SHOCK WAVES IN MAGNETOHYDRODYNAMICS

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Usp. Fiz. Nauk 72, 33-52 (September, 1960)

1. INTRODUCTION

APPLICATIONS of magnetohydrodynamics have become most widespread in recent years. These applications now cover many problems in the theory of liquid-metal coolants for nuclear reactors,[1] the theory of magnetohydrodynamic measuring instruments,[2] the physics of the sun,[3] geophysics and astrophysics,[4] the theory of gas discharge and plasma,[5-7] the theory of controllable thermonuclear reactions,[8] and magnetoaerodynamics.[9,10]

The equations of magnetohydrodynamics are nonlinear, so that many specific effects appear, particularly the formation of shock waves. Shock waves play an essential role in the compression and heating of a plasma.[11-13] They are one of the mechanisms of the production of interstellar magnetic fields.[14] Without shock waves it is impossible to analyze the supersonic flow about a body in a magnetic field.[15,16] Magnetohydrodynamic shock waves are essential also in the theory of pulsed discharges in a plasma.[17]

The present article is a review of many theoretical papers devoted to magnetohydrodynamic shock waves, the foundation for which was laid by the work by Hoffman and Teller.[18]

2. SIMPLE WAVES

An important class of nonlinear solutions of the equations of magnetohydrodynamics are simple waves (defined as waves in which all the magnetohydrodynamic quantities such as density ρ, pressure p, entropy s, fluid velocity **v**, and magnetic field **H** are functions of one of these quantities, for example ρ, which in turn depends on the coordinate x and on the time t). We confine ourselves to the case of plane one-dimensional simple waves. The reason why simple waves play a special role in magnetohydrodynamics is that in the absence of discontinuities they are the only waves that can border on the region of continuous flow.[19,20*]

A particular case of a simple wave is the self-similar wave, i.e., a wave in which the magnetohydrodynamic quantities depend on the ratio x/t. Self-similar waves are always produced when the initial conditions do not contain parameters with the dimension of length.

In magnetohydrodynamics we deal with three types of simple waves:[19,20]

1) The Alfven wave

$$\mathbf{v} = -\frac{\varepsilon \mathbf{H}}{\sqrt{4\pi\varrho}}, \qquad v_t \equiv \sqrt{v_y^2 + v_z^2} = \text{const},$$
$$H_t \equiv \sqrt{H_y^2 + H_z^2} = \text{const}, \quad \varrho = \text{const}, \quad s = \text{const}, \quad H_x = \text{const}. \tag{2.1}$$

$\epsilon = +1$ for waves propagating in the direction of positive x, and $\epsilon = -1$ for ways propagating in the opposite direction.

2) The magnetoacoustic wave

$$\frac{dv_x}{d\varrho} = \varepsilon\frac{U_\pm}{\varrho}, \quad \frac{d\mathbf{v}_t}{d\varrho} = -\varepsilon\frac{H_x\mathbf{H}_t U_\pm}{4\pi\varrho^2(U_\pm^2 - U_x^2)},$$
$$\frac{d\mathbf{H}_t}{d\varrho} = \frac{U_\pm^2\mathbf{H}_t}{\varrho(U_\pm^2 - U_x^2)}, \quad \frac{dp}{d\varrho} = c^2, \quad s = \text{const}, \quad H_x = \text{const}, \tag{2.2}$$

where U_x is the Alfven velocity, defined by the relation

$$\mathbf{U} = \frac{\mathbf{H}}{\sqrt{4\pi\varrho}},$$

$\mathbf{v}_t$ and $\mathbf{H}_t$ are the transverse components of the velocity of the magnetic field, c is the velocity of sound, and

$$U_\pm = \sqrt{\frac{[U^2 + c^2 \pm \sqrt{(U^2+c^2)^2 - 4c^2U_x^2}]}{2}};$$

the subscripts plus or minus in $U_\pm$ correspond to the fast or slow magnetoacoustic wave.

3) The entropy wave

$$d\varrho = (\partial\varrho/\partial s)_p\, ds, \quad p = \text{const}, \quad \mathbf{v} = \text{const}, \quad \mathbf{H} = \text{const}. \tag{2.3}$$

In the Alfven simple wave the quantities p, ρ, v_x, **v**, and **H** do not change, and the vectors **v** and **H** are turned about the x axis through an equal angle. The phase velocity of propagation of an Alfven wave is $v_x + \epsilon U_x$. Since the latter quantity does not change, the Alfven simple wave propagates without changing its shape. It follows from this, in particular, that Alfven waves cannot be self-similar.

The phase velocity of propagation of a magnetoacoustic wave is $v_x + \epsilon U_\pm$. If, as usual, the adiabatic compressibility decreases with increasing pressure

$$\left(\frac{\partial^2}{\partial p^2}\right)_s\left(\frac{1}{\varrho}\right) > 0,$$

then it follows from the differential equations (2.2) that

$$\frac{d}{d\varrho}(v_x + \varepsilon U_\pm) > 0. \tag{2.4}$$

*Mugibayashi's claim[21] of finding a solution which is not a simple wave and which borders on the region of continuous flow is in error. In order for the motion of the medium to be described by Mugibayashi's solution it is necessary to apply at the boundary a varying (and not a constant) external pressure, which varies in accordance with a definite law; this was noted, for example, in the paper by A. G. Kulikovskiĭ.[22]

The inequality (2.4) denotes that the profile of a moving magnetoacoustic wave becomes distorted. The slope of the wave decreases in the rarefaction regions ($d\rho/dt < 0$) and increases in the compression regions ($d\rho/dt > 0$); this leads in final analysis the formation of discontinuity (shock waves) in the compression regions.[26-29*] The solutions of the equations for simple magnetoacoustic waves are quite complicated.† We shall report only a few qualitative deductions that can be inferred directly from the differential equations. We note first that the pressure changes in the same direction as the density; in a fast magnetoacoustic wave the transverse magnetic field H_t changes in the same direction as the density, and in a slow one it changes in the opposite direction. Simple magnetoacoustic waves are plane: if $v_z = 0$ and $H_z = 0$ at the initial instant, these relations will hold for all time. If a self-similar wave propagates to the right ($\epsilon = +1$) and if the quantities H_x and H_y are of the same sign, then the longitudinal velocity v_x decreases, and the transverse velocity v_y increases in a self-similar wave and decreases in a slow one.

3. SHOCK WAVES. ZEMPLEN'S THEOREM ‡

We have already seen that these discontinuities are produced in the compression region of a simple wave as a result of distortion in the wave profile. On the discontinuity surface, the mass, momentum, and energy are conserved and the transverse electric field and the longitudinal magnetic field are continuous.[18,30,31] We confine ourselves to an analysis of discontinuities of constant amplitude.

A classification of all possible types of magnetohydrodynamic discontinuities was given by S. I. Syrovatskiĭ.[32,33]

We know[34,35] that if the conditions

$$\left(\frac{\partial^2}{\partial p^2}\right)_s \left(\frac{1}{\varrho}\right) > 0, \quad \left(\frac{\partial s}{\partial p}\right)_\varrho > 0 \tag{3.1}$$

are satisfied in ordinary hydrodynamics, then Zemplen's theorem takes place, by which the pressure and the density increase in the shock wave: $p_2 > p_1$ and $\rho_2 > \rho_1$; in other words, the shock waves are always compression waves.

L. D. Landau and E. M. Lifshitz[30] have shown that if conditions (3.1) are satisfied in magnetohydrodynamics, shock waves of small amplitude are compression waves (for arbitrary direction of the magnetic field). The inverse of the Zemplen theorem, stating that the entropy of a shock wave in an ideal gas increases, was proved by Hoffman and Teller.[18]

Zemplen's theorem was proved by Hoffman and Teller for a perpendicular shock of small amplitude, but it holds for any intensity and for any direction of the magnetic field.[36-38]

4. EVOLUTIONARY CONDITIONS OF THE DISCONTINUITIES

Knowledge of the boundary conditions on the discontinuity is not sufficient to determine uniquely the discontinuous solution. This difficulty is encountered also in ordinary hydrodynamics. Thus, for example, when a piston moves out of a tube, two types of waves are formally possible: 1) rarefaction self-similar wave and 2) rarefaction shock wave.[39] The second solution is discarded in ordinary hydrodynamics, since it contradicts Zemplen's theorem.

In magnetohydrodynamics, as noted earlier, rarefaction shock waves are also impossible. However, there are too many compression shock waves in magnetohydrodynamics and the problem of the motion of the medium under specified initial and boundary conditions has an infinite set of solutions.

Thus, for example, if an ideally conducting piston is moved into a magnetohydrodynamic medium at rest, with the magnetic field H_x normal to the piston, then if the following inequalities are satisfied

$$\frac{3(U_{0x}^2-c_0^2)}{4U_{0x}} < u < \frac{3(U_{0x}^2-c_0^2)}{\sqrt{4U_{0x}^2-3c_0^2}}, \quad U_{0x} > c_0$$

[$U_{0x} = H_{0x}/\sqrt{4\pi\rho_0}$, c_0 is the velocity of sound, the subscript "zero" pertains to the unperturbed medium, u is the velocity of the piston, and $\gamma = 5/3$], two types of solutions are possible:

1) the same compression shock wave as in the absence of a magnetic field,

2) two magnetohydrodynamic shock waves moving at the same velocity

$$D = \frac{2u}{3} + \sqrt{\frac{4u^2}{9} + c_0^2},$$

when the transverse magnetic field H_{1t} between these waves, is given by

$$H_{1t} = \sqrt{\frac{8\pi\varrho_0 (D^2 - U_{0x}^2)(4U_{0x}^2 - D^2 - 3c_0^2)}{3U_{0x}^2}}.$$

Both shock waves are compression waves. The density of the medium between the waves is

$$\varrho_1 = \frac{D^2\varrho_0}{U_{0x}^2} > \varrho_0,$$

the density of the medium behind the two waves is

$$\varrho_2 = \frac{U_{0x}^2\varrho_1}{D(D-u)} > \varrho_1;$$

the entropy increases on both shock waves.

We see thus that the condition that the entropy must increase, by which the "excessive" discontinuities could be excluded from ordinary hydrodynamics, is too weak in magnetohydrodynamics.

*Analogous results were obtained by S. A. Kaplan and K. P. Stanyukovich,[23] Segrè,[24] and Taniuti[25] for the case when the magnetic field is perpendicular to the direction of the wave propagation.

†It has been shown by Friedrichs (see reference 29) that the solution of the equations of simple waves can be reduced to quadratures.

‡Zemplen, Gyözö, "Besondere Ausführungen über unstetige Bewegungen in Flüssigkeiten" – Enz. d. Math. Wiss., Leipzig 1901-1908, pp. 281-323. – Tr.

Actually, however, not all shock waves on which the boundary conditions are satisfied and the entropy increases are feasible. It is necessary, in addition, that the solution be continuously dependent on the initial and boundary conditions, i.e., that an infinitesimal perturbation of the magnetohydrodynamic quantities produce an infinitesimal change in the solution. Following I. M. Gel'fand,[33] we shall call such discontinuities evolutionary.* In non-evolutionary discontinuities, an infinitesimal perturbation produces a finite change in the solution, namely the decomposition of the initial discontinuity into several discontinuities of finite magnitude.[40] Non-evolutionary discontinuities are thus unstable relative to splitting, and can therefore not exist. The evolutionary conditions for magnetohydrodynamics were derived in reference 40.

In order to determine whether a magnetohydrodynamic shock wave is evolutionary, it is necessary to add to the constant equilibrium values of the magnetohydrodynamic quantities, (p, ρ, v_x, v_y, v_z, H_y, and H_z) infinitesimally small perturbations δp, $\delta\rho$, δv_x, δv_y, δv_z, δH_y, and δH_z, which depend on the coordinates and on the time as

$$\exp i\,(kx - \omega t).$$

These perturbations can be represented as a superposition of waves of infinitesimal amplitude: magnetoacoustic, Alfven, and entropy waves; only waves that diverge from the discontinuity surface are considered. After linearizing the boundary conditions we obtain seven linear homogeneous algebraic equations in the amplitudes of the different waves on the two sides of the discontinuity surface. Account must be taken here of the fact that the shock-wave velocity D also acquires an infinitesimal increment δD. After eliminating δD, six independent equations (boundary conditions) remain. If the number of the diverging waves on the two sides of the discontinuity surface is also six, we obtain a system of six linear homogeneous algebraic equations with six unknowns (the amplitudes of the waves of infinitesimal intensity). The condition for the existence of a non-trivial solution (in which the wave amplitudes are different from zero) is that the determinant of the system vanish. This equality relates ω with k. If a real value of ω exists for any value of k the discontinuity is stable in the ordinary sense; if certain possible values of k correspond to values of ω with positive imaginary part, the discontinuity is unstable — the initial perturbation will build up exponentially with time. In either case, the discontinuity is evolutionary. Even if the perturbations grow exponentially, they may still remain small after a finite time interval.

In order for the discontinuity to be non-evolutionary, it is necessary that the perturbations, which are small at the initial instant of time $t = 0$, be no longer small for arbitrary $t > 0$. The linearization of the equations is no longer valid, and this in turn makes the number of the equations for the determination amplitudes of the infinitesimal intensity wave no longer equal to the number of unknowns, i.e., not equal to the number of waves that diverge on both sides of the discontinuity surface. Since the number of independent boundary conditions in magnetohydrodynamics is six, the evolutionarity condition is that the total number of diverging waves be six.

*The idea of evolutionarity was first advanced in connection with a study of discontinuities in ordinary hydrodynamics, see reference 34, p. 405, and reference 35, p. 215.

In magnetohydrodynamics there are fourteen different phase velocities of the propagation of infinitesimal amplitude waves:

$$v_{1x}+U_{1x},\ v_{1x}-U_{1x},\ v_{1x}+U_{1+},\ v_{1x}-U_{1+},\ v_{1x}+U_{1-},$$
$$v_{1x}-U_{1-},\ v_{1x},\ v_{2x}+U_{2x},\ v_{2x}-U_{2x},\ v_{2x}+U_{2+},\ v_{2x}-U_{2+},$$
$$v_{2x}+U_{2-},\ v_{2x}-U_{2-},\ v_{2x}$$

[the subscripts "1" and "2" pertain to the region ahead ($x < 0$), and behind the shock wave ($x > 0$); the coordinate system is chosen such that the discontinuity is at rest in it and is located in the plane $x = 0$; the x axis is so directed that the projection of the velocity of the medium on the x axis is positive]. The divergent waves have negative phase velocities in the region ahead of the shock wave and positive phase velocities behind the shock wave.

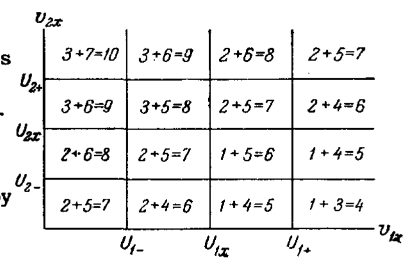


FIG. 1. Number of waves diverging from the discontinuity surface. First term – number of Alfven waves, second term – number of magnetoacoustic and entropy waves.

Four of the aforementioned fourteen phase velocities correspond to converging waves:

$$v_{1x}+U_{1x},\ v_{1x}+U_{1+},\ v_{1x}+U_{1-},\ v_{1x},$$

and four to diverging waves:

$$v_{2x}+U_{2x},\ v_{2x}+U_{2+},\ v_{2x}+U_{2-},\ v_{2x}$$

(all these phase velocities are definitely positive). The remaining waves will converge or diverge, depending on the relations between v_x and U_x, U_+ or U_-. Figure 1 shows the total number of diverging waves at the different values of v_{1x} and v_{2x}. Evolutionary waves correspond to those regions in the (v_{1x}, v_{2x}) plane where the number of diverging waves is six. As already indicated, a shock wave is evolutionary if the small-perturbation problem has a unique solution. For this purpose it is necessary that the number of equations (number of boundary equations minus one) be equal to the number of unknowns* (number of amplitudes of the diverging waves).

*The evolutionarity conditions so formulated were obtained by Lax[19] and by K. I. Babenko and I. M. Gel'fand.[41]

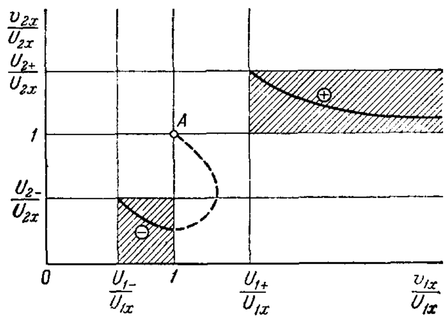


FIG. 2. Evolutionary regions of shock waves (shaded). The "+" sign denotes a fast wave and the "−" a slow wave. The evolutionary sections of the shock adiabat are designated by a solid heavy line, while the non-evolutionary section is shown by a dashed line. The letter A denotes the Alfven discontinuity.

However, mere equality of the number of equations to the number of unknowns is insufficient for the existence and for the uniqueness of the solution. The equations for the amplitudes of the diverging waves and the boundary conditions may break up into several isolated groups. In this case, the evolutionarity conditions (that the number of diverging waves equal the number of independent boundary conditions) must be satisfied not only for the totality of the variables, but also for each group separately. Such a division of the equations and boundary conditions into two isolated groups occurs in magnetohydrodynamics for waves propagating perpendicularly to the discontinuity surface.* In fact, in low-amplitude Alfven waves the quantities δv_z and δH_z differ from zero (the xy plane is oriented such that $H_z \equiv 0$), while in magnetoacoustic and entropy waves $\delta\rho$, δp, δv_x, δv_y, and δH_y differ from zero. The boundary conditions, linearized relative to small perturbations, also break up into similar two groups:

1) Alfven perturbations

$$\left\{\varrho v_x \delta v_z - \frac{H_x \delta H_z}{4\pi}\right\} = 0,$$
$$\{v_x \delta H_z - H_x \delta v_z\} = 0; \tag{4.1}$$

2) magnetoacoustic and entropy perturbations

$$\left.\begin{aligned} &\{\varrho(\delta v_x - \delta D) + v_x \delta\varrho\} = 0,\\ &\left\{\delta p + 2\varrho v_x(\delta v_x - \delta D) + v_x^2 \delta\varrho + \frac{H_y \delta H_y}{4\pi}\right\} = 0,\\ &\left\{\varrho v_x \delta v_y - \frac{H_x \delta H_y}{4\pi}\right\} = 0,\\ &\{H_x \delta v_y - H_y(\delta v_x - \delta D) - v_x \delta H_y\} = 0,\\ &\left\{\varrho v_x [v_x(\delta v_x - \delta D) + v_y \delta v_y + \delta w] + \frac{1}{4\pi}(v_x H_y - v_y H_x)\,\delta H_y\right\} = 0 \end{aligned}\right\} \tag{4.2}$$

(δD is the perturbation of the velocity of the shock wave, w is the heat function).

The boundary conditions (4.1) do not contain δD. They are therefore all independent and there should be two diverging Alfven waves. The boundary conditions (4.2) contain the perturbation δD of the shock-wave velocity; after this perturbation is eliminated, four independent boundary conditions remain. Consequently, the number of diverging magnetoacoustic and entropy waves should be four. It is seen from Fig. 4 that there exist two regions where the shock waves are evolutionary* (Fig. 2):

1) fast shock waves (marked "+" in Fig. 2), for which

$$U_{1+} < v_{1x}, \qquad U_{2x} < v_{2x} < U_{2+}, \tag{4.3}$$

2) slow shock waves (marked "−" in Fig. 2), for which

$$U_{1-} < v_{1x} < U_{1x}, \qquad v_{2x} < U_{2-}. \tag{4.4}$$

So far we have not considered the evolutionarity conditions with respect to perturbations that depend only on x and t. An account of perturbations of general form (which depend also on y and z) leads to the same evolutionarity conditions[44] (4.3) and (4.4).

It is necessary to emphasize the essential difference between non-evolutionarity and instability. Unstable states can occur when a magnetohydrodynamic medium moves under the influence of internal factors. They exist for a certain time, until the fluctuations reach a critical value, after which the unstable state is destroyed. Non-evolutionary discontinuities cannot arise by themselves. They can be produced only under the influence of external factors (i.e., collisions between gas masses) and can exist only for an instant as discontinuities in the initial conditions, after which they decompose immediately into several shock waves or self-similar waves. Such a decomposition of a non-evolutionary magnetohydrodynamic wave was considered in reference 45 (see Sec. 8).

It can be shown by the method developed in the present section that other magnetohydrodynamic discontinuities (contact, tangential, Alfven) are always evolutionary.

5. CONSEQUENCES OF THE EVOLUTIONARITY CONDITIONS

Conditions (4.3) and (4.4) lead to important consequences. First, if two shock waves of similar type (both fast or both slow) follow one another, then the rear wave will overtake the front wave. To prove this statement let us consider, for example, two slow waves. The velocity of the front wave relative to the medium contained between the two waves is v_{2x}, while the velocity of the rear wave is v_{1x}. As follows from inequalities (4.4), for waves of this kind $v_{1x} > U_{1-}$ and

*This circumstance was first noted by S. I. Syrovat-skiĭ.[42]

*References 43 and 40 contain an incorrect statement that there exists a third evolutionary region

$$U_{1x} < v_{1x} < U_{1+}, \quad U_{2-} < v_{2x} < U_{2x}.$$

This error is due to failure to take into account the aforementioned division of the boundary conditions into two isolated groups. (The same references use instead of the term "evolutionary" the less successful term "stable with respect to decomposition.")

$v_{2x} < U_{2-}$. Since the velocities U_{1-} and U_{2-} pertain to the same region in space, we have $U_{1-} = U_{2-}$ and consequently $v_{1x} > v_{2x}$.

In the case of waves of different types, it is easy to see that the Alfven discontinuity will overtake the slow shock wave, while a fast shock wave will overtake all other discontinuities. We can establish analogously that a shock wave overtakes a weak discontinuity if the latter is of the same type as the shock wave or of a slower type. A weak discontinuity overtakes a shock wave of the same type and shock waves of slower type.

Noting that a weak discontinuity is the boundary between the simple wave and the region of constant flow, we reach the conclusion that not more than three shock waves or three simple waves which do not overtake one another can move on each side: a fast wave (shock or simple magnetoacoustic) in front, followed by an Alfven wave (discontinuous or simple), and finally a slow wave (shock or simple).

Using the evolutionarity conditions and Zemplen's theorem, we can make definite conclusions concerning the change in the magnetic field in a fast and slow shock wave.[38] We use the relation

$$H_{2y} = \frac{H_{1y}\varrho_2 (v_{1x}^2 - U_{1x}^2)}{\varrho_1 v_{1x}^2 - \varrho_2 U_{1x}^2}, \tag{5.1}$$

which is a consequence of the boundary conditions. It follows from (5.1) that the transverse magnetic field H_y increases in fast shock waves and decreases in slow ones. In either case, the transverse magnetic field will not change direction.* Weak magnetic fields

$$\frac{H_x^2}{8\pi} < \frac{\varrho_1 v_{1x}^2}{2}$$

will become stronger upon passage of a shock wave, whereas strong magnetic fields

$$\frac{H_x^2}{8\pi} > \frac{\varrho_1 v_{1x}^2}{2}$$

will become weaker. This points to a certain equalizing effect of the shock waves. After passage of a large number of random shock waves, static equilibrium occurs when the magnetic and kinetic energies are equal:†

$$\frac{H_x^2}{8\pi} = \frac{\varrho_1 v_{1x}^2}{2}. \tag{5.2}$$

There are statements in the literature that weak magnetic fields

$$\frac{H_x^2}{8\pi} \ll p_1 \tag{5.3}$$

become stronger on passage of a shock wave [references 14, 52, and 53 (p. 253)]. Strictly speaking, two shock waves exist for any ratio of H_x and p_1, a fast one in which the magnetic field increases, and a slow one in which the magnetic field decreases. However, if inequality (5.3) is satisfied, the slow shock wave can have only an infinitesimally small intensity, as follows from the evolutionarity conditions (4.4),

$$0 < \frac{\Delta\varrho}{\varrho_1} < \frac{H_1^2}{8\pi p_1}. \tag{5.4}$$

If inequality (5.4) is violated, the slow shock wave is no longer evolutionary and decomposes. Thus, only fast shock waves, on which the magnetic field increases, can actually exist in a medium in which the magnetic pressure is considerably less than the hydrostatic pressure (5.3). It follows from the foregoing that magnetohydrodynamic waves are one of the mechanisms of production of interstellar magnetic fields.[14]

*The statements contained in references 46, 43, and 47 that the direction of transverse magnetic field can change in a shock wave are due to failure to take evolutionarity conditions into account.

†Relation (5.2) – the law of equipartition of energy – is characteristic of magnetohydrodynamics. It is derived from other considerations in references 48–51.

6. THE SHOCK ADIABAT

In magnetohydrodynamics the shock wave is characterized by the values of all the magnetohydrodynamic quantities ρ_1, p_1, v_1 and H_1 ahead of the shock wave and also by the values of one of these quantities, for example ρ_2, behind the shock wave. All the remaining quantities behind the shock wave, particularly the pressure p_2, are functions of ρ_2. The curve $p_2 = f(1/\rho_2)$ is called the shock adiabat.

If we choose a coordinate system in which the discontinuity is at rest and is located in the plane $x = 0$, while the projection of the magnetic field on the z axis vanishes and the velocity vector is parallel to the magnetic field, then the equation of the shock adiabat

$$p_2 = f\left(\frac{1}{\varrho_2}\right)$$

contains as parameters the quantities p_1, ρ_1, H_x, and H_{1y}:

$$p_2 = f\left(\frac{1}{\varrho_2};\ p_1,\ \varrho_1,\ H_x,\ H_{1y}\right). \tag{6.1}$$

Instead of the five dimensional parameters ρ_2, p_1, ρ_1, H_x, and H_{1y}, the shock wave can be characterized by three dimensionless parameters,[14] for example U_{1x}/c_1, U_{1y}/c_1, and v_{1z}/c_1, where **U** is the Alfven velocity and v_{1x} is the velocity of the shock wave relative to the fluid at rest.

To find the shock adiabat it is necessary to eliminate from the boundary conditions all the magnetohydrodynamic quantities that do not enter in (6.1).

The equation of the shock adiabat in magnetohydrodynamics has the following form:[30,31]

$$(\varepsilon_2 - \varepsilon_1) + \frac{p_2 + p_1}{2}\left(\frac{1}{\varrho_2} - \frac{1}{\varrho_1}\right) + \frac{1}{16\pi}\left(\frac{1}{\varrho_2} - \frac{1}{\varrho_1}\right)(H_{2y} - H_{1y})^2 = 0. \tag{6.2}$$

In this formula the internal ϵ_2 is expressed in terms of p_2 and ρ_2 by means of the equation of state, and H_{2y} must be replaced by its value given by (5.1).

Equation (6.2) is a third-degree algebraic equation in p_2 or ρ_2; consequently, there are three branches of the shock adiabat,[14,46,36,54] but only two of these are evolutionary.[55] In the particular case of a perpendicular shock wave ($H_x = 0$) only one evolutionary branch of the shock adiabat exists.

The variation of the magnetohydrodynamic quantities in a shock wave was first investigated by Helfer.[14] However, because of an unfortunate choice of parameters, the curves given by Helfer are exceedingly uninstructive. In addition, no distinction is made in this work between evolutionary and non-evolutionary shock waves.

A later paper by Lüst[46] describes an investigation of the changes on the shock wave of the following essential physical quantities: density, pressure, magnetic field, heat energy, and the angle between the magnetic field and the normal direction. The changes of these quantities are illustrated by a large number of curves obtained by means of an electronic computer. Unfortunately, the author does not take the evolutionarity conditions into account; the only physically realizeable portions of the curves given by Lust are those corresponding to evolutionary shock waves.

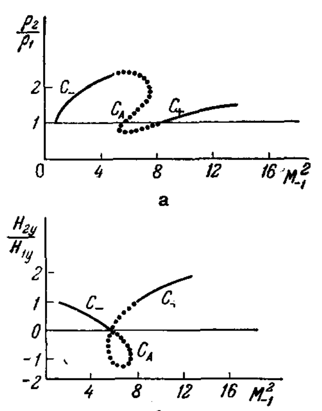


FIG. 3. Density (a) and magnetic field (b) on a shock wave as functions of the Mach number. The non-evolutionary portions are dotted: c_+ – fast shock wave, c_- – slow shock wave, c_A – non-existing non-evolutionary shock wave.

By way of an example, Figs. 3a and 3b show the dependence of the quantities ρ_2/ρ_1 and H_{2y}/H_{1y} on the Mach number, as obtained by Lust,

$$M_{-1}^2 \equiv \frac{v_{1x}^2}{U_{1-}^2};$$

$$\text{for } \beta_1 \equiv \frac{(3p_1/2)}{[(3p_1/2)+(H_1^2/8\pi)]} = 0{,}25.$$

$$\eta_1^2 \equiv \frac{H_x^2}{H_1^2} = 0.75.$$

The most complete investigation of the variation of the quantities in the magnetohydrodynamic shock wave is that of Bazer and Ericson.[47] But this investigation also fails to allow for the evolutionarity conditions. We shall supplement here the results of Bazer and Ericson by an investigation of the evolutionary parts of the shock adiabat.[55]

There exist fast and slow shock waves. Fast waves are always evolutionary. Slow shock waves of low intensity are also evolutionary. As the density jump $\Delta\rho \equiv \rho_2 - \rho_1$ increases, the transverse magnetic field H_{2y} decreases behind the slow shock wave and vanishes on the evolutionarity boundary. Moving further along the shock adiabat, we encounter the non-evolutionary portion of the slow shock wave, which is gradually transformed into an Alfven discontinuity, which in turn rotates the magnetic field by 180° (see Fig. 2).

In fast and slow shock waves the pressure and entropy jumps are monotonically increasing functions of the density jump (on the evolutionary portions). In a fast shock wave the maximum density jump $(\Delta\rho)_{max} = 2\rho_1/(\gamma - 1)$, where γ is the Poisson adiabatic index. The pressure and entropy jumps now become infinite. The dependence of the jump of the magnetic field $\Delta H_y \equiv H_{2y} - H_{1y}$ on the jump of the density of the fast wave can be of two types. In waves of the first type, which are realized if

$$\sin^2\theta_1 \geqslant \frac{(\gamma-1)(1-r_1)}{\gamma}$$

($r_1 = c_1^2/U_{1x}^2 \equiv 4\pi\gamma p_1/H_x^2$ and θ_1 is the angle between the direction of the magnetic field $\mathbf{H}_1$ and the normal to the discontinuity surface), the jump in the magnetic field gradually increases with increasing density jump from zero to a maximum value

$$(\Delta H_y)_{\max} = \frac{2H_{1y}}{\gamma-1}.$$

In waves of the second type, which are realized when

$$\sin^2\theta_1 < \frac{(\gamma-1)(1-r_1)}{\gamma},$$

a nonmonotonic dependence exists between the jump in the magnetic field and the jump in the density: as the density jump increases, the jump in the magnetic field first increases from zero to a certain maximum value, and then decreases to a value

$$\Delta H_y = \frac{2H_{1y}}{\gamma-1}.$$

On the evolutionary portion of the slow shock wave, the jump in the magnetic field always increases with increasing density jump.*

In the limiting case when $\theta_1 \to 0$ and $r_1 < 1$, the fast wave is the same as the absence of the magnetic field, while the slow wave has an infinitesimally small amplitude. In the case when $\theta_1 \to 0$ and $r_1 < 1$, the fast wave belongs to the second type. If the fast shock wave has a low intensity $\left(\frac{\rho_2}{\rho_1} < \frac{\gamma + 1 - 2r_1}{\gamma - 1}\right)$, the transverse magnetic field H_{2y} behind the wave is different from zero. When the intensity of the shock wave ρ_2/ρ_1 exceeds the value $(\lambda + 1 - 2r_1)/(\gamma - 1)$, the transverse magnetic field H_{2y} behind the wave vanishes and the shock wave becomes the same as in a magnetic field.

*In other words, slow shock waves always belong to the first type. The statement by Bazer and Ericson[47] that a nonmonotonic dependence of the jump of the magnetic field on the jump in the density is possible in slow shock waves is due to failure to take account of the evolutionarity conditions.

In the case when $\theta_1 \to 0$ and $r_1 < 1$, a slow wave on the evolutionary section will be the same as in the absence of a magnetic field.

In the limiting case $\theta_1 \to \pi/2$ the fast shock wave is of the first type while the slow shock wave is transformed into a tangential discontinuity.

The presence of a magnetic field increases the pressure jump at a fixed density jump.[47]*

In fast shock waves of high intensity ($p_2 \gg p_1 + H_1^2/8\pi$) the presence of a magnetic field is of no significance.[53,46] In particular, the greatest compression ρ_2/ρ_1 attainable reached in the shock wave[57,47] is $(\gamma+1)/(\gamma-1)$.

A slow shock wave cannot have an arbitrarily high intensity. Therefore when $p_2 \gg p_1 + H_1^2/8\pi$, only one (fast) shock wave exists.[36,46]

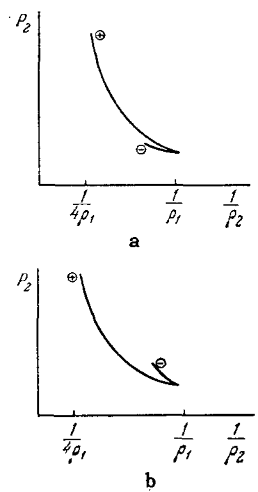


FIG. 4. Evolutionary portions of the shock adiabat in the variables $1/\rho_2$ and p_2: a) $H_{1y} \ll H_x$; b) $H_{1y} \gg H_x$. The symbol "+" denotes the fast shock wave, while "–" denotes the slow shock wave.

The evolutionary sections of the shock adiabat were determined[55] in terms of the variables $1/\rho_2$ and p_2 only in the limiting cases $H_{1y} \ll H_x$ (Fig. 4a) and $H_{1y} \gg H_x$ (Fig. 4b). As can be seen from Fig. 4a, when $H_{1y} \ll H_x$ the portion of the shock adiabat corresponding to the fast shock wave lies above the portion corresponding to the slow shock wave. This means that the fast shock wave is thermodynamically more favorable when $H_1 \ll H_x$, since its entropy increase is greater. In the case when $H_{1y} \ll H_x$, the slow shock wave is thermodynamically favored (Fig. 4b).

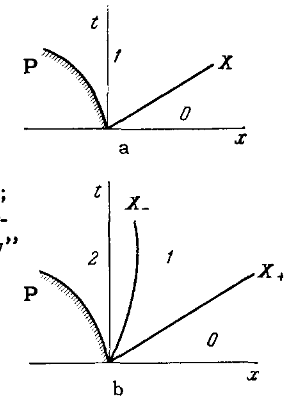
The difference in the relative placement of the part of the adiabat corresponding to the fast and slow shock waves in Figs. 4a and 4b can be explained in the following manner. In a shock wave the kinetic energy $\rho_1 v_{1x}^2/2$ is converted into magnetic energy $H_2^2/8\pi$ and heat energy $3p_2/2$. A fast shock wave corresponds to a greater transfer of kinetic energy. In the case when $H_{1y} \ll H_x$ there is little change in the magnetic energy, and a fast shock wave is therefore accompanied by greater heating. In the case when $H_{1y} \gg H_x$ a significant change takes place in the magnetic energy. Since the magnetic field increases in a fast shock wave and decreases in a slow one,[38] it becomes clear why less heating takes place in a fast shock wave when $H_{1y} \gg H_x$.

*This fact was noted by G. S. Golitsyn[56] for the case when $H_x = 0$.

7. THE PISTON PROBLEM

The treatment of the piston problem encounters in magnetohydrodynamics two principal difficulties not found in ordinary hydrodynamics. The first difficulty is that the boundary conditions on the discontinuity surfaces and on the piston do not define uniquely the continuous and shock waves produced. In the case of a perfect gas an infinite set of solutions corresponds to a specified piston motion, but only one of these solutions satisfies the evolutionarity conditions. This difficulty is therefore eliminated by excluding from consideration all the non-evolutionary discontinuities, and the piston problem becomes mathematically correct.*

FIG. 5. Waves formed when a piston moves a) in ordinary hydrodynamics, b) in magnetohydrodynamics. P – line of piston motion, X – ordinary hydrodynamic characteristic $dx/dt = v_x + c$; X_+ – "fast" magnetohydrodynamic characteristic $dx/dt = v_x + U_+$; X_- – "slow" magnetohydrodynamic characteristic $dx/dt = v_x + U_-$; 0 – region at rest, 1 – simple wave, 2 – non-simple wave.



The second difficulty is that the continuous flows produced by the motion of the piston are not always simple waves. It goes without saying that in the absence of discontinuities the region bordering on the constant flow is a simple wave. But this region may not reach the piston. Another region, in which the wave is not simple may lie between the piston and the simple wave (Fig. 5). In order to get around this difficulty we shall consider a piston moving with constant velocity.

Since the problem has in this case no parameter with dimension of length, the motion of the medium will be self-similar, i.e., all the waves produced will be either discontinuous or simple.

The motion induced in the medium by the piston is characterized by a train of shock and self-similar waves that follow one another. As we have already seen, there exist three types of evolutionary discontinuous waves that move relative to the medium — fast and slow shock waves and Alfven discontinuities. In addition, there are two types of continuous solutions

*This difficulty can be circumvented also without investigating the evolutionarity properties, by considering only piston motion in which no slow shock waves are produced.[58]

— fast and slow self-similar magnetoacoustic waves. As noted earlier, shock waves are compression waves while self-similar waves are rarefaction waves. The velocities of these waves are such that only the fast waves (shock or self-similar) can move in front, followed by the Alfven discontinuity and then by the slow wave (shock or self-similar). Since some of these waves may be missing, the medium may move in many qualitatively different manners under the influence of different piston velocities.

The obvious boundary condition satisfied on the surface of the piston is

$$v_x = u_x. \tag{7.1}$$

To obtain the remaining two conditions, it is necessary to change to a reference frame moving together with the piston; in this frame the boundary conditions $E'_y = 0$ and $E'_z = 0$ are satisfied on the surface of the conducting piston (the prime denotes that the corresponding quantity is measured in a coordinate system moving together with the piston). In view of the infinite conductivity of the medium $\mathbf{E}' = -\mathbf{v}' \times \mathbf{H}/c$ (c is the velocity of light) and, consequently,

$$v_y = u_y, \quad v_z = u_z \quad (H_x \neq 0). \tag{7.2}$$

Thus, on metallic surfaces the relative velocity of the conducting liquid is equal to zero (an exception is the case when the magnetic field has no normal component). A unique phenomenon of "electrodynamic viscosity" is obtained.[59]

If a rarefaction shock wave has a sufficiently large amplitude, the density of the medium behind the wave vanishes and cavitation sets in. In this case the magnetic field and the transverse component of the electric field should be continuous on the boundary with the vacuum, i.e., the following boundary conditions should be satisfied;

$$\begin{aligned} \varrho = 0, \ H_x (u_y - v_y) - H_y (u_x - v_x) = 0, \\ H_x (u_z - v_z) - H_z (u_x - v_x) = 0. \end{aligned} \tag{7.3}$$

Let us consider the most interesting case, when the magnetic field, the piston velocity, and the normal to the piston surface lie in a single plane (the xy plane). In this case v_z and H_z will vanish not only in the unperturbed medium but also in all the waves produced. Therefore the Alfven discontinuity can rotate the magnetic field only through 180°. We assume the unperturbed medium velocity v_0 to be equal to zero. To be definite, we shall assume the components of unperturbed magnetic field H_{0x} and H_{0y} to be positive. The types of waves arising during the motion depend on the piston velocity (u_x, u_y). This dependence[60] is shown in Fig. 6. When the amplitude of the slow rarefaction wave is sufficiently large, the density of the medium behind the wave vanishes and cavitation sets in. Compared with ordinary hydrodynamics, in which cavitation sets in when the piston moves with a velocity exceeding $2c_0/(\gamma - 1)$ (c_0 is the velocity of sound

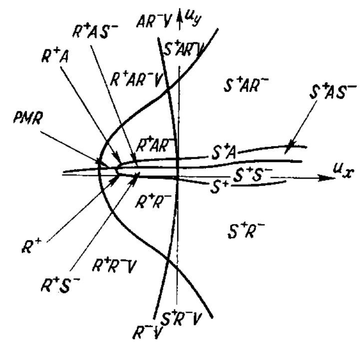


FIG. 6. Waves arising in the motion of the piston. The abscissas are the longitudinal components of piston velocity u_x, while the ordinates are the transverse components u_y. The letters S^+, S^-, R^+, R^-, and A denote the presence of a fast and slow shock wave, fast and slow rarefaction (self-similar) wave, Alfven discontinuity and the formation of a vacuum. PMR is the point of maximum rarefaction reached in a fast self-similar wave, V – presence of cavitation.

in the unperturbed medium), cavitation occurs in magnetohydrodynamics also at lower piston velocities, provided the velocity of the piston in the transverse direction is sufficiently large. If the piston moves only in the transverse direction, the cavitation sets in when the piston velocity is 3.67 times greater than the velocity of sound in the unperturbed medium[29] at $\gamma = 5/3$. Cavitation takes place also when the piston moves into the medium and moves simultaneously in the transverse direction. If the piston moves into the medium at supersonic velocity, cavitation takes place[6] if the angle between the piston velocity vector and the normal surface exceeds 70° ($\gamma = 5/3$). (In this case the difference between the rate of displacement of the boundary and the piston velocity is quite small.)

In contrast with the slow rarefaction wave, cavitation is impossible in a fast rarefaction wave. Whereas the Alfven velocity in the unperturbed medium is considerably lower than the velocity of sound, the Alfven velocity behind a fast rarefaction wave of maximum amplitude becomes close to the velocity of sound. If the condition $H_0^2/8\pi \ll p_0$ is satisfied, the density of the medium near the point of maximum rarefaction (point PMR in Fig. 6) will become quite small.

If the transverse velocity of the piston is supersonic, the magnetic field builds up from an infinitesimally small value to a finite value; the magnetic pressure then becomes comparable with or greater than the hydrostatic pressure. If the piston moves in and glides at supersonic velocities, the generated magnetic field is directly proportional to the longitudinal component of the piston velocity.

The topological structure shown in Fig. 6 can be obtained also without calculations, from qualitative considerations.[61] For this purpose let us determine the regions through which the ordinate axis ($u_x = 0$) passes.

Since the magnetic field lines are "glued" to the particles of the medium and to the piston, the mag-

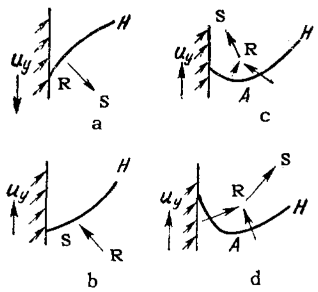


FIG. 7. Formation of a shock wave (S), Alfven discontinuity (A), and a rarefaction wave (R) in transverse motion of a piston, due to stretching of the magnetic force lines.

netic force lines are deformed when $u_y < 0$ (Fig. 7a). The bending of the magnetic force line results in quasielastic tension forces in the directions of the arrows in Fig. 7a.

Since $v_x = 0$ near the piston and at infinity, a compression (shock) wave is produced in front of the arrow and a rarefaction (self-similar) wave in the rear. No Alfven wave is produced in this sketch, since H_y has the same sign near the piston and at infinity (we recall that the sign of H_y remains unchanged in a shock and self-similar wave,[38] and that the sign is reversed in a 180° Alfven discontinuity). Thus, when $u_x = 0$ and $u_y < 0$, a shock wave travels in front, followed by a self-similar wave (combination S^+R^-).

Analogous arguments show that the combination R^+S^- is realized in the case shown in Fig. 7b. As the velocity u_y is increased, the sign of H_y changes near the piston, bringing about an Alfven wave (R^+AS^-) (Fig. 7c). Further increase in u_y makes the value of $|H_y|$ near the piston greater than the value of H_y at infinity; in this case the resultant of the tension forces is directed away from the piston (Fig. 7c), corresponding to a combination S^+AR^-.

When $|u_y|$ is sufficiently large, the amplitude of the rarefaction wave becomes so large, that the density of the medium behind the wave vanishes and cavitation sets in.

The motion of the medium at $u_x \neq 0$, $u_y \neq 0$, and $u_c = 0$ can be visualized by starting with the case $u_x = 0$ just considered. As u_x is increased the amplitude of the rarefaction wave decreases and the amplitude of the compression wave increases.*

At a certain value of u_x the rarefaction wave turns into a compression wave. Analogously, a reduction in u_x causes the compression wave to be transformed into a rarefaction wave. A further reduction in the value of u_x results in cavitation.

The piston motion gives rise to a resistance force F with two components; a longitudinal component F_x, which is the frontal resistance, and a transverse component F_y, the lifting force. In the case of a fast shock of large amplitude accompanied by a slow rarefaction wave in the region

$$u \gg c_0, \ |u_y| \ll u_x \ \text{ and } H_0^2/8\pi \ll p_0,$$

the frontal resistance is determined by the formula[60]

$$-F_x = \frac{(\gamma+1)^2 \varrho_0 u_x^2}{2(\gamma-1)}.$$

If we increase $|u_y|$ and keep u_x constant, the value of the frontal resistance decreases, and assumes on the cavitation line the value

$$-F_x = \frac{(\gamma+1)\varrho_0 u_x^2}{2}.$$

Further increase of $|u_y|$ does not change the frontal resistance.

The lifting force is determined respectively by the formulas

$$-F_y = \frac{(\gamma+1)\varrho_0 u_x u_y}{\gamma-1} \ \text{ when } |u_y| \ll u_x$$

and

$$-F_y = \sqrt{\gamma+1}\, \varrho_0 U_{0x} \operatorname{sign} u_y$$

in the presence of cavitation.

The case when the piston velocity does not lie in the plane of the magnetic field xy (i.e., when $u_z \neq 0$) differs from the preceding case in that the angle of rotation of the magnetic field in the Alfven discontinuity differs from 180°.

8. DECAY OF THE DISCONTINUITY

The question arises: what happens with the discontinuity of the initial conditions if the necessary boundary conditions (continuity of the fluxes of the mass, momentum, etc.) are not satisfied? Discontinuous initial conditions of this kind are obtained, for example, in collisions between gas masses or upon a sudden destruction of a partition between two gases at different pressures. This problem was proposed by Riemann, and solved for the case of ordinary hydrodynamics by N. E. Kochin.[62] An investigation has shown that the discontinuity breaks up into three waves,[34] one moving to the right (shock or self-similar), one moving to the left (shock or self-similar), and a *tangential discontinuity*.

In magnetohydrodynamics, the decay of the discontinuity of the initial conditions in the absence of a longitudinal magnetic field H_x will be qualitatively the same as in the absence of a magnetic field.[63] The problem of the decay of the discontinuity produced by a collision between two gas masses and at $H_x = 0$ was solved by T. F. Volkov[64] in connection with the problem of heating of a plasma to thermonuclear temperatures with the aid of shock waves.

The decay of discontinuity in a stationary plasma at $H_x = 0$ and $|\Delta p^*| \ll p^*$ ($p^* = p + H^2/8\pi$) was considered by Kato.[65] The solution obtained was used for

*An exception from this rule occurs in the region S^+S^- for a piston with supersonic velocity. When u_x is increased, a redistribution takes place in the amplitudes of the fast and slow shock waves; the increase in the amplitude of the fast shock is accompanied by a certain reduction in the amplitude of the slow shock wave.

qualitative consideration of pulsations of a plasma pinch in a longitudinal magnetic field.

If the longitudinal magnetic field H_x differs from zero, the decomposition of the initial discontinuity in magnetohydrodynamics will be qualitatively different from that in ordinary hydrodynamics. The initial discontinuity decomposes into seven waves, three moving to the right, three to the left, and a contact discontinuity between them at rest relative to the medium.

Each of these waves is characterized by a single parameter (amplitude, i.e., a jump in one of the magnetohydrodynamic quantities). On the other hand, the sum of the jumps of each of the hydrodynamic quantities on the seven resulting waves is equal to the initial jump. Since the number of magnetohydrodynamic quantities is seven (ρ, p, v_x, v_y, v_z, H_y, and H_z) we obtain a system of seven equations with seven unknowns, the solution of which yields the amplitudes of all the waves arising in the decay of the initial discontinuity.

No general solution of the problem of the decay of a magnetohydrodynamic discontinuity has been obtained because of the great mathematical difficulties. This problem has been solved[66] for the case when the initial discontinuity is quite small.*

In this case all the discontinuities produced are also small. Since the relations between the jumps of the magnetohydrodynamic quantities in a self-similar and a shock wave of low intensity are the same in first approximation, the only difference between a shock wave and a self-similar wave is that the density increases in the former and decreases in the latter. The density jumps on the fast and slow magnetoacoustic waves are determined by the formulas

$$\Delta_{\pm}^{(\varepsilon)}\varrho=\frac{1}{2R}\left\{\frac{c^2U_t^2\left[\Delta\varrho-\left(\frac{\partial\varrho}{\partial s}\right)_p\Delta s\right]}{U_{\pm}^2-U^2}-\frac{\Delta H_t^2}{8\pi}\right.$$
$$\left.+\frac{\varepsilon\varrho U_x^2}{U_{\pm}}\left[\frac{H_t\Delta v_t}{H_x}+\frac{U_t^2\Delta v_x}{U_{\pm}^2-U_x^2}\right]\right\}, \tag{8.1}$$

where $R=\sqrt{(U^2+c^2)^2-4c^2U_x^2}$; $\Delta\rho$, Δs, and Δv, and ΔH_t are the jumps in the density, entropy, velocity, and transverse magnetic field on the initial discontinuity; ϵ and the symbols "±" have the same meaning as in (2.2). This formula makes it possible to determine the signs of $\Delta_{\pm}^{(\epsilon)}\rho$ and ascertain thereby the waves into which the initial discontinuity decomposes. A shock wave corresponds to $\Delta_{\pm}^{(\epsilon)}\rho > 0$ and a self-similar wave corresponds to $\Delta_{\pm}^{(\epsilon)}\rho < 0$.

A discontinuity in the initial conditions has a certain similarity to a non-evolutionary shock wave. Although all the boundary conditions are satisfied in the latter case, an infinitesimally small perturbation is sufficient to upset the boundary conditions, causing the non-evolutionary shock wave to decompose into several diverging waves (discontinuous or continuous). Such a decomposition was considered in reference 45 for the case when the magnetic field on both sides of the shock wave makes a small angle with the normal to the surface of the discontinuity, and the velocity of propagation of the shock wave v_{1x} is close to the Alfven velocity U_{1x}, which in turn is greater than the velocity of sound c_1 (the subscript "1" pertains to the region ahead of the wave). The evolutionarity condition of the slow shock wave has in this case the form $v_{1x} < U_{1x}$. If this condition is violated, the shock wave becomes non-evolutionary. Such a non-evolutionary shock wave can be obtained if an ordinary stable hydrodynamic shock wave, in which $v_{1x} > c_1$ and $v_{2x} < c_2$ is placed in a magnetic field. If the transverse magnetic field H_y is equal to zero, this shock wave can decompose into two singular shock waves,[42] and the magnetohydrodynamic quantities in the region contained between these two waves are

$$\left.\begin{aligned}
&\tilde{\varrho}=\frac{\varrho_1 v_{1x}^2}{U_{1x}^2}, \quad \tilde{v}_x=\frac{U_{1x}^2}{v_{1x}},\\
&\tilde{p}=p_1+\frac{\varrho_1(v_{1x}^2-U_{1x}^2)(3c_1^2+v_{1x}^2-U_{1x}^2)}{3U_{1x}^2},\\
&\tilde{v}_y=\pm\left[\frac{2(v_{1x}^2-U_{1x}^2)(4U_{1x}^2-v_{1x}^2-3c_1^2)}{3v_{1x}^2}\right]^{1/2},\\
&\tilde{H}_y=\pm\left[\frac{8\pi\varrho_1(v_{1x}^2-U_{1x}^2)(4U_{1x}^2-v_{1x}^2-3c_1^2)}{3U_{1x}^2}\right]^{1/2}.
\end{aligned}\right\} \tag{8.2}$$

Such a decomposition is possible[45] only when the evolutionarity condition $v_{1x} < U_{1x}$ is not satisfied. This follows from the fact that when $v_{1x} < U_{1x}$ the expressions for v_y and H_y become imaginary.

If a small transverse magnetic field is taken into account, the initial non-evolutionary shock wave splits up into four waves [neglecting the waves whose amplitudes are of the order of αH_{1y}, $\alpha \equiv \sqrt{(v_{1x}-U_{1x})/U_{1x}} \ll 1$], viz., a fast shock wave traveling to the left [x axis directed to the right] with an amplitude on the order of α and a velocity on the order of $U_{1y} \equiv H_{1y}/\sqrt{4\pi\rho_1}$, a slow shock wave traveling to the left with an amplitude that differs little from the amplitude of the initial wave and with a velocity on the order of αH_{1y}, an Alfven discontinuity moving to the right and rotating the magnetic field by 180°, and a fast shock wave moving to the right with amplitude on the order of H_{1y}.

The decay of the discontinuity in the initial conditions is closely related with the question of transitions between magnetohydrodynamic discontinuities.* Allowance for the evolutionarity conditions changes somewhat the picture of the possible transitions. First, the statement that the Alfven discontinuity can be gradually transformed into a shock wave becomes incorrect. In fact, the Alfven discontinuity can coincide with a shock wave only when the magnetic field lies

*Lax[19] investigated the more general case, when the system of equations of magnetohydrodynamics is replaced by a hyperbolic system of n equations with n unknowns.

*This question was first considered in the paper by S. I. Syrovat-skiĭ.[32]

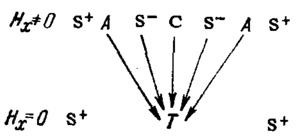


FIG. 8. Transitions between the magnetohydrodynamic discontinuities. The letters S^+, S^-, A, C, and T denote the fast and slow shock waves and the Alfven, contact, and tangential discontinuities.

in the same plane on both sides of the discontinuity, i.e., if the magnetic field is rotated in the Alfven discontinuity by 180°. The transverse component of the magnetic field reverses its sign on such a discontinuity, but not on a shock wave. Therefore no transition between the Alfven discontinuity and an evolutionary shock is possible, but the non-existent non-evoluionary slow shock wave gradually changes into an Alfven discontinuity[47] (see Fig. 2).

Continuous transitions are likewise impossible between fast and slow shock waves. This follows from the fact that the regions of existence of the fast and slow shock waves have no points of contact (see Fig. 2).

A fast shock wave cannot be gradually transformed into a tangential discontinuity, for this would contradict the conditions (4.3).

Thus, transitions are possible only between tangential and contact discontinuities, between tangential and Alfven discontinuities, and between tangential discontinuities and slow shock waves.

The meaning of the possible transitions between the magnetohydrodynamic discontinuities becomes clearer if we consider the problem of the decay of an arbitrary discontinuity of the initial conditions. If the normal magnetic field H_x differs from zero, then the discontinuity breaks up into seven waves, each characterized by one parameter.

On the other hand, if the normal magnetic field H_x is equal to zero, then the initial discontinuity breaks up into three waves: a fast shock wave moving to the right, a fast shock wave moving to the left, and a tangential discontinuity between the two. Each shock wave is characterized by a single parameter, while the tangential discontinuity is characterized by five parameters.[30] The total number of parameters is seven, i.e., the same as in the number of jumps in the magnetohydrodynamic quantities on the initial discontinuity. Thus, the tangential discontinuity is a merger of five discontinuities (two slow shock waves, two Alfven discontinuities, and a contact discontinuity). The transitions between the magnetohydrodynamic discontinuities are shown schematically in Fig. 8.

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Translated by J. G. Adashko