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*SPIN WAVES IN FERROMAGNETS AND ANTIFERROMAGNETS II**

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II: INTERACTION OF SPIN WAVES WITH ONE ANOTHER AND WITH LATTICE VIBRATIONS; RELAXATION AND KINETIC PROCESSES

10. Fusion and Splitting of Spin Waves. Scattering of Spin Waves by Spin Waves.

In the preceding sections we have obtained the energy spectrum of a ferromagnet and used it for calculating the spin specific heat and the magnetic moment of the ferromagnet. In doing this we started from the Hamiltonian for the ferromagnet, either in its simplest form (1.6), or in the form (2.3), which takes account of magnetic interaction and the anisotropy of the energy, and we kept in this Hamiltonian only terms which were quadratic in the operators for creation and annihilation of spin waves. The neglect of higher powers of products of these operators, which is justified for finding the energy spectrum of the ferromagnet close to its ground state, does not, however, enable us to study various kinetic and relaxation processes in ferromagnets which are due to interactions between the elementary excitations.

Proceeding now to the investigation of relaxation processes in ferromagnets, we must include both the interaction of the spin waves with one another, as well as their interaction with other elementary excitations which are characteristic of the ferromagnet. Here we shall limit ourselves to a treatment of ferrodielectrics only, and therefore, in addition to the interaction of spin waves with one another, we shall also consider their interaction with phonons.[1]

To obtain the Hamiltonian for the interaction, we must take the total Hamiltonian for the ferromagnet which takes into account lattice oscillations, and, after transforming to variables which describe the creation and annihilation of spin waves ($c_{\mathbf{k}}^{+}$, $c_{\mathbf{k}}$) and phonons ($b_{\mathbf{f}s}^{+}$, $b_{\mathbf{f}s}$), we must by expanding in series select those terms which contain the operators $c_{\mathbf{k}}^{+}$ and $c_{\mathbf{k}}$ to third and higher orders, as well as mixed terms containing both the operators $c_{\mathbf{k}}$ and the operators $b_{\mathbf{f}s}$. The entire collection of these terms is the interaction Hamiltonian in which we are interested.

First let us consider the interaction of spin waves with one another. The Hamiltonians for interaction of spin waves with one another can be obtained from $\mathcal{H}_s^{(3)}$ and $\mathcal{H}_s^{(4)}$ (cf. Sec. 3) if we change from the operators $a_{\mathbf{k}}^{+}$ and $a_{\mathbf{k}}$ to the operators $c_{\mathbf{k}}^{+}$ and $c_{\mathbf{k}}$ according to formulas (3.16). As a result they take the form:

$$\mathcal{H}_s^{(3)} = \sum_{123} \Phi'_{12;3} c_1^{+} c_2^{+} c_3 + \text{Herm. conj.}$$

$$\mathcal{H}_s^{(4)} = \sum_{1234} \Phi'_{12;34} c_1^{+} c_2^{+} c_3 c_4 + \sum_{1234} \Phi'_{1;234} c_1^{+} c_2 c_3 c_4 + \text{Herm. conj.} \qquad (10.1)$$

where the quantities Φ' are obtained from Φ by means of the transformation (3.16). The operators $c_{\mathbf{k}}^{+}$ and $c_{\mathbf{k}}$ which appear here are the true operators for the creation and absorption of spin waves (in contrast to $a_{\mathbf{k}}^{+}$ and $a_{\mathbf{k}}$). The expressions $\mathcal{H}_s^{(3)}$ and $\mathcal{H}_s^{(4)}$ which are defined by formulas (10.1) are the interaction Hamiltonians for the interaction of spin waves with one another. Clearly, $\mathcal{H}_s^{(3)}$ describes processes of splitting of one spin wave into two and fusion of two spin waves into one, while $\mathcal{H}_s^{(4)}$ describes processes of scattering of one spin wave by another, as well as the splitting of one spin wave into three waves and the fusion of three spin waves into one wave.

In the case of a uniaxial ferromagnet with strong anisotropy, when the inequality

$$\beta + \frac{H_0}{M_0} \gg 1$$

is satisfied, the quantity $u_{\mathbf{k}}$ is close to unity, and $v_{\mathbf{k}}$ is close to zero, i.e.,

$$c_{\mathbf{k}} \approx a_{\mathbf{k}}, \quad c_{\mathbf{k}}^{+} \approx a_{\mathbf{k}}^{+}.$$

Therefore in this case the variables $a_{\mathbf{k}}^{+}$ and $a_{\mathbf{k}}$ can be regarded as operators for creation and absorption of spin waves, and we can use the expressions (3.14)

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*For Part I cf. Usp. Fiz. Nauk 71, 533 (1960), Soviet Phys.—Uspekhi 3, 567 (1961).

and (3.15) as the interaction Hamiltonians. However, we note that such an approximation is already insufficient for the case of cubic crystals with small anisotropy constant, if they are in a weak magnetic field.

The total Hamiltonian for the interaction of spin waves with one another is a series in powers of the operators $c_{\mathbf{k}}^{+}$ and $c_{\mathbf{k}}$, with an expansion parameter which is the mean value of the deviation of the magnetic moment of the ferromagnet from its maximum value. This quantity is essentially small at low temperatures. Therefore, the Hamiltonian for the interaction of spin waves with one another can be used only over ranges of temperature which are low compared to the Curie temperature.

In the total Hamiltonian of the interaction, the terms of lowest order contain the operators $c_{\mathbf{k}}^{+}$ and $c_{\mathbf{k}}$ to the third power. They arise from the expansion of the magnetic interaction energy. However, in addition to these terms, one must generally also include terms containing $c_{\mathbf{k}}^{+}$ and $c_{\mathbf{k}}$ to fourth order, these terms arising from the expansion of the exchange energy and therefore containing an additional large factor. Terms which contain powers of the operators $c_{\mathbf{k}}^{+}$ and $c_{\mathbf{k}}$ higher than the fourth are only small corrections, and we shall therefore not include them.

Now let us calculate the probabilities of processes described by the Hamiltonians $\mathcal{H}_s^{(3)}$ and $\mathcal{H}_s^{(4)}$, i.e., the probabilities for fusion and splitting, as well as scattering, of spin waves. Let us begin with a determination of the probability for fusion and splitting of spin waves.

Using formula (3.20), which gives the matrix elements of the operators $c_{\mathbf{k}}^{+}$ and $c_{\mathbf{k}}$, we obtain the following expressions for the matrix elements of the processes of interest to us:

$$\left.\begin{aligned}&(n_1, n_2, n_3 | \mathcal{H}_s^{(3)} | n_1+1, n_2+1, n_3-1)\\ &\quad = 2\Phi_{12;3}\sqrt{(n_1+1)(n_2+1)n_3}\, e^{\frac{i}{\hbar}(\varepsilon_1+\varepsilon_2-\varepsilon_3)t}\Delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3),\\ &(n_1, n_2, n_3 | \mathcal{H}_s^{(3)} | n_1+1, n_2-1, n_3-1)\\ &\quad = 2\Phi_{23;1}\sqrt{(n_1+1)n_2n_3}\, e^{\frac{i}{\hbar}(\varepsilon_1-\varepsilon_2-\varepsilon_3)t}\Delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3),\end{aligned}\right\} \tag{10.2}$$

where $n_i = n_{\mathbf{k}_i}$, $\epsilon_i = \epsilon_{\mathbf{k}_i}$.

The probabilities for the processes are

$$\begin{aligned}W_{n_1+1, n_2+1, n_3-1}^{n_1, n_2, n_3} &= \frac{2\pi}{\hbar}|2\Phi_{12;3}|^2(n_1+1)(n_2+1)\\ &\quad\times n_3\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3)\Delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3),\\ W_{n_1+1, n_2-1, n_3-1}^{n_1, n_2, n_3} &= \frac{2\pi}{\hbar}|2\Phi_{23;1}|^2(n_1+1)\\ &\quad\times n_2n_3\delta(\varepsilon_2+\varepsilon_3-\varepsilon_1)\Delta(\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}_1),\end{aligned} \tag{10.2'}$$

where $\delta(x)$ is the Dirac δ function.

We see that, in the processes we are considering, the law of conservation of energy holds:

$$\varepsilon_3 = \varepsilon_1+\varepsilon_2, \quad \varepsilon_1 = \varepsilon_2+\varepsilon_3$$

as well as the law of conservation of the wave vector or the momentum:

$$\mathbf{k}_3 = \mathbf{k}_1+\mathbf{k}_2, \quad \mathbf{k}_1 = \mathbf{k}_2+\mathbf{k}_3.$$

We note that, in general, in crystals the latter relation is replaced by the more general relation

$$\mathbf{k}_1 \pm \mathbf{k}_2 - \mathbf{k}_3 = 2\pi\mathbf{b},$$

where $\mathbf{b}$ is a vector of the reciprocal lattice. However, in defining the probability averages we can limit ourselves henceforth to the case $\mathbf{b} = 0$, since the inclusion of scattering processes with $\mathbf{b} \neq 0$ (these are called "umklapp" processes), leads to small corrections.

Knowing the probabilities for fusion and splitting of spin waves, we can determine the change in the number of spin waves per unit time due to these processes:

$$\dot{n}_1^{c} = L_{\mathbf{k}_1}^{(3)}\{n\},$$

$$\begin{aligned}L_{\mathbf{k}_1}^{(3)}\{n\} = \frac{8\pi}{\hbar}\sum_{23}\{&|\Phi_{12;3}|^2[(n_1+1)(n_2+1)n_3 - n_1n_2(n_3+1)]\\ &\times\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3)\Delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3) + |\Phi_{13;2}|^2[(n_1+1)(n_3+1)n_2\\ &- n_1n_3(n_2+1)]\delta(\varepsilon_1+\varepsilon_3-\varepsilon_2)\Delta(\mathbf{k}_1+\mathbf{k}_3-\mathbf{k}_2)\\ &+|\Phi_{23;1}|^2[(n_1+1)n_2n_3 - n_1(n_2+1)(n_3+1)]\delta(\varepsilon_1-\varepsilon_2-\varepsilon_3)\\ &\times\Delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\},\end{aligned} \tag{10.3}$$

where $L_{\mathbf{k}}^{(3)}\{n\}$ denotes the collision operator associated with the Hamiltonian $\mathcal{H}_s^{(3)}$.

Using the expression (10.3), we shall calculate the mean time for splitting or fusion of spin waves.

We shall assume that the numbers of spin waves differ very little from their equilibrium values $n_{\mathbf{k}}^0 = \left(\exp\frac{\epsilon_{\mathbf{k}}}{T} - 1\right)^{-1}$:

$$n_{\mathbf{k}} = n_{\mathbf{k}}^0 + \delta n_{\mathbf{k}}, \quad |\delta n_{\mathbf{k}}| \ll n_{\mathbf{k}}^0,$$

and expand the collision operator $L_{\mathbf{k}}^{(3)}\{n\}$ in powers of $\delta n_{\mathbf{k}}$. The zeroth order terms in this expansion vanish since the equilibrium function makes the collision operator zero. The coefficient of $\delta n_{\mathbf{k}}$ taken with reversed sign can be regarded as the reciprocal of the lifetime of a spin wave with wave vector $\mathbf{k}$ with respect to processes of fusion and splitting. This quantity, which we shall denote by $1/\tau_{\mathbf{k}}^{(3)}$, is obviously equal to

$$\frac{1}{\tau_{\mathbf{k}}^{(3)}} = -\left(\frac{\delta L_{\mathbf{k}}^{(3)}\{n\}}{\delta n_{\mathbf{k}}}\right)_0, \tag{10.4}$$

where the subscript zero on the functional derivative means that the numbers of spin waves are replaced by the corresponding equilibrium functions. Averaging $1/\tau_{\mathbf{k}}^{(3)}$ over the equilibrium distribution $n_{\mathbf{k}}^0$, we find a quantity which is the reciprocal of the mean life of the spin wave with respect to the processes of fusion or splitting:

$$\frac{1}{\tau^{(3)}} \equiv w^{(3)} = \frac{\sum_{\mathbf{k}}\frac{1}{\tau_{\mathbf{k}}^{(3)}}n_{\mathbf{k}}^0}{\sum_{\mathbf{k}}n_{\mathbf{k}}^0}. \tag{10.5}$$

The calculation of the functional derivative gives

$$w^{(3)} = \frac{8\pi}{\hbar} \sum_{123} |\Phi_{12;\,3}|^2 (2n_1^0 n_2^0 + n_3^0)\, \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3) \times \Delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3) \frac{1}{\sum\limits_1 n_1^0}. \tag{10.5'}$$

Changing from summation over $\mathbf{k}$ to integration in accordance with formula (7.5′), we obtain:[1,2]

$$w^{(3)} = \frac{\pi}{5} \frac{\mu^3 M_0}{a^3 \hbar \theta_C} \left(\frac{T}{\theta_C}\right)^{1/2} F(\eta), \tag{10.6}$$

where

$$F(\eta) = \iint\limits_{xy \geqslant \eta^2} \left\{ \frac{1}{e^{x+y} - 1} + \frac{2}{(e^x - 1)(e^y - 1)} \right\} \times \left(\frac{1}{3} + \frac{\eta^2}{xy}\right) dx\, dy \left(\int\limits_0^\infty \frac{x^2 dx}{e^{x+2\eta} - 1} \right)^{-1},$$

$$\eta = \frac{\mu H^{(e)}}{T}, \quad H^{(e)} = H_0 + \beta M_0.$$

An explicit expression for the function $F(\eta)$ can be obtained only in the limiting cases of small and large η:

$$F(\eta) = \begin{cases} \dfrac{2}{3\zeta(3)} \ln^2 \eta, & \eta \ll 1 \\ 2\sqrt{\pi\eta}\, e^{-\eta}, & \eta \gg 1. \end{cases} \tag{10.6'}$$

Thus $w^{(3)}$ is defined by the following formulas:

$$w^{(3)} = \begin{cases} \dfrac{2\pi}{15\zeta(3)} \dfrac{\mu M_0}{\hbar} \dfrac{\mu^2}{a^3 \theta_C} \left(\dfrac{T}{\theta_C}\right)^{1/2} \ln^2 \dfrac{\mu H^{(e)}}{T}, & \mu H^{(e)} \ll T \\ \dfrac{2\pi^{3/2}}{5} \dfrac{\mu M_0}{\hbar} \dfrac{\mu^2}{a^3 \theta_C} \left(\dfrac{\mu H^{(e)}}{\theta_C}\right)^{1/2} e^{-\frac{\mu H^{(e)}}{T}}, & \mu H^{(e)} \gg T. \end{cases} \tag{10.7}$$

We see that in the regions of low effective magnetic field the average time for splitting is essentially inversely proportional to $\sqrt{T}$; in the case of strong fields it increases exponentially with decreasing temperature.

Now let us proceed to consider the processes of scattering of spin waves by spin waves, as well as the fusion and splitting of spin waves, which are described by the Hamiltonian $\mathcal{H}_S^{(4)}$.

The matrix element for the scattering of a spin wave by a spin wave has the form:

$$(n_1, n_2, n_3, n_4 | \mathcal{H}_s^{(4)} | n_1 + 1, n_2 + 1, n_3 - 1, n_4 - 1) = 4(\Phi_{12;\,34} + \Phi^*_{34;\,12}) \sqrt{(n_1 + 1)(n_2 + 1) n_3 n_4} \times e^{\frac{i}{\hbar}(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)t} \Delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4).$$

The probability of this scattering process is

$$W^{n_1, n_2, n_3, n_4}_{n_1+1, n_2+1, n_3-1, n_4-1} = \frac{32\pi}{\hbar} |\Phi_{12;\,34} + \Phi^*_{34;\,12}|^2 (n_1 + 1)(n_2 + 1) n_3 n_4 \times \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \Delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4). \tag{10.8}$$

The matrix element for the splitting of one spin wave into three spin waves, which is described by the Hamiltonian $\mathcal{H}_S^{(4)}$, has the form

$$(n_1, n_2, n_3, n_4 | \mathcal{H}_s^{(4)} | n_1 - 1, n_2 + 1, n_3 + 1, n_4 + 1) = 6\Phi^*_{1;\,234} \sqrt{n_1 (n_2 + 1)(n_3 + 1)(n_4 + 1)}\, e^{\frac{i}{\hbar}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 - \varepsilon_1)t} \times \Delta(\mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4 - \mathbf{k}_1).$$

The probability for this process is

$$W^{n_1, n_2, n_3, n_4}_{n_1-1, n_2+1, n_3+1, n_4+1} = \frac{72\pi}{\hbar} |\Phi_{1;\,234}|^2 n_1 (n_2 + 1)(n_3 + 1)(n_4 + 1) \times \delta(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 - \varepsilon_1) \Delta(\mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4 - \mathbf{k}_1). \tag{10.8'}$$

Let us now determine the change in the numbers of spin waves because of scattering processes and splitting and fusion of spin waves as described by the Hamiltonian $\mathcal{H}_S^{(4)}$:

$$\dot{n}_1^c \equiv L^{(4)}_{\mathbf{k}_1} \{n\},$$

$$\begin{aligned} L^{(4)}_{\mathbf{k}_1} = \frac{2\pi}{\hbar} \sum_{234} \{ & 48 |\Phi_{12;\,34} + \Phi^*_{34;\,12}|^2 [(n_1 + 1)(n_2 + 1) n_3 n_4 \\ & - n_1 n_2 (n_3 + 1)(n_4 + 1)]\, \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \\ & \times \Delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \\ & + 36 |\Phi_{1;\,234}|^2 [(n_1 + 1) n_2 n_3 n_4 - n_1 (n_2 + 1)(n_3 + 1)(n_4 + 1)] \\ & \times \delta(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \Delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \\ & + 108 |\Phi_{4;\,231}|^2 [(n_1 + 1)(n_2 + 1)(n_3 + 1) n_4 - n_1 n_2 n_3 (n_4 + 1)] \\ & \times \delta(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4) \Delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_4) \}. \end{aligned} \tag{10.9}$$

We represent the collision operator $L^{(4)}_{\mathbf{k}}\{n\}$ in the form:

$$L^{(4)}_{\mathbf{k}}\{n\} = L^{(e)}_{\mathbf{k}}\{n\} + L^{(r)}_{\mathbf{k}}\{n\}, \tag{10.10}$$

$$\begin{aligned} L^{(e)}_{\mathbf{k}}\{n\} = \frac{6\pi\mu^4}{\hbar V^2} \sum_{2,3,4} & \alpha(\mathbf{k}_1\mathbf{k}_2 + \mathbf{k}_3\mathbf{k}_4) \Big[\alpha(\mathbf{k}_1\mathbf{k}_2 + \mathbf{k}_3\mathbf{k}_4) + 4\beta \\ & + 2\pi(\sin^2\theta_1 + \sin^2\theta_2 + \sin^2\theta_3 + \sin^2\theta_4) \\ & - 4\pi \left(\frac{(\mathbf{k}_1 - \mathbf{k}_4)_z^2}{(\mathbf{k}_1 - \mathbf{k}_4)^2} + \frac{(\mathbf{k}_1 - \mathbf{k}_3)_z^2}{(\mathbf{k}_1 - \mathbf{k}_3)^2} + \frac{(\mathbf{k}_2 - \mathbf{k}_4)_z^2}{(\mathbf{k}_2 - \mathbf{k}_4)^2} + \frac{(\mathbf{k}_2 - \mathbf{k}_3)_z^2}{(\mathbf{k}_2 - \mathbf{k}_3)^2} \right) \Big] \\ & \times [(n_1 + 1)(n_2 + 1) n_3 n_4 - n_1 n_2 (n_3 + 1)(n_4 + 1)] \\ & \times \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \Delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4), \end{aligned} \tag{10.10'}$$

$$\begin{aligned} L^{(r)}_{\mathbf{k}_1}\{n\} = \frac{6\pi\mu^4}{\hbar V^2} \sum_{234} & \Big[2\beta + \pi(\sin^2\theta_1 + \sin^2\theta_2 + \sin^2\theta_3 + \sin^2\theta_4) \\ & - 2\pi \left(\frac{(\mathbf{k}_1 - \mathbf{k}_4)_z^2}{(\mathbf{k}_1 - \mathbf{k}_4)^2} + \frac{(\mathbf{k}_1 - \mathbf{k}_3)_z^2}{(\mathbf{k}_1 - \mathbf{k}_3)^2} + \frac{(\mathbf{k}_2 - \mathbf{k}_4)_z^2}{(\mathbf{k}_2 - \mathbf{k}_4)^2} + \frac{(\mathbf{k}_2 - \mathbf{k}_3)_z^2}{(\mathbf{k}_2 - \mathbf{k}_3)^2} \right) \Big]^2 \\ & \times [(n_1 + 1)(n_2 + 1) n_3 n_4 - (n_3 + 1)(n_4 + 1) n_1 n_2] \\ & \times \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \Delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \\ + \frac{2\pi^3\mu^4}{\hbar V^2} \sum_{234} & \{ |\sin^2\theta_2 e^{-2i\varphi_2} + \sin^2\theta_3 e^{-2i\varphi_3} + \sin^2\theta_4 e^{-2i\varphi_4}|^2 \\ & \times [(n_1 + 1) n_2 n_3 n_4 - n_1 (n_2 + 1)(n_3 + 1)(n_4 + 1)] \\ & \times \delta(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 - \varepsilon_1) \Delta(\mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4 - \mathbf{k}_1) + 3|\sin^2\theta_1 e^{-2i\varphi_1} \\ & + \sin^2\theta_2 e^{-2i\varphi_2} + \sin^2\theta_3 e^{-2i\varphi_3}|^2 \\ & \times [(n_1 + 1)(n_2 + 1)(n_3 + 1) n_4 - n_1 n_2 n_3 (n_4 + 1)] \\ & \times \delta(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4) \Delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_4) \}. \end{aligned} \tag{10.10''}$$

We note that in the temperature range $T \gg \mu M_0$ the operator $L^{(r)}_{\mathbf{k}}\{n\}$ can be regarded as a small correction to the operator $L_{\mathbf{k}}^{(e)}\{n\}$, since

$$\alpha k^2 \sim \frac{\varepsilon_{\mathbf{k}}}{\mu M_0} \sim \frac{T}{\mu M_0} \gg 2\pi\beta.$$

Therefore, in determining the average probability of scattering of a spin wave by a spin wave we can neglect the operator $L_{\mathbf{k}}^{(r)}\{n\}$, and in the operator $L_{\mathbf{k}}^{(e)}\{n\}$ we can omit terms associated with the anisotropy energy and the magnetic dipole interaction. Inclusion of this operator is necessary when one treats relaxation of the magnetic moment (cf. Sec. 12).

Using the expression (10.10′) for $L_{\mathbf{k}}^{(e)}\{n\}$ and proceeding in the same way as was done in finding $1/\tau^{(3)}$, we can determine the mean time for scattering of a spin wave by a spin wave:

$$\frac{1}{\tau^{(e)}} \equiv w^{(e)} = \frac{6\pi}{\hbar}\frac{\alpha^2\mu^4}{V^2}\sum_{1234}(\mathbf{k}_1\mathbf{k}_2+\mathbf{k}_3\mathbf{k}_4)^2 n_2^0 n_4^0 (n_3^0+1)$$
$$\times\Delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\,\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)\frac{1}{\sum_{\mathbf{k}} n_{\mathbf{k}}^0}. \tag{10.11}$$

Changing from summation over **k** to integration, we obtain, except for a numerical factor of order unity:[2,3]

$$w^{(e)} \approx \frac{\theta_C}{\hbar}\left(\frac{T}{\theta_C}\right)^4. \tag{10.11′}$$

The processes of scattering of spin waves by spin waves, which are associated with the exchange interaction, cannot change the total magnetic moment of the ferromagnet, since the magnetic moment of the body commutes with the exchange energy. These processes need not be included along with the splitting and fusion of spin waves, if $w^{(3)} \gg w^{(e)}$. If, however, $w^{(e)} \gg w^{(3)}$, as is the case for $T \gg \theta_C\left(\frac{\mu M_0}{\theta_C}\right)^{4/7}$, their role becomes important. In fact, just because of these processes, there is established a quasi-equilibrium Bose distribution of the spin waves (with a non-equilibrium magnetic moment) which gradually, because of the processes of splitting and fusion of spin waves and also of scattering caused by the anisotropy energy, goes over into the equilibrium distribution (cf. Sec. 12).

11. Interaction of Spin Waves with Lattice Vibrations

We now proceed to investigate processes of interaction between spin waves and lattice vibrations. In order to find a Hamiltonian describing these interactions, we consider the magnetostriction part of the total Hamiltonian of the ferromagnet, containing the spin variables and the deformation tensor:

$$\mathscr{H}'_{sl} = \int_V \gamma_{ik}(\mathbf{M})\,u_{ik}\,dv + \int_V \gamma_{iklmrs}(\mathbf{M})\frac{\partial M_r}{\partial x_i}\frac{\partial M_s}{\partial x_k}u_{lm}\,dv, \tag{11.1}$$

where $\gamma_{ik}(\mathbf{M})$ and $\gamma_{iklmrs}(\mathbf{M})$ are the tensors of the magnetostriction constants: the first of these describes magnetoelastic effects for homogeneous magnetization, and the second for inhomogeneous magnetization.

The terms written here exhaust all combinations which contain the deformation tensor u_{ik} linearly, and consequently contain the operators for creation and absorption of phonons linearly. We can restrict ourselves to these terms in treating low temperatures.

As compared to the Hamiltonian (6.1), we have here written an additional term $\gamma_{iklmrs}\frac{\partial M_r}{\partial x_i}\frac{\partial M_s}{\partial x_k}u_{lm}$, containing the derivatives of the magnetic moment quadratically and the deformation tensor linearly. Since this is a term of third order (with respect to M_i and u_{ik}), this term was omitted in considering coupled magnetoacoustic vibrations in Sec. 5.

In the expression $\gamma_{iklmrs}\frac{\partial M_r}{\partial x_i}\frac{\partial M_s}{\partial x_k}u_{lm}$ for the inhomogeneous magnetostrictive energy, it is sufficient to keep only the main part, which is due to exchange interaction, and omit the part which is of relativistic origin. The exchange part of the magnetostriction energy can be obtained from the exchange energy (1.2) if in it we expand the quantity $J(r_{lm})$, which depends on the distance between atoms, in a series of powers of the deformation tensor.

The term in this expansion which is linear with respect to u_{ik} has the form $\gamma_{iklm}\frac{\partial M_r}{\partial x_i}\frac{\partial M_r}{\partial x_k}u_{lm}$. This expression, which is invariant with respect to rotation of the moment **M**, is the exchange part of the magnetostrictive energy in which we are interested; it is characterized by a tensor γ_{iklm} which is fourth rank and not of sixth rank, and whose components can be assumed to be independent of **M**. If, for simplicity, we assume that the body is isotropic, we can write this tensor in the form:

$$\gamma_{iklm} = \frac{1}{2}\beta_1\frac{\theta_C a^2}{\mu M_0}(\delta_{il}\delta_{km}+\delta_{im}\delta_{kl}) + \beta_2\frac{\theta_C a^2}{\mu M_0}\delta_{ik}\delta_{lm}, \tag{11.2}$$

where $\beta_{1,2}$ are quantities of order unity. In this case the tensor $\gamma_{ik}(\mathbf{M})$ has the form:

$$\gamma_{ik}(\mathbf{M}) = \gamma M_i M_k + \gamma_0(M^2)\delta_{ik}, \tag{11.2′}$$

where the values of γ and γ_0 can be taken for $\mathbf{M} = \mathbf{M}_0$.

Substituting (11.2) and (11.2′) in the Hamiltonian (11.1), and noting that $\int u_{ii}\,dv = 0$, we obtain in the isotropic case, after expanding the moment **M** in powers of the deviation **m**:

$$\mathscr{H}'_{sl} = 2\gamma M_0\int_V (m_x u_{xz} + m_y u_{yz})\,dv$$
$$+\frac{1}{4}\gamma\int_V \{(m^+)^2(u_{xx}-u_{yy}-2iu_{xy}) - (m^-)^2(u_{xx}-u_{yy}+2iu_{xy})$$
$$+2m^+m^-(u_{xx}+u_{yy}) + 8M_0 m_z u_{zz}\}\,dv$$
$$+\frac{\theta_C a^2}{\mu M_0}\int_V\left(\beta_1\frac{\partial m^+}{\partial x_i}\frac{\partial m^-}{\partial x_k}u_{ik} + \beta_2\frac{\partial m^+}{\partial x_k}\frac{\partial m^-}{\partial x_k}u_{ll}\right)dv, \tag{11.3}$$

where we have limited ourselves to terms of first and second order in **m**.

The first term, containing the oscillations of the magnetic moment and the deformation tensor linearly, is of no importance in the investigation of processes of interaction of spin waves with lattice vibrations, and can therefore be omitted. In fact, this term must be included in the fundamental Hamiltonian of the ferromagnet, which contains the deviation of the magnetic moment and the deformation tensor quadratically. It leads to the existence of coupled magnetoacoustic waves. Waves of this type were considered by us in Sec. 5, and we saw there that the coupling between the magnetic and acoustic waves is determined by a small parameter. Therefore, in studying interactions between spin waves and lattice vibrations we need not include the formation of coupled waves, and can start from simple spin and simple sound waves.

Omitting the first term in (11.3), we get the Hamiltonian for the interaction of spin waves with the lattice, which we shall denote by $\mathcal{H}_{sl}$.

In order to determine the matrix elements for transitions caused by the Hamiltonian $\mathcal{H}_{sl}$, we must relate the deformation tensor u_{ik} to the operators for creation and annihilation of phonons. For this purpose we expand the operator for the displacement $\mathbf{u}(\mathbf{r}, t)$ in plane waves:

$$\mathbf{u}(\mathbf{r}, t) = \left(\frac{\hbar}{2\varrho V}\right)^{1/2} \sum_{\mathbf{f},\, s} \frac{\mathbf{e}_{\mathbf{f}s}}{\sqrt{\omega_{\mathbf{f}s}}} \left(b_{\mathbf{f}s} e^{i\mathbf{f}\mathbf{r}} + b_{\mathbf{f}s}^{+} e^{-i\mathbf{f}\mathbf{r}}\right), \qquad (11.4)$$

where $\mathbf{f}$ and $\omega_{\mathbf{f}s}$ are the wave vector and frequency of the vibration with polarization s, $\mathbf{e}_{\mathbf{f}s}$ is the unit polarization vector, $b_{\mathbf{f}s}^{+}$ and $b_{\mathbf{f}s}$ are the operators for creation and annihilation of a phonon with wave vector $\mathbf{f}$ and polarization s, and ρ is the density of the material.

The matrix elements of the operators for creation and annihilation of a phonon with wave vector $\mathbf{f}$ are equal to

$$(N_{\mathbf{f}s} - 1 \,|\, b_{\mathbf{f}s} \,|\, N_{\mathbf{f}s}) = \sqrt{N_{\mathbf{f}s}}\, e^{-i\omega_{\mathbf{f}s} t},$$
$$(N_{\mathbf{f}s} \,|\, b_{\mathbf{f}s}^{+} \,|\, N_{\mathbf{f}s} - 1) = \sqrt{N_{\mathbf{f}s}}\, e^{i\omega_{\mathbf{f}s} t}, \qquad (11.4')$$

where $N_{\mathbf{f}s}$ is the number of phonons of polarization s with wave vector $\mathbf{f}$.

Using expression (11.4) for the displacement operator and the expression (3.6) relating $\mathbf{m}$ to the operators for creation and annihilation of spin waves, we rewrite the Hamiltonian $\mathcal{H}_{sl}$ in the form

$$\mathcal{H}_{sl} = \sum_{123} \{\Psi_{12;\,3} a_1^{+} a_2^{+} b_3 \Delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{f}_3)$$
$$+ \Psi_{1;\,23} a_1^{+} a_2^{+} b_3 \Delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{f}_3)\}, \qquad (11.5)$$

where

$$\Psi_{12;\,3} = \frac{i}{2V^{1/2}} \gamma\mu M_0 \left(\frac{\hbar}{2\varrho\omega_3}\right)^{1/2} e_3^{-} f_3^{-},$$

$$\Psi_{1;\,23} = i\theta_C a^2 \sqrt{\frac{\hbar}{2\varrho V\omega_3}} [\beta_1 (\mathbf{e}_3\mathbf{k}_1)(\mathbf{f}_3\mathbf{k}_2)$$
$$+ \beta_1 (\mathbf{e}_3\mathbf{k}_2)(\mathbf{f}_3\mathbf{k}_1) + 2\beta_2 (\mathbf{k}_1\mathbf{k}_2)(\mathbf{e}_3\mathbf{f}_3)]$$
$$+ i\gamma\mu M_0 \sqrt{\frac{\hbar}{2\varrho V\omega_3}} (\mathbf{f}_3\mathbf{e}_3 - 3 f_{3z} e_{3z}).$$

(The subscript "3" serves to indicate the wave vector and polarization of the phonon.)

If we consider temperatures $T \gg 2\pi\mu M_0$, the second term in $\Psi_{1;23}$ can be neglected, since the ratio of the second term to the first term is equal in order of magnitude to

$$\frac{\theta_C (ak)^2}{\mu M_0} \sim \frac{T}{\mu M_0} \gg 1.$$

The Hamiltonian (11.5) obviously describes processes of creation and absorption of a phonon by the spin wave, as well as the process of transformation of a phonon into two spin waves. To the process of creation of a phonon by the spin wave there corresponds the momentum conservation law:

$$\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{f}_3,$$

and to the process of transformation of two spin waves into a phonon, the conservation law:

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{f}_3.$$

Let us first consider the creation and absorption of a phonon by a spin wave. These processes can be treated as Cerenkov radiation (absorption) of sound waves. The radiating particle here is the spin wave. Since the dispersion law for spin waves is the same (when we neglect magnetic interaction) as the dispersion law for ordinary, nonrelativistic free particles with mass $m_0 = \frac{\hbar^2}{2\theta_C a^2}$, the condition for radiating the phonon is that the velocity of the spin wave v must exceed the sound velocity s. Noting that $v = \hbar k/m_0$, and expressing k in terms of the energy of the spin wave, we can express the condition for radiation in the form:

$$\varepsilon_k \gg \frac{\theta_D^2}{4\theta_C},$$

where $\theta_D = \hbar s/a$ is the Debye temperature.

The matrix element for the process of creation of a phonon by a spin wave has the form

$$(n_1,\, n_2,\, N_3 \,|\, \mathcal{H}_{sl} \,|\, n_1 + 1,\, n_2 - 1,\, N_3 + 1)$$
$$= \Psi_{1,\,23}^{*} \sqrt{(n_1 + 1)\, n_2 (N_3 + 1)}\, \Delta(\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{f}_3)\, e^{\frac{i}{\hbar}(\varepsilon_1 - \varepsilon_2 + \hbar\omega_3) t},$$

according to (11.5), (1.11) and (11.4′), while the probability for "Cerenkov" radiation of the phonon is equal to

$$W_{n_1+1,\, n_2-1,\, N_3+1}^{n_1,\, n_2,\, N_3} = \frac{2\pi}{\hbar} |\Psi_{1;\,23}|^2 (n_1 + 1)$$
$$\times (N_3 + 1)\, n_2 \delta(\varepsilon_1 - \varepsilon_2 + \hbar\omega_3)\, \Delta(\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{f}_3). \qquad (11.6)$$

Let us find the change in the numbers of sound waves produced by processes of creation and absorption of a phonon by the spin wave:

$$\dot{n}_1^{c} \equiv L_{\mathbf{k}_1}^{(ls)} \{n,\, N\},$$

$$L_{\mathbf{k}_1}^{(ls)}(n, N) = \frac{2\pi}{\hbar} \sum_{23} \{|\Psi_{1;\,23}|^2 [(n_1 + 1)\, n_2 N_3 - n_1 (n_2 + 1)(N_3 + 1)]$$
$$\times \delta(\varepsilon_1 - \varepsilon_2 - \hbar\omega_3)\, \Delta(\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{f}_3) + |\Psi_{1;\,23}|^2 [(n_1 + 1)\, n_2$$
$$\times (N_3 + 1) - n_1 (n_2 + 1) N_3]\, \delta(\varepsilon_2 - \varepsilon_1 - \hbar\omega_3)\, \Delta(\mathbf{k}_2 - \mathbf{k}_1 - \mathbf{f}_3)\}. \qquad (11.7)$$

Proceeding in a fashion analogous to that of Sec. 10, we can determine the mean time for emission of longitudinal and transverse phonons by the spin wave:

$$\frac{1}{\tau_l} = -\frac{\sum\limits_{\mathbf{k}}\left(\frac{\delta L_{\mathbf{k}}^{(ls)}}{\delta n_{\mathbf{k}}}\right)_{0,\,l} n_{\mathbf{k}}^0}{\sum\limits_{\mathbf{k}} n_{\mathbf{k}}^0} = \frac{2\pi}{\hbar}\sum_{12\mathbf{f}} |\Psi_{12;\,\mathbf{f}l}|^2 [(N_{\mathbf{f}l}^0+1)\,n_1^0$$

$$+ N_{\mathbf{f}l}^0 n_2^0]\,\delta(\varepsilon_1-\varepsilon_2-\hbar\omega_l)\,\frac{1}{\sum\limits_{\mathbf{k}} n_{\mathbf{k}}^0},$$

$$\frac{1}{\tau_t} = -\frac{\sum\limits_{\mathbf{k}}\left(\frac{\delta L_{\mathbf{k}}^{(ls)}}{\delta n_{\mathbf{k}}}\right)_{0,\,t} n_{\mathbf{k}}^0}{\sum\limits_{\mathbf{k}} n_{\mathbf{k}}^0} = \frac{2\pi}{\hbar}\sum_{12\mathbf{f}s_t} |\Psi_{12;\,\mathbf{f}s_t}|^2 [(N_{\mathbf{f}t}^0+1)\,n_1^0$$

$$+ N_{\mathbf{f}t}^0 n_2^0]\,\delta(\varepsilon_1-\varepsilon_2-\hbar\omega_t)\,\frac{1}{\sum\limits_{\mathbf{k}} n_{\mathbf{k}}^0},$$

where $N_{\mathbf{f}s}^0 = \left(\exp\frac{\hbar\omega_{\mathbf{f}s}}{T} - 1\right)^{-1}$.

Changing from summation over **k** and **f** to integration, we obtain:

$$\frac{1}{\tau_l} \equiv w_l = \frac{1}{\pi^{3/2}\zeta\left(\frac{3}{2}\right)}\frac{\theta_C}{\hbar}\left(\frac{\theta_C}{\theta_l}\right)^{1/2}\frac{T}{\varrho a^3 s_l^2}\left(\frac{T}{\theta_l}\right)^{11/2} J_l(\alpha_l), \quad (11.8)$$

$$\frac{1}{\tau_t} \equiv w_t = \frac{\beta_1^2}{4\pi^{3/2}\zeta\left(\frac{3}{2}\right)}\frac{\theta_C}{\hbar}\frac{T}{\varrho a^3 s_t^2}\left(\frac{T}{\theta_C}\right)^{5/2} J_t(\alpha_t), \quad (11.8')$$

where

$$J_l(\alpha_l) = \int_0^\infty \frac{y^2\,dy}{e^y-1}\int_{\frac{(y+\alpha_l)^2}{4\alpha_l}}^\infty \Big[\frac{1}{4}\beta_1(y+\alpha_l)^2 + \beta_2\alpha_l x$$

$$-\frac{1}{2}(\beta_1+\beta_2)\,y\,(y+\alpha_l)\Big]^2\left(\frac{e^y}{e^x-1}+\frac{1}{e^{x-y}-1}\right)dx,$$

$$J_t(\alpha_t) = \int_0^\infty \frac{y^2\,dy}{e^y-1}\int_{\frac{(y+\alpha_t)^2}{4\alpha_t}}^\infty \left[x - \frac{(y+\alpha_t)^2}{4\alpha_t}\right]\left(\frac{e^y}{e^x-1}+\frac{1}{e^{x-y}-1}\right)dx$$

and

$$\alpha_l = \frac{\theta_l^2}{T\theta_C}, \quad \alpha_t = \frac{\theta_t^2}{T\theta_C}.$$

The integrals appearing here can be calculated for large and small values of the parameters α_l and α_t. As a result, we obtain:[1,4]

$$w_l = \begin{cases} 0.2\,(\beta_1+\beta_2)^2\,\frac{\theta_l}{\hbar}\,\frac{\theta_l}{\varrho a^3 s_l^2}\left(\frac{T}{\theta_C}\right)^{5/2} e^{-\frac{\theta_l^2}{4T\theta_C}}, & T \ll \frac{\theta_l^2}{\theta_C}, \\ 0.4\,(\beta_1^2+\beta_2^2+\beta_1\beta_2)\,\frac{\theta_C}{\hbar}\,\frac{T}{\varrho a^3 s_l^2}\left(\frac{T}{\theta_C}\right)^{5/2}, & T \gg \frac{\theta_l^2}{\theta_C}, \end{cases}$$

$$w_t = \begin{cases} 0.4\beta_1^2\,\frac{\theta_C}{\hbar}\,\frac{T}{\varrho a^3 s_l^2}\left(\frac{T}{\theta_C}\right)^{5/2} e^{-\frac{\theta_t^2}{4T\theta_C}}, & T \ll \frac{\theta_t^2}{\theta_C}, \\ 0.2\beta_1^2\,\frac{\theta_t}{\hbar}\,\frac{\theta_t}{\varrho a^3 s_t^2}\left(\frac{T}{\theta_C}\right)^{5/2}, & T \gg \frac{\theta_t^2}{\theta_C}. \end{cases} \quad (11.9)$$

Comparing the expressions obtained for w_l and w_t, we see that at low temperatures it is more probable to create phonons with transverse polarization ($\theta_t < \theta_l$, since $s_t < s_l$), while at high temperatures it is more probable to create phonons with longitudinal polarization.

If we are interested in the emission of a phonon with arbitrary polarization, we obtain for the mean probability for such a process the expression:[1,4]

$$w_p = \begin{cases} 0.4\beta_1^2\,\frac{\theta_C}{\hbar}\,\frac{T}{\varrho a^3 s_t^2}\left(\frac{T}{\theta_C}\right)^{5/2} e^{-\frac{\theta_t^2}{4T\theta_C}}, & T \ll \frac{\theta_D^2}{\theta_C}, \\ 0.4\,(\beta_1^2+\beta_2^2+\beta_1\beta_2)\,\frac{\theta_C}{\hbar}\,\frac{T}{\varrho a^3 s_l^2}\left(\frac{T}{\theta_C}\right)^{5/2}, & T \gg \frac{\theta_D^2}{\theta_C}. \end{cases} \quad (11.10)$$

The exponential dependence on temperature of the mean probability for emission of a phonon for $T \ll \theta_t^2/\theta_C$ is related to the fact that in the radiation processes there can participate only those spin waves whose energies are greater than $\theta_D^2/4\theta_C$.

Since at low temperatures this probability contains an exponentially small factor, we must, along with the processes of emission of a phonon by the spin wave, also consider processes of fusion of two spin waves into a phonon and splitting of a phonon into two spin waves. As is clear from the expression for the Hamiltonian (11.5), these processes are not described by an exchange interaction, but rather by the homogeneous magnetostrictive interaction. Therefore their probability is generally much less than the probability w_p for the processes of creation and absorption of a phonon by a spin wave, and may be comparable with w_p only at sufficiently low temperatures.

The matrix element for the transformation of two spin waves into a phonon has the form:

$$(n_1,\ n_2,\ N_3)\,|\,\mathscr{H}_{sl}\,|\,n_1-1,\ n_2-1,\ N_3+1)$$

$$= 2\Psi_{12;\,3}^{*}\sqrt{n_1 n_2 (N_3+1)}\,\Delta(\mathbf{f}_3-\mathbf{k}_1-\mathbf{k}_2)\,e^{\frac{i}{\hbar}(\hbar\omega_3-\varepsilon_1-\varepsilon_2)t},$$

while the probability for this process is

$$W_{n_1-1,\,n_2-1,\,N_3+1}^{n_1,\,n_2,\,N_3}$$

$$= \frac{8\pi}{\hbar}|\Psi_{12;\,3}|^2 n_1 n_2 (N_3+1)\,\Delta(\mathbf{f}_3-\mathbf{k}_1-\mathbf{k}_2)\,\delta(\varepsilon_1+\varepsilon_2-\hbar\omega_3).$$

From this we can, as before, find the average probability for fusion of two spin waves into a phonon:

$$w_p' = \begin{cases} \frac{\mu M_0}{\hbar}\,\frac{\gamma^2\mu M_0}{\varrho a^3 s_t^2}\left(\frac{T}{\theta_t}\right)^3, & T \ll \frac{\theta_D^2}{\theta_C}, \\ \frac{\hbar}{\varrho a^5}\left(\frac{\gamma\mu M_0}{\theta_C}\right)^2\left(\frac{T}{\theta_C}\right)^{1/2}, & T \gg \frac{\theta_D^2}{\theta_C}. \end{cases} \quad (11.11)$$

Comparison of formulas (11.10) and (11.11) shows that the inequality $w_p \gg w_p'$ holds down to temperatures of the order of a degree.

Let us now compare the probabilities $w^{(e)}$ and $w^{(3)}$ which characterize the intensities of the interactions of spin waves with one another with the probabilities w_p and w_p', which characterize the intensity of the interaction of spin waves with the lattice. It is easy to see that $w^{(e)} \gg w_p$, if $T \gg \theta_C\left(\frac{\mu M_0}{\theta_C}\right)^{4/7}$, and $w^{(3)}$

$\gg w_p$, if $\theta_C\left(\frac{\mu M_0}{\theta_C}\right)^{4/7} \gg T \gg 2\pi\mu M_0$. Thus, the interaction of spin waves with one another is stronger than the interaction of spin waves with the lattice. From this we may conclude that equilibrium in the spin wave system can be reached faster than equilibrium between the spin waves and the lattice. Therefore, the temperature of spin waves and lattice may, in general, differ from one another.[1] We shall treat the process of equilibrating of these temperatures, together with the problem of relaxation of the magnetic moment, in Sec. 12.

12. Relaxation of the Magnetic Moment in Ferrodielectrics

The probabilities we have found for processes of interaction of spin waves with one another and with phonons enables us to explain how relaxation of the magnetic moment proceeds in ferrodielectrics.[3]

Above, we determined the equilibrium value of the magnetic moment at a given temperature. The problem of relaxation of the magnetic moment consists in explaining how the non-equilibrium value of the magnetic moment approaches its equilibrium value.

We shall, to start with, assume that the anisotropy constant or the external magnetic field $\mathbf{H}_0$ is sufficiently large, and begin with the treatment of temperatures $\theta_C \gg T \gg \theta_C\left(\frac{\mu M_0}{\theta_C}\right)^{4/7}$. At such temperatures the strongest interactions are, as we have seen, the exchange interactions of the spin waves with one another. This results in the establishment of a Bose distribution of the spin waves. The exchange interaction does not change the magnetic moment of the system; therefore, the Bose distribution which is established does not, in general, correspond to the equilibrium value of the magnetic moment. On the contrary, since the Hamiltonian for the exchange interaction commutes with the total magnetic moment of the system $\mathfrak{M}$, the latter can be arbitrary both in magnitude and direction. The transition to the equilibrium value of the moment is caused by interactions which make possible a change in the magnetic moment of the system; i.e., it is due to magnetic dipole interaction, anisotropy energy, and the interaction between the spin waves and phonons. All these forms of interaction in this range of temperatures are weak compared with the exchange interaction between spin waves; therefore, the relaxation of the magnetic moment proceeds slowly compared with the process of establishing the Bose distribution with the given value of magnetic moment $\mathfrak{M}$.

In order to determine the time of relaxation of the magnetic moment and the time for equilibrating the temperatures of the spin waves T_s and of the lattice T_l, which generally may differ from one another, we shall start from the kinetic equation for the spin waves

$$\dot{n}_{\mathbf{k}} = \dot{n}_{\mathbf{k}}^{c} \equiv L_{\mathbf{k}}\{n, N\}, \tag{12.1}$$

where $L_{\mathbf{k}}\{n, N\}$ is the total operator for collision of spin waves:

$$L_{\mathbf{k}}\{n, N\} = L_{\mathbf{k}}^{(e)}\{n\} + L_{\mathbf{k}}^{(3)}\{n\} + L_{\mathbf{k}}^{(r)}\{n\} + L_{\mathbf{k}}^{(ls)}\{n, N\}$$

[The operators $L_{\mathbf{k}}^{(e)}\{n\}$, $L_{\mathbf{k}}^{(3)}\{n\}$, $L_{\mathbf{k}}^{(r)}\{n\}$ and $L_{\mathbf{k}}^{(ls)}\{n, N\}$ are related to the Hamiltonians $\mathcal{H}_S^{(4)}$, $\mathcal{H}_S^{(3)}$, $\mathcal{H}_{lS}$, and were defined in Secs. 9 and 10.]

In the temperature range $\theta_C \gg T \gg \theta_C\left(\frac{\mu M_0}{\theta_C}\right)^{4/7}$, the inequality $w^{(e)} \gg w^{(3)}$ holds, so that the largest term in $L_{\mathbf{k}}\{n, N\}$ will be $L_{\mathbf{k}}^{(e)}\{n\}$. The remaining terms

$$L_{\mathbf{k}}^{(3)}\{n\} + L_{\mathbf{k}}^{(r)}\{n\} + L_{\mathbf{k}}^{(ls)}\{n, N\} \equiv L'_{\mathbf{k}}\{n, N\}$$

can be treated as a small perturbation, and we can in the first approximation start in place of (12.1) from the equation

$$L_{\mathbf{k}}^{(e)}\{n\} = 0. \tag{12.2}$$

The general solution of this equation has the form:

$$n_{\mathbf{k}} = \begin{cases} n_0, & \mathbf{k} = 0, \\ \left(e^{\frac{\varepsilon_k - \zeta}{T_s}} - 1\right)^{-1}, & \mathbf{k} \neq 0, \end{cases} \tag{12.3}$$

where n_0 and ζ are arbitrary constants. They can be related to the initial values of the square of the magnetic moment perpendicular to the axis of easiest magnetization:

$$\mathfrak{M}^2 = \left(\int \mathbf{M}\, dv\right)^2 = (M_0V)^2 - 2\mu M_0 V \sum_{\mathbf{k}\neq 0} n_{\mathbf{k}},$$
$$\mathfrak{M}_\perp^2 = \mathfrak{M}_x^2 + \mathfrak{M}_y^2 = 2\mu M_0 V n_0. \tag{12.4}$$

We shall now take account of the weak interactions which are described in the kinetic equation by the operators $L_{\mathbf{k}}^{(3)}$, $L_{\mathbf{k}}^{(r)}$, and $L_{\mathbf{k}}^{(ls)}$. The distribution (12.3), which satisfies equation (12.2), will then no longer satisfy equation (12.1). Since, however, $w^{(e)} \gg w^{(3)}$, w_p, the distribution (12.3) with the slowly varying parameters ζ, n_0, T_s can approximately satisfy equation (12.1). If the body is thermally isolated from the surrounding medium, then the temperature of the phonon should also be treated as a slowly varying function of the time.

We note that, although $L_{\mathbf{k}}^{(r)}\{n\}$ is much less than $L_{\mathbf{k}}^{(3)}\{n\}$, the operator $L_{\mathbf{k}}^{(r)}\{n\}$ must nevertheless be retained in the kinetic equation (12.1). In fact, the time of relaxation of $\mathfrak{M}_\perp$ represents, as we see from formulas (12.4), the time for relaxation of spin waves with wave vector equal to zero. On the other hand, processes of fusion of two spin waves into one and splitting of one spin wave into two cannot change n_0, if the anisotropy constant or the external magnetic field are sufficiently large. Let us consider, for example, the splitting of a spin wave with wave vector $\mathbf{k} = 0$ into two spin waves with wave vectors $\mathbf{k}$ and

$-\mathbf{k}$. This process is impossible, since the energy conservation law:

$$\varepsilon_0 = 2\varepsilon_k,$$

where $\epsilon_k = \mu M_0\left(\alpha k^2 + \beta + \frac{H_0}{M_0}\right)$, is not satisfied in this case, as one can easily see. In precisely the same way it is impossible to have a process of fusion of a spin wave with wave vector $\mathbf{k}$ with a spin wave with wave vector 0. Therefore in calculating the time of relaxation of $\mathfrak{M}_\perp$ we must include processes involving a larger number of spin waves, where these processes must not be associated with conservation of the magnetic moment. Such processes are the scattering of a spin wave by a spin wave, caused by the anisotropy energy and the magnetic dipole interaction, as well as the fusion of three spin waves into one. *All these processes are described in the kinetic equation by the operator* $L_{\mathbf{k}}^{(r)}\{n\}$.

Our problem is to explain how $\mathfrak{M}$ and $\mathfrak{M}_\perp^2$ vary with time. To do this, according to (12.4), we must find the form of the functions $\zeta(t)$, $n_0(t)$ and $T_s(t)$. The quantity ζ is the chemical potential of the spin waves and is determined by the total number of such waves. Therefore, to find $\dot{\zeta}(t)$ we should use the equation

$$\sum_{\mathbf{k}} \dot{n}_{\mathbf{k}} = \sum_{\mathbf{k}} L_{\mathbf{k}}\{n, N\}. \qquad (12.1')$$

To obtain the complete system of equations which determine all the quantities ζ, n_0, T_s and T_l we *must add to this equation the equations:*

$$\left.\begin{aligned} \dot{n}_0 &= L_0\{n, N\}, \\ \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \dot{n}_{\mathbf{k}} + \sum_{\mathbf{f}s} \hbar\omega_{\mathbf{f}s} \dot{N}_{\mathbf{f}s} &= 0, \\ \sum_{\mathbf{f}s} \hbar\omega_{\mathbf{f}s} \dot{N}_{\mathbf{f}s} &= -\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} L_{\mathbf{k}}\{n\}, \end{aligned}\right\} \qquad (12.1'')$$

where

$$N_{\mathbf{f}s} = \frac{1}{e^{\frac{\hbar\omega_{\mathbf{f}s}}{T_l}} - 1}.$$

Substituting into these equations the expressions for the collision operator $L_{\mathbf{k}}\{n, N\}$, and linearizing with respect to the small quantities ζ, $\Delta T = T_s - T_l$ and $\eta = \epsilon_0(n_0/N)$, where N is the total number of atoms in the body (the quantity n_0/N is assumed to be small* but finite for $N \to \infty$), we obtain:

$$\left.\begin{aligned} \Delta\dot{T} + G_1\dot{\zeta} - \frac{1}{C_l}\dot{\eta} &= B_{\zeta\zeta}\zeta + B_{\zeta\eta}\eta + B_{\zeta T}\Delta T, \\ \Delta\dot{T} + G_2\dot{\zeta} + \frac{1}{C_s}\dot{\eta} &= B_{T\zeta}\zeta + B_{T\eta}\eta + B_{TT}\Delta T, \\ \dot{\eta} &= B_{\eta\eta}\eta, \end{aligned}\right\} \qquad (12.5)$$

where

$$G_1 = -\frac{1}{C_l}\left\{\frac{T}{N}\sum_{\mathbf{k}}\frac{\partial n_k^0}{\partial T} + (C_s + C_l)\frac{\sum_{\mathbf{k}}\frac{\partial n_k^0}{\partial \varepsilon_{\mathbf{k}}}}{\sum_{\mathbf{k}}\frac{\partial n_{\mathbf{k}}^0}{\partial T}}\right\}, \quad G_2 = \frac{T}{N}\frac{1}{C_l}\sum_{\mathbf{k}}\frac{\partial n_k^0}{\partial T},$$

$$B_{\zeta\zeta} = A\frac{T}{G_2}\sum_{\mathbf{k}}\left(\frac{\partial L'_{\mathbf{k}}}{\partial\zeta}\right)_0, \quad B_{T\zeta} = A\sum_{\mathbf{k}}\varepsilon_k\left(\frac{\partial L'_k}{\partial\zeta}\right)_0,$$

$$B_{\zeta\eta} = A\frac{T}{G_2}\sum_{\mathbf{k}}\left(\frac{\partial L'_{\mathbf{k}}}{\partial\eta}\right)_0, \quad B_{T\eta} = A\sum_{\mathbf{k}}\varepsilon_k\left(\frac{\partial L'_k}{\partial\eta}\right)_0,$$

$$B_{\zeta T} = A\frac{T}{G_2}\sum_{\mathbf{k}}\left(\frac{\partial L'_{\mathbf{k}}}{\partial\Delta T}\right)_0, \quad B_{TT} = A\sum_{\mathbf{k}}\varepsilon_k\left(\frac{\partial L'_k}{\partial\Delta T}\right)_0,$$

$A = \frac{C_s + C_l}{C_s C_l}\frac{1}{N}$; C_s and C_l are the specific heats of the spins and lattice, per atom; $n_k^0 = \left(\exp\frac{\epsilon_k}{T} - 1\right)^{-1}$, $\left(\frac{\partial L'_k}{\partial\zeta}\right)_0$, $\left(\frac{\partial L'_k}{\partial\eta}\right)_0$ and $\left(\frac{\partial L'_k}{\partial\Delta T}\right)_0$ are coefficients in the expansion

$$L'_{\mathbf{k}}\{n, N\} = \zeta\left(\frac{\partial L'_{\mathbf{k}}}{\partial\zeta}\right)_0 + \eta\left(\frac{\partial L'_{\mathbf{k}}}{\partial\eta}\right)_0 + \Delta T\left(\frac{\partial L'_{\mathbf{k}}}{\partial\Delta T}\right)_0.$$

(The subscript 0 means that the values of the derivatives of the collision integral are taken for $\zeta = \eta = \Delta T = 0$.)

Assuming that the quantities ζ, η and ΔT vary with time according to the law $e^{-\lambda t}$, we obtain the following three values for λ:

$$\begin{aligned} \lambda_{1,2} = \frac{1}{2(G_2 - G_1)}\{&G_1 B_{TT} + B_{\zeta\zeta} - 2G_2 B_{\zeta T} \\ &\mp [B_{\zeta\zeta}(B_{\zeta\zeta} - 4G_2 B_{\zeta T}) + G_1 B_{TT}(G_1 B_{TT} - 4G_2 B_{\zeta T}) \\ &+ 4G_1 G_2 B_{\zeta T}^2 - 2B_{\zeta\zeta} B_{TT}(G_1 - 2G_2)]^{1/2}\}, \\ \lambda_3 = -B_{\eta\eta}&. \end{aligned} \qquad (12.6)$$

The expressions for λ_1, λ_2 and λ_3 simplify greatly if $T \gg \theta_C\left(\frac{\mu M_0}{\theta_C}\right)^{4/7}$:

$$\left.\begin{aligned} &\lambda_1 \approx \frac{\mu M_0}{\hbar}\frac{\mu M_0}{\sqrt{\varepsilon_0\theta_C}}\frac{T}{\theta_C}, \\ &\lambda_2 \approx \begin{cases} \frac{\hbar}{\varrho a^5}\beta_1^2\left(\frac{T}{\theta_C}\right)^2 e^{-\frac{\theta_l^2}{4T\theta_C}}, & T \ll \frac{\theta_D^2}{\theta_C}, \\ \frac{\hbar}{\varrho a^5}[\beta_1^2 + 2(\beta_1 + \beta_2)^2]\left(\frac{T}{\theta_C}\right)^{5/2}, & T \gg \frac{\theta_D^2}{\theta_C}, \end{cases} \\ &\lambda_3 \approx \frac{\mu M_0}{\hbar}\frac{\mu M_0}{\theta_C}\left(\frac{T}{\theta_C}\right)^2, \end{aligned}\right\} \qquad (12.7)$$

where $\epsilon_0 = \mu(H_0 + \beta M_0)$.

We give a numerical estimate of the quantities λ_1, λ_2 and λ_3. Setting $\theta_C \sim 10^3$, $M_0 \approx 10^3$, $\rho \approx 10$, $a \approx 2 \times 10^{-8}$, $\theta_D \approx 10^2$, $T \approx 10^2$, we get $\lambda_1 \approx 10^7$, $\lambda_2 \approx 10^9$, $\lambda_3 \approx 10^5$. We note that for $T \gg \theta_D^2/\theta_C \sim 10$, the quantities $\lambda_{1,2,3}$ satisfy the inequality $\lambda_2 \gg \lambda_1 \gg \lambda_3$.

Now let us determine $\mathfrak{M}(t)$ and $\mathfrak{M}_\perp^2(t)$. Linearizing the expression (11.4) for $\mathfrak{M}$ with respect to ΔT, ζ, and η we obtain

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*Smallness of n_0/N corresponds to a small deviation of $\mathfrak{M}$ from the equilibrium direction.

$$\mathfrak{M} = \overline{\mathfrak{M}} - M_0 V \frac{G_2 c_s c_l}{(c_s + c_l) T} \left(\Delta T + G_1 \zeta - \frac{\eta}{c_l} \right),$$

$$\mathfrak{M}_\perp^2 = 2 (M_0 V)^2 \frac{\eta}{\varepsilon_0}, \tag{12.8}$$

where $\overline{\mathfrak{M}}$ is the equilibrium value of the magnetic moment of the body at the given temperature. Using next Eq. (12.5) we obtain finally:

$$\left.\begin{aligned} &\mathfrak{M}_\perp^2 = \mathfrak{M}_{\perp 0}^2 e^{-\lambda_3 t}, \\ &\frac{\mathfrak{M} - \overline{\mathfrak{M}}}{M_0 V} = \frac{\mathfrak{M}_0 - \overline{\mathfrak{M}}}{M_0 V} e^{-\lambda_1 t} + \frac{\Delta T_0}{T} \frac{\varepsilon_0}{\theta_C} \left(\frac{\lambda_1}{\lambda_2} \right)^2 \left(\frac{T}{\theta_C} \right)^{1/2} (e^{-\lambda_2 t} - e^{-\lambda_1 t}) \\ &\qquad + \frac{\mathfrak{M}_{\perp 0}^2}{(M_0 V)^2} \left(\frac{\beta}{8} \right)^2 \left(\frac{\varepsilon_0}{\theta_C} \right)^{1/2} \frac{T}{\theta_C} (e^{-\lambda_1 t} - e^{-\lambda_3 t}), \\ &\frac{\Delta T}{T} = \frac{\Delta T_0}{T} \frac{e^{-\lambda_2 t} + \alpha e^{-\lambda_1 t}}{1 + \alpha} + 10 \frac{\mathfrak{M}_0 - \overline{\mathfrak{M}}}{M_0 V} \left(\frac{\varepsilon_0 \theta_C}{T^2} \right)^{1/2} \frac{\theta_C}{T} (e^{-\lambda_1 t} - e^{-\lambda_2 t}) \\ &\qquad + \frac{\varepsilon}{2T} \frac{\mathfrak{M}_{\perp 0}^2}{(M_0 V)^2} \left[\frac{\lambda_3}{c_s \lambda_2} (e^{-\lambda_3 t} - e^{-\lambda_1 t}) + \beta^2 (e^{-\lambda_1 t} - e^{-\lambda_2 t}) \right], \end{aligned}\right\} \tag{12.9}$$

where $\mathfrak{M}_0$, $\mathfrak{M}_{\perp 0}$, and ΔT_0 are the initial values of the quantities $\mathfrak{M}$, $\mathfrak{M}_\perp$ and ΔT; $\alpha = 2 \left(\frac{\lambda_1}{\lambda_2} \right)^2 \left(\frac{\epsilon_0}{T} \right)^{1/2}$. These formulas are valid for $T \gg \frac{\theta_D^2}{\theta_C}$. [Setting $\theta_C \sim 10^3$, $\theta_D \sim 10^2$, we find $\frac{\theta_D^2}{\theta_C} \sim \theta_C \left(\frac{\mu M_0}{\theta_C} \right)^{4/7} \sim 10$.]

Since the smallest of the three quantities λ_1, λ_2, or λ_3 is λ_3, for sufficiently large values of t the variation with time of the whole deviation goes like $e^{-\lambda_3 t}$.

From formula (12.9) it is clear that the change in $\mathfrak{M}_\perp$ is related only to the constant λ_3. Thus the quantity $\tau_\perp = 2/\lambda_3$ can be treated as the relaxation time for the transverse component of the magnetic moment. According to (12.7)

$$\tau_\perp \approx 2 \frac{\hbar}{\mu M_0} \frac{\theta_C}{\mu M_0} \left(\frac{\theta_C}{T} \right)^2. \tag{12.10}$$

To clarify the physical meaning of the constant λ_1, we assume that $\Delta T_0 = \mathfrak{M}_{\perp 0} = 0$; then

$$\mathfrak{M} - \overline{\mathfrak{M}} = (\mathfrak{M}_0 - \overline{\mathfrak{M}}) e^{-\lambda_1 t}. \tag{12.9'}$$

Thus, for the assumed initial conditions the quantity $\tau = 1/\lambda_1$ determines the time for establishment of the equilibrium value of $\mathfrak{M}$. According to (12.7)

$$\tau \approx \frac{\hbar}{\mu M_0} \frac{(\varepsilon_0 \theta_C)^{1/2}}{\mu M_0} \frac{\theta_C}{T}. \tag{12.11}$$

We see that the establishment of the equilibrium value of the square magnitude of the magnetic moment occurs *faster than the change in the perpendicular* component of the magnetic moment. In other words, the equilibrium value for the magnitude of the magnetic moment is reached first, and then there still occurs a rotation of the magnetic moment into the axis of easiest magnetization. Such a relaxation process can be described phenomenologically by means of the relaxation term $\frac{\lambda}{M^2} \mathbf{M} \times [\mathbf{M} \times \mathbf{H}^{(e)}]$ in the Landau-Lifshitz equation (2.12).

Finally we consider the process of equalizing of the temperatures T_l and T_s. Assuming that $\mathfrak{M}_{\perp 0} = \mathfrak{M}_0 = 0$, we find

$$\Delta T = \Delta T_0 \frac{e^{-\lambda_2 t} + \alpha e^{-\lambda_1 t}}{1 + \alpha}. \tag{12.9''}$$

Since $\alpha \ll 1$ and $\lambda_2 \gg \lambda_1$, in the initial stage of equalization of the temperatures the first exponential is most important, while the second is important in the final stage.

Let us now go on to investigate the relaxation of the *magnetic moment in the temperature region* $T \ll \theta_C \left(\frac{\mu M_0}{\theta_C} \right)^{4/7}$, assuming as before that the anisotropy constant or the external magnetic field are sufficiently large.[5] For temperatures $T \ll \theta_C \left(\frac{\mu M_0}{\theta_C} \right)^{4/7}$, the inequality $w^{(3)} \gg w^{(e)}$ holds; i.e., processes of scattering of spin waves by spin waves due to exchange interaction are less important than processes of fusion and splitting of spin waves. This means that the operator $L_{\mathbf{k}}^{(3)}\{n\}$ plays the principal role in the kinetic equation (12.1), while the collision integrals $L_{\mathbf{k}}^{(e)}\{n\}$, $L_{\mathbf{k}}^{(r)}\{n\}$, $L_{\mathbf{k}}^{(ls)}\{n, N\}$ can be treated as a small perturbation.

Thus, for $T \ll \theta_C \left(\frac{\mu M_0}{\theta_C} \right)^{4/7}$ the kinetic equation can in first approximation be replaced by the equation

$$L_{\mathbf{k}}^{(3)} \{n\} = 0. \tag{12.12}$$

The solution of this equation has the form:

$$n_{\mathbf{k}} = \begin{cases} \left(e^{\frac{\varepsilon_k}{T_s}} - 1 \right)^{-1}, & k \neq 0 \\ n_0, & k = 0. \end{cases} \tag{12.12'}$$

We note that this solution also makes the collision integral $L_{\mathbf{k}}^{(e)}\{n\}$ go to zero. The time for establishment of the distribution (12.12′) is equal in order of magnitude to the mean time of fusion of two spin waves, $\tau^{(3)}$. During this same time there is established an equilibrium value of the absolute magnitude of the moment $\mathfrak{M}$.

The parameters T_s and n_0 of the distribution (12.12′) are determined once again by the energy of the spins and the transverse component of the magnetic moment. Establishment of complete thermodynamic equilibrium is the result of the weak interactions which are described in the kinetic equation (12.1) by the collision integrals $L_{\mathbf{k}}^{(r)}$ and $L_{\mathbf{k}}^{(ls)}$. Because of these interactions, the magnetic moment of the body slowly shifts to its equilibrium direction and the temperatures of the spin waves and of the lattice become equal.

Using the expression (12.12′) for the distribution function of the spin waves and the phonons, as well as the kinetic equation (12.1), we can obtain the following system of equations for determining the quantities $\eta = \epsilon_0 n_0 / N$ and $\Delta T = T_s - T_l$:

$$\Delta \dot{T} + \frac{1}{c_s} \dot{\eta} = B'_{T\eta} \eta + B'_{TT} \Delta T, \qquad \dot{\eta} = B'_{\eta\eta} \eta, \tag{12.13}$$

where

$$B'_{TT}=\left(\frac{1}{c_s}+\frac{1}{c_l}\right)\frac{1}{N}\sum_{\mathbf{k}}\varepsilon_{\mathbf{k}}\left(\frac{\partial L_{\mathbf{k}}}{\partial\Delta T}\right)_0,$$

$$B'_{T\eta}=\left(\frac{1}{c_s}+\frac{1}{c_l}\right)\frac{1}{N}\sum_{\mathbf{k}}\varepsilon_{\mathbf{k}}\left(\frac{\partial L_{\mathbf{k}}}{\partial\eta}\right)_0,$$

$$B'_{\eta\eta}=\frac{\varepsilon_0}{N}\left(\frac{\partial L_0}{\partial\eta}\right)_0$$

and N is the number of atoms in the body. If $\epsilon_0 \ll T \ll \theta_C\left(\frac{\mu M_0}{\theta_C}\right)^{4/7}$,

$$\left.\begin{aligned}&B'_{TT}\approx-\frac{\hbar\beta_1^2}{\varrho a^5}\left(\frac{T}{\theta_C}\right)^2 e^{-\frac{\theta_t^2}{4\theta_C T}},\\&B'_{T\eta}\approx-\frac{\hbar}{\pi^3\varrho a^5}\left(\frac{\gamma\mu M_0}{\theta_t}\right)^2\left(\frac{\varepsilon_0}{T}\right)^2\left(\frac{\theta_t^2}{T\theta_C}\right)^2 e^{-\frac{4\varepsilon_0^2\theta_C}{T\theta_t^2}},\\&B'_{\eta\eta}\approx\frac{\mu M_0}{\hbar}\frac{\mu M_0}{\theta_C}\left(\frac{T}{\partial_C}\right)^2.\end{aligned}\right\}\quad(12.13')$$

Assuming that the quantities η and ΔT change with time as $e^{-\lambda t}$, we get the following values of the relaxation constants:

$$\lambda'=-B'_{\eta\eta}\approx\frac{\mu M_0}{\hbar}\frac{\mu M_0}{\theta_C}\left(\frac{T}{\theta_C}\right)^2,$$

$$\lambda''=-B'_{TT}\approx\frac{\hbar\beta_1^2}{\varrho a^5}\left(\frac{T}{\theta_C}\right)^2 e^{-\frac{\theta_t^2}{4\theta_C T}}.\qquad(12.14)$$

The change with time of the quantities $\mathfrak{M}_\perp^2$ and ΔT is given by the formulas:

$$\mathfrak{M}_\perp^2=\mathfrak{M}_{\perp 0}^2 e^{-\lambda' t},$$

$$\Delta T=\Delta T_0 e^{-\lambda'' t}+\frac{1}{2}B'_{T\eta}\left(\frac{\varepsilon_0}{\lambda''}\right)^2\left(\frac{\mathfrak{M}_{\perp 0}}{M_0 V}\right)^2\left(e^{-\lambda' t}-e^{-\lambda'' t}\right).\qquad(12.15)$$

We see that λ' has a simple physical meaning: $2/\lambda'$ is the time of relaxation of the perpendicular component of the magnetic moment. We note that λ' coincides with the expression (12.7) for λ_3.

If the deviation from equilibrium is associated only with a difference between the temperatures of the spin system and the lattice, while $\mathfrak{M}_{\perp 0}=0$, the time for equalization of the temperatures is given by the quantity λ''. If $\Delta T_0=0$, while $\mathfrak{M}_{\perp 0}\neq 0$, then in the process of relaxation of the magnetic moment the temperatures of the spin waves and phonons will change because of a conversion into heat of the energy associated with the deviation of the magnetic moment from its equilibrium direction. The time for establishment of a common temperature under these conditions is equal to half the time of relaxation of the transverse component of the magnetic moment.

The results obtained refer to the case where the anisotropy constant or the magnetic field H_0 are sufficiently large. As already mentioned above, in this case processes of fusion and splitting of spin waves with wave vector $\mathbf{k}=0$ are impossible. In crystals with a small anisotropy constant (crystals with cubic symmetry are of this type) and for sufficiently weak fields, splitting of spin waves with wave vector $\mathbf{k}=0$ into two spin waves with wave vectors $\mathbf{k}$ and $-\mathbf{k}$ becomes possible. This process may make possible a rotation of the magnetic moment into the axis of easiest magnetization, and since it is more probable than the process of scattering of spin waves due to the anisotropy energy, we can, in studying the relaxation of the magnetic moment in crystals of cubic symmetry omit the operator $L_{\mathbf{k}}^{(r)}\{n\}$ in the kinetic equation for the spin waves.

Let us consider in more detail this case of relaxation.[6] Suppose that a ferrodielectric with cubic symmetry fills a half-space whose boundary is one of the crystallographic planes. The field $\mathbf{H}_0$ lies in this plane and is directed along a crystallographic axis. The energy of the spin wave with vector $\mathbf{k}=0$ is then, as we have seen in Section 4, equal to

$$\varepsilon_0=\mu\sqrt{(H_0+\beta M_0)(B_0+\beta M_0)}.$$

The splitting of such a wave into two spin waves with wave vectors $\mathbf{k}$ and $-\mathbf{k}$ is possible, as we see from the energy conservation law

$$\varepsilon_0=2\varepsilon_{\mathbf{k}},$$

when the inequality $\beta+\frac{H_0}{M_0}<\frac{4\pi}{3}$ is satisfied. The results obtained earlier refer to the case when $\beta+\frac{H_0}{M_0}>\frac{4\pi}{3}$. If $\beta+\frac{H_0}{M_0}<\frac{4\pi}{3}$, then we must make a special computation of the relaxation constants. We shall not do this here, but simply give the final results:

$$\lambda_1'\approx\begin{cases}\dfrac{\mu M_0}{\hbar}\left(\dfrac{\mu M_0}{\theta_C}\right)^{1/2}\dfrac{T}{\theta_C}\dfrac{1}{\ln\left[2\left(\beta+\dfrac{H_0}{M_0}\right)\right]^{-1}}, & \beta+\dfrac{H_0}{M_0}\ll 1,\\[2ex] \dfrac{\mu M_0}{\hbar}\left(\dfrac{\mu M_0}{\theta_C}\right)^{1/2}\dfrac{T}{\theta_C}, \quad \beta+\dfrac{H_0}{M_0}\sim\dfrac{4\pi}{3}, &\end{cases}$$

$$\lambda_2'\approx\begin{cases}\dfrac{\hbar}{\varrho a^5}\beta_1^2\left(\dfrac{T}{\theta_C}\right)^2 e^{-\frac{\theta_t^2}{4T\theta_C}}, & T\ll\dfrac{\theta_D^2}{\theta_C},\\[2ex] \dfrac{\hbar}{\varrho a^5}\left[\beta_1^2+2(\beta_1+\beta_2)^2\right]\left(\dfrac{T}{\theta_C}\right)^{5/2}, & T\gg\dfrac{\theta_D^2}{\theta_C},\end{cases}$$

$$\lambda_3'\approx\begin{cases}\dfrac{\mu M_0}{\hbar}\left(\dfrac{\mu M_0}{\theta_C}\right)^{1/2}\dfrac{T}{\theta_C}, \quad \beta+\dfrac{H_0}{M_0}\ll 1,\\[2ex] \dfrac{10^2\mu M_0}{\hbar}\left(\dfrac{\mu M_0}{\theta_C}\right)^{1/2}\dfrac{T}{\theta_C}\left(\dfrac{4\pi}{3}-\beta-\dfrac{H_0}{M_0}\right)^{5/2},\\[2ex] \left|\beta+\dfrac{H_0}{M_0}-\dfrac{4\pi}{3}\right|\ll 1.\end{cases}\qquad(12.16)$$

It can be shown that the quantity $2/\lambda_3'$ has the meaning of a relaxation time for the transverse component of the magnetic moment of the body, while $1/\lambda_1'$ is the time of relaxation of the absolute value of the magnetic moment. The quantity λ_3', as one might expect, goes to zero for $\beta+\frac{H_0}{M_0}=\frac{4\pi}{3}$.

We give estimates of the quantities λ_1', λ_2', and λ_3'. Setting $\theta_C\sim 10^3$, $\theta_D\sim 10^2$, $\beta+\frac{H_0}{M_0}\sim 1$, $M_0\sim 10^3$, $T\sim 10^2$, we obtain $\lambda_1'\sim\lambda_3'\sim 10^7\ \mathrm{sec}^{-1}$, $\lambda_2'\sim 3\times 10^8\ \mathrm{sec}^{-1}$. Thus, if $\beta+\frac{H_0}{M_0}$ is not too close to $\frac{4\pi}{3}$, the times of relaxation of the absolute value and of the transverse component of the magnetic moment have the same order of magnitude. This is the distinction between the relaxation for the case of small anisotropy and weak fields and the relaxation for the case of large anisotropy or strong fields. In the latter case, the establishment of the equilibrium value of the magnetic moment occurs much more rapidly than the rotation of the moment toward the equilibrium direction.

In the temperature region $T \ll \theta_C \left(\frac{\mu M_0}{\theta_C}\right)^{4/7}$, the main role in the kinetic equation is played by the operator $L_{\mathbf{k}}^{(3)}\{n\}$. Therefore in crystals with low magnetic anisotropy, in which one can have a decay of a spin wave with wave vector $\mathbf{k} = 0$ into two spin waves with wave vectors $\mathbf{k}$ and $-\mathbf{k}$, simultaneously with the establishment of the Bose distribution of the spin waves, we have an establishment of the equilibrium value of the magnetic moment.

13. Dispersion of the Magnetic Permeability of a Ferromagnetic Dielectric

In the preceding paragraph we considered the relaxation of the magnetic moment. The complex character of the relaxation imposes some fundamental features on the frequency dependence of the magnetic susceptibility of a ferrodielectric.[1,7] We shall consider the case where the variable magnetic field is polarized along the direction of the equilibrium magnetic moment, i.e., we shall calculate the longitudinal component of the magnetic permeability $\chi_3 \equiv \chi_{zz}$. We recall that the static value of this quantity can be obtained by differentiating the magnetic moment per unit volume (3.23) with respect to the magnetic field.

If the frequency of the variable magnetic field $h_0 e^{-i\omega t}$ is very much less than the reciprocal of the relaxation time τ_{ss} in the spin system, i.e.,

$$\omega\tau_{ss} \ll 1, \tag{13.1}$$

the field can be treated classically as the cause for the system of spin waves to deviate from its equilibrium situation.

The condition $\omega\tau_{ss} \ll 1$ allows us to represent the energy of the spin wave in the form

$$\varepsilon_{\mathbf{k}} = \varepsilon_0 + \theta_C (ak)^2 + \mu h_0 e^{-i\omega t}. \tag{13.2}$$

This expression for the energy of the spin wave is valid, as already pointed out, for any temperatures $T \ll \theta_C$, if $\epsilon_0 \gg 2\pi\mu M_0$. If $\epsilon_0 \lesssim 2\pi\mu M_0$, formula (13.2) can be used for $\theta_C \gg T \gg 2\pi\mu M_0$.

The kinetic equation (12.1) enables us to determine the distribution function of the spin waves $n_{\mathbf{k}}(t)$.

We consider low temperatures, $\epsilon_0 \ll T \ll T_0$, where $T_0 = \theta_C \left(\frac{\mu M_0}{\theta_C}\right)^{4/7}$. In this temperature range, according to Sec. 11, the most probable processes are those of fusion of two spin waves into one and splitting of one spin wave into two. This means that $\tau_{ss} \sim \tau_3$ (cf. Eq. 10.7).

According to Sec. 11, the solution of the kinetic equation (12.1) for $\epsilon_0 \ll T \ll T_0$ should have the form

$$n_{\mathbf{k}} = n^0\left(\frac{\varepsilon_{\mathbf{k}}}{T_s}\right) + n'_{\mathbf{k}} \quad |n'_{\mathbf{k}}| \ll n^0_{\mathbf{k}}. \tag{13.3}$$

We note that the quantities $\epsilon_{\mathbf{k}}$ and T_s are slowly varying functions of the time. Such a form of the solution means that in the system of spin waves a quasi-equilibrium state is "able" to establish itself with its own temperature, different from the temperature of the lattice. The dispersion of the magnetic susceptibility in this case is associated with two basically different mechanisms: first with the dependence on the time of the spin temperature T_s, in which the energy dissipation is caused by transfer of energy to phonons, and second with the deviation of the distribution function from its equilibrium value. In accordance with this we shall write χ_3 in the form

$$\chi_3 = \chi_{3,1} + \chi_{3,2}, \tag{13.4}$$

where $\chi_{3,1}$ is caused by the first mechanism, and $\chi_{3,2}$ by the second. The characteristic time for the first mechanism is the time for equalizing of the temperatures, $\tau_{sl} \approx 1/\lambda_2$ (cf. Eq. 12.7). The characteristic time for the second mechanism is the spin-spin relaxation time τ_{ss}. Since $\tau_{ss} \ll \tau_{sl}$, for frequencies $\omega\tau_{sl} \ll 1$ the main part in the dispersion of the magnetic susceptibility is played by the spin-phonon interaction, while for frequencies $1/\tau_{sl} \ll \omega \ll 1/\tau_{ss}$ the main effect is that of the deviation of the distribution function from its equilibrium value.

For the calculation of the time dependence of the temperature T_s we must use the equation of heat balance, which can be obtained from the kinetic equation (12.1) using expressions (12.3) and (13.3):

$$\dot{T}_s + \frac{1}{\tau_{sl}}(T_s - T_l) = q\mu\dot{h}, \qquad q \approx 0{,}4, \tag{13.5}$$

where

$$\frac{1}{\tau_{sl}} = \frac{64\beta_1^2}{5\pi^{5/2}\zeta(5/2)} \frac{\theta_C}{\hbar} \frac{T^{7/2}}{\varrho a^3 s_t^2 \theta_C^{5/2}} \exp\left(-\frac{\theta_t^2}{4\theta_C T}\right).$$

An analogous equation can be obtained for the phonon temperature T_l. Here, however, the question of the interaction of the phonons with the external medium arises. If the thermal contact is very good, the temperature of the lattice will not change. We shall restrict ourselves to this case.*

From (13.5) we have

$$T_s = T_l + T', \quad T' = -q\frac{i\omega\tau_{sl}}{1 - i\omega\tau_{sl}}\mu h. \tag{13.6}$$

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*The difference between τ_{sl} and $1/\lambda_2$ (cf. Eq. 12.7) is related to the fact that in Sec. 12 the equalization of the temperatures of of spins and lattice was considered for the case where the system was thermally isolated.

Now we shall use the expression for the magnetic moment (3.23). If we assume $n_{\mathbf{k}} = n^0\left(\frac{\epsilon_{\mathbf{k}}}{T_S}\right)$, then

$$\chi_{3,1}=\frac{\partial M}{\partial h}+\frac{\partial M}{\partial T_s}\frac{\partial T_s}{\partial h}\,. \tag{13.7}$$

From this and from expression (3.23) we have

$$\chi_{3,1}=(\chi_3)_{\mathrm{stat}}-\frac{\partial M}{\partial T_s}\frac{i\omega\tau_{sl}}{1-i\omega\tau_{sl}}q\mu. \tag{13.8}$$

We shall now calculate the quantity $\chi_{3,2}$ which is related to the deviation of the distribution function from its equilibrium value. This mechanism of dissipation is important for relatively high frequencies $\omega\tau_{sl} \gg 1$, for which, however, we still may use formula (13.2), in view of the validity of condition (13.1). In the kinetic equation (12.1), we can then neglect the term $L_{\mathbf{k}}^{sl}\{n\} = L_{\mathbf{k}}^{sl}\{n_0\} + L_{\mathbf{k}}^{sl}\{n'\}$. In fact, the first term $L_{\mathbf{k}}^{sl}\{n\}$ is small compared to $\partial n_{\mathbf{k}}^0/\partial t$, while the second term $L_{\mathbf{k}}^{sl}\{n'\}$ is small compared to $L_{\mathbf{k}}^{(3)}\{n'\}$, since $\tau_{ss} \ll \tau_{sl}$. Noting also that $L_{\mathbf{k}}^{(3)}\{n_0\} = 0$ and $\partial n'/\partial t \ll L_{\mathbf{k}}^{(3)}\{n'\}$, since $\omega\tau_{ss} \ll 1$, the kinetic equation (12.1) can be written as follows:

$$\frac{\partial n_{\mathbf{k}}^0}{\partial t}=L_{\mathbf{k}}^{(3)}\{n'\}. \tag{13.9}$$

Neglecting the term $L_{\mathbf{k}}^{sl}\{n_0\}$ in the kinetic equation corresponds to neglect of the heat loss in the heat balance equation (13.5). Therefore for $\omega\tau_{sl} \gg 1$ we have $\dot{T}_S = q\mu\dot{h}$, and (13.9) can be written as follows:

$$\left(1-q\frac{\varepsilon}{T}\right)\frac{\partial n_{\mathbf{k}}^0}{\partial\left(\frac{\varepsilon}{T}\right)}\frac{\mu\dot{h}}{T}=L_{\mathbf{k}}\ \{n'\}\equiv\frac{1}{\tau_{ss}}L\{n'\}, \tag{13.10}$$

where L is the dimensionless collision operator, which is easily obtained from formula (10.3) if we go over from summation to integration with respect to the wave vector $\mathbf{k}$, and introduce the dimensionless wave vector $\mathbf{x} = \left(\frac{\theta_C}{T}\right)^{1/2} a\mathbf{k}$, and

$$\frac{1}{\tau_{ss}}=\frac{\mu M_0}{\hbar}\frac{\mu M_0}{\theta_C}\left(\frac{T}{\theta_C}\right)^{1/2}. \tag{13.11}$$

From (13.10) we have

$$n'_{\mathbf{k}}=\tau_{ss}\mu\dot{h}\varphi(\mathbf{x}), \tag{13.12}$$

where $\varphi(\mathbf{x})$ is a dimensionless function of a dimensionless argument which cannot be determined without an exact solution of (13.10). This function should be of the order of unity, since the equation for it contains neither small nor large parameters.

Knowing the dependence of n' on temperature, we can find the dependence of $\chi_{3,2}$ on temperature and frequency. To do this we can again make use of the expression for the magnetic moment (3.23). From (3.23) and (13.12) we have, to within a numerical factor:[7]

$$\chi_{3,2}\sim i\frac{\mu^2}{a^3\theta_C}\left(\frac{T}{\theta_C}\right)^{1/2}\omega\tau_{ss}. \tag{13.13}$$

As we see from the last expression, in this case $\operatorname{Re}\chi_{3,2} = 0$, which is related to the neglect of the term $\partial n'/\partial t$ in the kinetic equation (12.1). If we do not neglect this term and replace the collision integral $L_{\mathbf{k}}^{(3)}$ by the expression n'/τ_{ss}, we have for $\chi_{3,2}$:

$$\chi_{3,2}\sim i\frac{\mu^2}{a^3\theta_C}\left(\frac{T}{\theta_C}\right)^{1/2}\frac{\omega\tau_{ss}}{1-i\omega\tau_{ss}}\,. \tag{13.14}$$

This expression is, of course, less accurate than expression (13.8) for $\chi_{3,1}$.

Comparing formulas (13.8) and (13.14), we can verify that $\chi_{3,1}$ and $\chi_{3,2}$ coincide for $\omega \approx (\tau_{sl}\tau_{ss})^{-1/2}$. For lower frequencies $|\chi_{3,1}| > |\chi_{3,2}|$, for higher frequencies $|\chi_{3,1}| < |\chi_{3,2}|$.

We shall now consider the case of relatively high temperatures:[7] $T_0 \ll T \ll \theta_C$. In this temperature range the exchange forces play the main part in the interaction of spin waves with one another.

Nevertheless, in the kinetic equation we cannot drop the small relativistic terms because precisely these terms are responsible for the dispersion of the magnetic susceptibility.

The solution of (12.1) must be sought for in the form (cf. Sec. 12):

$$n=n_0^{(0)}\left(\frac{\varepsilon-\zeta}{T_s}\right)+n',\quad |n'|\ll n_0, \tag{13.15}$$

realizing, in doing this, that the energy ϵ, the chemical potential ζ, and the temperature of the spin waves T_S depend on the time. In the sequel we shall use formula (13.2) for the energy of the spin wave, assuming it to be valid also for $\mathbf{k} = 0$. In other words, the formulas found below are valid for $\epsilon_0 \gg 2\pi\mu M_0$.

From Eq. (12.1), with account of (13.5), it is easy to find the relation between the variable part of the temperature T'_S, the chemical potential ζ, and the variable magnetic field:

$$\begin{aligned}&(A_{T\zeta}-i\omega qC_s)\,T'_s+(TA_{\zeta\zeta}-i\omega B)\,\zeta=B\mu\dot{h},\\&(A_{TT}-i\omega C_s)\,T'_s+(A_{T\zeta}-i\omega C_s)\,\zeta=qC_s\mu\dot{h}.\end{aligned} \tag{13.16}$$

Here

$$\begin{aligned}&A_{T\zeta}=\frac{\hbar\gamma_1}{6\pi^2\varrho a^5}\left(\frac{\mu M_0}{\theta_C}\right)^2\left(\frac{T}{\theta_C}\right)^2,\\&A_{TT}=\frac{2\pi^2 h}{45\varrho a^5}\left[2\beta_1^2+(2\beta_1+\beta_2)^2\right]\left(\frac{T}{\theta_C}\right)^4,\\&A_{\zeta\zeta}=\frac{\chi}{\hbar}\left(\frac{\mu M_0}{\theta_C}\right)^2\frac{T^2}{\varepsilon_0\theta_C},\quad \chi\approx 0.1,\\&B=\frac{1}{8\pi}\frac{T^2}{\varepsilon_0^{1/2}\theta_C^{3/2}}\,.\end{aligned} \tag{13.17}$$

By means of (13.16) we determine T_S and ζ, and knowing them, from the formula for the magnetic moment (3.23) we determine the magnetic susceptibility of the ferromagnetic dielectric:

$$\chi(\omega)=\chi_{\mathrm{stat}}\left\{1+\frac{i\omega\tau_2(1-i\omega\tau_2)}{1-i\omega\tau_1-\omega^2\tau_1\tau_2}\right\}-\left(\frac{\partial M}{\partial T}\right)_H\frac{i\omega\tau_2(1+iq^*\omega\tau_1)}{1-i\omega\tau_1-\omega^2\tau_1\tau_2},$$

$$\tau_1=\frac{1}{\lambda_1},\quad \tau_2=\frac{1}{\lambda_2},\quad q^*\approx\frac{3}{2}\,. \tag{13.18}$$

It can be shown that there is still another mechanism which results in a dispersion of the magnetic susceptibility, namely the deviation of the distribution function from its equilibrium value (analogous to the second mechanism for low temperatures). However, it is easy to see that this mechanism is always unimportant, since the exchange forces, which play the primary role in this temperature range, do not change the size of the magnetic moment, while the relativistic interactions are included in (13.16).

We now turn to the treatment of the high frequency case $\omega \gg 1/\tau_{ss}$. With increasing frequency, the magnetic susceptibility tends to unity.[8] Therefore, at sufficiently high frequencies it is meaningful to speak, not of a calculation of the magnetic susceptibility, but rather of the absorption coefficient for photons Γ, defined as the difference of the probabilities for all processes of absorption and emission of a photon.

It can be shown[7] that the absorption of a photon occurs because of the decay of the latter into two spin waves, the probability for this process being

$$4\pi^2 V\omega\mu^2 \,|\, v_{\mathbf{k}}u_{\mathbf{k}'} + v_{\mathbf{k}'}u_{\mathbf{k}} \,|^2 (n_{\mathbf{k}} + 1)(n_{\mathbf{k}'} + 1)\,\delta(\varepsilon_{\mathbf{k}} + \varepsilon_{\mathbf{k}'} - \omega\hbar).$$

From this, the absorption coefficient Γ has the form:

$$\Gamma = \sum_{\mathbf{k}} 4\pi^2 V\omega\mu^2 (n_{\mathbf{k}} + n_{-\mathbf{k}} + 1)\,|\, v_{\mathbf{k}}u_{-\mathbf{k}} + u_{\mathbf{k}}v_{-\mathbf{k}} \,|\, \delta(\varepsilon_{\mathbf{k}} + \varepsilon_{-\mathbf{k}} - \hbar\omega). \tag{13.19}$$

In obtaining the last formula, we have used the fact that the photon momentum is considerably less than the momenta of the spin waves, and have set it equal to zero.

The splitting of a photon into two spin waves can occur only when the photon energy $\hbar\omega$ is greater than $2\epsilon_0$. For $\hbar\omega < 2\epsilon_0$, absorption of the energy of the magnetic field is associated in the main with inelastic scattering of the photon by a spin wave. This process is extremely improbable since it occurs only in the second approximation of perturbation theory.

From expression (13.19), noting that $\epsilon_{\mathbf{k}}$, $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are even functions of the wave vector, we find:

$$\Gamma = \begin{cases} \dfrac{\mu^2 (2\pi\mu M_0)^{3/2}}{2\theta_C^{3/2} a^3 \hbar} \coth \dfrac{\hbar\omega}{4T}\, I(\nu, \eta), & \nu > \eta, \\ 0, & \nu < \eta, \end{cases} \tag{13.20}$$

where

$$\nu = \frac{\hbar\omega}{4\pi\mu M_0}, \qquad \eta = \frac{\varepsilon_0}{2\pi\mu M_0},$$

and

$$I = \int_0^{\sigma(\nu, \eta)} \frac{x^2 [\sqrt{x^2 + \nu^2} - (x + \eta)]^{1/2}\, dx}{\sqrt{(1 - x)(x^2 + \nu^2)}},$$

$$\sigma = \begin{cases} 1, & \nu > \sqrt{\eta^2 + 2\eta}, \\ \dfrac{\nu^2 - \eta^2}{2\eta}, & \eta < \nu < \sqrt{\eta^2 + 2\eta}. \end{cases}$$

We give the asymptotic values of the integral I:

$$I \approx \begin{cases} \dfrac{16}{15}\sqrt{\nu}, & \nu \gg \eta, \\ \dfrac{16}{105}\dfrac{(\nu - \eta)^3}{\eta}, & \nu \gtrsim \eta \gg 1, \\ \dfrac{\pi}{8\sqrt{2}}\sqrt{\eta}\,(\nu - \eta)^2, & \nu \gtrsim \eta, \quad \eta \ll 1. \end{cases} \tag{13.21}$$

Substituting these expressions in formula (13.20), we can determine the frequency dependence of the absorption coefficient of a photon in all the limiting cases. We give it only for the case of high frequencies:*

$$\Gamma \approx \frac{16}{15}\frac{\mu^2}{a^3\hbar}\left(\frac{2\pi\mu M_0}{\theta_C}\right)^{3/2}\left(\frac{2\pi\mu M_0}{\hbar\omega}\right)^{1/2} \coth\frac{\hbar\omega}{4T}, \quad h\omega \gg 2\pi\mu M_0\,\varepsilon_0. \tag{13.22}$$

For photon energies large compared to the temperature ($\hbar\omega \gg 4T$), *the absorption coefficient falls off* proportionally to $\omega^{-3/2}$.

14. Thermal Conductivity of Ferrodielectrics

Knowing the Hamiltonian for the interaction of spin waves with one another and also for the interaction of spin waves with phonons and phonons with one another, we can compute the thermal conductivity of a ferrodielectric.

For this purpose let us write the kinetic equations which determine the change in the numbers of spin waves $n_{\mathbf{k}}$ and of photons $N_{\mathbf{ks}}$ in the presence of a weak temperature gradient ∇T. These equations have the following form:

$$\begin{aligned} \dot{n}_{\mathbf{k}}^{c} &\equiv L_{\mathbf{k}}^{ss}\{n\} + L_{\mathbf{k}}^{sl}\{n, N\} = n_{\mathbf{k}}^0 (n_{\mathbf{k}}^0 + 1)\frac{\varepsilon_{\mathbf{k}}}{T^2}(\mathbf{v}_{\mathbf{k}}, \nabla T), \\ \dot{N}_{\mathbf{k}s}^{c} &\equiv L_{\mathbf{k}s}^{ll}\{N\} + L_{\mathbf{k}s}^{sl}\{n, N\} = N_{\mathbf{k}s}^0 (N_{\mathbf{k}s}^0 + 1)\frac{\hbar\omega_{\mathbf{k}s}}{T^2}(s_{\mathbf{k}s}, \nabla T), \end{aligned} \tag{14.1}$$

where $\mathbf{v}_{\mathbf{k}} = \dfrac{1}{\hbar}\dfrac{\partial\epsilon_{\mathbf{k}}}{\partial\mathbf{k}}$ and $s_{\mathbf{ks}} = \dfrac{\partial\omega_{\mathbf{ks}}}{\partial\mathbf{k}}$; the operators $L_{\mathbf{k}}^{ss}\{n\}$ and $L_{\mathbf{k}}^{sl}\{n, N\}$ describe the collisions of spin waves with one another and of spin waves with phonons, while the operators $L_{\mathbf{ks}}^{ll}\{N\}$ and $L_{\mathbf{ks}}^{ls}\{n, N\}$ describe *collisions of phonons with one another and with spin* waves.

The operators $L_{\mathbf{k}}^{ss}\{n\} = L_{\mathbf{k}}^{(e)}\{n\} + L_{\mathbf{k}}^{(3)}\{n\}$ and $L_{\mathbf{k}}^{sl}\{n, N\}$ are given by formulas (10.3), (10.10′) and (11.7), in which, however, we must include umklapp processes. This means that in these formulas the *function* $\Delta(\Sigma\mathbf{k})$ *must be replaced by the function* $\Delta(\Sigma\mathbf{k} - 2\pi n\tau)$, where τ is a vector of the reciprocal lattice and n is an integer, and we must carry out a summation over all values of n.

The operator $L_{\mathbf{ks}}^{ls}\{n, N\}$ can be obtained by using the Hamiltonian (11.5), and has the following form:

$$L_1^{ls}\{n, N\} = \frac{2\pi}{\hbar}\sum_{23;\, n} |\Psi_{1,\,23}|^2 [(N_1 + 1)(n_2 + 1)\, n_3 - N_1 n_2 (n_3 + 1)] \times \delta(\hbar\omega_1 + \varepsilon_2 - \varepsilon_3)\Delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - 2\pi n\tau). \tag{14.2}$$

The operator $L_{\mathbf{ks}}^{ll}\{N\}$, when we include only processes of fusion of two phonons into one and splitting of *one phonon into two, is given by the following expression:*

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*Of course, we are assuming $\hbar\omega \ll \theta_C$.

$$L^{\mathrm{ll}}_{\mathbf{k}_1}\{N\} = \frac{8\pi}{\hbar}\sum_{2\,3;\,n}\{|X_{12,3}|^2[(N_1+1)(N_2+1)N_3$$
$$-N_1N_2(N_3+1)]\,\delta(\hbar\omega_1+\hbar\omega_2-\hbar\omega_3)\,\Delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-2\pi n\boldsymbol{\tau})$$
$$+|X_{13,2}|^2[(N_1+1)(N_3+1)N_2-N_1N_3(N_2+1)]$$
$$\times\delta(\hbar\omega_1+\hbar\omega_3-\hbar\omega_2)\,\Delta(\mathbf{k}_1+\mathbf{k}_3-\mathbf{k}_2-2\pi n\boldsymbol{\tau})$$
$$+|X_{23,1}|^2[(N_1+1)N_2N_3-N_1(N_2+1)(N_3+1)]$$
$$\times\delta(\hbar\omega_2+\hbar\omega_3-\hbar\omega_1)\,\Delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3-2\pi n\boldsymbol{\tau})\}, \qquad (14.3)$$

where the quantity $X_{12,3}$ is equal to

$$X_{12,3} = A_{12,3}\frac{\hbar^{3/2}s^2}{\varrho^{1/2}V^{1/2}}\frac{k_1k_2k_3}{\sqrt{\omega_1\omega_2\omega_3}} \qquad (14.4)$$

and $A_{12,3}$ is a dimensionless quantity of order unity, depending on the polarization and direction of the wave vectors of the phonons participating in the collision process.

Setting $\dot{n}^c_{\mathbf{k}}$ and $\dot{N}^c_{\mathbf{k}s}$ equal to zero, we obtain equations for determining the stationary distributions of the spin waves and phonons.

If we do not include umklapp processes in the collision integrals the stationary distributions will have the following form:

$$\tilde{n}_{\mathbf{k}} = \{e^{\frac{\varepsilon_{\mathbf{k}}+\mathbf{kg}}{T}}-1\}^{-1}, \qquad \tilde{N}_{\mathbf{k}s} = \{e^{\frac{\hbar\omega_{\mathbf{k}s}+\mathbf{kg}}{T}}-1\}^{-1}, \qquad (14.5)$$

where $\mathbf{g}$ is an arbitrary constant vector, which is determined by the total momentum of spin waves and phonons.

To this distribution there corresponds obviously a heat flux,

$$S = \sum_{\mathbf{k}s}\hbar\omega_{\mathbf{k}s}\mathbf{s}_{\mathbf{k}s}\tilde{N}_{\mathbf{k}s} + \sum_{\mathbf{k}}\varepsilon_{\mathbf{k}}\mathbf{v}_{\mathbf{k}}\tilde{n}_{\mathbf{k}},$$

which, for $\mathbf{g} \neq 0$, is different from zero in the absence of a temperature gradient.* Such a conclusion is related to the fact that, in defining the stationary distribution of spin waves and phonons, we did not include umklapp processes, for which the law of conservation of quasimomentum is violated. This means that if we do not include umklapp processes, we shall obtain an infinite value for the thermal conductivity.[9]

When umklapp processes are included, the equations

$$\dot{n}^c_{\mathbf{k}} = 0, \quad \dot{N}^c_{\mathbf{k}s} = 0$$

are satisfied only if $\mathbf{g} = 0$, and therefore for $\nabla T = 0$ the thermal flux goes to zero.

We shall look for solutions of (14.1) in the form:

$$n_{\mathbf{k}} = n^0_{\mathbf{k}} + n^0_{\mathbf{k}}(n^0_{\mathbf{k}}+1)\frac{\varphi_{\mathbf{k}}}{T},$$
$$N_{\mathbf{k}s} = N^0_{\mathbf{k}s} + N^0_{\mathbf{k}s}(N^0_{\mathbf{k}s}+1)\frac{\Phi_{\mathbf{k}}}{T}. \qquad (14.6)$$

If the temperature gradient ∇T is small, the corrections to the equilibrium functions will also be small. Therefore the collision integrals in equations (14.1) may be linearized with respect to $\varphi_{\mathbf{k}}$ and $\Phi_{\mathbf{k}}$. As a result, we obtain the following equations for determining $\varphi_{\mathbf{k}}$ and $\Phi_{\mathbf{k}}$:

$$(n^0_1+1)\,n^0_1\frac{\varepsilon_1}{T}(\mathbf{v}_1,\nabla T) = -\frac{96\pi}{\hbar}\sum_{234,\,n}|\Phi_{12,34}+\Phi^*_{34,12}|$$
$$\times(n_1+1)(n_2+1)n_3n_4(\varphi_1+\varphi_2-\varphi_3-\varphi_4)\,\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)$$
$$\times\Delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4-2\pi n\boldsymbol{\tau}) - \frac{8\pi}{\hbar}\sum_{23,\,n}\{|\Phi_{12,3}|^2(n_1+1)$$
$$\times(n_2+1)n_3(\varphi_1+\varphi_2-\varphi_3)\,\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3)$$
$$\times\Delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-2\pi n\boldsymbol{\tau}) + |\Phi_{13,2}|^2(n_1+1)(n_3+1)$$
$$\times n_2(\varphi_1+\varphi_3-\varphi_2)\,\delta(\varepsilon_1+\varepsilon_3-\varepsilon_2)\,\Delta(\mathbf{k}_1+\mathbf{k}_3-\mathbf{k}_2-2\pi n\boldsymbol{\tau})$$
$$+|\Phi_{23,1}|^2(n_1+1)n_2n_3(\varphi_1-\varphi_2-\varphi_3)\,\delta(\varepsilon_1-\varepsilon_2-\varepsilon_3)$$
$$\times\Delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3-2\pi n\boldsymbol{\tau})\} - \frac{2\pi}{\hbar}\sum_{23,\,n}\{|\Psi_{12,3}|^2(n_1+1)$$
$$\times n_2N_3(\varphi_1-\varphi_2-\Phi_3)\,\delta(\varepsilon_1-\varepsilon_2-\hbar\omega_3)\,\Delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3-2\pi n\boldsymbol{\tau})$$
$$+|\Psi_{12,3}|^2(n_1+1)n_2(N_3+1)(\varphi_1+\Phi_3-\varphi_2)$$
$$\times\delta(\varepsilon_1+\hbar\omega_3-\varepsilon_2)\,\Delta(\mathbf{k}_1+\mathbf{k}_3-\mathbf{k}_2-2\pi n\boldsymbol{\tau})\}, \qquad (14.7)$$

$$(N_1+1)N_1\frac{\hbar\omega_1}{T}(\mathbf{s}_1,\nabla T) = -\frac{8\pi}{\hbar}\sum_{2\,3;\,n}\{|X_{12,3}|^2(N_1+1)(N_2+1)$$
$$\times N_3(\Phi_1+\Phi_2-\Phi_3)\,\delta(\hbar\omega_1+\hbar\omega_2-\hbar\omega_3)\,\Delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-2\pi n\boldsymbol{\tau})$$
$$+|X_{13,2}|^2(N_1+1)(N_3+1)N_2(\Phi_1+\Phi_3-\Phi_2)$$
$$\times\delta(\hbar\omega_1+\hbar\omega_3-\hbar\omega_2)\,\Delta(\mathbf{k}_1+\mathbf{k}_3-\mathbf{k}_2-2\pi n\boldsymbol{\tau})$$
$$+|X_{23,1}|^2(N_1+1)N_2N_3(\Phi_1-\Phi_2-\Phi_3)$$
$$\times\delta(\hbar\omega_1-\hbar\omega_2-\hbar\omega_3)\,\Delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3-2\pi n\boldsymbol{\tau})\}$$
$$-\frac{2\pi}{\hbar}\sum_{23,\,n}|\Psi_{23,1}|^2(N_1+1)(n_2+1)n_3(\Phi_1+\varphi_2-\varphi_3)$$
$$\times\delta(\hbar\omega_1+\varepsilon_2-\varepsilon_3)\,\Delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-2\pi n\boldsymbol{\tau}). \qquad (14.8)$$

We shall show that the main part of the functions $\varphi_{\mathbf{k}}$ and $\Phi_{\mathbf{k}}$ has the following form:[10]

$$\varphi^0_{\mathbf{k}} = \Phi^0_{\mathbf{k}} = \mathbf{kg}, \qquad (14.9)$$

where $\mathbf{g}$ is a constant vector which is proportional to the temperature gradient. This main part of the functions $\varphi_{\mathbf{k}}$ and $\Phi_{\mathbf{k}}$ corresponds to stationary functions of the form (14.5), in which an expansion is made in powers of $\mathbf{g}$ and the first two terms are retained.

In order to verify that the corrections to $\varphi^0_{\mathbf{k}}$ and $\Phi^0_{\mathbf{k}}$ are small, we shall write the system of equations (14.7) and (14.8) schematically in the form of an equation with one unknown function;

$$L_0(\varphi) + \xi L_u(\varphi) = n(n+1)\frac{\varepsilon}{T}(\mathbf{v},\nabla T). \qquad (14.10)$$

where L_0 is the collision operator omitting umklapp processes, L_u is the collision operator describing the umklapp processes, and ξ is the small parameter characterizing the probability for umklapp processes. The operator L_0 has the property that $L_0(\varphi_0) = 0$ if $\varphi_0 = \mathbf{gk}$.

We shall look for a solution of (14.10) in the form $\varphi = \mathbf{gk} + \varphi'$ and select $\mathbf{g}$ from the condition

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*The body as a whole is assumed to be at rest.

$$\xi \sum_{\mathbf{k}} \mathbf{k} L_u (\mathbf{g}\mathbf{k}) = \sum_{\mathbf{k}} n (n+1) \frac{\varepsilon_{\mathbf{k}}}{T} \mathbf{k} (\mathbf{v}_{\mathbf{k}}, \nabla T). \qquad (14.11)$$

The left side of this relation gives the change in momentum of the system per unit time as a result of collisions, while the right side gives the change in momentum of the system because of the presence of the temperature gradient. From (14.11) it follows that $\mathbf{g} \sim \frac{1}{\xi} \nabla T$. We now show that $\varphi' \ll \varphi_0$. For $\xi \ll 1$, the equation for determining φ' has the form:

$$L_0 (\varphi') = n (n+1) \frac{\varepsilon}{T} (\mathbf{v}, \nabla T) + \xi L_u (g\mathbf{k}).$$

Since this equation does not contain a small parameter ($\xi g \sim 1$), φ' will be of the order of magnitude of unity, whereas φ_0 is inversely proportional to ξ.

Let us now turn to the initial system (14.7) and (14.8). In order to find the vector $\mathbf{g}$, we should, in accordance with condition (14.11), multiply the kinetic equations (14.7) and (14.8) by the vector $\mathbf{k}$ and add them. As a result, we obtain the following equation for determining the vector $\mathbf{g}$:

$$\mathbf{R}_l + \mathbf{R}_s = A\mathbf{g}, \qquad (14.12)$$

where

$$\mathbf{R}_s = \frac{1}{V} \sum_{\mathbf{k}} n_{\mathbf{k}} (n_{\mathbf{k}} + 1) \frac{\varepsilon_{\mathbf{k}}}{T} \mathbf{k} (v_{\mathbf{k}}, \nabla T),$$

$$\mathbf{R}_l = \frac{1}{V} \sum_{\mathbf{k}s} N_{\mathbf{k}s} (N_{\mathbf{k}s} + 1) \frac{\hbar\omega_{\mathbf{k}s}}{T} \mathbf{k} (\mathbf{s}_{\mathbf{k}s}, \nabla T).$$

$$A = L^{ss} + L^{sl} + L^{ll},$$

$$L^{ss} = \frac{24\pi}{\hbar V} (2\pi\tau)^2 \sum_{1234} |\Phi_{12,34} + \Phi^*_{34,12}|^2 (n_1 + 1)(n_2 + 1)$$

$$\times n_3 n_4 \delta (\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \Delta (\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4 - 2\pi\boldsymbol{\tau})$$

$$+ \frac{8\pi}{\hbar V} (2\pi\tau)^2 \sum_{123} |\Phi_{12,3}|^2 (n_1 + 1)(n_2 + 1) n_3 \delta (\varepsilon_1 + \varepsilon_2 - \varepsilon_3)$$

$$\times \Delta (\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - 2\pi\boldsymbol{\tau}),$$

$$L^{sl} = \frac{2\pi}{\hbar V} (2\pi\tau)^2 \sum_{123} |\Psi_{12,3}|^2 (n_1 + 1) n_2 N_3 \delta (\varepsilon_1 - \varepsilon_2 - \hbar\omega_3)$$

$$\times \Delta (\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3 - 2\pi\boldsymbol{\tau}),$$

$$L^{ll} = \frac{8\pi}{\hbar V} (2\pi\tau)^2 \sum_{123} |X_{12,3}|^2 (N_1 + 1)(N_2 + 1)$$

$$\times N_3 \delta (\hbar\omega_1 + \hbar\omega_2 - \hbar\omega_3) \Delta (\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - 2\pi\boldsymbol{\tau}).$$

In the expressions L we include terms with lowest n different from zero, i.e., n = 1. The quantities $\mathbf{R}_s$ and $\mathbf{R}_l$ are easily calculated:

$$\mathbf{R}_s = \frac{2}{3a^2} \frac{T}{\hbar} C_s \nabla T, \quad \mathbf{R}_l = \frac{1}{3a^3} \frac{T}{\hbar} C_l \nabla T. \qquad (14.13)$$

Therefore $\mathbf{g}$ is equal to

$$\mathbf{g} = \frac{C_l + 2C_s}{3A} \frac{T}{\hbar} \nabla T. \qquad (14.14)$$

Using (14.6) for $n_{\mathbf{k}}$ and $N_{\mathbf{k}s}$ and (14.14) for $\mathbf{g}$, we get the following expression for the heat flux:

$$S = \sum \varepsilon_{\mathbf{k}} n_{\mathbf{k}} v_{\mathbf{k}} + \sum \hbar\omega_{\mathbf{k}s} N_{\mathbf{k}s} \mathbf{s}_{\mathbf{k}s}$$

$$= -\frac{V}{a^3} \left(\frac{T}{\hbar}\right)^2 \left(\frac{2}{3} C_s + \frac{1}{3} C_l\right)^2 \frac{1}{A} \nabla T.$$

On the other hand,

$$\mathrm{S} = -V\varkappa\nabla T,$$

where κ is the thermal conductivity coefficient. Therefore

$$\varkappa = \left(\frac{T}{\hbar}\right)^2 \left(\frac{2}{3} C_s + \frac{1}{3} C_l\right)^2 \frac{1}{a^3 A}. \qquad (14.15)$$

Let us now estimate the quantity A. Since when quasimomentum is not conserved we cannot simultaneously have all the components of the quasimomenta of the colliding particles be small, the quantities L must contain exponential factors. The exponents of these factors are obviously determined by the lowest values of the quantities $\epsilon_1 + \epsilon_2$, $\hbar\omega_1 + \hbar\omega_2$, and $\hbar\omega_1 + \epsilon_2$ in the corresponding regions of integration. To find these minimum values in general is not possible, since to do this it would be necessary to know the exact dispersion laws for the spin waves and phonons in the region of large wave vectors. We can, however, assert that, in order of magnitude, the minimum values of $\epsilon_1 + \epsilon_2$ and $\hbar\omega_1 + \hbar\omega_2$ are equal to θ_C and θ_D. Therefore the quantities L^{ss} and L^{ll} can be written in the form:

$$L^{ss} = \alpha_s \frac{\theta_C}{\hbar} \frac{V}{a^5} \left(\frac{T}{\theta_C}\right)^s e^{-\gamma_s \frac{\theta_C}{T}}, \qquad (14.16)$$

$$L^{ll} = \alpha_l \frac{\hbar}{\varrho a^5} \frac{V}{a^5} \left(\frac{T}{\theta_D}\right)^l e^{-\gamma_l \frac{\theta_D}{T}}, \qquad (14.17)$$

where γ_s, γ_l and α_s, α_l are of the order of magnitude of unity, while the power of the factor before the exponentials will depend on where the minimum values of the quantities $\epsilon_1 + \epsilon_2$, $\hbar\omega_1 + \hbar\omega_2$ are reached within the integration region, when both of the quasimomenta $\mathbf{k}_1$ and $\mathbf{k}_2$ are large, or for $k_1 = \pi/b$, $k_2 \ll \pi/b$. If we assume that $\epsilon(\mathbf{k}) = \theta_C (ak)^2$ and $\omega_{\mathbf{k}s} = s_s k$ down to $k = \pi/a$, the quantities s and l will be equal to $s = 4$, $l = 2$, and $\gamma_s = 1/2$, $\gamma_l = 1$.

We shall now evaluate the quantity L^{sl} in the two limiting cases when $\theta_D \gg \theta_C$ and $\theta_D \ll \theta_C$. If $\theta_D \ll \theta_C$, then, as follows from the conservation laws,

$$\varepsilon_1 + \hbar\omega_2 = \varepsilon_3 \text{ и } \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + 2\pi\boldsymbol{\tau},$$

$$-\pi\tau \leqslant k_j \leqslant \pi\tau, \ k_1 \approx k_3 \approx \pi\tau, \ k_1 \ll \pi\tau$$

and therefore

$$\varepsilon_1 + \hbar\omega_2 \approx \theta_C.$$

Thus, for $\theta_C \gg \theta_D$ the exponent of the exponential in the expression for L^{sl} is of the order of magnitude of θ_C, and L^{sl} has the form:

$$L^{sl} \simeq \alpha \frac{\hbar}{\varrho a^5} \frac{V}{a^5} \left(\frac{T}{\theta_C}\right)^n \left(\frac{T}{\theta_D}\right)^m e^{-\gamma \frac{\theta_C}{T}} \qquad (14.18)$$

We note that, if the quadratic dispersion law for the spin waves and the linear dispersion law for phonons were valid down to $k = \pi\tau$, the quantities n, m and γ would have the following values:

$$n = 1, \quad m = 2, \quad \gamma = \frac{1}{4}.$$

It is easy to show that in the limiting case of $\theta_D \gg \theta_C$ the radiation of a phonon by the spin wave is impossible; therefore for $\theta_D \gg \theta_C$ the quantity $L^{sl} = 0$.

Comparison of the expressions (14.16), (14.17) and (14.18) for L^{ss}, L^{ll}, and L^{ls} shows that if $\theta_C \gg \theta_D$, $L^{ll} \gg L^{ss}$, L^{sl}; if $\theta_C \ll \theta_D$, $L^{ss} \gg L^{ll}$, L^{ls}. Therefore in these limiting cases the thermal conductivity is determined by the following formulas:

$$\varkappa \approx \frac{4}{9}\frac{T}{a\hbar} e^{\frac{\theta_C}{2T}} C_s^2, \quad \theta_D \gg \theta_C. \tag{14.19}$$

If $\theta_D \ll \theta_C$,

$$\varkappa \approx \frac{ms^2}{9a\hbar} e^{\frac{\theta_D}{T}} \cdot \left| \begin{array}{ll} 4C_s^2, & T \ll \frac{\theta_D^2}{\theta_C}, \\ C_l^2, & \theta_D \gg T \gg \frac{\theta_D^2}{\theta_C}. \end{array} \right. \tag{14.20}$$

We note that for $T \ll \theta_D^2/\theta_C$ the main part in the thermal conductivity is played by the spin waves, since then $C_s \gg C_l$. But if $\theta_D \gg T \gg \theta_D^2/\theta_C$, the heat is transferred by the phonons, since then $C_s \ll C_l$.

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