LINEAR ELECTROMAGNETIC PHENOMENA IN A PLASMA

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THE theory of linear electromagnetic processes in a plasma is now in a state of intense flux. These processes can, to a certain extent, be set in correspondence with oscillations and waves, but these, naturally, do not cover all the possible electromagnetic processes in a plasma. We are deeply convinced that the theory of linear processes is the foundation of plasma electrodynamics, and the theory of many nonlinear electromagnetic phenomena is based on it. In the present review we pay principal attention to those linear processes which, in our opinion, are the basis for the nonlinear electrodynamics of a plasma.

The spatial dispersion of the dielectric constant* is important in many cases of linear plasma electrodynamics. The general problems involved in linear electrodynamics of media with spatial dispersion are discussed in our earlier review.[3]

Certain linear electromagnetic processes in an equilibrium Maxwellian electron-ion plasma are treated in [1] on the basis of such an electrodynamic theory. In the present review we pay little attention to such a plasma, but consider the electromagnetic properties of plasma under rather general assumptions concerning the particle-velocity distribution function. Particular attention will be paid here to linear electromagnetic phenomena in a nonequilibrium plasma.

1. THE COMPLEX DIELECTRIC CONSTANT TENSOR OF A PLASMA

In linear electrodynamics, electromagnetic properties of a medium are defined with the aid of the material equation[1,3]

$$D_i'(\mathbf{r}, t) = \int_{-\infty}^{t} dt' \int d\mathbf{r}' \hat{\varepsilon}_{ij}(t-t', \mathbf{r}, \mathbf{r}') E_j(\mathbf{r}', t'). \quad (1.1)$$

For spatially-homogeneous media, the kernel of the integral equation (1.1) depends on the difference $\mathbf{r}-\mathbf{r}'$. It is convenient in this case to represent the electromagnetic field by a Fourier integral in the form of a set of plane monochromatic waves $\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)$. For such waves, relation (1.1) becomes

$$D_i'(\omega, \mathbf{k}) = \varepsilon_{ij}(\omega, \mathbf{k}) E_j(\omega, \mathbf{k}), \quad (1.2)$$

where

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \int_0^{\infty} dt \int d\mathbf{r} e^{-i\mathbf{k}\mathbf{r}+i\omega t} \hat{\varepsilon}_{ij}(t, \mathbf{r}) \quad (1.3)$$

is the complex dielectric-constant tensor, which characterizes the electromagnetic properties of the medium. The dependence of the tensor $\epsilon_{ij}(\omega, \mathbf{k})$ on the frequency defines the frequency dispersion of the dielectric constant, while the dependence on the wave vector defines the spatial dispersion.

To calculate the dielectric constant tensor explicitly we must specify a definite model of the medium. A particular advantage of plasma from the theoretical point of view is that it comprises a system of weakly-interacting particles:

$$e^2 N^{1/3} \ll \langle \mathscr{E} \rangle \sim \varkappa T.$$

The presence of the small parameter makes it possible to formulate the plasma theory. A plasma can be conveniently described by means of a kinetic equation with self-consistent field

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\frac{\partial f_\alpha}{\partial \mathbf{r}} + e_\alpha \left\{ \mathbf{E} + \frac{1}{c}[\mathbf{vB}] \right\} \frac{\partial f_\alpha}{\partial \mathbf{p}} = \left(\frac{\partial f_\alpha}{\partial t} \right)_{st}. \quad (1.4)^*$$

Using the kinetic equation (1.4) we can obtain an expression for the dielectric constant tensor of the plasma. In the case of a plasma without strong fields, the dielectric constant tensor has, when particle collisions can be neglected, the form†

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \delta_{ij} + \sum_\alpha \frac{4\pi e_\alpha^2}{\omega} \int d\mathbf{p} \frac{v_i}{\omega - \mathbf{kv}} \frac{\partial f_{0\alpha}}{\partial P_l} \left\{ \left(1 - \frac{\mathbf{kv}}{\omega}\right) \delta_{jl} + \frac{v_j k_l}{\omega} \right\}, \quad (1.5)$$

where $f_{0\alpha}(\mathbf{p})$ is the distribution function of particles of kind α in the ground state of the plasma. The integral in the right half of (1.5) is of the Cauchy type, and therefore $\epsilon_{ij}(\omega, \mathbf{k})$ has, as a function of the complex variable ω, singularities only on the integration contour,‡ i.e., for real values of ω.

For a plasma in a strong magnetic field $\mathbf{B}_0$, neglecting particle collisions, we obtain from (1.4)[1]

*Naturally, spatial dispersion can be neglected for many phenomena that can occur in a plasma. The electrodynamics of such phenomena is treated elsewhere (see, for example, the books [1,2]) and will therefore not be discussed in detail here.

*$[\mathbf{vB}] = \mathbf{v} \times \mathbf{B}$.

†No general expression whatever has been derived so far for $\varepsilon_{ij}(\omega, \mathbf{k})$ with account of the particle collisions. A consistent account of particle collision can be made only when the spatial dispersion can be neglected.[1,4]

‡Strictly speaking, this situation prevails only if account is taken of the fact that the particle velocity cannot exceed the velocity of light. On the other hand, say in the case of a nonrelativistic Maxwellian distribution, $\varepsilon_{ij}(\omega, \mathbf{k})$ has a singularity in the vicinity of an infinitely remote point.

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \delta_{ij} - \sum_\alpha \frac{4\pi e_\alpha^2}{\omega} \int dp v_i \int_0^\infty d\tau e^{i\mathbf{k}\mathbf{R}(\tau)+i\omega\tau}$$

$$\times \left\{ \left(1 - \frac{\mathbf{k}\mathbf{V}(\tau)}{\omega}\right)\delta_{jl} + \frac{V_j(\tau)k_l}{\omega} \right\} \frac{\partial f_{0\alpha}(\mathbf{P})}{\partial P_l}, \quad (1.6)$$

where

$$\mathbf{P}(\tau) = \frac{\mathscr{E}_\alpha}{c^2}\mathbf{V}(\tau), \quad \mathbf{R}(\tau) = -\int_0^\tau \mathbf{V}(\tau')\, d\tau', \quad \Omega_\alpha = \frac{e_\alpha c B_0}{\mathscr{E}_\alpha},$$

$$\mathbf{V}(\tau) = \frac{\mathbf{B}_0(\mathbf{v}\mathbf{B}_0)}{B_0^2} + \frac{[\mathbf{B}_0[\mathbf{v}\mathbf{B}_0]]}{B_0^2}\cos\Omega_\alpha\tau - \frac{[\mathbf{v}\mathbf{B}_0]}{B_0}\sin\Omega_\alpha\tau.$$

In the absence of an external magnetic field $(\mathbf{B}_0 \to 0)$ we have

$$\mathbf{V}(\tau) = \mathbf{v}, \quad \mathbf{R}(\tau) = -\mathbf{V}\tau, \quad \mathbf{P}(\tau) = \mathbf{p}.$$

The dielectric constant tensor (1.6) coincides in this case with (1.5). With the aid of (1.5) and (1.6) we can calculate specific expressions for the dielectric constant of a plasma, using the various expressions for the particle distribution functions $f_{0\alpha}(\mathbf{p})$.

2. ELECTROMAGNETIC PROPERTIES OF AN ISOTROPIC PLASMA

In an isotropic plasma the particle velocity distribution function in the ground state, $f_{0\alpha}(\mathbf{p})$, is a function of the absolute value of the momentum p. The dielectric constant tensor of the plasma (1.5) can be represented in this case by

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\varepsilon^{tr}(\omega, k) + \frac{k_i k_j}{k^2}\varepsilon^l(\omega, k). \quad (2.1)$$

The longitudinal and transverse dielectric constants, ϵ^l and ϵ^{tr} respectively, are given in accord with (1.5) by

$$\left.\begin{aligned} \varepsilon^l(\omega, k) &= 1 + \sum_\alpha \frac{4\pi e_\alpha^2}{\omega k^2} \int dp \frac{\partial f_{0\alpha}}{\partial \mathscr{E}_\alpha}(\mathbf{kv})^2 \left[\frac{P}{\omega - \mathbf{kv}} - i\pi\delta(\omega - \mathbf{kv})\right], \\ \varepsilon^{tr}(\omega, k) &= 1 + \sum_\alpha \frac{2\pi e_\alpha^2}{\omega k^2} \int dp \frac{\partial f_{0\alpha}}{\partial \mathscr{E}_\alpha}(\mathbf{kv})^2 \left[\frac{P}{\omega - \mathbf{kv}} - i\pi\delta(\omega - \mathbf{kv})\right], \end{aligned}\right\} \quad (2.2)$$

where the symbol P denotes that the singularity at the point $\omega = \mathbf{k}\cdot\mathbf{v}$ must be taken in the sense of principal value. Expressions (2.2) can be readily integrated with respect to the angles. We give here the formulas for the imaginary parts of $\epsilon^l(\omega, k)$ and $\epsilon^{tr}(\omega, k)$, characterizing the absorption of the longitudinal and transverse waves in a plasma

$$\left.\begin{aligned} \varepsilon^{l''}(\omega, k) &= -\sum_\alpha \frac{8\pi^3 e_\alpha^2 \omega}{|k|^3 c^4} \begin{cases} \int_{\mathscr{E}_{1\alpha}}^\infty d\mathscr{E}_\alpha \mathscr{E}_\alpha^2 \frac{\partial f_{0\alpha}}{\partial \mathscr{E}_\alpha} & \text{for } \omega^2 < k^2c^2, \\ 0 & \text{for } \omega^2 > k^2c^2, \end{cases} \\ \varepsilon^{tr''}(\omega, k) &= \\ -\sum_\alpha \frac{4\pi^3 e_\alpha^2 m_\alpha^2 c^2}{\omega |k|} &\begin{cases} \int_{\mathscr{E}_{1\alpha}}^\infty d\mathscr{E}_\alpha \frac{\partial f_{0\alpha}}{\partial \mathscr{E}_\alpha}\left[\frac{\mathscr{E}_\alpha^2}{\mathscr{E}_{1\alpha}^2} - 1\right] & \text{for } \omega^2 < k^2c^2, \\ 0 & \text{for } \omega^2 > k^2c^2, \end{cases} \end{aligned}\right\} \quad (2.3)$$

where $\mathscr{E}_{1\alpha} = m_\alpha c^2/\sqrt{1 - \omega^2/k^2c^2}$. It follows from these formulas, in particular, that electromagnetic waves with phase velocities greater than the velocity of light c will not be absorbed in an isotropic plasma, since $\epsilon^{l''}$ and $\epsilon^{tr''}$ vanish when $\omega^2 > k^2c^2$.

In the low-frequency limit $\omega/k \to 0$ we obtain from (2.2)

$$\left.\begin{aligned} \varepsilon^l(0, k) &= 1 + \frac{1}{k^2 r_{scr}^2}, \quad r_{scr}^{-2} = -\sum_\alpha 4\pi e_\alpha^2 \int dp \frac{\partial f_{0\alpha}}{\partial \mathscr{E}_\alpha}, \\ \varepsilon^{tr}(0, k) &= 1 + \frac{4\pi i}{\omega}\frac{C}{|k|}, \\ C &= -\sum_\alpha \pi^2 m_\alpha^2 e_\alpha^2 c^2 \int_{m_\alpha c^2} d\mathscr{E}_\alpha \frac{\partial f_{0\alpha}}{\partial \mathscr{E}_\alpha}\left(\frac{\mathscr{E}_\alpha}{m_\alpha c^2} - 1\right). \end{aligned}\right\} \quad (2.4)$$

The function $\epsilon^l(0, k)$ characterizes the behavior of the static field of a charge in the plasma at large distances. The quantity r_{scr}, on the other hand, is the screening radius of the scalar potential of the field of a point charge [1]

$$\Phi(\mathbf{r}) = \frac{e}{|\mathbf{r} - \mathbf{r}_0|} e^{-\frac{|\mathbf{r} - \mathbf{r}_0|}{r_{scr}}}. \quad (2.5)$$

In the case of a Maxwellian plasma, r_{scr} coincides with the Debye screening radius. [3]

It follows from (2.4) that in the absence of particle collisions the transverse dielectric constant of an isotropic plasma has a pole $\sim 1/\omega$ at low frequencies $(\omega \to 0)$. This corresponds to the presence of a finite conductivity, due to the Cerenkov dissipation of the waves in the plasma. We note that the transverse dielectric constant (2.4) corresponds to the frequency range of the anomalous skin effect in a plasma. [5,6]

At high frequencies $k/\omega \to 0$ $(\omega \gg k\langle v_{e,i}\rangle)$, when spatial dispersion can be neglected, we obtain

$$\varepsilon^l(\omega, 0) = \varepsilon^{tr}(\omega, 0) = 1 - \frac{\omega_0^2}{\omega^2}, \quad (2.6)$$

where

$$\omega_0^2 = \sum_\alpha \omega_{0\alpha}^2 = -\sum_\alpha \frac{4\pi e_\alpha^2}{3} \int dp v^2 \frac{\partial f_{0\alpha}}{\partial \mathscr{E}_\alpha}.$$

In the limit k = 0 the imaginary part of the dielectric constant tensor vanishes. However, if we take into account the weak spatial dispersion due to the thermal motion of the particles in the plasma, we obtain from (2.3) a small but nonvanishing imaginary part for the tensor $\epsilon_{ij}(\omega, \mathbf{k})$, and this leads to absorption of electromagnetic waves in the plasma.

As is well known, both longitudinal and transverse waves, which vary with the coordinates and with the time as $\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)$, are possible in an isotropic medium. The dispersion equations that establish the connection between the frequency and the wave vector $\mathbf{k}$ in longitudinal and transverse waves have respectively the form [1,3]

$$\varepsilon^l(\omega, k) = 0, \quad (2.7)$$

$$k^2 - \frac{\omega^2}{c^2}\varepsilon^{tr}(\omega, k) = 0. \quad (2.8)$$

At high frequencies, when the spatial dispersion of the dielectric constant is weak, we obtain for the spectra of the longitudinal and transverse oscillations

$$\omega^2 = \omega_0^2, \tag{2.9}$$

$$\omega^2 = \omega_0^2 + k^2c^2. \tag{2.10}$$

It follows therefore that in an isotropic plasma the phase velocity of the transverse waves is greater than the velocity of light, and these waves are consequently not attenuated (neglecting particle collisions). As regards the longitudinal waves, they likewise do not attenuate when $\omega_0^2 > k^2c^2$, but do attenuate when $\omega_0^2 < k^2c^2$, and in the case of the spectrum (2.9) the decrement is $(\omega \to \omega - i\gamma)$

$$\gamma = \frac{\varepsilon^{l''}(\omega, k)}{\frac{\partial}{\partial\omega}\varepsilon^{l'}(\omega, k)} = \frac{1}{2}\omega_0\varepsilon^{l''}(\omega_0, k). \tag{2.11}$$

In accordance with (2.3), the imaginary part of the longitudinal dielectric constant $\epsilon^{l''}(\omega, k)$ is determined by the particle distribution in the plasma. We can therefore determine in principle the particle velocity distribution by measuring the decrement of the longitudinal waves. It must be noted that the ions make practically no contribution to the imaginary part $\epsilon^{l''}(\omega, k)$. This method therefore can yield only the plasma electron distribution function.

At low frequencies, when the spatial dispersion of the dielectric constant plays a decisive role, the electromagnetic waves are greatly attenuated in the isotropic plasma. There is, however, a case when low-frequency longitudinal waves attenuate little in an isotropic plasma. We refer here to non-isothermal electron-ion plasma, in which the mean thermal energy of the electrons is much greater than that of the ions. In this case, at phase velocities intermediate between the ion and electron thermal velocities $(k\langle v_i\rangle \ll \omega \ll k\langle v_e\rangle)$, we have

$$\varepsilon^{l'}(\omega, k) = 1 - \frac{\omega_{0i}^2}{\omega^2} + \frac{\omega_{-1e}^2}{k^2\langle v_e^2\rangle}, \tag{2.12}$$

where

$$\omega_{-1e}^2 = -4\pi e^2\langle v_e^2\rangle \int dp\frac{\partial f_{0e}}{\partial \mathcal{E}}; \quad \langle v_e^m\rangle = \int = dpv^m f_{0e}.$$

The imaginary part of the longitudinal dielectric constant is in this case a small quantity

$$\varepsilon^{l''}(\omega, k) = \frac{\pi}{2}\frac{\omega\tilde{\omega}_{0e}^2}{|k^3|\langle v_e^3\rangle}, \tag{2.13}$$

where

$$\tilde{\omega}_{0e}^2 = -4\pi e^2\langle v_e^3\rangle \int \frac{dp}{v}\frac{\partial f_{0e}}{\partial \mathcal{E}}.$$

We note that in accordance with (2.13) the wave absorption in the non-isothermal plasma is due to the thermal motion of the electrons. The expressions for the frequency and for the decrement of the waves are

$$\omega = \frac{\omega_{0i}k\langle v_e^2\rangle^{1/2}}{\sqrt{\omega_{-1e}^2 + k^2\langle v_e^2\rangle}}, \quad \gamma = \frac{\omega^3}{2\omega_{0i}^2}\varepsilon^{l''}(\omega, k). \tag{2.14}$$

The corresponding formulas for a Maxwellian plasma are given in [1].

In conclusion, we discuss briefly the concept of permeability of a plasma. It is frequently stated that the permeability of a plasma is equal to unity. Actually, however, the concept of permeability is not fully defined. The permeability of an isotropic medium is [1,3]

$$\mu(\omega, k) = \left\{1 - \frac{\omega^2}{c^2k^2}[\varepsilon^l(\omega, k) - \varepsilon^{tr}(\omega, k)]\right\}^{-1}. \tag{2.15}$$

The static permeability of a medium is defined here as the limit

$$\mu_k(0, 0) = \lim_{k\to 0}\lim_{\omega/k\to 0}\mu(\omega, k).$$

We can readily see that $\mu(\omega, k)$ for an isotropic plasma differs from unity, whereas $\mu_k(0, 0) = 1$. That the concept of permeability has a restricted meaning can be seen from the following formula, which holds true for a Maxwellian electron plasma:

$$\frac{\mu''(\omega, k)}{|\mu(\omega, k)|^2} = \sqrt{\frac{\pi}{2}}\frac{\omega_{0e}^2\omega}{c^2|k|^3}\sqrt{\frac{m_e}{\varkappa T_e}}\left(1 - \frac{\omega^2 m_e}{k^2\varkappa T_e}\right)e^{-\frac{m_e\omega^2}{2k^2\varkappa T_e}} \tag{2.16}$$

At low frequencies, when $\omega < k\sqrt{\kappa T_e/m_e}$, the imaginary part of the permeability is positive, $\mu''(\omega, k) > 0$ (as is usually assumed in electrodynamics when spatial dispersion is neglected [7,8]). To the contrary, at high frequencies $(\omega > k\sqrt{\kappa T_e/m_e})$, corresponding to weak spatial dispersion, the imaginary part of the permeability is negative, $\mu''(\omega, k) < 0$. This is not surprising, however, since the imaginary part of the permeability does not determine the absorption in the medium at all.

3. ANISOTROPIC PLASMA IN THE ABSENCE OF STRONG FIELDS

In the absence of strong fields produced by external sources, a plasma can be anisotropic because the particle velocity distribution is anisotropic in the ground state. One can readily visualize a great variety of examples of anisotropic plasma without external strong fields. We consider here the case of greatest practical interest, that of beam anisotropy of a plasma. By this we mean a plasma in which a definite quasi-neutral group of particles has along with thermal motion also directed motion with constant velocity **u**. This involves frequently interaction between an unbounded quasi-neutral beam of charged particles and the plasma. The dielectric constant tensor of such a plasma can be calculated with the aid of (1.5). We give here, however, a different method of determining the dielectric constant tensor of an anisotropic plasma consisting of an unbounded beam of charged particles (moving plasma) in a plasma at rest. Using the additivity of the conductance as a function of the plasma density, we can write

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \varepsilon_{ij}^{(1)}(\omega, \mathbf{k}) + \varepsilon_{ij}^{(2)}(\omega, \mathbf{k}) - \delta_{ij}, \qquad (3.1)$$

where $\epsilon_{ij}^{(1)}(\omega, \mathbf{k})$ and $\epsilon_{ij}^{(2)}(\omega, \mathbf{k})$ are the dielectric constant tensors of the beam and of the stationary plasma, respectively, in a laboratory frame fixed in the stationary plasma. The expression for $\epsilon_{ij}^{(2)}(\omega, \mathbf{k})$ is obviously determined from (2.1) and (2.2). To obtain the dielectric constant of the beam in the laboratory frame we use the Lorentz transformation formulas for the induced current and charge densities. As a result we obtain[1,9]

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \frac{\omega'}{\omega} \alpha_{im} \{\varepsilon_{ml}^{(1)\prime}(\omega', \mathbf{k}') - \delta_{ml}\} \beta_{lj} + \varepsilon_{ij}^{(2)}(\omega, \mathbf{k}), \qquad (3.2)$$

where $\epsilon_{ij}^{(1)}(\omega', \mathbf{k}')$ is the dielectric constant tensor of the beam in the moving system of coordinates connected with the beam, determined by formulas such as (2.1) and (2.2). It must be borne in mind that in a moving system of coordinates the particle density is determined with account of the Lorentz contraction of the volume. The quantities ω' and $\mathbf{k}'$ are the transformed frequency and wave vector

$$\omega' = \frac{\omega - \mathbf{uk}}{\sqrt{1 - u^2/c^2}},$$

$$\mathbf{k}' = \mathbf{k} + \mathbf{u} \frac{\mathbf{uk}\,(1 - \sqrt{1 - u^2/c^2}) - \omega u^2/c^2}{u^2 \sqrt{1 - u^2/c^2}}. \qquad (3.3)$$

The tensors α_{ij} and β_{ij} have the form

$$\alpha_{ij} = \delta_{ij} + \frac{1}{\sqrt{1 - u^2/c^2}} \left[\frac{u_i u_j}{u^2} (1 - \sqrt{1 - u^2/c^2}) + \frac{u_i k_j'}{\omega'} \right],$$

$$\beta_{ij} = \frac{\omega'}{\omega} \delta_{ij} + \frac{1}{\sqrt{1 - u^2/c^2}} \left[\frac{u_i u_j}{u^2} (\sqrt{1 - u^2/c^2} - 1) + \frac{k_i u_j}{\omega} \right]. \qquad (3.4)$$

In the nonrelativistic beam velocity limit, $u \ll c$, formulas (3.3) and (3.4) simplify to

$$\omega' \to \omega - \mathbf{uk}, \qquad \mathbf{k}' \to \mathbf{k},$$

$$\alpha_{ij} \to \delta_{ij} + \frac{u_i k_j}{\omega'}, \quad \beta_{ij} \to \frac{\omega'}{\omega} \delta_{ij} + \frac{k_i u_j}{\omega}.$$

For the dielectric constant tensor of the plasma we obtain in this limit

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \varepsilon_{ij}^{(2)}(\omega, \mathbf{k}) + \frac{\omega'^2}{\omega^2} [\varepsilon_{ij}^{(1)\prime}(\omega', \mathbf{k}) - \delta_{ij}]$$
$$+ \frac{k^2 u_i u_j}{\omega^2} \left(\frac{k_l \varepsilon_{lm}^{(1)\prime}(\omega', \mathbf{k})\, k_m}{k^2} - 1 \right) + \frac{\omega'}{\omega} [u_i k_l \varepsilon_{lj}^{(1)\prime}(\omega', \mathbf{k})$$
$$+ u_j k_l \varepsilon_{jl}^{(1)\prime}(\omega', \mathbf{k}) - u_i k_j - u_j k_i]. \qquad (3.6)$$

Expression (3.6) is also readily derived from (2.5).

A characteristic feature of our anisotropic plasma, which contains beams of charged particles, is instability with respect to small electromagnetic oscillations of the system. As is well known, a charged particle moving in a medium radiates electromagnetic waves. If not one charged particle but a whole beam moves in the medium, the collective radiation of the beam particles may cause the electromagnetic field in the medium to increase with time. This means that a system comprising a medium and a beam of charged particles (in our case, the anisotropic plasma) is unstable and decays in time.

In determining the interaction between a beam of charged particles and a plasma it is customary to solve in each specific case a system of kinetic equations for the plasma and for the beam. There is no need for this, however. It is sufficient to substitute in the dispersion equation for the electromagnetic waves in an anisotropic medium[1,3]

$$\left| k^2 \delta_{ij} - k_i k_j - \frac{\omega^2}{k^2} \varepsilon_{ij}(\omega, \mathbf{k}) \right| = 0 \qquad (3.7)$$

the expression (3.2) for the dielectric constant tensor. Unstable electromagnetic oscillations of the system correspond to solutions of the dispersion equation (3.7) with positive imaginary part, $\operatorname{Im} \omega(\mathbf{k}) > 0$, which is also called the buildup increment of the waves in the plasma. In the absence of strong fields the dispersion equation (3.7) for our anisotropic plasma breaks up into two equations

$$k^2 - \frac{\omega^2}{c^2} \varepsilon^{(2)\mathrm{tr}}(\omega, k) - \frac{\omega'^2}{c^2} [\varepsilon^{(1)\prime\mathrm{tr}}(\omega', k') - 1] = 0, \qquad (3.8)$$

$$\left\{ k^2 - \frac{\omega^2}{c^2} \varepsilon^{(2)\mathrm{tr}}(\omega, k) - \frac{\omega'^2}{c^2} [\varepsilon^{(1)\prime\mathrm{tr}}(\omega', k') - 1] \right\} [\varepsilon^{(2)\mathrm{l}}(\omega, k)$$
$$+ \varepsilon^{(1)\prime\mathrm{l}}(\omega', k') - 1] - \frac{k^2 u^2 - (\mathbf{uk})^2}{c^2 (1 - u^2/c^2)} \Big[\varepsilon^{(1)\prime\mathrm{l}}(\omega', k')$$
$$- 1 + \frac{\omega'^2}{c^2 k'^2} \left(\varepsilon^{(1)\prime\mathrm{tr}}(\omega', k') - \varepsilon^{(1)\prime\mathrm{l}}(\omega', k') \right) \Big]$$
$$\times \left[\varepsilon^{(2)\mathrm{l}}(\omega, k) - 1 + \frac{\omega^2}{c^2 k^2} \left(\varepsilon^{(2)\mathrm{tr}}(\omega, k) - \varepsilon^{(2)\mathrm{l}}(\omega, k) \right) \right] = 0. \qquad (3.9)$$

The first of these equations is the analog of the dispersion equation for transverse waves (in the absence of a beam it goes over into the equation for transverse waves), while the second is the analog of the equation for longitudinal waves in a plasma. It can be shown that the only solutions of (3.8) correspond to damped oscillations of the system.[1] Being interested in unstable oscillations only we proceed now to analyze (3.9). For the sake of simplicity we confine ourselves to nonrelativistic beam velocities $u \ll c$.* Equation (3.9) assumes in this limit the form

$$\varepsilon^{(1)\mathrm{l}}(\omega - \mathbf{uk}, k) + \varepsilon^{(2)\mathrm{l}}(\omega, k) - 1 = 0. \qquad (3.10)$$

Unstable oscillations must obviously be expected first at frequencies ω ($\omega' = \omega - \mathbf{u} \cdot \mathbf{k}$) for which the imaginary parts of $\epsilon^{(1)l}(\omega', k)$ and $\epsilon^{(2)l}(\omega, k)$ are negligibly small compared with their real parts. It is precisely in this region of frequencies ω and ω' that weakly attenuating longitudinal waves, the excitation of which can cause the system to become unstable, exist in the beam and in the stationary plasma. This situation obtains at high frequencies, when the thermal motion of the particles can be neglected, and

*See [1] for the interaction between a relativistic beam and a plasma.

also in a non-isothermal plasma at frequencies $k\langle v_i\rangle \ll \omega$ and $\omega' \ll k\langle v_e\rangle$, when the thermal motion of the electrons is significant. Expressions such as (2.6) and (2.12) for the beam and for the stationary plasma determine the entire manifold of unstable oscillations of the system in the absence of an external magnetic field. It is possible to obtain here both high-frequency instabilities, corresponding to excitation of electron oscillations, and low-frequency instabilities, corresponding to excitation of plasma sound in the system.

At high frequencies ω and ω', when the particle thermal motion can be neglected, Eq. (3.10) assumes the form

$$1-\frac{\omega_{01}^2}{(\omega-\mathbf{uk})^2}-\frac{\omega_{02}^2}{\omega^2}=0, \qquad (3.11)$$

where ω_{01} and ω_{02} characterize the densities of the beam and of the plasma, respectively.* In the case of low beam densities, when $\omega_{01} \ll \omega_{02}$, we obtain from (3.11)

$$\omega=\mathbf{uk}\pm\omega_{01}\left(1-\frac{\omega_{02}^2}{(\mathbf{uk})^2}\right)^{-1/2}. \qquad (3.12)$$

We see therefore that when $|\mathbf{u}\cdot\mathbf{k}| < \omega_{02}$ the plasma oscillations become unstable. The maximum instability takes place when $|\mathbf{u}\cdot\mathbf{k}| \sim \omega_{02}$ and corresponds to excitation of high-frequency electron oscillations, described by the dispersion equation (2.9), in the stationary plasma.

In the case when the beam and the plasma densities are comparable ($\omega_{01} \sim \omega_{02}$), unstable high-frequency oscillations of the system correspond to the following solutions of (3.11):

$$\omega=\mathbf{uk}\frac{1\pm i\frac{\omega_{01}}{\omega_{02}}}{1+\left(\frac{\omega_{01}}{\omega_{02}}\right)^2}, \qquad (3.12')$$

which are valid when $|\mathbf{u}\cdot\mathbf{k}| \ll \omega_{01,2}$.

We now consider the case of a cold beam moving in a non-isothermal hot plasma. Neglecting the thermal motion of the beam particles, we take into account the thermal motion of the electrons in the stationary plasma. We assume here that the following inequalities are satisfied:

$$\omega' \gg k\langle v_{e,i}\rangle_1, \qquad k\langle v_i\rangle_2 \ll \omega \ll k\langle v_e\rangle_2.$$

In this case (3.10) assumes the form

$$1+\frac{\omega_{-1e2}^2}{k^2\langle v_e^2\rangle_2}-\frac{\omega_{0i2}^2}{\omega^2}-\frac{\omega_{01}^2}{(\omega-\mathbf{uk})^2}=0. \qquad (3.13)$$

In the case of small beam densities we get

$$\omega=\mathbf{uk}\pm\omega_{01}\left(1+\frac{\omega_{-1e2}^2}{k^2\langle v_e^2\rangle_2}-\frac{\omega_{0i2}^2}{(\mathbf{uk})^2}\right)^{-1/2}. \qquad (3.14)$$

It follows therefore that the system oscillations become unstable if

$$1+\frac{\omega_{-1e2}^2}{k^2\langle v_e^2\rangle_2}<\frac{\omega_{0i2}^2}{(\mathbf{uk})^2}.$$

*The index 1 pertains throughout to the beam, and the index 2 to the plasma.

We readily see that these instabilities correspond to excitation of low-frequency (sound) oscillations, described by the dispersion equation (2.14), in the stationary plasma. These oscillations can also occur at beam velocities lower than the electron thermal velocity in the stationary plasma.

In conclusion we consider a beam, representing a non-isothermal plasma, moving in a non-isothermal but stationary plasma. The thermal motion of the ions will be neglected both in the beam and in the stationary plasma and the following inequalities are assumed satisfied

$$k\langle v_i\rangle_1 \ll \omega' \ll k\langle v_e\rangle_1,$$
$$k\langle v_i\rangle_2 \ll \omega \ll k\langle v_e\rangle_2.$$

In this case the dispersion equation (3.10) must be written in the form

$$1+\frac{\omega_{-1e1}^2}{k^2\langle v_e^2\rangle_1}+\frac{\omega_{-1e2}^2}{k^2\langle v_e^2\rangle_2}-\frac{\omega_{0i2}^2}{\omega^2}-\frac{\omega_{0i1}^2}{(\omega-\mathbf{uk})^2}=0. \qquad (3.15)$$

In the case of low beam densities the approximate solution of (3.15) has the form

$$\omega=\mathbf{uk}\pm\omega_{0i1}\left(1+\frac{\omega_{-1e2}^2}{k^2\langle v_e^2\rangle_2}-\frac{\omega_{0i2}^2}{(\mathbf{uk})^2}\right)^{-1/2}. \qquad (3.16)$$

If

$$1+\frac{\omega_{-1e2}^2}{k^2\langle v_e^2\rangle_2}<\frac{\omega_{0i2}^2}{(\mathbf{uk})^2}$$

expression (3.16) describes unstable oscillations corresponding to the excitation of low-frequency waves in a stationary plasma. In this case the thermal motion of the electrons in the beam does not play an appreciable role. When the beam and plasma densities are commensurate, to the contrary, the character of the wave excitation depends appreciably on the thermal motion of the beam electrons. Indeed, unstable oscillations of the system correspond in this case to solutions of the form

$$\omega=\mathbf{uk}\left(1\pm i\frac{\omega_{0i1}}{\omega_{0i2}}\right)\left(1+\frac{\omega_{0i1}^2}{\omega_{0i2}^2}\right)^{-2}, \qquad (3.17)$$

which are valid in the region

$$1+\frac{\omega_{-1e2}^2}{k^2\langle v_e^2\rangle_2}+\frac{\omega_{-1e1}^2}{k^2\langle v_e^2\rangle_1}\ll\frac{\omega_{0i2}^2}{(\mathbf{uk})^2}.$$

It is easily seen that these instabilities are the consequence of excitation of low-frequency (sound) waves in the entire system as a whole.

The foregoing instabilities can be called hydrodynamic, since they are not connected with the imaginary part of the dielectric constant and can be described by using the equations of two-liquid and single-liquid hydrodynamics of a plasma, with dissipation neglected. One must not think that the electromagnetic oscillations of the plasma are stable if there are no hydrodynamic instabilities. The solutions of the dispersion equation (3.10), with the imaginary parts $\epsilon^{(1)l''}(\omega', k)$ and $\epsilon^{(2)l''}(\omega, k)$ neglected, are then real. An account of the imaginary parts of these quantities leads to the following expression for the attenuation decrement (or buildup increment) of the electromagnetic waves in the plasma

$$\gamma = \frac{\operatorname{Im}[\varepsilon^{(1)l}(\omega - \mathbf{uk}, k) + \varepsilon^{(2)l}(\omega, k)]}{\frac{\partial}{\partial\omega}\operatorname{Re}[\varepsilon^{(1)l}(\omega - \mathbf{uk}, k) + \varepsilon^{(2)l}(\omega, k)]}. \tag{3.18}$$

When $\gamma > 0$ the electromagnetic waves attenuate, and when $\gamma < 0$ they increase with time, and the system is unstable. These instabilities, which depend essentially on the imaginary part of the dielectric constant tensor, are called kinetic. Without going into a detailed analysis of the kinetic instabilities of the plasma,* we note merely that in accord with (2.3), (2.6), (2.12), and (2.13) the values of $\operatorname{Im}\epsilon^l(\omega, \mathrm{k})$ and $\frac{\partial}{\partial\omega}\operatorname{Re}\epsilon^l(\omega, \mathrm{k})$ are positive when $\omega > 0$ and are odd functions of the frequency. It follows therefore that the value of γ determined from (3.18) can become negative only at frequencies $\omega \lesssim \mathbf{k}\cdot\mathbf{u}$, i.e., when the beam velocity is either greater than or of the same order as the phase velocity of the wave.

From the foregoing brief analysis of the interaction between a plasma and a low-density beam of charged particles it follows that an electromagnetic wave is excited in the plasma when the beam velocity is of the same order as the phase velocity of the wave. As a consequence there are two instability regions, one at high frequencies, corresponding to excitation of electron oscillations, and one at low frequencies, corresponding to excitation of sound in the plasma.

4. ELECTROMAGNETIC WAVES IN A PLASMA SITUATED IN A STRONG MAGNETIC FIELD

In the preceding two sections we considered the electromagnetic properties of a plasma in the absence of strong fields. Under real conditions the plasma is frequently in a rather strong magnetic field. We shall therefore proceed to investigate such a plasma. We begin our analysis with a case in which the particle velocity distribution functions $f_{0\alpha}(p)$ are isotropic. In this case the dielectric constant tensor of the plasma (1.6) can be transformed into [1,10]

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \delta_{ij} + \sum_\alpha \frac{4\pi e_\alpha^2}{\omega}\int d\mathbf{p}\frac{\partial f_{0\alpha}}{\partial \varepsilon_\alpha}\sum_m \pi_{ij}^{(m)} \times\left\{\frac{P}{\omega - m\Omega_\alpha - k_z v_z} - i\pi\delta(\omega - m\Omega_\alpha - k_z v_z)\right\}, \tag{4.1}$$

where

$$\pi_{ij}^{(m)} = \begin{pmatrix} v_\perp^2\left(m\frac{J_m}{b_\alpha}\right)^2, & iv_\perp^2\left(m\frac{J_m}{b_\alpha}\right)J'_m, & v_\perp v_z m\frac{J_m^2}{b_\alpha}, \\ -iv_\perp^2\left(m\frac{J_m}{b_\alpha}\right)J'_m, & v_\perp^2(J'_m)^2, & -iv_\perp v_z J_m J'_m, \\ v_\perp v_z m\frac{J_m^2}{b_\alpha}, & iv_\perp v_z J_m J'_m, & v_z^2 J_m^2 \end{pmatrix}.$$

*For details see [40].

Here J_m and J'_m are the Bessel function of order m and its derivative, the argument being

$$b_\alpha = \frac{k_\perp v_\perp}{\Omega_\alpha}.$$

As the independent coordinate axes we choose $(x, y, z) = (\mathbf{B}\times[\mathbf{k}\times\mathbf{B}]/kB^2,\ [\mathbf{B}\times\mathbf{k}]/kB,\ \text{and}\ \mathbf{B}/B)$, and $k_\perp$ and $v_\perp$ are the projections of the corresponding vectors on the plane perpendicular to the vector **B**.

Many papers have been devoted to the investigation of the electromagnetic properties of a magnetoactive plasma (see [1] and the literature cited there). One of the most interesting phenomena in a magnetoactive plasma is cyclotron absorption of electromagnetic waves. Kudryavtsev and Trubnikov [11-13] have shown that cyclotron absorption and radiation of a plasma is essential both for the energy balance of the hot plasma and for the particle velocity distribution.

We shall consider below the electromagnetic properties of a magnetoactive plasma under rather general assumptions concerning the form of the particle velocity distribution, and indicate when the distribution functions can be determined from experiments on the absorption of electromagnetic waves in a plasma. We shall henceforth be interested essentially in a nonrelativistic plasma, when

$$\frac{k\langle v_\alpha\rangle}{\omega} = n\frac{\langle v_\alpha\rangle}{c} \ll 1,$$

where n is the refractive index. However, for a correct description of the absorption of waves with phase velocities that are comparable with the velocity of light, we shall use the exact relativistic formulas for the anti-Hermitian part of the dielectric constant tensor. With this restriction, the investigation of the dispersion equation of the electromagnetic waves

$$\left|k^2\delta_{ij} - k_i k_j - \frac{\omega^2}{c^2}\varepsilon_{ij}(\omega, \mathbf{k})\right| = 0 \tag{4.2}$$

still remains complicated, since the diagonal as well as all the nondiagonal terms of the dielectric constant remain significant. However, if the thermal motion is a small effect, or, to the contrary, the decisive effect, the situation becomes much simpler. Indeed, in the two limiting cases, when

$$\frac{k_z\langle v_\alpha\rangle}{\omega - m\Omega_\alpha} = \frac{\omega}{\omega - m\Omega_\alpha}\frac{n\langle v_\alpha\rangle}{c}\cos\vartheta \ll 1 \tag{4.2a}$$

or when

$$\frac{k_z\langle v_\alpha\rangle}{\omega - m\Omega_\alpha} = \frac{\omega}{\omega - m\Omega_\alpha}\frac{n\langle v_\alpha\rangle}{c}\cos\vartheta \gg 1, \tag{4.2b}$$

we can use in the investigation of (4.2) the following expression for the dielectric constant tensor: [10]

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \varepsilon_{ij}^{H} + \varepsilon_{ij}^{a} = \begin{pmatrix} \varepsilon_1 & ig & 0 \\ -ig & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_2 \end{pmatrix}, \tag{4.3}$$

where ϵ_{ij}^{H} and ϵ_{ij}^{a} are respectively the Hermitian and anti-Hermitian parts of the dielectric constant tensor. In our case of nonrelativistic plasma ($\Omega_\alpha = e_\alpha B/m_\alpha c$) and subject to condition (4.2a), we obtain from (4.1)*

$$\left.\begin{aligned}
\varepsilon_1^{\rm H} &= 1-\sum_\alpha \frac{\omega_{0\alpha}^2}{\omega^2-\Omega_\alpha^2} - \sum_\alpha\sum_{m>1}\frac{\omega_{m-1,\alpha}^2\langle v_\alpha^{2m-2}\rangle}{\omega(\omega-m\Omega_\alpha)}\left(\frac{k_\perp}{\Omega_\alpha}\right)^{2m-2}\frac{m^2}{2^m m!\,(2m-1)!!},\\
g^{\rm H} &= -\sum_\alpha \frac{\omega_{0\alpha}^2\Omega_\alpha}{\omega(\omega^2-\Omega_\alpha^2)} - \sum_\alpha\sum_{m>1}\frac{\omega_{m-1,\alpha}^2\langle v_\alpha^{2m-2}\rangle}{\omega(\omega-m\Omega_\alpha)}\left(\frac{k_\perp}{\Omega_\alpha}\right)^{2m-2}\frac{m^2}{2^m m!\,(2m-1)!!},\\
\varepsilon_2^{\rm H} &= 1-\sum_\alpha \frac{\omega_{0\alpha}^2}{\omega^2} - \sum_\alpha\sum_{m>0}\frac{\omega_{m\alpha}^2\langle v_\alpha^{2m}\rangle}{\omega(\omega-m\Omega_\alpha)}\left(\frac{k_\perp}{\Omega_\alpha}\right)^{2m}\frac{m^2}{2^m m!\,(2m+1)!!},
\end{aligned}\right\} \quad (4.4)$$

where

$$\omega_{m\alpha}^2 = -\frac{4\pi e_\alpha^2}{2m+3}\frac{1}{\langle v^{2m}\rangle}\int d\mathbf{p}\, v^{2m+2}\frac{\partial f_{0\alpha}}{\partial \varepsilon_\alpha},$$

$$\langle v_\alpha^{2m}\rangle = \int d\mathbf{p}\, v^{2m} f_{0\alpha}.$$

For an equilibrium Maxwellian plasma we have

$$\omega_{m\alpha}^2 = \frac{4\pi N_\alpha e_\alpha^2}{m_\alpha},$$

$$\langle v_\alpha^{2m}\rangle = (2m+1)!!\left(\frac{\varkappa T_\alpha}{m_\alpha}\right)^m.$$

Terms that take into account the spatial dispersion in (4.4) are significant only near the resonant frequencies $\omega \sim m\Omega_\alpha$. In the opposite limiting case, when condition (4.2b) is satisfied, the correct expression for the Hermitian part of the dielectric constant is obtained for $m > 1$ from (4.4) by taking the limit $k_\perp = 0$. On the other hand, near the first absorption line $m = 1$ we have

$$\left.\begin{aligned}
\varepsilon_1^{\rm H} &= 1-\frac{\omega_{0\alpha}^2}{2\omega(\omega+\Omega_\alpha)},\\
g^{\rm H} &= -\frac{\omega_{0\alpha}^2}{2\omega(\omega+\Omega_\alpha)} + \frac{\omega_{0e}^2}{\omega\Omega_e}\delta_{\alpha i},\\
\varepsilon_2^{\rm H} &= 1-\frac{\omega_{0e}^2}{\omega^2}.
\end{aligned}\right\} \quad (4.5)$$

It is impossible to expand in powers of k_z [condition (4.2a)] or k_z^{-1} [condition (4.2b)], in the calculation of the anti-Hermitian part of the dielectric constant tensor, which determines the cyclotron absorption line shape, for then the required information concerning the shape of the absorption line in the plasma will be lost. Using only expansion in powers of $k_\perp$, we obtain from (4.1)

$$\begin{pmatrix}\varepsilon_1^{\rm a}\\ g^{\rm a}\end{pmatrix} = -i\sum_\alpha\frac{8\pi^3 e_\alpha^2 m_\alpha^2}{\omega}\sum_{m\neq 0}\begin{pmatrix}1\\ \frac{m}{|m|}\end{pmatrix}\frac{1}{2^{2|m|}}\left(\frac{k_\perp}{\Omega_\alpha}\right)^{2|m|-2} \times \frac{(k_z^2c^2-\omega^2)^{|m|}}{|k_z|^{2|m|+1}[(|m|-1)!]^2} I_{m\alpha},$$

$$\varepsilon_2^{\rm a} = -i8\pi^3 e^2\omega m_e^2\sum_{m\neq 0}\frac{1}{2^{2|m|}}\left(\frac{k_\perp}{\Omega_e}\right)^{2|m|}\frac{(k_z^2c^2-\omega^2)^{|m|}}{|k_z|^{2|m|+3}(|m|!)^2} I_{1m} - i\frac{8\pi^3 e^2\omega}{|k_z|^3c^4}\begin{cases}\displaystyle\int_{\mathscr{E}_{1e}}^{\infty} d\mathscr{E}\,\mathscr{E}^2\frac{\partial f_{0e}}{\partial \varepsilon} & \text{for } \omega^2 < k_z^2c^2,\\ 0 & \text{for } \omega^2 > k_z^2c^2.\end{cases} \quad (4.6)$$

We have introduced here the following notation

$$I_{m\alpha} = \begin{cases}\theta(m)\displaystyle\int_{\mathscr{E}_{-\alpha}(m)}^{\mathscr{E}_{+\alpha}(m)} d\mathscr{E}\frac{\partial f_{0\alpha}}{\partial \varepsilon}\left[\left(\frac{\mathscr{E}-\mathscr{E}_{+\alpha}}{m_\alpha c^2}\right)\left(\frac{\mathscr{E}-\mathscr{E}_{-\alpha}}{m_\alpha c^2}\right)\right]^{|m|} & \text{for } \omega^2 < k_z^2c^2,\\ \displaystyle\int_{\mathscr{E}_{-\alpha}(m)}^{\infty} d\mathscr{E}\frac{\partial f_{0\alpha}}{\partial \varepsilon}\left[\left(\frac{\mathscr{E}-\mathscr{E}_{+\alpha}}{m_\alpha c^2}\right)\left(\frac{\mathscr{E}-\mathscr{E}_{-\alpha}}{m_\alpha c^2}\right)\right]^{|m|} & \text{for } \omega^2 > k_z^2c^2,\\ 0 & \text{for } \omega^2 > m^2\Omega_\alpha^2 + k_z^2c^2;\end{cases}$$

$$I_{1m} = \begin{cases}\theta(m)\displaystyle\int_{\mathscr{E}_{-1e}(m)}^{\mathscr{E}_{+e}(m)} d\mathscr{E}\frac{\partial f_{0e}}{\partial \varepsilon}\left[\left(\frac{\mathscr{E}-\mathscr{E}_{+e}}{m_e c^2}\right)\left(\frac{\mathscr{E}-\mathscr{E}_{-e}}{m_e c^2}\right)\right]^{|m|}\left(\frac{\mathscr{E}}{m_e c^2}-\frac{m\Omega_e}{\omega}\right)^2 & \text{for } \omega^2 < k_z^2c^2,\\ \displaystyle\int_{\mathscr{E}_{-e}(m)}^{\infty} d\mathscr{E}\frac{\partial f_{0e}}{\partial \varepsilon}\left[\left(\frac{\mathscr{E}-\mathscr{E}_{+e}}{m_e c^2}\right)\left(\frac{\mathscr{E}-\mathscr{E}_{-e}}{m_e c^2}\right)\right]^{|m|}\left(\frac{\mathscr{E}}{m_e c^2}-\frac{m\Omega_e}{\omega}\right)^2 & \text{for } \omega^2 > k_z^2c^2,\\ 0 & \text{for } \omega^2 > m^2\Omega_e^2 + k_z^2c^2;\end{cases}$$

$$\mathscr{E}_{1e} = \frac{m_e c^2}{\sqrt{1-\dfrac{\omega^2}{k_z^2c^2}}}, \quad \theta(m) = \begin{cases}1 & \text{for } m > 0,\\ 0 & \text{for } m < 0,\end{cases}$$

$$\mathscr{E}_{\pm\alpha}(m) = \frac{m_\alpha c^2(m^2\Omega_\alpha^2 + k_z^2c^2)}{m\omega\Omega_\alpha \mp \sqrt{k_z^2c^2(m^2\Omega_\alpha^2 + k_z^2c^2-\omega^2)}}.$$

We note that the contribution of the ion terms to the expression for $\epsilon_2^{\rm a}$ is small. We therefore confine ourselves in (4.6) only to an account of the electron terms. In the case of purely perpendicular propagation the formulas (4.6) become appreciably simplified.

We have

$$\varepsilon_1^{\rm a} = g^{\rm a} = \begin{cases}\displaystyle -i\sum_\alpha 16\pi^3\Omega_\alpha^3 m_\alpha^2 e_\alpha^2\sum_{m>0}\left(\frac{m_\alpha c}{\omega}\right)^{2m+3}\frac{k^{2m-2}}{2^m m!\,(2m+1)!!} \times\left(1-\frac{\omega^2}{m^2\Omega_\alpha^2}\right)^{m+\frac12} f'_{0\alpha}\left(m\Omega_\alpha\frac{m_\alpha c^2}{\omega}\right) & \text{for } \omega < m\Omega_\alpha,\\ 0 & \text{for } \omega > m\Omega_\alpha;\end{cases}$$

$$\varepsilon_2^{\rm a} = \begin{cases}\displaystyle -i16\pi^3\Omega_e^3 m_e^2 e^2\sum_{m>0}\left(\frac{m_e c}{\omega}\right)^{2m+5}\frac{k^{2m}}{2^m m!\,(2m+3)!!} \times\left(1-\frac{\omega^2}{m^2\Omega_\alpha^2}\right)^{m+\frac32} f'_{0e}\left(m m_e c^2\frac{\Omega_e}{\omega}\right) & \text{for } \omega < m\Omega_e,\\ 0 & \text{for } \omega > m\Omega_e.\end{cases} \quad (4.7)$$

*Formulas (4.4) do not take the transverse Doppler effect into account, and are therefore valid when $\omega - m\Omega_\alpha \ll m\Omega_\alpha\langle v_\alpha^2\rangle/c^2$. Concerning the dielectric constant for the opposite limiting case see [10].

The resultant formulas for the anti-Hermitian part of the dielectric constant are relativistic, and are therefore suitable also for the description of absorption of waves with phase velocities comparable with the velocity of light. In the range of nonrelativistic plasma temperatures it is necessary to retain in the expressions for ϵ_2^a, $I_{m\alpha}$, and I_{im} the higher-order terms in $\mathscr{E}_1$ and in $\mathscr{E}_{\pm\alpha}(m)$. This corresponds to an account of the higher-order terms of the expansion in the powers of $\langle v^2\rangle/c^2$. In the case of a nonrelativistic plasma we have

$$\begin{pmatrix} \varepsilon_1^a \\ g^a \end{pmatrix} = i \sum_\alpha \frac{8\pi^3 e_\alpha^2 m_\alpha}{\omega |k_z|} \sum_{m\neq 0} \left(\frac{-1}{2}\right)^{|m|} \binom{|m|}{m} \left(\frac{k_\perp}{\Omega_\alpha \sqrt{m_\alpha}}\right)^{2|m|-2} \times F_{|m|\alpha}\left(\frac{m_\alpha(\omega - m\Omega_\alpha)^2}{2k_z^2}\right),$$

$$\varepsilon_2^a = i\frac{8\pi^3 e^2 m_e^2}{\omega |k_z|^3} \sum_m \left(-\frac{1}{2}\right)^{|m|} \frac{(\omega - m\Omega_e)^2}{|m|!} \left(\frac{k_\perp}{\Omega_e \sqrt{m_e}}\right)^{2|m|} \times F_{|m|e}\left(\frac{m_e(\omega - m\Omega_e)^2}{2k_z^2}\right), \tag{4.8}$$

where the function $F_{|m|\alpha}(x)$ is defined by the relation

$$\frac{\partial^{|m|}}{\partial x^{|m|}} F_{|m|\alpha}(x) = f_{0\alpha}(x).$$

For a nonrelativistic Maxwellian plasma we have

$$F_{|m|\alpha}(\mathscr{E}) = (-1)^{|m|} (\varkappa T_\alpha)^{|m|} f_{0\alpha}(\mathscr{E}).$$

Having obtained general formulas for the dielectric constant tensor, let us investigate the propagation of electromagnetic waves in a magnetoactive plasma. Substituting the dielectric constant (4.3) in (4.2) we obtain the following equations for the ordinary and extraordinary waves in the plasma

$$n^2 = \frac{(\varepsilon_1^2 - g^2 - \varepsilon_1\varepsilon_2)\sin^2\vartheta + 2\varepsilon_1\varepsilon_2 \mp \sqrt{(\varepsilon_1^2 - g^2 - \varepsilon_1\varepsilon_2)^2 \sin^4\vartheta + 4g^2\varepsilon_2^2\cos^2\vartheta}}{2(\varepsilon_1 \sin^2\vartheta + \varepsilon_2\cos^2\vartheta)}. \tag{4.9}$$

At frequencies for which the anti-Hermitian part of the dielectric constant tensor is small compared with the Hermitian part, the absorption of the waves is relatively weak. This occurs when condition (4.2a) is satisfied and the anti-Hermitian part of the dielectric constant is exponentially small, and also under condition (4.2b) with $m > 1$, when the ratio of the anti-Hermitian part to the Hermitian part is of the order of $(n\langle v_\alpha\rangle/c)^{2|m|-1}$. In these frequency regions we can neglect in the first approximation the anti-Hermitian part $\epsilon_{ij}^a(\omega, \mathbf{k})$ in (4.9). If the small anti-Hermitian part $\epsilon_{ij}^a(\omega, \mathbf{k})$ is taken into account with accuracy to linear terms, using (4.9), then the coefficients of wave absorption are expressed in terms of the $\epsilon_{ij}^a(\omega, \mathbf{k})$. In accordance with (4.6)—(4.8), the $\epsilon_{ij}^a(\omega, \mathbf{k})$ are expressed in turn in terms of the particle velocity distribution functions. By measuring the coefficient of wave absorption we can therefore determine in principle the plasma particle distribution functions. We note that these functions can be obtained only for $n^2 > 0$, i.e., in the transparency region of the medium.

In the range of frequencies defined by (4.2b), the anti-Hermitian part of the dielectric constant tensor is greater than the Hermitian part when $m = 1$. This causes the absorption of the electromagnetic wave to occur over a distance on the order of the wavelength. Consequently a correct analysis of electromagnetic waves in a plasma in this frequency region, which corresponds to the region of anomalous skin effect, calls for a solution of a boundary value problem.[1,5,6]

At high frequencies $(\omega \gg \sqrt{m_i/m_e}\,\Omega_i)$ the ion motion in the plasma can be neglected and the plasma regarded as electronic. It is then necessary to include only the electronic terms in (4.4)—(4.8). The expressions for the refractive indices for the absorption coefficients of waves propagating in an electronic plasma at an angle to the magnetic field are quite cumbersome[10] and are not essential here. We give only the formulas for the angles $\vartheta = 0$ and $\vartheta = \pi/2$. When $\vartheta = 0$ we obtain for the ordinary and extraordinary waves

$$n_{1,2}^2 = \varepsilon_1 \mp g, \qquad \varkappa_{1,2} = \frac{-i(\varepsilon_1^a \mp g^a)}{2n_{1,2} - \frac{\partial\varepsilon_1}{\partial n}(1 \mp 1)}. \tag{4.10}$$

When $\vartheta = \pi/2$ we have

$$n_1^2 = \varepsilon_2, \qquad \varkappa_1 = \frac{-i\varepsilon_2^a}{2n_1 - \frac{\partial\varepsilon_2}{\partial n}},$$

$$n_2^2 = \varepsilon_1 - \frac{g^2}{\varepsilon_1}, \qquad \varkappa_2 = \frac{-i\varepsilon_1^a\left(1 - \frac{g}{\varepsilon_1}\right)^2}{2n_2 - \frac{\partial\varepsilon_1}{\partial n}\left(1 - \frac{g}{\varepsilon_1}\right)^2}. \tag{4.11}$$

We note that formulas (4.10) and (4.11) for the absorption coefficients κ are valid only at those frequencies for which $n^2 > 0$.

For ordinary and extraordinary waves propagating at an angle $\vartheta = 0$ to the magnetic field at frequencies defined by the condition (4.2a) we have near the first resonance absorption line

$$n_{1,2}^2 = 1 - \frac{\omega_{0e}^2}{\omega(\omega \pm \Omega_e)}, \qquad \varkappa_{1,2} = \frac{-i(\varepsilon_1^a \mp g^a)}{2n_{1,2}}. \tag{4.12}$$

It follows therefore that for the ordinary wave $n_1^2 > 0$ when $\omega_{0e}^2 < 2\Omega_e^2$, i.e., in the case of a sufficiently dilute plasma, while for the extraordinary wave we have $n_2^2 > 0$ when $\omega < \Omega_e$. These are precisely the conditions under which we can use (4.12) to determine the electron distribution function in the plasma. We note that when $\vartheta = 0$ we can use for the anti-Hermitian part of the dielectric constant tensor the nonrelativistic formulas (4.8):

$$\begin{pmatrix} \varepsilon_1^a \\ g^a \end{pmatrix} = -i\frac{4\pi^3 e^2 m_e c}{\omega^2 |n|}\left[F_{1e}\left(\frac{m_e c^2(\omega - \Omega_e)^2}{2\omega^2 n^2}\right) \pm F_{1e}\left(\frac{m_e c^2(\omega + \Omega_e)^2}{2\omega^2 n^2}\right)\right]. \tag{4.13}$$

An account of the relativistic effects of thermal motion on the electrons leads in this case to inessential corrections. To the contrary, for waves propagating at an angle $\vartheta = \pi/2$ to the magnetic field, the absorption is due entirely to the relativistic effect of the thermal electron motion. In the nonrelativistic analysis there is no wave absorption in the plasma at all when $\vartheta = \pi/2$, since $\epsilon^a_{ij}(\omega, \mathbf{k}) = 0$. Near the first resonance absorption line for the ordinary and extraordinary waves propagating at an angle $\vartheta = \pi/2$ to the magnetic field we obtain

$$n_1^2 = \frac{1 - \frac{\omega_{0e}^2}{\omega^2}}{1 + \frac{\langle v_e^2 \rangle}{c^2} \frac{\omega_{1e}^2 \omega}{\Omega_e^2 (\omega - \Omega_e)}}, \quad \varkappa_1 = \frac{-in_1\varepsilon_2^a}{2\left(1 - \frac{\omega_{0e}^2}{\omega^2}\right)},$$

$$n_2^2 = \frac{1 - \frac{\omega_{0e}^2}{\omega^2 - \Omega_e^2}\left(2 - \frac{\omega_{0e}^2}{\omega^2}\right)}{1 - \frac{\omega_{0e}^2}{\omega^2 - \Omega_e^2}}, \quad \varkappa_2 = \frac{-i\varepsilon_1^a}{2n_2}\left(1 + \frac{\Omega_e}{\omega} \frac{\omega_{0e}^2}{\omega^2 - \Omega_e^2 - \omega_{0e}^2}\right), \tag{4.14}$$

where

$$\varepsilon_1^a = g^a = \begin{cases} -i\frac{8}{3}\frac{\pi^3\Omega_e^3 m_e^2 e^2 c^5}{\omega^5}\left(1 - \frac{\omega^2}{\Omega_e^2}\right)^{3/2} f'_{0e}\left(m_e c^2 \frac{\Omega_e}{\omega}\right) & \text{for } \omega < \Omega_e, \\ 0 & \text{for } \omega > \Omega_e; \end{cases}$$

$$\varepsilon_2^a = \begin{cases} -i\frac{8}{15}\frac{\pi^3\Omega_e^3 m_e^2 e^2 c^5}{\omega^5}\left(1 - \frac{\omega^2}{\Omega_e^2}\right)^{5/2} f'_{0e}\left(m_e c^2 \frac{\Omega_e}{\omega}\right) & \text{for } \omega < \Omega_e, \\ 0 & \text{for } \omega > \Omega_e. \end{cases}$$

It follows from these formulas that when $\omega \sim \Omega_e$ the refractive index of the ordinary wave is small and reverses sign. For the extraordinary wave $n_2^2 > 0$ only in the case of a dilute plasma, when $\omega_{0e}^2 < 2\Omega_e^2$.

In order to determine the electron distribution function from measurements of the absorption coefficients it is necessary that the absorption in the plasma be appreciable. In the nonrelativistic analysis the electromagnetic waves propagating at an angle $\vartheta = \pi/2$ to the magnetic field are not absorbed in the plasma. However, an account of the relativistic effects of thermal motion of the particles leads to considerable wave absorption even at nonrelativistic temperatures. Thus, for example, in a Maxwellian plasma we have $\omega = \Omega_e/2$, $T_e \sim 10^9$, $B \sim 10^4$, $N \sim 10^{13}$, and $n_{1,2} \sim 1$. In this case $\kappa_1 \sim 10^{-3}$ and $\kappa_2 \sim 10^{-2}$, i.e., the ordinary wave is completely absorbed over a length $\delta \sim 100$ cm, while the extraordinary wave is absorbed within $\delta \sim 10$ cm (the wavelength in this case is $\lambda \sim 1$ cm). We note that when [10]

$$10^{30} e^{-\frac{10^{10}}{T_e}} \gg NT_e$$

cyclotron absorption, described by formulas (4.13) predominates over the absorption due to particle collisions in the plasma. When $T_e \gtrsim 3 \times 10^8$ this condition is well satisfied over a wide range of variation of B and N ($B \sim 10^3$–10^5, $N \sim 10^{13}$–10^{15}).

An investigation of the propagation of electromagnetic waves in the frequency range defined by the condition (4.2a) with $m > 1$ is of no interest from the point of view of determining the electron distribution function in a nonrelativistic plasma. In this frequency region the anti-Hermitian part of the dielectric constant tensor contains along with an exponentially small factor also a factor on the order of $(\langle v_e \rangle/c)^{2m}$. Therefore cyclotron absorption of waves in a plasma is practically always negligible in this frequency range compared with the absorption due to particle collision.

So far we have discussed frequency regions defined by condition (4.2a). As already noted, when condition (4.2b) is satisfied, it becomes meaningful to investigate the waves near resonance frequencies with $m > 1$ by means of (4.9)–(4.11), using for the Hermitian part of the dielectric constant tensor the expression

$$\varepsilon_1^H = 1 - \frac{\omega_{0e}^2}{\omega^2 - \Omega_e^2}, \quad g^H = -\frac{\omega_{0e}^2 \Omega_e}{\omega(\omega^2 - \Omega_e^2)}, \quad \varepsilon_2^H = 1 - \frac{\omega_{0e}^2}{\omega^2}. \tag{4.15}$$

Of particular interest is the frequency region near the second resonant absorption line. At these frequencies the anti-Hermitian part of the dielectric constant tensor is small compared with the Hermitian part, and differs from it only by an amount $n\langle v_e \rangle/c$. We can therefore expect rather intense wave absorption in the plasma. Actually, when $\omega = 2\Omega_e$, in accord with (4.9) and (4.15), electromagnetic waves can propagate only in a sufficiently dilute plasma, in which $\omega_{0e}^2 \leq 4\Omega_e^2$. In this case cyclotron absorption predominates over collision absorption, provided the following condition is satisfied [10]

$$0.1N \ll BT_e^2.$$

This condition is well satisfied over a wide range of N, T_e, and B, both in a laboratory plasma and in ionospheric plasma [19] (F layer of the ionosphere). In a plasma for which $N \sim 10^{13}$, $T_e \sim 10^7$, and $B \sim 10^4$ we have for both waves $n \sim 1$ and $\kappa \sim 0.01$, i.e., they are completely absorbed within $\delta \sim 10$ cm.

We now consider electromagnetic waves at frequencies close to the ion resonance frequencies $\omega \sim m\Omega_i$. The investigation of electromagnetic wave propagation in a plasma at ion cyclotron frequencies is simplified if conditions (4.2a) and (4.2b) are satisfied, because the inequality $\epsilon_1^2 - g^2 \ll \epsilon_1\epsilon_2$ holds true. Taking this into account, we obtain from the dispersion equation (4.9) the following equations for the ordinary and extraordinary waves:

$$n_1^2 = \frac{\varepsilon_1\varepsilon_2(1 + \cos^2\vartheta)}{\varepsilon_1 \sin^2\vartheta + \varepsilon_2\cos^2\vartheta}, \quad n_2^2 = \frac{\varepsilon_1^2 - g^2}{\varepsilon_1(1 + \cos^2\vartheta)}. \tag{4.16}$$

The component ϵ_1 of the dielectric constant tensor, and also the component g^a, contain only ionic terms in this case; the component g^H contains along with ionic terms also the term $\omega_{0e}^2/\omega\Omega_e$, due to the motion of the electrons. [14] Finally, for the component ϵ_2 we obtain at ion cyclotron resonance frequencies the following expression [see formulas (4.8)]

$$\varepsilon_2 = -\frac{\omega_{0e}^2}{\omega^2} + i\frac{8\pi^3 e^2 m_e^2}{\omega^{-1}|k_z|^3} f_{0e}\left(\frac{m_e\omega^2}{2k_z^2}\right). \tag{4.17}$$

Equation (4.16) can be readily investigated for all angles ϑ. At frequencies $\omega \sim \Omega_i$ we obtain under condition (4.2a) the following expression for the refractive indices of the waves:

$$n_1^2 = \frac{\omega_{0e}^2}{\omega^2} \frac{1+\cos^2\vartheta}{\frac{\omega_{0e}^2}{\omega_{0i}^2}\frac{\Omega_i^2-\omega^2}{\omega^2}\cos^2\vartheta - \sin^2\vartheta}, \quad n_2^2 = -\frac{\omega_{0i}^2}{\omega^2}\frac{1}{1+\cos^2\vartheta}. \quad (4.18)$$

We see therefore that $n_2^2 < 0$, i.e., the extraordinary wave cannot propagate in a plasma near the first ion resonance frequency $\omega \sim \Omega_i$. For the ordinary wave we have $n_1^2 > 0$ with $\tan^2\vartheta < \frac{\omega_{0e}^2}{\omega_{0i}^2}\frac{\Omega_i^2-\omega^2}{\omega^2}$. This obviously takes place when $\omega < \Omega_i$ and for all angles ϑ that are not too close to $\pi/2$. For the absorption coefficient we now obtain

$$\varkappa_1 = \frac{n_1^3}{2(1+\cos^2\vartheta)}\left[\frac{-i\varepsilon_1^a\cos^2\vartheta}{\left(\frac{\omega_{0i}^2}{\omega^2-\Omega_i^2}\right)^2} + \frac{-i\varepsilon_2^a\sin^2\vartheta}{\left(\frac{\omega_{0e}^2}{\omega^2}\right)^2}\right]. \quad (4.19)$$

Since ϵ_1^a is expressed directly in terms of the ion distribution function, measurements of κ at $\vartheta = 0$ enable us to determine the ion distribution function of the plasma. We note that for ions we can practically always neglect the relativistic effects and use (4.8) for ϵ^a. In a sufficiently hot plasma, cyclotron absorption of waves by the plasma ions can become appreciable. Thus, for example, in a Maxwellian plasma with $T_i \sim 10^7$, $B \sim 10^4$, and $N \sim 10^{14}$ we have for a wave propagating at an angle $\vartheta = 0$ near the first resonant frequency $n_1 \sim 100$ and $\kappa_1 \sim 1$, i.e., such a wave is completely absorbed by the plasma ions within $\delta \sim 100$ cm (the wavelength is $\lambda \sim 1$ cm).*

Near the ion resonance frequencies, with $m > 1$ and condition (4.2a) satisfied, the anti-Hermitian part of the dielectric constant tensor (4.8) contains, along with the exponentially small term, also a small factor of order $(\langle v_i \rangle/c)^{2m}$. Consequently the cyclotron absorption is always smaller than the collision absorption.

At the frequencies determined by condition (4.2b), greatest interest attaches, as in the case of an electronic plasma, to the region $\omega \sim 2\Omega_i$. In this frequency region, in a plasma in which $\omega_{0i}^2 > 3\Omega_i^2$, the quantities ϵ_1^H, g^H, and ϵ_2^H are negative, and therefore $n_1^2 < 0$, i.e., the ordinary wave cannot propagate in the plasma. On the other hand, in a dilute plasma, in which $\omega_{0i}^2 < 3\Omega_i^2$ we have $n_1 \sim 1$ and $\kappa_1 \sim \langle v_i \rangle/c$. As to the extraordinary wave, we obtain

$$n_2^2 = \frac{\Omega_i^2\omega^2 - (\omega_{0i}^2+\Omega_i^2)^2}{\Omega_i^2(\omega^2-\Omega_i^2-\omega_{0i}^2)(1+\cos^2\vartheta)},$$
$$\varkappa_2 = \frac{-i\varepsilon_1^a(\omega\Omega_i+\Omega_i^2+\omega_{0i}^2)^2(\omega-\Omega_i)^2}{2n_2(1+\cos^2\vartheta)\Omega_i^2(\omega^2-\Omega_i^2-\omega_{0i}^2)^2}. \quad (4.20)$$

It follows therefore that $n_2^2 > 0$ when $\omega_{0i}^2 > 3\Omega_i^2$. In a sufficiently dense plasma the cyclotron absorption, described by formula (4.20), can become appreciable and exceed the collision absorption in the plasma. [14] Thus, for example, when $T_i \sim 10^7$, $N \sim 10^{14}$, and $B \sim 10^4$ we have $n_2 \sim 100$ and $\kappa_2 \sim 10$. Such a wave is completely absorbed within $\delta \sim 10$ cm (the wavelength here is on the order of $\lambda \sim 1$ cm) and can be used to heat ions in modern laboratory apparatus.

*For more details on the absorption of waves in the region of ion cyclotron resonance frequencies see [14].

All the foregoing was based on the smallness of the parameter $n\langle v_\alpha \rangle/c \ll 1$. However, as shown by Dnestrovskii and Kostomarov, [15] in the region of large refractive indices this condition may be violated. When $n\langle v_\alpha \rangle/c \sim 1$, an analytic investigation of the electromagnetic waves is practically impossible and it becomes necessary to solve the dispersion equations by numerical means. [15] In the limiting case when $n\langle v_\alpha \rangle/c \gg 1$, however, the situation simplifies greatly because the asymptotic representations of the Bessel functions of large argument can be used in the expression (4.1) for the dielectric constant tensor. As shown in [16], this inequality can be satisfied near the resonant frequencies $\omega \sim m\Omega_\alpha$ under condition (4.2a) for both electrons and ions, and furthermore within a very narrow interval of angles near $\vartheta \sim \pi/2$. The dielectric constant tensor of a Maxwellian electron-ion plasma has in the transparency region the form

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix}, \quad (4.21)$$

where

$$\left.\begin{aligned} \varepsilon_{11} &= 1 - \frac{m^2}{\sqrt{2\pi}}\frac{\omega_{0\alpha}^2\Omega_\alpha^3}{\omega(\omega-m\Omega_\alpha)}\left(\frac{m_\alpha}{k_\perp^2 T_\alpha}\right)^{3/2}\left[1 - i\sqrt{\frac{\pi}{2}}\beta_{m\alpha}e^{-\frac{\beta_{m\alpha}^2}{2}}\right], \\ \varepsilon_{22} &= 1 - \frac{1}{\sqrt{2\pi}}\frac{\omega_{0\alpha}^2\Omega_\alpha}{\omega(\omega-m\Omega_\alpha)}\left(\frac{m_\alpha}{k_\perp^2 T_\alpha}\right)^{1/2}\left[1 - i\sqrt{\frac{\pi}{2}}\beta_{m\alpha}e^{-\frac{\beta_{m\alpha}^2}{2}}\right], \\ \varepsilon_{33} &= 1 - \frac{1}{\sqrt{2\pi}}\frac{\omega_{0\alpha}^2\Omega_\alpha}{\omega(\omega-m\Omega_\alpha)}\left(\frac{m_\alpha}{k_\perp^2 T_\alpha}\right)^{1/2}\left[1 - i\sqrt{\frac{\pi}{2}}\beta_{m\alpha}^3 e^{-\frac{\beta_{m\alpha}^2}{2}}\right]. \end{aligned}\right\} \quad (4.22)$$

The dispersion equation (4.2) breaks up in this case into three equations, which are the dispersion equations for the ordinary, extraordinary, and plasma waves, respectively:

$$n^2\sin^2\vartheta = \varepsilon_{33}, \quad n^2 = \varepsilon_{22}, \quad \varepsilon_{11} = 0. \quad (4.23)$$

From this we obtain the refractive indices and the absorption coefficients of the electromagnetic waves that are in resonance with the gyro frequency of one definite kind of particles:

$$\left.\begin{aligned} n_1 &= \left(\frac{\omega_{0\alpha}^2\Omega_\alpha c\sqrt{m_\alpha}}{\sqrt{2\pi}\,\omega^2(m\Omega_\alpha-\omega)\sqrt{T_\alpha}}\frac{1}{\sin^3\vartheta}\right)^{1/3}, \quad \varkappa_1 \approx -\sqrt{\frac{\pi}{2}}n_1\beta_{m\alpha}^3 e^{-\frac{\beta_{m\alpha}^2}{2}}, \\ n_2 &= \left(\frac{\omega_{0\alpha}^2\Omega_\alpha c\sqrt{m_\alpha}}{\sqrt{2\pi}\,\omega^2(m\Omega_\alpha-\omega)\sqrt{T_\alpha}}\frac{1}{\sin\vartheta}\right)^{1/3}, \quad \varkappa_2 \approx -\sqrt{\frac{\pi}{2}}n_2\beta_{m\alpha}e^{-\frac{\beta_{m\alpha}^2}{2}}, \\ n_3 &= \left(\frac{\omega_{0\alpha}^2\Omega_\alpha^3 c^3 m_\alpha^{3/2} m^2}{\sqrt{2\pi}\,\omega^4(\omega-m\Omega_\alpha)T_\alpha^{3/2}\sin^3\vartheta}\right)^{1/3}, \quad \varkappa_3 \approx \frac{1}{3}\sqrt{\frac{\pi}{2}}n_3\beta_{m\alpha}e^{-\frac{\beta_{m\alpha}^2}{2}}, \end{aligned}\right\} \quad (4.24)$$

where

$$\beta_{m\alpha}=\frac{\omega-m\Omega_\alpha}{n\,|\cos\vartheta\,|\,\omega}\frac{c\sqrt{m_\alpha}}{\sqrt{T_\alpha}}\gg 1.$$

Thus, in the limit considered here, the ordinary and extraordinary waves can propagate in a plasma when $\omega < m\Omega_\alpha$, while the plasma wave can propagate when $\omega > m\Omega_\alpha$. This result agrees with the qualitative investigation of [15].

Worthy of particular analysis is the frequency range $\omega \ll \Omega_i$, when the so-called magnetohydrodynamic waves propagate in the plasma. We can consider two cases: 1) cold plasma, when the thermal motion of the particles can be neglected (limit of two-fluid hydrodynamics), and 2) non-isothermal plasma, when an account of the thermal motion of the electrons is important (the limit of single-fluid hydrodynamics).[17] In the first case the anti-Hermitian part of the dielectric constant tensor (4.8), which describes the absorption of waves in a collisionless plasma, is exponentially small, and for the Hermitian part of $\epsilon_{ij}(\omega, \mathbf{k})$ we have

$$\varepsilon_{ij}(\omega,\mathbf{k})=\begin{pmatrix}\varepsilon_1 & 0 & 0\\ 0 & \varepsilon_1 & 0\\ 0 & 0 & \varepsilon_2\end{pmatrix}, \qquad (4.25)$$

where

$$\varepsilon_1^s=1+\frac{\omega_{0i}^2}{\Omega_i^2},\quad \varepsilon_2^s=1-\frac{\omega_{0e}^2}{\omega^2}\approx-\frac{\omega_{0e}^2}{\omega^2}.$$

From the dispersion equation (4.2), which in this limit separates into three equations,

$$k^2=\frac{\omega^2}{c^2}\varepsilon_1,\quad \varepsilon_1\sin^2\vartheta+\varepsilon_2\cos^2\vartheta=0,$$

$$k^2=\frac{\omega^2}{c^2}\frac{\varepsilon_1\varepsilon_2}{\varepsilon_1\sin^2\vartheta+\varepsilon_2\cos^2\vartheta}, \qquad (4.26)$$

we obtain the following spectrum of the magnetohydrodynamic and Alfven waves (it is customary in hydrodynamics to express the frequency and the decrement in terms of the wave vector):*

$$\omega^2=\frac{k^2v_A^2}{1+\frac{v_A^2}{c^2}},\quad \gamma=-i\frac{\omega\varepsilon_1^a}{\varepsilon_1^s}, \qquad (4.27a)$$

$$\omega^2=\frac{\omega_{0e}^2}{\omega_{0i}^2}\operatorname{ctg}^2\vartheta,\quad \gamma=-i\frac{\omega^3}{2\omega_{0e}^2}\ \frac{(\varepsilon_1^a\sin^2\vartheta+\varepsilon_2^a\cos^2\vartheta)}{\cos^2\vartheta}, \qquad (4.27b)$$

$$\omega^2=\frac{k^2v_A^2\cos^2\vartheta}{1+\frac{v_A^2}{c^2}+\frac{k^2c^2}{\omega_{0e}^2}\left(\sin^2\vartheta+\frac{v_A^2}{c^2}\right)},$$

$$\gamma=-i\frac{(k^2c^2\sin^2\vartheta-\omega^2\varepsilon_2^s)\,\varepsilon_1^a+(k^2c^2\cos^2\vartheta-\omega^2\varepsilon_1^s)\,\varepsilon_2^a}{(k^2c^2\cos^2\vartheta-\omega^2\varepsilon_1^s)\frac{\omega_{0e}^2}{2\omega^3}-\omega\varepsilon_1^s\varepsilon_2^s}. \qquad (4.27c)$$

Here $v_A = c\Omega_i/\omega_{0i}$ is the Alfven velocity; the values of ϵ_1^a and ϵ_2^a are given by (4.8), with the electronic terms making the main contribution. It should be noted that in accordance with the requirement $\omega \ll \Omega_i$ the solution (4.27b) is valid only in a very narrow interval of angles near $\vartheta \sim \pi/2$.

In conclusion we consider electromagnetic waves in a non-isothermal plasma, in which the mean thermal energy of the electrons is considerably greater than the ion energy, at low frequencies $k\sin\vartheta\langle v_i\rangle \ll \omega \ll k\cos\vartheta\langle v_e\rangle$.

From the general expression for the dielectric constant tensor (4.1) we obtain in this limit[18]

$$\varepsilon_{ij}(\omega,\mathbf{k})=\begin{pmatrix}\varepsilon_{11} & 0 & 0\\ 0 & \varepsilon_{22} & \varepsilon_{23}\\ 0 & -\varepsilon_{23} & \varepsilon_{33}\end{pmatrix}, \qquad (4.28)$$

where the following expressions hold true for the Hermitian and anti-Hermitian parts of ϵ_{ij}:†

$$\varepsilon_{11}^H=\frac{\omega_{0i}^2}{\Omega_i^2},\quad \varepsilon_{11}^a=0,$$

$$\varepsilon_{22}^H=\frac{\omega_{0i}^2}{\Omega_i^2},\quad \varepsilon_{22}^a=-i\frac{\pi^2e^2k\sin^2\vartheta}{\omega\,|\cos\vartheta\,|}\int d\mathbf{p}\frac{\partial f_{0e}}{\partial\mathscr{E}}v^3,$$

$$\varepsilon_{33}^H=-\frac{\omega_{0i}^2}{\omega^2}+\frac{\omega_{-1e}^2}{k^2\cos^2\vartheta\,\langle v_e^2\rangle},\quad \varepsilon_{33}^a=-i\frac{2\pi^2e^2\omega}{k^3\,|\cos^3\vartheta\,|}\int d\mathbf{p}\frac{1}{v}\frac{\partial f_{0e}}{\mathscr{E}},$$

$$\varepsilon_{23}^H=i\frac{\omega_{0i}^2}{\omega\Omega_i}\operatorname{tg}\vartheta,\quad \varepsilon_{23}^a=-\frac{\pi^2e^2\sin\vartheta}{k\cos^2\vartheta\,\Omega_e}\int d\mathbf{p}\frac{\partial f_{0e}}{\partial\mathscr{E}}v.$$

We note that only the electron terms contribute to the anti-Hermitian part of $\epsilon_{ij}(\omega, \mathbf{k})$, i.e., the wave absorption is due to the plasma electrons.

The dispersion equation (4.2) separates in this case into the following two equations

$$k^2\cos^2\vartheta-\frac{\omega^2}{c^2}\varepsilon_{11}=0,\quad \left(k^2-\frac{\omega^2}{c^2}\varepsilon_{22}\right)\varepsilon_{33}+\varepsilon_{23}^2\frac{\omega^2}{c^2}=0. \qquad (4.29)$$

From this we obtain the spectrum of the magnetohydrodynamic and magnetic-sound waves propagating in a non-isothermal plasma in the limit of single-fluid hydrodynamics

$$\omega^2=k^2v_A^2\cos^2\vartheta,\quad \gamma=0, \qquad (4.30a)$$

$$\omega_\pm^2=\frac{1}{2}k^2\left\{v_A^2+v_H^2\pm\sqrt{(v_A^2+v_H^2)^2-4v_A^2v_H^2\cos^2\vartheta}\right\},$$

$$\gamma_\pm=-i\frac{\omega_\pm}{2k^2}\frac{1}{\varepsilon_{33}^H}\left\{\left(k^2-\frac{\omega_\pm^2}{c^2}\varepsilon_{22}^H\right)\varepsilon_{33}^a-\frac{\omega_\pm^2}{c^2}\varepsilon_{33}^H\varepsilon_{22}^a+2\varepsilon_{23}^H\frac{\omega_\pm^2}{c^2}\varepsilon_{23}^a\right\}, \qquad (4.30b)$$

where $v_s = \omega_{0i}\sqrt{\langle v_e^2\rangle}/\omega_{-1e}$ is the velocity of sound in the non-isothermal plasma. In the case of a Maxwellian plasma formulas (4.30) assume the customary form.[17-18]

5. INTERACTION BETWEEN A BEAM OF CHARGED PARTICLES AND A MAGNETOACTIVE PLASMA

In the presence of an external magnetic field, the character of the beam instabilities in the plasma becomes highly varied, and any detailed examination of this question is outside the scope of the present review.* To illustrate the simplicity of the general theory developed in Sec. 3, we shall consider here only

*ctg = cot.

*The interaction between a beam of charged particles and a magnetoactive plasma has been treated in a tremendous number of papers. Soon to be published is a book by Ya. B. Faĭnberg[20], devoted to this problem, where a detailed bibliography can be found (see also [1,40]).

†tg = tan.

low-frequency instabilities, corresponding to the excitation of magnetohydrodynamic and magnetic-sound waves by a nonrelativistic beam in a plasma, especially because this problem was solved most recently.[21,22]

Unstable solutions of the dispersion equation (3.7) should be sought primarily in the region of the frequencies ω and ω' where the absorption is small, and therefore the anti-Hermitian parts of the dielectric constant tensors of the beam $\epsilon_{ij}^{(1)\prime}(\omega', \mathbf{k}')$ and of the plasma $\epsilon_{ij}^{(2)}(\omega, \mathbf{k})$, which are responsible for the absorption of the electromagnetic waves, are negligibly small compared with their Hermitian parts. This frequency region includes, on the one hand, the oscillations for which thermal motion of the particles is insignificant (these are the high-frequency oscillations, the electron and ion cyclotron waves, and also the low-frequency oscillations corresponding to the limit of the two-fluid hydrodynamics), and on the other hand, low-frequency oscillations of non-isothermal plasma, which depend appreciably on the thermal motion of the electrons and correspond to the limit of the single-fluid hydrodynamics of a plasma.[17] In the former frequency region the dielectric constant tensor $\epsilon_{ij}(\omega, \mathbf{k})$ is determined by expressions (4.3)—(4.5), while in the latter it is given by (4.28). Expressions of this type for the dielectric constants of the beam and the plasma enable us to find the entire manifold of unstable oscillations in the system in the presence of an external magnetic field.

As already indicated above, the electromagnetic wave is excited in the plasma at low beam density, only when the speed of the beam is of the order of the phase velocity of the wave. Therefore, without resorting to an investigation of the dispersion equation (3.7), we can use the results of the preceding section to determine the conditions imposed on the beam velocity u (under which the oscillations in the plasma will be unstable).* It is easy to see that these conditions have the form $u \sim c/n$, where n is the refractive index of the wave excited in the plasma. Only solution of the dispersion equation (3.7) can yield the frequencies and buildup increments of the oscillations and permits investigation of the case when the beam and plasma densities are comparable.

We proceed now to a more detailed investigation of the excitation of magnetohydrodynamic and magnetic-sound waves ($\omega, \omega' \ll \Omega_i$) in a plasma by a nonrelativistic beam of charged particles. We confine ourselves to the case when the thermal motion of the particles in the beam can be neglected, i.e., when $\omega' \gg k\langle v_\alpha \rangle_1$.

*The results of Sec. 3 remain in force if the beam moves along the magnetic field. In the case of a homogeneous plasma, only this formulation of the problem has physical meaning.

In the limit of two-fluid hydrodynamics, when thermal motion of the particles can be neglected both in the beam and in the stationary plasma, the dispersion equation (3.7) separates into two equations

$$k^2 - \frac{\omega^2}{c^2}\left[\frac{(\omega - \mathbf{uk})^2}{\omega^2}\frac{\omega_{0i1}^2}{\Omega_i^2} + \frac{\omega_{0i2}^2}{\Omega_i^2}\right] = 0, \qquad (5.1)$$

$$\left\{k^2\cos^2\vartheta - \frac{\omega^2}{c^2}\left[\frac{(\omega - \mathbf{uk})^2}{\omega^2}\frac{\omega_{0i1}^2}{\Omega_i^2} + \frac{\omega_{0i2}^2}{\Omega_i^2}\right]\right\}$$
$$\times\left\{k^2\sin^2\vartheta - \frac{\omega^2}{c^2}\left[1 - \frac{\omega_{0i2}^2}{\omega^2} - \frac{\omega_{0i1}^2}{(\omega - \mathbf{uk})^2} + \frac{k^2\sin^2\vartheta u^2}{\omega^2}\frac{\omega_{0i1}^2}{\Omega_i^2}\right]\right\}$$
$$- \left[k^2\sin\vartheta\cos\vartheta + \frac{uk\sin\vartheta(\omega - \mathbf{uk})}{c^2}\frac{\omega_{0i1}^2}{\Omega_i^2}\right]^2 = 0. \qquad (5.2)$$

From (5.1) we obtain

$$\omega = \frac{v_{A2}^2}{v_{A1}^2 + v_{A2}^2}\left\{\mathbf{uk} \pm \sqrt{(\mathbf{uk})^2\left(1 - \frac{v_{A1}^2 + v_{A2}^2}{v_{A2}^2}\right)^2 + k^2 v_{A1}^2\frac{v_{A1}^2 + v_{A2}^2}{v_{A2}^2}}\right\}, \qquad (5.3)$$

where v_{A1} and v_{A2} are the Alfven velocities in the beam and in the stationary plasma, respectively. From (5.3) it follows that when

$$(\mathbf{uk})^2\left(\frac{v_{A1}^2 + v_{A2}^2}{v_{A2}^2} - 1\right) > k^2 v_{A1}^2\frac{v_{A1}^2 + v_{A2}^2}{v_{A2}^2} \qquad (5.4)$$

the system oscillations become unstable. These instabilities correspond to excitation of magnetohydrodynamic waves, described by the dispersion equation (4.27a). It is easy to see that when $\mathbf{u} \cdot \mathbf{k} = 0$, i.e., for waves propagating transversely to the magnetic field, there are no such instabilities in the plasma. In the case of small beam density ($v_{A1} \gg v_{A2}$) the condition (5.4) assumes the form $(\mathbf{u} \cdot \mathbf{k})^2 > k^2 v_{A2}^2$.[21,22]

An investigation of (5.2) for arbitrary angles ϑ and for arbitrary beam densities is quite complicated. When $\vartheta = \pi/2$, Eq. (5.2) has only solutions corresponding to damped oscillations in the system. When $\vartheta = 0$, (5.2) separates into two equations, one of which coincides with (5.1) and the other with (3.11). In the case of a beam of low density we obtain from (5.2) the following approximate solution

$$\omega = \mathbf{uk} \pm i\frac{\omega_{0e1}}{-c}\operatorname{ctg}\vartheta \cdot \frac{v_{A2}}{c}\sqrt{\frac{u^2 - v_{A2}^2}{2v_{A2}^2 - u^2}}, \qquad (5.5)$$

which for $v_{A2} < u < \sqrt{2}\, v_{A2}$ describes unstable oscillations corresponding to the excitation of magnetohydrodynamic waves in a stationary plasma.

Finally, in the case of a hot non-isothermal plasma, when thermal motion of the beam particles can be neglected, but the thermal motion of the electrons in the stationary plasma must be taken into account, the dispersion equation (3.7) has for $\vartheta = \pi/2$ only solutions that correspond to damped oscillations of the system. When $\vartheta = 0$ this solution breaks up into two equations, one coinciding with (5.1) and describing the excitation of magnetohydrodynamic waves, and the other coinciding with (3.13) and describing the excitation of plasma sound in the system (for more details see [22]).

6. PARTICLE COLLISIONS IN A PLASMA

We now discuss another question which can be fully answered with the aid of linear electrodynamics of a plasma. We refer to the collision problem and to radiation and absorption of electromagnetic waves by plasma particles. The separation of radiation and absorption of electromagnetic waves in a plasma from particle collisions is only arbitrary. Consequently, strictly speaking, both processes must be considered in unified fashion, although there are many examples in which these processes can be separated.

Collisions of particles in a fully ionized plasma were considered by Landau.[23] He assumed that the electrons and ions collide in a plasma in the same manner as in vacuum. Actually, the collisions of charged particles in a plasma differ from collisions in vacuum because polarization of the medium makes the field of charged particles in the plasma different from the field of a particle in vacuum. The polarization of the medium is described by the dielectric constant tensor. Thus, for example, for the field of a point charge at rest in a plasma we can write the following expressions [1,3]

$$\mathbf{E}(\mathbf{r}) = -\frac{\partial}{\partial \mathbf{r}} \int d\mathbf{k} \frac{e}{2\pi^2} \frac{e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}_0)}}{k_i \varepsilon_{ij}(0, \mathbf{k}) k_j}. \qquad (6.1)$$

However, this particle field can be used to determine the scattering of two charged particles only if one of these particles is infinitely heavy. An account of the motion of the particles, naturally, complicates the matter. The Coulomb field of a nonrelativistic charge moving in a medium with a velocity $v \ll c$ is determined by the following expression [1,3]

$$\mathbf{E}(\mathbf{r}) = -\frac{e}{2\pi^2} \frac{\partial}{\partial \mathbf{r}} \int d\mathbf{k} \frac{e^{i\mathbf{k}(\mathbf{r}-\mathbf{v}t)}}{k_i \varepsilon_{ij}(\mathbf{k}\mathbf{v}, \mathbf{k}) k_j}. \qquad (6.2)$$

Here $\mathbf{k}\cdot\mathbf{v}$ plays the role of the frequency, and $\hbar\mathbf{k}$ plays the role of the momentum of the electromagnetic wave radiated by the particle (provided, of course, that these quantities correspond to the transparency region). It follows therefore that if only Coulomb interaction is considered in the scattering of nonrelativistic charged particles it becomes in fact necessary to calculate the following matrix elements:*

$$4\pi e_\alpha e_\beta \int d\mathbf{k} \frac{\langle \mathbf{p}'_\alpha | e^{i\mathbf{k}\mathbf{r}_\alpha} | \mathbf{p}_\alpha \rangle \langle \mathbf{p}'_\beta | e^{i\mathbf{k}\mathbf{r}_\beta} | \mathbf{p}_\beta \rangle}{k_i \varepsilon_{ij}\left(\frac{E_{\mathbf{p}'_\alpha} - E_{\mathbf{p}_\alpha}}{\hbar}, k\right) k_j}, \qquad (6.3)$$

where $E_{\mathbf{p}_\alpha}$, $\mathbf{p}_\alpha$, $\mathbf{p}_\beta$, and $E_{\mathbf{p}'_\alpha}$, $\mathbf{p}'_\alpha$, $\mathbf{p}'_\beta$ are the energy and momenta of the colliding particles before and after collisions. In calculating the scattering matrix elements one must bear in mind the energy and momentum conservation laws.

If there is no strong magnetic field in the plasma and the wave functions of the particles in the plasma can be regarded as plane waves, the particle scattering probability assumes the form

$$w_{\alpha\beta}(\mathbf{p}_\alpha, \mathbf{p}'_\alpha) = \frac{2\pi}{\hbar} \left| \frac{4\pi e_\alpha e_\beta \hbar^2}{(\mathbf{p}_\alpha - \mathbf{p}'_\alpha)_i \varepsilon_{ij}\left(\frac{p'^2_\alpha - p^2_\alpha}{2m_\alpha \hbar}, \frac{\mathbf{p}'_\alpha - \mathbf{p}_\alpha}{\hbar}\right)(\mathbf{p}'_\alpha - \mathbf{p}_\alpha)_j} \right|^2$$

Using the scattering probability $w_{\alpha\beta}(\mathbf{p}_\alpha, \mathbf{p}'_\alpha)$, as was done in [25], we can write down for the collision integral of the plasma particles

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_{st} = \sum_\beta \int \frac{d\mathbf{p}'_\alpha}{(2\pi\hbar)^3} d\mathbf{p}_\beta d\mathbf{p}'_\beta \, \delta(\mathbf{p}_\alpha + \mathbf{p}_\beta - \mathbf{p}'_\alpha - \mathbf{p}'_\beta)$$
$$\times \delta\left[\frac{p'^2_\alpha}{2m_\alpha} - \frac{p^2_\alpha}{2m_\alpha} + \frac{p'^2_\beta}{2m_\beta} - \frac{p^2_\beta}{2m_\beta}\right] w_{\alpha\beta}(\mathbf{p}_\alpha, \mathbf{p}'_\alpha)\, [f_\alpha(\mathbf{p}'_\alpha) f_\beta(\mathbf{p}'_\beta) - f_\alpha(\mathbf{p}_\alpha) f_\beta(\mathbf{p}_\beta)]. \qquad (6.4)$$

In the classical limit ($\hbar \to 0$) the particle collision integral can be reduced to the form proposed by Landau

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_{st} = \frac{\partial}{\partial p^i_\alpha} \sum_\beta \int d\mathbf{p}_\beta I^{ij}_{\alpha\beta}(\mathbf{v}_\alpha, \mathbf{v}_\beta) \left[\frac{\partial f_\alpha}{\partial p^j_\alpha} f_\beta - f_\alpha \frac{\partial f_\beta}{\partial p^j_\beta}\right], \qquad (6.5)$$

where

$$I^{ij}_{\alpha\beta}(\mathbf{v}_\alpha, \mathbf{v}_\beta) = \frac{(4\pi e_\alpha e_\beta)^2 \pi}{(2\pi)^3} \int d\mathbf{k}\, \delta(\mathbf{k}\mathbf{v}_\alpha - \mathbf{k}\mathbf{v}_\beta) \frac{k_i k_j}{|k_i \varepsilon_{ij}(\mathbf{k}\mathbf{v}_\alpha, \mathbf{k}) k_j|^2}. \qquad (6.6)$$

A similar collision integral was obtained by many authors.[25-29] The most important feature of this collision integral is that it takes into account both the screening of the field at large distances and the radiation of electromagnetic waves in the plasma. If we put $\epsilon_{ij} \equiv \delta_{ij}$ (vacuum), expression (6.6) goes over into the well known Landau expression [23]

$$I^{ij}_{\alpha\beta}(\mathbf{v}_\alpha, \mathbf{v}_\beta) = (2\pi e_\alpha e_\beta)^2 \frac{1}{\pi} \frac{u^2 \delta_{ij} - u_i u_j}{u^3} L, \qquad (6.7)$$

where $\mathbf{u} = \mathbf{v}_\alpha - \mathbf{v}_\beta$ and

$$L = \int \frac{dk}{k} \approx \ln \frac{k_{max}}{k_{min}} = \ln \frac{\varrho_{min}}{\varrho_{max}}$$

is the so-called Coulomb logarithm. In such an analysis, of course, there is no screening of the field and the integral with respect to $d\mathbf{k}$ must be artificially cut off at small $k \geq k_{min}$ (which corresponds to cutting off the Coulomb particle interaction at large distances ρ_{max}, owing to the Debye screening). We note that the artificial cutoff of the interaction must be made at small distances $\rho \geq \rho_{min}$ (i.e., at large

*The quantum method of analyzing the scattering of particles is clearer than the classical method.

$k \leq k_{max}$). In the classical limit $\rho_{min} \sim e^2/\kappa T$, while in the quantum limit $\rho_{min} \sim e^2/\hbar\langle v\rangle$. If polarization of the medium is taken into account, the interaction of the particles is cut off at large distances automatically, because of the presence of the dielectric-constant tensor in formula (6.6); at small distances, on the other hand,* it is also necessary to introduce in this case artificial cutoff at $\rho = \rho_{min}$.

The expressions obtained above for the plasma-particle collision integral are nonrelativistic and do not take into account the relativistic effects of thermal motion of particles in the plasma. To obtain a relativistic expression for the collision integral we must start not from (6.2) but from the formula

$$E_i(\mathbf{r}) = \frac{e}{2\pi^2}\int d\mathbf{k}e^{i\mathbf{k}(\mathbf{r}-\mathbf{v}t)}\frac{\mathbf{kv}}{c^2}A_{ij}^{-1}v_j, \tag{6.8}$$

where

$$A_{ij} = \left(\frac{\mathbf{kv}}{c}\right)^2\varepsilon_{ij}(\mathbf{kv},\mathbf{k}) - k^2\delta_{ij} + k_ik_j,$$

representing the field of a relativistically charged particle moving with velocity **v** in a medium with dielectric constant $\epsilon_{ij}(\omega, \mathbf{k})$ (see [1,3]). If (6.8) is used for a plasma in the absence of a strong magnetic field, the collision integral is written in the form (6.5), where[25]

$$I_{\alpha\beta}^{ij}(\mathbf{v}_\alpha, \mathbf{v}_\beta) = \left(\frac{4\pi e_\alpha e_\beta}{c^2}\right)^2\int\frac{d\mathbf{k}}{(2\pi)^3}\pi k_ik_j\delta(\mathbf{kv}_\alpha - \mathbf{kv}_\beta) \times |v_\alpha^r A_{re}^{-1}(\mathbf{kv}_\alpha)v_\beta^e|^2. \tag{6.9}$$

In the nonrelativistic limit $(c \rightarrow \infty)$ the expression (6.9) goes over into (6.6). In the particular case of an isotropic plasma, when the dielectric constant tensor has the form (2.1), we obtain

$$I_{\alpha\beta}^{ij}(\mathbf{v}_\alpha, \mathbf{v}_\beta) = \frac{(4\pi e_\alpha e_\beta)^2\pi}{(2\pi)^3}\int d\mathbf{k}\frac{k_ik_j}{k^4}\delta(\mathbf{kv}_\alpha - \mathbf{kv}_\beta) \times \left|\frac{1}{\varepsilon^l(\mathbf{kv}_\alpha, k)} + \frac{k^2\mathbf{v}_\alpha\mathbf{v}_\beta - (\mathbf{kv}_\alpha)^2}{(\mathbf{kv}_\alpha)^2\varepsilon^{tr}(\mathbf{kv}_\alpha, k) - k^2c^2}\right|^2. \tag{6.10}$$

If we put in this expression $\epsilon^l = \epsilon^{tr} = 1$ (vacuum), we obtain the well-known formula of Belyaev and Budker.[31]

It follows from the foregoing that the radiation and absorption of electromagnetic waves in a plasma fits consistently in the kinetics of particle "collisions." A similar conclusion was proposed earlier (see [32,33]), but only for a plasma in a state that differs little from thermal equilibrium. At the present time more and more attention is paid to plasma far from equilibrium, for which the scheme proposed to account for oscillations and waves in the plasma is particularly suitable.

A similar analysis of the collision integral was carried out also for a plasma in a strong magnetic field.[34,35] In this case the scattering matrix elements (6.3) must be calculated by using the wave functions of the charged particles in a strong magnetic field. The nonrelativistic particle collision integral has in this case the form

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_{st} = \sum_{\beta m, n}\int_{-\infty}^{+\infty}dp_{\beta z}\int_0^\infty 2\pi p_{\perp\beta}\,dp_{\perp\beta}\int\frac{d\mathbf{k}}{(2\pi)^3}\left(k_z\frac{\partial}{\partial p_{\alpha z}} + \frac{m\Omega_\alpha}{v_{\perp\alpha}}\frac{\partial}{\partial p_{\perp\alpha}}\right)\pi\delta(k_zv_{z\alpha} + m\Omega_\alpha - k_zv_{z\beta} - n\Omega_\beta)$$
$$\times\left|\frac{4\pi e_\alpha e_\beta}{k_ik_j\varepsilon_{ij}(m\Omega_\alpha + k_zv_{z\alpha}, k_\perp, k_z)}\right|^2 J_m^2\left(\frac{k_\perp v_{\perp\alpha}}{\Omega_\alpha}\right)J_n^2\left(\frac{k_\perp v_{\perp\beta}}{\Omega_\beta}\right)$$
$$\times\left\{k_z\frac{\partial}{\partial p_{z\alpha}} + \frac{m\Omega_\alpha}{v_{\perp\alpha}}\frac{\partial}{\partial p_{\perp\alpha}} - k_z\frac{\partial}{\partial p_{z\beta}} - \frac{n\Omega_\beta}{v_{\perp\beta}}\frac{\partial}{\partial p_{\perp\beta}}\right\} \times f_\alpha(p_{z\alpha}, p_{\perp\alpha})f_\beta(p_{z\beta}, p_{\perp\beta}), \tag{6.11}$$

where $J_n(x)$ is the Bessel function.

In conclusion we point out that the collision integral (6.5) can be written in a form corresponding to the Fokker-Planck equation:

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_{st} = \frac{\partial}{\partial p_\alpha^i}\left[D_{ij}\frac{\partial f_\alpha}{\partial p_\alpha^j} - A_if_\alpha\right], \tag{6.12}$$

where the quantities D_{ij} and A_i, called the diffusion and friction coefficients, respectively, are given by the following expressions

$$D_{ij} = \sum_\beta\int d\mathbf{p}_\beta I_{\alpha\beta}^{ij}(\mathbf{v}_\alpha, \mathbf{v}_\beta)f_\beta,$$
$$A_i = \sum_\beta\int d\mathbf{p}_\beta I_{\alpha\beta}^{ij}(\mathbf{v}_\alpha, \mathbf{v}_\beta)\frac{\partial f_\beta}{\partial p_\beta^j}. \tag{6.13}$$

7. FLUCTUATIONS OF ELECTROMAGNETIC FIELD IN A PLASMA

The theory of thermal fluctuations in media with spatial distribution, particularly in a plasma, can by now be regarded as fully developed.* According to this theory, the thermal fluctuations of electrodynamic quantities in a plasma are described by the dielectric constant tensor. We can thus conclude that the complex dielectric constant tensor describes completely all the linear electromagnetic properties of the plasma. We can thus state that the principal aspect of linear plasma electrodynamics has been actually developed. The remaining problem is to investigate the specific electromagnetic properties of the plasma for a specified particle distribution function and to calculate the dielectric constant tensor. Actually, however, such a statement applies only to an equilibrium plasma. In a dilute plasma, where particle collisions are exceedingly rare, the distribution function may deviate from equilibrium for a long time. Of course, the theory of thermal fluctuations, developed in [3], cannot be used for such a non-equilibrium plasma. We therefore describe in this section briefly the theory developed in [36] for fluctuations in a non-equilibrium plasma

*The quantum expression (6.4), of course, does not diverge at large momentum transfers (i.e., at small distances). Concerning the cutoff of interaction at small distances in the classical limit see [30].

*A review of such a theory and the corresponding bibliography are contained in [1] (see also [1]).

(see also [37]), and show that knowledge of the dielectric constant tensor is not always sufficient.

We introduce the Fourier components of the electric-field operator

$$\hat{\mathbf{E}}(\omega, \mathbf{k}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} dt \int d\mathbf{r}\, \hat{\mathbf{E}}(\mathbf{r}, t)\, e^{i\omega t - i\mathbf{k}\mathbf{r}}. \tag{7.1}$$

To determine the fluctuations of the electric field we must calculate the quantum-mechanical mean of the operator

$$\frac{1}{2}\{\hat{E}_j(\omega, \mathbf{k})\, \hat{E}_i(\omega', \mathbf{k}') + \hat{E}_i(\omega', \mathbf{k}')\, \hat{E}_j(\omega, \mathbf{k})\}. \tag{7.2}$$

To determine the mean value of the operator (7.2) it is sufficient to know the matrix elements of the electric-field operator, which we determine by starting from the matrix elements of the particle current density operator.

We consider the case when there is no strong magnetic field in the plasma and the wave functions of the particles can be regarded as plane. The matrix element of the particle current density α, corresponding to the transition from state n to state m, has in this case the form

$$e_\alpha \alpha_{mn} \exp\left\{\frac{i}{\hbar}\left[(\mathbf{p}_n - \mathbf{p}_m, \mathbf{r}) - (E_n - E_m)\, t\right]\right\}, \tag{7.3}$$

where $\mathbf{p}_n$ and E_n are the momentum and energy of the particle in the state n, while α_{mn} are the Dirac matrices. Substituting (7.3) into Maxwell's equations, we obtain the matrix elements of the electric-field operator

$$\langle m | \hat{E}_i(\omega, \mathbf{k}) | n\rangle = -\frac{4\pi i\omega}{c} e_\alpha A_{ij}^{-1}(\omega, \mathbf{k})\, \alpha_{mn}^j \times \delta\left(\mathbf{k} - \frac{\mathbf{p}_n - \mathbf{p}_m}{\hbar}\right) \delta\left(\omega - \frac{E_n - E_m}{\hbar}\right), \tag{7.4}$$

where

$$A_{ij}(\omega, \mathbf{k}) = \frac{\omega^2}{c^2} \varepsilon_{ij}(\omega, \mathbf{k}) - k^2\delta_{ij} - k_i k_j.$$

Formula (7.4) enables us to calculate the unknown quantum-mechanical mean value of the operator (7.2). Assuming the the distribution function of the particles of kind α with respect to the momenta $\mathbf{p}_\alpha$ is defined by the function $f_\alpha(\mathbf{p}_\alpha)$, we can write[36]

$$\overline{\frac{1}{2}[\hat{E}_j(\omega, \mathbf{k})\, \hat{E}_i(\omega', \mathbf{k}') + \hat{E}_i(\omega', \mathbf{k}')\, \hat{E}_j(\omega, \mathbf{k})]} = \delta(\omega + \omega')\, \delta(\mathbf{k} + \mathbf{k}')\, (E_i E_j)_{\omega, \mathbf{k}}, \tag{7.5}$$

Here the superior bar denotes quantum-mechanical averaging, and

$$\begin{aligned}(E_i E_j)_{\omega, \mathbf{k}} = {} & \frac{1}{4} \sum_\alpha (4\pi e_\alpha)^2 \frac{\omega^2}{c^2} A_{ie}^{-1}(\omega, \mathbf{k})\, A_{jr}^{*-1}(\omega, \mathbf{k}) \\ & \times \int d\mathbf{p}_\alpha f_\alpha(\mathbf{p}_\alpha) \left\{\delta\left[\omega - \frac{1}{\hbar}\{E(\mathbf{p}_\alpha + \hbar\mathbf{k}) - E(\mathbf{p}_\alpha)\}\right]\right. \\ & \times \left(\frac{c^2 p_\alpha^r (p_\alpha + \hbar \mathbf{k})^l}{E(\mathbf{p}_\alpha)\, E(\mathbf{p}_\alpha + \hbar\mathbf{k})} - \delta_{lr} \frac{[E(\mathbf{p}_\alpha + \hbar\mathbf{k}) - E(\mathbf{p}_\alpha)]^2 - c^2 k^2 \hbar^2}{2E(\mathbf{p}_\alpha)\, E(\mathbf{p}_\alpha + \hbar\mathbf{k})}\right. \\ & \left. + \frac{c^2 p_\alpha^l (p_\alpha + \hbar\mathbf{k})^r}{E(\mathbf{p}_\alpha)\, E(\mathbf{p}_\alpha + \hbar\mathbf{k})}\right) + \delta\left[\omega + \frac{1}{\hbar}\{E(\mathbf{p}_\alpha - \hbar\mathbf{k}) - E(\mathbf{p}_\alpha)\}\right] \\ & \times \left(\frac{c^2 p_\alpha^r (p_\alpha - \hbar\mathbf{k})^l}{E(\mathbf{p}_\alpha)\, E(\mathbf{p}_\alpha - \hbar\mathbf{k})} + \frac{c^2 p_\alpha^l (p_\alpha - \hbar\mathbf{k})^r}{E(\mathbf{p}_\alpha)\, E(\mathbf{p}_\alpha - \hbar\mathbf{k})}\right. \\ & \left.\left. - \delta_{lr} \frac{[E(\mathbf{p}_\alpha - \hbar\mathbf{k}) - E(\mathbf{p}_\alpha)]^2 - c^2\hbar^2 k^2}{2E(\mathbf{p}_\alpha)\, E(\mathbf{p}_\alpha - \hbar\mathbf{k})}\right)\right\}.\end{aligned} \tag{7.6}$$

In the classical limit, as $\hbar \to 0$, we therefore obtain

$$(E_i E_j)_{\omega, \mathbf{k}} = \sum_\alpha \left(4\pi e_\alpha \frac{\omega}{c}\right)^2 A_{ie}^{-1}(\omega, \mathbf{k})\, A_{jr}^{*-1}(\omega, \mathbf{k}) \times \int d\mathbf{p}_\alpha f_\alpha(\mathbf{p}_\alpha) \frac{v_\alpha^l v_\alpha^r}{c^2} \delta(\omega - \mathbf{k}\mathbf{v}_\alpha). \tag{7.7}$$

In the particular case of an isotropic particle distribution $f_\alpha(\mathbf{p}_\alpha)$, when the dielectric constant tensor of the plasma is given by (2.1), formula (7.7) becomes

$$(E_i E_j)_{\omega, \mathbf{k}} = \sum_\alpha \left(\frac{4\pi e_\alpha}{k^2}\right)^2 \int d\mathbf{p}_\alpha f_\alpha(\mathbf{p}_\alpha)\, \delta(\omega - \mathbf{k}\mathbf{v}_\alpha) \times \left\{\frac{k_i k_j}{|\varepsilon^l(\omega, k)|^2} + \frac{1}{2} \frac{\omega^2}{c^4} \frac{[\mathbf{k}\mathbf{v}_\alpha]^2 (k^2\delta_{ij} - k_i k_j)}{\left|\frac{\omega^2}{c^2}\varepsilon^{tr}(\omega, k) - \frac{k^2}{c^2}\right|}\right\}. \tag{7.8}$$

In the case of an equilibrium Maxwellian particle distribution $f_\alpha(\mathbf{p}_\alpha)$, formulas (7.6)—(7.8) go over into the well-known formulas of the theory of thermal fluctuations. [1,3]

Using Maxwell's equations we can readily obtain, with the aid of the formulas derived for the electric-field fluctuations, expressions for the fluctuations of the magnetic induction, the electric induction, the charge and current densities induced in the plasma, etc. We give here an expression for the flucutations of the Lorentz force acting on a particle of kind α: [36]

$$\begin{aligned}\left\{e_\alpha^2\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}_\alpha\mathbf{B}]\right)_i \left(\mathbf{E} + \frac{1}{c}[\mathbf{v}_\alpha\mathbf{B}]\right)_j\right\}_{\omega, \mathbf{k}} = {} & \sum_\beta \left(\frac{4\pi e_\alpha e_\beta \omega}{c^2}\right)^2 \times \int d\mathbf{p}_\beta f_\beta(\mathbf{p}_\beta)\, \delta(\omega - \mathbf{k}\mathbf{v}_\beta)\, v_\beta^l v_\beta^r \left\{\delta_{is}\left[1 - \frac{\mathbf{k}\mathbf{v}_\alpha}{\omega}\right]\right. \\ & \left. + \frac{k_i v_\alpha^s}{\omega}\right\} A_{sl}^{-1}(\omega, \mathbf{k})\, A_{tr}^{*-1}(\omega, \mathbf{k}) \left\{\delta_{jt}\left[1 - \frac{\mathbf{k}\mathbf{v}_\alpha}{\omega}\right] + \frac{k_j v_\alpha^t}{\omega}\right\}.\end{aligned} \tag{7.9}$$

The theory of linear fluctuations in a non-equilibrium plasma is directly connected with the kinetic "collisions" of the particles in the plasma (radiation and absorption of waves in the plasma), and consequently also with the recently developing quasi-linear approach to the plasma theory. [38,39] Actually, comparing formulas (7.9), (6.9), and (6.13) we can write for the diffusion and friction coefficients contained in the collision integral (6.12)

$$\left.\begin{aligned} D_{ij} &= \pi \int \frac{d\mathbf{k}}{(2\pi)^3} \left\{e_\alpha^2 \left(\mathbf{E} + \frac{1}{c}[\mathbf{v}_\alpha\mathbf{B}]\right)_i \times \left(\mathbf{E} + \frac{1}{c}[\mathbf{v}_\alpha\mathbf{B}]\right)_j\right\}_{\mathbf{k}\mathbf{v}_\alpha, \mathbf{k}}, \\ A_i &= \frac{i}{2} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{k_i}{4\pi} \left\{\varepsilon_{tr}(\mathbf{k}\mathbf{v}_\alpha, \mathbf{k}) - \varepsilon_{tr}^*(\mathbf{k}\mathbf{v}_\alpha, \mathbf{k})\right\} E_r^{(\alpha)*}(\mathbf{k}\mathbf{v}_\alpha, \mathbf{k})\, E_t^{(\alpha)}(\mathbf{k}\mathbf{v}_\alpha, \mathbf{k}), \end{aligned}\right\} \tag{7.10}$$

where

$$E_i^{(\alpha)}(\mathbf{k}\mathbf{v}_\alpha, \mathbf{k}) = -\frac{4\pi i \mathbf{k}\mathbf{v}_\alpha}{c^2} e_\alpha A_{ij}^{-1}(\mathbf{k}\mathbf{v}_\alpha, \mathbf{k})\, v_j^\alpha$$

is the field produced by particle α, moving with velocity $\mathbf{v}_\alpha$ in the plasma. If only the Coulomb field is taken into account, the coefficient D_{ij} corresponds to the coefficient obtained in [29].

In conclusion we give an expression for the fluctuations of the Coulomb field in a plasma situated in a strong homogeneous magnetic field:[35]

$$(E_i E_j)_{\omega, \mathbf{k}} = \sum_{\alpha} \left(\frac{4\pi l e_{\alpha} \omega}{c} \right)^2 A_{il}^{-1}(\omega, \mathbf{k}) A_{jr}^{*-1}(\omega, \mathbf{k})$$

$$\times \int d\mathbf{p}_{\alpha} f_{\alpha}(\mathbf{p}_{\alpha}) \sum_m \delta(\omega - m\Omega_{\alpha} - k_z v_z^{\alpha}) \pi_{ij}^{(m)}, \qquad (7.11)$$

where $\pi_{ij}^{(m)}$ is the tensor that appears in the theory of the dielectric constant of a plasma [see expression (4.1)].

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