LOCAL INVARIANCE AND THE THEORY OF COMPENSATING FIELDS

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1. INTRODUCTION

ALL phenomena in nature are independent of the inertial coordinate system in which they are regarded. This fact has its expression in the statement that the equations describing such phenomena are invariant with respect to Lorentz transformations. The wave functions of charged particles (for example, electrons) are complex. Then the Lagrangian and the dynamical quantities are invariant with respect to the transformation $\psi \rightarrow \psi e^{i\alpha}$ and $\psi^* \rightarrow \psi^* e^{-i\alpha}$. The experimentally observed charge independence of nuclear forces led to the need to introduce the concept of isotopic spin. The mathematical expression of this fact is the independence of the equations describing strong interaction with respect to rotations in isotopic space. Thus, it is very characteristic of contemporary physical theories that the equations and dynamical variables referring to different phenomena are invariant with respect to different types of transformations. Here it is usually understood that the parameters determining the transformation, for example, the phase factor or the angles in a Lorentz transformation, are constant over all configuration space and, once given, are preserved over all the interval of time during which the problem is treated.

We now propose that we go on to a stronger requirement, that of local invariance; i.e., we demand that the parameters which determine the transformation be independent at each point in space-time. Thus we go over from parameters ϵ_i to functions $\epsilon_i(x, y, z, t)$. Is such an invariance possible? It turns out that it is, but we are then forced to introduce a new compensating field which guarantees invariance under the conditions of a space-time dependence $\epsilon_i(x_\mu)$. One of the examples of such a compensating field is very well known. This is the electromagnetic field which must be introduced if we assume that the phase factor α is a function of the coordinates and time. In a 1954 paper by Yang and Mills[1] it was shown that from the requirement of invariance with respect to locally independent rotations in isotopic spin space there follows the necessity for the existence of a certain 12-component B-field, which is a 4-vector in ordinary space and has three isotopic components. In a paper of Utiyama[2] (1956) the general approach was given to the problem of the compensating field. In this paper it was shown that if there is a field whose action is invariant with respect to a certain group of transformations, depending on one or more parameters α_i, then when we make the transition from group parameters which are numbers to parameters which depend on the coordinates $\alpha_i \rightarrow \alpha_i(x_\mu)$, it becomes necessary to introduce a compensating field. Utiyama also showed that the gravitational field can be obtained as the compensating field if we introduce a dependence of the parameters determining the Lorentz group on the coordinates. The introduction of the concept of local invariance has a meaning only for continuous transformations. Discrete transformations like space or charge reflections depend on a single constant parameter which is not varied. The concept of the compensating field was used by Salam and Ward[3] for introducing an intermediate vector meson into the theory of weak interactions. The question of the place of the principle of local invariance in contemporary theory is discussed in a paper of Sakurai.[4]

2. THE ELECTROMAGNETIC FIELD

First let us consider the well-known example of the electromagnetic field. The transformation $\psi \rightarrow \psi e^{i\alpha}$, which the wave functions of charged particles admit, is usually called a gauge transformation of the first kind. The invariant with respect to this transformation is just the charge (electrical or baryonic, i.e., nuclear). Invariance with respect to this transformation guarantees conservation of charge in the sense of an algebraic sum of negative and positive charges.[5] For example, we consider an interaction as a result of which a doubly charged particle ψ decays into two particles φ_1 and φ_2. In order for the interaction Lagrangian to be gauge invariant, it is necessary, when we make the replacement $\psi \rightarrow \psi e^{2ie\alpha}$, also to make the replacement $\varphi_1^* \rightarrow \varphi_1^* e^{-ie\alpha}$ and $\varphi_2^* \rightarrow \varphi_2^* e^{-ie\alpha}$, i.e., the doubly charged particle decays into two particles, each of which carries a unit charge of the same sign.

We note that charge conservation is guaranteed if the phase factor is assumed to be constant, i.e., independent of coordinates and time. But if we assume that the choice of the phase factor is arbitrary at each point in space-time, i.e., $\alpha = \alpha(x, y, z, t)$, then it is necessary to introduce a compensating field in order to achieve local invariance of the Lagrangian, i.e., an invariance which is satisfied independently at each point. In fact, the Lagrangian of the free electron field

$$L = \frac{1}{2}\overline{\psi}\left(i\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right)\psi - \frac{1}{2}\left(i\frac{\partial\overline{\psi}}{\partial x_\mu}\gamma_\mu - m\overline{\psi}\right)\psi,$$

though invariant with respect to the transformation $\psi \to \psi e^{i\epsilon\alpha}$ and $\bar{\psi} \to \bar{\psi}e^{-i\epsilon\alpha}$, does not remain invariant under the transformation $\psi \to \psi e^{i\epsilon\alpha(x_\mu)}$ and $\bar{\psi} \to \bar{\psi}e^{-i\epsilon\alpha(x_\mu)}$, because of the appearance of terms of the form $\partial\alpha/\partial x_\mu$. But the Lagrangian, which includes the electromagnetic field

$$L = \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2i}\left[\bar{\psi}\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - i\varepsilon A_\mu\right)\psi + im\bar{\psi}\psi\right]$$

$$+ \frac{1}{2i}\left[\left(\frac{\partial\bar{\psi}}{\partial x_\mu} - i\varepsilon A_\mu\bar{\psi}_\mu\right)\gamma_\mu\psi - im\bar{\psi}\psi\right],$$

is invariant with respect to the transformation $\psi \to \psi e^{i\epsilon\alpha(x_\mu)}$ and $\bar{\psi} \to \bar{\psi}e^{-ie\alpha(x_\mu)}$ when we make a simultaneous replacement $A_\mu \to A_\mu + \partial\alpha/\partial x_\mu$, which is easily verified by substitution. Here $F_{\mu\nu}$ is the electromagnetic field tensor. Thus local invariance exists with respect to the transformation $\psi \to \psi e^{i\epsilon\alpha(x_\mu)}$ if there is a simultaneous gauge transformation of the electromagnetic field (a gauge transformation of the second kind). We note that if the electromagnetic field Lagrangian contained a term with the photon mass, μ^2A^2, the invariance would not hold.

3. GENERAL THEORY OF LOCAL INVARIANCE

a. Formulation of the Problem

Let us now consider, following Utiyama,[2] the situation with local invariance and compensating field in its most general form. Let us assume that there are some fields $Q^A(x)$ $(A = 1, 2, \ldots N)$. The superscript A may, for example, run through values from 1 to 4, and the Q^A may be simply the components of a single field.

We introduce the Lagrangian density $L(Q^A, Q^A_\mu)$ where $Q^A_\mu = \partial Q^A/\partial x^\mu$ and the action $I = \int_\Omega L(Q^A, Q^A_\mu)\, d^4x$ where Ω is some arbitrary 4-volume.

We assume that the action I remains invariant when a transformation of the field components $S(\epsilon_\alpha)$ is made which depends on n parameters ϵ_α $(\alpha = 1, 2, \ldots n)$. These may be, for example, the three parameters defining a rotation, or the six parameters defining a Lorentz transformation,

$$Q' = S(\varepsilon_\alpha)Q, \text{ or in components, } Q^{A'} = S^A_B(\varepsilon_\alpha)Q^B. \quad (3.1)$$

The transformation S can be expanded in a Taylor series in the parameters ϵ_α,

$$Q' = \left(I + \frac{\partial S}{\partial\varepsilon_\alpha}\bigg|_{\varepsilon_\alpha=0}\varepsilon_\alpha + \ldots\right)Q,$$

where I is the identity transformation, and

$$T_\alpha = \frac{\partial S}{\partial\varepsilon_\alpha}\bigg|_{\varepsilon_\alpha=0} \quad (3.2)$$

is the operator for an infinitesimal transformation, or, as we say, an infinitesimal operator. The small change of the quantities Q is expressed directly in terms of the infinitesimal operators, the set of which completely determine the transformation:

$$\delta Q = T_\alpha\varepsilon_\alpha Q, \text{ or in components, } \delta Q^A = T^A_{\alpha B}\varepsilon_\alpha Q^B. \quad (3.3)$$

From the fact that the action I is invariant with respect to the transformation Q′ = SQ and that this invariance holds for an arbitrary volume Ω, it follows that the Lagrangian density is also invariant:

$$\delta L = \frac{\partial L}{\partial Q^A}\delta Q^A + \frac{\partial L}{\partial Q^A_\mu}\delta Q^A_\mu = 0. \quad (3.4)$$

Substituting (3.3) in (3.4), we get

$$\frac{\partial L}{\partial Q^A} T^A_{\alpha B}Q^B + \frac{\partial L}{\partial Q^A_\mu} T^A_{\alpha B}Q^B_\mu = 0. \quad (3.5)$$

Noting that the second term in (3.4) can be written as

$$\frac{\partial L}{\partial Q^A_\mu}\delta Q^A_\mu = \frac{\partial}{\partial x_\mu}\left[\frac{\partial L}{\partial Q^A_\mu}\delta Q^A\right] - \frac{\partial}{\partial x_\mu}\frac{\partial L}{\partial Q^A_\mu}\delta Q^A,$$

we obtain

$$\left\{\frac{\partial L}{\partial Q^A} - \frac{\partial}{\partial x_\mu}\left(\frac{\partial L}{\partial Q^A_\mu}\right)\right\}\delta Q^A + \frac{\partial}{\partial x_\mu}\left[\frac{\partial L}{\partial Q^A_\mu}\delta Q^A\right] = 0. \quad (3.6)$$

The first term goes to zero because it is simply the field equation obtained from the principle of least action. There then follow from (3.4) the conservation laws

$$\frac{\partial J^\mu_\alpha}{\partial x_\mu} = 0, \text{ and } J^\mu_\alpha = \frac{\partial L}{\partial Q^A_\mu} T^A_{\alpha B}Q^B \quad (3.7)$$

may be regarded as the current vector.

Let us now assume that the parameters ϵ_α defining the transformation are no longer numbers, but functions of the coordinates and time:

$$\varepsilon_\alpha = \varepsilon_\alpha(x_\mu). \quad (3.8)$$

Then $\delta Q^A = \delta Q^A(x_\mu) = T^A_{\alpha B}\epsilon_\alpha(x_\mu)Q^B$. Here the infinitesimal operators $T^A_{\alpha B}$ which define the transformation of course remain the same at all points in space. Now in the expression for the variation of the Lagrangian there is added a new term, and we can no longer set it equal to zero:

$$\delta L \equiv \left\{\frac{\partial L}{\partial Q^A} T^A_{\alpha B}Q^B + \frac{\partial L}{\partial Q^A_\mu} T^A_{\alpha B}Q^B_\mu\right\}\varepsilon_\alpha(x_\mu) + \frac{\partial L}{\partial Q^A_\mu} T^A_{\alpha B}Q^B\frac{\partial\varepsilon_\alpha}{\partial x_\mu},$$

or

$$\delta L \equiv \frac{\partial L}{\partial Q^A_\mu} T^A_{\alpha B}Q^B\frac{\partial\varepsilon_\alpha}{\partial x_\mu} \quad (3.9)$$

on the basis of (3.5). In order to retain invariance of the Lagrangian under the conditions (3.8), we must introduce a field

$$A^J(x_\mu),\ J = 1, 2, \ldots, M,$$

to compensate for the additional term arising in the variation of the Lagrangian (3.9). For this purpose we must consider a new Lagrangian which also contains the field A:

$$L'(Q^A, Q^A_\mu, A^J).$$

b. The Locally Invariant Lagrangian and Transformation Properties of the A-Field

We set the infinitesimal change of the field A equal to

$$\delta A^J = U^J_{\alpha K} A^K \varepsilon_\alpha (x_\mu) + C^J_{\alpha\mu} \frac{\partial \varepsilon_\alpha}{\partial x_\mu}, \qquad (3.10)$$

while the infinitesimal changes of the field are as before equal to $\delta Q = T_\alpha \epsilon_\alpha Q$. U and C are as yet unknown matrices which are to be determined from the condition of invariance of the Lagrangian L′,

$$\delta L' \equiv \frac{\partial L'}{\partial Q^A} \delta Q^A + \frac{\partial L'}{\partial Q^A_\mu} \delta Q^A_\mu + \frac{\partial L'}{\partial A'^J} \delta A'^J \equiv 0.$$

Substituting from (3.10) and collecting coefficients of $\epsilon_\alpha(x_\mu)$ and $\partial\epsilon_\alpha/\partial x_\mu$, we equate them to zero, since the identity $\delta L' \equiv 0$ should hold for any choice of $\epsilon_\alpha(x)$. We thus obtain two relations:

$$\frac{\partial L'}{\partial Q^A} T^A_{\alpha B} Q^B + \frac{\partial L'}{\partial Q^A_\mu} T^A_{\alpha B} Q^B_\mu + \frac{\partial L'}{\partial A'^J} U^J_{\alpha K} A^K \equiv 0, \qquad (3.11)$$

$$\frac{\partial L'}{\partial Q^A_\mu} T^A_{\alpha B} Q^B + \frac{\partial L'}{\partial A'^J} C^J_{\alpha\mu} \equiv 0. \qquad (3.12)$$

The set (3.12) has a total of 4n equations. (To each of the n parameters of the transformation there correspond four equations, since $\mu = 1, 2, 3, 4$.) Thus, in order that the dependence of the Lagrangian L′ on the field A, which is determined by the partial derivatives $\partial L'/\partial A'^J$, be unique, we need M = 4n equations, where M is the largest value of J, i.e., it must be equal to the number of unknowns.

To the matrix C we can make correspond an inverse matrix C^{-1} which is defined by the conditions

$$C^{J\mu}_\alpha C^{-1\alpha}_{\mu K} = \delta^J_K \quad \text{and} \quad C^{-1\alpha}_{\mu J} C^{J\nu}_\beta = \delta^\alpha_\beta \delta^\nu_\mu .$$

Using the inverse matrix, we go over from A'^J to $A^\alpha_\mu = C^{-1\alpha}_{\mu J} A'^J$. Then $\partial L'/\partial A'^J = (\partial L'/\partial A^\alpha_\mu) C^{-1\alpha}_{\mu J}$. Substituting this expression in (3.12), we get

$$\frac{\partial L'}{\partial A^\alpha_\mu} + \frac{\partial L'}{\partial Q^A_\mu} T^A_{\alpha B} Q^B \equiv 0. \qquad (3.13)$$

To satisfy this identity it is necessary that the field A^α_μ be contained in the Lagrangian in the combination

$$\nabla_\mu Q^A \equiv \frac{\partial Q^A}{\partial x_\mu} - T^A_{\alpha B} Q^B A^\alpha_\mu . \qquad (3.14)$$

In this case we have

$$\frac{\partial L'}{\partial A^\alpha_\mu} \equiv - \frac{\partial L'}{\partial Q^A_\mu} T^A_{\alpha B} Q^B .$$

In going over from A^J to A^α_μ, formula (3.10) which describes the transformation properties of the A field goes over into the expression

$$\delta A^a_\mu = M^{a\nu}_{c\mu\beta} A^\beta_\nu \varepsilon^c (x) + \frac{\partial \varepsilon^a}{\partial x_\mu}, \qquad (3.15)$$

where

$$M^{a\nu}_{c\mu\beta} = C^{-1a}_{\mu J} U^J_{cK} C^{K\nu}_\beta .$$

Now we can write $L'(Q^A, Q^A_\mu, A^\alpha_\mu) = L''(Q^A, \nabla_\mu Q^A)$ where $\nabla_\mu Q^A$ is defined by formula (3.14). Then we find the following relations between the derivatives:

$$\frac{\partial L'}{\partial Q^A} = \frac{\partial L''}{\partial Q^A}\bigg|_{\nabla Q=\mathrm{const}} - \frac{\partial L''}{\partial \nabla_\mu Q^B}\bigg|_{Q=\mathrm{const}} T^B_{\alpha A} A^\alpha_\mu ,$$

$$\frac{\partial L'}{\partial Q^A_\mu} = \frac{\partial L''}{\partial \nabla_\mu Q^A}\bigg|_{Q=\mathrm{const}}, \quad \frac{\partial L'}{\partial A'^J} = - \frac{\partial L''}{\partial \nabla_\mu Q^A}\bigg|_{Q=\mathrm{const}} T^A_{\beta B} Q^B C^{-1\beta}_{\nu J}.$$

If we substitute these expressions in (3.11), go over from A'^J to A^α_μ and use relation (3.15), we get

$$\frac{\partial L''}{\partial Q^A}\bigg|_{\nabla Q=\mathrm{const}} T^A_{\alpha B} Q^B + \frac{\partial L''}{\partial \nabla_\mu Q^A}\bigg|_{Q=\mathrm{const}} T^A_{\alpha B} \nabla_\mu Q^B$$
$$+ \frac{\partial L''}{\partial \nabla_\mu Q^A}\bigg|_{Q=\mathrm{const}} Q^B A^\beta_\nu \{[T_\alpha T_\beta]^A_B \delta^\nu_\mu - S^{d\nu}_{\alpha\mu\beta} T^A_{dB}\} = 0. \qquad (3.16)$$

Here

$$[T_\alpha T_\beta]^A_B = T^A_{\alpha c} T^c_{\beta B} - T^A_{\beta c} T^c_{\alpha B}.$$

Since the Lagrangian $L''(Q^A, \nabla_\mu Q^A)$ is obtained from the initial Lagrangian $L'(Q^A, Q^A_\mu)$ by replacing the derivative $\partial Q^A/\partial x_\mu$ by the "invariant" derivative,

$$\nabla_\mu Q^A = \frac{\partial Q}{\partial x_\mu} - T^A_{\alpha B} Q^B A^\alpha_\mu,$$

just as one introduces the electromagnetic field by means of the transition $\partial/\partial x \to \partial/\partial x - i\epsilon A_X$, then according to the identity (3.5) the first two terms in (3.16) vanish, while the last two terms can be written in the form

$$\frac{\partial L}{\partial \nabla_\mu Q^A}\bigg|_{Q=\mathrm{const}} Q^B A^\beta_\nu \{f^d_{\alpha\beta} \delta^\nu_\mu - M^{d\nu}_{c\mu\beta}\} T^A_{dB} = 0. \qquad (3.17)$$

Here we have made use of the well-known commutation relations between the infinitesimal operators, which follow from group theory:

$$[T_a T_b]^A_B = T^A_{ac} T^c_{bB} - T^A_{bc} T^c_{aB} = f^c_{ab} T^A_{cB}. \qquad (3.18)$$

The numbers f^a_{bc} have the properties

$$f^m_{ab} f^c_{ma} + f^m_{bc} f^c_{ma} + f^m_{ca} f^c_{mb} \equiv 0, \quad f^c_{ab} = - f^c_{ba}. \qquad (3.19)$$

From the identity (3.17) we can express the unknown matrix M in terms of the coefficients f, which are defined for any transformation:

$$M^{c\nu}_{a\mu b} = \delta^\nu_\mu f^c_{ab}. \qquad (3.20)$$

Thus we have obtained the matrix M which determines the law of transformation of the A-field as a function of the transformation to which the field Q is subjected. Now, the infinitesimal change of the field A takes the form

$$\delta A^a_\mu = f^a_{bc} A^c_\mu \varepsilon^b (x) + \frac{\partial \varepsilon^a}{\partial x} . \qquad (3.21)$$

In addition, we obtain the rule according to which the transition to the locally invariant Lagrangian is accomplished by replacing the ordinary derivative by the invariant derivative (3.14).

c. General Properties of the Free Compensating Field

Now we can investigate the properties of the free A-field. We denote the Lagrangian of the free field by L_0:

$$L_0(A_\mu^a,\ A_{\mu,\nu}^a),\quad A_{\mu\nu}^a = \frac{\partial A_\mu^a}{\partial x_\nu}.$$

Let us consider the consequences of the invariance of the Lagrangian L_0 under the action of the transformations (3.15) applied to the field A:

$$\delta L_0 = \frac{\partial L_0}{\partial A_\mu^a}\,\delta A_\mu^a + \frac{\partial L_0}{\partial A_{\mu,\nu}^a}\,\delta A_{\mu,\nu}^a = 0.$$

Substituting

$$\delta A_\mu^a = f_{bc}^a A_\mu^c \varepsilon^b(x) + \frac{\partial \varepsilon^a}{\partial x_\mu},$$

$$\delta A_{\mu,\nu}^a = f_{bc}^a A_{\mu,\nu}^c \varepsilon^b(x) + f_{bc}^a A_\mu^c \frac{\partial \varepsilon^b}{\partial x_\nu} + \frac{\partial^2 \varepsilon^a}{\partial x_\mu\,\partial x_\nu}$$

and collecting coefficients of ϵ and its derivatives, we get

$$\frac{\partial L_0}{\partial A_\mu^a} f_{bc}^a A_\mu^c + \frac{\partial L_0}{\partial A_{\mu,\nu}^a} f_{bc}^a A_{\mu\nu}^c \equiv 0, \tag{3.22}$$

$$\frac{\partial L_0}{\partial A_\mu^a} + \frac{\partial L_0}{\partial A_{\nu,\mu}^b} f_{bc}^a A_\mu^c \equiv 0, \tag{3.23}$$

$$\frac{\partial L_0}{\partial A_{\mu,\nu}^a} + \frac{\partial L_0}{\partial A_{\nu,\mu}^a} \equiv 0. \tag{3.24}$$

In order for the last identity to be satisfied, the derivatives of A must enter into L_0 in the combinations

$$A_{[\mu,\nu]}^a = \frac{\partial}{\partial x_\mu} A_\nu^a - \frac{\partial}{\partial x^\nu} A_\mu^a = A_{\nu,\mu}^a - A_{\mu,\nu}^a.$$

Since $\partial L_0/\partial A_{\mu,\nu}^a = -\partial L_0/\partial A_{\nu,\mu}^a = -\partial L_0/\partial A_{[\mu,\nu]}^a$, Eq. (3.23) can be written in the form $\partial L_0/\partial A_\mu^a = (\partial L_0/\partial A_{[\nu,\mu]}^c) f_{ab}^c A_\nu^b$; from this it follows that $A_{[\mu,\nu]}^a$ enters into L_0 in combination with the expression $\frac{1}{2} f_{bc}^a [A_\mu^b A_\nu^c - A_\nu^b A_\mu^c]$ in the form

$$F_{\mu\nu}^a = \frac{\partial A_\nu^a}{\partial x_\mu} - \frac{\partial A_\mu^a}{\partial x_\nu} - \frac{1}{2} f_{bc}^a [A_\mu^b A_\nu^c - A_\nu^b A_\mu^c]. \tag{3.25}$$

Using (3.23) and the relation (3.24), one can show that the Lagrangian of the free field L_0 is a function only of $F_{\mu\nu}^a$.

We may note the analogy between the quantity $F_{\mu\nu}^a$ and the field intensity in electrodynamics. This quantity is invariant with respect to the transformation (3.15):

$$A_\mu^{\prime a} \rightarrow A_\mu^a + \delta_\mu^\nu f_{ab}^c A_\nu^b \varepsilon^c(x) + \frac{\partial \varepsilon^a}{\partial x_\mu},$$

which, in analogy to electrodynamics, can be called a gauge transformation. The Lagrangian of the free A field, being a function only of $F_{\mu\nu}^a$, contains no term with a rest mass, which would destroy the gauge invariance. Thus the A-field should not have a rest mass.

One can also introduce the vector

$$J_a^\mu = -\left(\frac{\partial L}{\partial \nabla_\mu Q^A} T_{aB}^A Q^B + \frac{\partial L_0}{\partial F_{\mu\nu}^b} f_{ac}^b A_\nu^c\right) \tag{3.26}$$

and show that it satisfies a conservation law of the form

$$\frac{\partial J_a^\mu}{\partial x_\mu} = 0. \tag{3.27}$$

In the absence of the compensating field, the expression (3.26) goes over into the current vector (3.7). We note that the properties of the free compensating field do not depend on the infinitesimal operators which establish the transformation of the initial field, but on the commutation relations between them, which, as we know, are the same for all representations of a particular group. This means that the compensating field is the same for different fields which are subjected to the action of the same transformation.

4. THE GROUP OF PHASE TRANSFORMATIONS AND THE ELECTROMAGNETIC FIELD

Let us now turn once again to the example of the electromagnetic field and obtain it as a special case of compensating field arising under phase transformations. Let us consider a charged field described by the function ψ and the complex conjugate function ψ^*. The Lagrangian is invariant with respect to phase transformation, i.e., with respect to the transformation

$$\psi' = \psi e^{ie\alpha} \text{ and } \psi^{*\prime} = \psi^* e^{-ie\alpha}.$$

The operator S in formula (3.1), which determines the transformation, is equal in this case to simply $e^{ie\alpha}$ or $e^{-ie\alpha}$. The infinitesimal operator $T = \left.\frac{\partial S}{\partial \alpha}\right|_{\alpha=0} = ie$ for ψ and $T = -ie$ for ψ^*:

$$\delta\psi = ie\alpha\psi \text{ and } \delta\psi^* = -ie\alpha\psi^*.$$

The group of phase transformations, like every one-parameter group, is commutative, i.e., the order of operations in such a group is irrelevant. This also follows from the form of the operator $S = e^{ie\alpha}$. There is a single infinitesimal operator. Therefore the number f, which is associated with the commutation relation between the infinitesimal operators according to formula (3.19), is equal to zero. If now we require local invariance for the charged field ψ and consequently assume $\alpha = \alpha(x)$, the compensating field will have very simple properties. In this case $M = 0$, [cf. (3.20)] and according to (3.15)

$$\delta A_\mu = \frac{\partial \alpha}{\partial x_\mu}. \tag{4.1}$$

The new Lagrangian is

$$L' = L(\psi,\ \psi^*,\ \nabla_\mu\psi,\ \nabla_\mu\psi^*),$$

where $\nabla_\mu\psi = \partial\psi/\partial x_\mu - ieA_\mu\psi$, $\nabla_\mu\psi^* = \partial\psi^*/\partial x_\mu + ieA_\mu\psi^*$ according to the general formula (3.14). Thus the charge, which guaranteed conservation of particles in the algebraic sense, turns out in addition to be the coupling constant between the original and the compensating fields.

The Lagrangian of the free field is $L_0 = L_0(F_{\mu\nu})$ where $F_{\mu\nu} = \partial A_\nu/\partial x_\mu - \partial A_\mu/\partial x_\nu$ according to the general formula (3.25).

5. CONSERVATION OF BARYONIC CHARGE AND THE CORRESPONDING PHASE TRANSFORMATION

All reactions involving heavy particles (baryons), i.e., nucleons and hyperons, occur in such a way that the algebraic sum of the numbers of particles and anti-particles is conserved. This means that there exists a conserved quantity, the baryonic charge, which is positive for baryons and negative for anti-baryons. Thus, there is complete analogy with the conservation of electrical charge. The wave functions of the baryons must also be complex and be invariant with respect to phase transformations, just like the wave functions of electrons.

Thus with respect to baryons, for example, proton and neutron, we may write

$$\psi_p \to e^{i\eta\alpha}\psi_p, \quad \psi_p^* \to e^{-i\eta\alpha}\psi_p^*, \quad \psi_n \to e^{i\eta\alpha}\psi_n, \quad \psi_n^* \to e^{-i\eta\alpha}\psi_n^*;$$

where η is the baryonic charge.

Just as in the case of electrodynamics, when we make the transition from constant phase α to variable phase $\alpha(x, y, z, t)$ it becomes necessary to introduce a compensating field.[6] This field must have the same properties as the electromagnetic field, with the one difference that it interacts only with "baryonically" charged particles, just as the electromagnetic field interacts with electrically charged particles. If such a field actually exists, then between baryons, and in particular between nucleons, there must exist, in addition to the nuclear forces, forces of Coulomb type. Lee and Yang[6] have called attention to this point. Since all bodies consist of nucleons and do not contain antinucleons, these forces must be repulsive in character. Between two massive bodies there should thus act the force

$$F = -\gamma\frac{M_1M_2}{R^2} + \eta^2\frac{A_1A_2}{R^2}.$$

Here the first term corresponds to the gravitational attraction, and the second term to quasi-Coulomb repulsion, γ is the gravitational constant, M_1, M_2 and A_1, A_2 are the inertial masses and the nucleon numbers, i.e., baryonic charges, of the two bodies respectively. To clarify the problem of the magnitude of such a quasi-Coulomb repulsion, we consider the force acting between a proton and some other body. It can be written in the form

$$F = -\gamma\frac{M_p\mathfrak{M}}{R^2} = -\gamma\frac{M_pM_i}{R^2} + \eta^2\frac{A}{R^2},$$

where $\mathfrak{M}$ is the effective gravitating mass of the body, including its quasi-Coulomb interaction, M_i is the inertial mass of the body, M_p is the mass of the proton. Then

$$\frac{\mathfrak{M}}{M_i} = 1 + \frac{\eta^2}{\gamma}\frac{A}{M_iM_p};$$

M_i can be written in the form

$$M_i = MM_p,$$

where M is the atomic weight expressed in terms of the proton mass; the value of A/M changes from substance to substance because of the difference in mass defects, which are $\sim 10^{-3}$; consequently $\mathfrak{M}/M_i$ should vary by $\sim 10^{-3}(\eta^2M_i/\gamma M_p^2)$. From experiment we know $\mathfrak{M}/M_i$ to an accuracy of 10^{-8}. Consequently, $\eta^2/\gamma M_p^2 < 10^{-5}$ and $\eta^2 < 10^{-5}\gamma M_p^2 \sim 10^{-58}$. Thus the charge η is at least 10^{19} times smaller than the electric charge.

If we admit the existence of a rest mass for the compensating field, this destroys the gauge invariance of the field (3.15), and thus the corresponding forces acting between baryons will not have a quasi-Coulomb long-range interaction. These would now be short-range forces, whose range of action is given by a factor $e^{-r/\Lambda}$, where $\Lambda = h/mc$ is the Compton wavelength of the particles with mass m. The considerations of order of magnitude which were given above would not be applicable to such forces. In this case, the particles of such a neutral vector field would serve as the carriers of strong interaction between baryons, as was proposed by Kobzarev and Okun'.[7]

6. THE GROUP OF ROTATIONS IN ISOTOPIC SPACE AND THE COMPENSATING FIELD OF YANG AND MILLS[1]

Let ψ be a two-component field function of a field with isotopic spin $\frac{1}{2}$, and S the two-by-two unitary matrix which produces the transformation of the function ψ under rotation of the three-dimensional isotopic space. All processes in which isotopic spin is conserved must be described by equations which are invariant with respect to rotation in isotopic space. Usually it is assumed that the rotation can be arbitrary, but must be the same at all points in space and at every moment in time. The reason for this restriction is that, having once oriented the isotopic axes, we define the isotopic components of the field, for example $I_z = \frac{1}{2}$, as a proton, and $I_z = -\frac{1}{2}$ as a neutron, and regard this definition as necessary over all space and at all times. The breakdown of this restriction would mean a spontaneous charge-exchange in the course of time, or in the movement of a particle from one point to another.

If we do not make this restriction and thus extend the principle of local invariance to phenomena associated with the isotopic spin then, in accordance with the general theory of local invariance and in a way analogous to that in electrodynamics, it becomes necessary to introduce a compensating field.

The matrix S depends on the three parameters determining the rotation in isotopic space. Usually we choose for these parameters the angles of rotation around the coordinate axes. Then the infinitesimal change of one of the components of the function ψ will be equal to

$$\delta\psi^{\alpha}=i\sum_{c=1}^{3}\delta\varphi^{c}\tau^{\alpha}_{c\beta}\,\psi^{\beta}, \tag{6.1}$$

or in matrix form

$$\delta\psi=i\sum_{c=1}^{3}\delta\varphi^{c}\tau_{c}\psi, \tag{6.2}$$

where τ_1, τ_2, τ_3 are the isotopic spin matrices of the nucleon, which are the infinitesimal operators for rotation around the coordinate axes in isotopic space. Now if we assume the rotation in isotopic space to be independent at each point in configuration space, we must introduce a compensating field. Such a field was introduced by Yang and Mills.[1] According to the general rule (3.14), the compensating field is introduced by making the transition from the ordinary derivative to the invariant derivative:

$$\frac{\partial}{\partial x_{\mu}}\rightarrow\frac{\partial}{\partial x_{\mu}}-i\tau_{c}B^{c}_{\mu}. \tag{6.3}$$

From this we see that the field B has three isotopic components and $\tau_c B^c_\mu$ is nothing but the scalar product in isotopic space. In addition, B^c_μ is a 4-vector in ordinary space. It is obvious that B^c_μ is a 12-component field which is a 4-vector in ordinary space-time and a vector in isotopic space, i.e., it possesses isotopic spin 1. Each isotopic component describes a vector particle: positive, negative, and neutral, which form an isotopic triplet similar to the π mesons, which also have an isotopic spin equal to one.

According to the general formula (3.25), one can form a quantity analogous to the field intensity:

$$F^{a}_{\mu\nu}=\frac{\partial B^{a}_{\nu}}{\partial x_{\mu}}-\frac{\partial B^{a}_{\mu}}{\partial x_{\nu}}-\frac{1}{2}\left[B^{b}_{\mu}B^{c}_{\nu}-B^{b}_{\nu}B^{c}_{\mu}\right]; \tag{6.4}$$

F and B are vectors in isotopic space. Making use of this fact, we can write (6.4) in more compact form

$$\mathbf{F}_{\mu\nu}=\frac{\partial\mathbf{B}_{\nu}}{\partial x_{\mu}}-\frac{\partial\mathbf{B}_{\mu}}{\partial x_{\nu}}-\frac{1}{2}\left[\mathbf{B}_{\mu}\times\mathbf{B}_{\nu}\right]. \tag{6.5}$$

The Lagrangian of the free B field can be written by analogy with electrodynamics in the form

$$L_{0}=\frac{1}{4}\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}. \tag{6.6}$$

The total Lagrangian including the B-field and nucleonic field can be written by making the replacement in the Lagrangian of the free nucleon field (replacement of the ordinary derivative by the invariant derivative) and adding the free B-field:

$$L=-\frac{1}{4}\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}-\bar{\psi}\gamma_{\mu}\left(\frac{\partial}{\partial x_{\mu}}-i\tau\mathbf{B}_{\mu}\right)\psi-m\bar{\psi}\psi. \tag{6.7}$$

From formulas (6.4) and (6.5) it follows that the B-field is nonlinear, i.e., one can have a direct interaction of isotopic components of a field with one another. The isotopic spin of the system consisting of the ψ- and B-fields obviously is given by the sum of the charges of these fields. As already pointed out above, the independence of the orientations of the isotopic axes at different points in space-time means that it is possible to have a charge-exchange in movement from one point in space to another. For example, a charged particle moving from x to x′ becomes neutral, or a neutral particle at rest is converted to a charged particle. The introduction of the compensating B-field enables us to explain charge-exchange as the emission of charged particles—isotopic components of the B-field.

If we choose as the initial field, which is a vector in isotopic space, the π^+, π^0 and π^- mesons, we get this same compensating field B. This is a consequence of the general situation pointed out in Sec. 3 that to each transformation there corresponds one compensating field, which interacts with all fields admitting this transformation. But the invariant derivative will look somewhat different for different fields. For the π meson field

$$\frac{\partial}{\partial x_{\mu}}\rightarrow\frac{\partial}{\partial x_{\mu}}-i\mathbf{T}\mathbf{B}_{\mu},$$

and the Lagrangian of the π meson field interacting with the B-field is

$$L=\left[\frac{\partial\varphi}{\partial x_{\mu}}-i(\mathbf{T}\mathbf{B}_{\mu})\varphi\right]\left[\frac{\partial\varphi}{\partial x^{\mu}}-i(\mathbf{T}\mathbf{B}^{\mu})\varphi\right]+m^{2}\varphi\varphi,$$

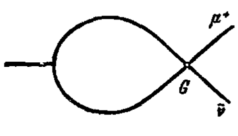
where T is the three-by-three matrix of the isotopic spin of the π meson.

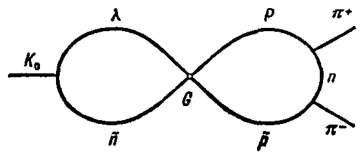
7. THEORY OF COMPENSATING FIELDS AND THE INTERMEDIATE VECTOR MESON IN WEAK INTERACTIONS

a. The Hypothesis of an Intermediate Meson in Weak Interactions

Weak interaction is the term for processes which occur with an interaction constant $G = 1.4 \times 10^{-49}$ erg-cm^3. Another general feature of these interactions is nonconservation of parity. These processes involve an even number of fermions. In such processes as, for example, β decay or meson decay, four fermions participate. In the decay of the π meson $\pi \rightarrow \mu + \nu$ or $\pi \rightarrow e + \nu$, there are two fermions involved. Finally there are processes as, for example, the decay of K mesons $K^0 \rightarrow \pi^+ + \pi^-$ or $K^+ \rightarrow \pi^+ + \pi^+ + \pi^-$ in which no fermions participate. According to the Sakata model, π mesons and K mesons are not truly elementary particles, but consist of two fermions. Thus every weak interaction is a four-fermion interaction. According to this scheme, the decays of the π and K meson have the form shown in the figure.

Another important feature of weak interactions is the fact that of the four fermions participating two are





charged and two are neutral. The four-fermion interaction is described as the interaction of two currents

$$L \sim J_{\alpha}^{*} J_{\alpha}.$$

The current J_α contains four terms:

$$J_{\alpha} = J_{\alpha}^{e} + J_{\alpha}^{\mu} + J_{\alpha}^{n} + J_{\alpha}^{\Lambda}, \qquad J_{\alpha}^{e} = \bar{\psi}_{e}\Gamma_{\alpha}\psi_{\nu}, \quad J_{\alpha}^{\mu} = \bar{\psi}_{\mu}\Gamma_{\alpha}\psi_{\nu},$$

$$J_{\alpha}^{n} = \bar{\psi}_{n}\Gamma_{\alpha}\psi_{p}, \quad J_{\alpha}^{\Lambda} = \bar{\psi}_{\Lambda}\Gamma_{\alpha}\psi_{p}, \quad \Gamma_{\alpha} = \gamma_{\alpha}(1+\gamma_{5}),$$

where the superscripts denote: e—electron, ν—neutrino, μ—muon, n—neutron, p—proton, Λ—lambda hyperon.

Each such current contains one charged and one neutral particle. In a process of weak interaction there occurs a sort of "charge exchange." Such a charge exchange is most easily described by assuming, for example, that the neutral particle is converted to a charged particle by emitting an intermediate charged particle which then decays into two fermions, one charged and one neutral. Such a scheme guarantees the forbiddenness of processes which do not occur in nature, in which all four fermions are charged, as for example $\mu \to e^- + e^+ + e^-$ and $\mu^- + p \to p + e^-$, or those in which both of the leptons created are charged: $K^0 \to e^+ + e^-$. The hypothesis of the existence of such an intermediate particle has been proposed by many authors.[8,9]

The introduction of an intermediate particle brings nonlocality into the interaction between the four fermions. This is a nonlocality of the order of the Compton wave length of the particle which carries the interaction between the fermion pairs, $l \sim h/mc$. The lighter the intermediate particle, the greater the nonlocality which it introduces. For $m \to 0$ there is a long-range interaction which is characteristic of the Coulomb interaction. It is obvious that the intermediate meson in weak interactions should have a large mass, at least of the order of the nucleon mass, in order that it not carry highly nonlocal effects which are not observed in experiment. The hypothesis of a particle transferring the weak interaction meets with serious difficulties associated with the non-occurrence of the process $\mu \to e + \gamma$, which should occur if there exists an intermediate charged particle.[10] To avoid this difficulty there has been proposed an additional hypothesis of two different neutrinos,[11] one of which participates in the muon current and the other in the electronic current. A more detailed presentation of this type of question can be found in the survey by Okun'.[12]

b. Application of the Principle of Local Invariance to the Lepton Field

Let us now consider the scheme proposed by Salam and Ward[3] of applying the idea of a compensating field to the theory of weak interaction. They used not isotopic space, but a charged space proposed in a paper by D'Espagnat, Prentky, and Salam. Isotopic space and the concepts associated with it are unsatisfactory, according to these authors, because they do not have a natural place for the introduction of leptons. The charge space proposed by them is three-dimensional. The particles e^+, ν, e^- form a charged triplet L_1 and are described by the components $\psi^+ = (1/\sqrt{2})\times(\psi_X + i\psi_Y)$, $\psi^0 = \psi_Z$ and $\psi^- = (1/\sqrt{2})(\psi_X - i\psi_Y)$ of a vector ψ in this charge space. The particles, since they form a triplet, have the same "bare" mass. The difference in the real masses occurs as a result of the differences in interaction among the components of the triplet. The Lagrangian of the free lepton field is described by

$$L = -\bar{\psi}\gamma_{\mu}\left(\frac{\partial}{\partial x_{\mu}} - m\right)\psi; \tag{7.1}$$

since ψ is a vector in the charge space, each term in the Lagrangian is a sum of three terms for the charge components of the ψ-field; m is the mass of the "bare" lepton (possibly equal to zero).

The Lagrangian (7.1) is invariant with respect to rotations in the charge space. If we now postulate invariance of the Lagrangian with respect to locally independent rotations, then, as in the preceding cases, the derivatives in the Lagrangian (7.1) must be replaced by invariant derivatives according to formula (6.3):

$$\frac{\partial}{\partial x} \to \nabla_{\mu} = \frac{\partial}{\partial x_{\mu}} - i\varepsilon\mathbf{T}\mathbf{B}.$$

The total Lagrangian, including the B-field, takes the form:

$$L = -\frac{1}{4}\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} - \bar{\psi}\,\gamma_{\mu}\left(\frac{\partial}{\partial x_{\mu}} - i\varepsilon\mathbf{T}\mathbf{B}\right)\psi - m\bar{\psi}\psi, \tag{7.2}$$

where $\mathbf{T}$ is a vector corresponding to spin 1 in the charged space. The B-field is the field of Yang and Mills, since the charge and isotopic spaces have the same geometric properties.

We now make use of the relation which exists for an operator corresponding to spin 1 to give the interaction term in the Lagrangian a different form:

$$[\hat{g} \times \hat{f}]_{\alpha} = i g_{\beta}\, s_{\alpha}\, f_{\beta},$$

where s_α is the α component of the spin operator s. We may then write

$$\bar{\psi}\mathbf{T}\psi = -i\,[\bar{\psi} \times \psi].$$

Thus the interaction term in (7.2) can be written in the form

$$\mathbf{JB} = \varepsilon [\psi \times \gamma \psi] \mathbf{B}. \quad (7.3)$$

We now expand the product:

$$[\overline{\psi} \times \psi] \mathbf{B} = (\overline{\psi}^+ \psi^- - \overline{\psi}^- \psi^+) B^0 + (\overline{\psi}^0 \psi^- - \overline{\psi}^+ \psi^0) B^+ + (\overline{\psi}^- \psi^0 - \overline{\psi}^0 \psi^-) B^-.$$

We identify the neutral component of the B-field with the electromagnetic field. We introduce the notation $\psi^+ \to e^+$, $\psi^- \to e^-$, $\psi^0 \to \nu$ and $B^0 \to A$, and write the interaction between the electron-neutrino and B-fields:

$$\varepsilon \{[\bar{e}^+ \gamma_\mu e^+ - \bar{e}^- \gamma_\mu e^-] A + [\bar{e}^+ \gamma_\mu \nu - \bar{\nu} \gamma_\mu e^-] B^+ + [\bar{\nu}^- \gamma_\mu e^+ - \bar{e}^- \gamma_\mu \nu] B^-\}. \quad (7.4)$$

The first term is the usual electromagnetic interaction. The second and third terms give the weak interaction of leptons of the charge triplet $L_1(e^+, \nu, e^-)$ via the charged vector particles B^+ and B^-, where the interaction constant is the same for the electromagnetic and weak interactions.

As a consequence of the fact that the rest mass of the neutrino is equal to zero, the neutrino Lagrangian and also interactions involving the neutrino must be invariant with respect to the following transformation of the neutrino function:

$$\nu \longrightarrow \gamma_5 \nu.$$

The interaction between the leptonic and vector meson fields takes a form which is invariant with respect to this transformation, if we go over from (7.4) to the expression

$$\varepsilon \{[\bar{e}^+ \gamma_\mu e^+ - \bar{e}^- \gamma_\mu e^-] A + [\bar{e}^+ \gamma_\mu (1 + \gamma_5) \nu - \bar{\nu} \gamma_\mu (1 + \gamma_5) e^-] B^+ + [\bar{\nu} \gamma_\mu (1 + \gamma_5) e^+ - \bar{e}^- \gamma_\mu (1 + \gamma_5) \nu] B^-\}. \quad (7.5)$$

From (7.5) it follows that interactions with the charged components of the B-field (i.e., weak interactions) lead to a parity violation. Thus the invariance of the neutrino with respect to the transformation $\nu \to \gamma_5 \nu$ has the consequence that one axis in the charge space becomes distinguished. The component $B_3 = A$ interacts with the leptons without parity violation, while the other components of the B-field interact with the leptons with parity violation.

Since we have no indications of the mass of the charged particles B^+ and B^- from the method by which they were introduced into interaction with the lepton field, we will assume that their mass is determined by equating the coupling constants of the electromagnetic and weak interactions according to the formula[9]

$$4\pi e^2 = 2\sqrt{2} G m^2, \quad (7.6)$$

which gives $m_B \approx 56$ nucleon masses.

All the arguments given above can be applied to the second group of leptons: $L_2(\mu^+, \nu, \mu^-)$. The neutrino entering in the leptonic triplet L_2 may not be identical with the neutrino of the triplet $L_1(e^+, \nu, e^-)$. The interaction of the B-field with mesons and hyperons can also be constructed by going over in the Lagrangian of these fields from $\partial/\partial x$ to the locally invariant derivative

$$\nabla = \frac{\partial}{\partial x} - i\varepsilon \mathbf{TB}.$$

We thus see that if we represent the electron and neutrino as a charge triplet and apply the principle of local invariance with respect to rotations in charge space, the charged components of the compensating field may be identified with intermediate particles in weak interaction, and the neutral component with the electromagnetic field. However, in order to bring this scheme close to actuality as it exists in weak and electromagnetic interactions, we must ascribe a mass to the charged components of the B-field and introduce nonconservation of parity, which destroys the invariance with respect to rotations in charge space.

8. THE LORENTZ GROUP AND THE GRAVITATIONAL FIELD

We shall now show, as was done by Utiyama[2] in his work, that if we assume that the parameters of the Lorentz group depend on the coordinates, then in order to preserve invariance we must, as in the general case, have a compensating field which is just the gravitational field. Let us consider some field $Q^A(x)$, whose action $I = \int L(Q^A, Q^A{}_{,k}) d^4x$ is invariant with respect to Lorentz transformations. (Here we use the usual abbreviated notation $Q^A{}_{,k} = \partial Q^A / \partial x_k$.) In addition to the inertial coordinate system x_k, we introduce the curvilinear system u_μ. Then the interval is given in both systems by the metric tensor

$$ds^2 = g_{ik} dx^i dx^k = g_{\mu\nu} du^\mu du^\nu.$$

Latin indices refer to the inertial system and Greek indices to the curvilinear:

$$g_{11} = g_{22} = g_{33} = -g_{44} = 1,$$

$$g_{ik} = 0 \quad \text{for} \quad i \neq k \text{ and } g_{\mu\nu} = \frac{\partial x^i \partial x^k}{\partial u_\mu \partial u_\nu} g_{ik}.$$

We introduce, as functions of the curvilinear coordinates, the quantities

$$h^k_\mu(u) = \frac{\partial x^k}{\partial u_\mu} \text{ and } h^\mu_k(u) = \frac{\partial u^\mu}{\partial x^k}. \quad (8.1)$$

In curvilinear coordinates the action is written as

$$I = \int L(Q^A(u), Q^A{}_{,\mu}\, h^\nu_\mu) h d^4u, \text{ where } h = \det |h^k_\mu|.$$

These quantities enable us at each point, with given curvilinear coordinates u^μ, to go over to the inertial system. At each such point we can carry out a Lorentz transformation

$$x^k \longrightarrow x^k + \varepsilon^k_l x^l, \quad \varepsilon^{kl} = -\varepsilon^{lk}; \quad (8.2)$$

where

$$h^\mu_k \longrightarrow h^\mu_k + \delta h^\mu_k, \quad \delta h^\mu_k = -\varepsilon^l_k h^\mu_l, \quad \delta h^k_\mu = \varepsilon^k_l \delta h^l_\mu,$$

while the field components are subjected to the transformation

$$\delta Q^A = \frac{1}{2} T^A_{klB} Q^B \varepsilon^{kl}. \quad (8.3)$$

Here T^A_{klB} is an infinitestimal operator corresponding to one of the representations of the Lorentz group, depending on the nature of the field Q. The infinitesimal operators satisfy the commutation relations

$$[T_{kl} T_{mn}] = \frac{1}{2} f^{ab}_{kl\,mn} T_{ab}. \quad (8.4)$$

The coefficients f^{ab}_{klmn} characterize the Lorentz group and do not depend on the representation. The quantities h^k_μ are related by the equations

$$\frac{\partial h^k_\mu}{\partial u^\nu} = \frac{\partial u^k_\nu}{\partial h_\mu}, \quad (8.5)$$

which are simply the equating of the mixed derivatives $\partial^2 x^k/\partial u^\mu \partial u^\nu$, which are preserved under Lorentz transformations. Suppose now that the parameters ϵ^{ik} depend on the curvilinear coordinates of the point at which we carry out the Lorentz transformation, $\epsilon^{ik} = \epsilon^{ik}(u)$. Correspondingly

$$\delta Q^A = \frac{1}{2} \varepsilon^{kl}(u) T^A_{klB} Q^B \text{ and } \delta h^k_\mu = \varepsilon^k_l(u) h^l_\mu. \quad (8.6)$$

The field Lagrangian will no longer be invariant with respect to such transformations and we must introduce, in accordance with the general rule, a compensating field $A^{kl}_\mu(u)$. We note immediately that $A^{kl}_\mu(u) = -A^{kl}_\mu(u)$, since the indices k and l number the axes of the inertial coordinate system, and therefore A^{kl}_μ and A^{lk}_μ correspond to the same rotation, but in opposite directions. The transformation properties of the A-field are obtained by applying the general formula (3.21):

$$\delta A^{kl}_\mu = \frac{1}{4} f^{kl}_{hg} A^{ml}_\mu \varepsilon^{ab}(u) A^{hg}_\mu + \frac{\partial \varepsilon^{kl}}{\partial u_\mu}.$$

Using the properties of the structure constants of the Lorentz group (antisymmetry in all three pairs of indices, equality to zero when there are four different indices, and equality to one or minus one when in each pair there is one index from each of the other two pairs), we get

$$\delta A^{kl}_\mu = \varepsilon^k_\mu A^{ml}_\mu + \varepsilon^l_\mu A^{km}_\mu + \frac{\partial \varepsilon^{kl}}{\partial u_\mu}. \quad (8.7)$$

The Lagrangian of the field Q remains invariant under the action of transformation (8.6), if we go over in the Lagrangian from the ordinary derivatives $\partial/\partial u_\mu$ to invariant derivatives in accordance with the general rule (3.1):

$$\frac{\partial Q^A}{\partial u_\mu} \to \frac{\partial Q^A}{\partial u_\mu} - \frac{1}{2} A^{kl}_\mu T^A_{klB} Q^B. \quad (8.8)$$

When we have a spatial dependence $\epsilon^{ik}(u)$, the relation (8.5) is no longer satisfied. Satisfying this condition would mean that we can always go over from curvilinear to inertial coordinate systems, i.e., although we may have used the curvilinear coordinate system, the space is actually flat. (All the components of the Riemann-Christoffel curvature tensor $R_{ij,kl} = 0$.) Violation of condition (8.5) means that the space is curved. The curvature of the space, as we know, indicates the presence of a gravitational field. The metric of a space curved because of the dependence $\epsilon^{ik}(u)$ is given by the metric tensor

$$g_{\mu\nu}(u) = h^k_\mu(u) h_{k\nu}(u).$$

It remains only to find the connection between the metric tensor $g_{\mu\nu}$ and the components of the compensating field. To do this we consider the tensor field Q^{kl}. We form the invariant derivative

$$\nabla_\mu Q^{kl} = \frac{\partial Q^{kl}}{\partial u^\mu} - A^{km}_\mu Q^l_m - A^{lm}_\mu Q^k_m. \quad (8.9)$$

Now we go over from the tensor components Q^{kl} to the components $Q^{\rho\nu} = h^\rho_k h^\nu_l Q^{kl}$. Substituting $Q^{kl} = h^k_\rho h^l_\nu Q^{\rho\nu}$ in the right-hand side of (8.9), remembering that h depends on u_μ, and then multiplying by $h^\rho_k h^\nu_l$ and using the relation $h^\mu_k h^k_\nu = \delta^\mu_\nu$, we get the expression

$$\nabla_\mu Q^{\varrho\nu} = \frac{\partial Q^{\varrho\nu}}{\partial u^\mu} + \Gamma^\varrho_{\sigma\mu} Q^{\sigma\nu} + \Gamma^\nu_{\sigma\mu} Q^{\varrho\sigma}, \; \nabla_\mu Q^{kl} = h^k_\varrho h^l_\nu \nabla_\mu Q \;, \quad (8.10)$$

where we have introduced the notation

$$\Gamma^\varrho_{\sigma\mu} = h^\varrho_l \frac{\partial h^l_\nu}{\partial u^\mu} - A^\varrho_{\nu\mu}, \; A^\varrho_{\nu\mu} = h^\varrho_k h_{l\nu} A^{kl}_\mu. \quad (8.11)$$

The invariant derivative (8.10) has the form of the usual covariant derivative with the Christoffel tensors $\Gamma^\rho_{\sigma\mu}$. We show that the expressions (8.11) were correctly called Christoffel symbols:

$$\nabla_\mu g^{\varrho\nu} = h^\varrho_k h^\nu_l \nabla_\mu g^{kl}.$$

From this equation

$$\nabla_\mu g^{\varrho\nu} = \frac{\partial g^{\varrho\nu}}{\partial u_\mu} + \Gamma^\varrho_{\sigma\mu} g^{\sigma\mu} + \Gamma^\nu_{\sigma\mu} g^{\varrho\sigma} = 0.$$

If we assume symmetry of $\Gamma^\rho_{\sigma\mu}$ in the lower indices, then we arrive at the usual expression for the Christoffel symbols in terms of the metric tensor

$$\Gamma^\varrho_{\mu\nu} = \frac{1}{2} g^{\varrho\sigma} \left(\frac{\partial g_{\sigma\mu}}{\partial u^\nu} + \frac{\partial g_{\nu\sigma}}{\partial u^\mu} - \frac{\partial g_{\mu\nu}}{\partial u^\sigma} \right). \quad (8.12)$$

Thus we have shown that a dependence of the parameters determining the Lorentz group on the curvilinear coordinates u_i leads to the necessity for introducing a compensating field A^{kl}_μ for preserving the invariance. On the other hand, this dependence results in a curvature of the space. The metric tensor and the Christoffel symbols determining this spatial curvature and characterizing the gravitational field are related to the compensating field A^{kl}_μ by the relations (8.11) and (8.12). Thus the compensating field A^{kl}_μ is nothing other than the gravitational field. Introducing the Lagrangian and the action of the field A, Utiyama shows that one can, by using variational principles, also obtain the gravitational field equation.[2]

9. CONCLUSION

Thus the principle of local invariance leads to the appearance of compensating fields. To each type of transformation there corresponds its own field. These fields have two general properties. They must have a tensor dimensionality which is not lower than unity, i.e., they cannot be scalar, but must be vector or tensor fields. They must not have a rest mass, since a term containing the rest mass would violate the local invariance.

The very principle of local invariance is a logical requirement on the Lagrangian and the field equations. In fact, why should one retain invariance with respect to transformations which are the same over all space-time? Why should the transformation at one point have to depend on how the transformation is carried out at some other point, especially if these points are separated by a space-like interval? Invariance with respect to phase transformations means non-measurability of the phase of the wave function of a charged particle. Local invariance means, in this case, as pointed out by Yang and Lee, that the phase difference is not measurable for wave functions at different points in space-time, i.e., these quantities are obviously no more observable than the phase itself.

In the scheme of compensating fields one includes in a natural way the gravitational and electromagnetic fields. These fields correspond to transformations with respect to which we have an absolutely exact invariance. Application of the principle of local invariance to transformations of the type of isotopic transformations leads to the introduction of charged fields without rest mass. In order that they may serve as carriers of the already known interactions, weak or strong, they must have a rest mass. But then we violate the local invariance. The situation is analogous to that which arises in the case of the spirality transformation $\psi \to \gamma_5\psi$. Only the free neutrino field is invariant with respect to this transformation. The weak interactions are also invariant with respect to it, but only when we omit the expressions which contain the mass of the weakly interacting particles. The absence of a rest mass for the compensating fields would mean that these fields are completely analogous to the Coulomb field and are in no way related to the short-range forces of weak and strong interactions, and in general, as pointed out by Lee and Yang (cf. Sec. 5) have practically no effect.

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