] A. I. Akhiezer and Ya. B. Faĭnberg, J. Tech. Phys. (U.S.S.R.) **25**, 2516 (1955).

[15] W. Anderson and E. H. Belcher, Brit. J. Appl. Phys. **5**, 53 (1954).

[16] Anisimova, Bernashevskii, Vystavkin, and Lomize,

Радиотехника и электроника (Radio Engineering and Electronics) **5**, 969 (1960).
[17] G. A. Askar'yan, JETP **29**, 388 (1955), Soviet Phys. JETP **2**, 334 (1956).
[18] G. A. Askar'yan, JETP **30**, 584 (1956), Soviet Phys. JETP **3**, 613 (1956).
[19] A. Ascoli Balzanelli and R. Asoli, Nuovo cimento **10**, 1345 (1953).
[20] A. Ascoli Balzanelli and R. Asoli, Nuovo cimento **11**, 562 (1954).
[21] A. Ascoli Balzanelli and R. Asoli, Nuovo cimento **6**, 1392 (1957).
[22] Auerbach, Bernardini, Filosofo, Hanson, Odian, and Yamagata, CERN Symposium, Vol. 2, p. 291 (1956).
[23] Baccalin, Bassi, and Manduchi, Nuovo cimento **1**, 657 (1955).
[24] N. L. Balazs, Am. J. Phys. **24**, 185 (1956).
[25] N. L. Balazs, Phys. Rev. **104**, 1220 (1956).
[26] F. Barclay and J. V. Jelley, Nuovo cimento **1**, 657 (1955).
[27] F. Barclay and J. V. Jelley, Oxford Conference on Extensive Air Showers, A.E.R.E., Harwell, p. 74 (1956).
[28] Barker, Binnie, Hyams, Ront, and Shepherd, Phil. Mag. **46**, 307 (1955).
[29] Barker, Binnie, Hyams, Ront, and Shepherd, Phil. Mag. **46**, 307 (1955).
[30] V. G. Bar'yakhtar and M. I. Kaganov, JETP **35**, 766 (1958), Soviet Phys. JETP **8**, 532 (1959).
[31] K. A. Barsukov, JETP **37**, 1106 (1958), Soviet Phys. JETP **10**, 787 (1960).
[32] K. A. Barsukov, JETP **36**, 1485 (1959), Soviet Phys. JETP **9**, 1052 (1959).
[33] K. A. Barsukov and A. A. Kolomenskii, ZhTF **29**, 954 (1959), Soviet Phys.-Tech. Phys. **4**, 868 (1960).
[34] K. A. Barsukov and B. M. Bolotovskii, op. cit. ref. 37, **3**, 336 (1960).
[35] K. A. Barsukov, ZhTF **30**, 1337 (1960), Soviet Phys.-Tech. Phys. **5**, 1268 (1961).
[36] K. A. Barsukov, Dissertation, V. I. Lenin Moscow State Polytech. Inst., 1960.
[37] K. A. Barsukov and L. G. Baryshkina, Изв. вузов (Радиофизика) (News of the Universities—Radiophysics) **4**, No. 3 (1961).
[38] P. Bassi, Nuovo cimento **8**, 807 (1951).
[39] Bassi, Bianchi, and Manduchi, Nuovo cimento **9**, 861 (1952).
[40] P. Bassi and A. M. Bianchi, Nuovo cimento **9**, 1087 (1952).
[41] A. Battig, Dissertation, Instituto de Fisika Universidad Nacional del Tucuman, Argentina, Publication No. 591, 30 (1951).
[42] Bay, Cleland, and McLernon, Phys. Rev. **87**, 901 (1952).
[43] G. Beck, Phys. Rev. **74**, 795 (1948).
[44] G. A. Begiashvili and É. V. Gedalin, JETP **35**, 1513 (1958), Soviet Physics JETP **8**, 1059 (1959).
[45] G. A. Begiashvili and É. V. Gedalin, JETP **36**, 1939 (1959), Soviet Phys. JETP **9**, 1379 (1959).
[46] G. A. Begiashvili and É. V. Gedalin, JETP **38**, 1738 (1960), Soviet Phys. JETP **11**, 1254 (1960).
[47] A. I. Bedritskii, JETP **36**, 339 (1959), Soviet Phys. JETP **9**, 234 (1959).
[48] E. H. Belcher, Proc. Roy. Soc. (London) **A216**, 90 (1953).
[49] E. H. Belcher, Nature **170**, 571 (1952).
[50] Belyakov, Vovenko, Kirillov, Kulakov, Lyubimov, Matulenko, and Savin, Preprint No. R-476, High Energy Lab., Joint Institute for Nuclear Research, Dubna.
[51] E. A. Bernashevskii and V. Ya. Éĭdman, op. cit. ref. 37, **4**, 253 (1961).
[52] Bernashevskii, Vystavkin, and Lomize, Proceedings of the Symposium on Millimeter Waves, Polytechnic Institute of Brooklyn, March 31—April 1, 2 (1959), p. 169.
[53] R. W. Birge, Phys. Rev. **85**, 766 (1952).
[54] P. M. S. Blacket, Rep. Conf. Gassiot. Comm. Phys. Soc., 1948, p. 34; also Phys. Abstr. **52**, No. 4347 (1949).
[55] P. V. Bliokh, op. cit. ref. 37, **2**, 63 (1959).
[56] D. I. Blokhintsev and V. L. Indenbom, JETP **20**, 1123 (1950).
[57] V. I. Bobrinev and V. G. Braginskii, DAN SSSR **123**, 634 (1958).
[58] L. S. Bogdankevich, JETP **36**, 835 (1959), Soviet Phys. JETP **9**, 589 (1959).
[59] L. S. Bogdankevich, ZhTF **28**, 1505 (1958), Soviet Phys.-Tech. Phys. **3**, 1392 (1959).
[60] L. S. Bogdankevich, ZhTF **29**, 1086 (1959), Soviet Phys.-Tech. Phys. **4**, 992 (1960).
[61] L. S. Bogdankevich, Dissertation, V. I. Lenin Moscow State Polytech. Inst., 1960.
[62] L. S. Bogdankevich, ZhTF **31**, 311 (1961) [sic!].
[63] L. S. Bogdankevich and B. M. Bolotovskii, JETP **32**, 1421 (1957), Soviet Phys. JETP **5**, 1157 (1957).
[64] B. M. Bolotovskii, Dissertation, P. N. Lebedev Physics Institute, Acad. Sci. U.S.S.R., 1954.
[65] B. M. Bolotovskii, UFN **62**, 201 (1957).
[66] B. M. Bolotovskii, UFN **67**, 163 (1959), Soviet Phys.-Uspekhi **2**, 165 (1959).
[67] B. M. Bolotovskii and A. A. Kolomenskii, DAN SSSR **85**, 59 (1952).
[68] B. M. Bolotovskii and A. A. Rukhadze, JETP **37**, 1346 (1959), Soviet Phys. JETP **10**, 958 (1960).
[69] Booth, Hutchinson, and Leadley, Proc. Phys. Soc. (London) **71**, 293 (1958).
[70] N. E. Booth and B. Leadley, J. Nuclear Inst. **1**, 345 (1957).
[71] N. Bohr, Penetration of Atomic Particles Through Matter, Danske Vidensk. Selsk. Mat.-fys. Medd. **18**, No. 8 (1948).
[72] O. Bohr, Effect of Atomic Interactions on Penetration of Atomic Particles Through Matter, [Supplement to reference 71].

[73] Brabant, Moyer, and Wallace, Rev. Sci. Instr. **28**, 421 (1957).

[74] Brabant, Cork, Horwitz, Moyer, Murray, Wallace, and Wentzel, Phys. Rev. **101**, 498 (1956).

[75] Brabant, Cork, Horwitz, Moyer, Murray, Wallace, and Wentzel, Phys. Rev. **102**, 1622 (1956).

[76] Brabant, Kenney, and Wallace, Bull. Am. Phys. Soc. **1**, 251 (1956).

[77] V. B. Braginskii, op. cit. ref. 16, **1**, 225 (1956).

[78] V. B. Braginskii and E. P. Mustel', op. cit. ref. 37, No. 3, 124 (1958).

[79] P. Budini, Nuovo cimento **10**, 236 (1953); Phys. Rev. **89**, 1147 (1953).

[80] P. Budini and L. Taffara, Nuovo cimento **10**, 1489 (1953).

[81] Budini, Taffara, and Viola, Nuovo cimento **18**, 864 (1960).

[82] Budini, Taffara, and Viola, Nuovo cimento **20**, 265 (1961).

[83] M. Cassels, CERN Symposium, Vol. 1, p. 74 (1956).

[84] Cassels, Fidecaro, Wetherell, and Wormald, Proc. Phys. Soc. (London) **A70**, 505 (1957).

[85] Chamberlain, Wiegand, Segre, and Ypsilantis, Phys. Rev. **100**, 947 (1955).

[86] O. Chamberlain and C. Wiegand, CERN Symposium, Vol. 2, p. 82 (1956).

[87] N. L. Tsintsadze and A. D. Pataraya, ZhTF **30**, 1178 (1960), Soviet Phys.-Tech. Phys. **5**, 1113 (1961).

[88] V. N. Tsytovich, Vestnik, Moscow State University **2**, 27 (1951).

[89] V. N. Tsytovich, JETP **34**, 1646 (1958), Soviet Phys. JETP **7**, 1131 (1958).

[90] V. N. Tsytovich, JETP **35**, 1407 (1958), Soviet Phys. JETP **8**, 983 (1959).

[91] P. A. Cerenkov, DAN SSSR **2**, 451 (1934).

[92] P. A. Cerenkov, DAN SSSR **3**, 12, 413 (1936).

[93] P. A. Cerenkov, Phys. Rev. **52**, 378 (1937).

[94] P. A. Cerenkov, Izv. AN SSSR ser. fiz. No. 4—5, 455 (1937).

[95] P. A. Cerenkov, DAN SSSR **14**, 99 (1937).

[96] P. A. Cerenkov, DAN SSSR **14**, 103 (1937).

[97] P. A. Cerenkov, Izv. AN SSSR, OMEN (Div. Math. Nat. Sci.) 760 (1938).

[98] P. A. Cerenkov, DAN SSSR **20**, 653 (1938).

[99] P. A. Cerenkov, DAN SSSR **21**, 117 (1938).

[100] P. A. Cerenkov, DAN SSSR **21**, 323 (1938).

[101] P. A. Cerenkov, Trudy, Physics Institute, Acad. Sci. U.S.S.R. **2**, No. 4 (1944).

[102] Cerenkov, Nobel Lecture, UFN **68**, 377 (1959).

[103] Cerenkov, Tamm, and Frank, Нобелевские лекции (Nobel Lectures), Fizmatgiz, Moscow, 1960.

[104] Cziz, Ericson, and Glashow, Nuclear Phys. **13**, 516 (1959).

[105] W. Cziz and S. L. Glashow, Nuclear Phys. **20**, 309 (1960).

[106] Z. A. Chizhikova, Оптика и спектроскопия (Optics and Spectroscopy) **7**, 223 (1960).

[107] P. D. Coleman and C. Enderby, J. Appl. Phys. **31**, 1695 (1960).

[108] G. Collins and V. Reiling, Phys. Rev. **54**, 499 (1938).

[109] R. T. Cox, Phys. Rev. **66**, 196 (1944).

[110] A. E. Chudakov and N. M. Nesterova, Nuovo cimento Suppl. 8, Ser. X, 606 (1958).

[111] Chudakov, Nesterova, Zatsepin, and Tukish, Proc. Moscow Conf. on Cosmic Rays, AN SSSR 1960, Vol. II, p. 47.

[112] V. L. Dadykin, Приборы и техника эксперимента (Instruments and Experimental Techniques) (1961).

[113] F. S. Dainton, Ann. Rep. Chem. Soc. **45**, 5 (1949).

[114] Danos, Geschwind, Lashinsky, and Van Trier, Phys. Rev. **92**, 828 (1953).

[115] M. Danos and H. Lashinsky, IRE Trans. P.G.M.M.T. **2**, 21 (1952).

[116] M. Danos, J. Appl. Phys. **26**, 2 (1955).

[117] P. Debye, Physik. Bl. No. 11, 568 (1960).

[118] K. G. Dedrick, Phys. Rev. **87**, 891 (1952).

[119] P. I. Dee and W. T. Richards, Nature **168**, 736 (1951).

[120] R. H. Dicke, Phys. Rev. **71**, 737 (1947).

[121] A. N. Didenko, JETP **35**, 655 (1958), Soviet Phys. JETP **8**, 455 (1959).

[122] Yu. N. Dnestrovskii and D. P. Kostomarov, DAN SSSR **116**, 377 (1957), Soviet Phys.-Doklady **2**, 442 (1958).

[123] Yu. N. Dnestrovskii and D. P. Kostomarov, ibid. **124**, 792 (1959), transl. **4**, 132 (1959).

[124] Yu. N. Dnestrovskii and D. P. Kostomarov, ibid. **124**, 1026 (1959), transl. **4**, 158 (1959).

[125] Yu. N. Dnestrovskii and D. P. Kostomarov, op. cit. ref. 16, **4**, 303 (1959).

[126] T. Duerden and B. D. Hyams, Phil. Mag. **43**, 717 (1952).

[127] F. X. Eder, Funk und Ton. No. 3, 67 (1949).

[128] V. Ya. Éĭdman, Trudy, Gorkii Physics-Engineering Research Institute and Gorkii State University, Radiophysics Faculty (Science Notes XXX), Phys. Ser., 229 (1959).

[129] V. Ya. Éĭdman, JETP **34**, 131 (1958), Soviet Phys. JETP **7**, 91 (1958); Correction to this paper, JETP **36**, 1335 (1959), Soviet Phys. JETP **9**, 947 (1959).

[130] V. Ya. Éĭdman, op. cit. ref. 37, **3**, 192 (1960).

[131] Cerenkov Effect and Polarization Losses. (Coll. of Translations) Проблемы современной физики (Problems of Contemporary Physics) No. 7, (1953).

[132] G. R. Ellis, J. Atom. Terr. Phys. **10**, 302 (1957).

[133] Ya. B. Faĭnberg and N. A. Khizhnyak, JETP **32**, 883 (1957), Soviet Phys. JETP **5**, 720 (1957).

[134] U. Fano, Phys. Rev. **103**, 1202 (1956).

[135] L. B. Felsen and A. Hessel, Nuovo cimento **19**, 1065 (1961).

[136] E. Fermi, Phys. Rev. **56**, 1242 (1939).

[137] E. Fermi, Phys. Rev. **57**, 485 (1940).

[138] I. Filosofo, Nuovo cimento **6**, 701 (1957).

[139] I. Filosofo and T. Yamagata, CERN Symposium, Vol. 1, p. 85 (1956).
[140] Finocchiaro, Finzi, and Mezzetti, Nuovo cimento 15, 317 (1960).
[141] Fisher, March, and Marshall, Phys. Rev. 109, 533 (1958).
[142] V. Fitch and R. Motley, Phys. Rev. 101, 496 (1956).
[143] G. N. Fowler and G. M. D. Jones, Proc. Phys. Soc. (London) A66, 597 (1953).
[144] I. M. Frank, Izv. AN SSSR, ser. fiz. 6, 3 (1942).
[145] I. M. Frank, J. Phys. (U.S.S.R.) 7, 49 (1943).
[146] I. M. Frank, UFN 30, 149 (1946).
[147] I. M. Frank, DAN SSSR 42, 354 (1944).
[148] I. M. Frank, Сб. Памяти С. И. Вавилова (Memorial volume to S. I. Vavilov), AN SSSR, 1952, p. 172.
[149] I. M. Frank, UFN 58, 11 (1955).
[150] I. M. Frank, Nobel Lecture, UFN 68, 387 (1959).
[151] I. M. Frank, JETP 36, 823 (1959), Soviet Phys. JETP 9, 580 (1959); preprint Joint Inst. Nuc. Res. R-237, Dubna.
[152] I. M. Frank, Сборник Памяти акад. Г. С. Ландсберга (Collection In memory of Academician G. S. Landsberg), AN SSSR, Moscow 1959.
[153] I. M. Frank, Lecture at Brussels International Exposition, Aug. 13, 1958, Znanie Press, Moscow, 1959.
[154] I. M. Frank, JETP 38, 1751 (1960), Soviet Phys. JETP 11, 1263 (1960).
[155] I. M. Frank and V. L. Ginzburg, J. Phys. (U.S.S.R.) 9, 353 (1945).
[156] F. A. Fuertes, Instituto de Fisica, Universidad Nacional del Tucuman, Argentina. Publication No. 465.
[157] Gabriel, Garwin, and York, University of Chicago Report No. 325, Chicago, Illinois.
[158] W. Galbraith and J. V. Jelley, Nature 171, 349 (1953).
[159] W. Galbraith and J. V. Jelley, J. Atm. Terr. Phys. 6, 250 (1955).
[160] G. M. Garibyan, Izv. AN SSSR ser. fiz. 9, 46 (1946).
[161] G. M. Garibyan, Izv. AN Arm.S.S.R. 11, 7 (1958).
[162] G. M. Garibyan, JETP 35, 1435 (1958), Soviet Phys. JETP 8, 1003 (1959).
[163] G. M. Garibyan, JETP 37, 527 (1959), Soviet Phys. JETP 10, 372 (1960).
[164] G. M. Garibyan, JETP 38, 1814 (1960), Soviet Phys. JETP 11, 1306 (1960).
[165] G. M. Garibyan, JETP 39, 332 (1960), Soviet Phys. JETP 12, 237 (1961).
[166] G. M. Garibyan and I. I. Goldman, DAN Arm.S.S.R. 31, 219 (1960).
[167] G. M. Garibyan and G. A. Chalikyan, JETP 35, 1282 (1958), Soviet Phys. JETP 8, 894 (1959).
[168] G. M. Garibyan and G. A. Chalikyan, Izv. AN Arm. S.S.R. 11, 49 (1959).
[169] G. M. Garibyan and I. Ya. Pomaranchuk, JETP 37, 1828 (1959), Soviet Phys. JETP 10, 1290 (1959).
[170] G. M. Garibyan and O. S. Mergelyan, Izv. AN Arm.S.S.R. 12, 91 (1959).
[171] G. M. Garibyan and O. S. Mergelyan, ibid. 13, 123 (1960).
[172] I. A. Getting, Phys. Rev. 71, 123 (1947).
[173] V. L. Ginzburg, DAN SSSR 24, 131 (1939).
[174] V. L. Ginzburg, JETP 10, 608 (1939).
[175] V. L. Ginzburg, J. Phys. (U.S.S.R.) 3, 101 (1940).
[176] V. L. Ginzburg, JETP 10, 589 (1940).
[177] V. L. Ginzburg, J. Phys. (U.S.S.R.) 2, 441 (1940).
[178] V. L. Ginzburg, JETP 12, 425 (1942).
[179] V. L. Ginzburg, J. Phys. (U.S.S.R.) 7, 115 (1943).
[180] V. L. Ginzburg, J. Phys. (U.S.S.R.) 5, 47 (1941).
[181] V. L. Ginzburg, DAN SSSR 56, 145 (1947).
[182] V. L. Ginzburg, DAN SSSR 56, 253 (1947).
[183] V. L. Ginzburg, Izv. AN SSSR, ser. fiz. 11, 165 (1947).
[184] V. L. Ginzburg, op. cit. ref. 148, p. 193.
[185] V. L. Ginzburg, JETP 34, 1593 (1958), Soviet Phys. JETP 7, 1096 (1958).
[186] V. L. Ginzburg, UFN 69, 537 (1959), Soviet Phys.-Uspekhi 2, 874 (1960).
[187] V. L. Ginzburg, JETP 16, 15 (1946).
[188] V. L. Ginzburg and I. M. Frank, DAN SSSR 56, 583 (1947).
[189] V. L. Ginzburg and I. M. Frank, DAN SSSR 56, 699 (1947).
[190] V. L. Ginzburg and I. M. Frank, JETP 35, 817 (1958), Soviet Phys. JETP 8, 567 (1959).
[191] V. L. Ginzburg and V. Ya. Éĭdman, JETP 35, 1508 (1958), Soviet Phys. JETP 8, 1055 (1959).
[192] V. L. Ginzburg and V. Ya. Éĭdman, op. cit. ref. 37, 2, 331 (1959).
[193] V. L. Ginzburg and V. Ya. Éĭdman, JETP 36, 1823 (1959), Soviet Phys. JETP 9, 1300 (1959).
[194] V. I. Gol'danskii and G. B. Zhdanov, JETP 26, 405 (1954).
[195] I. I. Gol'dman, JETP 38, 1866 (1960), Soviet Phys. JETP 11, 1866 (1960).
[196] P. Goldsmith and J. V. Jelley, Phil. Mag. 4, 836 (1959).
[197] Grienfield, Norman, Dowdley, and Kratz, J. Opt. Soc. Am. 43, 42 (1953).
[198] Grushin, Zapevalov, and Leĭkin, op. cit. ref. 112, No. 2, 27 (1960).
[199] G. Gyorgyi, Magyar Fizikai Folyoirat 3, 369 (1955).
[200] O. Halpern and H. Hall, Phys. Rev. 73, 477 (1948).
[201] R. J. Hanson and D. C. Moore, Nuovo cimento 4, 1558 (1956).
[202] J. M. Harding and K. E. Henderson, Phys. Rev. 74, 1560 (1948).
[203] S. Hayakawa and K. Kitao, Progr. Theoret. Phys. 16, 131 (1956).
[204] E. Heiberg and J. Marshall, Rev. Sci. Instr. 27, 618 (1956).
[205] N. A. Khizhnyak, op. cit. ref. 16, 5, 654 (1960).
[206] N. A. Khizhnyak and V. P. Shestopalov, Trudy, Radiophysics Faculty, Khar'kov State University 3, 69 (1959).

[207] N. Hokkyo, Nucl. Fusion Res. (Tokyo) 6, No. 2, 69 (1961).
[208] R. Hofstadter, CERN Symposium, Vol. 1, p. 75 (1956).
[209] N. Horwitz, Phys. Rev. 98, 165 (1955).
[210] P. Hubert, Atomes (Paris) No. 112, 224 (1955).
[211] M. Huybrechts and C. Schonberg, Nuovo cimento 9, 764 (1952).
[212] D. D. Ivanenko and V. S. Gurgenidze, DAN SSSR 67, 997 (1949).
[213] D. D. Ivanenko and V. S. Gurgenidze, Vestnik, Moscow State University, No. 2, 269 (1950).
[214] D. D. Ivanenko and A. A. Sokolov, Классическая теория поля (Classical Theory of Fields), Gostekhizdat 1951, Sec. 27.
[215] D. D. Ivanenko and V. N. Tsytovich, JETP 28, 291 (1955), Soviet Phys. JETP 1, 135 (1955).
[216] V. L. Indenbom, JETP 21, 737 (1951).
[217] P. Janssens and M. Huybrechts, Centre de physique nucleaire, Universite libre de Bruxelles, No. 29, April 1951.
[218] J. M. Jauch and K. M. Watson, Phys. Rev. 74, 1485 (1948).
[219] J. M. Jauch and K. M. Watson, Phys. Rev. 75, 1249 (1949).
[220] V. V. Zheleznyakov, op. cit. ref. 37, 2, 14 (1959).
[221] B. L. Zhelnov, JETP 40, 170 (1961), Soviet Phys. JETP 13, 117 (1961).
[222] J. V. Jelley, Proc. Phys. Soc. (London) A46, 82 (1951).
[223] J. V. Jelley, Atomics 4, 81 (1953).
[224] J. V. Jelley, Progr. in Nuclear Phys., Frisch, ed., 1953, Vol. 3, p. 84.
[225] J. V. Jelley, AERE Report NP/R1770.
[226] J. V. Jelley, CERN Symposium, Vol. 1, p. 76 (1956).
[227] J. V. Jelley, Cerenkov Radiation, Pergamon, 1958.
[228] J. V. Jelley and W. Galbraith, Phil. Mag. 44, 619 (1953).
[229] J. V. Jelley and W. Galbraith, J. Atm. Terr. Phys. 6, 304 (1955).
[230] M. H. L. Jester, University of California, Radiation Laboratory Report, URCL-2990.
[231] Jones, Kratz, and Rouvina, Rev. Sci. Instr. 28, 167 (1957).
[232] M. I. Kaganov, ZhTF 23, 507 (1953).
[233] M. I. Kaganov, Trudy, Physics Department of the Physics-Mathematics Faculty, Khar'kov State University 4, 67 (1953).
[234] M. I. Kaganov, JETP 23, 523 (1953).
[235] M. I. Kaganov, JETP 23, 514 (1953).
[236] M. I. Kaganov, Dissertation, Khar'kov State University, 1954.
[237] S. P. Kapitza, JETP 39, 1367 (1960), Soviet Phys. JETP 12, 954 (1961).
[238] A. Kantz and R. Hofstadter, Nuclear Phys. 12, 36 (1954).
[239] K. Kitao, Progr. Theoret. Phys. (Kyoto) 23, 759 (1960).
[240] B. B. Kinsey and W. Wentzel, CERN Symposium, Vol. 2, p. 68 (1956).
[241] N. Klepikov, Vestnik, Moscow State University, No. 8, 61 (1951).
[242] Yu. L. Klimontovich, JETP 35, 1276 (1958), Soviet Phys. JETP 8, 891 (1959).
[243] Yu. L. Klimontovich, and V. P. Silin, UFN 70, 247 (1960), Soviet Phys.-Uspekhi 3, 84 (1960).
[244] A. A. Kolomenskii, Report, Physics Institute Acad. Sci., 1951.
[245] A. A. Kolomenskii, Report, Physics Institute, Acad. Sci., 1952.
[246] A. A. Kolomenskii, DAN SSSR 86, 1097 (1952).
[247] A. A. Kolomenskii, JETP 24, 167 (1953).
[248] A. A. Kolomenskii, DAN SSSR 106, 982 (1956), Soviet Phys.-Doklady 1, 133 (1956).
[249] N. A. Korkhmazyan, Izv. AN Arm.S.S.R. 10, 29 (1957).
[250] N. A. Korkhmazyan, ibid. 11, 87 (1958).
[251] N. A. Korkhmazyan, ibid. 13, 139 (1960).
[252] V. A. Korshunov and G. V. Skrotskii, Изв. вузов (Физика) (News of the Universities, Physics) No. 4, 56 (1960).
[253] Kruse, Marshall, and Platt, Astrophys. J. 124, 601 (1956).
[254] A. B. Kukanov, op. cit. ref. 106, 10, 289 (1961).
[255] V. N. Kurdyumov, DAN SSSR 134, 563 (1960), Soviet Phys.-Doklady 5, 1023 (1961).
[256] M. A. Lampert, Phys. Rev. 102, 299 (1956).
[257] L. D. Landau, Foreword to translation of reference 71.
[258] L. D. Landau and E. M. Lifshitz, Электродинамика сплошных сред (Electrodynamics of Continuous Media), Gostekhizdat, 1957.
[259] A. I. Larkin, JETP 37, 264 (1959), Soviet Phys. JETP 10, 186 (1960).
[260] H. Lashinsky, J. Appl. Phys. 27, 631 (1956).
[261] H. Lashinsky, Proceedings of the Symposium on Millimeter Waves, Polytechnic Institute of Brooklyn, March 31—April 1, 1959, p. 181.
[262] J. D. Lawson, Phil. Mag. 45, 748 (1948).
[263] M. L. Levin, JETP 20, 381 (1950).
[264] V. M. Lenchenko, Dissertation, Central Asiatic State University, Tashkent 1955.
[265] Li Meng-Ha, JETP 38, 934 (1960), Soviet Phys. JETP 11, 673 (1960).
[266] Li Meng-Ha, Vestnik, Moscow State University, No. 6, 78 (1960).
[267] Yin-yuan Li, Phys. Rev. 80, 104 (1950).
[268] Yin-yuan Li, Phys. Rev. 82, 281 (1951).
[269] S. I. Lindenbaum and A. Pevsner, Rev. Sci. Instr. 25, 285 (1954).
[270] I. Lindhart, Dansk. Mat. Fys. Medd. 28, No. 8, (1954).
[271] I. Lindhart, in "Niels Bohr and the Development of Physics," ed. W. Pauli, (Russ. Transl.), IL, Moscow, 1958.
[272] J. G. Linhart, J. Appl. Phys. 26, 527 (1955).

[273] J. G. Linhart, L'onde electrique 36, 979 (1956).
[274] J. G. Linhart, Research 8(10), 402 (1955).
[275] I. Linsley, Phys. Rev. 93, 899 (1954).
[276] I. Linsley, Phys. Rev. 97, 1292 (1955).
[277] I. Linsley, Phys. Rev. 101, 826 (1956).
[278] I. Linsley and N. Horwitz, Rev. Sci. Instr. 26, 557 (1955).
[279] L. G. Lomize, op. cit. ref. 16, 5, 707 (1960).
[280] L. G. Lomize, op. cit. ref. 16, 5, 1546 (1960).
[281] L. G. Lomize, ZhTF 31, 301 (1961), Soviet Phys.-Tech. Phys. 6, 215 (1961).
[282] Yu. M. Loskutov, Научн. доклады высш. шк., сер. физ.-мат. (Science Reports of the Higher Schools, Mathematics-Physics Series) 4, 103 (1958).
[283] Yu. M. Loskutov and A. B. Kukanov, JETP 34, 477 (1958), Soviet Phys. JETP 7, 328 (1958).
[284] A. Lundby, CERN Symposium, Vol. 2, p. 68 (1956).
[285] L. Mallet, Compt. rend. 183, 274 (1926); 187, 222 (1929); 188, 445 (1929).
[286] M. Mando, Nuovo cimento 12, 5 (1954).
[287] J. Marshall, Phys. Rev. 81, 275 (1951).
[288] J. Marshall, Phys. Rev. 86, 685 (1952).
[289] J. Marshall, Ann. Rev. Nuclear Sci. 4, 141 (1954).
[290] L. Marshall, Astrophys. J. 124, 469 (1956) (See reference 253).
[291] J. Marshall, CERN Symposium, Vol. 1, p. 63 (1956).
[292] G. Marx and G. Gyorgyi, Ann. Phys. 16, 241 (1955) (See reference 199).
[293] R. L. Mather, Phys. Rev. 84, 181 (1951).
[294] J. W. Mather and E. A. Martinelli, Phys. Rev. 92, 780 (1953).
[295] J. W. Mather and E. A. Martinelli, University of California, Radiation Lab. Report, No. 1646.
[296] Matulenko, Savin, and Stavinskii, op. cit. ref. 112, No. 3, 44 (1956).
[297] F. B. McDonald, Phys. Rev. 104, 1723 (1956).
[298] E. Maurer and H. Kolz, Z. Angew. Physik 2, 223 (1950).
[299] O. S. Mergelyan, Izv. AN Arm.S.S.R. 13, 107 (1960).
[300] S. Mikhalyak, Dissertation, Moscow State University, 1961.
[301] C. H. Miller and E. P. Hincks, Can. J. Phys. 35, 363 (1957).
[302] Миллиметровые и субмиллиметровые радиоволны (Millimeter and Submillimeter Radio Waves). Collection of Translations, IL 1959.
[303] J. Moffat and M. W. Stringfellow, J. Sci. Instr. 35, 18 (1958).
[304] A. I. Morozov, Vestnik, Moscow State University, Mathematics Physics Series, No. 1, 72 (1957).
[305] A. I. Morozov, Dissertation, Moscow State University, 1957.
[306] A. I. Morozov, JETP 31, 1079 (1956), Soviet Phys. JETP 4, 920 (1957).
[307] A. I. Morozov, JETP 32, 1261 (1957), Soviet Phys. JETP 5, 1029 (1957).
[308] A. I. Morozov, JETP 33, 933 (1957), Soviet Phys. JETP 6, 717 (1958).
[309] A. I. Morozov, Физика плазмы и проблема управляемых термоядерных реакций (Plasma Physics and the Problem of a Controlled Thermonuclear Reaction) AN SSSR, 1958, p. 331.
[310] A. I. Morozov and L. S. Solov'ev, ibid., p. 391.
[311] W. E. Mott and R. B. Sutton, Handbuch d. Physik, 1958, Vol. 45, p. 164.
[312] H. Motz and L. I. Schiff, Am. J. Phys. 21, 258 (1953).
[313] H. Motz, J. Appl. Phys. 22, 527 (1951).
[314] H. Motz and M. Nakamura, Ann. Phys. 7, 1, 84 (1959).
[315] H. Motz, Trans. IRE AP-4, No. 3, 374 (1956).
[316] G. Mourier, L'onde electrique No. 371, 132 (1958).
[317] C. Muzicar, Czech. J. Phys. 5, 1, 9 (1955).
[318] C. Muzicar, JETP 39, 163 (1960), Soviet Phys. JETP 12, 117 (1961).
[319] C. Muzicar and V. E. Pafomov, Preprint R-604, Joint Inst. Nuc. Rsch., Dubna 1960; Czech. J. Phys. (in press).
[320] V. D. Nag and A. M. Sayied, Proc. Roy. Soc. (London) A235, 544 (1951).
[321] S. M. Neamtam, Phys. Rev. 92, 1362 (1953).
[322] I. M. Nesterova and A. E. Chulakov, JETP 28, 384 (1955), Soviet Phys. JETP 1, 389 (1955).
[323] K. W. Ogilvie, Can. J. Phys. 33, 9, 555 (1955).
[324] R. H. Pantell, Proc. of the Symposium on Millimeter Waves, Polytechnic Institute of Brooklyn, March 31—April 1, 1959, p. 301.
[325] Pantuev, Khachaturyan, and Chuvilo, op. cit. ref. 112, No. 1, 19 (1960).
[326] V. E. Pafomov, Diploma Thesis, Moscow State University, Physics Faculty, 1952.
[327] V. E. Pafomov, JETP 30, 761 (1956), Soviet Phys. JETP 3, 596 (1956).
[328] V. E. Pafomov, JETP 32, 366 (1957), Soviet Phys. JETP 5, 307 (1957).
[329] V. E. Pafomov, JETP 32, 610 (1957), Soviet Phys. JETP 5, 504 (1957).
[330] V. E. Pafomov, JETP 33, 1074 (1957), Soviet Phys. JETP 6, 829 (1958).
[331] V. E. Pafomov, Dissertation, Institute of Physics, Acad. Sci., 1957.
[332] V. E. Pafomov, JETP 36, 1853 (1959), Soviet Phys. JETP 9, 1321 (1959).
[333] V. E. Pafomov, JETP 39, 134 (1960), Soviet Phys. JETP 12, 97 (1961).
[334] V. E. Pafomov, DAN SSSR 133, 1316 (1960), Soviet Phys.-Doklady 5, 850 (1961).
[335] J. R. Pierce, J. Appl. Phys. 26, 627 (1955).
[336] I. V. Polubarinov, Diploma Thesis, Moscow State University, Physics Faculty, 1953.
[337] A. P. Polikarov, Trans. Bulgarian Academy of Sciences 7, 2, 29 (1954).
[338] D. Pines, Revs. Modern Phys. 28, 184 (1956).

[339] B. T. Price, Repts. Progr. Phys. 18, 52 (1955).
[340] N. A. Porter, Nuovo cimento 5, 526 (1957).
[341] N. Porter and A. C. Sherwood, The Oxford Conference on Extensive Air Showers, p. 71 (1956).
[342] Rich, Slovacek, and Studer, J. Opt. Soc. Am. 43, 750 (1953).
[343] A. Roberts, Preprint No. VO 9031, University of Rochester, Rochester, N. Y., 1960.
[344] S. Robin, J. phys. radium 11, January (1950).
[345] M. I. Ryazanov, JETP 32, 1244 (1957), Soviet Phys. JETP 5, 1013 (1957).
[346] M. I. Ryazanov, in Некоторые вопросы теоретической физики (Certain Problems in Theoretical Physics), Atomizdat, 1958, p. 75.
[347] G. S. Saakyan, DAN Arm.S.S.R. 28, 121 (1959).
[348] G. S. Saakyan, JETP 38, 843 (1960), Soviet Phys. JETP 11, 610 (1960).
[349] K. Saito and K. Suga, Nuovo cimento 11, 600 (1959).
[350] A. M. Sayied, Proc. Phys. Soc. (London) No. 459, 398 (1958).
[351] L. I. Schiff, Quantum Mechanics, McGraw Hill, N.Y. (1949).
[352] M. Schonberg, Bull. centre phys. nucl. Universite de Bruxelles 20, 1 (1950); Nuovo cimento 8, 3, 159 (1951); 9, 2, 210 (1952); 9, 4, 372 (1952).
[353] G. Schmidt, Magyar Fizikai Folyoirat 4, 453 (1956).
[354] V. A. Shakhbazyan, Izv. AN Arm.S.S.R. 9, 5, 91 (1956).
[355] V. D. Shafranov, JETP 34, 1475 (1958), Soviet Phys. JETP 7, 1019 (1958).
[356] M. Ya. Shirobokov, JETP 19, 481 (1949).
[357] V. P. Silin, JETP 37, 273 (1959), Soviet Phys. JETP 10, 192 (1959).
[358] A. G. Sitenko, Dissertation, Khar'kov State University, 1952.
[359] A. G. Sitenko, DAN SSSR 98, 377 (1954).
[360] A. G. Sitenko, ZhTF 23, 2200 (1953).
[361] A. G. Sitenko, Уч. зап. Харьк. ун-та 64, Тр. физ. отд. физ.-мат. фак-та (Sci. Notes Khar'kov University 64, Proc. Div. Phys. Math. Sci. Fac.) 6, 17 (1955).
[362] A. G. Sitenko and M. I. Kaganov, DAN SSSR 100, 681 (1955).
[363] A. G. Sitenko and A. A. Kolomenskii, JETP 30, 511 (1956), Soviet Phys. JETP 3, 410 (1956).
[364] A. G. Sitenko and V. S. Tkalich, ZhTF 29, 1074 (1959), Soviet Phys.-Tech. Phys. 4, 981 (1960).
[365] K. Skarvag and A. Lundby, JENER Report, No. 22.
[366] T. Smith and E. M. Purcell, Phys. Rev. 92, 1069 (1953).
[367] A. Sokolov, DAN SSSR 28, 415 (1940).
[368] A. Sokolov and D. Ivanenko, Квантовая теория поля (Quantum Theory of Fields) Gostekhizdat, 1952.
[369] A. A. Sokolov and Yu. M. Loskutov, JETP 32, 630 (1957), Soviet Phys. JETP 5, 523 (1957).
[370] A. A. Sokolov and Yu. M. Loskutov, JETP 34, 1022 (1958), Soviet Phys. JETP 7, 706 (1958).
[371] A. Sommerfeld, Proc. Amst. Acad. 26, (1904).
[372] A. Sommerfeld, Gott. Nachr. p. 201 (1905).
[373] A. Sommerfeld, Optics (Russ. Transl.) IL, 1953.
[374] V. N. Zrelov and G. D. Stoletov, JETP 36, 659 (1959), Soviet Phys. JETP 9, 461 (1959).
[375] R. M. Sternheimer, Phys. Rev. 88, 851 (1952).
[376] R. M. Sternheimer, Phys. Rev. 89, 1148 (1953).
[377] R. M. Sternheimer, Phys. Rev. 91, 256 (1953).
[378] Sutton, Fields, Fox, Kane, Mott, and Smallwood, Phys. Rev. 97, 783 (1950).
[379] W. F. G. Swann, J. Franklin Inst. 226, 598 (1938).
[380] C. E. Swartz, Nucleonics 14, 44 (1956).
[381] C. E. Swartz and J. W. De Wire, Phys. Rev. 98, 1164 (1955).
[382] M. S. Svirskii, Физ. мет. и металловед (Phys. Metals and Metallography) 2, 397 (1956).
[383] I. E. Tamm, J. Phys. (U.S.S.R.) 1, 439 (1939).
[384] I. E. Tamm, Nobel Lecture, UFN 68, 387 (1959).
[385] I. E. Tamm and I. M. Frank, DAN SSSR 14, 107 (1939).
[386] I. E. Tamm and I. M. Frank, Тр. Физ. ин-та им. П. Н. Лебедева АН СССР (Trans. P. N. Lebedev Phys. Inst.) 2, No. 4 (1947).
[387] Tamm, Frank, and Cerenkov, Izv. AN SSSR, OMEN (Div. Math. Nat. Sci.) p. 30 (1938).
[388] K. Tanaka, Phys. Rev. 93, 459 (1954).
[389] T. Taniuti, Progr. Theoret. Phys. (Kyoto) 6, 207 (1951).
[390] M. L. Ter-Mikaelyan, JETP 38, 895 (1958), Soviet Phys. JETP 11, 646 (1960).
[391] M. L. Ter-Mikaelyan, Izv. AN Arm.S.S.R. 12, 141 (1959).
[392] M. L. Ter-Mikaelyan, DAN SSSR 134, 318 (1960), Soviet Phys.-Doklady 5, 1015 (1961).
[393] M. L. Ter-Mikaelyan, Dissertation, Phys. Inst. Acad. Sci., Moscow, 1961.
[394] M. L. Ter-Mikaelyan, Izv. AN Arm.S.S.R. 14, 103 (1961).
[395] M. L. Ter-Mikaelyan and A. D. Gazazyan, JETP 39, 1693 (1960), Soviet Phys. JETP 12, 1183 (1961).
[396] D. D. Ternovskii, JETP 39, 171 (1960), Soviet Phys. JETP 12, 123 (1961).
[397] D. D. Ternovskii, JETP 39, 491 (1960), Soviet Phys. JETP 12, 344 (1961).
[398] Thomas, Sun, and Marusich, Nucleonics 15, 98 (1957).
[399] D. A. Tidaman, Nuovo cimento 3, 503 (1956).
[400] D. A. Tidman, Nuclear Phys. 2, 289 (1956).
[401] A. V. Tollestrup and W. A. Wentzel, Phys. Rev. 93, 950 (1954).
[402] G. Toraldo di Francia, Nuovo cimento 16, 61 (1960).
[403] W. W. Wada, Phys. Rev. 75, 981 (1949).
[404] R. Warnecke, Convegno di electronice televisione (Milan) 2, 706 (1954).
[405] S. I. Vavilov, DAN SSSR 2, 457 (1934).
[406] S. I. Vavilov, Микроструктура света (Microstructure of Light) AN SSSR, 1952.
[407] V. I. Veksler, CERN Symposium, Vol. 1, p. 80 (1956); Атомная энергия (Atomic Energy) 5, 427 (1957).

[408] W. R. Webber, Nuovo cimento **4**, 1285 (1956).
[409] W. R. Webber and F. B. McDonald, Phys. Rev. **100**, 1460 (1955).
[410] P. B. Weiss and B. L. Anderson, Phys. Rev. **72**, 431 (1947).
[411] J. Winckler, Phys. Rev. **85**, 1054 (1952).
[412] J. R. Winckler, Phys. Rev. **87**, 241 (1952).
[413] J. R. Winckler and K. Anderson, Rev. Sci. Instr. **23**, 765 (1952).
[414] J. Winckler and K. Anderson, Phys. Rev. **93**, 596 (1954).
[415] Winckler, Mitchell, Anderson, and Peterson, Phys. Rev. **98**, 1411 (1955).
[416] A. N. Vystavkin and S. S. Shakhudzhanov, op. cit. ref. 16, **5**, 1404 (1960).
[417] H. Wyckoff and J. E. Henderson, Phys. Rev. **64**, 1 (1943).
[418] I. E. Tamm, Proc. Second Intern. Conf. on Peaceful Uses of Atomic Energy (Geneva, 1958) Vol. 1, p. 408.
[418] H. Lashinsky, Advances in Electronics and Electron Physics, Vol. 14, 1961, p. 265.

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