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THE DYNAMICAL PROPERTIES OF ELEMENTARY PARTICLES AND THE THEORY OF THE SCATTERING MATRIX

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INTRODUCTION

THE dynamical properties of elementary particles, which manifest themselves when the particles interact, are very complex, since there are an infinite variety of transformations of particles into one another in which the numbers and character of the particles change. To describe the structure of these interactions one must give a set of functions, called scattering amplitudes, or generalized form factors. Finding these scattering amplitudes is the main problem of the theory of elementary particles. At present there exists no system of equations for determining the amplitudes. There are only isolated relations between them and a general understanding of how many functions of what variables are of interest to us, and how these functions are connected with experimentally observed phenomena.

The simplest types of amplitudes are those which describe electromagnetic and certain other properties of particles. Electromagnetic form factors are functions of a single variable (the momentum or wave vector), so that they can be related to a definite spatial structure by using the Fourier transform. In this way we get an illustrative interpretation.

We begin our presentation with the nonrelativistic

example of the description of the simplest fixed system of charges, which can serve as a model for the atom. After introducing with this example the main quantities which are needed for the formal apparatus of the theory (Sec. 1), we go on in Sec. 2 to the general formulation of the problem of the relativistic theory of elementary particles and in Sec. 3 to the general properties of scattering amplitudes and the theory of dispersion relations.

The theory of elementary particles was developed as a field theory, following the example of quantum electrodynamics. Gradually, however, with the development of the physics of strongly interacting particles it became clear that the connection between fields and particles is not so direct. It is only in the realm of weak interactions that we can, from the types of particles observed in experiment, reconstruct the fields associated with them. For this reason two lines of development began. The first was the construction of more complicated (i.e., less naive) models of fields. The second was to free oneself altogether from the concepts of field theory. Actually these lines of attack are not mutually contradictory. The point is that in field theory there are really no methods for solving problems, aside from perturbation theory. Except for it, one can obtain only general relations based on general properties of fields. The idea naturally arises of starting the development of the theory from this stage, by formulating some general properties for quantities which have a more direct physical interpretation than the field. The elements of the scattering matrix are quantities of this sort. An outline of the theory of the scattering matrix, in which field theory is not used directly, is the main content of this article.

The remaining two sections are devoted to applications.

1. FORM FACTORS AND THEIR SPECTRAL REPRESENTATION

1. Form factors. The simplest picture of a particle is a small sphere with some stationary distribution of electric charge. Its structure is determined by the dependence of the charge density on distance to the center of the particle. We shall denote this function by $e\rho(\mathbf{r})$ or, in order to emphasize the assumed spherical symmetry of the charge distribution, by $e\rho(\mathbf{r}^2)$ (where e is the charge on the electron, and is introduced here as a convenient unit for measurement of charge).

Formally, to describe the structure of such an electrostatic system we can equally well use quantities related to the function $\rho(\mathbf{r}^2)$ via Fourier or Laplace transformations.

The quantity

$$f(\mathbf{q}) = \int \varrho(\mathbf{r}) e^{i\mathbf{q}\mathbf{r}}\, d\mathbf{r} \tag{1.1}$$

is called the form factor of the system. If the charge distribution is spherically symmetric, $\rho = \rho(\mathbf{r}^2)$, the form factor is a function of the single variable $\mathbf{q}^2$. We shall denote it by $f = f(-\mathbf{q}^2)$. Then (1.1) can be written as

$$qf(-\mathbf{q}^2) = 4\pi \int_0^\infty r\varrho(r^2) \sin qr\, dr, \tag{1.2}$$

$$r\varrho(r^2) = \frac{1}{2\pi^2} \int_0^\infty qf(-\mathbf{q}^2) \sin qr\, dq \qquad (q = \sqrt{\mathbf{q}^2}). \tag{1.3}$$

We note that f is a real quantity, as is obvious from (1.2).

We also introduce the quantity l, which is related to ρ via the Laplace transformation:

$$r\varrho(r^2) = \frac{1}{2\pi^2} \int_0^\infty \alpha l(\alpha^2) e^{-\alpha r}\, d\alpha. \tag{1.4}$$

From (1.2) and (1.4) it follows that

$$f(-\mathbf{q}^2) = \frac{1}{\pi} \int_0^\infty \frac{l(\alpha^2)}{\alpha^2 + \mathbf{q}^2}\, d\alpha^2. \tag{1.5}$$

The relation (1.5) is called the spectral representation of the form factor, while the function $l(\alpha^2)$ is the spectral density.

We express the total charge of the system

$$e \int \varrho\, d\mathbf{r} \equiv eQ$$

and the moments of the charge distribution

$$\int r^{2n} \varrho(r^2)\, d\mathbf{r} = Q_n$$

in terms of the form factor and the spectral density. From these definitions it follows that

$$Q = f(0) = \frac{1}{\pi} \int_0^\infty \frac{l(\alpha^2)\, d\alpha^2}{\alpha^2}, \tag{1.6}$$

$$Q_n = \frac{(2n+1)!}{n!} f^{(n)}(0) = \frac{(2n+1)!}{\pi} \int_0^\infty \frac{l(\alpha^2)\, d\alpha^2}{\alpha^{2(n+1)}}. \tag{1.7}$$

2. Asymptotic properties and types of representations. Using the spectral density $l(\alpha^2)$, one can conveniently describe the simplest types of charge distributions.

Suppose, for example, that $l(\alpha^2)$ is different from zero only in the neighborhood of the value $\alpha = 1/a$ (i.e., $l(\alpha^2) \to (\pi/a^2)\,\delta(\alpha^2 - 1/a^2)$). Then according to (1.4) and (1.6)

$$r\varrho(r^2) = \frac{Q}{4\pi a^2} e^{-r/a}. \tag{1.8}$$

Such a distribution corresponds to the simplest model of the electron shell of an atom of radius a. The expansion (1.4) can thus be interpreted as a description of the charge distribution as a set of "shells" with radii $a = 1/\alpha$.

When $a \to 0$, the distribution (1.8) describes a point charge:

$$r\varrho(r^2) = \left(\frac{Q}{4\pi a^2} e^{-r/a}\right)_{a\to 0} = \frac{Q}{4\pi r}\,\delta(r). \tag{1.9}$$

From (1.4) we see that small values of α are important for the distribution at large r. Suppose that $l(\alpha^2)$ is different from zero only for $\alpha > \mu$. Then

$$r\varrho(r^2) = \frac{1}{4\pi^2}\int_{\mu^2}^{\infty} e^{-\alpha r} l(\alpha^2)\, d\alpha^2 = \frac{e^{-\mu r}}{2\pi^2}\int_0^{\infty} e^{-\beta r}(\mu+\beta)\, l\left((\mu+\beta)^2\right) d\beta.$$

For $r \to \infty$ we can expand the integrand of the last expression in powers of β. If

$$l\left((\mu+\beta)^2\right) = \lambda\beta^{\sigma},$$

then

$$r\varrho(r^2) = \frac{\mu\lambda}{2\pi^2}\Gamma(\sigma+1)\frac{e^{-\mu r}}{r^{\sigma+1}}. \tag{1.10}$$

We see that asymptotically the charge density falls off exponentially and that the character of the falloff is determined by the threshold argument of the spectral density, the value μ. (For $\mu = 0$, the charge density falls off according to a power law.)

Conversely, the asymptotic behavior of the spectral density and the form factor is determined by the behavior of $\rho(r^2)$ as $r \to 0$.

If $r\rho(r^2)$ is finite, then according to (1.4) the integral

$$L = \int_{\mu^2}^{\infty} l(\alpha^2)\, d\alpha^2 = 4\pi^2 (r\varrho)_{r=0},$$

should exist, i.e., $l(\alpha^2)$ falls off like $1/\alpha^{2\lambda}$, where $\lambda > 1$. From (1.5) we find that the asymptotic expression for the form factor has the following form:

$$f(-\mathbf{q}^2)_{\mathbf{q}^2\to\infty} = \frac{L}{\pi\mathbf{q}^2}. \tag{1.11}$$

For the case where $\rho(0)$ is finite, i.e., L = 0, the form factor falls off faster:

$$f(-\mathbf{q}^2)_{\mathbf{q}^2\to\infty} = -\frac{L'}{\pi q^4}, \tag{1.12}$$

where

$$L' = \int l(\alpha^2)\,\alpha^2\, d\alpha^2 = 4\pi^2\left(\frac{d^2(r\varrho)}{dr^2}\right)_{r=0}.$$

If $r\rho(r^2)$ goes to infinity for $r \to 0$, i.e., if the integral L does not exist, then $f(-\mathbf{q}^2)$ falls off for $\mathbf{q}^2 \to \infty$ more slowly than according to formula (1.11). In fact, in this case, starting from (1.5), we can write the asymptotic expression for $f(-\mathbf{q}^2)$ in the following form:

$$f(-\mathbf{q}^2) = \frac{1}{\pi}\left\{\int_0^a l(\xi\mathbf{q}^2)\, d\xi + \int_a^b \frac{l(\xi\mathbf{q}^2)\, d\xi}{1+\xi} + \int_b^{\infty} \frac{l(\xi\mathbf{q}^2)}{\xi}\, d\xi\right\},$$

where a and b are numbers chosen so that $a \ll 1 \ll b$. Suppose that the asymptotic behavior of $l(\alpha^2)$ is given by a power law, so that

$$l(\xi\mathbf{q}^2) = l(\xi)\frac{1}{(\mathbf{q}^2)^{\lambda}},$$

then

$$f(-\mathbf{q}^2) \sim \frac{C}{(\mathbf{q}^2)^{\lambda}}. \tag{1.13}$$

From the condition $L = \infty$ and the finiteness of $\int \frac{l(\alpha^2)\, d\alpha^2}{\alpha^2} \equiv L_1$ it follows that the exponent is contained within the following limits:

$$0 < \lambda \leqslant 1.$$

The coefficient C in (1.13) can depend weakly (for example, logarithmically) on $\mathbf{q}^2$.

We see that every distribution which can be represented in the form of the spectral resolution (1.5) leads to $f(-\infty) = 0$, if the total charge of the system is finite. If the form factor tends to a constant for $\mathbf{q}^2 \to \infty$, $f(-\infty) = \text{const}$, we can, by subtracting this value, get back to the previous case:

$$f(-\mathbf{q}^2) - f(-\infty) = \frac{1}{\pi}\int_0^{\infty} \frac{l(\alpha^2)\, d\alpha^2}{\alpha^2 + \mathbf{q}^2}. \tag{1.14}$$

Taking the Fourier transform, we have

$$r\varrho(r^2) = \frac{f(-\infty)}{4\pi r}\delta(r) + \frac{1}{4\pi^2}\int_0^{\infty} l(\alpha^2)\, e^{-\alpha r}\, d\alpha^2. \tag{1.15}$$

This means that the case of $f(-\infty) \neq 0$ means that there is a point charge $Q_0 = f(-\infty)$ at the center of the particle. From (1.14) and (1.15) it follows that the total charge of the system is

$$Q = f(0) = f(-\infty) + \frac{1}{\pi}\int_0^{\infty} \frac{l(\alpha^2)\, d\alpha^2}{\alpha^2} = Q_0 + L_1. \tag{1.16}$$

Eliminating $f(-\infty)$ from (1.14) and (1.16), we can write the spectral representation of the form factor in the following form:

$$f(-\mathbf{q}^2) - f(0) = -\frac{\mathbf{q}^2}{\pi}\int_0^{\infty} \frac{l(\alpha^2)\, d\alpha^2}{\alpha^2(\alpha^2+\mathbf{q}^2)}. \tag{1.17}$$

The representation (1.17) allows us to treat the case where $f(-\mathbf{q}^2) \to \infty$. This will occur if $L_1 = \int \frac{d\alpha^2 l(\alpha^2)}{\alpha^2} = \infty$, while the integral $L_2 = \int \frac{l(\alpha^2)\, d\alpha^2}{\alpha^4}$ exists. We then also have $Q_0 = \infty$, i.e., the system contains an infinite point charge which compensates the distributed charge described by the spectral density $l(\alpha^2)$, where in the limit $\alpha \to \infty$, $l(\alpha^2)$ either tends toward a constant or goes to infinity, but in such a way that the total charge f(0) of the system is finite. As we shall see later, such a system is a model for the electron in quantum electrodynamics.

3. Dispersion relations. We present another formal procedure which will be very important in what follows. First we rewrite the fundamental formula (1.15) for the spectral representation of the form factor in the following form:

$$f(t) = \frac{1}{\pi}\int_{t_0}^{\infty} \frac{l(t')\, dt'}{t'-t}. \tag{1.18}$$

We note that, from the meaning of the form factor, the function f(t) is defined for negative values of the argument $t = -\mathbf{q}^2 < 0$, and the function $l(t')$ for positive

values of the argument $t' = \alpha^2 > 0$. For convenience we have introduced a lower integration limit t_0 in (1.18), assuming that $l(t') = 0$ for $t' < t_0$ (where, in particular, t_0 may be equal to zero).

Now we introduce the complex variable $t = x + iy$ and use formula (1.18) to define the function $f(t)$ over the whole complex t plane. Let us find the properties of this function. First we find the value of Im $f(t)$. From (1.18) it immediately follows that

$$\operatorname{Im} f(t) = \frac{1}{\pi} y \int_{t_0}^{\infty} \frac{l(x')\,dx'}{(x'-x)^2+y^2},$$

from which we see that:

1) for $y \neq 0$, i.e., off the real axis, Im $f \neq 0$;

2) for $y = 0$ and $x < t_0$, the denominator of the integrand does not vanish within the range of integration, the integral is finite and therefore Im $f(t) = 0$, in agreement with the meaning of the definition (1.4) of the form factor;

3) for $x > t_0$ and $y \to 0$, Im $f(t)$ is different from zero because of the region near the zero of the denominator of the integrand, $x' \to x$. Therefore

$$\operatorname{Im} f(x+iy)_{\substack{y\to 0\\ t>t_0}} = \lim_{y\to 0} \frac{y}{\pi} l(x) \int_{-\infty}^{\infty} \frac{d\xi}{\xi^2+y^2} = \frac{y}{|y|} l(x).$$

This same result could have been obtained directly from (1.18) by using the well-known relation

$$\operatorname{Im} \frac{1}{x'-x-iy} = \pi \frac{y}{|y|} \delta(x'-x). \tag{1.19}$$

Thus the function $f(t)$ has discontinuities along the line which is part of the real axis, from $x = t_0$ to $x = \infty$. As we approach this line from above and below, Im $f(t)$ takes on different values:

$$\operatorname{Im} f(x \pm i0) = \pm l(x) \qquad (x > t_0). \tag{1.20}$$

Furthermore we see from (1.18) that if the function $f(t)$ exists for real $t < 0$, then it exists for all complex t, where $f(\infty) = 0$. Thus $f(t)$ is an analytic function in the t plane with a cut along the branch line $x > t_0$. The function $f(t)$ has no singularities except those associated with the branch line.

Using (1.20), we can rewrite (1.18) as follows:

$$f(t) = \frac{1}{\pi} \int_{t_0}^{\infty} \frac{\operatorname{Im} f(x'+i0)}{x'-t}\,dx'. \tag{1.21}$$

Integral relations like (1.21), which express an analytic function in terms of its imaginary part on a branch line, are usually called dispersion relations. The dispersion relation (1.21) is equivalent to the assertion that the function $f(t)$ is analytic in this domain. In fact it can be obtained from the Cauchy formula by deforming the contour of integration to the form shown in Fig. 1. We note that the integral along the circle of infinite radius goes to zero because $l(\infty) = 0$, which in turn follows from the finiteness of the charge of the system, $L_1 = \frac{1}{\pi}\int \frac{l(\alpha^2)\,d\alpha^2}{\alpha^2}$.

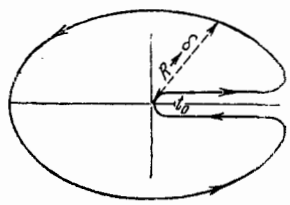

FIG. 1

Relation (1.20) remains valid also for the case where the form factor is not representable in the form (1.5), but where the spectral representation (1.17) exists. In this case the analytic function with the stated properties is the function

$$\frac{f(t)-f(0)}{t} = \frac{1}{\pi} \int_{t_0}^{\infty} \frac{\operatorname{Im} f(x'+i0)}{x'(x'-t)}\,dx'. \tag{1.22}$$

4. Form factors and scattering amplitudes. The form factor of a particle is more directly related to quantities that are measured experimentally than is the density distribution of the charge in space; it is therefore a more fundamental quantity in the theory of elementary particles. Actually the structure of a particle can be studied by measuring the scattering of test particles.

Let us consider the formulation of such a problem in nonrelativistic quantum mechanics. Suppose the test particle (we shall call it an electron) with point charge e and mass m is scattered by the particle with mass M and charge distribution $e\rho(r^2)$. We denote by $\mathbf{k}_1$ and $\mathbf{k}_2$ the initial and final momenta of the electron, by $\mathbf{p}_1$ and $\mathbf{p}_2$ the initial and final momenta of the particle, and by $\hbar\mathbf{q}$ the change in momentum in the scattering,

$$\hbar\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_2 = \mathbf{k}_2 - \mathbf{k}_1. \tag{1.23}$$

We refer all quantities to the center of inertia of the colliding particles:

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{k}_1 + \mathbf{k}_2 = 0, \quad \hbar^2\mathbf{q}^2 = 2\mathbf{p}_1^2(1-\cos\theta),$$

where θ is the scattering angle in the center-of-mass system.

Suppose that the scattering occurs under conditions for which perturbation theory (Born approximation) is applicable. Then the scattering amplitude F, which is related to the differential cross section $d\sigma$ by the formula

$$d\sigma = |F^2|\,do, \tag{1.24}$$

where do is the element of solid angle in the c.m.s., is given by the Born formula

$$F = -\frac{\bar{m}}{2\pi\hbar^2} U(\mathbf{q}), \tag{1.25}$$

where $\bar{m}$ is the reduced mass $\bar{m} = \frac{mM}{m+M}$, and

$$U(\mathbf{q}) = \int U(\mathbf{r})\, e^{i\mathbf{q}\mathbf{r}}\,d\mathbf{r}$$

is the corresponding Fourier component of the potential energy of the electron in the field of the scatterer $U(\mathbf{r})$. We can represent the function $U(\mathbf{r})$ in the form

$$U(\mathbf{r}) = e\Phi(\mathbf{r}),$$

where $\Phi(\mathbf{r})$ is the electrical potential of the scatterer. Since

$$\Delta\Phi(\mathbf{r}) = -e\varrho(r^2),$$

we have

$$\Phi(\mathbf{q}) = \frac{ef(-\mathbf{q}^2)}{\mathbf{q}^2},$$

where $f(-\mathbf{q}^2)$ is the form factor of the scatterer. Thus

$$U(\mathbf{q}) = e\Phi(\mathbf{q}) = \frac{e^2 f(-\mathbf{q}^2)}{\mathbf{q}^2} \qquad (1.26)$$

and

$$F = -\frac{\bar{m}e^2}{2\pi\hbar^2}\frac{f(-\mathbf{q}^2)}{\mathbf{q}^2}. \qquad (1.27)$$

We note that formula (1.26) can also be written as

$$U(\mathbf{q}) = ef(-\mathbf{q}^2)\varphi(\mathbf{q}), \qquad (1.28)$$

where $\varphi(\mathbf{q}) = e/\mathbf{q}^2$ is the Fourier component of the potential of a point charge, $\varphi(\mathbf{r}) = e/r$.

Formula (1.28) holds when the question is so formulated that the test particle itself does not appear, but only its field, in which the particle under investigation is scattered. Formulas (1.25) and (1.28) are also valid for the scattering of particles with distributed charge in an arbitrary external potential φ (in the Born approximation).

2. THE SCATTERING AMPLITUDE

1. Formulation of the problem in relativistic quantum theory. In nonrelativistic theory one can consider a particle of finite size with a given spatial distribution of charge. One can assign to the particle either three degrees of freedom (point mass) or six degrees of freedom (rigid body). This is impossible in a relativistic theory, since the theory of relativity does not admit the existence of absolutely rigid bodies. The distribution of charge and mass must depend on the state of motion of the particle, i.e., it is a dynamical quantity. Then the particle must be described as a system with an infinite number of degrees of freedom (a density at each point). The difficulties associated with this fact already came to light in the development of the theory of the electron and remained unresolved within the framework of the classical electrodynamics of Lorentz. For this reason classical relativistic theory is restricted to treating point particles. This treatment makes sense in the limit where one can avoid the familiar paradoxes associated with point charges (infinite energy of the electron, etc). Actually the necessity for including quantum effects arises before the internal contradictions of the classical theory of the point electron begin to have an effect.

In relativistic quantum theory the situation is at one and the same time more complicated and simpler. Relativistic quantum theory from the very outset deals with processes in which the number of particles is not conserved (radiation and absorption of photons and mesons, pair formation, etc). Thus, in relativistic quantum theory, objects with a fixed number of degress of freedom are impossible in principle. It deals with systems with an infinite number of degrees of freedom; these are usually described by fields.

On the other hand, the existence of processes in which the number of particles is not conserved, i.e., processes in which some particles are transformed into others, shows that the particles cannot be point-like. This is a direct consequence of the uncertainty relation. Let us give some simple examples.

1) A hydrogen atom can decay into a proton and an electron. We know the energy I required for exciting this process (the binding energy). It can be written as $I \sim mv^2/2$, where m is the electron mass and v is its orbital velocity. The uncertainty relation enables us to determine the radius of the hydrogen atom, $a \sim \hbar/mv$.

2) A neutron can decay into a proton and a π meson. The energy required for this process is $I \sim \mu c$, where μ is the π-meson mass. From the uncertainty relation it follows that the neutron radius is $a \sim \hbar/\mu c$.

From the general point of view it is not significant that we usually regard the hydrogen atom as being a composite particle and the neutron an elementary particle. This is related only to the fact that in the first case the binding energy is small compared to the rest energy of the electron mc^2, while in the second case it is almost equal to the rest energy μc^2. The difference is only a quantitative one.

This discussion suggests a natural approach to the problem of the structure of a particle in relativistic quantum theory. There is no need to ascribe to the particle any internal degrees of freedom. The structure of the particle is determined by those same degrees of freedom of the system (fields) which determine the processes of scattering and transformation of particles. The particle itself is only one of the states of this system. The simplest process of elastic scattering of the particle in a given electromagnetic field is also actually a complex phenomenon, in which there participate, in principle, all the degrees of freedom of the system. Let us suppose that we have found this amplitude and that its form, corresponding to formulas (1.25) and (1.28), is

$$F(\mathbf{q}) \sim f(\mathbf{q})\varphi(\mathbf{q}),$$

i.e., proportional to the corresponding Fourier component of the field $\varphi(\mathbf{q})$. Then $f(\mathbf{q})$ will, by definition, be the form factor of the particle, and the corresponding spatial structure $\rho(\mathbf{r})$ is determined by the Fourier transform

$$\varrho(\mathbf{r}) = \frac{1}{(2\pi)^3}\int f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} d^3q.$$

Thus the problem of the structure of particles reduces completely to the problem of scattering of particles. Here each particle in the scattering problem is characterized only by its momentum and energy (and its spin state), i.e., by those variables which are attributed to the point particle in quantum mechanics.

2. The scattering matrix and the invariant amplitude. The general problem of scattering in relativistic quantum theory is formulated as follows. We know that, at time $t = -\infty$, the system is in state a. We are required to find the probability that, at time $t = +\infty$, it will be in state b. This probability is expressed in terms of the probability amplitude S_{ba}:

$$w_{ba} = |S_{ba}|^2. \tag{2.1}$$

The set of quantities S_{ba} is called the scattering matrix S.

The states a and b are characterized by definite numbers of particles N_a and N_b (different in general), the momenta of these particles, their polarizations, and other properties. In practice, two types of initial states are of interest: 1) $N_a = 1$, where w_{ba} determines the probability of decay of the particle into several particles in state b; 2) $N_a = 2$, where w_{ba} determines the probability of collision of the two particles in which they are converted into a set of particles in state b. In particular, if $N_b = 2$ and the particles in state b are the same as those in state a, w_{ba} determines the probability for elastic scattering (for example, of an electron by a proton).

The scattering matrix is usually written in the form

$$S = 1 + iT, \tag{2.2}$$

where 1 is the unit matrix. The matrix T indicates the occurrence of the process; if $T = 0$, there is no scattering (for example, when $N_a = 1$, $T = 0$ means that the particle is stable).

The first fundamental property of the scattering matrix is its relativistic invariance. But this assertion assumes that the states a and b are also assigned in an invariant way. Actually in practice, for simplicity, one usually introduces certain noninvariant quantities in the definition of the state of a particle, for example, a normalization volume Ω within which the system is contained. These noninvariant quantities appear only in the intermediate stages of the calculation of cross sections and life times and vanish in the final expressions which have physical significance. It is therefore convenient to extract the corresponding normalization factors from the matrix elements of T.

First, using the conservation of energy and momentum, we write

$$T_{ba} = (2\pi)^4 \delta(p_b - p_a)\,\Gamma_{ba}, \tag{2.3}$$

where p_a and p_b are the 4-momenta of the initial and final states and δ is the four-dimensional δ function. We also write Γ_{ba} in the following form:*

$$\Gamma_{ba} = U_{ba}\prod_\alpha \frac{\xi_\alpha}{\sqrt{2\Omega\varepsilon_\alpha}}, \tag{2.4}$$

where Ω is the normalization volume, ϵ_α the energy of the particle, ξ_α an arbitrary factor; the index α runs over all particles in both the initial and final states.†

According to (2.1), (2.2), and (2.3), the probability of transition per unit time has the form‡

$$\dot{w}_{ba} = (2\pi)^4 |\Gamma_{ba}|^2 \delta(p_b - p_a)\Omega. \tag{2.5}$$

In order to find the differential probability $d\dot{w}$ for transition into an infinitesimal final-state interval characterized by having the particle momenta in the interval $\mathbf{p}_\beta$ to $\mathbf{p}_\beta + d\mathbf{p}_\beta$, we must multiply (2.5) by the statistical weight ρ_b of the final state

$$\varrho_b = \prod_\beta \frac{\Omega\, d^3p_\beta}{(2\pi)^3} = \prod_\beta \frac{\Omega}{(2\pi)^3} 2\varepsilon_\beta d^4p_\beta \delta(p_\beta^2 - m_\beta^2)\theta(\varepsilon_\beta), \tag{2.6}$$

where the subscript β refers to the particles in the final state, m_β is the mass of the particle and

$$\theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

From (2.5), (2.4), and (2.6), we get

$$d\dot{w} = \frac{\Omega^{1-N_a}}{(2\pi)^{3N_b-4}} |U_{ba}|^2 \prod_\alpha \frac{\xi_\alpha^2}{2\varepsilon_\alpha} \prod_\beta \xi_\beta^2 d^4p_\beta \delta(p_\beta^2 - m_\beta^2)\delta(p_b - p_a)\theta(\varepsilon_\beta), \tag{2.7}$$

where the index α refers to particles in the initial state, and β to those in the final state.

We shall apply formula (2.7) to the two fundamental problems of scattering theory.

1) Decay of a particle, $N_a = 1$. The quantity of interest is the invariant $d\tau^{-1}$, the differential probability for decay in the rest system of the particle. Then $\epsilon_\alpha = \epsilon_a = m_a$ and

$$d\tau^{-1} = \frac{\xi_a^2}{2(2\pi)^{3N_b-4}} |U_{ba}|^2 \delta(p_b - p_a) \prod_\beta \xi_\beta^2 d^4p_\beta \delta(p_\beta^2 - m_\beta^2)\theta(\varepsilon_\beta). \tag{2.8}$$

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*The factors in (2.4) can be understood as follows. Suppose the particle is described by the wave function $\psi = Ae^{ikx}$. If the particle has spin zero, the normalization condition has the form

$$i\int\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right)d^3x = |A|^2 2\varepsilon\Omega = 1,$$

i.e. $A = \xi/\sqrt{2\Omega\varepsilon}$, $\xi = 1$. If the particle has spin ½, the normalization condition can be chosen in the form $\int\psi^*\psi\, d^3x = \bar{A}A\frac{\varepsilon}{m}\Omega = 1$ and $A = \frac{\xi}{\sqrt{2\Omega\varepsilon}}u$, where $\xi = \sqrt{2m}$, and u is a four-component spinor, normalized so that $\bar{u}u = 1$ ($\bar{u} = u^*\gamma_4$).

†From now on we use the natural system of units in which $\hbar = c = 1$. Then the momentum, energy, and mass have the same dimensions, those of inverse length (or inverse time).

‡Here we have used the relation

$$[\delta(p_b - p_a)]^2 = \frac{1}{(2\pi)^4}\delta(p_b - p_a)\,\Omega t|_{t\to\infty}.$$

2) Collision of two particles, $N_a = 2$. The quantity of interest is the differential cross section

$$d\sigma = \frac{1}{j} d\dot{w}, \tag{2.9}$$

where j is the current density of the colliding particles. In determining it we can use the system of the center of mass of the colliding particles. Then

$$j = \frac{1}{\Omega}(v_1 + v_2) = \frac{p(\varepsilon_1+\varepsilon_2)}{\Omega\varepsilon_1\varepsilon_2}, \tag{2.10}$$

where v_1 and v_2 are the particle velocities and p is their common momentum. The numerator of (2.10) is an invariant:

$$p(\varepsilon_1+\varepsilon_2) \equiv I = \sqrt{(p_1p_2)^2 - m_1^2m_2^2} \tag{2.11}$$

and

$$j = \frac{I}{\Omega\varepsilon_1\varepsilon_2}. \tag{2.12}$$

From (2.9), (2.7), and (2.12) we get

$$d\sigma = \frac{\xi_1^2\xi_2^2}{(2\pi)^{3N_b-4}} |U_{ba}|^2 \frac{1}{4I} \delta(p_b - p_a) \prod_\beta \xi_\beta^2 d^4p_\beta \delta(p_\beta^2 - m_\beta^2)\,\theta(\varepsilon_\beta). \tag{2.13}$$

From formulas (2.8) and (2.13), which give the expressions for the invariant quantities $d\tau^{-1}$ and $d\sigma$, we see that the quantities U_{ba} are also invariant. They are called the invariant scattering amplitudes. We note that the dimensionality of U_{ba} depends on the choice of the factors ξ_i.

In the special case of elastic scattering, formula (2.13) becomes

$$d\sigma = \frac{\lambda^2}{4\pi^2 I} |U_{ba}|^2 \delta(p_1' + p_2' - p_1 - p_2)\, d^4p_1' d^4p_2' \delta(p_1'^2 - m_1^2)$$
$$\times \delta(p_2'^2 - m_2^2), \tag{2.14}$$

where p_1' and p_2' are the momenta of the particles after scattering, and $\lambda = \xi_1^2\xi_2^2$.

In the center of mass system, we find from (2.14)

$$d\sigma = \frac{\lambda^2}{16(2\pi)^2(\varepsilon_1+\varepsilon_2)^2} |U_{ba}|^2 do. \tag{2.15}$$

If we choose $\xi_1 = \xi_2 = 1$, then $\lambda = 1$ and U_{ba} is dimensionless. If $\xi_1 = \sqrt{2m_1}$, $\xi_2 = \sqrt{2m_2}$, then $\lambda = 4m_1m_2$ and U has the dimensions of $(\text{length})^2$, i.e., the same dimensions as the Fourier component of the potential energy. We note that if we write (2.15) in the form (1.24):

$$d\sigma = |F|^2 do, \tag{2.16}$$

then

$$F = -\frac{m_1m_2}{2\pi(\varepsilon_1+\varepsilon_2)} U_{ba}, \tag{2.17}$$

which corresponds precisely to the form of (1.25), since in the nonrelativistic limit $m_1m_2/(\epsilon_1+\epsilon_2) = \overline{m}$.

3. Kinematic invariants. Let us first consider the kinematic characteristics of the simplest process which can occur for stable particles: two particles collide and are changed into two (usually different) particles. Such a process is described by the diagram shown in Fig. 2, which we shall call a tetrode. For simplicity we shall treat the case where all the particles are spinless. The generalization to the case of particles with spin will be given later (see item 6).

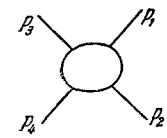

FIG. 2

The momentum and energy conservation law can be written as follows:

$$p_1 + p_2 + p_3 + p_4 = 0, \tag{2.18}$$

where p_1 and p_2 are the 4-momenta of the colliding particles and $-p_3$ and $-p_4$ are the momenta of the particles formed in the collision. In addition to the four equations (2.18), the 4-momenta p_i are connected by the relations

$$p_i^2 = m_i^2, \tag{2.19}$$

which express the energy in terms of the momentum (m_i is the mass of the particle). Thus the 16 components of the four 4-momenta are determined by 8 independent quantities. Of these, six are determined by the choice of the reference frame, since the general Lorentz transformation contains six parameters (the three angles which determine the orientation of the space axes, and the three components of the velocity of the reference system). Thus the physical process of scattering is determined by a pair of invariant quantities.*

It is easy to show directly, by using (2.18) and (2.19), that only two invariants can be formed from the four vectors p_i.

Thus the condition of relativistic invariance of the scattering amplitude U means that it is a function of the two invariants. To make the treatment symmetric, it is convenient to consider three rather than two invariants, which are given by

$$\left.\begin{aligned} s &= (p_1+p_2)^2 = (p_3+p_4)^2, \\ t &= (p_1+p_3)^2 = (p_2+p_4)^2, \\ u &= (p_1+p_4)^2 = (p_2+p_3)^2. \end{aligned}\right\} \tag{2.20}$$

It is easy to show, using (2.18) and (2.19), that these three invariants are related by the formula

$$s + t + u = h = m_1^2 + m_2^2 + m_3^2 + m_4^2. \tag{2.21}$$

We shall write the invariant amplitude U as $U(s,t,u)$.

*It is easy to obtain in similar fashion the number of independent invariants determining the collision process in which the total number of particles before and after collision is equal to $n \gg 4$. The number of components of 4-momenta is 4n. Equations (2.18) and (2.19) give n + 4 relations. Counting the six parameters for Lorentz transformations, we are left with 3n – 10 invariants.

The invariants s, t, u can be expressed in terms of the momentum p_s and z_s, the cosine of the scattering angle in the c.m. system of the colliding particles. Let the time and space components of p_i be given as follows:

$$\left.\begin{aligned} p_1 &= (\varepsilon_1, \mathbf{p}_s), & \varepsilon_1^2 &= p_s^2 + m_1^2, \\ p_2 &= (\varepsilon_2, -\mathbf{p}_s), & \varepsilon_2^2 &= p_s^2 + m_2^2, \\ p_3 &= (-\varepsilon_3, -\mathbf{p}'_s), & \varepsilon_3^2 &= p_s'^2 + m_3^2, \\ p_4 &= (-\varepsilon_4, \mathbf{p}'_s), & \varepsilon_4^2 &= p_s'^2 + m_4^2, \end{aligned}\right\}$$
$$z_s = (\mathbf{p}_s \mathbf{p}'_s) / p_s p'_s. \qquad (2.22)$$

Then

$$s = W_s^2, \qquad (2.23)$$
$$W_s = \varepsilon_1 + \varepsilon_2 = \varepsilon_3 + \varepsilon_4,$$
$$4sp_s^2 = [s - (m_1 + m_2)^2][s - (m_1 - m_2)^2], \qquad (2.24)$$
$$4sp_s'^2 = [s - (m_3 + m_4)^2][s - (m_3 - m_4)^2], \qquad (2.25)$$
$$\left.\begin{aligned} 2t &= h - s + 4p_s p'_s z_s - \frac{(m_1^2 - m_2^2)(m_3^2 - m_4^2)}{s}, \\ 2u &= h - s - 4p_s p'_s z_s + \frac{(m_1^2 - m_2^2)(m_3^2 - m_4^2)}{s}. \end{aligned}\right\} \qquad (2.26)$$

In the case of elastic scattering ($m_1 = m_3$, $m_2 = m_4$, $p'_S = p_S$) formulas (2.26) become

$$\left.\begin{aligned} t &= -2p_s^2(1 - z_s), \\ u &= -2p_s^2(1 + z_s) + (\varepsilon_1 - \varepsilon_2)^2. \end{aligned}\right\} \qquad (2.27)$$

4. Symmetry of the scattering matrix. In addition to Lorentz transformations, we should still consider discrete transformations: space inversion P, time reversal T and charge conjugation C, as well as combinations of them—combined inversion CP and "total (or strong) reflection" CPT. All amplitudes for scattering processes satisfy the requirement of invariance with respect to T and CPT (and consequently also CP). Invariance with respect to P and C holds for strong and electromagnetic interactions. In the case of the tetrode (a process of transformation of two spinless particles into two spinless particles) invariance with respect to P is automatically satisfied. In fact the process is characterized only by the pair of space vectors $\mathbf{p}_S$ and $\mathbf{p}'_S$. Reflection means the transformation $\mathbf{p}_S \to -\mathbf{p}_S$, $\mathbf{p}'_S \to -\mathbf{p}'_S$, which is equivalent to rotation through π about an axis perpendicular to the $(\mathbf{p}_S, \mathbf{p}'_S)$ plane.* This also follows from the fact that s, t, u are invariant with respect to the reflection $\mathbf{p}_i \to -\mathbf{p}_i$.*

The transformation T means a reversal of the process, i.e., replacement of the initial states by final states and vice versa, and change in the signs of the momenta p_i. This corresponds to replacing ϵ_i by $-\epsilon_i$ in (2.22), where now p_3 and p_4 are the 4-momenta of the initial states and $-p_1$ and $-p_2$ those of the final states. Since, from the above remarks and from P-invariance, we can also replace $\mathbf{p}_i$ by $-\mathbf{p}_i$, T (or TP) means reversing the 4-momenta, $p_i \to -p_i$. The invariants s, t, u are left unchanged. Consequently the requirement of T-invariance implies a symmetry of the scattering matrix elements, i.e.,

$$U_{ba}(s, t, u) = U_{ab}(s, t, u). \qquad (2.28)$$

The transformation CPT* is an improper rotation of four-dimensional space-time (i.e., one associated with a crossing of the light cone), in which all the components of a four-vector change sign. It is equivalent to the reflection $p_\alpha \to -p_\alpha$ of all the components of the 4-momentum. How do we interpret the negative sign of the time (energy) component of the 4-momentum? The equations of relativistic quantum field theory always contain solutions with both positive frequencies $\omega = \epsilon$ and negative, $\omega = -\epsilon$. To the first of these there correspond operators for absorption of particles with energy ϵ, and to the second operators for emission of particles with energy ϵ. Thus the transformation CPT means replacing particles by antiparticles and initial states (i.e., absorption) by final states (emission). Thus the requirement of CPT-invariance, which can be proved rigorously in field theory for a quite wide class of interactions, means that

$$U_{ab}(s, t, u) = U_{\bar{b}\bar{a}}(s, t, u), \qquad (2.29)$$

where $\bar{a}$ and $\bar{b}$ denote states of antiparticles with the same momenta as in the particle states a and b.

Thus the single amplitude U(s, t, u) describes four different processes: $a \to b$, $b \to a$, $\bar{b} \to \bar{a}$, $\bar{a} \to \bar{b}$.

After treating P, T, and CPT, it is obviously unnecessary to treat charge conjugation separately.

5. The principle of universality. The treatment of CPT-invariance suggests the possibility of generalizing it. Let us consider the conservation law (2.18). Two of the four-momenta in that equation (p_1 and p_2) have positive time components $p_{10} = \epsilon_1$, $p_{20} = \epsilon_2$, while two (p_3 and p_4) have negative time components $p_{30} = -\epsilon_3$, $p_{40} = -\epsilon_4$. The process $\bar{b} \to \bar{a}$ differs from the process $a \to b$ in having the roles of p_1, p_2 and p_3, p_4 interchanged in this sense. But Eq. (2.18) can always be satisfied if two of the four momenta have positive, and two have negative, time components. They correspond to processes in which the colliding particles are those for which $p_{i0} > 0$, and the scattered particles are those for which $p_{i0} < 0$; in addition, particles should be replaced by antiparticles if the sign of p_{i0} is opposite to that indicated in (2.22).

We use the term reaction to mean the set of all processes contained in this sense in Eq. (2.18) and the diagram of Fig. 2. The process $a \to b$, together with those which are equivalent to it according to (2.28) and (2.29), we call the s-channel of the reaction. The diagram of Fig. 2 thus contains three channels:

*The requirement of invariance with respect to P (conservation of parity) imposes a restriction on the internal parity of the particles. The product of the internal parities of all four particles should equal + 1. If this is not the case, the process is forbidden ($U \equiv 0$).

*See, for example, [1].

1) channel s: $1 + 2 \to 3 + 4$,
2) channel u: $1 + \bar{4} \to 3 + \bar{2}$,
3) channel t: $1 + \bar{3} \to \bar{2} + 4$.

The meaning of the notation is obvious: the numbers correspond to the indices for the momentum, the bar over a number denotes the antiparticle; the colliding particles are on the left ($p_{i0} > 0$), the scattered particles on the right ($p_{i0} < 0$). The kinematic formulas which we found in item 3 referred to the s-channel. Keeping the definition (2.20), we can express the invariants s, t, u in terms of the center-of-mass momenta p_t, p_t', p_u, p_u' corresponding to the t and u channels, and the cosines of the scattering angles, z_t, z_u. We obtain formulas analogous to (2.26) (substituting $s \to u$, $t \to t$, $2 \to 4$):

$$\left.\begin{aligned}
&p_1 = (\varepsilon_1, \mathbf{p}_u),\ p_4 = (\varepsilon_4, -\mathbf{p}_u),\ p_3 = (-\varepsilon_3, -\mathbf{p}_u'),\ p_2 = (-\varepsilon_2, \mathbf{p}_u'),\\
&z_u = (\mathbf{p}_u \mathbf{p}_u')/p_u p_u',\quad W_u = \varepsilon_1 + \varepsilon_4 = \varepsilon_2 + \varepsilon_3,\quad u = W_u^2,\\
&4up_u^2 = [u - (m_1 + m_4)^2][u - (m_1 - m_4)^2],\\
&4up_u'^2 = [u - (m_2 + m_3)^2][u - (m_2 - m_3)^2],\\
&2t = h - u + 4p_u p_u' z_u - \frac{(m_1^2 - m_4^2)(m_3^2 - m_2^2)}{u},\\
&2s = h - u - 4p_u p_u' z_u + \frac{1}{u}(m_1^2 - m_4^2)(m_3^2 - m_2^2)
\end{aligned}\right\} \tag{2.30}$$

and (substituting $2 \to 3$, $s \to t$, $u \to u$):

$$\left.\begin{aligned}
&p_1 = (\varepsilon_1, \mathbf{p}_t),\ p_3 = (\varepsilon_3, -\mathbf{p}_t),\ p_2 = (-\varepsilon_2, -\mathbf{p}_t'),\ p_4 = (-\varepsilon_2, \mathbf{p}_t'),\\
&z_t = (\mathbf{p}_t\ \mathbf{p}_t')/p_t p_t',\quad W_t = \varepsilon_1 + \varepsilon_3 = \varepsilon_2 + \varepsilon_4,\quad t = W_t^2,\\
&4tp_t^2 = [t - (m_1 + m_3)^2][t - (m_1 - m_3)^2],\\
&4tp_t'^2 = [t - (m_2 + m_4)^2][t - (m_2 - m_4)^2],\\
&2s = h - t - 4p_t p_t' z_t - \frac{1}{t}(m_1^2 - m_3^2)(m_2^2 - m_4^2),\\
&2u = h - t + 4p_t p_t' z_t + \frac{1}{t}(m_1^2 - m_3^2)(m_2^2 - m_4^2).
\end{aligned}\right\} \tag{2.31}$$

The variables s, t, u for the different reaction channels have different physical meanings and different ranges of variation. Thus, for the s-channel the invariant s represents the square of the energy in the center-of-mass system, while t and u are the squares of the momentum transfers (i.e., the squares of the difference of the 4-momenta of one of the incident and one of the scattered particles). In the u channel, the invariant u is the square of the energy, and in the t channel, t is the energy squared; in each case the other two invariants are the squares of the momentum transfers. The range of the variables in the s-channel is determined by the conditions

$$m_3 + m_4 \leqslant W_s \geqslant m_1 + m_2, \qquad |z_s| < 1,$$

and similarly for the u and t channels

$$\begin{aligned}
&m_2 + m_3 \leqslant W_u \geqslant m_1 + m_4, \qquad |z_u| < 1,\\
&m_2 + m_4 \leqslant W_t \geqslant m_1 + m_3, \qquad |z_t| < 1.
\end{aligned}$$

The conditions $|z_r| < 1$ ($r = s, t, u$) can be expressed as a single condition determining the range of the variables corresponding to all channels. It has the form [2]

$$sut > as + bu + ct, \tag{2.32}$$

where

$$\begin{aligned}
ha &= (m_1^2 m_2^2 - m_3^2 m_4^2)(m_1^2 + m_2^2 - m_3^2 - m_4^2),\\
hb &= (m_1^2 m_4^2 - m_2^2 m_3^2)(m_1^2 + m_4^2 - m_2^2 - m_3^2),\\
hc &= (m_1^2 m_3^2 - m_2^2 m_4^2)(m_1^2 + m_3^2 - m_2^2 - m_4^2).
\end{aligned}$$

For a graphical description of the ranges of the variables corresponding to the different channels (the "physical regions"), it is convenient to use trilinear coordinates in the plane. We construct (Fig. 3) an equilateral triangle with altitude h, and regard the sides of the triangle as coordinate axes, perpendicular to which we measure the values of s, u, t (the inside of the triangle being the positive direction). It is easily seen that for every point in the plane the sum of the three trilinear coordinates is equal to the altitude of the triangle, i.e., s + u + t = h. The Cartesian coordinates of a point in the plane are $\nu = \frac{1}{\sqrt{3}}(s - u)$ (abscissa) and t (ordinate). The equation corresponding to an equality sign in (2.32) determines the boundary of the three physical regions, which do not intersect. The asymptotes of the curves are the axes

$$s = 0, \quad u = 0, \quad t = 0, \tag{2.33}$$

and the boundary curves intersect the asymptotes along the line

$$as + bu + ct = 0. \tag{2.34}$$

Figure 4 gives examples of the physical regions for some reactions.

If the mass of one of the particles is greater than the sum of the other three, $m_1 > m_2 + m_3 + m_4$, a fourth reaction channel is possible, corresponding to the decay

$$1 \to \bar{2} + 3 + 4.$$

For this decay channel, in the rest system of the decaying particle,

$$\left.\begin{aligned}
&p_1 = (m_1, 0),\ p_2 = (-\varepsilon_2 - \mathbf{p}_2),\ p_3 = (-\varepsilon_3 - \mathbf{p}_3),\ p_4 = (-\varepsilon_4 - \mathbf{p}_4),\\
&\varepsilon_2 + \varepsilon_3 + \varepsilon_4 = m_1,\quad \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 = 0,\\
&s = m_1^2 + m_2^2 - 2m_1\varepsilon_2,\\
&u = m_1^2 + m_4^2 - 2m_1\varepsilon_4,\\
&t = m_1^2 + m_3^2 - 2m_1\varepsilon_3.
\end{aligned}\right\} \tag{2.35}$$

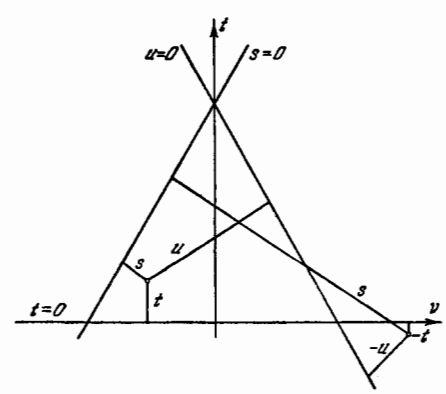

FIG. 3

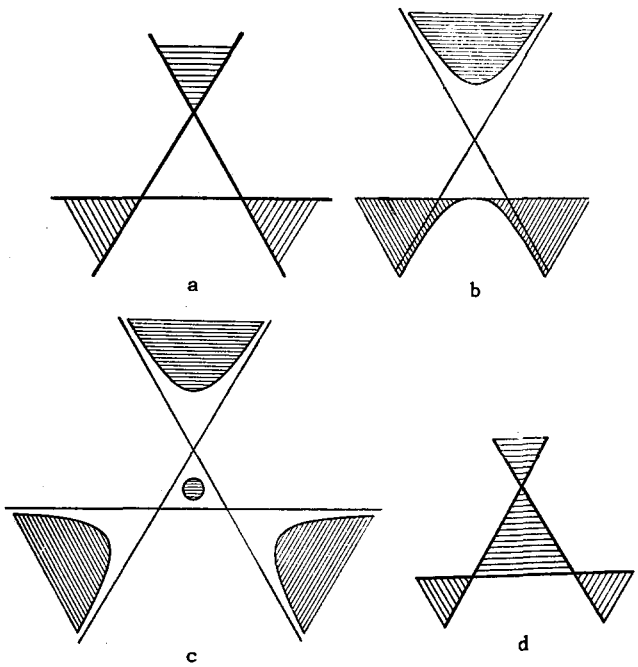

FIG. 4. Physical regions for the reactions: a) $\pi + \pi \to \pi + \pi$; b) $N + \pi \to N + \pi$; c) $K + \pi \to \pi + \pi$; d) $\mu + \nu \to e + \nu$

All three invariants are positive. The physical region is in the interior of the triangle and is inscribed in the hexagon

$$\left.\begin{aligned} s&=(m_1-m_2)^2, & s&=(m_3+m_4)^2,\\ t&=(m_1-m_3)^2, & t&=(m_2+m_4)^2,\\ u&=(m_1-m_4)^2, & u&=(m_2+m_3)^2. \end{aligned}\right\} \qquad (2.36)$$

If some of the masses are zero, the hexagon degenerates into a polygon with a smaller number of sides.

The generalization of the CPT-theorem, which we mentioned earlier, arises from the fact that if we apply the CPT-transformation not to all the particles participating in the reaction, but to each of them independently, the expression for the invariant amplitude does not change. Thus the same amplitude U(s,t,u) describes all the channels of the reaction. We may refer to this as the principle of universality of the reaction amplitude. There exists no rigorous proof of this theorem. An argument in its favor is that in any Feynman diagram describing the given process, any of the external lines may be regarded either as a particle in the initial (final) state, or as an antiparticle in the final (initial) state, without changing the internal structure of the diagram.

It is important to note that, strictly speaking, the assertion of the principle of universality is an empty statement if we do not make some basic remarks. Actually the regions of variation of the variables s, u, t are different in the different reaction channels and do not overlap. What is the difference between a single function U, defined in all three regions, and three different functions, each defined in its own region? In making our assertion for the single function U(s,u,t) we have naively assumed that it is given by some definite analytic expression which is meaningful for all values of the variables (which is actually the case if we consider individual Feynman diagrams). Thus, to express ourselves more precisely, in formulating the principle of universality we simultaneously assume that the reaction amplitude is an analytic function; consequently, assigning it in one of the physical regions determines its value over the whole domain of analyticity, including both the other physical regions and also nonphysical regions, among them the regions of complex values of the variables. Thus an important question is that of the domain of analyticity of the function U, i.e., the location and nature of its singularities.

We note that the principle of universality can be proved, starting from the equations of quantum field theory, to the same extent and in the same cases where one can prove the analyticity of the amplitude.[3]

6. Generalization to particles with spin and isospin. So far we have assumed that the state of any particle can be completely determined by assigning its momentum. If the particles possess a spin, the scattering amplitude U is a matrix in the spin quantum numbers. If we denote the projection of the spin of the i-th particle by μ_i and its spin by s_i, then for the case of the tetrode, for example, this matrix has $(2s_1+1)(2s_2+1)(2s_3+1)(2s_4+1)$ elements $U_{\mu_1\mu_2\mu_3\mu_4}$. But the elements of this matrix are not independent, since we must satisfy the condition of invariance of the scattering matrix under rotations. If we introduce the spin amplitudes u_i, whose transformation properties are known, the problem reduces to finding the linearly independent invariants I_k which can be constructed from the amplitudes u_i and the momenta p_i. Then

$$U=\sum_{k=1}^{\nu} U_k(s,\ u,\ t)\,I_k, \qquad (2.37)$$

where the U_k are invariant functions and ν is the number of linearly independent invariants I_k. Thus the difference between the cases of particles with spin and without spin reduces to the fact that the scattering is now characterized by ν amplitudes U_k instead of one invariant amplitude U.

In practice, in finding the number ν, it is convenient to use the law of conservation of angular momentum, after which it is not difficult to construct the necessary number of invariants I_k. Let us now consider some of the most important tetrode amplitudes, using the notation introduced in item 5.

1) Particles 1 and 3 are nucleons (spin $\frac{1}{2}$), 2 and 4 are pions (spin 0); the s-channel is the scattering of a pion by a nucleon, the t-channel the annihilation of a nucleon-antinucleon pair into a pion pair. We consider a state with definite angular momentum j in the t-channel. Then the pion pair has orbital angular momentum j and parity $(-1)^j$. The nucleon pair can be either in a singlet state with orbital angular momentum j and parity $(-1)^{j+1}$ (since the internal parities of nucleon and antinucleon are opposite), or in a

triplet state with orbital angular momentum $l = j, j \pm 1$ and parity $(-1)^{l+1}$. Of the four states for a nucleon pair, in two (the triplet states with $l = j \pm 1$) the parities coincide with the parity of the nucleon pair. Consequently $\nu = 2$ and

$$U = A(s, u, t) I_1 + B(s, u, t) I_2. \tag{2.38}$$

Now it is easy to construct the invariants

$$\left.\begin{aligned} I_1 &= \bar{u}(-p_3)\, u(p_1), \\ I_2 &= \bar{u}(-p_3)(\hat{p}_2 - \hat{p}_4)\, u(p_1), \end{aligned}\right\} \tag{2.39}$$

where u(p) is the four-component spinor amplitude for a nucleon.

2) Particles 1 and 3 are nucleons, 2 is a photon and 4 a pion (channel s is photoproduction of pions on nucleons, t is the annihilation of a nucleon pair into a photon and a pion). Let us consider the t-channel. For a given angular momentum j, the photon, and consequently the photon-pion system, can be in two states with opposite parities (electric or magnetic multipole fields). As we saw above, the nucleon pair can be in four states, two even and two odd. Consequently $\nu = 4$.

3) Particles 1 and 3 are nucleons, 2 and 4 are photons (channel s is the Compton scattering on a nucleon, channel t is the pair annihilation into two photons). Let us consider the t-channel. For a given angular momentum, the two-photon system can have at most two even states and one odd, while the pair can have two even and two odd states. Consequently $\nu = 6$.

4) All four particles have spin $\frac{1}{2}$. For a given angular momentum, there are 4 initial and 4 final states. We can treat the amplitude as a matrix $U_{\alpha\beta}$, where the subscripts α and β take on values 1, 2, 3, 4, where 1 denotes the singlet state and 2, 3, 4 are the triplet states with $l = j, j+1, j-1$. In the general case, the process is described by 16 amplitudes. However, in practice this number is reduced because of additional symmetries. Let us look at some examples:

a) Elastic scattering (a and b are the same particles). It then follows from T-invariance that $U_{\beta\alpha} = U_{\alpha\beta}$, $\nu = 10$.

b) Parity conservation. Then the transitions $l \to l \pm 1$ are absent if the particles a and b have the same internal parity, while the transition $l \to l$ is out if they have opposite parity; $\nu = 8$.

c) Elastic scattering with parity conservation: $\nu = 6$.

d) Elastic scattering of identical particles: $\nu = 7$ if there is no parity conservation; $\nu = 5$ if parity is conserved (e.g., proton-proton scattering).

e) Processes involving a neutrino (for example, β decay and the accompanying scattering). Because the neutrino has only one polarization, $\nu = 8$.

f) Processes involving two neutrinos (for example, μ decay): $\nu = 4$.

Let us also look at processes involving three particles (a "triode").

1) Suppose that two particles (of momenta p_1 and p_3) with spin $\frac{1}{2}$ are transformed into one spinless particle (with momentum $-p_2$). In the rest system of the final particle, the particles which annihilate have angular momentum 0. There are two such states: $^1S_0$ ($s = 0, l = 0, g = -$) and $^3P_0$ ($s = 1, l = 1, g = +$) (where s is the spin and g the parity of the state). The corresponding invariants have the form

$$\left.\begin{aligned} I_1 &= \bar{u}(-p_3)\, u(p_1), \\ I_2 &= \bar{u}(-p_3)\, \gamma_5\, u(p_1). \end{aligned}\right\} \tag{2.40}$$

Thus we have two amplitudes if parity is not conserved, or one if it is conserved. The invariant I_2 corresponds to the transformation of a nuclon pair into a pion or (the other channel of the same reaction) of a nuclon and a pion into a nucleon),* i.e.,

$$U = g(t) I_2, \quad (t = (p_1 + p_3)^2). \tag{2.41}$$

2) Suppose that two particles with spin $\frac{1}{2}$ (p_1, p_3) are transformed into a particle of spin 1 ($-p_2$). In the system of the center of mass of the annihilating particles, their angular momentum is unity. There are four such states: two odd, $^3S_1$, $^3D_1$, and two even, $^3P_1$ and $^1P_1$. If parity is conserved, the process is determined by two amplitudes. If the internal parity of the produced particles is negative (vector field), the corresponding invariants can be chosen in the form†

$$\left.\begin{aligned} I_1 &= \bar{u}(-p_3)\, \gamma_\mu u(p_1)\, a_\mu(-p_2), \\ I_2 &= \bar{u}(-p_3)(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\, u(p_1)\, a_\mu(-p_2)(p_1 + p_3)_\nu, \\ I_3 &= \bar{u}(-p_3)\, u(p_1)\, a_\mu(-p_2)(p_1 - p_3)_\mu, \end{aligned}\right\} \tag{2.42}$$

where a_μ are the amplitudes for the vector particle, which form a vector satisfying the condition

$$a_\mu(-p_2)\, p_{2\mu} = 0; \tag{2.43}$$

I_1, I_2 and I_3 are connected by the relation

$$4mI_1 - I_2 - 2I_3 = 0, \tag{2.44}$$

where m is the mass of particles 1 and 3.

3) Suppose that particles 1 and 3 are spinless, while particle 2 has spin 1; then only one state of the annihilating particles has spin 1 ($l = 1, g = -1$). There is one invariant

$$I = a_\mu(p_1 - p_3)_\mu. \tag{2.45}$$

In a similar way one can treat the scattering problem when the particles have isospin. The number of invariant amplitudes can be counted by using conservation of the isospin T, and then one can construct the corresponding invariants in isospin space.

*Concerning the significance of processes with three stable particles, cf. Sec. 3, item 4.

†The matrices γ_μ are defined so that the Dirac equation has the form $\hat{p}u(p) = mu(p)$ ($\hat{p} = \gamma_\mu p_\mu$).

1) Scattering of a π meson by a π meson. The isospin of the pion is unity. The two pions have isospin T = 0, 1, 2. Consequently the scattering is determined by three amplitudes

$$U = U_1 I_1 + U_2 I_2 + U_3 I_3.$$

If χ is the pion amplitude (a vector in isospin space), the invariants can be written as:

$$\left.\begin{aligned} I_1 &= (\chi_1\chi_3)(\chi_2\chi_4), \\ I_2 &= (\chi_1\chi_4)(\chi_2\chi_3), \\ I_3 &= (\chi_1\chi_2)(\chi_3\chi_4). \end{aligned}\right\} \tag{2.46}$$

2) Pion-nuclon scattering (or annihilation of nucleons into pions). The nucleons have isospin $\frac{1}{2}$, so the two nucleons can have isospin T = 0 and 1. Consequently the scattering is determined by two amplitudes. If χ_1 and χ_2 are the meson amplitudes, and v_1 and v_2 the nucleon amplitudes (spinors in isospin space), the invariants have the form

$$\left.\begin{aligned} I_1 &= (\chi_1\chi_2)(v_2^{\cdot}v_1), \\ I_2 &= \chi_1 \times \chi_2 (v_2^{\cdot}\tau v_1), \end{aligned}\right\} \tag{2.47}$$

where τ_i are the Pauli matrices.

3) A triode of two nucleons and one pion. There is one invariant

$$I = \chi v_2^{\cdot}\tau v_1. \tag{2.48}$$

3. UNITARITY AND ANALYTICITY

1. The unitarity relation. So far we have used only the properties of invariance of the scattering amplitude under Lorentz transformations and discrete transformations. These invariance requirements may be called the kinematical principles of the theory. A dynamic principle of the theory is the requirement of unitarity of the scattering matrix:

$$SS^* = 1, \tag{3.1}$$

where S^+ denotes the matrix which is the Hermitian conjugate of S, and 1 is the unit matrix. The unitarity of the matrix S guarantees conservation of normalization and orthogonality of states in the reaction. The diagonal elements of relation (3.1) express the simple fact that the sum of the probabilities of transition from a given initial state to any final state is equal to unity:

$$\sum_b |S_{ba}|^2 = 1.$$

In order to make clear why this simple property is put forth as a dynamical principle, we note that one might also formulate the dynamical principles of classical mechanics and quantum mechanics in similar fashion. In classical mechanics the equations of motion are a canonical transformation of the generalized coordinates and momenta. The same holds in quantum mechanics, where the canonical transformation is accomplished by means of a unitary matrix. Thus, if by $S(t_2, t_1)$ we understand a matrix which transforms a state at an arbitrary instant of time t_1 into a state at another arbitrary instant t_2, then the unitarity relation relating two infinitesimally close instants in time is equivalent to the usual (Hamiltonian) equations of motion in quantum mechanics, which are also used (taking into account the relativistic kinematical principles) in the quantum theory of fields. The scattering matrix $S = S(\infty, -\infty)$ transforms the state at the initial time $t_1 = -\infty$ ("incident particles") into the state at the final time $t_2 = \infty$ ("scattered particles"). Thus the unitarity relation (3.1) is a less restrictive (i.e., more general) statement than the equations of motion of the quantum theory of fields. Heisenberg[4] proposed the idea that in general the only observable physical quantities which should appear in the theory of elementary particles are the states at $t = -\infty$ and $t = \infty$. If this idea is correct, then relation (3.1) may pretend to the role of the fundamental equation of the theory. In the following we shall consider the consequences which can be obtained from the unitarity relation (3.1).

2. The imaginary part of the amplitude. The optical theorem. If we write the matrix S in the form (2.2), relation (3.1) takes the form

$$i(T - T^*) = -TT^*, \tag{3.2}$$

or

$$i(T_{ba} - T^*_{ab}) = -\sum_n T_{bn} T^*_{an}, \tag{3.3}$$

where n denotes those states of the system into which states a and b can transform. We shall call them intermediate states. Furthermore, if we take account of the symmetry of the T matrix (2.28), i.e.,

$$T_{ba} - T^*_{ab} = 2i \operatorname{Im} T_{ba},$$

then (3.3) takes the form

$$2 \operatorname{Im} T_{ba} = \sum_n T_{bn} T^*_{an}. \tag{3.4}$$

A state n is characterized by: 1) the number of particles, which we shall denote by ν; 2) the momenta of the particles p_i $(i = 1 \ldots \nu)$; 3) the internal quantum numbers of the particles, including their masses, spins, etc. In this paragraph we shall, for simplicity, assume that a given number ν of particles corresponds only to one set of particles and that the spins of all the particles are zero. Then

$$\sum_n \ldots = \sum_\nu \prod_{i=1}^{\nu} \int \frac{d^3 q_i}{(2\pi)^3 \Omega}. \tag{3.5}$$

Using the expressions (2.3) and (2.4) for T_{ba} in terms of the amplitudes U_{ba}, and taking the normalization volume to be $\Omega = 1$, we get

$$\operatorname{Im} U_{ba} = \frac{(2\pi)^4}{2} \sum_\nu \frac{\lambda_\nu}{(2\pi)^{3\nu}} \int U_{b\nu} U^*_{a\nu} \delta(p_\nu - p_a) \times \prod_{i=1}^{\nu} d^4 q_i \delta(p_i^2 - m_i^2)\, \theta(\varepsilon_i), \tag{3.6}$$

where

$$\lambda_\nu = \prod_{i=1}^{\nu} \xi_i^2.$$

From (3.6) we can easily get the well-known relation called the optical theorem. Let a and b be the same two-particle state. Then U_{aa} represents the amplitude for elastic scattering at zero angles. Comparing (3.6) with the general expressions for the cross section (2.13), we obtain

$$\operatorname{Im} U_{aa} = \frac{2I}{\lambda_a}\sigma, \tag{3.7}$$

where σ is the total scattering cross section, (i.e., the sum of the cross sections for elastic scattering and all cross sections for transformation of particles a into any other particles), $\lambda = \xi_1^2\xi_2^2$, and I is defined by formula (2.11). If we normalize the scattering amplitude according to (2.16), the optical theorem takes the form

$$\operatorname{Im} F(0) = \frac{p}{4\pi}\sigma. \tag{3.8}$$

Relation (8) is valid both in the center-of-mass system as well as in the rest system of one of the colliding particles. Correspondingly p denotes the momentum of the incident particle in one or the other system.

3. The principle of analyticity. Relation (3.6) is very remarkable. It may claim to be the fundamental dynamical system of equations describing the scattering amplitude. In fact, it relates the imaginary part of the amplitude for a given process with the square of an expression in terms of the amplitudes for other processes. For each of these amplitudes in turn one can write a relation of the form (3.6). This would give a system of equations for determining the amplitudes if the unknown quantities appeared everywhere in the form of amplitudes. However, the left side of (3.6) contains not the amplitude, but its imaginary part. Consequently we still need a relation connecting the imaginary part with the real part. Such relations exist for analytic functions and can be expressed by means of Cauchy's formula. They are usually referred to in the theory of elementary particles as dispersion relations, in analogy to the relations first obtained by Kramers and Kronig for the dielectric constant as a function of frequency. We take as our fundamental postulate that the scattering amplitudes are analytic functions of those invariants r on which they depend. There exists no complete proof of analyticity of the amplitudes, but in certain cases it can be proven on the basis of the general equations of the quantum theory of fields.[3,5]

An analytic function is determined by the location and nature of its singularities. We shall assume that the amplitude U_{ba} has those singularities whose presence follows, as we shall see later, from (3.6), and has no others. Then in principle it becomes possible to obtain a system of equations for the amplitudes starting from the fundamental principles of invariance, universality, unitarity, and analyticity.*

*This idea was first put forward by M. Gell-Mann and developed by L. D. Landau.[6]

However, their practical construction in complete form meets with serious difficulties. These difficulties are related to the complexity of the analytic properties of the amplitudes, the nature of which can be explained in the following way: the unitarity property (3.1) is a relation connecting the real scattering amplitudes, i.e., the amplitudes as functions of the kinematic invariants whose values lie in physical regions. But the analytic properties of the amplitudes are determined by all their singularities, including those which lie in regions of non-physical values of the invariants, including also complex values. Thus a complete investigation of the singularities requires the continuation (extrapolation) of relation (3.6) into the region of non-physical values of the invariants. The function defined by (3.6) then ceases to be a real function, and the question of its analytic properties arises.

We may say that the principle of analyticity compensates (though we cannot say whether this compensation is complete or only partial) for the incompleteness of the unitarity requirement (3.1) as a dynamical principle. Above it was pointed out that the Hamiltonian form of quantum mechanics means the examination of the matrix $S(t_2, t_1)$, whereas (3.1) contains only $S(\infty, -\infty)$. A treatment of continuous time development, i.e., the matrix $S(t_2, t_1)$ in quantum mechanics or quantum field theory leads to the necessity for treating intermediate virtual states. The construction of a theory here on the basis of relation (3.1) apparently does not recognize virtual states: the intermediate states in (3.1) are real. But in practice the use of the analytic properties of amplitudes requires one to consider non-physical regions which, to a certain extent, is equivalent to bringing in the treatment of virtual states.

The development of the principle of analyticity looks schematically as follows: Let us consider one of the channels of a reaction described by the amplitude U. Let $r = p_a^2$ be the energy invariant for this channel ($r = W_r^2$). Let M_r be the minimum total mass of particles in states appearing on the right side of (3.6). Then the imaginary part of U, considered as a function of the invariant r, is equal to zero for $r < M_r^2$ and is different from zero for $r > M_r^2$. Then we can, as explained in Sec. 1, item 3, represent U as a function of the complex variable r in the form (1.21)

$$U(r) = \frac{1}{\pi}\int_{M_r^2}^{\infty} \frac{\operatorname{Im} U(r')}{r' - r}\, dr' + C, \tag{3.9}$$

where C is independent of r [or in the form (1.22)]. The point $r = M_r^2$ is a singular point of the function U(r), the segment of the real axis from the point $r = M_r^2$ to $r = \infty$ is a branch line. The value of U(r) for real r should be considered as the limiting value on the upper side of the cut, i.e.,

$$U(r) = U(r + i\varepsilon)|_{|\varepsilon| \to 0}. \tag{3.10}$$

Formula (3.9) is not completely correct; it explains

only the scheme for expressing the amplitudes in terms of their imaginary parts. Since the amplitudes describe a series of channels for the reaction, the imaginary part is also determined by applying the unitarity relation with respect to all the channels. Even if we treat the amplitude as a function of one of the variables r for fixed values of the others, we cannot restrict our treatment to channel r alone, since the number of channels is greater than the number of independent variables and the energy variables for the different channels are not independent. Let us make this clear on the example of a tetrode amplitude, which is a function of the three invariants s, u, t, connected by equation (2.21). We shall consider U(s, u, t) as a function of s for fixed t, denoting it by $U_t(s)$. The imaginary part of $U_t(s)$ is determined by two channels: in one of them the energy variable is s, while in the other it is u = h − s − t. Therefore

$$U_t(s) = c(t) + \frac{1}{\pi}\int\limits_{M_s^2}^{\infty} \frac{\operatorname{Im} U^s}{s'-s}\,ds' + \frac{1}{\pi}\int\limits_{M_u^2}^{\infty} \frac{\operatorname{Im} U^u}{u'-u}\,du', \qquad (3.11)$$

or

$$U_t(s) = c(t) + \frac{1}{\pi}\int\limits_{M_s^2}^{\infty} \frac{\operatorname{Im} U^s}{s'-s}\,ds' - \frac{1}{\pi}\int\limits_{-\infty}^{h-t-M_u^2} \frac{\operatorname{Im} U^u}{s'-s}\,ds'. \qquad (3.12)$$

Thus the representation of the function $U_t(s)$ (the dispersion relation in the variable s) contains integrals over both edges of the cut in the s plane: one (on the right) from the point $s' = M_s^2$ to $s' = \infty$, and the second (on the left) from $s' = -\infty$ to $s' = h - t - M_u^2$. We note that when $s > M_s^2$, i.e., on the right cut, the value of $U_t(s)$ for real s is defined as $U_t(s) = U_t(s + i0)$, but the corresponding value on the left cut is $U_t(s) = U_t(s)_{u=u+i0} = U_t(s - i0)$, i.e., it is the limiting value of the function on the lower edge of the cut.

It is important to note that Im U^s (or Im U^u) is determined according to (3.6) in the physical region of the corresponding channel, while the limits of the physical region with respect to s (or u) depend on t. Therefore, the integration in (3.11), even for the case where s and t lie in the physical region, may include non-physical domains. The problem arises of continuing Im U into these regions. The solution of this problem requires a more detailed treatment of the structure of the unitarity relation.

4. Poles of the amplitude. Let U be the amplitude for some tetrode. Let us consider one of its channels, r, in which there is a one-particle state among the intermediate states. In other words, this means that there exists a particle whose internal quantum numbers coincide with the corresponding quantum numbers of the initial (a) and final (b) states of the two particles in channel r of the tetrode. We extract the corresponding term in (3.6) ($\nu = 1$), denoting it by $U^{(1)}$ i.e.,

$$\operatorname{Im} U^{(1)} = \pi\lambda_1 \int U_{b1} U_{a1}^* \delta(q - p_a)\,\delta(q^2 - \mu^2)\,d^4q, \qquad (3.13)$$

where q is the 4-momentum of the intermediate particle, μ its mass, p_a the total 4-momentum of the initial state ($p_a^2 = r$). Using the notation

$$\left.\begin{aligned} (U_{ba})_{r=\mu^2} = g_b, \quad (U_{a1}^*)_{r=\mu^2} = g_a, \\ R = \lambda_1 g_a g_b, \end{aligned}\right\} \qquad (3.14)$$

we get from (3.13)

$$\operatorname{Im} U^{(1)} = \pi R \delta(r - \mu^2). \qquad (3.15)$$

From (3.15), on the basis of the analyticity principle [cf. (3.9) or (1.20)], we obtain

$$U^{(1)} = \frac{R}{\mu^2 - r}. \qquad (3.16)$$

Formula (3.16) is valid for any complex values of r; for $r \to u^2$ we assume that one takes the limit from above, i.e., $r \to r + i0$, or $\mu^2 = \mu^2 - i0$.

The pole character of the amplitude $U^{(1)}$ is specific to a one-particle intermediate state. For $\nu > 1$, because of the appearance of more integrations in (3.6) in comparison with (3.13), the intermediate states will not lead to an expression of the type (3.15). Therefore, as a rule many-particle states must lead to singularities of the form of branch points. The nature of the singularities will depend essentially on the behavior of of the amplitudes appearing in the integrands.

Let us discuss the structure of the expression for R (3.14). Each of the amplitudes U_{a1}, U_{b1} describes a "triode" (Fig. 5), i.e., the decay of a particle (of mass μ) into two (m_1 and m_2). The triode amplitude (cf. Fig. 5) can be considered as a function of the three invariants p_1^2, p_2^2, and $r = (p_1 + p_2)^2$ or, for $p_1^2 = m_1^2$, $p_2^2 = m_2^2$, as a function of the single variable r. The conservation laws

$$q + p_1 + p_2 = 0$$

can be satisfied only when

$$\mu > m_1 + m_2, \qquad (3.17)$$

where the physical region of the variable r contains only a single point

$$r = \mu^2.$$

However, we can consider the triode amplitude $U_{a1}(r)$ in the case where condition (3.17) is not satisfied, i.e., where particle μ is stable. In fact, we actually do consider the amplitude as a function of its arguments over the whole domain of variation of the latter, both in physical as well as in non-physical regions, and the conservation laws will still be taken into account in the uni-

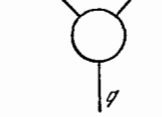

FIG. 5

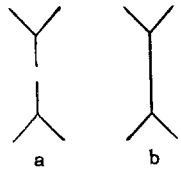

FIG. 6

tarity relation (3.6). Thus, despite the fact that, for a stable particle, the triode (cf. Fig. 5) does not describe a real process, it determines the pole part of the tetrode amplitude. We may say that $U_{a1}(\mu^2) = g_a$ is the analytic continuation of the decay amplitude as a function of μ^2 from the region of real decay (3.17) into the region of a stable intermediate particle.

Formulas (3.15) and (3.16) can be illustrated by the diagrams shown in Fig. 6. The upper part of Fig. 6a shows the amplitude of the triode U_{a1}, the lower for U_{b1}. The diagram of Fig. 6b shows the pole part of the amplitude of the tetrode $U^{(1)}$. The upper and lower vertices have the same meaning as in the diagram of Fig. 6a; the line joining them corresponds to the pole factor $1/(\mu^2 - r)$, which represents simply the Green's function of a free particle of mass μ. Thus the diagram of Fig. 6b is identical in meaning to a Feynman diagram of second order constructed on the basis of the primary interaction described by the three-prong vertices of Fig. 6a. We note that g_a and g_b are equivalent to the renormalized coupling constants (charges) in the usual field theory and μ is the renormalized mass.

If we assume that many-particle intermediate states give a small contribution to Im U compared with one-particle states, the pole part of the amplitude is a first approximation. Then, in second approximation we may include two-particle intermediate states, substituting the first (pole) approximation for them in (3.6). Thus, iterating (3.6) successively and using the relation of U to Im U (dispersion relations), we can in principle obtain the perturbation series. This will be a series in powers of the constants g.

Let us illustrate, with the pole amplitudes as an example, the important role played by the universality principle in the method of constructing the theory being developed here. Suppose, for example, that channel s of the tetrode we are considering describes the elastic scattering of particle α by particle β. Then channel t corresponds to the annihilation of a pair $\alpha + \bar{\alpha}$ with formation of a pair $\beta + \bar{\beta}$. Suppose that there exists a one-particle state in channel t, but no such states in other channels. Then, according to (3.16),

$$U^{(1)}(t) = \frac{R}{\mu^2 - t}.$$

This amplitude describes channel s, in which $t = -2p_c^2(1-\cos\theta) < 0$ (p_c is the momentum in the center-of-mass system of α, β), and can be interpreted as the Born amplitude, i.e., as the Fourier transform of the potential

$$U^{(1)}(r) = \frac{Re^{-\mu r}}{r} \tag{3.18}$$

(the Yukawa potential, which for $\mu = 0$ becomes the Coulomb potential). Thus we may say that the intermediate state of the annihilating channel generates the potential for the elastic channel.

For simplicity we have here treated the case of spinless particles. In the presence of spin, formula (3.13) will also contain a summation over the spin states of the intermediate particle. As a result, the expression $1/(\mu^2 - r)$ is replaced by the appropriate Green's function for this particle, which will also have a pole at $r = \mu^2$.

Let us give examples of the location of poles of the tetrode amplitude.

1) Scattering of an electron by a proton or muon (channel s). There is a one-particle state (photon) in the annihilation channel t. There is a pole for $t = 0$.

2) Electron-electron scattering. Annihilation of the pair into a photon occurs in channels t and u, i.e., there are poles for $t = 0$, $u = 0$.

3) Scattering of a π^+ meson by a proton (channel s). The single-particle state (neutron) occurs in channel u ($\pi^- + p \to n$). There is a pole at $u = m_n$.

4) Neutron-proton scattering. There are poles at $t = m_\pi$ ($p + \bar{p} \to \pi^0$) and $u = m_{\pi^+}$ ($p + \bar{n} \to \pi^+$). In examples 3 and 4 we have not included electromagnetic forces (photon poles).

5. Two-particle intermediate states. Let us consider now the part of the tetrode amplitude which is associated with two-particle intermediate states. Denoting it by $U^{(2)}$, we get from (3.6)

$$\operatorname{Im} U^{(2)} = \frac{\lambda_2}{8\pi^2} \int U_{b2} U_{a2}^* \delta(p_a - q_1 - q_2)\, \delta(q_1^2 - m_1^2)$$

$$\times\, \delta(q_2^2 - m_2^2)\, d^4q_1\, d^4q_2. \tag{3.19}$$

The notation here is the same as in (3.6): U_{b2} and U_{a2} are the amplitudes for transitions from states b and a, which are the final and initial state for the given channel (to be specific, we shall call it the t channel), q_1 and q_2 are the four-momenta of the particles in the intermediate state, and m_1 and m_2 are their masses.

Working in the center-of-mass system, we can rewrite (3.19) as follows:

$$\operatorname{Im} U^{(2)} = \frac{\lambda_2}{32\pi^2} \frac{q}{W} \int U_{b2} U_{a2}^*\, do, \tag{3.20}$$

where W is the total energy, q is the value of the momentum of the intermediate particle ($W = \sqrt{t} = \sqrt{q^2+m_1^2} + \sqrt{q^2+m_2^2}$) and do is the element of solid angle in the center-of-mass system.

From (3.20) we see that Im $U^{(2)}$ is finite and goes to zero for $q = 0$ ($W = m_1 + m_2$), i.e., at the threshold of the reaction with formation of two intermediate par-

FIG. 7

FIG. 8

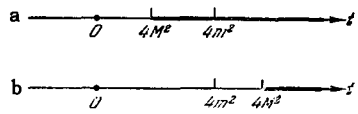

FIG. 9

ticles. From this it follows, as we have already pointed out earlier, that two-particle states can lead to singularities of the amplitude in the form of branch points, and to dispersion relations of the type (3.9).

Let us consider in more detail the structure of expression (3.20). Each of the amplitudes U_{b2}, U_{a2} and $U^{(2)} = U^{(2)}_{ba}$ which appears there is a tetrode amplitude and is a function of two variables, for which we may choose $t = W^2$ and the cosine of the scattering angle in the center-of-mass system. Let us denote the cosines of the angles between the initial and final momenta by z, between initial and intermediate, by z_1, and between final and intermediate by z_2, i.e.,

$$U^{(2)} = U^{(2)}(t, z), \quad U_{a2} = U_{a2}(t, z_1), \quad U_{b2} = U_{b2}(t, z_2), \quad do = dz_2\, d\varphi. \tag{3.21}$$

As the simplest special case, let U_{a2} be independent of scattering angle: $U_{a2} = U_{a2}(t)$. Such a case is realized, for example, when U_{a2} is described by the pole amplitude $U_{a2} = R/(\mu^2 - t)$, or when one of the particles interacts weakly and is described by first-order perturbation theory, while the second has arbitrarily strong interaction (for example, scattering of an electron by a proton or a π meson), so that (cf. the diagram in Fig. 7)

$$U_{a2} = \frac{F_{a2}(t)}{\mu^2 - t}. \tag{3.22}$$

In this case it follows from (3.20) that $U^{(2)}$ is also independent of scattering angle:

$$\operatorname{Im} U^{(2)} = \frac{\lambda_2}{16\pi} \frac{q}{W} U_{a2}(t)\, U^0_{b2}(t), \tag{3.23}$$

where

$$U^0_{b2}(t) = \int_{-1}^{1} U_{b2}(t, z_2)\, dz_2. \tag{3.24}$$

We note that U^0_{b2} is proportional to the partial amplitude corresponding to orbital angular momentum $l = 0$.

If U_{a2} has the form (3.22), then $U^{(2)}$ for real $t > \mu^2$ can be written in the form

$$U^{(2)} = \frac{F(t)}{\mu^2 - t},$$

where

$$\operatorname{Im} F(t) = \frac{\lambda_2}{16\pi} \frac{q}{W} F_{a2}(t)\, U^0_{b2}(t).$$

This relation corresponds to the triode diagram of Fig. 8.

In the general case where both amplitudes depend on angles, it is convenient to choose z_1 and z_2 as integration variables. To do this we must make use of the relation

$$z_1 = zz_2 + \sqrt{1 - z^2}\sqrt{1 - z_2^2}\cos\varphi.$$

Formula (3.20) then takes on the following form:

$$\operatorname{Im} U^{(2)}(t, z) = \frac{\lambda_2}{16\pi^2} \frac{q}{W} \int_{-1}^{1} dz_2 \int_{z_1^-}^{z_1^+} \frac{U_{b2}(t, z_2)\, U^*_{a2}(t, z_1)\, dz_1}{\sqrt{K(z, z_1, z_2)}}, \tag{3.25}$$

where

$$K = (1 - z^2)(1 - z_2^2)\sin^2\varphi = 1 - z^2 - z_2^2 - z_1^2 + 2zz_1z_2. \tag{3.26}$$

The limits of the inner integration are the roots of the function K

$$z_1^{\pm} = zz_2 \pm \sqrt{(1 - z_2^2)(1 - z^2)}, \tag{3.27}$$

i.e., the values of z_1 for $\varphi = 0$ and $\varphi = \pi$. Here the dependence on z appears explicitly in the integrand (3.20).

6. Triangle diagrams. We now go on to discuss the complications that may arise in constructing the dispersion relations because of the form of the unitarity relation. Let us begin with the simplest case, where we are dealing with a function of a single variable t, and the imaginary part of the amplitude is given by formulas (3.23). For simplicity we shall assume that the masses of the particles in the final state are identical and equal to M, while in the intermediate state they are identical and equal to m. The masses of the particles in the initial state will be assumed to be smaller than M and m.

The establishment of the form of the dispersion relation for $U^{(2)}(t)$ is simple for the case where $M < m$. Then the physical region for the intermediate reaction $2 \to b$ lies at $t > 4m^2$, i.e., it is further than the physical region for the reaction $a \to b$, which occurs for $t > 4M^2$ (Fig. 9a). The branch point for the amplitude $U^{(2)}$, i.e., the threshold for formation of the intermediate particles, $t = 4m^2$, lies within the physical region of the reaction $a \to b$. Therefore the dispersion relation has the very simple form*

$$U(t) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{A(t')}{t' - t}\, dt', \tag{3.28}$$

where

$$A(t) = \operatorname{Im} U(t). \tag{3.29}$$

*The function A in formula (3.28) and other similar functions are usually called the absorptive part of the amplitude, this term being retained even when A is not given by formula (3.29) and, in particular, may be complex (cf. below).

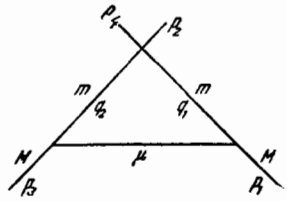

FIG. 10

The situation is different when M is greater than m, so that the threshold for the reaction of formation of the intermediate particles lies outside the physical range of the reaction a → b (Fig. 9b), and the unitarity relation does not have any immediate physical meaning in the region $4m^2 < t < 4M^2$. To establish the form of the dispersion relation for this case, we shall consider the parameter M^2 as the independent variable and require that the function $U(t) \equiv U(t, M^2)$ be an analytic function of the variable M^2.* Then the problem reduces to the analytic continuation of the function $U(t, M^2)$, defined for $M^2 < m^2$ by formulas (3.28), (3.29), (3.23), and (3.24), to the region $M^2 > m^2$.

Let us consider an example where the amplitude U_{ba} is determined by a pole in the s channel. (The amplitude $U^{(2)}$ corresponds to the triangle Feynman diagram of Fig. 10.†) Thus, let

$$U_{b2} = \frac{R}{\mu^2 - s_2}, \tag{3.30}$$

where

$$s_2 = (p_1 - q_1)^2 = M^2 + m^2 - 2W^2 + 2pqz_2, \tag{3.31}$$

p_1 and q_1 are the four-momenta of the particles in the final and intermediate states, $p = \sqrt{(t/4) - M^2}$, $q = \sqrt{(t/4) - m^2}$ are the corresponding momenta in the center-of-mass system. In the expression for the function $A = \mathrm{Im}\, U^{(2)}$ (3.23), the dependence on M^2 is contained only in the last factor U^0_{b2}. According to (3.24), (3.30), and (3.31),

$$U^0_{b2} = -\frac{R}{2pq}\, a_0(t, M^2),$$

where

$$a_0(t, M^2) = \int_{-1}^{1} \frac{dz_2}{z_2 - \zeta(t, M^2)} \tag{3.32}$$

and

$$\zeta(t, M^2) = \frac{t - 4m^2 + 2(M_c^2 - M^2)}{\sqrt{(t-4m^2)(t-4M^2)}}, \qquad M_c^2 = m^2 + \mu^2. \tag{3.33}$$

*We note that $M^2 = p_1^2$ is a kinematic variable completely equivalent to $t = (p_1 + p_3)^2$. It is natural to assume that the amplitude is an analytic function of all kinematic invariants, including those which we previously assumed to be given parameters. Thus, M^2 may be regarded as a complex variable, and the value of the amplitude for a given particle mass M_0 as the limit when $M^2 \to M_0^2 + i0$.

†The investigation of the analytic properties of the amplitude for triangle and square diagrams was first carried out by Karplus, Sommerfield, and Wichmann.[7] The method described here, based on the unitarity relation, is due to Mandelstam.[8]

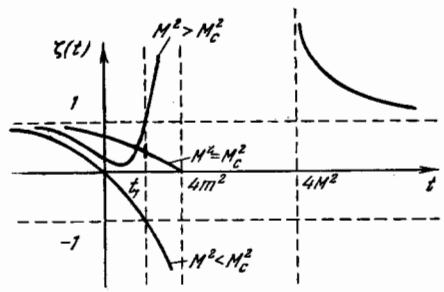

FIG. 11

The problem reduces to investigating the analytic properties of the function $a_0(t, M^2)$.* For $|\zeta(t, M^2)| > 1$, the denominator of the integrand in (3.32) is different from zero and the function a_0 is regular. A singularity appears at that value $t = t_1$ for which $\zeta(t, M^2)$ enters the region of integration $(-1, +1)$. From the condition $\zeta^2 = 1$, we find

$$t_1 = 4m^2 - \frac{(M^2 - M_c^2)^2}{\mu^2}. \tag{3.34}$$

In Fig. 11 we show the dependence of ζ on t for $M^2 > m^2$. For given M^2, the point t_1 corresponds to the values

$$\zeta(t_1, M^2) = \begin{cases} -1 & (M^2 < M_c^2), \\ +1 & (M^2 > M_c^2), \\ 0 & (M^2 = M_c^2). \end{cases}$$

Let us follow the variation of t_1 as M^2 increases, starting from values $M^2 < m^2$. For $M^2 < m^2$, $t_1 < 4m^2$ and the singularity of $a_0(t)$ lies to the left of the lower integration limit in (3.28); $a_0(t)$ and $A(t)$ are real inside the integration region, as they should be, because $A(t)$ is the imaginary part of the amplitude in the physical region. As M^2 increases, t_1 approaches the point $t = 4m^2$, remaining to the left of it. Thus, when M^2 reaches the value m^2 and exceeds it, $a_0(t)$ ceases to be regular within the region of integration in (3.28), and consequently we retain for $U(t)$ the form of the dispersion relations (3.28) in which the non-physical region $4m^2 < t < 4M^2$ appears, where $A(t)$ is the analytic continuation of the function $\mathrm{Im}\, U(t)$. However, when M^2 reaches the "anomalous" value

$$M^2 = M_c^2 \equiv m^2 + \mu^2, \tag{3.35}$$

the point t_1 reaches the lower limit of the integral (3.28), $t_0 = 4m^2$, and its further continuation requires an additional investigation of the behavior of $a_0(t, M^2)$ in the neighborhood of the point $M^2 = M_c^2$, despite the fact that, with further increase in M^2, t_1 once again goes out of the region, to the left of the point $t = 4m^2$.

The point t_1 is a branch point for the function $a_0(t)$, and the latter is regular in the t plane with a cut extending from the point t_1 toward the left to infinity. But the point t_1 itself is a function of the complex

*The results presented here can, of course, be obtained by carrying out the explicit integration in (3.32).

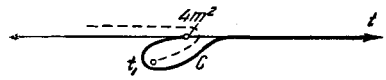

FIG. 12

variable M^2. Let $M^2 = M_c^2 + \epsilon$ ($\epsilon = \alpha + i\beta$, $\beta > 0$). Then $t_1 = 4m^2 - (1/\mu^2)(\alpha^2 - \beta^2 + 2i\alpha\beta)$. With increasing M^2, keeping an infinitesimal positive imaginary part, the point t_1 moves along the curve shown by the dotted line in Fig. 12. When $\alpha < 0$ ($M^2 < M_c^2$), it moves along the upper side of the real axis; for $\alpha > 0$, it goes over onto the left edge surrounding the point $t = 4m^2$. Thus $a_0(t)$ for $M^2 > M_c^2$ is continued analytically as a function in the t plane with a cut from the point t_1 along the dashed curve in Fig. 12. In this case ($M^2 > M_c^2$), the integral (3.28) becomes meaningless, since the integration contour, the real axis, intersects the branch line. The analytic continuation of the integral (3.28) can be accomplished by deforming the integration contour into the contour C which is shown by the solid line in Fig. 12. In fact, for $M^2 < M_c^2$ the contour C is equivalent to the section of the real axis $4m^2 \leq t < \infty$, while for $M^2 > M_c^2$ it does not intersect the branch line.

Thus in the "anomalous" case ($M^2 > M_c^2$) the dispersion relation has the form

$$U(t) = \frac{1}{\pi} \int_C \frac{A(t')\,dt'}{t' - t}\,dt'.$$

This integral can also be written as an integral along the real axis

$$U(t) = \int_{t_1}^{\infty} \frac{A'(t')\,dt'}{t' - t}, \qquad (3.36)$$

where

$$A'(t) = \begin{cases} A(t) & (t > 4m^2), \\ A(t - i0) - A(t + i0) & (t < 4m^2). \end{cases}$$

In Sec. 1 we explained the relation between the form factor of a particle, i.e., a quantity proportional to the scattering amplitude U(t), and its spatial structure. According to (1.10), the characteristic dimension of the particle R_0 is determined by the lower limit of the dispersion integral. In the normal case it is simply related to the mass of the intermediate particles

$$R_0 \sim \frac{1}{m}$$

For example, in the case of a nucleon, m is the pion mass. In the anomalous case

$$R \sim \frac{1}{\sqrt{t_1}},$$

where t_1 may be significantly smaller than m^2, and the radius of the particle correspondingly larger.

A good illustration of the anomalous case is that of the deuteron. The triangle diagram showing the scattering of an electron by a deuteron is given in Fig. 13a; the letter on each line denotes the corresponding particle. It is easy to see that here we are dealing with the anomalous case where

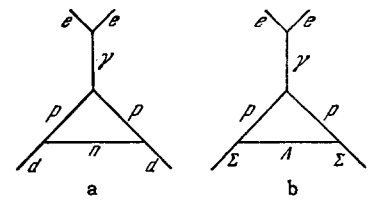

FIG. 13

$$t_1 = M\omega,$$

where M is the nucleon mass, and

$$\omega = M_n + M_p - M_d$$

is the binding energy of the deuteron. The expression for the deuteron radius

$$R_0 \sim \frac{1}{\sqrt{t_1}} \sim \frac{1}{\sqrt{M\omega}}$$

gives the well-known relation from the elementary theory of the deuteron.

Another example of the anomalous case, which already deals with a "truly elementary" particle, the Σ hyperon, is explained in Fig. 13b.

We have considered the case where the amplitude U_{b2} has the form (3.30). It is not difficult to consider the more general case where U_{b2} has the structure of a dispersion integral of type (3.9) in the variable s_2, i.e.,

$$U(t, s_2) = \frac{1}{\pi} \int_{s_2^0}^{\infty} \frac{A_s(t, s_2')\,ds_2'}{s_2' - s_2}. \qquad (3.37)$$

Then

$$U_{b2}^0(t) = -\frac{1}{2pq} \int_{s_2^0}^{\infty} a_s^0(t, s_2')\,ds_2'$$

and

$$a_s^0(t, s_2) = \int_{-1}^{1} \frac{dz_2}{z_2 - \zeta(t, s_2)}.$$

It is obvious that the dispersion relation for U(t) retains the form (3.28) for the "normal" case, where $s_2^0 < M_c^2$.

We would have arrived at the same results by considering the case where U_{b2} is determined by the pole in channel u, i.e.,

$$U_{b2} = \frac{R}{\mu^2 - u_2},$$

or where U_{b2} has the representation (3.9).

We have restricted ourselves to the case of equal masses of particles in the intermediate ($m_1 = m_2 = m$) and final ($M_1 = M_2 = M$) states. In the case of unequal masses, the general picture is similar. If $M_1 > m_1 + \mu$, i.e., if the particle M_1 is unstable, the singular point t_1 becomes complex.

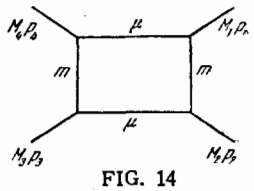

FIG. 14

7. Square diagrams. The unitarity relation, applied to channels t and u, allows us to establish the form of the dispersion relation for the amplitude U(s,t) considered as a function of the variable t for a given value of s. Here, naturally, t may take on all values, in particular, those referring to the physical region of channel s or to non-physical regions. The value of s itself is assumed to be fixed in the physical region of channels t or u (i.e., $|z| < 1$, for example, $s < 0$ for elastic scattering). Now our problem is the continuation of the dispersion relation into the region of arbitrary values of s, referring to the physical region or non-physical region of channel s. By carrying this out we obtain a representation for U(s,t) as a function of both variables.

Our problem is the following. Let the amplitude U(t,z) be expressed by the dispersion integral

$$U(t, z)=\frac{1}{\pi}\int\frac{A(t', z)}{t'-t}dt',$$

where A(t,z) is real in the physical region ($z^2 \leq 1$) and is determined by the unitarity relation (3.20). We must find the continuation of the functions A(t,z) and U(t,z) for arbitrary z.*

Let us consider relation (3.25) for a very simple example, applying it to the dependence of U on two variables. (This amplitude corresponds to the Feynman diagram of Fig. 14.) Let U_{b2} and U_{a2} be determined by the poles in the s channel:

$$U_{b2}=\frac{R_1}{\mu^2-s_1},\quad U_{b2}=\frac{R_2}{\mu^2-s_2}. \tag{3.38}$$

Then (3.25) takes the following form:

$$A(t, z)=\frac{\lambda_2}{16\pi^2}\frac{R_1}{2pq}\frac{R_2}{2p'q}a(z, t),$$

where

$$a(z, t)=\int_{-1}^{1}\frac{dz_2}{z_2-\zeta_2}\int_{z_1^-}^{z_1^+}\frac{dz_1}{(z_1-\zeta_1)\sqrt{K(z, z_1, z_2)}}. \tag{3.39}$$

K is determined by formula (3.26), $z_1^\pm$ by formula (3.27), p, q, and p′ are respectively the center-of-mass momenta of the final, intermediate, and initial states, ζ_1 and ζ_2 are given by formula (3.33) for M = M_1 or M_2 (the mass of each of the two particles in the initial or final state).

In the inner integral of (3.39), by introducing the factor ½, the integration along the real axis may be replaced by an integration along the contour C of

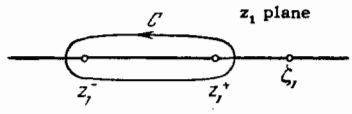

FIG. 15

Fig. 15, circling the branch line of the function $\sqrt{K}$. For $z^2 < 1$ we have $\zeta_{1,2} > 1$ and $z_1^\pm < 1$ in the absence of anomalous mass relations. The integral over z_1 reduces to the residue at the point $z_1 = \zeta_1$:

$$\left.\begin{aligned}\int_C\frac{dz_1}{(z_1-\zeta_1)\sqrt{K(z, z_1, z_2)}}&=-\frac{2\pi}{\sqrt{-K(z, \zeta_1, z_2)}},\\ a(z, t)&=-\pi\int_{-1}^{1}\frac{dz_2}{z_2-\zeta_2}\frac{1}{\sqrt{-K(z, \zeta_1, z_2)}}.\end{aligned}\right\} \tag{3.40}$$

Now the problem reduces to investigating the analytic properties of a(z,t) for $z > 1$. The singularities of the integrand are the following: $z_2 = \zeta_2$ (a pole) and $z_2 = z_2^\pm$ (branch points), where

$$z_2^\pm=z\zeta_1\pm\sqrt{(z^2-1)(\zeta_1^2-1)}. \tag{3.41}$$

Since $\zeta_2 > 1$ (for $t > 4m^2$ and in the absence of anomalous mass relations), the first singularity is outside the region of integration (3.40). From (3.41) it follows that for $z^2 < 1$ the branch points are complex. For $z = 1$, $z_2^\pm = \zeta_1 = 1$. Furthermore, for $z > 1$, z_2^+ moves to the right and z_2^- to the left of the point ζ_1. z_2^- takes on its minimum value and touches the integration contour $z_2^- = 1$ for $z = \zeta_1$. To continue a(z,t) into the region $z > \zeta_1$, it is necessary (just as in item 3 for the triangle diagram) to investigate the behavior of the function in the neighborhood of the point $z = \zeta_1$ for complex values of z. It turns out that the branch line circles the point $z_2 = 1$, and thus the continuation of (3.40) is an integral along the deformed contour C (Fig. 16). Here the point $z = \zeta_1$ is not a singular point of the function a(z,t), which remains real also for $z > \zeta_1$. A singularity appears when z_2^- reaches the values $z_2^- = \zeta_2$. Then the integration contour C encircles the pole of the integrand $z_1 = \zeta_1$. Since

$$\operatorname{Im}\frac{1}{z_2-\zeta_2}=-\pi\delta(z_2-\zeta_2),$$

for $z_2^- > \zeta_2$,

$$\operatorname{Im}a(z, t)=\frac{\pi^2}{\sqrt{-K(z, \zeta_1, \zeta_2)}}. \tag{3.42}$$

According to (3.41), z_2^- reaches the value ξ_2 for $z = z_c$ where

$$z_c=\zeta_1\zeta_2+\sqrt{(\zeta_1^2-1)(\zeta_2^2-1)}. \tag{3.43}$$

Since ζ_1 and ζ_2 are functions of t, Eq. (3.43) is a curve in the z,t plane (or the s,t plane) on which the function A(t,z) has a singularity. Thus A(t,z) can be represented as a dispersion integral in the variable z or s:

$$A(t, s)=\frac{1}{\pi}\int_{z_c}^{\infty}\frac{\varrho(t, z')}{z'-z}dz'=\frac{1}{\pi}\int_{s_c}^{\infty}\frac{\varrho(t, s')}{s'-s}ds', \tag{3.44}$$

*The method of investigating the analytic properties of amplitudes on the basis of the unitarity relation is due to Mandelstam.[9] In our presentation we follow Gribov.[10]

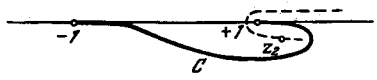

FIG. 16

where $\rho = \operatorname{Im} A$, while s is related to z by the linear relation (2.31). Here we designate the same quantity by $A(t,z)$, $\rho(t,z)$ or $A(t,s)$, $A(t,s)$. Thus the amplitude $U(t,s)$ in the example considered here has the following double spectral representation:

$$U(t,s) = \frac{1}{\pi^2}\int\int \frac{\varrho(t',s')}{(t'-t)(s'-s)}\,dt'\,ds', \qquad (3.45)$$

where ρ is real and different from zero in the region with the boundary given by curve (3.43). This region is shaded in Fig. 17. We note that this is a non-physical region within which the two variables s and t are positive and greater than $4\mu^2$ and $4m^2$ respectively.

The results obtained above no longer hold in the case of the anomalous mass relation $M^2 > m^2 + \mu^2$. We shall not treat this case here.

8. The Mandelstam representation. Now we can formulate the general statement, due to Mandelstam,[9] concerning the analytic representation of the tetrode amplitude. It is expressed by the following formula:

$$U_i(s,u,t) = \frac{1}{\pi^2}\left\{\int\int \frac{\varrho_1(s',t')\,ds'\,dt'}{(t'-t)(s'-s)} + \int\int \frac{\varrho_2(s',u')\,ds'\,du'}{(s'-s)(u'-u)} + \int\int \frac{\varrho_3(u',t')\,du'\,dt'}{(u'-u)(t'-t)}\right\}, \qquad (3.46)$$

where ρ_i are real functions of the corresponding variables, which are different from zero within the region of positive values of their arguments, bounded by certain curves.

By $U(s,u,t)$ we here mean that part of the amplitude which depends on both independent variables. In addition, the amplitude may contain a part depending on only one of the variables s, u, t, which is expressed by pole terms of type (3.15) and one-dimensional dispersion integrals of the type (3.28).

A rigorous proof of the Mandelstam representation based on field theory or on the principles of analyticity, universality, and unitarity has not been obtained as yet. We here present those arguments which make this representation extremely natural.

First of all, let us consider the generalizations which follow directly from the results of our investigation of the simplest example (square diagram) obtained above in item 4. There we used two-particle intermediate states in the unitarity relation for the t-channel. The amplitudes of the intermediate processes were determined by poles in the s-channel. Along with these, we can consider the intermediate amplitudes with pole terms from the channel and, in addition, we can use in similar fashion the unitarity relations for the s-channel (with intermediate pole amplitudes for the u and t channels) or the u-channel (pole amplitudes correspondingly for s and t). We would obviously obtain the same expression (3.45) with s and t replaced

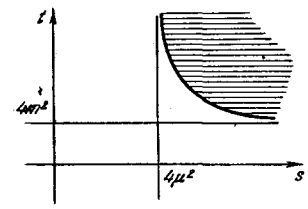

FIG. 17

by another pair of variables (s,u or u,t). The limit of the integration region in (3.45) would be determined by (3.43), in which the corresponding values of z would be determined for the other channels and where, in place of the masses m and μ, there would appear the corresponding masses of the intermediate particles for these other channels. Thus, the complete expression for the amplitude would have the form (3.46) with the value ρ_i determined by the function (3.42) (of the appropriate variables). We note that this amplitude corresponds to a set of square diagrams, i.e., the diagrams of Fig. 14 and others obtained from it by interchanging the external lines.

Furthermore, we can consider the case where the intermediate amplitudes in the two-particle term of the unitarity relation, for example, for the channel t, are expressed not by pole terms, but by a dispersion integral over s for a given t of the type (3.11). This generalization is similar to that which was made in par. 3 [cf. (3.37) et seq.]. We then again arrive at the Mandelstam representation in which ρ is defined not by (3.42), but by the formula[9]

$$\operatorname{Im} a(z,t) = \int\int \frac{A_1(t,\zeta_1)\,A_2^*(t,\zeta_2)}{\sqrt{-K(z,\zeta_1,\zeta_2)}}\,d\zeta_1\,d\zeta_2, \qquad (3.47)$$

where A_1 and A_2 are the corresponding absorptive parts of the intermediate amplitudes. The lower limit of integration in (3.46) is determined again by (3.43) where, in place of μ^2, we must substitute the value of the lower limit in the dispersion integrals for the intermediate amplitudes. Figure 18 gives a schematic picture of the boundary curves and regions of definition of the functions ρ_i for the reaction of scattering of a pion by a nucleon.

The last step which remains to be done for justifying the Mandelstam representation consists in the assumption that inclusion of many-particle ($\nu > 2$) states in the unitarity relation, in which the total mass of the state is greater than the mass of the two-particle state, leads to the same results except with a boundary which is farther out. Then we arrive at formula (3.46), in which the boundary of integration is determined by the two-particle states, and where the function ρ in some region is given by (3.42) and (3.47) and by additional, more complicated expressions.

Essentially the assertion expressed in the Mandelstam representation reduces to the fact that the function expressing the imaginary part of the amplitude in the physical region is an analytic function of z, whose

singularities are determined by the minimum masses of intermediate particles in the corresponding channel.

Within the framework of perturbation theory this assertion can be verified directly by investigating the analytic properties of the integrals corresponding to the Feynman diagrams. For the triangle and square diagrams this was done by Karplus, Sommerfield, and Wichmann.[7] Landau[11] proposed a general method for investigating the singularities of arbitrary Feynman amplitudes. In those cases for which Landau's method has been applied to specific diagrams the results turn out to be in agreement with the Mandelstam representation.

From the representation (3.46) one can, of course, obtain the usual (i.e., single-variable) dispersion relations. For example, for fixed t

$$U(t, s) = \frac{1}{\pi}\int\limits_{M_s}^{\infty} \frac{A_1(s', t)\,ds'}{s'-s} + \frac{1}{\pi}\int\limits_{M_u^2}^{\infty} \frac{A_2(u', t)\,du'}{u'-u}, \qquad (3.48)$$

where M_s and M_u are the minimum intermediate mass in channels s and u,

$$\left.\begin{aligned} A_1(s, t) &= \frac{1}{\pi}\int\limits_{t_1(s)}^{\infty} \frac{\varrho_1(s, t')\,dt'}{t'-t} - \frac{1}{\pi}\int\limits_{-\infty}^{t_2(s)} \frac{\varrho_2(s, t')\,dt'}{t'-t}, \\ A_2(u, t) &= \frac{1}{\pi}\int\limits_{t_1(u)}^{\infty} \frac{\varrho_3(u, t')\,dt'}{t'-t} - \frac{1}{\pi}\int\limits_{-\infty}^{t_2(u)} \frac{\varrho_2(u, t')\,dt'}{t'-t}; \end{aligned}\right\} \qquad (3.49)$$

$t_i(s)$ and $t_i(u)$ are defined by the equations for the corresponding boundary curves.

From (3.46) one can also obtain dispersion relations for other quantities, for example, the partial amplitudes.

Formula (3.46) assumes a definite behavior of the spectral functions ρ_i at infinity in order that the corresponding integrals exist. Just as in treating one-dimensional representations, formula (3.46) can be generalized to the case where ρ_i tends to a constant or increases according to a power law. Then one can form, in place of (3.46), a corresponding representation with subtractions. It is assumed that the functions ρ_i cannot increase faster than a power law, since this would mean that the amplitude has an essential singularity at infinity.

We should emphasize once more that the Mandelstam representation holds only in the absence of anomalous mass relations. If this is not the case, the analytic properties of the absorptive parts are more complicated[12] and do not lead to formula (3.46). For this same reason, a simple generalization of (3.46) to the case of amplitudes describing reactions in which more than four particles participate is impossible. In fact, if we were to combine, for example, two or more particles, we would then have a situation analogous to the triode or tetrode in which one of the particles has a mass equal to W —the energy of the whole group of particles. With increasing W, we would encounter anomalous and complex singularities.

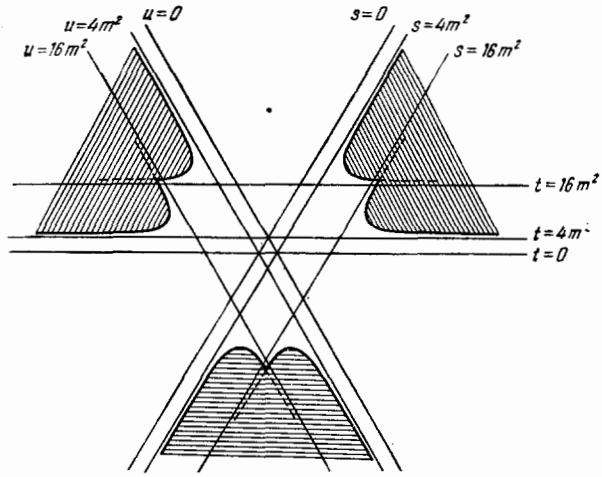

FIG. 18a. Region of definition of the functions ρ_i in the reaction $\pi + \pi \to \pi + \pi$.

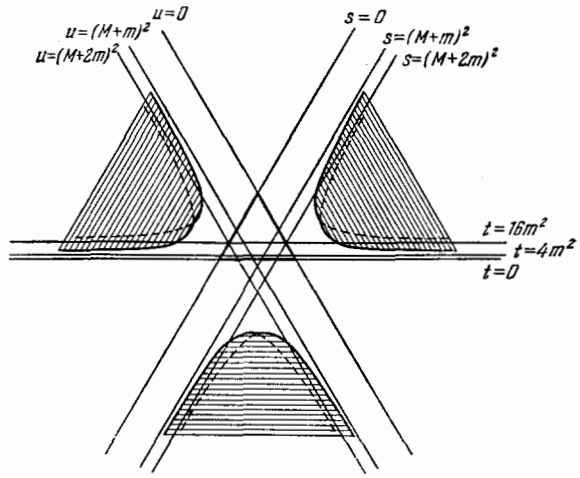

FIG. 18b. Region of definition of the functions ρ_i in the reaction $\pi + N \to \pi + N$.

4. ELECTROMAGNETIC STRUCTURE OF PARTICLES

1. Scheme of construction of perturbation theory. The only satisfactory part of the theory of elementary particles is quantum electrodynamics, i.e., the theory of interaction of electrons and photons. We want to show that quantum electrodynamics can be constructed on the basis of the principles presented above of a dynamical theory of the scattering matrix, without using the apparatus of quantum field theory. Such a construction contains no new physical results and therefore cannot pretend to replace the previous theory and to eliminate the fundamental concepts of field theory, especially since the principles of the S-matrix theory, and in particular the universality principle, are essentially based on results of field theory. However, it is of interest for several reasons. First, its apparatus and method of calculation of higher approximations is somewhat simpler. Secondly, we do not introduce such concepts

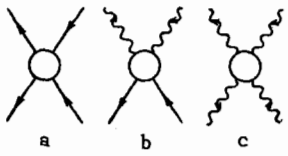

FIG. 19.

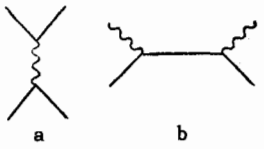

FIG. 21

as "bare" particles, which together with the concept of a vacuum lead to infinities which are then eliminated by a process of renormalization. The theory involves only finite quantities (physical charges and masses) and the construction of the amplitude on the basis of the unitarity relation allows one to understand clearly the significance of renormalization in field theory. In the third place, it is important in principle to be able to have another description, showing that the development and generalization of the theory need not necessarily stay within the framework of field theory.

Thus the starting point for us is the existence of electrons (particles and anti-particles) and photons with their assigned properties (mass, spin, etc.). The simplest reactions from the point of view of the number of particles participating are shown in Fig. 19 by the three tetrodes, where the straight lines correspond to electrons and the wavy ones to photons. These are the reactions: a) electron-electron scattering, b) scattering of a photon by an electron and annihilation of a pair into two photons, and c) photon-photon scattering.

FIG. 20

To find the amplitudes we use perturbation theory, which means the following: the larger the number of particles participating in a process, the smaller its amplitude. This means that in the first approximation we may keep, in the unitarity relation, only the one-particle intermediate states. The higher approximations can be obtained by iteration.

As we have seen (Sec. 3, item 4), the amplitudes in this approximation are expressed in terms of triode amplitudes. From electrons and photons we can construct only one triode, which is shown in Fig. 20. The other three triodes that one might imagine cannot occur: they do not, for example, satisfy the law of conservation of momentum.*

Thus in first approximation the amplitudes of tetrodes will be expressed by pole terms of type (3.16). The tetrode amplitude in Fig. 19a has poles in the t and u channels which correspond to the Feynman diagram of Fig. 21a (the intermediate particle is a photon). The tetrode amplitude of Fig. 19b has poles in the s and u channels (Fig. 21b) (where the intermediate particle is an electron). The tetrode amplitude of Fig. 19c has no poles and, correspondingly, photon-photon scattering does not occur in first approximation.

Let us now turn to the structure of the triode of Fig. 20. According to (2.42) it is described by two constants

$$\left.\begin{aligned}(U_{a1})_{t=0} &= [g_1\bar{u}(p_2)\gamma_\mu u(p_1) + g_2\bar{u}(p_2)\sigma_{\mu\nu}u(p_1)q_\nu]a_\mu,\\ \sigma_{\mu\nu} &= \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu).\end{aligned}\right\} \quad (4.1)$$

Electrodynamics is obtained if we set

$$g_1 = e, \quad g_2 = 0, \quad (4.2)$$

where e is the electron charge. The arbitrariness in the choice of the form of the triode amplitude contained in (4.1) corresponds exactly to that which exists in the usual construction of the theory based on the interaction Lagrangian.*

We obtain the second approximation if we retain the two-particle states in the unitarity relation and substitute for the amplitudes of the intermediate tetrodes the first (pole) approximation. This will give the absorptive parts of the amplitudes. The expressions for the amplitudes in terms of their absorptive part A is given by the dispersion integral

$$U(r) = \frac{1}{\pi}\int \frac{A(r')}{r'-r}\,dr', \quad (4.3)$$

if A(r) tends to zero as $r \to \infty$, so that this integral exists. This case corresponds to the absence of divergences in the Feynman integrals. If, however, A tends to a constant or increases slowly (logarithmically), the amplitude is given in the form of a dispersion relation "with subtractions":

$$U(r) = U(r_0) + \frac{r-r_0}{\pi}\int \frac{A(r')\,dr'}{(r'-r_0)(r'-r)}. \quad (4.4)$$

This case corresponds to a logarithmic divergence of the Feynman integral, and (4.4) is equivalent to the usual regularization. Here one must introduce the constant $U(r_0)$. For a power law behavior $A \sim r^n$, one must introduce $n+1$ subtractions, i.e., one must assign $n+1$ constants, for example, the value of the amplitude and its n derivatives at the point r_0.

*The electron spin is ½, that of the photon 1; the two photons cannot have angular momentum ½.

*The most general permissible form of the Lagrangian is the following:

$$L = g_1 j_\mu A_\mu + g_2\sigma_{\mu\nu}F_{\mu\nu},$$

where A_μ is the vector potential, $F_{\mu\nu}$ the electromagnetic field tensor, j_μ the current density, and $\sigma_{\mu\nu}$ the density of electromagnetic moment.

In quantum electrodynamics and other renormalizable field theories, we meet only logarithmic divergences. This means that for them we can use dispersion relations with one subtraction. The absorptive parts of the amplitudes constructed by iteration will, in the higher approximations, have the same behavior at infinity, which means that they will be renormalizable. Renormalizable theories lead to such a behavior of A that each iteration again leads to the necessity for introducing additional constants, which shows that it is impossible to construct the amplitude in the form of a perturbation series.

Later we shall give examples of the detailed construction of quantum electrodynamic amplitudes in the simplest case, where they depend on only a single variable.

2. Vacuum polarization. Let us consider a reaction described by the tetrode of Fig. 19a, with the one difference that we have not electrons, but two distinct charged particles a and b. The s channel of this reaction describes the elastic scattering of particle a by particle b, while the t channel describes the annihilation of an a pair with conversion to a b pair. The values of the particle spins will not be specified, and we shall write the amplitude of the fundamental electrodynamic triode (cf. Fig. 20) in general form:

$$U_{a1}=Z_a e\alpha_\mu a_\mu(q), \quad U_{b1}=Z_b e\beta_\mu a_\mu(q), \tag{4.5}$$

where α_μ and β_μ are the characteristic 4-vectors (currents) for particles a and b, which satisfy the condition (continuity equation)

$$\alpha_\mu q_\mu=0, \quad \beta_\mu q_\mu=0, \tag{4.6}$$

$Z_a e$ and $Z_b e$ are the charges of the particles, which are assumed to be small in order that we satisfy the requirements for the applicability of perturbation theory (the iteration method).

The amplitude of the reaction we are considering has a pole in the t channel (Fig. 21a). Thus, in first approximation, according to (3.14)—(3.16) and (4.5),

$$U^{(1)}=-\frac{Z_a Z_b e^2}{t}\alpha_\mu\beta_\mu. \tag{4.7}$$

To obtain a second approximation we consider the two-particle part of the unitarity relation. As intermediate particles we may consider only electron pairs.* We denote the amplitude in second approximation by $U^{(P)}$ and its absorptive part by $A^{(P)}$. According to (3.20) we get

$$A^{(P)}=\frac{\lambda_2}{64\pi^2}\sqrt{\frac{t-4m^2}{t}}\int\Sigma U_{b2}^{(1)}U_{a2}^{(1)*}\,do. \tag{4.8}$$

Here Σ denotes a summation over electron spins, m is the electron mass, do is the element of solid angle in the system of the c.m.s. of the pair. The amplitudes for annihilation of an electron pair into a pair of particles a or b also have pole behavior in the t channel:

$$\left.\begin{aligned} U_{b2}^{(1)}&=-\frac{Z_b e^2\beta_\mu\bar v(k_1)\gamma_\mu v(k_2)}{t},\\ U_{a2}^{(1)*}&=-\frac{Z_a e^2\alpha_\mu\bar v(k_2)\gamma_\mu v(k_1)}{t}, \end{aligned}\right\} \tag{4.9}$$

where $v(k_i)$ are the four-component spinor amplitudes, $-k_1$ and k_2 are the 4-momenta of the electron pair. If we choose the normalization

$$\bar v(k_i)\,v(k_i)=2m,$$

the coefficient λ_2 in (4.8) will be equal to unity. Thus

$$A^{(P)}=\frac{Z_a Z_b e^2}{64\pi^2}\alpha_\nu\beta_\mu\frac{1}{t^2}\sqrt{\frac{t-4m^2}{t}}\int\Sigma\,(\bar v(k_1)\gamma_\mu v(k_2))\times(\bar v(k_2)\gamma_\nu v(k_1))\,do.$$

The summation over polarization is carried out in the usual way, and the integration is elementary. Also using relation (4.5), we get

$$A^{(P)}(t)=Z_a Z_b e^2\alpha_\mu\beta_\mu\frac{\alpha}{3}\frac{t+2m^2}{t^2}\sqrt{\frac{t-4m^2}{t}}, \tag{4.10}$$

where $\alpha=e^2/4\pi=1/137$.

Formula (4.10) shows that $U^{(P)}$ differs from $U^{(1)}$ by a factor which is a universal function of t. In fact, comparing (4.7) with (4.10), we can write

$$U^{(P)}=U^{(1)}\frac{P(t)}{t}, \tag{4.11}$$

where

$$\operatorname{Im}\frac{P(t)}{t^2}=-\frac{\alpha}{3t^2}(t+2m^2)\sqrt{\frac{t-4m^2}{t}}. \tag{4.12}$$

The quantity P coincides with the vacuum polarization operator which is defined in quantum electrodynamics. We note that from (4.7) and (4.11) it follows that (since $U^{(P)}\ll U^{(1)}$)

$$U^{(1)}+U^{(P)}=\frac{Z_a Z_b e^2\alpha_\mu\beta_\mu}{t-P(t)}, \tag{4.13}$$

i.e., the correction reduces to replacing 1/t by 1/[t − P(t)] in the photon Green's function.

From (4.12) one can obtain P(t) directly by using the dispersion relation

$$\frac{P(t)}{t^2}=\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{\operatorname{Im}P(t')/t'^2}{t'-t}\,dt'. \tag{4.14}$$

Substituting (4.12) and integrating gives the familiar expression for the renormalized vacuum polarization operator*

$$P(t)=\frac{\alpha}{\pi}t\left[\frac{1}{9}-(1-\theta\operatorname{ctg}\theta)\frac{2t+4m^2}{3t}\right], \tag{4.15}\dagger$$

where $\sin^2\theta=t/4m^2$.

Formula (4.11) can be interpreted as follows: The second approximation correction reduces to replacing the current α_μ in $U^{(1)}$ by $\alpha_\mu+\delta\alpha_\mu$, where

*The particles having the lowest mass are the most important, as will be seen from the result.

*Cf., for example, [13].

†ctg = cot

$$\delta\alpha_\mu(t)=\alpha_\mu\frac{P(t)}{t}.$$

This means that particle a has a structure given by the form factor

$$F(t)=1+F^{(P)}(t),$$

where

$$F^{(P)}(t)=\frac{P(t)}{t}. \tag{4.16}$$

The first term corresponds to a point charge, the second to a distribution which has a universal character for all particles. We have seen [cf. (1.4) and (1.20)] that the imaginary part of the form factor is related to the charge distribution by a Laplace transformation. Thus from (4.12) and (1.4) we find for the distributed part of the charge $\delta\rho(r^2)$:

$$r\delta\varrho=\frac{\alpha 4m^2}{6\pi^2}\int_1^\infty e^{-2mr\zeta}\left(1+\frac{1}{2\zeta^2}\right)(\zeta^2-1)^{1/2}d\zeta. \tag{4.17}$$

From this we see that the effective radius of the distribution $R_0 \sim 1/m$. Integrating, it is not difficult to obtain the familiar formula for the charge contained within a small sphere r for $r \ll 1/m$:

$$e(r)=e+\delta e=e\left(1+\frac{\alpha}{3\pi}\ln\frac{1}{(mr)^2}\right). \tag{4.18}$$

Of course, this formula is applicable only so long as $\delta e \ll e$.

3. Structure of the electron. If one of the particles (for example, b) is an electron, the amplitude $U_{b2}^{(1)}$ has two poles corresponding to the t and s channels (the Moeller formula):

$$U_{b2}^{(1)}=-e^2\left\{\frac{[\bar v(p_2)\gamma_\nu v(p_1)][\bar v(k_1)\gamma_\nu v(k_2)]}{t}\right.$$
$$\left.-\frac{[\bar v(p_2)\gamma_\nu v(k_2)][\bar v(k_1)\gamma_\nu v(p_1)]}{s_2}\right\}, \tag{4.19}$$

where p_1 and $-p_2$ are the momenta corresponding to electrons in state b, $s_2=(p_1-k_1)^2$.

As we have seen, the first term in (4.19) determines a universal effect of vacuum polarization. The presence of the second term is specific for the electron. It brings about part of the amplitude for scattering of the electron by particle a (second approximation), which we denote by $U^{(e)}$, while its absorptive part will be called $A^{(e)}$.

According to (4.19) and (4.8), we obtain [cf. (3.23), (3.32)]

$$A^{(e)}=\frac{1}{32\pi^2}\sqrt{\frac{1}{t(t-4m^2)}}\frac{Z_a e^2}{t}\alpha_\mu\int\Sigma(\bar v(p_2)\gamma_\nu v(k_2))(\bar v(k_2)\gamma_\mu v(k_1))$$
$$\times(\bar v(k_1)\gamma_\nu v(p_1))\frac{d\sigma}{z-\zeta(t)}, \tag{4.20}$$

where [cf. (3.33)] $\zeta=1+\frac{2\mu^2}{t-4m^2}$, z is the cosine of the angle between the electron in the intermediate state and state b in the center-of-mass system, and μ is the "photon mass."*

*This is the quantity introduced to eliminate "infrared" problems; cf., for example,[13].

Formula (4.20) can be written in the form

$$A^{(e)}=\frac{-Z_a e^2\alpha_\mu}{t}f_\mu(t) \tag{4.21}$$

and correspondingly the amplitude for scattering of the electron is

$$U^{(e)}=-\frac{Z_a e^2\alpha_\mu}{t}F_\mu^{(e)}(t).$$

$F_\mu^{(e)}$ can be interpreted as the vector form factor of the electron which, as explained in Sec. 1, can be related to the spatial structure of the particle. The general expression for the vector form factor (cf. Sec. 1, item 6) must have the form

$$F_\mu^{(e)}(t)=\bar v(p_2)\left[F^{(e)}(t)\gamma_\mu+\frac{1}{2m}G^{(e)}(t)\sigma_{\mu\nu}q_\nu\right]v(p_1) \tag{4.22}$$

and similarly

$$f_\mu(t)=\bar v(p_2)\left[f(t)\gamma_\mu+\frac{1}{2m}g(t)\sigma_{\mu\nu}q_\nu\right]v(p_1). \tag{4.23}$$

The two invariant form factors $F^{(e)}(t)$ and $G^{(e)}(t)$ determine the distribution of charge and magnetic moment of the electron.* f(t) and g(t) are the corresponding absorptive parts of the form factors,

$$\left.\begin{aligned}\operatorname{Im}F^{(e)}(t)&=f(t),\\ \operatorname{Im}G^{(e)}(t)&=g(t).\end{aligned}\right\} \tag{4.24}$$

Carrying out the summation over spin states of the intermediate pair in (4.20) in the usual fashion and doing the elementary integration, we obtain, comparing with (4.21) and (4.23),

$$f(t)=\frac{\alpha}{4}\frac{1}{\sqrt{t(t-4m^2)}}\left[3t-8m^2-(2t-4m^2)\ln\frac{t-4m^2}{\mu^2}\right], \tag{4.25}$$

$$g(t)=\frac{\alpha}{4}\frac{4m^2}{\sqrt{t(t-4m^2)}}. \tag{4.26}$$

Furthermore, on the basis of (4.24), using the dispersion relations,

$$F^{(e)}(t)=\frac{t}{\pi}\int_{4m^2}^\infty\frac{f(t')}{(t'-t)t'}dt', \tag{4.27}$$

$$G^{(e)}(t)=\frac{1}{\pi}\int_{4m^2}^\infty\frac{g(t')}{t'-t}dt'. \tag{4.28}$$

We arrive at expressions for the form factors which give for $F_\mu^{(e)}$ the familiar expression which in quantum electrodynamics is called the renormalized vertex part for the electron (second approximation). In particular, it is easy to see that

$$G^{(e)}(0)=\frac{\alpha}{4\pi}\int_1^\infty\frac{dx}{x\sqrt{x(x-1)}}=\frac{\alpha}{2\pi}, \tag{4.29}$$

which states the familiar result due to Schwinger.

In conclusion, we make a remark concerning the character of the renormalization of charge in the scheme presented here for developing quantum elec-

*More precisely, the "anomalous moment." The "normal," i.e., Dirac magnetic moment of the electron, is, of course, contained in the expression $\bar v(p_2)\gamma_\mu v(p_1)$.

trodynamics. At first glance, there appears to be no renormalization here, since throughout we have dealt only with finite quantities which admit of a direct physical interpretation. However, a renormalization is actually contained in the form of the dispersion relations which we are using. Thus, for $F^{(e)}(t)$ and $F^{(P)}(t)$ we applied the dispersion relations (4.27) and (4.14) "with one subtraction." The dispersion relation without subtractions is not admissible here, since the corresponding integral for f(t) given by the expression (4.25) diverges [since $f(\infty) = \text{const}$]. We chose the value of $F^{(e)}(0)$ in (4.27) to be equal to zero since this constant, which does not contain the expansion parameter α, is equivalent to the form factor $F^{(1)}(0) = 1$, which is contained in the first approximation. A more general expression may also contain F(0). Then the condition

$$F^{(1)}(0) + F^{(e)}(0) + F^{(P)}(0) = 1 \qquad (4.30)$$

is equivalent to a charge renormalization.

We note that, with respect to P(t), (4.14) is a dispersion relation "with two subtractions." To it we can add two terms, $P(0) + tF^{(P)}(0)$. The second term is absorbed in (4.30), while the condition $P(0) = 0$ corresponds to a renormalization of the photon mass, since the amplitude (4.13) (or the Green's function) of the photon should have a pole at $t = 0$. For $G^{(e)}(t)$ the first approximation gives a zero value, in accordance with the postulates of quantum electrodynamics (4.2). Therefore $G^{(e)}(0)$ should be determined uniquely by the second approximation. It is remarkable that the form of g(t) [$g(\infty) = 0$] in (4.25) gives the possibility of using a dispersion relation without subtraction (4.28).

4. The pion form factor. One of the most important problems, which goes beyond the framework of quantum electrodynamics, is the problem of the electromagnetic structure of strongly interacting particles. The simplest of these is the problem of the electromagnetic structure of the lightest of the strongly interacting particles, the π meson. It cannot be completely solved, but it is interesting to treat the formulation of this problem and see what can be obtained on the basis of the two-particle part of the unitarity relation. [14]

Let us consider the same tetrode (cf. Fig. 19a) in which particle b is a π meson and particle a an electron. The s channel of this reaction represents scattering of the meson by the electron, and the t channel the conversion of an electron pair into a meson pair. Radiative corrections associated with electromagnetic interactions are small and can be neglected. Then the general expression for the amplitude, in first approximation in the electromagnetic interactions and exact with respect to the strong interactions, should have the following form, which is determined by the photon pole in the t channel:

$$U(t) = -e^2 \alpha_\mu \beta_\mu \frac{F(t)}{t}, \qquad (4.31)$$

where, according to (4.1), (4.2), and (2.44),

$$\left.\begin{aligned} \alpha_\mu &= \bar{v}(p_2')\gamma_\mu v(p_1'), \\ \beta_\mu &= p_\mu \equiv (p_1 + p_2)_\mu \end{aligned}\right\} \qquad (4.32)$$

(p_i' and p_i are the momenta of electrons and pions, corresponding to the s channel). The quantity F(t) is the electric form factor of the pion.

We now consider the unitarity relation in the t channel. Since the internal parity of the pion is negative, the parity conservation law and the angular momentum conservation law forbid the transformation of two pions into a single pion.* Thus, the first term in a unitarity relation will be a two-particle term corresponding to conversion of the pion pair into an intermediate pair. As intermediate particles, we shall consider particles with minimum mass, i.e., again pions. The threshold of the reaction for formation of an intermediate pair is $t = 4m^2$, where m is the pion mass, which will coincide with the threshold of the physical region of the t channel for this reaction. The threshold for intermediate reactions with formations of pairs of other particles (K mesons or baryons) lies at $t = 4M^2$, where M is the mass of the corresponding particle. Since $M > 2m$, it is meaningless to consider such two-particle intermediate states without simultaneously treating states of four or more pions.†

Thus, if we include in the unitarity relation only pion pairs, we obtain an exact expression for the imaginary part of the amplitude in the region $4m^2 < t < 16m^2$.

According to (3.20) (for $\lambda_2 = 1$) we have in this region

$$\operatorname{Im} U = \frac{1}{64\pi^2} \sqrt{\frac{t-4m^2}{t}} \int U_{b2} U_{a2}^* \, do. \qquad (4.33)$$

U_{a2} is the amplitude for transformation of an electron pair into an intermediate meson pair, i.e., it is given by the same expression (4.31), but with the initial momenta p_i replaced by the intermediate ones k_i ($k = k_1 + k_2$):

$$U_{a2} = -\frac{e^2 \alpha_\mu}{t} k_\mu F(t). \qquad (4.34)$$

U_{b2} is the amplitude for meson-meson scattering, which we shall denote by U_π.

From formulas (4.31)—(4.34) we obtain

$$p_\mu \operatorname{Im} F(t) = \frac{1}{64\pi^2} \sqrt{\frac{t-4m^2}{t}} F^*(t) \int k_\mu U_\pi \, do,$$

*The transformation of a pair of pions into a K meson is forbidden by strangeness conservation, and the transformation into one of the other strongly interacting particles (baryons) is forbidden by conservation of angular momentum and other quantum numbers.

†Transformation of two pions into three pions is also forbidden by the laws of conservation of isospin and charge parity.

or

$$\operatorname{Im} F(t)=\frac{1}{32\pi}\sqrt{\frac{t-4m^2}{t}}\,F^*(t)\int_{-1}^{1} zU_\pi(t,z)\,dz, \quad (4.35)$$

where z is the cosine of the scattering angle in the center-of-mass system.

The integral which appears in (4.35) is proportional to the scattering amplitude in the p-state ($l = 1$).* We use the well-known representation for the scattering amplitude

$$U_\pi(t,z)=16\pi\sqrt{\frac{t}{t-4m^2}}\sum_l (2l+1)\,a_l(t)\,P_l(z), \quad (4.36)$$

where the P_l are Legendre polynomials, a_l is the partial amplitude, which is related to the scattering phase δ_l by

$$a_l(t)=\frac{1}{2i}\left(e^{2i\delta_l}-1\right), \quad (4.37)$$

the coefficients in (4.36) are easily found from a comparison of formulas (2.15) and (2.16).

Substituting (4.36) in (4.35), we obtain

$$\operatorname{Im} F(t)=F^*(t)\,a_1(t). \quad (4.38)$$

Unfortunately, this interesting relation holds only in the region $t < 16m^2$, in which there is only elastic meson-meson scattering. If it were valid for all t,† then we could directly express F(t) in terms of $\delta_1(t)$ by means of (4.38).

In fact, by virtue of the reality of expression (4.38), the phase F(t) should coincide with the phase of a_1, i.e., should be equal to δ_1. Representing F(t) in the form

$$F(t)=e^{u(t)},$$

we see that

$$\operatorname{Im} u(t)=\begin{cases}\delta_1(t), & t>4m^2,\\ 0, & t<4m^2.\end{cases}$$

Thus u(t) can be represented as a dispersion integral. Taking into account the normalization of the pion charge,

$$F(0)=1, \quad \text{i.e.,} \quad u(0)=0,$$

we obtain

$$F(t)=e^{\frac{t}{\pi}\int_{4m^2}^{\infty}\frac{\delta_1(t')}{t'(t'-t)}dt'}. \quad (4.39)$$

The theory of the pion form factor presented above obviously has only illustrative character. It is extremely interesting since it shows that one can formulate problems outside the framework of perturbation theory and indicates what restrictions are met in solving them in specific cases. From the practical point of view we can compute the absorptive parts only including two-particle states, in our particular example, in the region $t < 16m^2$. The inclusion of intermediate states with three or more particles requires that we treat amplitudes for processes in which more than four particles participate (pentodes, hexodes, and more generally polyodes), whose analytic properties and spectral representations are too complex.

5. SOME APPLICATIONS OF THE THEORY OF DISPERSION RELATIONS

1. Dispersion relations for meson-nucleon scattering. Here we present some important physical results obtained by using the theory of dispersion relations. These results refer to the scattering of π mesons by nucleons. It is therefore desirable to have a specific form of the dispersion relations for the pion-nucleon tetrode amplitude which gives the explicit dependence on spin and isospin.

Let p_1 and k_1 denote the initial momenta of a nucleon and meson, p_2 and k_2 their final momenta, corresponding to channel s (scattering). We use u_1 and u_2 to denote the amplitudes for nucleon states, containing their dependence on spin and isospin variables, and χ_1 and χ_2 for the isovector amplitudes of the mesons. According to (2.47) we can write the amplitude for the reaction U in the form

$$U=\bar{u}_2\chi^*_{2\beta}\left(U^+\delta_{\beta\alpha}+\frac{1}{2}(\tau_\beta\tau_\alpha-\tau_\alpha\tau_\beta)\,U^-\right)\chi_{1\alpha}u_1, \quad (5.1)$$

where U^+ and U^- are matrices which, in accordance with (2.38) and (2.39), have the following form:

$$U^\pm=a^\pm(s,u,t)+\frac{1}{2}b^\pm(s,u,t)\,\hat{k}, \quad (5.2)$$

where

$$\left.\begin{aligned} k&=k_1+k_2,\\ s=(p_1+k_1)^2,\quad u&=(p_1-k_2)^2,\quad t=(p_1-p_2)^2,\end{aligned}\right\} \quad (5.3)$$

and a and b are invariant functions.

Thus the process of scattering of a pion by a nucleon (and processes which represent other channels of the same reaction) are described by the four amplitudes $a^\pm$, $b^\pm$. These functions have definite symmetry with respect to permutation of the variables s and u, which is called crossing symmetry.

This symmetry is a consequence of the fact that the s and u channels of this reaction are identical processes: scattering of a pion by a nucleon. Therefore, when we interchange the momenta $k_1 \rightleftarrows -k_2$, and consequently $s \rightleftarrows u$, and simultaneously interchange the isospin indices $\alpha \rightleftarrows \beta$, the amplitude U should not change. Then it follows from (5.1) that $U^+ \to \pm U^\pm$ for $k \to -k$, $s \rightleftarrows u$, and, further, from (5.2)

$$\left.\begin{aligned} a^\pm(s,u,t)&=\pm\,a^\pm(u,s,t),\\ b^\pm(s,u,t)&=\mp\,b^\pm(u,s,t).\end{aligned}\right\} \quad (5.4)$$

*We note that in a p-state (i.e., one which is antisymmetric with respect to interchange of the two mesons) we can have mesons with isospin T = 1.

†One can imagine a situation where the only region which is important in the dispersion integral is the region $t < 16m^2$. This case is assumed to be realized when one proposes the hypothesis of a strong resonance in the meson-meson scattering in p-states.

We note that in the special case of scattering of a $\pi^{\pm}$ meson by a proton, the corresponding amplitudes $U_{\pi^{\pm}}$ are expressed according to (5.1) as follows:

$$U_{\pi\pm} = U^{+} \mp U^{-},\ U^{\pm} = \frac{1}{2}(U_{\pi^-} \pm U_{\pi^+}). \tag{5.5}$$

In order to establish the form of the dispersion relations which these amplitudes satisfy, we must know their following characteristics. First, the location of the poles and the values of the residues at the poles, which are determined by the masses of the one-particle intermediate states and the amplitudes of the corresponding triodes. Second, the location of the nearest branch points, which are determined by the masses of the two-particle states. Third, the nature of the behavior of the amplitudes at infinity.

The single-particle intermediate state (nucleon) occurs in channels s and u. Using the results of Sec. 3, item 4 and Sec. 2, item 6, we get for the pole part of the amplitude

$$U_{\text{pol}} = \bar{u}_2 \chi^{*}_{2\beta} \frac{\hat{k}}{2} g^2 \left(\frac{\tau_\beta \tau_\alpha}{M^2 - s} - \frac{\tau_\alpha \tau_\beta}{M^2 - u} \right) \chi_{1\beta} u_1, \tag{5.6}$$

where M is the nucleon mass, g is a constant [$g = g(M^2)$] in (2.41).* Comparing (5.6) with (5.1) and (5.2) gives for the pole part of the matrix $U^{\pm}$

$$U^{\pm}_{\text{pol}} = g^2 \frac{\hat{k}}{2} \left(\frac{1}{M^2 - s} \mp \frac{1}{M^2 - u} \right), \tag{5.7}$$

i.e., for the pole part of the amplitudes

$$\left.\begin{aligned} a^{\pm}_{\text{pol}} &= 0, \\ b^{\pm}_{\text{pol}} &= g^2 \left(\frac{1}{M^2 - s} \mp \frac{1}{M^2 - u} \right). \end{aligned}\right\} \tag{5.8}$$

Two-particle states represent a nucleon and pion (mass M + m) in the channels s and u and two pions (mass 2m) in the channel t. Thus for the amplitude considered as a function of s for given t, the start of the branch line is at the points $s = (M + m)^2$ and $u = (M + m)^2$.

Regarding the behavior of the amplitude at infinity, on which the form of the dispersion relations depends in the sense of the required number of subtraction terms, we have no information except that which may be obtained from comparison with experiment.

The dispersion relations without subtractions have the form†

$$\left.\begin{aligned} a^{\pm}(s, u, t) &= \frac{1}{\pi} \int_{(M+m)^2}^{\infty} \frac{a_1^{\pm}(x, t)}{x - s}\, dx + \frac{1}{\pi} \int_{(M+m)^2}^{\infty} \frac{a_2^{\pm}(x, t)}{x - u}\, dx, \\ b^{\pm}(s, u, t) &= g^2 \left(\frac{1}{M^2 - s} \mp \frac{1}{M^2 - u} \right) + \frac{1}{\pi} \int_{(M+m)^2}^{\infty} \frac{b_1^{\pm}(x, t)}{x - s}\, dx \\ &\quad + \frac{1}{\pi} \int_{(M+m)^2}^{\infty} \frac{b_2^{\pm}(x, t)}{x - u}\, dx. \end{aligned}\right\} \tag{5.9}$$

*The role of g in the nucleon-meson triode is the same as that of the charge e in the electron-photon triode. Therefore g is called the nucleon-meson coupling constant.

†In relation (5.9) there may also occur terms c(t) depending only on t.

In order to obtain dispersion relations with one subtraction, one should obviously, on the basis of (5.9), form the difference

$$a^{\pm}(s, u, t) - a^{\pm}(s_0, u_0, t)$$

and

$$b^{\pm}(s, u, t) - a^{\pm}(s_0, u_0, t),$$

where s_0 and u_0 are certain fixed values of the variables for given t.

The absorptive parts $a^{\pm}_{1,2}$ and $b^{\pm}_{1,2}$ in the physical regions are equal to the imaginary parts of the corresponding amplitudes $a^{\pm}(s,u,t)$ and $b^{\pm}(s,u,t)$:

$$\left.\begin{aligned} a_1^{\pm}(x, t) &= \operatorname{Im} a^{\pm}(x, h - x - t, t), \\ a_2^{\pm}(x, t) &= \operatorname{Im} a^{\pm}(h - x - t, x, t) \\ &\quad (h = 2M^2 + 2m^2), \end{aligned}\right\} \tag{5.10}$$

and similarly for $b_{1,2}$. From this, as a consequence of the symmetry properties (5.4),

$$\left.\begin{aligned} a_1^{\pm}(x, t) &= \pm a_2^{\pm}(x, t), \\ b_1^{\pm}(x, t) &= \mp b_2^{\pm}(x, t). \end{aligned}\right\} \tag{5.11}$$

2. Dispersion relations for forward scattering. Coupling constants of strong interactions. The simplest thing from the point of view of comparison with experimental data is to consider the dispersion relations for the pion-nucleon scattering amplitude at zero angle,[3] i.e., for t = 0. Then the whole integration region is in the physical region of the variables. Furthermore the imaginary part of the amplitude is expressed, according to the optical theorem, in terms of the total cross section, and is thus measured independently.

Suppose that ω denotes the meson energy (both initial and final, which coincide for the case of scattering at zero angle) in the rest system of the nuclei. Then

$$\left.\begin{aligned} s &= m^2 + M^2 + 2M\omega, \\ u &= m^2 + M^2 - 2M\omega. \end{aligned}\right\} \tag{5.12}$$

Correspondingly we shall use the notation

$$\left.\begin{aligned} a^{\pm}(s, u, 0) &\equiv a^{\pm}(\omega), \\ b^{\pm}(s, u, 0) &\equiv b^{\pm}(\omega) \end{aligned}\right\} \tag{5.13}$$

and according to (5.10)

$$\left.\begin{aligned} a_1^{\pm}(x, 0) &= \operatorname{Im} a^{\pm}(\nu), \\ b_1^{\pm}(x, 0) &= \operatorname{Im} b^{\pm}(\nu), \end{aligned}\right\} \tag{5.14}$$

where

$$x = m^2 + M^2 + 2M\nu. \tag{5.15}$$

The symmetry property (5.11) takes the form

$$\left.\begin{aligned} a_2^{\pm}(x, 0) &= \pm \operatorname{Im} a^{\pm}(\nu), \\ b_2^{\pm}(x, 0) &= \mp \operatorname{Im} b^{\pm}(\nu). \end{aligned}\right\} \tag{5.16}$$

We note that the arguments ν are assumed to have an infinitesimal positive imaginary part ($\nu \to \nu + i0$).

Using all of these notations, we can on the basis of (5.9)–(5.11) construct the dispersion relations directly for the amplitudes

$$U^{\pm}(s, u, 0) \equiv U^{\pm}(\omega).$$

We note that for a nucleon at rest

$$\bar{u}_2 \frac{\hat{k}}{2} u_1 = \omega \bar{u}_2 u_1$$

so that

$$U^{\pm}(\omega) = a^{\pm}(\omega) + \omega b^{\pm}(\omega). \tag{5.17}$$

We write these dispersion relations in the following form:*

$$U^{(-)}(\omega) = \frac{2f^2}{\omega^2 - \omega_0^2} + \frac{2\omega}{\pi} \int_m^\infty \frac{\operatorname{Im} U^-(\nu)}{\nu^2 - \omega^2} d\nu, \tag{5.18}$$

$$U^{(-)}(\omega) = \frac{\omega}{m} U^-(m) - \frac{2f^2 g^2 \omega}{(m^2 - \omega_0^2)(\omega^2 - \omega_0^2)} + \frac{2q^2\omega}{\pi} \int_m^\infty \frac{\operatorname{Im} U^-(\nu)\, d\nu}{(\nu^2 - m^2)(\nu^2 - \omega^2)}, \tag{5.19}$$

$$U^{(+)}(\omega) = U^+(m) + \frac{2f^2 \omega_0 q^2}{(m^2 - \omega_0^2)(\omega^2 - \omega_0^2)} + \frac{2q^2}{\pi} \int_m^\infty \frac{\nu \operatorname{Im} U^+(\nu)\, d\nu}{(\nu^2 - m^2)(\nu^2 - \omega^2)}, \tag{5.20}$$

where $\omega_0 = m^2/2M$ (the energy of the meson at the pole), $f = g(m/2M)$, $q = \sqrt{\omega^2 - m^2}$ (momentum of the meson).

Formula (5.18) is a dispersion relation without subtractions. It obviously has a meaning if $\operatorname{Im} U^-(\nu)$ increases more slowly than $\sim \nu$ for $\nu \to \infty$.† Formula (5.19) is obtained from (5.18) by subtracting the quantity $(\omega/m)U(m)$. Formula (5.20) is a dispersion relation with one subtraction. For its existence we require that $\operatorname{Im} U^+(\nu)$ increase more slowly than ν^2 for $\nu \to \infty$.

To apply the optical theorem, we use expression (5.5) relating $U^{\pm}$ with the amplitudes for scattering of $\pi^{\pm}$ mesons. Applying formula (3.7) to each of them (in this formula $I = qM$, $\lambda_a = 2M$), we get

$$\operatorname{Im} U^{\pm}(\omega) = \frac{1}{2} q (\sigma_{\pi^-} \pm \sigma_{\pi^+}), \tag{5.21}$$

where $\sigma_{\pi^{\pm}}$ is the total cross section for interaction of a $\pi^{\pm}$ meson with a proton.

If in accordance with the present data we assume that as $\omega \to \infty$

$$\sigma_{\pi^{\pm}}(\omega) \to \sigma_{\pi^{\pm}}^{\infty}, \tag{5.22}$$

where $\sigma_{\pi}^{\infty\pm}$ are constants, the integrals appearing in (5.19) and (5.20) converge. These relations have been used for an experimental test of the dispersion relations. An important result obtained in this way was the determination of the constant f. They gave

$$f^2 = 0{,}082 \pm 0.015,$$

i.e.,

$$g^2 \approx 15. \tag{5.23}$$

This number is a measure of the "strength" of meson-nucleon interaction. (For electromagnetic interactions the corresponding quantity is the constant $e^2/4\pi = 1/137$.)

3. Pomeranchuk's theorem. We have assumed that the behavior of the imaginary part of the amplitude $U^{\pm}$ for large ω is given according to (5.21) and (5.22) by the law

$$U^{\pm}(\omega) = C^{\pm}\omega, \tag{5.24}$$

where

$$C^{\pm} = \sigma_{\pi^-}^{\infty} \pm \sigma_{\pi^+}^{\infty}. \tag{5.25}$$

The dispersion relations permit us to find from these the behavior of the real part of the amplitude at large ω. From (5.19) and (5.20) we have for large ω

$$\operatorname{Re} U^+(\omega) < \underset{\omega\to\infty}{\text{const}} \cdot \omega, \quad \operatorname{Re} U^-(\omega) \underset{\omega\to\infty}{\longrightarrow} \omega\left(\text{const} + C^- \ln \frac{\omega}{n}\right). \tag{5.26}$$

We see that if $C^- \neq 0$, the real part of the amplitude increases faster than the imaginary part. But it is difficult to make such a behavior agree with the constancy of the cross section. For example, in the natural model which leads to a constant cross section and a finite interaction radius, the imaginary part of the amplitude for elastic scattering is greater than the real part.* To eliminate this contradiction we must assume that $C^- = 0$, i.e.,

$$\sigma_{\pi^+}^{\infty} = \sigma_{\pi^-}^{\infty}. \tag{5.27}$$

This assertion is the content of Pomeranchuk's theorem.[15] We note that because of (5.27), the simplest dispersion relation (5.18) has a meaning.

The Pomeranchuk theorem can be formulated as a general relation between the asymptotic values of the total cross sections for collision of a particle with a certain particle and with its antiparticle, for the case where these cross sections are constant and the real part of the amplitude increases no faster than the imaginary part. In fact, under these conditions the scattering amplitude for the particle at zero angle for large energies ω has the form

$$U = C\omega. \tag{5.28}$$

*If in (5.9) we add a term c(t), then in (5.18) there appear the additional terms $c_1 + c_2\omega$.

†We assume a power law increase. For slower increase (for example, $\sim \nu/\ln\nu$) equation (5.18) is not applicable.

*A finite radius R for the interaction means that in expanding the amplitude in partial waves of the type

$$U(\omega) = \frac{1}{\omega} \sum_l (2l+1) a_l$$

the important effects come from $l < l_0 \sim \omega R$. Since $|a_l| < 1$, it then follows that $|U(\omega)| \leq R^2\omega$, i.e., ReU always increases no faster than $\sim\omega$. Since

$$\operatorname{Im} a_l = \frac{1}{2}(1 - \cos 2\delta_l e^{-2\eta_l}), \quad \operatorname{Re} a_l = \frac{1}{2} e^{-2\eta_l} \sin 2\delta_l$$

(the factor $e^{-2\eta_l}$ is related to the presence of inelastic processes), i.e., $\operatorname{Im} a_l$ contains a constant term, and it is natural to expect that $\operatorname{Im} a_l \gg \operatorname{Re} a_l$ (as, for example, in the case of diffraction.)

Expression (5.28) is the limiting value of the function of a complex variable U(z) for $z = \omega + i0$ and $\omega \to \infty$. But then, over the whole upper half plane (since the singularities of U lie on the real axis), the amplitude must have the form

$$U(z) = Cz \quad (|z| \to \infty)$$

and consequently,

$$U(-\omega + i0) = -C\omega.$$

Using crossing symmetry, according to which

$$U(-z) = \overline{U}(z),$$

where $\overline{U}$ is the amplitude for scattering of the antiparticle. The argument of this amplitude in the physical region is $z = \omega + i0$. Thus

$$\overline{U}(\omega + i0) = U(-\omega - i0) = U^*(-\omega + i0) = -C^*\omega,$$

i.e.,

$$\operatorname{Im}\overline{U} = \operatorname{Im} U. \tag{5.29}$$

4. Peripheral collisions. An important consequence of the principle of universality is the equivalence of the analytic properties of the amplitude for different variables, which play essentially different roles kinematically in a given channel of the reaction. In the preceding we have considered dispersion relations with respect to the variable s in channel s. Now let us look at dispersion relations in the variable t in channel s.

The amplitude U(t, s) as a function of t for given s we write in the form [cf. (3.11) and (3.16)]

$$U(t, s) = \frac{R}{\mu^2 - t} + \frac{1}{\pi}\int_{M_1^2}^{\infty} \frac{A_1(s, t')\,dt'}{t' - t} + \frac{1}{\pi}\int_{M_2^2}^{\infty} \frac{A_2(s, u')\,du'}{u' - u}, \tag{5.30}$$

where M_1^2 and M_2^2 are the masses of two-particle intermediate states of the corresponding channels. For simplicity we have here assumed that the amplitude has no pole in the u variable and the particle has no spin. Furthermore for simplicity* we shall assume that the s channel corresponds to elastic scattering and that the masses of the particles are the same, so that

$$t = -2p^2(1 - z), \quad u = -2p^2(1 + z), \tag{5.31}$$

where z is the cosine of the scattering angle, p the momentum in the center-of-mass system. If we use (5.31) and introduce the notation $U(t, s) \equiv U(z, p)$ and similarly for A_1 and A_2, we can represent (5.30) as follows:

$$U(z, p) = \frac{R}{2p^2(z - z_0)} + \frac{1}{\pi}\int_{1+\frac{M_1^2}{2p^2}}^{\infty} \frac{A_1(z', p)\,dz'}{z' - z} + \frac{1}{\pi}\int_{1+\frac{M_2^2}{2p^2}}^{\infty} \frac{A_2(z', p)\,dz'}{z' - z}, \tag{5.32}$$

where $z_0 = 1 + \dfrac{M^2}{2p^2}$.

Let us find the partial amplitude corresponding to scattering in the state with angular momentum l:

$$a_l(p) = \frac{1}{2}\int_{-1}^{1} U(z, p) P_l(z)\,dz. \tag{5.33}$$

The amplitude (5.33) differs from the amplitude a_l in (4.36) by a normalization factor:

$$a_l = 8\pi\frac{W}{p} a_l. \tag{5.34}$$

Substituting (5.32) in (5.33), we find*

$$a_l(p) = \frac{R}{2p^2} Q_l(z_0) + \frac{1}{\pi}\int_{1+\frac{M_1^2}{2p^2}}^{\infty} Q_l(z') A_1(z', p)\,dz' + \frac{(-1)^l}{\pi}\int_{1+\frac{M_2^2}{2p^2}}^{\infty} Q_l(z') A_2(z', p)\,dz', \tag{5.35}$$

where Q_l is the Legendre function of the second kind,

$$Q_l(z') = \frac{1}{2}\int_{-1}^{1} \frac{P_l(z)}{z' - z}\,dz.$$

$Q_l(z')$ falls off rapidly for sufficiently large l as z' increases ($z' > 1$). Thus the main contribution to α_l will come from the first term in (5.35), associated with the pole part in the amplitude.[16] This result has a simple physical interpretation. Since the absorptive part A(t) determines the interaction with radius $1/\sqrt{t}$, for large angular momenta, i.e., large impact parameters, the important role is played by the largest radius, i.e., the smallest mass for the intermediate state (a one-particle state, if there are such).

Suppose, for example, that $p/\mu \ll 1$ (nonrelativistic case). Then

$$Q_l\left(1 + \frac{\mu^2}{2p^2}\right) \sim \left(\frac{p}{\mu}\right)^{2l+2},$$

and according to (5.35) and (5.34) we have

$$a_l \sim p^{2l+1},$$

a well-known result of quantum mechanics. For $\mu/p \ll 1$, but $l\mu/p \gg 1$,

$$Q_l\left(1 + \frac{\mu^2}{2p^2}\right) \sim e^{-l\frac{\mu}{p}}.$$

The partial amplitudes fall off exponentially with increasing l.

Corrections to the first term (or the main contribution when there are no one-particle intermediate states) are contained in the integral term in (5.35). Because of the drop-off of $Q_l(z')$, we may choose such conditions that the main part in the integral is due to the

*All of these assumptions are not essential. In the case of particles with spin we should understand by u one of the invariant amplitudes.

*Of course, we are assuming here that the integrals (5.32) exist, so that in the double integral obtained one can interchange the order of integration. The results are not changed if we use dispersion relations with subtractions.

gion in the neighborhood of the lower limit. Then to find α_l it is sufficient to know the absorptive parts of the amplitudes only in the region where they are determined solely by two-particle intermediate states. This poses the problem of calculating partial amplitudes for large l.[17]

The main problem of this paper was the presentation of the general theory of the scattering matrix, not based on the apparatus of field theory. We have seen that this scheme is equivalent to field theory in the region of applicability of perturbation theory. The successes of the present theory outside the realm of perturbation theory are as yet only episodic in character. In principle, this theory contains a small parameter, not connected with the smallness of interaction, which allows us to hope that it can be extended. This is the ratio of the masses of intermediate states to the smallest masses present. On this basis, numerous attempts have been made to construct a closed theory in the region of low energies (cf. [18,19]) using only two-particle intermediate states. So far these attempts have not met with success. It appears that the amplitudes at low energy depend essentially on the properties of the processes at high energy. The treatment of many-particle states meets with difficulties associated with the complicated analytic properties of polyodes. At present there exists no general algorithm similar to the Feynman rules for constructing the part of the amplitude which is related to a given number of intermediate particles.

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