SPIN WAVES IN FERROMAGNETS AND ANTIFERROMAGNETS, I

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THE energy spectrum of a macroscopic body determines all its thermodynamic and kinetic properties. For example, the phonon spectrum of sound vibrations determines the specific heat and the thermal conductivity of the simplest dielectric crystals, the electron spectrum of metals determines their electrical, magnetic, and thermal properties. Therefore, the finding of the energy spectrum is the most important problem of a microscopic theory of macroscopic bodies. However, this problem is fraught with great mathematical difficulties, and its solution is known only for some of the simplest systems. In addition to the phonon spectrum mentioned above, which applies for any solid body, there have been investigated theoretically also the energy spectrum of ferromagnets near saturation, the spectrum of helium II, and the energy spectrum of superconductors.

In the present summary we shall consider the fundamental properties of the ferromagnetic energy spectrum in the neighborhood of magnetic saturation. This spectrum determines at low temperatures the dependence of the magnetization of the ferromagnet on temperature and external magnetic field, the thermal properties of the ferromagnet, the relaxation of the magnetic moment, and the behavior of the ferromagnet in variable electromagnetic and acoustic fields.

The ferromagnetic energy spectrum appears in crystals, in which the exchange interaction between atoms plays a fundamental role. Under these conditions, the deviation of the magnetic moment of any atom from the prevailing direction is not localized at a definite place in the crystal lattice, but is propagated in the form of a wave with a wave vector of definite frequency dependence.

If we multiply the frequency of this wave, which is called a spin wave,* by the quantum constant $\hbar$, we obtain the energy of the elementary excitation associated with the spin wave. In the neighborhood of the maximum value of the magnetic moment, the magnetic part of the energy of the ferromagnet is a sum of energies of such elementary excitations.

A macroscopic body can in general have various energy spectra, which can be regarded as different branches of the single energy spectrum of the body. Each such branch is characterized by its own dispersion law, i.e., by the dependence of the energy on the wave vector. The main point is that the weakly excited energy states of macroscopic bodies can be treated as a set of gases of elementary excitations or quasi-particles.†

The properties of these gases, aside from the dispersion law, are determined by the statistics to which the quasi-particles are subjected and also by the nature of the interaction of the quasi-particles with one another.

At sufficiently low temperatures the interaction between quasi-particles is weak. If we neglect these in-

*Spin waves were first introduced and investigated by F. Bloch.[1]

†This assertion is not fully rigorous for systems with a Fermi spectrum (cf. reference 2).

teractions, i.e., if we start from the picture of an ideal gas of quasi-particles, then, knowing the dispersion law and the statistics of the quasi-particles, we can obtain all the thermodynamic properties of the body. To study the kinetic properties of the body, ideal gas models are not sufficient, since these properties are essentially determined by the interaction between the quasi-particles.

The problems of the theory of ferromagnetism which are presented below are linked by the general idea of spin waves. We shall, for the most part, consider the properties of ferro-dielectrics, i.e., bodies which have a spontaneous magnetic moment and do not possess conductivity. Ferrites at low temperatures are very similar in their properties to such bodies.

I. ENERGY SPECTRA. THERMAL AND MAGNETIC PROPERTIES OF FERROMAGNETS AND ANTIFERROMAGNETS.

1. Spin Waves in Ferromagnets

The introduction of spin waves can be accomplished in two ways: by starting from a microscopic model of the ferromagnet, and by a phenomenological argument. Let us start with a presentation of the microscopic theory of spin waves.[1,3]

In the simplest model of a ferromagnet it is assumed that in the ground state the spins of all the atoms are oriented in one direction. The dependence of the energy of the system on the orientation of the spins is determined not by the magnetic interaction of the atoms, which is very small, but is a purely quantum effect related to the indistinguishability of the particles. Because of this effect and also because of the Pauli principle, spatial wave functions with different symmetries will correspond to different values of the total spin of the system and will give different energy values. Thus we can say that some peculiar interaction between the particles leads to a dependence of the energy on the total spin of the system. This interaction is called the exchange interaction (cf., for example, reference 4).

The Hamiltonian for the exchange interaction of two particles can be represented as

$$\mathcal{H} = -J(r_{12})\,\mathbf{s}_1\mathbf{s}_2, \tag{1.1}$$

where $\mathbf{s}_1$ and $\mathbf{s}_2$ are the spins of the particles, and $J(r_{12})$ is some function of the distance r_{12} between the particles; $J(r_{12})$ is called the exchange integral. It is easily verified that $\mathbf{s}_1\mathbf{s}_2 = -\frac{3}{4}$ for a singlet state (total spin $S = 0$) and $\mathbf{s}_1\mathbf{s}_2 = \frac{1}{4}$ for a triplet state ($S = 1$); thus for positive $J(r_{12})$, the triplet state has lower energy.

Since the interaction between the particles is basically electrostatic in nature, the exchange integral has the same order of magnitude as the electrostatic interaction energy of the atoms.

In the exchange model of a ferromagnet,* we start from a Hamiltonian which is a generalization of the Hamiltonian (1.1),

$$\mathcal{H} = -\frac{1}{2}\sum_{l,m} J(r_{lm})\,\mathbf{s}_l\mathbf{s}_m. \tag{1.2}$$

where $\mathbf{s}_l$ is the spin of the l-th atom, $J(r_{lm})$ is the exchange integral between the l-th and m-th atoms, $r_{lm} = |\mathbf{r}_l - \mathbf{r}_m|$, and $\mathbf{r}_m$ is the radius vector of the m-th atom; the summation extends over all atoms in the crystal. The quantity $J(r_{lm})$ is assumed to be positive so that in the ground state all the atomic spins have the same orientation. However, this orientation is not a distinguished direction since the function $J(r_{lm})$ is a scalar.

In the presence of an external magnetic field H_0, which is directed along the z axis, we must add to the Hamiltonian (1.2) the energy of the spins in the external field, which is equal to $-2\mu_0\sum_l s_l^z H_0$, where μ_0 is the Bohr magneton.

Thus the Hamiltonian operator in the presence of an external magnetic field has the form

$$\mathcal{H} = -\frac{1}{2}\sum_{l,m} J(r_{lm})\,\mathbf{s}_l\mathbf{s}_m - 2\mu_0 H_0\sum_l s_l^z. \tag{1.3}$$

Our problem is to find the eigenvalues of the Hamiltonian (1.3) for the case where the magnetic moment is close to saturation. For this purpose it is convenient to introduce the circular spin projections

$$s_l^+ = s_l^x + is_l^y, \quad s_l^- = s_l^x - is_l^y,$$

which satisfy the commutation conditions

$$s_l^+ s_{l'}^- - s_{l'}^- s_l^+ = 2s_l^z\delta_{ll'},$$

and in addition go over to the operators a_l and a_l^+, which are related to the operators s_l^+, s_l^-, and s_l^z, by the following relations:†

$$\left.\begin{aligned} s_l^+ &= (2s)^{1/2}\,a_l^+\left(1-\frac{a_l^+a_l}{2s}\right)^{1/2},\\ s_l^- &= (2s)^{1/2}\left(1-\frac{a_l^+a_l}{2s}\right)^{1/2}a_l,\\ s_l^z &= s - a_l^+a_l. \end{aligned}\right\} \tag{1.4}$$

It is easy to verify that the operators a_l and a_l^+ satisfy the commutation relations

$$a_l a_m^+ - a_m^+ a_l = \delta_{lm}. \tag{1.5}$$

From these relations it follows that the eigenvalues of the operator $\mathfrak{N}_l = a_l^+ a_l$ are positive integers. The quantity $\mathfrak{N}_l$ obviously determines the deviation of the z-projection of the spin of the l-th atom from its maximum value.

We note that, according to the commutation relations (1.5), $\mathfrak{N}$ runs through all values from zero to infinity,

*The exchange model of a ferromagnet was proposed by Ya. I. Frenkel', Ya. G. Dorfman[5] and W. Heisenberg.[6]

†Here we follow the paper of Holstein and Primakoff.[3]

whereas, according to (1.4), it cannot exceed 2s. This contradiction, however, is unimportant, since in the sequel we shall be interested only in states which are close to the ground state, for which case the main role is played by small deviations of the spins $\mathfrak{N}_l$. For this case, replacing the square root in (1.4) by unity:

$$\left.\begin{aligned} s_l^+ &\approx (2s)^{1/2} a_l, \\ s_l^- &\approx (2s)^{1/2} a_l^+, \\ s_l^z &= s - a_l^+ a_l, \end{aligned}\right\} \qquad (1.4')$$

and substituting these formulas in (1.3), we obtain the following approximate expression for the Hamiltonian of the ferromagnet:

$$\mathcal{H} = -2s \sum_{l,m} J(r_{lm}) (a_l^+ a_m - a_l^+ a_l) + 2\mu_0 H_0 \sum_l a_l^+ a_l + C, \qquad (1.6)$$

where

$$C = -\sum_{l,m} J(r_{lm}) s^2 - 2\mu_0 sNH_0$$

and N is the total number of atoms in the body. The quantity C is the energy of the ground state of the system when all the spins are oriented along the z axis.

In order to find the eigenvalues of the Hamiltonian (1.6), we go over from the operators a_l and a_l^+ to their Fourier transforms $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^+$:

$$\begin{aligned} a_{\mathbf{k}} &= \frac{1}{\sqrt{N}} \sum_l e^{-i\mathbf{k}\mathbf{r}_l} a_l, \qquad a_{\mathbf{k}}^+ = \frac{1}{\sqrt{N}} \sum_l e^{i\mathbf{k}\mathbf{r}_l} a_l^+, \\ a_l &= \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}_l} a_{\mathbf{k}}, \qquad a_l^+ = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{r}_l} a_{\mathbf{k}}^+. \end{aligned} \qquad (1.7)$$

It is easy to verify that the operators $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^+$ satisfy the same commutation relations as a_l and a_m^+:

$$a_{\mathbf{k}} a_{\mathbf{k}'}^+ - a_{\mathbf{k}'}^+ a_{\mathbf{k}} = \delta_{\mathbf{k}\mathbf{k}'}. \qquad (1.8)$$

Substituting (1.7) into (1.6) we obtain

$$\mathcal{H} = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} a_{\mathbf{k}}^+ a_{\mathbf{k}} + C, \qquad (1.9)$$

where

$$\varepsilon_{\mathbf{k}} = \sum_h 2sJ(r_h)(1 - e^{i\mathbf{k}\mathbf{r}_h}) + 2\mu_0 H_0. \qquad (1.9')$$

It follows from (1.8) that the eigenvalues of the operator $\mathcal{H}$ are equal to

$$\mathcal{E} = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} n_{\mathbf{k}} + C, \qquad (1.10)$$

where $n_{\mathbf{k}} = 0, 1, 2 \ldots$

Thus we see that the energy of weakly excited states of the ferromagnet can be represented as a sum of energies of individual particles — the elementary excitations. The energy of each excitation is equal to $\epsilon_{\mathbf{k}}$. These excitations are called spin waves.

In the representation in which the Hamiltonian operator (1.9) is diagonal, the non-zero matrix elements of the operators $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^+$ have the following form:

$$\begin{aligned} (n_{\mathbf{k}} - 1 \,|\, a_{\mathbf{k}} \,|\, n_{\mathbf{k}}) &= \sqrt{n_{\mathbf{k}}}\, e^{-i\frac{\varepsilon_{\mathbf{k}} t}{\hbar}}, \\ (n_{\mathbf{k}} \,|\, a_{\mathbf{k}}^+ \,|\, n_{\mathbf{k}} - 1) &= \sqrt{n_{\mathbf{k}}}\, e^{i\frac{\varepsilon_{\mathbf{k}} t}{\hbar}}. \end{aligned} \qquad (1.11)$$

Thus the operators $a_{\mathbf{k}}^+$ and $a_{\mathbf{k}}$ can be interpreted as the operators for emission and absorption of spin waves with wave vector $\mathbf{k}$.

In the case of weakly excited states the quantities $\epsilon_{\mathbf{k}}$ must be small compared to the exchange integral for neighboring atoms. Therefore, as we see from formula (1.9'), we must limit ourselves in this theory to long-wave oscillations, for which $ak \ll 1$, where a is the lattice constant. In this case it is easy to determine the dependence of $\epsilon_{\mathbf{k}}$ on $\mathbf{k}$. Expanding the exponent (1.9') in powers of $\mathbf{k}$ and making use of the rapid decrease of the exchange integral with distance, we get

$$\varepsilon_{\mathbf{k}} = 2sJ(ak)^2 + 2\mu_0 H_0, \qquad (1.12)$$

where J is the exchange integral for neighboring atoms (for simplicity it is assumed that the crystal has the symmetry of a simple cubic lattice).

The exchange integral J coincides in order of magnitude with the Curie temperature T_C.

In the Hamiltonian (1.2) we do not consider magnetic interaction between the spins of the atoms and spin-orbit interaction. Although these interactions have a relativistic character and are therefore weak compared with the exchange interaction, it is necessary to consider them when one investigates kinetic and relaxation processes, and also when one computes thermodynamic quantities for ferromagnets in the region of very low temperatures. Relativistic interactions are most easily considered within the framework of a phenomenological theory of spin waves, to the presentation of which we now turn.

2. Phenomenological Theory of Spin Waves

We shall show that spin waves can be introduced purely phenomenologically as oscillations of the magnetic moment of the ferromagnet.*

Let us first write the expression for the classical Hamiltonian of a ferromagnet (cf. reference 10). The classical analog of the Hamiltonian for the exchange interaction can be obtained in the case of long wave oscillations, when the magnetic moment is a slowly varying function of the coordinates, from the expression (1.2).

Noting that the exchange integral falls off rapidly (exponentially) with distance between the atoms, we can in the Hamiltonian $\mathcal{H} = -\frac{1}{2} \sum_{l,h} J(\mathbf{r}_h)\, \mathbf{s}_l \cdot \mathbf{s}_{l+h}$ take the summation over h only over nearest neighbors and replace the summation over l by an integration

*This problem was investigated rigorously by Dyson.[7]

*Such a treatment is due to E. Lifshitz;[8] cf. also the paper of Herring and Kittel.[9]

over the volume. Changing from $\mathbf{s}$ to the magnetic moment per unit volume $\mathbf{M}(\mathbf{r}) = \mu_0 \mathbf{s}/a^3$ and expanding $\mathbf{s}_{l+h}$ in powers of h, we obtain, omitting terms proportional to s^2 which are unimportant for what follows, the following expression for the Hamiltonian of the exchange interaction:

$$\mathcal{H} = -\frac{1}{2}\int \alpha_{ih}\frac{\partial^2 M_l}{\partial x_i\,\partial x_h} M_l\, dv.$$

where the quantity α_{ik} is proportional to J (terms proportional to $\frac{\partial}{\partial x_k} M_l$ obviously vanish if the crystal has a center of inversion). Integrating the last expression by parts, we finally obtain:

$$\mathcal{H} = \frac{1}{2}\int \alpha_{ih}\frac{\partial M_l}{\partial x_i}\frac{\partial M_l}{\partial x_k}\, dv. \tag{2.1}$$

This expression can be regarded as the expansion of the exchange interaction energy in powers of the gradient of the magnetic moment. We note that in general this expression should be written in the form

$$\mathcal{H} = \frac{1}{2}\int \alpha_{iklm}\frac{\partial M_l}{\partial x_i}\frac{\partial M_m}{\partial x_k}\, dv,$$

where α_{iklm} is some 4'th rank tensor. Since, however, the exchange interaction is invariant under rotation of the spins, the tensor α_{iklm} must have the form $\alpha_{ik}\delta_{lm}$, and we arrive at the expression (2.1).

In the case of a cubic crystal, $\alpha_{ik} = \alpha\delta_{ik}$ and

$$\mathcal{H} = \frac{1}{2}\int \alpha\left(\frac{\partial M_l}{\partial x_i}\right)^2 dv. \tag{2.2}$$

We shall write the quantity α in the form:

$$\alpha = \frac{\theta_C}{\mu_0 M_0}\, a^2, \tag{2.2'}$$

where θ_C coincides in order of magnitude with the Curie temperature and M_0 is the saturation magnetic moment.

In addition to the exchange energy, the Hamiltonian of a ferromagnet must contain a part which depends not on the derivatives of the magnetic moment, but directly on its components. This part of the Hamiltonian which is called the anisotropy energy, is due to the relativistic spin-spin and spin-orbit interactions. The anisotropy energy, which we shall denote by $\beta(\mathbf{M})$, depends on the orientation of the magnetic moment with respect to the crystallographic axes. In the case of a uniaxial crystal, $\beta(\mathbf{M})$ has the form:[10]

$$\beta(\mathbf{M}) = -\frac{1}{2}\beta(\mathbf{Mn})^2,$$

where $\mathbf{n}$ is a unit vector along the direction of the axis of easiest magnetization, and β is a number independent of $\mathbf{n}$. In cubic crystals[10]

$$\beta(\mathbf{M}) = \frac{\beta}{2M_0^2}(M_x^2M_y^2 + M_x^2M_z^2 + M_y^2M_z^2)$$

(the axes x, y, z are along the crystallographic axes).

Finally, the Hamiltonian of the ferromagnet must also contain the electromagnetic field energy, whose density is equal to $\frac{1}{8\pi}(\mathbf{H}^2 + \mathbf{ED})$ where H and E are the magnetic and electric field strengths, and D is the electric induction.

Thus the classical Hamiltonian of the ferromagnet has the form;

$$\mathcal{H} = \int_V \left\{\frac{1}{2}\alpha_{ih}\frac{\partial M_l}{\partial x_i}\frac{\partial M_l}{\partial x_h} + \beta(\mathbf{M}) + \frac{\mathbf{H}^2 + \mathbf{ED}}{8\pi}\right\} dv. \tag{2.3}$$

In addition to the Hamiltonian (2.3) we must know the equation of "motion" of the magnetic moment, i.e., the law of variation of the magnetic moment with time. If we neglect dissipative processes, this change is determined by the equation*

$$\frac{d\mathbf{M}}{dt} = g[\mathbf{M} \times \mathbf{H}^{(e)}], \tag{2.4}$$

where $\mathbf{H}^{(e)}$ is the effective field acting on the magnetic moment, and g is the gyromagnetic ratio which here is regarded as some experimentally determined constant. This equation corresponds to the assumption of "rigidity" of the magnetic moment, i.e., that its absolute magnitude is invariant.

The effective field $\mathbf{H}^{(e)}$ can be defined as the negative of the functional derivative of the Hamiltonian $\mathcal{H}$ with respect to the magnetic moment M for fixed induction B. The variation of the field and the magnetic moment are connected here by the relation

$$\delta\mathbf{H} = -4\pi\delta\mathbf{M},$$

and consequently

$$\mathbf{H}^{(e)} = -\left(\frac{\delta\mathcal{H}}{\delta\mathbf{M}}\right)_{\mathbf{B}} = \mathbf{H} - \frac{\partial\beta}{\partial\mathbf{M}} + \alpha_{ih}\frac{\partial^2\mathbf{M}}{\partial x_i\,\partial x_h}. \tag{2.5}$$

In the case of a uniaxial crystal

$$\mathbf{H}^{(e)} = \mathbf{H} + \beta\mathbf{n}(\mathbf{Mn}) + \alpha_{ik}\frac{\partial^2\mathbf{M}}{\partial x_i\,\partial x_k}.$$

Adding to (2.5) the Maxwell equations

$$\left.\begin{aligned} &\operatorname{curl}\mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t},\\ &\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},\\ &\operatorname{div}\mathbf{B} = 0,\\ &\operatorname{div}\mathbf{D} = 4\pi\varrho, \end{aligned}\right\} \tag{2.6}$$

where j and ρ are the current and charge densities, and, assigning the relation between D and E, we obtain the complete system of equations for determining all the quantities in which we are interested.

The expression for $\mathbf{H}^{(e)}$ can also be obtained by computing the total derivative of $\mathcal{H}$ with respect to the time.[12] Using (2.5) and (2.6) we find:

$$\frac{d\mathcal{H}}{dt} = \int_V \left\{\alpha_{ik}\frac{\partial\mathbf{M}}{\partial x_i}\frac{\partial^2\mathbf{M}}{\partial x_k\partial t} + \frac{\partial\beta}{\partial\mathbf{M}}\frac{\partial\mathbf{M}}{\partial t} + \frac{1}{4\pi}\left(\mathbf{H}\frac{\partial\mathbf{H}}{\partial t} + \mathbf{E}\frac{\partial\mathbf{D}}{\partial t}\right)\right\} dv$$

$$= \int_V \left\{-\frac{\partial\mathbf{M}}{\partial t}\left(\alpha_{ik}\frac{\partial^2\mathbf{M}}{\partial x_i\,\partial x_k} + \mathbf{H} - \frac{\partial\beta}{\partial\mathbf{M}}\right) - \mathbf{jE}\right\} dv$$

$$-\int_S \left\{\frac{c}{4\pi}[\mathbf{E}\times\mathbf{H}]_k - \alpha_{ik}\frac{\partial\mathbf{M}}{\partial x_i}\frac{\partial\mathbf{M}}{\partial t}\right\} ds_k,$$

*This equation, together with the expression (2.5) for $\mathbf{H}^{(e)}$, was established by L. Landau and E. Lifshitz.[11]

where $d\mathbf{s}$ is an element of the surface S surrounding the volume of the ferromagnet. If we neglect dissipative processes, the volume integral must obviously vanish, i.e.,

$$-\int_V g[\mathbf{M}\times\mathbf{H}^{(e)}]_l\left(\alpha_{ik}\frac{\partial^2 M_l}{\partial x_i\,\partial x_k}-\frac{\partial\beta}{\partial \mathbf{M}_l}+\mathbf{H}_l\right)dv=0.$$

This equation is satisfied identically if we assume that

$$\mathbf{H}^{(e)}=\mathbf{H}-\frac{\partial\beta}{\partial\mathbf{M}}+\alpha_{ik}\frac{\partial^2\mathbf{M}}{\partial x_i\,\partial x_k}.$$

Thus we again arrive at expression (2.5) for the effective field $\mathbf{H}^{(e)}$.

Thus

$$\frac{d\mathcal{H}}{dt}=-\int_V \mathbf{j}\mathbf{E}\,dv-\int_S\left\{\frac{c}{4\pi}[\mathbf{E}\times\mathbf{H}]_k-\alpha_{ik}\frac{\partial\mathbf{M}}{\partial x_i}\frac{\partial\mathbf{M}}{\partial t}\right\}ds_k. \quad (2.7)$$

We see that the density of the energy flux in the ferromagnet has the form:[12]

$$\Pi_k=\frac{c}{4\pi}[\mathbf{E}\times\mathbf{H}]_k-\alpha_{ik}\frac{\partial M_l}{\partial x_i}\frac{\partial M_l}{\partial t}. \quad (2.8)$$

Here we have, in addition to the usual Poynting vector, an additional vector $-\alpha_{ik}\dfrac{\partial \mathrm{M}_l}{\partial \mathrm{x_i}}\dfrac{\partial \mathrm{M}_l}{\partial \mathrm{t}}$ which corresponds to the energy flux transferred by spin waves.

In the quasi-static case (curl $\mathbf{H}=0$), the Poynting vector $\dfrac{c}{4\pi}[\mathbf{E}\times\mathbf{H}]$ goes over, as one easily shows, to $\dfrac{1}{4\pi}\varphi\dot{\mathbf{B}}$, where φ is the potential of the magnetic field, $\mathbf{H}=-\nabla\varphi$; thus, in this case the energy flux density has the form

$$\Pi_k=\frac{1}{4\pi}\frac{\partial}{\partial t}\dot{B}_k\varphi-\alpha_{ik}\frac{\partial}{\partial x_i}M_l\frac{\partial}{\partial t}M_l. \quad (2.8')$$

Let us now proceed to a classical treatment of the small oscillations of the magnetic moment around the equilibrium value $\mathbf{M}_0$ corresponding to the ground state of the ferromagnet. We shall assume that a constant, homogeneous magnetic field $\mathbf{H}_0$ acts in the interior of the ferromagnet. If the external magnetic field is directed along the axis of easiest magnetization, $\mathbf{M}_0$ will also be directed along this axis. If this is not the case, the direction of the vector $\mathbf{M}_0$ is determined from the condition

$$\frac{d}{dt}\mathbf{M}=g[\mathbf{M}\times\mathbf{H}^{(e)}]=0,$$

i.e.,

$$\mathbf{H}_0-\frac{\partial\beta}{\partial\mathbf{M}}+\nu\mathbf{M}=0, \quad (2.9)$$

where ν is a parameter which is found from the condition that the square of the magnetic moment be fixed. In the case of a uniaxial crystal, we obtain in this way the following relation between the direction of the magnetic moment and the direction of the external magnetic field $\mathbf{H}_0$:

$$H_0\sin(\varphi-\psi)-\frac{1}{2}\beta M_0\sin 2\psi=0, \quad (2.9')$$

where φ is the angle between $\mathbf{H}_0$ and $\mathbf{n}$, and ψ is the angle between $\mathbf{M}_0$ and $\mathbf{n}$ ($\mathbf{n}$ is a unit vector along the axis of easiest magnetization); the vector $\mathbf{M}_0$ lies in the plane of $\mathbf{n}$ and $\mathbf{H}_0$.

Let us set $\mathbf{M}=\mathbf{M}_0+\mathbf{m}$, $\mathbf{H}=\mathbf{H}_0+\mathbf{h}$, where $\mathbf{m}$ and $\mathbf{h}$ are small additions to $\mathbf{M}_0$ and $\mathbf{H}_0$, and let us write the linearized equation of motion of the magnetic moment and the Maxwell equations. If the ferromagnet, which we shall assume to be a dielectric, is uniaxial, and the magnetic field $\mathbf{H}_0$ is along the axis of easiest magnetization, these equations have the form

$$\left.\begin{aligned}\frac{\partial\mathbf{m}}{\partial t}&=gM_0\left[\mathbf{n},\ \alpha_{ik}\frac{\partial^2\mathbf{m}}{\partial x_i\,\partial x_k}-\frac{1}{M_0}(H_0+\beta M_0)\,\mathbf{m}+\mathbf{h}\right],\\ \operatorname{curl}\mathbf{h}&=\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t},\\ \operatorname{curl}\mathbf{E}&=-\frac{1}{c}\frac{\partial\mathbf{b}}{\partial t},\end{aligned}\right\} \quad (2.10)$$

where $\mathbf{b}=\mathbf{h}+4\pi\mathbf{m}$.

Let us consider first the low-frequency oscillations. In this case we can neglect the displacement current and set curl $\mathbf{h}=0$. The solution of these equations in the form of plane waves $[e^{-i(\omega t-\mathbf{k}\cdot\mathbf{r})}]$ gives

$$-i\omega\mathbf{m}=-gM_0\left[\mathbf{n},\ \alpha_{ik}k_ik_k\mathbf{m}+\frac{1}{M_0}(H_0+\beta M_0)\,\mathbf{m}-\mathbf{h}\right],$$

$$\mathbf{h}=-4\pi\frac{1}{k^2}\mathbf{k}\,(\mathbf{mk}),$$

from which*

$$\hbar\omega_{\mathbf{k}}=\sqrt{A_{\mathbf{k}}^2-B_{\mathbf{k}}^2}, \quad (2.11)$$

where

$$\left.\begin{aligned}A_{\mathbf{k}}&=\mu M_0\left(\alpha_{ij}k_ik_j+\beta+\frac{H_0}{M_0}+2\pi\sin^2\theta_{\mathbf{k}}\right),\quad \mu=g\hbar,\\ B_{\mathbf{k}}&=2\pi\mu M_0\sin^2\theta_{\mathbf{k}}e^{2i\varphi_{\mathbf{k}}}\end{aligned}\right. \quad (2.11')$$

and $\theta_{\mathbf{k}}$ and $\varphi_{\mathbf{k}}$ are the polar angles of the vector $\mathbf{k}$.

The quantity $\epsilon_{\mathbf{k}}=\hbar\omega_{\mathbf{k}}$ is the energy of the spin wave.

If in formula (2.11) we neglect the anisotropy constant β and the quantity $2\pi\sin^2\theta_{\mathbf{k}}$, which results from inclusion of the magnetic field arising as a result of the oscillations of the magnetic moment, we obtain the formula (1.12) given earlier for the energy of the spin wave.

In the general case, when the direction of the magnetic field $\mathbf{H}_0$ does not coincide with the direction of easiest magnetization, the frequency of oscillations of the magnetic moment is given by formula (2.11) where[13]

$$\left.\begin{aligned}A_{\mathbf{k}}&=\mu M_0\left[\alpha_{ik}k_ik_k+\beta\cos^2\psi+\frac{H_0}{M_0}\cos(\varphi-\psi)\right.\\ &\quad\left.-\frac{1}{2}\beta\sin^2\psi+2\pi\sin^2\theta_{\mathbf{k}}\right],\\ B_{\mathbf{k}}&=\mu M_0\left(2\pi\sin^2\theta_{\mathbf{k}}e^{2i\varphi_{\mathbf{k}}}+\frac{1}{2}\beta\sin^2\psi\right).\end{aligned}\right. \quad (2.11'')$$

We note that, in the case where the external field is directed along the axis of easiest magnetization, the anisotropy energy can be expressed in terms of an effective anisotropy magnetic field equal to

$$\mathbf{H}_0=\beta\mathbf{M}_0.$$

So far we have not included processes leading to damping of the spin waves. Such processes are related to the presence of conductivity of the medium and

*The structure of this formula was obtained in reference 3.

relaxation of the magnetic moment. Including these processes makes it possible to clarify the conditions for existence of spin waves.[12]

We can include relaxation of the magnetic moment by adding to the right side of the equation of motion of the magnetic moment a dissipative force in addition to the rotational force. This dissipative force takes account of the change in angle between the magnetic moment and the effective magnetic field.

The simplest expression for such a force is the expression $-\lambda/M^2[\mathbf{M}[\mathbf{M}, \mathbf{H}^{(e)}]]$, where λ is a constant which we may call the "relaxation constant." We shall therefore start from the following equation of motion of the magnetic moment:[11]

$$\frac{d\mathbf{M}}{dt} = g[\mathbf{M} \times \mathbf{H}^{(e)}] - \frac{\lambda}{M^2}[\mathbf{M} \times [\mathbf{M} \times \mathbf{H}^{(e)}]]. \quad (2.12)$$

An estimate of the value of λ will be given in Sec. 12, starting from a microscopic theory of the relaxation process.

Let us now determine the damping of the spin waves assuming that it is sufficiently small. In this case, the damping coefficient can be defined by

$$\Gamma = -\frac{1}{2}\frac{1}{\overline{\mathcal{H}}}\frac{d\overline{\mathcal{H}}}{dt}, \quad (2.13)$$

where $\frac{d\overline{\mathcal{H}}}{dt}$ is the average energy loss per unit time, and $\overline{\mathcal{H}}$ is the average value of the energy of the system in the absence of absorption. The quantity $\frac{d\overline{\mathcal{H}}}{dt}$ can be found from the Hamiltonian (2.3) by using the Maxwell equations and the equation of motion[12] of the magnetic moment (2.12)

$$\frac{d\mathcal{H}}{dt} = -\int_V \frac{\lambda}{M^2}[\mathbf{M} \times \mathbf{H}^{(e)}]^2 dv - \int_V \sigma^{(e)} \mathbf{E}^2 dv. \quad (2.14)$$

Here the first term gives the losses associated with relaxation of the magnetic moment, and the second term gives the Joule heat; $\sigma^{(e)}$ is the effective conductivity of the medium. Since we are interested in low temperatures, we should in general take into account the fact that the mean free path of the conduction electron may be of the order of or even much greater than the wave length of the oscillations of the magnetic moment. In the limiting cases of large and small mean free paths the quantity $\sigma^{(e)}$ is given by the following formulas:

$$\sigma^{(e)}(k) \approx \begin{cases} \sigma_0, & l \ll \delta, \\ \frac{\sigma_0}{kl}, & l \gg \delta, \end{cases} \quad (2.15)$$

where σ_0 is the static electric conductivity, σ is the skin depth, equal to $\sigma = \left(\frac{c^2}{2\pi\omega\sigma_0}\right)^{1/2}$ [We note that the quantity $\sigma^{(e)}(k)$ relates the Fourier components of the current and the field.]

Neglecting the displacement current and using (2.13), we obtain, on the assumption that $\sigma^{(e)} \ll \frac{c^2k^2}{\omega} \sim \frac{c^2\hbar}{\theta_c}$, the following value for Γ:

$$\Gamma = 8\pi^2 \frac{\sigma^{(e)} gM_0}{\hbar (ck)^2}(A_k + |B_k|) + (A_k - |B_k|)\cos^2\theta_k) + \frac{\lambda}{\mu M_0}(A_k - |B_k|) + 2\pi\mu M_0 \sin^2\theta_k). \quad (2.16)$$

The absorption of the spin waves will be small if

$$\lambda \ll gM_0, \quad \sigma^{(e)} \ll \frac{(ck)^2}{gM_0}. \quad (2.17)$$

These inequalities are the conditions for existence of spin waves.

The first condition is satisfied over a wide range of temperatures (this follows from the fact that ferromagnetic resonance exists, since the frequency of ferromagnetic resonance is of the order of gM_0, while the width of the line is of the order of λ). From the second condition in (2.17) it follows that, for the existence of a spin wave, its wave vector must be greater than a certain limiting value k_0, equal to

$$k_0 = \begin{cases} \frac{1}{\delta_0}, & l \ll \delta_0 \\ (\delta_0^2 l)^{-1/3}, & l \gg \delta_0, \end{cases} \quad (2.17')$$

where $\delta_0 = \left(\frac{c^2}{2\pi\sigma_0 gM_0}\right)^{1/2}$.

If $k \ll k_0$, formulas (2.11), (2.11'), and (2.11'') are no longer valid, and the character of the energy spectrum changes markedly. This corresponds to the case of large $\sigma^{(e)}$, when the inequality $\sigma^{(e)} \gg \frac{c^2k^2}{\omega_k}$ is satisfied. We must therefore treat separately the limiting case $\sigma^{(e)} = \infty$. In this case, as for $\sigma^{(e)} = 0$, there is no dissipation of energy associated with Joule heating. The electric field is equal to zero, and $\mathbf{h} + 4\pi\mathbf{m} = 0$. From the equation of motion of the magnetic moment, it follows that the energy of an elementary excitation is equal to[12]

$$\varepsilon_k = \hbar\omega_k = \mu M_0\left(ak^2 + \beta + \frac{B_0}{M_0}\right), \quad B_0 = H_0 + 4\pi M_0. \quad (2.18)$$

This spectrum should be used for $k \ll k_0$, which corresponds to temperatures $T \ll T_0$, $T_0 = \mu B_0\left(\frac{a^2\sigma_0\theta C}{\hbar c^2}\right)^{1/2}$. On the other hand, for $k \to 0$, the energy of the spin wave (2.18) tends to a finite limit $\epsilon_0 = \mu(B_0 + \beta M_0)$. As a result of the presence of this activation energy, at very low temperatures we get an exponential dependence of the type $e^{-\epsilon_0/T}$ for all the thermodynamic quantities which are determined by the spectrum of spin waves. Therefore, their contribution to the heat capacity of the body is negligibly small in this temperature region.

In deriving the formulas for the energy of a spin wave, we have not considered the mobility of the carriers of the spins. Including this does not lead to any change[14] in the fundamental dispersion relation (2.11).

In this phenomenological procedure for introducing spin waves we understood by $\mathbf{M}_0$ the saturation magnetic moment of the ferromagnet corresponding to a temperature of absolute zero. However, as is clear from the derivation, we can treat in a similar fashion the small oscillations of the magnetic moment of the

ferromagnet for any temperature, where by $\mathbf{M}_0$ we mean the equilibrium value of the moment at the particular temperature T. We shall thus obtain magnetic waves whose dispersion properties for $\sigma^{(e)} = 0$ will not differ from the dispersion properties of the spin waves (2.11).

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These magnetic waves correspond to omitting the displacement current in the linearized equations (2.10). If we do not neglect the displacement current, we obtain electromagnetic waves of a complicated type whose dispersion relation has the form:[12]

$$\left(1+\frac{u\xi^2}{x^2-\xi^2}\right)\left(1+\frac{u\xi^2}{x^2-\xi^2}\cos^2\theta_{\mathbf{k}}\right)-x^2=0, \quad (2.19)$$

where

$$x=u\frac{\omega}{4\pi gM_0}, \quad u=\frac{4\pi\mu M_0}{A_{\mathbf{k}}-|B_{\mathbf{k}}|+4\pi\mu M_0}, \quad \xi^2=\frac{u^2}{\epsilon}\left(\frac{ck}{4\pi gM_0}\right)^2,$$

$\theta_{\mathbf{k}}$ is the angle between $\mathbf{k}$ and $\mathbf{M}_0$, and ϵ is the dielectric constant.

If $\xi^2 \gg 1$, this equation has the solutions

$$\omega^2=\frac{c^2k^2}{\epsilon}\left(1\pm\frac{4\pi gM_0}{\omega}\cos\theta_{\mathbf{k}}\right), \qquad \hbar\omega=\sqrt{A_{\mathbf{k}}^2-|B_{\mathbf{k}}|^2}. \quad (2.20)$$

The first of these determines the frequency of electromagnetic waves propagating in a medium with a gyrotropic magnetic permeability, which, for the two types of waves corresponding to right and left circular polarizations, is equal to

$$\mu^+(\omega)=1-\frac{4\pi gM_0}{\omega}\cos\theta_{\mathbf{k}}, \qquad \mu^-(\omega)=1+\frac{4\pi gM_0}{\omega}\cos\theta_{\mathbf{k}}. \quad (2.20')$$

The second solution gives a pure magnetic wave whose dispersion properties are identical with the dispersion properties of the spin waves (2.11) and (2.11′).

For $\xi^2 \ll 1$, Eq. (2.19) gives

$$\omega=ck\sqrt{\frac{1-u}{\epsilon}}, \quad \omega=ck\sqrt{\frac{1-u\cos^2\theta_{\mathbf{k}}}{\epsilon}}, \quad (2.21)$$

$$\omega=\frac{4\pi gM_0}{u}=g(B_0+\beta M_0+\alpha k^2M_0). \quad (2.21')$$

Here (2.21) determines the frequencies of the electromagnetic waves, and (2.21′) the frequency of magnetic waves.

3. Quantization of Spin Waves

We shall now show how, starting from the phenomenological Hamiltonian of the ferromagnet (2.3), we can construct a quantum theory of spin waves.[13] Such a theory will differ from the theory of spin waves which corresponds to the simplest exchange Hamiltonian of the ferromagnet (1.2) (cf. Sec. 1) in that it will include the magnetic interaction between the spins and the anisotropy energy, which were not included in (1.2).

In order to carry out the quantization of the spin waves, starting from the Hamiltonian (2.3), we must obviously relate the components of the magnetic moment of the ferromagnet with operators of creation and absorption of spin waves.

The components of the total magnetic moment of the body $\mathfrak{M}_x$, $\mathfrak{M}_y$, $\mathfrak{M}_z$ satisfy the following commutation relations:

$$\left.\begin{aligned}\mathfrak{M}_x\mathfrak{M}_y-\mathfrak{M}_y\mathfrak{M}_x&=-i\mu\mathfrak{M}_z,\\ \mathfrak{M}_z\mathfrak{M}_x-\mathfrak{M}_x\mathfrak{M}_z&=-i\mu\mathfrak{M}_y,\\ \mathfrak{M}_y\mathfrak{M}_z-\mathfrak{M}_z\mathfrak{M}_y&=-i\mu\mathfrak{M}_x,\end{aligned}\right\} \quad (3.1)$$

where $\mu = g\hbar$ and g is the gyromagnetic ratio. Transforming to components of the magnetic moment density $\mathbf{M}(\mathbf{r}, t)$, we obtain the commutation relations:

$$\left.\begin{aligned}M_x(\mathbf{r},t)M_y(\mathbf{r}',t)-M_y(\mathbf{r}',t)M_x(\mathbf{r},t)&=-i\mu M_z(\mathbf{r},t)\delta(\mathbf{r}-\mathbf{r}'),\\ M_z(\mathbf{r},t)M_x(\mathbf{r}',t)-M_x(\mathbf{r}',t)M_z(\mathbf{r},t)&=-i\mu M_y(\mathbf{r},t)\delta(\mathbf{r}-\mathbf{r}'),\\ M_y(\mathbf{r},t)M_z(\mathbf{r}',t)-M_z(\mathbf{r}',t)M_y(\mathbf{r},t)&=-i\mu M_x(\mathbf{r},t)\delta(\mathbf{r}-\mathbf{r}').\end{aligned}\right\} \quad (3.2)$$

For the circular projections $\mathbf{M}^{\pm} = \mathbf{M}_x \pm i\mathbf{M}_y$, these relations have the form:

$$M^-(\mathbf{r},t)M^+(\mathbf{r}',t)-M^+(\mathbf{r}',t)M^-(\mathbf{r},t)=\mu M_z(\mathbf{r},t)\delta(\mathbf{r}-\mathbf{r}'). \quad (3.3)$$

We now introduce operators $a(\mathbf{r}, t)$ and $a^+(\mathbf{r}, t)$, analogous to the operators a_l and a_l^+ (cf. Sec. 1), which are related to M^+, M^- and M_Z by the relations:

$$\left.\begin{aligned}M^+&=\sqrt{2\mu M_0}\,a^+\left(1-\frac{\mu a^+a}{2M_0}\right)^{1/2},\\ M^-&=\sqrt{2\mu M_0}\left(1-\frac{\mu a^+a}{2M_0}\right)^{1/2}a,\\ M_z&=M_0-\mu a^+a.\end{aligned}\right\} \quad (3.4)$$

The operators a^+ and a satisfy the commutation relations

$$a(\mathbf{r},t)a^+(\mathbf{r}',t)-a^+(\mathbf{r}',t)a(\mathbf{r},t)=\delta(r-r'). \quad (3.5)$$

If we consider states close to saturation, the average values of the quantities $\frac{\mu}{2M_0}a^+a$ will be small compared to unity. Therefore for such states we can expand the square root in powers of $\frac{\mu}{2M_0}a^+a$. Keeping only the first two terms in the expansion, which is all we need for the following, we rewrite (3.4) in the form:

$$\left.\begin{aligned}M^+\equiv m^+&=\sqrt{2\mu M_0}\left(a^+-\frac{\mu}{4M_0}a^+a^+a\right),\\ M^-\equiv m^-&=\sqrt{2\mu M_0}\left(a-\frac{\mu}{4M_0}a^+aa\right),\\ m_z\equiv M_z-M_0&=-\mu a^+a.\end{aligned}\right\} \quad (3.6)$$

We now go over to the Fourier components of the quantities a, a^+, m:

$$a(\mathbf{r}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} a_{\mathbf{k}}(t) e^{i\mathbf{kr}}, \quad a^+(\mathbf{r}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} a^+_{\mathbf{k}}(t) e^{-i\mathbf{kr}},$$
$$\mathbf{m}(\mathbf{r}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \mathbf{m}_{\mathbf{k}}(t) e^{i\mathbf{kr}}. \tag{3.7}$$

It is easy to see that the operators $\mathbf{a_k}$ and $\mathbf{a^+_k}$ satisfy the commutation relations

$$a_{\mathbf{k}}(t) a^+_{\mathbf{k}'}(t) - a^+_{\mathbf{k}'}(t) a_{\mathbf{k}}(t) = \Delta(\mathbf{k} - \mathbf{k}'), \tag{3.8}$$

where

$$\Delta(\mathbf{k}) = \begin{cases} 1, & \mathbf{k} = 0 \\ 0, & \mathbf{k} \neq 0. \end{cases}$$

Using (3.6), we find

$$\left.\begin{aligned} m^+_{\mathbf{k}} &= \sqrt{2\mu M_0}\left(a^+_{-\mathbf{k}} - \frac{\mu}{4M_0 V} \sum_{123} a^+_1 a^+_2 a_3 \Delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3)\right), \\ m^-_{\mathbf{k}} &= \sqrt{2\mu M_0}\left(a_{\mathbf{k}} - \frac{\mu}{4M_0 V} \sum_{123} a^+_1 a_2 a_3 \Delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\right), \\ m^z_{\mathbf{k}} &= -\frac{\mu}{\sqrt{V}} \sum_{12} a^+_1 a_2 \Delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2). \end{aligned}\right\} \tag{3.9}$$

where $a_i = a_{\mathbf{k}i}$ and the summation is extended over the corresponding wave vectors.

Let us now turn to the classical Hamiltonian of the ferromagnet (2.3), which we shall denote by $\mathcal{H}_S$:

$$\mathcal{H}_s = \int_V \left(\frac{1}{2} \alpha_{ik} \frac{\partial M_l}{\partial x_i} \frac{\partial M_l}{\partial x_k} - \frac{1}{2} \beta (\mathbf{Mn})^2 + \frac{\mathbf{H}^2}{8\pi}\right) dv,$$

and consider states close to equilibrium. Expanding the Hamiltonian in the small deviations of the magnetic moment and field $\mathbf{m} = \mathbf{M} - \mathbf{M}_0$ and $\mathbf{h} = \mathbf{H} - \mathbf{H}_0$, we obtain, keeping terms up to fourth order, the following expression for $\mathcal{H}_S$:

$$\mathcal{H}_s = \int_V \left(\frac{1}{2} \alpha_{ik} \frac{\partial m_l}{\partial x_i} \frac{\partial m_l}{\partial x_k} - (\beta M_0 + H_0) m_z + \frac{\mathbf{h}^2}{8\pi} - \frac{1}{2} \beta m_z^2\right) dv. \tag{3.10}$$

[We note that m_x and m_y are first order quantities, while m_z is second order, since $(\mathbf{M}_0 + \mathbf{m})^2 = M_0^2$.]

Considering as before only the low frequency oscillations and going over to Fourier components, we rewrite $\mathcal{H}_S$ in the form

$$\mathcal{H}_s = \sum_{\mathbf{k}} \left\{\frac{1}{2} \alpha_{ik} k_i k_k \mathbf{m}_{\mathbf{k}} \mathbf{m}_{-\mathbf{k}} - V^{1/2} (\beta M_0 + H_0) m^z_{\mathbf{k}} \Delta(\mathbf{k}) + 2\pi \frac{(\mathbf{km}_{\mathbf{k}})(\mathbf{km}_{-\mathbf{k}})}{k^2} - \frac{1}{2} \beta m^z_{\mathbf{k}} m^z_{-\mathbf{k}}\right\}. \tag{3.11}$$

(We have used the relation $\mathbf{h_k} = -\frac{4\pi}{\mathbf{k}^2} \mathbf{k}(\mathbf{m_k k})$.)

In order to obtain the quantized Hamiltonian from the classical Hamiltonian (3.11), we shall assume that $m^x_{\mathbf{k}}$, $m^y_{\mathbf{k}}$ and $m^z_{\mathbf{k}}$ in formula (3.11) are operators related to the operators $a_{\mathbf{k}}$ and $a^+_{\mathbf{k}}$ by the relations (3.9). In the variables $a_{\mathbf{k}}$ and $a^+_{\mathbf{k}}$ the Hamiltonian $\mathcal{H}_S$ has the form:

$$\mathcal{H}_s = \mathcal{H}_s^{(0)} + \mathcal{H}_s^{(3)} + \mathcal{H}_s^{(4)}, \tag{3.12}$$

where

$$\mathcal{H}_s^{(0)} = \sum_{\mathbf{k}} \left(A_{\mathbf{k}} a^+_{\mathbf{k}} a_{\mathbf{k}} + \frac{1}{2} B_{\mathbf{k}} a_{\mathbf{k}} a_{-\mathbf{k}} + \frac{1}{2} B^*_{\mathbf{k}} a^+_{\mathbf{k}} a^+_{-\mathbf{k}}\right), \tag{3.13}$$

$$\mathcal{H}_s^{(3)} = \sum_{123} \Phi_{12;3} a^+_1 a^+_2 a_3 \Delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3) + \text{compl. conj.} \tag{3.14}$$

$$\mathcal{H}_s^{(4)} = \sum_{1234} \Phi_{12;34} a^+_1 a^+_2 a_3 a_4 \Delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) + \sum_{1234} \Phi_{1;234} a^+_1 a_2 a_3 a_4 \Delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) + \text{compl. conj.} \tag{3.15}$$

$$\Phi_{12;3} = -\frac{\pi\mu}{2\sqrt{V}} \sqrt{2\mu M_0} (\sin 2\theta_1 e^{-i\varphi_1} + \sin 2\theta_2 e^{-i\varphi_2}), \tag{3.14'}$$

$$\Phi_{12;34} = -\frac{\mu^2}{8V}\Big(\alpha(\mathbf{k}_1\mathbf{k}_2 + \mathbf{k}_3\mathbf{k}_4) + 2\beta + \pi(\sin^2\theta_1 + \sin^2\theta_2 + \sin^2\theta_3 + \sin^2\theta_4) - 2\pi\Big[\frac{(\mathbf{k}_1 - \mathbf{k}_4)_z^2}{(\mathbf{k}_1 - \mathbf{k}_4)^2} + \frac{(\mathbf{k}_1 - \mathbf{k}_3)_z^2}{(\mathbf{k}_1 - \mathbf{k}_3)^2} + \frac{(\mathbf{k}_2 - \mathbf{k}_4)_z^2}{(\mathbf{k}_2 - \mathbf{k}_4)^2} + \frac{(\mathbf{k}_2 - \mathbf{k}_3)_z^2}{(\mathbf{k}_2 - \mathbf{k}_3)^2}\Big]\Big), \tag{3.15'}$$

$$\Phi_{1;234} = -\frac{\pi\mu^2}{6V} (\sin^2\theta_2 e^{2i\varphi_2} + \sin^2\theta_3 e^{2i\varphi_3} + \sin^2\theta_4 e^{2i\varphi_4}) \tag{3.15''}$$

[the first term in (3.15') is related to the exchange interaction, while the remaining terms are due to the anistropy energy and the magnetic dipole interaction.]

The operator $\mathcal{H}_S^{(0)}$ which is quadratic in $a_{\mathbf{k}}$ and $a^+_{\mathbf{k}}$ is the fundamental Hamiltonian for free spin waves. The operators $\mathcal{H}_S^{(3)}$ and $\mathcal{H}_S^{(4)}$ describe the interaction between the spin waves and will be considered in more detail in Sec. 9. The eigenvalues of the operator $\mathcal{H}_S^{(0)}$ give the possible energy values of the ferromagnet.

Expression (3.13) differs from the Hamiltonian for the exchange interaction (1.9) in including additional terms proportional to $B_{\mathbf{k}}$ and containing the products of the operators $a_{\mathbf{k}}$ $a_{-\mathbf{k}}$ and $a^+_{\mathbf{k}} a^+_{-\mathbf{k}}$. These terms, which are associated with dipole interaction of the spins, do not commute with $a^+_{\mathbf{k}} a_{\mathbf{k}}$ and therefore, in diagonalizing the Hamiltonian $\mathcal{H}_S^{(0)}$, we must first carry out a canonical transformation of the variables $a_{\mathbf{k}}$ and $a^+_{\mathbf{k}}$.*

In place of $a_{\mathbf{k}}$ and $a^+_{\mathbf{k}}$ let us introduce new variables $c_{\mathbf{k}}$ and $c^+_{\mathbf{k}}$:

$$\left.\begin{aligned} a_{\mathbf{k}} &= u_{\mathbf{k}} c_{\mathbf{k}} + v^*_{\mathbf{k}} c^+_{-\mathbf{k}}, \\ a^+_{\mathbf{k}} &= u^*_{\mathbf{k}} c^+_{\mathbf{k}} + v_{\mathbf{k}} c_{-\mathbf{k}}, \end{aligned}\right\} \tag{3.16}$$

where $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are c-numbers. In order for the operators $c_{\mathbf{k}}$ and $c^+_{\mathbf{k}}$ to satisfy the commutation relation

$$c_{\mathbf{k}} c^+_{\mathbf{k}'} - c^+_{\mathbf{k}'} c_{\mathbf{k}} = \Delta(\mathbf{k} - \mathbf{k}'),$$

we must subject the quantities $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ to the condition

$$|u_{\mathbf{k}}|^2 - |v_{\mathbf{k}}|^2 = 1.$$

Let us now select $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ so that the Hamiltonian $\mathcal{H}_S^{(0)}$ is diagonal in the variables $c_{\mathbf{k}}$ and $c^+_{\mathbf{k}}$. To do this we make use of the equation of motion

$$\dot{a}_{\mathbf{k}} = \frac{i}{\hbar} [\mathcal{H}_s^{(0)}, a_{\mathbf{k}}] = -\frac{i}{\hbar} A_{\mathbf{k}} a_{\mathbf{k}} - \frac{i}{\hbar} B^*_{\mathbf{k}} a^+_{-\mathbf{k}}.$$

*In the following we use a method developed by N. N. Bogolyubov and S. V. Tyablikov.[15]

Now substituting (3.16), we get

$$\dot{a}_{\mathbf{k}} = -\frac{i}{\hbar}(A_{\mathbf{k}}u_{\mathbf{k}} + B_{\mathbf{k}}^{*}v_{-\mathbf{k}})\,c_{\mathbf{k}} - \frac{i}{\hbar}(A_{\mathbf{k}}v_{\mathbf{k}}^{*} + B_{\mathbf{k}}^{*}u_{-\mathbf{k}}^{*})\,c_{-\mathbf{k}}^{+}. \quad (3.17)$$

On the other hand

$$\dot{a}_{\mathbf{k}} = u_{\mathbf{k}}\dot{c}_{\mathbf{k}} + v_{\mathbf{k}}^{*}\dot{c}_{-\mathbf{k}}^{+}. \quad (3.17')$$

Since we want the Hamiltonian $\mathcal{H}_S^{(0)}$ to be diagonal in the variables $c_{\mathbf{k}}$, the operators $c_{\mathbf{k}}$ should change with time according to the law $e^{-i\epsilon_{\mathbf{k}}t/\hbar}$. Therefore formula (3.17′) can be rewritten as

$$\dot{a}_{\mathbf{k}} = -\frac{i}{\hbar}u_{\mathbf{k}}\varepsilon_{\mathbf{k}}c_{\mathbf{k}} + \frac{i}{\hbar}v_{\mathbf{k}}^{*}\varepsilon_{\mathbf{k}}c_{-\mathbf{k}}^{+}$$

(Here it is assumed that $\epsilon_{\mathbf{k}} = \epsilon_{-\mathbf{k}}$; this will be verified in the sequel). Comparing this expression with (3.17), we obtain:

$$(A_{\mathbf{k}} - \varepsilon_{\mathbf{k}})\,u_{\mathbf{k}} + B_{\mathbf{k}}^{*}v_{-\mathbf{k}} = 0,$$
$$B_{\mathbf{k}}u_{\mathbf{k}} + (A_{\mathbf{k}} + \varepsilon_{\mathbf{k}})\,v_{\mathbf{k}} = 0,$$

from which[13]

$$\varepsilon_{\mathbf{k}} = \sqrt{A_{\mathbf{k}}^2 - |B_{\mathbf{k}}|^2},$$
$$u_{\mathbf{k}} = \sqrt{\frac{A_{\mathbf{k}}+\varepsilon_{\mathbf{k}}}{2\varepsilon_{\mathbf{k}}}}, \quad v_{\mathbf{k}} = -\frac{B_{\mathbf{k}}}{|B_{\mathbf{k}}|}\sqrt{\frac{A_{\mathbf{k}}-\varepsilon_{\mathbf{k}}}{2\varepsilon_{\mathbf{k}}}} \quad (3.18)$$

(The arbitrary phase factor in $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ has been chosen equal to unity).

The expression obtained for $\epsilon_{\mathbf{k}}$ is identical with the formula found earlier for the energy of a spin wave.

In the variables $c_{\mathbf{k}}$ and $c_{\mathbf{k}}^{+}$, the Hamiltonian (3.13) has the form

$$\mathcal{H}_s^{(0)} = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}}c_{\mathbf{k}}^{+}c_{\mathbf{k}}. \quad (3.13')$$

From this it is clear that its eigenvalues are

$$\mathcal{E} = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}}n_{\mathbf{k}}, \quad (3.19)$$

where $n_{\mathbf{k}}$ are arbitrary positive integers. The quantity $n_{\mathbf{k}}$ represents the number of spin waves with wave vector $\mathbf{k}$.

In the representation in which the operator $\mathcal{H}_S^{(0)}$ is diagonal, the non-zero matrix elements of $c_{\mathbf{k}}$ and $c_{\mathbf{k}}^{+}$ are

$$(n_{\mathbf{k}} - 1\,|\,c_{\mathbf{k}}\,|\,n_{\mathbf{k}}) = \sqrt{n_{\mathbf{k}}}\,e^{-i\frac{\varepsilon_{\mathbf{k}}t}{\hbar}},$$
$$(n_{\mathbf{k}}\,|\,c_{\mathbf{k}}^{+}\,|\,n_{\mathbf{k}} - 1) = \sqrt{n_{\mathbf{k}}}\,e^{i\frac{\varepsilon_{\mathbf{k}}t}{\hbar}}. \quad (3.20)$$

Thus $c_{\mathbf{k}}^{+}$ can be interpreted as a creation operator, and $c_{\mathbf{k}}$ an annihilation operator for the spin waves with wave vector $\mathbf{k}$.

We note that it is the operators $c_{\mathbf{k}}^{+}$ and $c_{\mathbf{k}}$, and not the operators $a_{\mathbf{k}}^{+}$ and $a_{\mathbf{k}}$, which are the creation and annihilation operators for the spin waves.

The difference between $c_{\mathbf{k}}$ and $a_{\mathbf{k}}$ is due to the magnetic interaction and also to the anisotropy energy, which was not taken into account in the Hamiltonian (1.9).

Let us now express, in terms of the variables $c_{\mathbf{k}}$ and $c_{\mathbf{k}}^{+}$, the operators for the projections of the total magnetic moment of the ferromagnet $\mathfrak{M}_z$ and $\mathfrak{M}^{\pm}$ (the z axis is taken along the equilibrium direction of the magnetic moment). According to (3.6),

$$\mathfrak{M}_z = \int_V M_z\,dv = M_0V - \mu\sum_{\mathbf{k}} a_{\mathbf{k}}^{+}a_{\mathbf{k}} = M_0V - \mu\sum_{\mathbf{k}}|v_{\mathbf{k}}|^2$$
$$-\mu\sum_{\mathbf{k}}(|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2)\,c_{\mathbf{k}}^{+}c_{\mathbf{k}} - \mu\sum_{\mathbf{k}}(u_{\mathbf{k}}v_{\mathbf{k}}c_{-\mathbf{k}}c_{\mathbf{k}} + u_{\mathbf{k}}^{*}v_{\mathbf{k}}^{*}c_{\mathbf{k}}^{+}c_{-\mathbf{k}}^{+}),$$
$$\mathfrak{M}^{+} = \int_V M^{+}\,dv = \sqrt{2\mu M_0V}\,(u_0^{*}c_0^{+} + v_0c_0),$$
$$\mathfrak{M}^{-} = \int_V M^{-}\,dv = \sqrt{2\mu M_0V}\,(u_0c_0 + v_0^{*}c_0^{+}), \quad (3.21)$$

where c_0 and c_0^{+} are the values of $c_{\mathbf{k}}$ and $c_{\mathbf{k}}^{+}$ when $\mathbf{k} = 0$.

In a state with a definite value of the energy, the average values of the operators $\mathfrak{M}_z$ and $\mathfrak{M}^{\pm}$ are

$$\langle\mathfrak{M}_z\rangle = MV - \sum_{\mathbf{k}}\mu_{\mathbf{k}}n_{\mathbf{k}},$$
$$\langle\mathfrak{M}^{\pm}\rangle = 0, \quad (3.22)$$

where $n_{\mathbf{k}}$ is the number of spin waves with wave vector $\mathbf{k}$, and

$$M = M_0 - \frac{\mu}{V}\sum_{\mathbf{k}}|v_{\mathbf{k}}|^2 = M_0 - \frac{\mu}{2(2\pi)^3}\int\frac{A_{\mathbf{k}}-\varepsilon_{\mathbf{k}}}{\varepsilon_{\mathbf{k}}}\,d\mathbf{k},$$
$$\mu_{\mathbf{k}} = -\mu(|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2) = -\frac{\partial\varepsilon_{\mathbf{k}}}{\partial H_0}. \quad (3.23)$$

The expression for $\mathfrak{M}_z$ shows that the quantity $\mu_{\mathbf{k}}$ can be interpreted as the average value of the projection on the z axis of the magnetic moment of the spin wave with wave vector $\mathbf{k}$. This quantity differs from μ, the difference arising apparently from the dipolar interaction of the spins and the anisotropy energy. Because of these same interactions, the average value of the projection of the magnetic moment of the ferromagnet on the z axis differs from the quantity M_0 even in the absence of spin waves, i.e., at absolute zero.[3]

4. High Frequency Properties of Ferromagnets and Ferromagnetic Resonance

Up to this point we have studied the properties of spin waves in an infinite ferromagnet. However, many effects are due to the finite dimensions of the body. These effects make themselves felt primarily in the high frequency properties of ferromagnets. We shall now proceed to study them.

The high frequency properties of a ferromagnet are described by the system of Maxwell equations (2.6) and the equation of motion of the magnetic moment (2.12). The latter equation gives the connection between the magnetic moment and the magnetic field. In the case of plane monochromatic waves, this equation can be solved in the linear approximation for the magnetic moment:

$$m_i = \chi_{ik}h_k, \quad (4.1)$$

where

$$\chi_{ik}=\frac{gM_0\Omega}{\Omega^2-\left(\omega-i\Omega\frac{\lambda}{gM_0}\right)^2}\begin{pmatrix}1-i\frac{\lambda}{gM_0}\frac{\omega}{\Omega}+\left(\frac{\lambda}{gM_0}\right)^2, & i\frac{\omega}{\Omega}, & 0\\ -i\frac{\omega}{\Omega}, & 1-i\frac{\lambda}{gM_0}\frac{\omega}{\Omega}+\left(\frac{\lambda}{gM_0}\right)^2, & 0\\ 0 & 0 & 0\end{pmatrix} \tag{4.2}$$

and

$$\Omega=gM_0\left(\alpha k^2+\beta+\frac{H_{(0)}^{(i)}}{M_0}\right) \tag{4.3}$$

[the constant magnetic field $H_0^{(i)}$ inside the body is assumed to be along the axis of easiest magnetization].

In the general case, for $\lambda \neq 0$, the tensor χ_{ik} is not Hermitian. Its anti-Hermitian part $\chi_{ik} - \chi_{ki}^*$ determines[16] the rate of loss of magnetic energy in the body. For $\lambda = 0$, i.e., when we neglect dissipative processes, the tensor χ_{ik} will be Hermitian, $\chi_{ik} = \chi_{ki}^*$. Gyrotropy, i.e. the presence of imaginary non-diagonal components $\chi_{xy} = -\chi_{yx}$, is due to the rotation of the magnetic moment around the effective magnetic field.

The tensor χ_{ik} depends not only on the frequency ω, but also on the wave vector $\mathbf{k}$. This property is called spatial dispersion. It is caused by the exchange interaction between the atoms.

In treating the thermal and magnetic properties of a ferromagnet, the dependence of the frequency of the spin wave on wave vector plays a decisive role, whereas the spatial dispersion very rarely has an influence on its high frequency properties. This is related to the fact that the term αk^2 is practically always negligibly small, since $\alpha k^2 = \frac{\theta_C}{\mu M_0}(ak)^2 \ll 1$, if the wave length, which is equal to $2\pi/k$, is considerably greater than several hundreds of atomic distances. The spatial dispersion can be neglected when the frequency is not too close to the frequency $\omega_0 = gM_0\left(\beta + \frac{H_{(0)}^{i}}{M_0}\right)$. For $\omega \approx \omega_0$ the term $gM_0\alpha k^2$ cannot be neglected. One may suppose that including spatial dispersion will lead to the same peculiarities in the propagation of electromagnetic waves as occur in dielectrics in the neighborhood of an exciton absorption line.[17,18]

For $\lambda = 0$ the components of the tensor χ_{ik} can become infinite (at the frequency $\omega = \omega_0$), which shows that resonance absorption of energy from the magnetic field by the ferromagnet is possible. Actually the resonant absorption is observed not at the frequency ω_0, but at the frequencies of normal vibrations of the magnetic moment of a ferromagnet of finite dimensions. The difference between these frequencies and ω_0 is associated with the inclusion of the boundary conditions at the surface of the body (effect of shape).

The complete system of equations describing the proper oscillations of the magnetic moment has the form

$$\begin{aligned}\operatorname{curl}\mathbf{h}&=0,\\ \operatorname{div}(\mathbf{h}+4\pi\hat{\chi}\mathbf{h})&=0\end{aligned} \tag{4.4}$$

inside the body, and

$$\begin{aligned}\operatorname{curl}\mathbf{h}&=0,\\ \operatorname{div}\mathbf{h}&=0,\end{aligned} \tag{4.5}$$

outside the body, where the tensor χ_{ik} is defined by the expression (4.2), in which we should set $\lambda = 0$, and also omit the term $gM_0\alpha k^2$:

$$\chi_{ik}=\frac{gM_0\omega_0}{\omega_0^2-\omega^2}\begin{pmatrix}1 & i\frac{\omega}{\omega_0} & 0\\ -i\frac{\omega}{\omega_0} & 1 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{4.6}$$

It is permissible to neglect spatial dispersion for not too short waves and not too small dimensions of the body, when the inequalities $L, \lambda \gg \left(\frac{\theta_C}{\mu M_0}\right)^{1/2} a$ are satisfied (L gives the dimensions of the body).

We use magnetostatic equations here since we assume that the eigen-frequencies are not too large $\left(\frac{c}{\omega} \gg L\right)$, and also assume that the ferromagnet is a dielectric, i.e., $j = 0$.

Moreover, we can speak of natural vibrations only when the damping is sufficiently small. If δ is the damping length, which is easily obtained by using formulas (2.16) and (2.17'), we should also make sure that the condition $\delta \gg L$ holds.

To the system (4.4) and (4.5) we must add the boundary conditions: continuity of the tangential components of the magnetic field $\mathbf{h}$ and the normal components of the magnetic induction $\mathbf{b} = \mathbf{h} + 4\pi\hat{\chi}\mathbf{h}$ at the surface of the body, as well as the condition that the magnetic field goes to zero far from the body.

The problem which we have formulated can be solved for a ferromagnet of ellipsoidal shape (L. Walker[19]). Among the natural vibrations of the ellipsoid there is a vibration with a uniform field in the interior of the ellipsoid. The frequency of such a vibration is called the frequency of homogeneous resonance.* Let us determine this frequency (C. Kittel[20]).

It is known[16] that the uniform field $\mathbf{h}^{(i)}$ inside the ellipsoid is related to the field at infinity $\mathbf{h}^{(0)}$, by the following relation:

$$h_l^{(i)} + 4\pi n_{lk}\chi_{ks}h_s^{(i)} = h_l^{(0)}, \tag{4.7}$$

where n_{lk} is the tensor of the demagnetizing factors of the ellipsoid, which is determined by the ratios between the axes, but does not depend on the dimensions of the body. Since $\mathbf{h}^{(0)} = 0$, the condition for the existence of a non-zero magnetic field inside the ellipsoid is that the determinant

*In order to observe homogeneous resonance a ferromagnetic ellipsoid is placed in a homogeneous constant and variable magnetic field, and one determines the energy loss as a function of the steady external magnetic field for a fixed frequency of the variable field. When the eigenfrequency coincides with the frequency of the variable field, one observes a sharp increase in the energy loss.

$$\det|\delta_{ik}+4\pi n_{is}\chi_{sk}|=0 \qquad (4.8)$$

be equal to zero. From this, using expression (4.6) for χ_{ik}, it is easy to find the value of the frequency of homogeneous resonance[20]

$$\omega_r = g\sqrt{[H_0^{(e)}+4\pi M_0(n_1-n_3)][H_0^{(e)}+4\pi M_0(n_2-n_3)]}, \qquad (4.9)$$

where $H_0^{(e)} = H_0 + \beta M_0$, H_0 is the constant field far from the ellipsoid, n_j are the principal values of the tensor n_{ik}, and $\sum_j n_j = 1$.

In obtaining (4.9) we assumed that one of the axes of the ellipsoid (axis 3) coincides with the axis of easiest magnetization and that the external constant field is also parallel to this axis.

In the case of a sphere, $n_1 = n_2 = n_3 = 1/3$ and

$$\omega_r = gH_0^{(e)}. \qquad (4.10)$$

For a cylinder whose axis is along the axis of easiest magnetization, $n_1 = n_2 = 1/2$, $n_3 = 0$ and

$$\omega_r = g(H_0^{(e)}+2\pi M_0). \qquad (4.11)$$

For a cylinder whose axis is perpendicular to the axis of easiest magnetization, $n_1 = 0$, $n_2 = n_3 = 1/2$ and

$$\omega_r = g\sqrt{H_0^{(e)}(H_0^{(e)}-2\pi M_0)}. \qquad (4.12)$$

We recall that for the magnetization of a cylindrical sample to saturation in this case it is necessary that the external constant magnetic field be greater than $2\pi M_0$.

For a plate whose surface is parallel to the axis of easiest magnetization, $n_1 = n_3 = 0$, $n_2 = 1$ (the axis 2 is directed into the body) and

$$\omega_r = g\sqrt{H_0^{(e)}(H_0^{(e)}+4\pi M_0)}. \qquad (4.13)$$

Finally, for a plate whose axis of easiest magnetization is perpendicular to the surface, $n_1 = n_2 = 0$, $n_3 = 1$ and

$$\omega_r = g(H_0^{(e)}-4\pi M_0). \qquad (4.14)$$

In this case the plate is magnetized to saturation if $H_0^{(e)} > 4\pi M_0$.

The observation of homogeneous resonance is an important method for determining various constants characterizing the ferromagnet, in particular, the quantity g and the relaxation time.*

In addition to the homogeneous oscillation, there also exist natural vibrations for which the magnetic field inside the body is inhomogeneous. The frequencies of these vibrations are called frequencies of inhomogeneous resonance.†

*The effect of shape, i.e., the dependence of the resonance frequency on the demagnetizing factors, was first detected by Griffiths[21] on a plate in a constant field parallel to its surface.

†The resonance absorption of energy at these frequencies was discovered experimentally.[22] To observe inhomogeneous resonance, a ferromagnetic ellipsoid is placed in an inhomogeneous quasistatic magnetic field.

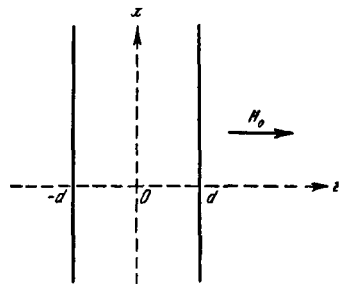

FIG. 1

The calculation of the frequencies of inhomogenous resonance for the case of an ellipsoid is extremely complicated, and we shall not carry it out here (cf. references 19 and 23). In order to illustrate the features of the spectrum of frequencies of inhomogeneous resonance, we consider magnetic vibrations of a plate placed in a constant field which is directed perpendicular to its surface (so that the axis of easiest magnetization coincides with the direction of the magnetic field). According to (4.14) the frequency of homogeneous resonance in this case is $\omega_r = gM_0\left(\frac{H_0^{(i)}}{M_0}+\beta\right)$.

If we introduce a potential ψ related to the magnetic field $\mathbf{h}$ by the formula $\mathbf{h} = -\nabla\psi$, and assume that the dependence of the field on coordinates in the plane of the plate has the form $e^{i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}}$ where $\boldsymbol{\rho}$ is a two-dimensional vector with components x and while y, $\boldsymbol{\kappa}$ is the wave vector lying in the (x, y) plane (cf. Fig. 1), Eqs. (4.4) and (4.5) can be written as follows:

$$\frac{d^2\psi}{dz^2}+\varkappa_{(i)}^2\psi=0, \quad |z|<d, \qquad (4.15)$$

$$\frac{d^2\psi}{dz^2}-\varkappa^2\psi=0, \quad |z|>d, \qquad (4.15')$$

where

$$\varkappa_{(i)}^2 = -\varkappa^2\frac{\omega^2-\omega_i^2}{\omega^2-\omega_0^2}, \qquad (4.16)$$

and

$$\omega_i^2 = g^2(H_0^{(i)}+\beta M_0)(H_0^{(i)}+4\pi M_0+\beta M_0). \qquad (4.17)$$

The boundary conditions in this case reduce to continuity of the function and its derivative with respect to z at the boundary of the plate (i.e., at $z = \pm d$) and the requirement that ψ vanish as $z \to \pm\infty$.

Because of the symmetry of the problem, the solutions are subdivided into two classes: symmetric and antisymmetric. From (4.15') and the boundary conditions at infinity we see that outside the plate the field falls off exponentially with increasing |z|. Inside the plate the symmetric solution has the form

$$\psi_s^{(i)} = A\cos\varkappa_i z,$$

while the antisymmetric solution is

$$\psi_a^{(i)} = B\sin\varkappa_i z.$$

The conditions at the boundaries of the plate give the dispersion equation, i.e., the relation between the

frequency ω and the wave vector κ. For the symmetric solution the dispersion equation has the form:

$$\cot v = \frac{v}{u}, \tag{4.18}$$

while for the antisymmetric solution

$$\tan v = -\frac{v}{u}, \tag{4.19}$$

where $u = \kappa d$, and $v = \kappa_i d$.

Since $u > 0$, solutions of the dispersion equations (4.18) and (4.19) exist only for $\kappa_i^2 > 0$, i.e., all the frequencies of inhomogeneous resonance lie between ω_0 and ω_1 [cf. (4.16)].

In accordance with (4.16), the resonance frequency can be represented as

$$\omega^2 = \frac{\omega_0^2 v^2 + \omega_1^2 u^2}{u^2 + v^2}, \tag{4.20}$$

where u and v are related by (4.18) for the symmetric solution, and by (4.19) for the antisymmetric solution.

Equations (4.18) and (4.19) have an infinite number of solutions, each of which is a continuous function of the wave vector κ. We shall therefore denote the frequencies of inhomogeneous resonance by $\omega_n^s(\kappa)$ and $\omega_n^a(\kappa)$, where n is the number of the solution and the indices s and a refer respectively to symmetric and antisymmetric solutions.

Let us consider some of the properties of the frequencies of inhomogeneous resonance. From formulas (4.18), (4.19), and (4.20) we see that

$$\omega_n^{(s,a)}(\varkappa) \approx \begin{cases} \omega_0, & \varkappa d \ll n\pi, \\ \omega_1, & \varkappa d \gg n\pi, \end{cases}$$

i.e., with increasing number of the solution and for fixed κ, the frequencies approach the value ω_0, while with increasing κ for fixed n, they approach ω_1. For small values of κ, the expansion in powers of κ begins for $\omega_1^s(\kappa)$ with the linear term, while for the other frequencies of inhomogeneous resonance the expansion begins with the quadratic term in κ.

The spectrum of eigenfrequencies of a bounded object, for example, an ellipsoid, is discrete,[19] and each frequency is characterized by three discrete indices. In order to obtain a picture of the nature of the spectrum in the case of a bounded body, let us consider the vibrations of a plate, where we impose a condition of periodicity along the x and y axes, i.e., assume that the components of the wave vector κ_x and κ_y are equal to

$$\varkappa_x = \frac{2\pi}{L} n_x, \quad \varkappa_y = \frac{2\pi}{L} n_y,$$

where n_x and n_y are integers, and L is the length of the period. Using these values of κ_x and κ_y we can represent the resonance frequency of the plate in the form

$$\omega = \omega_n^{(s,a)}\left(2\pi\sqrt{n_x^2 + n_y^2}\,\frac{d}{L}\right).$$

We see that as the dimensions of the plate increase (L, d $\to \infty$), the discrete nature of the spectrum is preserved so long as the ratio d/L remains finite.

The properties of the spectrum of frequencies of inhomogeneous resonance which we have considered for the example of a plate — the presence of two points of accumulation $\omega = \omega_0$ and $\omega = \omega_1$, and the fact that the frequencies depend on the ratio of the dimensions — are true also for the ellipsoid.[19]

We note that the eigenfrequencies of inhomogeneous resonance for the case of a plate are the frequencies of the spin waves for $k \to 0$. In fact, neglecting in formula (2.11 - 11') for the frequency of a spin wave the quantity αk^2 compared to $\frac{H_0^{(e)}}{M_0}$, and replacing $\sin^2\theta_{\mathbf{k}}$ by $\frac{\kappa^2}{\kappa^2 + \kappa_i^2}$, we obtain formula (4.20). Thus, taking account of the boundary conditions, reduces to finding the value of $\sin^2\theta_{\mathbf{k}}$ for $k \to 0$.

In calculating the frequencies of inhomogeneous resonance, we have not taken account of spatial dispersion, which is permissible so long as $\alpha k^2 = \alpha(\kappa^2 + \kappa_i^2) \ll 1$. This condition can be rewritten as follows:

$$\varkappa^2 + n^2\frac{\pi^2}{d^2} \ll \frac{\mu M_0}{\theta_C}\frac{1}{a^2}.$$

The last inequality is violated for $n \neq 0$, if the plate is sufficiently thin; in this case the spatial dispersion must be taken into account.

Let us calculate the eigenfrequencies of a plate, including spatial dispersion.[24] We shall again assume that the axis of easiest magnetization and the external constant magnetic field are perpendicular to the surface of the plate. In addition, we shall assume for simplicity that in the plane of the plate the field and the magnetic moment are homogeneous, i.e., all quantities depend only on the coordinate z. Since the projection of the moment m_z is zero, div $\mathbf{m}$ is equal to zero in this case. Therefore the magnetic field $\mathbf{h}$ satisfies the equations

$$\operatorname{div}\mathbf{h} = 0, \quad \operatorname{curl}\mathbf{h} = 0 \tag{4.21}$$

over the whole space. These equations together with the boundary conditions $\mathbf{h} = 0$ for $|z| \to \infty$ obviously have only the trivial solution $\mathbf{h} = 0$. From this it follows that a non-trivial solution for the magnetic moment, which is related to the magnetic field $\mathbf{h}$ by the relation

$$\mathbf{m} = \chi\mathbf{h} \text{ or } \mathbf{h} = \hat{\chi}^{-1}\mathbf{m},$$

will exist only for those frequencies for which the determinant becomes zero:

$$\det|\chi_{ik}^{-1}| = 0,$$

i.e., for those frequencies for which the determinant of χ_{ik} becomes infinite. Using formula (4.2) for χ_{ik} and setting $\lambda = 0$, we obtain the following expression for the eigenfrequency of the plate:

$$\omega = \Omega = \omega_0 + \frac{\theta_C}{\hbar}\left(\frac{a}{d}\right)^2 v^2, \quad v = k_z d. \tag{4.22}$$

This expression for ω naturally coincides with the frequency of the spin wave (2.11 - 2.11') for $\theta_{\mathbf{k}} = 0$.

The permissible values of the wave vector k_z must be determined from the boundary conditions. In addition to the usual boundary conditions for $\mathbf{h}$ and $\mathbf{b} = \mathbf{h} + 4\pi\mathbf{m}$, which are satisfied automatically for $\mathbf{h} = 0$, and $m_z = 0$, we must also include the boundary conditions for the magnetic moment $\mathbf{m}$, since in the presence of spatial dispersion the equation of motion of the magnetic moment (2.4) contains derivatives with respect to the coordinates.

To derive the boundary conditions we must consider the motion of the magnetic moment near the surface of the ferromagnet. Since the equations of motion of the magnetic moment contain spatial derivatives of second order, the boundary condition consists in setting equal to zero a linear combination of the moment and its derivative along the normal to the surface. If exchange forces are of primary importance, the boundary conditions have the form:[25]

$$\left.\frac{\partial \mathbf{m}}{\partial x_n}\right|_s = 0; \tag{4.23}$$

if at the surface there is a large additional anisotropy energy,[24]

$$\mathbf{m}|_s = 0. \tag{4.24}$$

Because of the symmetry of the problem the solutions split into two classes: symmetric ($\sim \cos k_z d$) and antisymmetric ($\sim \sin k_z d$).

The permissible values of k_z, on which the frequency ω depends, are determined from the boundary conditions (4.23) and (4.24).

If condition (4.23) holds, then for the symmetric solution

$$\sin v = 0, \tag{4.25}$$

from which $v = n\pi$; for the antisymmetric solution

$$\cos v = 0, \tag{4.26}$$

from which $v = (n + \frac{1}{2})\pi$.

If condition (4.24) holds, then for the symmetric solution

$$\cos v = 0, \tag{4.27}$$

so that $v = (n + \frac{1}{2})\pi$, while for the antisymmetric solution

$$\sin v = 0, \tag{4.28}$$

so that $v = n\pi$.

Thus the eigenfrequencies of vibration of the magnetic moment in a ferromagnetic plate, when we include spatial dispersion, are determined by the expression (4.22), in which v is a solution of equations (4.25) - (4.28). These frequencies are obviously the frequencies of standing spin waves. Resonance at these frequencies is naturally referred to as resonance on standing spin waves.

As we see from formula (4.22), the separation between neighboring frequencies is equal in order of magnitude to $\frac{\theta C}{\hbar}\left(\frac{a}{d}\right)^2 \pi n$. Therefore for the observation of resonance on standing spin waves one uses thin plates[26] ($d \sim 5600$ A). The measurement of the distance between resonance frequencies enables one experimentally to determine the exchange interaction constant. The value obtained in reference 26 for the exchange interaction constant $A = \theta C a^2 M_0/\mu$ in permalloy (80 - 20%) is equal to $\sim 2 \times 10^{-6}$ erg/cm.

5. Surface Impedance of Ferromagnets

In the preceding section we considered the proper vibrations of the magnetic moment in bounded samples. In doing this the characteristic dimensions of the body L were assumed to be much smaller than the damping length δ. In bulk ferromagnetic metal this condition is not satisfied because of the skin effect and we have the opposite limiting case, which we now proceed to consider. If the skin depth is not only much smaller than the dimensions of the body, but also much smaller than the wave length in vacuum, $\delta \ll \frac{c}{\omega}$, the high-frequency properties of the body are conveniently described by the surface resistance (impedance) tensor ζ_{ik} which is defined as follows:[16]

$$\mathbf{E}_t = \hat{\zeta}[\mathbf{H}_t \times \mathbf{n}], \tag{5.1}$$

where $\mathbf{n}$ is the vector of the external normal to the body, and $\mathbf{E}_t$ and $\mathbf{H}_t$ are the tangential components of the electric and magnetic fields at the surface of the body. We note that, because of the condition $\delta \ll \frac{\omega}{c}$, the impedance is practically independent of the angle of incidence of the electromagnetic waves,[27] which enables us to compute the impedance for the simplest case of normal incidence of the wave on the surface of the metal.

The high-frequency properties of a ferromagnetic metal are described by the Maxwell equations, in which the variable part of the induction $\mathbf{b}$ is related to the variable field $\mathbf{h}$ by

$$b_i = \mu_{ik} h_k, \tag{5.2}$$

and the current density j_i is equal to

$$j_i = \sigma_{ik} E_k. \tag{5.3}$$

Since the ferromagnet has the property of gyrotropy, the magnetic susceptibility tensor and the electric conductivity tensor have the form

$$\mu_{ik} = \delta_{ik} + 4\pi\chi_{ik} = \begin{pmatrix} \mu_1 & i\mu_2 & 0 \\ -i\mu_2 & \mu_1 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \tag{5.4}$$

$$\sigma_{ik} = \begin{pmatrix} \sigma_1 & i\sigma_2 & 0 \\ -i\sigma_2 & \sigma_1 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}. \tag{5.5}$$

The values of the quantities μ_1, μ_2, and μ_3 are easily obtained by using expression (4.6) for χ_{ik}:

$$\left.\begin{aligned}\mu_1 &= \frac{\Omega\left(\Omega+4\pi gM_0\right)-\left(\omega-i\Omega\frac{\lambda}{gM_0}\right)^2-4\pi i\omega\lambda}{\Omega^2-\left(\omega-i\Omega\frac{\lambda}{gM_0}\right)^2},\\ \mu_2 &= \frac{4\pi gM_0\omega}{\Omega^2-\left(\omega-i\Omega\frac{\lambda}{gM_0}\right)^2},\\ \mu_3 &= 1.\end{aligned}\right\} \quad (5.4')$$

We note that the "skew" components of the tensor σ_{ik} ($\sigma_{xy} = -\sigma_{yx}$) describe the Hall effect.[16]

We shall determine the surface resistance for the case where the constant magnetic field H_0 is directed along the axis of easiest magnetization, parallel to the surface of the metal which fills the half-space $y > 0$ (cf. Fig. 2). In this case the Maxwell equations are expressed as follows:

$$\frac{\partial E_z}{\partial y}+i\frac{\omega}{c}\left(\mu_1-\frac{\mu_2^2}{\mu_1}\right)H_x=0, \quad (5.6)$$

$$\frac{\partial H_x}{\partial y}+4\pi\frac{\sigma_3}{c}E_z=0,$$

$$\frac{\partial E_x}{\partial y}-i\frac{\omega}{c}\mu_3 H_z=0,$$

$$\frac{\partial H_z}{\partial y}-\frac{4\pi}{c}\left(\sigma_1-\frac{\sigma_2^2}{\sigma_1}\right)E_x=0. \quad (5.7)$$

We assume that all quantities depend only on the y coordinate and vary with time as $e^{i\omega t}$. Finding the solutions of equations (5.6) and (5.7), which go to zero for $y \to \infty$, and using the definition (5.1), we get

$$\zeta_{xx}=\left(\frac{i\omega\mu_3}{4\pi\tilde{\sigma}}\right)^{1/2}, \quad \zeta_{zz}=\left(\frac{i\omega\tilde{\mu}}{4\pi\sigma_3}\right)^{1/2}, \quad (5.8)$$

where

$$\tilde{\sigma}=\frac{\sigma_1^2-\sigma_2^2}{\sigma_1}, \quad \tilde{\mu}=\frac{\mu_1^2-\mu_2^2}{\mu_1}. \quad (5.9)$$

Since $\mu_3 = 1$, the specific high frequency properties of the ferromagnet appear only in the component ζ_{zz}. This is related to the fact that the electromagnetic wave, in which the magnetic field is polarized along $\mathbf{M}_0$, does not give rise to a rotation of the magnetic moment. (We recall that ζ_{zz} connects E_z and H_x).

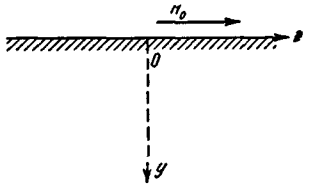

FIG. 2

Formulas (5.8) and (5.4') show that the dependence of the surface resistance on frequency (or on external magnetic field) has resonance character. In the absence of dissipation ($\lambda = 0$), for $\omega = g\sqrt{H_0^{(e)}B_0^{(e)}}$ the impedance ζ_{zz} becomes infinite, while for $\omega = gB_0^{(e)}$ it becomes zero. For $\lambda \neq 0$ the real part of the impedance ζ_{zz} has a maximum for $\omega = g\sqrt{H_0^{(e)}B_0^{(e)}}$, while for $\omega = gB_0^{(e)}$ it has a minimum.

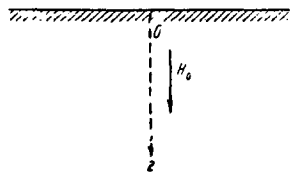

FIG. 3

We now calculate the surface resistance of a ferromagnet for the case where the constant magnetic field, parallel to the axis of easiest magnetization, is perpendicular to the surface of the metal (cf. Fig. 3). If we introduce the quantities $E^{\pm} = E_x \pm iE_y$, $h^{\pm} = h_x \pm ih_y$, and $\mu^{\pm} = \mu_1 \mp i\mu_2$, $\sigma^{\pm} = \sigma_1 \mp i\sigma_2$, the Maxwell equations take the form:

$$\pm i\frac{\partial h^{\pm}}{\partial z}=\frac{4\pi}{c}\sigma^{\pm}E^{\pm},$$
$$\pm i\frac{\partial E^{\pm}}{\partial z}=\frac{i\omega}{c}\mu^{\pm}h^{\pm}. \quad (5.10)$$

From this

$$\zeta^{\pm}=\pm\frac{E^{\pm}}{h^{\pm}}=-\sqrt{\frac{i\omega\mu^{\pm}}{4\pi\sigma^{\pm}}}, \quad (5.11)$$

where

$$\mu^{\pm}=\mu_1\pm\mu_2, \quad \sigma^{\pm}=\sigma_1\pm\sigma_2. \quad (5.12)$$

We see that only the quantity ζ^- can become zero or infinite when there is no dissipation. Inclusion of dissipation results in the fact that for $\omega = gH_0^{(e)}$ the real part of the impedance has a maximum, while for $\omega = g\sqrt{H_0^{(e)}B_0^{(e)}}$ it has a minimum. The presence of a sharp maximum in the impedance, with a width which is the smaller the smaller the relaxation constant λ, and the excitation of eigenvibrations in bodies of finite dimensions is called ferromagnetic resonance.

We note that in the case considered here the line width of ferromagnetic resonance is determined only by the relaxation constant λ and does not depend on the electrical conductivity σ_{ik}. As we shall see, this is related to the neglect of the spatial dispersion of the magnetic susceptibility and electrical conductivity, which was not included in obtaining formulas (5.8) and (5.11). The spatial dispersion of the magnetic susceptibility, as was pointed out in the preceding paragraph, is related to the exchange interaction between atoms. The magnetic permeability when we include the spatial dispersion is given by formula (4.2).

The spatial dispersion of the electrical conductivity manifests itself when the mean free path of the conduction electrons l is of the order of, or much greater than, the skin depth δ. In these cases ($l \gtrsim \delta$) the skin effect is said to be anomalous.[28]

We shall first give the results for the surface resistance of a ferromagnet in the case where one takes into account the spatial dispersion of μ_{ik} (Ament and Rado,[25] Gurevich[30]).

If the constant magnetic field and the axis of easiest magnetization are parallel to the surface of the metal, the impedance is given by the formula

$$\zeta_{zz}=\left(\frac{i\omega\tilde{\mu}}{4\pi\sigma_3}\right)^{1/2}. \tag{5.13}$$

For $H_0^{(e)} \ll 4\pi M_0$, $\omega_1 \ll 4\pi g M_0$, $\lambda \ll gM_0$

$$\frac{\theta_C}{\hbar}\left(\frac{a}{\delta}\right)^2 \ll \left(\frac{H_0^{(e)}}{4\pi M_0}\right)^{1/2}\lambda, \quad \text{where } \delta=\frac{c}{\sqrt{2\pi\sigma\omega}},$$

$$\omega_1 = g\,(H_0^{(e)}4\pi M_0)^{1/2};$$

Near resonance $\tilde{\mu}$ is given by the formula*

$$\tilde{\mu}=\frac{(4\pi gM_0)^2\left[\omega_1^2-\omega^2+4\pi i\omega\lambda+2\,(4\pi gM_0)^2\left(\frac{\theta_C}{4\pi\mu M_0}\right)^{1/2}\frac{a}{\delta}(1+i)\right]}{\left[\omega_1^2-\omega^2+4\pi i\omega\lambda+(4\pi gM_0)^2\left(\frac{\theta_C}{4\pi\mu M_0}\right)^{1/2}\frac{a}{\delta}\,(1+i)\right]^2}. \tag{5.14}$$

If the constant magnetic field is perpendicular to the surface of the sample, then under the same conditions we have[29]

$$\zeta^-=\left(\frac{i\omega\tilde{\mu}^-}{4\pi\sigma^-}\right)^{1/2}, \tag{5.15}$$

$$\tilde{\mu}^-=\frac{4\pi gM_0\left[gH_0\left(1+i\frac{\lambda}{gM_0}\right)-\omega+8\pi gM_0(1+i)\left(\frac{\theta_C}{4\pi\mu M_0}\right)^{1/2}\frac{a}{\delta}\right]}{\left[gH_0\left(1+i\frac{\lambda}{gM_0}\right)-\omega+4\pi gM_0(1+i)\left(\frac{\theta_C}{4\pi\mu M_0}\right)^{1/2}\frac{a}{\delta}\right]^2}. \tag{5.16}$$

We give the expressions for ζ_{zz} in the first case and for ζ^- in the second case since they contain the resonant dependence on frequency.

As we see from formulas (5.14) and (5.16), spatial dispersion results in an additional broadening of the ferromagnetic resonance line of the order of $4\pi gM_0 \times\left(\frac{\theta_C}{4\pi\mu M_0}\right)^{1/2}\frac{a}{\delta}$ and to a slight shift in the resonance frequency.

When we include the spatial dispersion of μ_{ik}, as pointed out in the previous paragraph, in addition to the usual electrodynamic conditions we must use boundary conditions for the magnetic moment. Formulas (5.14) and (5.16) are obtained on the assumption that $\frac{\partial \mathbf{m}}{\partial \kappa_n}=0$ at the boundary of the metal.

As we have already stated, the spatial dispersion of the conductivity causes the skin effect to be anomalous. We give the formula for the surface resistance in the limiting case of anomalous skin effect ($\delta \ll l$) where the spatial dispersion of the magnetic susceptibility is not important:[29,30]

$$\zeta=\frac{4}{9}\left(\frac{\sqrt{3}\,\tilde{\mu}^2\omega^2 l}{c\sigma}\right)^{1/2}, \tag{5.17}$$

where $\tilde{\mu}$ is given by formula (5.9), if the constant magnetic field is parallel to the metal surface, and by formula (5.12) if it is perpendicular to the metal surface. In the first case ζ stands for ζ_{zz}, and in the second for ζ^-.

*In the case where the conditions given here are not satisfied, i.e., for arbitrary fields and frequencies, the expression for $\tilde{\mu}$ is very complicated. It is contained in reference 25.

Finally, we mention that in formulas (5.13) – (5.17) for the surface resistance we do not take account of the gyrotropy of the electrical conductivity of the metal, which is justified if the radius of the orbit of the electron in the effective magnetic field* is much greater than the mean free path.

6. Coupled Magnetoelastic Waves. Ferroacoustic Resonance.

In an elastically deformed ferromagnet, because of magnetostriction and ponderomotive action due to the spontaneous magnetization, there should be a coupling of the magnetic waves with elastic waves. If the medium has a high conductivity, the coupled magnetoelastic waves resulting from this are analogous to the magnetoelastic waves which can propagate in metals in the presence of an external magnetic field and to the magnetohydrodynamic waves in liquid conductors. The coupling between magnetic and elastic waves makes it possible to excite magnetic waves by means of sound, where such excitation should be especially intense when the frequencies and wave vectors of the magnetic and sound waves coincide.

The interaction between magnetic and elastic waves results in a dependence of the sound velocity in the ferromagnet on the external magnetic field and the spontaneous magnetization. In addition, this interaction should lead to an additional sound absorption in ferromagnets, depending on the electrical conductivity of the medium and on relaxation processes (A. Akhiezer, V. Bar'yakhtar and S. Peletminskiĭ[12], C. Kittel[32]).

In order to investigate coupled magnetoelastic waves, we must include in the Hamiltonian of the ferromagnet both the elastic and the magnetostrictive energy. When these energies are included, the Hamiltonian of the ferromagnet has the form

$$\mathcal{H}=\int_V\left\{\frac{1}{2}\alpha_{ik}\frac{\partial M_l}{\partial x_i}\frac{\partial M_l}{\partial x_k}+\beta(\mathbf{M})+\frac{1}{8\pi}(\mathbf{H}^2+\mathbf{ED})\right.$$
$$\left.+\frac{1}{2}\rho\dot{u}_i^2+\frac{1}{2}\lambda_{iklm}u_{ik}u_{lm}+\gamma_{ik}(\mathbf{M})\,u_{ik}\right\}dv, \tag{6.1}$$

where ρ is the density of the matter, u is the vector of the elastic displacement, $u_{ik}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_k}\right)$ is the strain tensor, λ_{iklm} is the tensor of the elastic constants, and $\gamma_{ik}(\mathbf{M})$ is the magnetostriction tensor; the term proportional to λ_{iklm} represents the elastic energy, while the term containing $\gamma_{ik}(\mathbf{M})$ is the magnetostriction energy.

We must now write the equation of motion of the magnetic moment, the equations of elasticity, and the Maxwell equations. With regard to the Maxwell equations [cf. (2.6)], we understand here by the current density in these equations the expression

*The question of the effective magnetic field acting on the conduction electrons has not been solved. It is obviously not equal to the induction $\mathbf{B}_0$, as is shown by experiments on the Hall effect in ferromagnets.[31]

$$\mathbf{j}=\sigma\left(\mathbf{E}+\frac{1}{c}[\dot{\mathbf{u}}\times\mathbf{B}]\right).$$

The equation of motion of the magnetic moment and the equations of elasticity have the usual structure:

$$\frac{\partial\mathbf{M}}{\partial t}+\frac{\partial}{\partial x_k}(M\dot{u}_k)=g[\mathbf{M}\times\mathbf{H}^{(e)}]-\frac{\lambda}{M^2}[\mathbf{M}\times[\mathbf{M}\times\mathbf{H}^{(e)}]],$$
$$\varrho\ddot{\mathbf{u}}=\mathbf{f}, \tag{6.2}$$

where $\mathbf{f}$ is the force acting on unit volume of the medium and $\mathbf{H}^{(e)}$ is the effective magnetic field. (The second term on the left side of the equation for $\mathbf{M}$ describes the change in $\mathbf{M}$ due to a change in the density of matter.) However, the effective field $\mathbf{H}^{(e)}$ and the force $\mathbf{f}$ differ from their usual values. The simplest way for finding the modified values of $\mathbf{H}^{(e)}$ and $\mathbf{f}$ is to calculate the time derivative of the total energy of the ferromagnet:

$$\frac{d\mathcal{H}}{dt}=\int_V\left\{-\left(g[\mathbf{M}\times\mathbf{H}^{(e)}]-\frac{\lambda}{M^2}[\mathbf{M}\times[\mathbf{M}\times\mathbf{H}^{(e)}]]\right)\right.$$
$$\times\left(\mathbf{H}-\frac{\partial\beta}{\partial\mathbf{M}}+\alpha_{ik}\frac{\partial^2\mathbf{M}}{\partial x_i\partial x_k}-u_{lm}\frac{\partial\gamma_{lm}}{\partial\mathbf{M}}\right)-\frac{1}{\sigma}j^2+\dot{u}_i\left[f_i-\frac{\partial\sigma_{ik}}{\partial x_k}\right.$$
$$\left.\left.-\frac{1}{c}[\mathbf{jB}]_i-\mathbf{M}\frac{\partial}{\partial x_i}\left(\mathbf{H}-\frac{\partial\beta}{\partial\mathbf{M}}+\alpha_{ik}\frac{\partial^2\mathbf{M}}{\partial x_i\partial x_k}-u_{lm}\frac{\partial\gamma_{lm}}{\partial\mathbf{M}}\right)\right]\right\}dv$$
$$+\int_S\left\{\frac{c}{4\pi}[\mathbf{H}\times\mathbf{E}]_k+\alpha_{ik}\frac{\partial\mathbf{M}}{\partial x_i}\frac{\partial\mathbf{M}}{\partial t}+\dot{u}_i\left[\sigma_{ik}+\delta_{ik}\left(\mathbf{H}-\frac{\partial\beta}{\partial\mathbf{M}}\right.\right.\right.$$
$$\left.\left.\left.+\alpha_{lm}\frac{\partial^2\mathbf{M}}{\partial x_l\partial x_m}-u_{lm}\frac{\partial\gamma_{lm}}{\partial\mathbf{M}}\right)\mathbf{M}\right]\right\}ds_k, \tag{6.3}$$

where

$$\sigma_{ik}=\lambda_{iklm}u_{lm}+\gamma_{ik}(\mathbf{M}).$$

For $\lambda=\sigma=0$, the volume integral should go to zero. From this it is easy to conclude that the effective magnetic field $\mathbf{H}^{(e)}$ and the volume force $\mathbf{f}$ should be given by the following formulas:

$$\mathbf{H}^{(e)}=\mathbf{H}-\frac{\partial\beta}{\partial\mathbf{M}}+\alpha_{ik}\frac{\partial^2\mathbf{M}}{\partial x_i\partial x_k}-u_{lm}\frac{\partial\gamma_{lm}}{\partial\mathbf{M}},$$
$$f_i=\frac{\partial\sigma_{ik}}{\partial x_k}+\frac{1}{c}[\mathbf{j}\times\mathbf{B}]_i+\mathbf{M}\frac{\partial}{\partial x_i}\mathbf{H}^{(e)}. \tag{6.4}$$

We see that the effective field $\mathbf{H}^{(e)}$ differs from (2.5) by the presence of the term $u_{lm}\frac{\partial\gamma_{lm}}{\partial\mathbf{M}}$, which is proportional to the strain tensor. In the expression for the volume force f_i, in addition to the usual elastic force $\lambda_{iklm}\frac{\partial}{\partial x_k}u_{lm}$, there appear terms $\frac{\partial}{\partial x_k}\gamma_{ik}(\mathbf{M})$ $+\mathbf{M}\frac{\partial}{\partial x_i}\mathbf{H}^{(e)}+\frac{1}{c}[\mathbf{j}\times\mathbf{B}]_i$. The third term is related to the conduction current, while the first two result from the existence of a spontaneous magnetic moment.

Using the expressions (6.4) for $\mathbf{H}^{(e)}$ and $\mathbf{f}$, we obtain from (6.3) the following expressions for $d\mathcal{H}/dt$ for non-zero (but sufficiently small) values of λ and σ:

$$\frac{d\mathcal{H}}{dt}=-\int_V\frac{1}{\sigma}\mathbf{j}^2\,dv-\int_V\frac{\lambda}{M^2}[\mathbf{M}\times\mathbf{H}^{(e)}]^2\,dv$$
$$+\int_S\left\{\frac{c}{4\pi}[\mathbf{H}\times\mathbf{E}]_k+\alpha_{ik}\frac{\partial\mathbf{M}}{\partial x_i}\frac{\partial\mathbf{M}}{\partial t}+\dot{u}_i(\sigma_{ik}+\delta_{ik}\mathbf{H}^{(e)}\mathbf{M})\right\}dS_k. \tag{6.5}$$

In this expression the first two integrals are taken over the volume of the body and represent the dissipation of energy caused by the conductivity of the medium and by relaxation processes. (Dissipation of energy due to thermal conductivity and internal friction are not contained here, since we have omitted the corresponding terms in the elastic equations.) The last three terms (6.5) represent the energy flux in the elastic ferromagnet.

We now proceed to consider magnetoelastic waves. First let us consider the case $\lambda=0$ and $\sigma=0$, where we assume for simplicity that the medium is isotropic with respect to both elastic and magnetostrictive properties. Then

$$\gamma_{ik}(\mathbf{M})=\gamma_0M^2\delta_{ik}+\gamma M_iM_k,$$

where γ_0 and γ are the magnetostriction constants.

We set $\mathbf{M}=\mathbf{M}_0+\mathbf{m}$ and $\mathbf{H}=\mathbf{H}_0+\mathbf{h}$, where $\mathbf{M}_0$ is the equilibrium value of the magnetic moment at the given temperature, $\mathbf{H}_0$ is the constant field in the body, and $\mathbf{m}$ and $\mathbf{h}$ are small corrections to $\mathbf{M}_0$ and $\mathbf{H}_0$; $\mathbf{M}_0$ and $\mathbf{H}_0$ are assumed to be directed along the axis of easiest magnetization $\mathbf{n}$. The linearized equations for $\mathbf{m}$ and $\mathbf{u}$ have the form:

$$\left.\begin{aligned}&\frac{\partial\mathbf{m}}{\partial t}+\mathbf{M}_0\operatorname{div}\mathbf{u}=g[\mathbf{M}_0\times\mathbf{H}^{(e)}],\\ &\ddot{\mathbf{u}}=s_l^2\Delta\mathbf{u}+(s_l^2-s_t^2)\nabla\operatorname{div}\mathbf{u}+2\gamma_0\nabla(\mathbf{M}_0\mathbf{m})\\ &\quad+\frac{\gamma}{\varrho}(\mathbf{M}_0\operatorname{div}\mathbf{m}+(\mathbf{M}_0,\nabla)\mathbf{m})+\frac{1}{c\varrho}[\mathbf{j}\times\mathbf{B}_0]+\frac{1}{\varrho}\nabla(\mathbf{M}_0,\mathbf{H}^{(e)}),\end{aligned}\right\} \tag{6.6}$$

where

$$\mathbf{H}^{(e)}=\mathbf{h}-\beta\mathbf{m}_\perp-\mathbf{m}\frac{H_0}{M_0}-\delta\frac{\mathbf{M}_0(\mathbf{mM}_0)}{M_0^2}+\alpha\Delta\mathbf{m}$$
$$-2\gamma_0\mathbf{M}_0\operatorname{div}\mathbf{u}-\gamma[(\mathbf{M}_0\nabla)\mathbf{u}+\nabla(\mathbf{M}_0\mathbf{u})],\quad\mathbf{M}_0\|\mathbf{H}_0\|\mathbf{n},$$

$\delta=4\mathbf{M}_0^2\beta''(\mathbf{M}_0^2)$, s_l and s_t are the velocities of longitudinal and transverse sound waves, given by the elastic tensor. Assuming that all quantities vary as $e^{i(\mathbf{kr}-\omega t)}$, we obtain the dispersion equation

$$(v^2-s_t^2)^2(v^2-s_l'^2)\left(v^2-\frac{\Omega\Omega_1}{k^2}\right)-\zeta(v^2-s_t^2)f_1-\zeta^2f_2=0, \tag{6.7}$$

where

$$v=\frac{\omega}{k},\quad\zeta=\frac{M_0^2}{\varrho}\gamma^2,\quad s_l'^2=s_l^2+\frac{M_0^2}{\varrho}\left[4(\gamma-\pi)\cos^2\theta\right.$$
$$\left.+4\gamma_0-\delta-\frac{H_0}{M_0}-\alpha k^2\right],$$
$$f_1=\frac{gM\Omega_1}{k^2}\left\{v^2-s_t^2+(s_l^2-s_t^2)\cos^2 2\theta+\frac{\Omega}{\Omega_1}(v^2-s_t^2)\cos^2\theta\right.$$
$$\left.+\frac{4\pi^2}{\gamma^2}\left(1-\frac{\gamma}{\pi}\right)(v^2-s_t^2)\sin^2 2\theta\right\},$$
$$\Omega=gM_0\left(\alpha k^2+\beta+\frac{H_0}{M_0}+4\pi\sin^2\theta\right),$$
$$\Omega_1=gM_0\left(\alpha k^2+\beta+\frac{H_0}{M_0}\right),$$

θ is the angle between $\mathbf{k}$ and $\mathbf{M}_0$.

In equation (6.7) the coupling parameter for elastic and magnetic vibrations is the quantity ζ. If we set $\zeta=0$, the roots of equation (6.7) will be

$$v_1=s_t,\quad v_{2,3}=s_l,\quad v_4=\frac{\sqrt{\Omega\Omega_1}}{k},$$

The last root corresponds to a magnetic wave. In first approximation in the coupling parameter, the velocities of the sound waves are given by the formulas

$$v_1^2 = s_l^2\left\{1+\frac{gM_0^2}{\rho s_l^2}(\gamma-2\pi)^2\frac{\Omega_1}{\omega^2-\Omega\Omega_1}\sin^2 2\theta\right\},$$

$$v_{2,3}^2 = s_t^2\left\{1+\frac{gM_0^2}{2\rho s_t^2}\frac{\gamma^2}{\omega^2-\Omega\Omega_1}\times[\Omega_1\cos^2 2\theta + \Omega\cos^2\theta \pm \sqrt{(\Omega_1\cos^2 2\theta-\Omega\cos^2\theta)^2+4\omega^2\cos^2\theta\cos^2 2\theta}]\right\}. \quad (6.8)$$

These formulas determine the velocities of longitudinal (the first formula) and transverse sound waves, as modified by the weak coupling to the magnetic waves. They are valid if ω is not too close to $\sqrt{\Omega\Omega_1}$.

If $\omega \sim \sqrt{\Omega\Omega_1}$, an "entanglement" of the magnetic and sound branches of the vibrations[33] occurs, which we shall trace for $\theta = 0$. In this case

$$v = \begin{cases} s_t, \\ s_t - \frac{1}{2}\frac{\zeta}{s_t}\frac{gM_0}{s_t k+\Omega_1} \\ \frac{1}{2}\left(s_t+\frac{\Omega_1}{k}-\frac{1}{2}\frac{\zeta}{s_t}\frac{gM_0}{s_t k+\Omega_1}\right)+\frac{1}{2}\left[\left(s_t-\frac{\Omega_1}{k}\right)^2+\zeta\frac{gM_0}{s_t k}\frac{3ks_t+\Omega_1}{ks_t+\Omega_1}\right]^{1/2}, \\ \frac{1}{2}\left(s_t+\frac{\Omega_1}{k}-\frac{1}{2}\frac{\zeta}{s_t}\frac{gM_0}{s_t k+\Omega_1}\right)-\frac{1}{2}\left[\left(s_t-\frac{\Omega_1}{k}\right)^2+\zeta\frac{gM_0}{s_t k}\frac{3ks_t+\Omega_1}{ks_t+\Omega_1}\right]^{1/2}. \end{cases} \quad (6.9)$$

For $ks_t < \Omega_1$, the third root determines the phase velocity of the magnetic wave, and the fourth root the velocity of the transverse sound wave; if $ks_t > \Omega_1$, the third root determines the phase velocity of the transverse sound wave and the fourth that of the magnetic wave.

It can be shown that the transverse waves are elliptically polarized and that the ratio of the semi-axes of the ellipse is equal to

$$\frac{a}{b} = \frac{\Omega_1\cos 2\theta}{\omega\cos\theta}\left\{1+\frac{2\cos^2\theta}{\Omega_1^2}\times\frac{\omega^2-\Omega\Omega_1}{\cos^2 2\theta+\frac{\Omega}{\Omega_1}\cos^2\theta\pm\left[\left(\cos^2 2\theta-\frac{\Omega}{\Omega_1}\cos^2\theta\right)^2+4\frac{\omega^2}{\Omega_1^2}\cos^2\theta\cos^2 2\theta\right]^{1/2}}\right\}. \quad (6.10)$$

[The semi-axis a is in the (**n**, **k**) plane and is directed perpendicular to **k**.]

For $\theta = 0$ we get two circularly-polarized waves, while for $\theta = \pi/2$ we have two linearly-polarized waves for which the vector **u** is along $\mathbf{M}_0$ and $\mathbf{M}_0 \times \mathbf{k}$.

In a similar way one can treat coupled magnetoacoustic vibrations in the limiting case of large conductivity, $\sigma \gg \frac{\omega c^2}{s_t^2}$.

Here we give only the formulas for the phase velocities of sound waves for $\theta = 0$ and $\theta = \pi/2$. If $\theta = 0$,

$$v_1 = s_l\left\{1-\frac{M_0^2}{2\rho s_l^2}\left(4\gamma_0+4\gamma-\delta-4\pi-\frac{H_0}{M_0}\right)\right\},$$

$$v_2 = v_3 = s_t\left\{1+\frac{M_0^2}{2\rho s_t^2}\left[\frac{B_0^2}{4\pi M_0^2}-\frac{1}{2}\frac{gM_0}{\Omega_1+4\pi gM_0}\left(\gamma-\frac{B_0}{M_0}\right)\right]\right\}. \quad (6.11)$$

for $\theta = \pi/2$,

$$\left.\begin{aligned} v_1 &= s_l\left(1-\frac{2M_0^2}{\rho s_l^2}\left(\gamma-\frac{H_0}{8\pi M_0}\right)^2\right), \\ v_2 &= s_t\left(1-\frac{M_0^2}{8\rho s_t^2}\frac{gM_0}{\Omega_1+4\pi gM_0}\right), \\ v_3 &= s_t. \end{aligned}\right\} \quad (6.12)$$

We note that setting $M_0 = 0$ and $\gamma_0 = \gamma = 0$ in these formulae, we find the velocities of sound vibrations in a metal with very high conductivity and which is in an external magnetic field H_0. If $H_0^2/8\pi\rho s_t^2 \ll 1$, these formulae have the following form;

for $\theta = 0$,

$$v_1 = s_l,\ v_2 = v_3 = s_t\left(1+\frac{H_0^2}{8\pi\rho s_t^2}\right),$$

for $\theta = \pi/2$,

$$v_1 = s_l\left(1+\frac{H_0^2}{8\pi\rho s_t^2}\right),\ v_2 = v_3 = s_t. \quad (6.13)$$

Experimentally the change in velocity of longitudinal and transverse sound waves was observed in tin and aluminum.[56]

Let us now turn to the determination of the damping coefficient of magnetoacoustic vibrations. For this, we should, according to (6.1) and (6.5), calculate the quantities $\overline{\mathcal{H}}$ and $d\overline{\mathcal{H}}/dt$ for values of the field corresponding to $\lambda = 0$ and $\sigma = 0, \infty$. Here we shall give only the final formulae which determine the absorption of the sound vibrations.

For small values of σ, the damping coefficients for longitudinal and transverse waves are equal to

$$\left.\begin{aligned} \Gamma^{(l)} &= \lambda\omega^2\frac{M_0^2}{\rho s_l^2}(2\pi-\gamma)^2\frac{\omega^2+\Omega_1^2}{(\omega^2-\Omega\Omega_1)^2}\sin^2\theta \\ &+\sigma\frac{M_0^2}{\rho c^2}\left\{\left[\frac{H_0}{M_0}+8\pi(2\pi-\gamma)gM_0\Omega_1\frac{\cos^2\theta}{\omega^2-\Omega\Omega_1}\right]^2\right. \\ &\left.+\left[8\pi(2\pi-\gamma)\frac{gM_0\omega_1}{\omega^2-\Omega\Omega_1}\cos\theta\right]^2\right\}\sin^2\theta, \\ \Gamma^{(t)} &= \lambda\omega^2\frac{M_0^2}{\rho s_t^2}\gamma^2\frac{\eta^2+\cos^2 2\theta}{\eta^2+\cos^2\theta}\frac{\cos^4\theta}{(\omega\eta-\Omega_1\cos 2\theta)^2} \\ &+\frac{\sigma B_0^2}{\rho c^2}\frac{\cos^2\theta}{\eta^2+\cos^2\theta}\left\{\left[1-\frac{4\pi M_0\gamma}{B_0}\frac{gM_0\cos^2\theta}{\eta\omega-\Omega_1\cos 2\theta}\right]^2\eta^2\right. \\ &\left.+\left[1-\frac{4\pi M_0}{B_0}\frac{\gamma gM_0\cos 2\theta}{\eta\omega-\Omega_1\cos 2\theta}\right]^2\cos^2\theta\right\}, \end{aligned}\right\} \quad (6.14)$$

where $\eta = \frac{a}{b}\cos\theta$ [cf. (5.10)].

We see that the damping of sound has a markedly anisotropic character and that it is especially large at resonance, when the frequencies of sound vibrations and magnetic vibrations coincide.

To determine the coefficient for damping of sound near resonance, we must use the exact dispersion equation which takes into account the conductivity σ and the relaxation constant λ. It can be shown that in this case the damping coefficient is given by formulas (6.14), if we replace the denominator $(\omega^2 - \Omega\Omega_1)^2$ by $(\omega^2 - \Omega\Omega_1)^2 + \Gamma^2\Omega\Omega_1$, where Γ is the coefficient for damping of magnetic waves which is given by formula (2.16). (This replacement must also be made for the quantity η.)

Here we shall give only the final formula for the damping coefficient of longitudinal sound waves at resonance:

$$\Gamma^{(l)}=\frac{M_0^2}{\varrho s_l^2}(2\pi-\gamma)^2\frac{\omega_0^2}{\pi\lambda},\quad \omega_0^2=\Omega\Omega_1. \tag{6.15}$$

Let us compare this quantity with the damping coefficient due to thermal conductivity,

$$\Gamma_\varkappa=\frac{\omega_0^2}{c_v^2}\varkappa\alpha_T^2\varrho T,$$

where κ is the coefficient of heat conduction, T is the temperature in degrees, c_V is the heat capacity per unit volume, and α_T is the thermal expansion coefficient. Setting

$$\alpha_T=10^{-5}\ 1/\text{deg}.\quad c_v=10^6\frac{\text{erg}}{\text{g. deg}},\ \varkappa=10^6\frac{\text{erg}}{\text{cm. sec. deg.}},$$

$$T=100^\circ\text{K},\quad \varrho=10\frac{\text{g}}{\text{cm}^3},\quad \frac{\lambda}{gM_0}=10^{-1},\ \gamma=1,\ s_l=5\cdot10^5\frac{\text{cm}}{\text{sec}},$$

$M_0=10^3$ cgs emu,

we get

$$\Gamma^{(l)}\approx\Gamma_\varkappa.$$

Finally, we give formulas determining the absorption of sound in ferromagnets which have high conductivity, $\sigma\gg\frac{c^2\omega}{s_t^2}$.

For $\theta=0$,

$$\Gamma_{2,3}^{(t)}=\lambda\left(\gamma-\frac{B_0}{M_0}\right)^2\frac{M_0}{\varrho s_t^2}\left(\frac{M_0}{B_0+\beta M_0}\right)^2\left(\frac{\omega}{gM_0}\right)^2$$
$$+\frac{\omega^2}{\sigma}\frac{M_0^2}{\varrho s_t^2}\frac{c^2}{s_t^2}\left[\frac{B_0}{4\pi M_0}+\frac{M_0}{B_0+\beta M_0}\left(\gamma-\frac{B_0}{M_0}\right)\right]^2. \tag{6.16}$$

For $\theta=\pi/2$,

$$\Gamma_3^{(t)}=\lambda\omega^2\frac{M_0^2}{\varrho s_t^2}\frac{\gamma^2}{\Omega^2},$$
$$\Gamma^{(l)}=\frac{\omega^2}{16\pi^2\sigma}\frac{c^2}{s_t^2}\frac{H_0^2}{\varrho s_t^2}. \tag{6.16'}$$

The damping coefficients $\Gamma^{(l)}_{\theta=0}$ and $\Gamma^{(t)}_{2\theta=\pi/2}$ are equal to zero in this approximation. We see that for large values of σ the sound damping does not have resonance character.

In conclusion we consider the problem of excitation of magnetic waves by an external sound field.

Suppose that the half space $z>0$ is filled with a ferromagnet at whose external surface ($z=0$) there is assigned a displacement $u=u_0e^{-i\omega t}$ (u_0 is assumed to be constant). We are required to determine $\mathbf{m}(\mathbf{r}, t)$ and $\mathbf{u}(\mathbf{r}, t)$.

From formula (6.8), for $\theta=0$ only the interaction of transverse sound with a magnetic wave has resonance character. We shall therefore assume that $\mathbf{u}$ is in the plane (x, y) in which the vector $\mathbf{M}_0$ also lies.

In the present case the fundamental equations (5.6) can be rewritten as

$$\left.\begin{aligned}&\dot{\mathbf{m}}=gM_0\left[\mathbf{n}\times\mathbf{h}^{(s)}-\frac{\omega_0}{gM_0}\mathbf{m}\right]+\lambda\left(\frac{\omega_0}{gM_0}\mathbf{m}-\mathbf{h}^{(s)}\right),\\&\ddot{\mathbf{u}}-s_t^2\frac{\partial^2\mathbf{u}}{\partial z^2}-\gamma\frac{M_0}{\varrho}\frac{\partial\mathbf{m}}{\partial z}=0,\end{aligned}\right\} \tag{6.17}$$

where

$$\mathbf{h}^{(s)}=-\gamma M_0\frac{\partial\mathbf{u}}{\partial z},\quad \omega_0=gM_0\left(\beta+\frac{H_0}{M_0}\right).$$

From the first equation it follows that

$$\mathbf{m}=\hat{\chi}\mathbf{h}^{(s)}, \tag{6.18}$$

where the tensor $\hat{\chi}$ has the form

$$\chi=\begin{pmatrix}\chi_{xx},&\chi_{xy}\\ \chi_{yx},&\chi_{yy}\end{pmatrix}$$
$$=-\frac{gM_0\omega_0}{\omega_0^2-\left(\omega-i\omega_0\frac{\lambda}{gM_0}\right)^2}\begin{pmatrix}1-i\frac{\lambda}{gM_0}\frac{\omega}{\omega_0}+\left(\frac{\lambda}{gM_0}\right)^2,& i\frac{\omega}{\omega_0}\\ -i\frac{\omega}{\omega_0},& 1-i\frac{\lambda}{gM_0}\frac{\omega}{\omega_0}+\left(\frac{\lambda}{gM_0}\right)^2\end{pmatrix}.$$

Assuming that $\mathbf{u}$ and $\mathbf{m}$ are proportional to $e^{-i\omega t}$ we obtain

$$\omega^2\mathbf{u}+(s_t^2-\zeta\hat{\chi})\frac{\partial^2\mathbf{u}}{\partial z^2}=0, \tag{6.18'}$$

from which

$$\mathbf{u}=\mathbf{C}_1e^{ik_1z}+\mathbf{C}_2e^{ik_2z},$$

where $\mathbf{C}_1$ and $\mathbf{C}_2$ are integration constants and

$$k_{1,2}^2=\frac{\omega^2}{s_t^2-\zeta(\chi_{xx}\mp\sqrt{\chi_{xy}\chi_{yx}})}.$$

If $\mathbf{u}_{Z=0}=\mathbf{u}_0$ is specified,

$$C_{1x}=C_{2x}=iC_{1y}=-iC_{2y}=\frac{1}{2}u_0$$

(the x axis is taken along $\mathbf{u}_0$) and

$$\mathbf{u}(z,\ t)=\mathbf{u}_0\cos(\omega t-kz),$$

$$\left.\begin{aligned}m_x=&\frac{\gamma\omega\omega_0gM_0^2u_0}{s_t\left[(\omega^2-\omega_0^2)^2+4\omega^2\omega_0^2\left(\frac{\lambda}{gM_0}\right)^2\right]}\\&\times\left\{(\omega^2-\omega_0^2)\sin(\omega t-kz)-2\omega\omega_0\frac{\lambda}{gM_0}\cos(\omega t-kz)\right\},\\ m_y=&\frac{\gamma\omega^2gM_0^2u_0}{s_t\left[(\omega^2-\omega_0^2)^2+4\omega^2\omega_0^2\left(\frac{\lambda}{gM_0}\right)^2\right]}\\&\times\left\{(\omega^2-\omega_0^2)\cos(\omega t-kz)+2\omega\omega_0\frac{\lambda}{gM_0}\sin(\omega t-kz)\right\}\end{aligned}\right\} \tag{6.19}$$

(We have set $k_1\sim k_2\sim k=\frac{\omega}{s_t}$.)

At resonance

$$\frac{m}{M_0}\approx2\gamma(gM_0)^2\frac{u_0}{\lambda s_t}. \tag{6.19'}$$

Assuming that $\frac{\lambda}{gM_0}\sim10^{-1}$, $\omega_0\sim10^7$, we get $\left(\frac{m}{M_0}\right)_{\text{res}}\approx10^{-2}$.

7. Energy Spectrum of Antiferromagnets.

The phenomenological method which we have used for finding the energy spectrum of a ferromagnet can also be used for obtaining the energy spectrum of antiferromagnets.

Antiferromagnets are bodies consisting of several magnetized sub-lattices with quasi-independent magnetic moments which compensate one another.

One must remember that the microscopic description of an antiferromagnet as a body in which the spins

of neighboring atoms are oriented opposite to one another is not valid, since a state with such an orientation of the spins is not an eigenstate of the Hamiltonian containing the exchange interaction, which plays the basic role in antiferromagnets as it does in ferromagnets. In fact, since the Hamiltonian does not change under an interchange of neighboring atoms, the eigenfunctions of this Hamiltonian must either not change, or simply change sign for such a permutation, and the state with alternating spins does not satisfy this condition: when we interchange neighboring atoms, we destroy the checkerboard pattern.

However, one can give a macroscopic description of an antiferromagnet as an aggregate of magnetized sublattices, each of which is characterized by its angular momentum at each point in space. Such a description leads to results which are in good agreement with experimental data on magnetic and thermal properties and with neutron diffraction studies of antiferromagnets.

In the following for simplicity we shall consider an antiferromagnet consisting of two sublattices with magnetic moments $\mathbf{M}_1(\mathbf{r}, t)$ and $\mathbf{M}_2(\mathbf{r}, t)$, whose equilibrium values in the absence of an external magnetic field are equal in magnitude and opposite in direction. MnF_2 has such a magnetic structure.

The Hamiltonian of the antiferromagnet in the case where we have two mirror sublattices can be written as follows:

$$\mathcal{H} = \int dv \left\{ \frac{1}{2} \alpha \left(\frac{\partial \mathbf{M}_1}{\partial x_i} \frac{\partial \mathbf{M}_1}{\partial x_i} + \frac{\partial \mathbf{M}_2}{\partial x_i} \frac{\partial \mathbf{M}_2}{\partial x_i} \right) + \alpha_{12} \frac{\partial \mathbf{M}_1}{\partial x_i} \frac{\partial \mathbf{M}_2}{\partial x_i} + \delta \mathbf{M}_1 \mathbf{M}_2 - \frac{1}{2} \beta [(\mathbf{n}\mathbf{M}_1)^2 + (\mathbf{n}\mathbf{M}_2)^2] - (\mathbf{M}_1 + \mathbf{M}_2)\mathbf{H} - \frac{H^2}{8\pi} \right\}. \tag{7.1}$$

The first two terms here are the analogue of the energy of inhomogeneous exchange interaction in the Hamiltonian (2.3), while the term $\delta \mathbf{M}_1 \mathbf{M}_2$ is related to the exchange interaction of the uniformly magnetized sublattices. The fourth term is the anisotropy energy, written on the assumption that there is a preferred axis in the body, which we denote by n. Since the magnetic anisotropy is associated with relativistic interactions, $|\beta| \ll \delta$. In addition, we shall assume that $\beta > 0$. This means that in the absence of a magnetic field the magnetic moments are directed along the preferred axis. Finally, the last two terms give the magnetic energy. In treating quasi-static oscillations of the magnetic moments they are equivalent to the expression $+\frac{\mathbf{H}^2}{8\pi}$ which is used for studying spin waves in a ferromagnet. In fact, the difference of these two expressions, which is equal to

$$\int \left\{ \frac{\mathbf{H}^2}{4\pi} + \mathbf{H}(\mathbf{M}_1 + \mathbf{M}_2) \right\} dv = \int \frac{\mathbf{HB}}{4\pi} dv,$$

becomes a surface integral in the quasi-static case, when $\mathbf{H} = \nabla\varphi$, and can be dropped in determining the frequencies of vibration of the magnetic moments.

In writing the Hamiltonian (7.1) we start from the assumption that the minimum energy of the antiferromagnet corresponds to uniformly magnetized sublattices. This means that the quadratic form

$$\frac{1}{2} \alpha \left(\frac{\partial \mathbf{M}_1}{\partial x_i} \frac{\partial \mathbf{M}_1}{\partial x_i} + \frac{\partial \mathbf{M}_2}{\partial x_i} \frac{\partial \mathbf{M}_2}{\partial x_i} \right) + \alpha_{12} \frac{\partial \mathbf{M}_1}{\partial x_i} \frac{\partial \mathbf{M}_2}{\partial x_i}$$

must be essentially positive, from which it follows that $\alpha > 0$ and $\alpha_{12} < \alpha$. The quantities α and α_{12}, like the corresponding quantity α in ferromagnets, are proportional to the exchange integral between neighboring atoms. Therefore α and α_{12} have the same order of magnitude,

$$\alpha \sim \alpha_{12} \sim \frac{T_C}{\mu M_0} a^2,$$

where T_C is the Curie-Néel temperature of the antiferromagnet, i.e. the temperature at which the antiferromagnet makes a transition to the paramagnetic state. The constant δ is equal in order of magnitude to $T_C/\mu M_0$.

The magnetic field $\mathbf{H}$ is made up of the external constant homogeneous field $\mathbf{H}_0$ and the magnetic field of the spin waves $\mathbf{h}$, which is determined in the case of low frequency oscillations by the equations

$$\operatorname{curl} \mathbf{h} = 0, \quad \operatorname{div} \mathbf{h} = -4\pi \operatorname{div} (\mathbf{M}_1 + \mathbf{M}_2) \tag{7.2}$$

(The antiferromagnet is assumed to be a dielectric, so that we do not give a term containing the conduction current.)

Let us write the equations of motion of the magnetic moments, analogous to (2.4):

$$\frac{d\mathbf{M}_j}{dt} = g[\mathbf{M}_j \times \mathbf{H}_j^{(e)}], \tag{7.3}$$

where $\mathbf{H}_j^{(e)}$ is the effective field acting on the moment $\mathbf{M}_j$:

$$\mathbf{H}_j^{(e)} = -\frac{\delta \mathcal{H}}{\delta \mathbf{M}_j}. \tag{7.4}$$

(We shall not consider relaxation forces here.) Using the expression (7.1) for $\mathcal{H}$, we get

$$\begin{aligned} \mathbf{H}_1^{(e)} &= \mathbf{H} - \delta \mathbf{M}_2 - \beta \mathbf{M}_{1\perp} + \alpha \Delta \mathbf{M}_1 + \alpha_{12} \Delta \mathbf{M}_2, \\ \mathbf{H}_2^{(e)} &= \mathbf{H} - \delta \mathbf{M}_1 - \beta \mathbf{M}_{2\perp} + \alpha \Delta \mathbf{M}_2 + \alpha_{12} \Delta \mathbf{M}_1, \end{aligned} \tag{7.5}$$

where $\mathbf{M}_{j\perp} = \mathbf{M}_j - \mathbf{n}(\mathbf{n} \cdot \mathbf{M}_j)$.

To find the energy spectrum of the antiferromagnet, we must consider small oscillations of the magnetic moments around their equilibrium values $\mathbf{M}_{10}$ and $\mathbf{M}_{20}$. These values correspond to the minimum energy of the antiferromagnet, and are independent of coordinates. To determine the directions of $\mathbf{M}_{10}$ and $\mathbf{M}_{20}$ we must find the minimum of the expression

$$E = \delta \mathbf{M}_{10} \mathbf{M}_{20} - \frac{1}{2} \beta [(\mathbf{n}\mathbf{M}_{10})^2 + (\mathbf{n}\mathbf{M}_{20})^2] - (\mathbf{M}_{10} + \mathbf{M}_{20}, \mathbf{H}_0), \tag{7.6}$$

where $\mathbf{H}_0$ is a given external field, with the additional conditions $M_{10}^2 = M_{20}^2 = M_0^2$. As a result we obtain the following formulas which determine the direction of the magnetic moments $\mathbf{M}_{10}$ and $\mathbf{M}_{20}$:

$$\sin(\theta_2-\theta_1)+\frac{H_0}{\delta M_0}\sin(\theta_1-\varphi)+\frac{\beta}{\delta}\sin\theta_1\cos\theta_1=0,$$
$$\sin(\theta_1-\theta_2)+\frac{H_0}{\delta M_0}\sin(\theta_2-\varphi)+\frac{\beta}{\delta}\sin\theta_2\cos\theta_2=0, \quad (7.7)$$

where θ_1 and θ_2 are the angles between $\mathbf{n}$ and $\mathbf{M}_{10}$, $\mathbf{M}_{20}$: φ is the angle between $\mathbf{n}$ and $\mathbf{H}_0$ [the vectors $\mathbf{M}_{10}$ and $\mathbf{M}_{20}$ lie in the plane $(\mathbf{n}\cdot\mathbf{H}_0)$].

Equations (7.7) show that the orientation of the magnetic moments is essentially determined by the external magnetic field $\mathbf{H}_0$. Let us first consider the case where the external magnetic field is parallel to the preferred axis $\mathbf{n}$. If $H_0 < \sqrt{(2\delta+\beta)\beta}M_0$, then

$$\mathbf{M}_{10}=-\mathbf{M}_{20}, \quad \mathbf{M}_{10}\parallel\mathbf{H}_0. \quad (7.8)$$

If $(2\delta-\beta)M_0 \geq H_0 \geq \sqrt{(2\delta+\beta)\beta}\,\frac{2\delta-\beta}{2\delta+\beta}M_0$, the angle between the moments $\mathbf{M}_{10}$ and $\mathbf{M}_{20}$ is different from π. The angles formed by the moments $\mathbf{M}_{10}$ and $\mathbf{M}_{20}$ with the axis of easiest magnetization are given by the formulas

$$\cos\theta_1=\frac{H_0}{H_e}, \quad \theta_1=-\theta_2, \quad (7.9)$$

where $(2\delta-\beta)M_0 \gg H_0 \gg \sqrt{(2\delta+\beta)\beta}\,\frac{2\delta-\beta}{2\delta+\beta}M_0$.

In the range of fields

$$\sqrt{(2\delta+\beta)\beta}\,\frac{2\delta-\beta}{2\delta+\beta}M_0 \leq H_0 \leq \sqrt{(2\delta+\beta)\beta}\,M_0$$

there correspond to the minimum energy (7.6) the two configurations (7.8) and (7.9) (cf. Figs. 4-5). If the field satisfies

$$H_0 < \sqrt{(2\delta+\beta)\beta\,\frac{2\delta-\beta}{2\delta+\beta}}M_0,$$

then the deeper minimum, i.e., the ground state, corresponds to the configuration (7.8); if the field satisfies

$$H_0 > \sqrt{(2\delta+\beta)\beta\,\frac{2\delta-\beta}{2\delta+\beta}}M_0,$$

then the ground state will be the state with configuration (7.9). We note that for

$$H_0=\sqrt{(2\delta+\beta)\beta\,\frac{2\delta-\beta}{2\delta+\beta}}M_0$$

there is a readjustment of the ground state of the antiferromagnet: the magnetic moments rotate to an angle close to $\pi/2$ with respect to the preferred axis. For $H_0 > H_e$ the magnetic moments align themselves along the axis $\mathbf{n}$.

If the magnetic field $\mathbf{H}_0$ is perpendicular to the axis $\mathbf{n}$ and $H_0 < H'_e$, where $H'_e = (2\delta+\beta)M_0 \approx 2\delta M_0$, the magnetic moments lie in the plane $(\mathbf{n}, \mathbf{H})$ and

$$\sin\theta_1=\frac{H_0}{H'_e}, \quad \theta_2=\pi-\theta_1. \quad (7.10)$$

For $H_0 > H'_e$ both magnetic moments are oriented along the magnetic field, i.e. at $H_0 = H'_e$ the antiferromagnet, so to speak, goes over into the ferromagnetic state.

Let us denote the deviations of the magnetic moments $\mathbf{M}_1$ and $\mathbf{M}_2$ from their equilibrium values $\mathbf{M}_{10}$ and $\mathbf{M}_{20}$ by $\mathbf{m}_1$ and $\mathbf{m}_2$. Assuming that $|\mathbf{m}_1|$, $|\mathbf{m}_2| \ll M_0$, we obtain, after linearization, the following system of equations for $\mathbf{m}_1$ and $\mathbf{m}_2$:

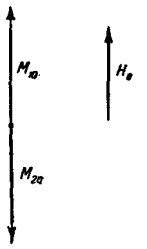

FIG. 4

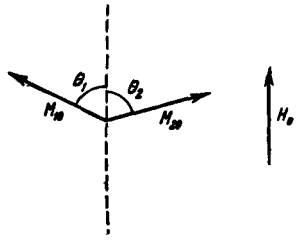

FIG. 5

$$\left.\begin{aligned}\frac{\partial\mathbf{m}_1}{\partial t}&=g\mathbf{M}_{10}\times[\mathbf{h}-\delta\mathbf{m}_2-\beta\mathbf{m}_{1\perp}+\alpha\Delta\mathbf{m}_1+\alpha_{12}\Delta\mathbf{m}_2]\\&\quad+g\,\mathbf{m}\times[\mathbf{H}_0-\delta\mathbf{M}_{20}-\beta\mathbf{M}_{10\perp}],\\\frac{\partial\mathbf{m}_2}{\partial t}&=g\mathbf{M}_{20}\times[\mathbf{h}-\delta\mathbf{m}_1-\beta\mathbf{m}_{2\perp}+\alpha\Delta\mathbf{m}_2+\alpha_{12}\Delta\mathbf{m}_1]\\&\quad+g\,\mathbf{m}_2\times[\mathbf{H}_0-\delta\mathbf{M}_{10}-\beta\mathbf{M}_{20\perp}],\end{aligned}\right\} \quad (7.11)$$

where $\mathbf{m}_{j\perp}=\mathbf{m}_j-\mathbf{n}(\mathbf{m}_j\mathbf{n})$, $\mathbf{M}_{0\perp}=\mathbf{M}_0-\mathbf{n}(\mathbf{n}\mathbf{M}_0)$.

The solution of these equations in the form of plane monochromatic waves [$\exp(i\mathbf{kr}-i\omega t)$] leads, after the elimination of the magnetic field $\mathbf{h}$, to the equations

$$\left.\begin{aligned}-i\omega\mathbf{m}_1&=g\mathbf{M}_{10}\times\Big[-\frac{4\pi\mathbf{k}}{k^2}(\mathbf{k},\ \mathbf{m}_1+\mathbf{m}_2)-\delta\mathbf{m}_2-\beta\mathbf{m}_{1\perp}-\alpha k^2\mathbf{m}_1\\&\quad-\alpha_{12}k^2\mathbf{m}_2\Big]+g\,\mathbf{m}_1\times[\mathbf{H}_0-\delta\mathbf{M}_{20}-\beta\mathbf{M}_{10\perp}],\\-i\omega\mathbf{m}_2&=g\mathbf{M}_{20}\times\Big[-\frac{4\pi\mathbf{k}}{k^2}(\mathbf{k},\ \mathbf{m}_1+\mathbf{m}_2)-\delta\mathbf{m}_1-\beta\mathbf{m}_{2\perp}-\alpha k^2\mathbf{m}_2\\&\quad-\alpha_{12}k^2\mathbf{m}_1\Big]+g\,\mathbf{m}_2\times[\mathbf{H}_0-\delta\mathbf{M}_{10}-\beta\mathbf{M}_{20\perp}].\end{aligned}\right\} \quad (7.12)$$

Equating the determinant of this system to zero, we find the relation between ω and $\mathbf{k}$. It turns out that there are two branches of the vibrations. They differ in their dispersive behavior in the presence of an external magnetic field. When the field goes to zero, the two branches coalesce, if we neglect the magnetic interaction of the spins (i.e., if we set $\mathbf{h}=0$). Neglecting this is always justified since the magnetic interaction is very much smaller than the exchange interaction. We note that in ferromagnets the magnetic interaction can be neglected only for sufficiently large k, when the inequality $ak \gg \left(\frac{4\pi\mu M_0}{\theta C}\right)^{1/2}$ is satisfied. Under this condition, the exchange interaction energy, which contains spatial derivatives of the moments, will be much greater than the magnetic interaction energy. In the case of antiferromagnets, magnetic interaction can be neglected for all wave vectors. The reason for this is that the energy of the antiferromagnet (7.1) contains the term $\delta\mathbf{M}_1\cdot\mathbf{M}_2$, which describes the exchange interaction of the uniformly magnetized sublattices. However, one must remember that the weak magnetic interaction results in a slight difference between the two branches of the energy spectrum of the antiferromagnet.

We shall not carry out the detailed calculations, but give only the final formulas for the energy of spin waves in the most interesting cases.

If the external magnetic field $\mathbf{H}_0$ is parallel to the preferred axis, when $H_0 < \sqrt{(2\delta + \beta)\beta}\, M_0$ then[34]

$$\varepsilon_{1,2} = \hbar\omega_{1,2} = \sqrt{(\mu H_a)^2 + \theta_C^2 (ak)^2} \pm \mu H_0, \qquad (7.13)$$

where $\mu = g\hbar$, $\theta_C = \sqrt{2\delta(\alpha - \alpha_{1,2})}\,\frac{1}{a}$ and $H_a = \sqrt{(2\delta + \beta)\beta}\, M_0 \approx \sqrt{2\delta\beta}\, M_0$ (θ_C is of the same order of magnitude as the Curie-Néel temperature).

We see that an increase in magnetic field results in a reduction in the energy of one of the branches of the spectrum.

If $H_e \gg H_0 > \sqrt{(2\delta + \beta)\beta}\,\frac{2\delta - \beta}{2\delta + \beta}$, then[38]

$$\varepsilon_1 = \theta_C (ak),$$
$$\varepsilon_2 = \sqrt{\mu^2 (H_0^2 - H_a^2) + \theta_C^2 (ak)^2}. \qquad (7.14)$$

We note that in this case ϵ_1 tends to zero with $\mathbf{k}$.

Next we consider the case where the magnetic field is perpendicular to the axis $\mathbf{n}$. If $H_0 \ll H_e'$, then[34]

$$\varepsilon_1 = \sqrt{(\mu H_a)^2 + \theta_C^2 (ak)^2};\quad \varepsilon_2 = \sqrt{(\mu H_a)^2 + (\mu H_0)^2 + \theta_C^2 (ak)^2}. \qquad (7.15)$$

If $H_0 > H_e'$, then[38]

$$\varepsilon_1 = \sqrt{[\theta_C (ak)^2 + \mu (H_0 - H_e')]\,[\theta_C (ak)^2 + \mu (H_0 - H_e)]}, \qquad (7.16)$$
$$\varepsilon_2 = \theta_{1C} (ak)^2 + \mu H_0,$$

where $\theta_{1C} = \frac{\mu M_0}{a^2}(\alpha + \alpha_{12})$.

If $\mathbf{H}_0 = 0$,

$$\varepsilon_{1,2} \approx \begin{cases} \theta_C (ak), & ak \gg \frac{\mu H_a}{\theta_C}, \\ \mu H_a + \frac{\theta_C}{2\mu H_a}(ak)^2, & ak \ll \frac{\mu H_a}{\theta_C}. \end{cases} \qquad (7.17)$$

We have considered oscillations of the magnetic moments in the simplest magnetic systems: in ferromagnets which have one set of magnetic moments, and in antiferromagnets with two mirror magnetic sublattices. Aside from these simplest systems there exist a whole variety of bodies having more complicated magnetic structure. If in a complex magnetic system there is not complete (or almost complete, as, for example, in $\alpha - Fe_2O_3$, $MnCO_3$ etc.) compensation of the magnetic moments, then all the branches of the energy spectrum except one contain an activation energy of the order of the Curie-Néel temperature. An analogous situation exists in the vibrations of complex crystal lattices in which, of the 3n (where n is the number of atoms in a unit cell) vibrations, three have no energy of activation and represent sound waves, while $3n - 3$ are optical vibrations. In the case of vibrations of magnetic moments there correspond to the activation branches oscillations with change of the angles between the moments of the sublattices. To the branches which do not contain an activation energy there correspond vibrations without change in angle between the moments. For example, for not too long waves, when we can neglect the magnetic interaction, the frequency of each non-activation wave in a system with two antiparallel moments $\mathbf{M}_1$ and $\mathbf{M}_2$ which are, however, not equal in absolute magnitude, is given by the formula

$$\hbar\omega = \frac{\mu_1\mu_2 (M_1^2 + M_2^2)}{\mu_2 M_1 - \mu_1 M_2}\alpha k^2 - 2\frac{\mu_1\mu_2 M_1 M_2}{\mu_2 M_1 - \mu_1 M_2}\alpha_{12} k^2 + \frac{\mu_1\mu_2 (M_1 - M_2) H_0}{\mu_2 M_1 - \mu_1 M_2}, \quad \mu_2 M_1 - \mu_1 M_2 > 0.$$

Now let us look at the case of an antiferromagnet with weak ferromagnetism.* The occurrence of weak ferromagnetism in antiferromagnets may be associated with two causes: either the magnetic moments of the sublattices are not strictly antiparallel, but are tilted at an angle which is close to 180°; or the magnetic moments of the sublattices are antiparallel, but are slightly different in magnitude because of a difference in the g factors. In the first case, the slight deviation of the magnetic moments from the antiparallel arrangement is caused by the specific form of the anisotropy energy which is characteristic for crystals with rhombohedral symmetry ($\alpha - Fe_2O_3$, $MnCO_3$ etc.). In such crystals the spontaneous magnetic moment is perpendicular to the axis of antiferromagnetism ("transverse" weak ferromagnetism). The second case can occur because the sublattices are made up of atoms of different elements, so that $N_1 s_1 = N_2 s_2$, while $g_1 \neq g_2$. Here N_1, s_1, g_1 and N_2, s_2, g_2 are the numbers of atoms, their spins and g factors respectively for the first and second sublattices. Besides, even for identical atoms, the g factors may be different if the atoms occupy non-equivalent positions in the crystal lattice. In this case, the resulting spontaneous magnetic moment of the crystal is parallel to the axis of antiferromagnetism ("longitudinal" weak ferromagnetism). Longitudinal weak ferromagnetism was discovered in deformed crystals of CoF_2 (A. Borovik-Romanov[40]).

Let us first consider the energy spectrum of the spin waves of antiferromagnets with "transverse" weak ferromagnetism (A. Borovik-Romanov,[39,40] E. Turov[42]).

The Hamiltonian for antiferromagnets whose crystal lattice has rhombohedral symmetry differs from the Hamiltonian for a uniaxial antiferromagnet (7.1) by the presence of a term

$$d (M_{1x} M_{2y} - M_{1y} M_{2x}), \qquad (7.18)$$

which is responsible for the weak ferromagnetism. In addition, in these crystals as a rule the magnetic anisotropy constant β is negative. Therefore, in the absence of the term (7.18) the magnetic moments in the ground state are in a plane perpendicular to the axis $\mathbf{n}$ for $\mathbf{H}_0 = 0$. Because of the specific energy (7.18), the magnetic moments are turned at an angle close to 180° (Fig. 6), as a result of which a weak magnetic moment appears in the antiferromagnet, equal in order of magnitude to $2\frac{d}{\delta}M_0$. Since the energy (7.18) is of

*A detailed investigation of weak ferromagnetism is contained in the papers of A. Borovik-Romanov[39,40] and I. Dzyaloshinskiĭ.[41]

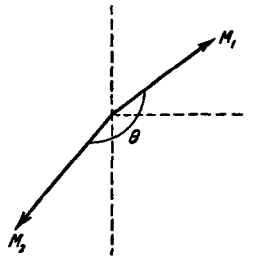

FIG. 6

relativistic origin and not exchange origin, $d \ll \delta$. In $MnCO_3$ the spontaneous magnetic moment is of the order of 200 cgs emu/mole, i.e. $d/\delta \approx 0.007$.

Carrying out calculations similar to those which were done earlier, we now get

$$\varepsilon_1 = \sqrt{\mu^2 (H_d + H_0) H_0 + \theta_C^2 (ak)^2},$$
$$\varepsilon_2 = \sqrt{\mu^2 (H_a^2 + H_d (H_d + H_0)) + \theta_C^2 (ak)^2}, \qquad (7.19)$$

where $H_d = dM_0$, and $H_a = \sqrt{2\delta |\beta| M_0}$.

The energy spectrum of spin waves of an antiferromagnet with longitudinal weak ferromagnetism can be obtained from the Hamiltonian of a ferromagnet with two magnetic sublattices by assuming that $M_1 g_2 = M_2 g_1$. Then there can exist in the body a magnetic moment equal to

$$M(0) = M_1 - M_2 = \nu M,$$

where

$$\nu = \frac{g_1 - g_2}{g_1 + g_2}, \quad M = \frac{1}{2}(M_1 + M_2).$$

Since the difference in the g factors is a relativistic effect, $M(0) \ll M$. Under these assumptions the spectrum of spin waves has the form (E. Turov[42]):

$$\varepsilon_{1,2} = \sqrt{\mu^2 (\nu H_e H_0 + \nu^2 H_0^2) + \theta_C^2 (ak)^2} \pm \mu H_0, \qquad (7.20)$$

where $\mu = \frac{1}{2}\hbar (g_1 + g_2)$, and H_0 is assumed to be directed along the axis of antiferromagnetism and to be sufficiently small in absolute value ($H_0 < \nu H_e$).

8. Thermal and Magnetic Properties of Ferromagnets.

A knowledge of the spectrum of spin waves enables us to determine the magnetic moment of a ferromagnet as a function of temperature and magnetic field for $T \ll T_C$, and also to find the contribution of the spin waves to the heat capacity and other thermodynamic characteristics of the ferromagnet.

In order to find the equilibrium value of the magnetic moment of the ferromagnet at a given temperature and a given external magnetic field, we shall start from formula (3.22) which determines the quantum-mechanical average value of the magnetic moment:

$$\langle \mathfrak{M}_z \rangle = MV - \sum_{\mathbf{k}} \mu_{\mathbf{k}} n_{\mathbf{k}},$$
$$\langle \mathfrak{M}^2 \rangle \cong (MV)^2 - 2MV \sum_{\mathbf{k} \neq 0} \mu_{\mathbf{k}} n_{\mathbf{k}}, \qquad (8.1)$$

where M and $\mu_{\mathbf{k}}$ are given by formulas (3.23) and the z axis coincides with the axis of easiest magnetization.

We see that these quantities are determined by the numbers of spin waves $n_{\mathbf{k}}$. Replacing the quantities $n_{\mathbf{k}}$ by their equilibrium values, which are given by the formula $n_{\mathbf{k}}^0 = \left(\exp \frac{\epsilon_{\mathbf{k}}}{T} - 1\right)^{-1}$, where $\epsilon_{\mathbf{k}}$ is the energy of the spin wave and T is the temperature in ergs, we obtain the following expression for the equilibrium magnetic moment of the ferromagnet:

$$\overline{\mathfrak{M}}_z = MV - \sum_{\mathbf{k}} \frac{\mu_{\mathbf{k}}}{e^{\varepsilon_{\mathbf{k}}/T} - 1},$$
$$\overline{\mathfrak{M}^2} \approx (\overline{\mathfrak{M}}_z)^2. \qquad (8.2)$$

The equilibrium distribution function of the spin waves also determines the heat capacity C_S of the ferromagnet associated with the spin waves:

$$C_s = \frac{d}{dT} \sum_{\mathbf{k}} \frac{\varepsilon_{\mathbf{k}}}{e^{\varepsilon_{\mathbf{k}}/T} - 1}. \qquad (8.3)$$

At absolute zero,

$$(C_s)_{T=0} = 0,$$
$$(\overline{\mathfrak{M}}_z)_{T=0} = MV. \qquad (8.3')$$

The magnetic moment at T = 0 depends in a complicated way on the applied field H_0. Here we shall give the expression for M in the two limiting cases of low and high fields (Holstein and Primakoff[3]):

$$M = \begin{cases} M_0 - \frac{\mu}{2a^3}\left(\frac{2\pi\mu M_0}{\theta_C}\right)^{3/2}, & H_0 + \beta M_0 \ll 2\pi M_0, \\ M_0 - \frac{\mu}{2a^3}\left(\frac{2\pi\mu M_0}{\theta_C}\right)^{3/2}\left(\frac{2\pi M_0}{H_0 + \beta M_0}\right)^{1/2}, & H_0 + \beta M_0 \gg 2\pi M_0. \end{cases} \qquad (8.4)$$

Now let us turn to the calculation of the spin specific heat and magnetic moment of a ferromagnet at finite temperatures. For this purpose it is convenient to introduce the thermodynamic potential Ω of the gas of spin waves, defined by the formula

$$\Omega = T \sum_{\mathbf{k}} \ln\left(1 - e^{-\frac{\varepsilon_{\mathbf{k}}}{T}}\right). \qquad (8.5)$$

The free energy F of the body is related to the potential Ω by the relation[43]

$$\Omega = F - \zeta N, \qquad (8.5')$$

where ζ is the chemical potential per particle and N is the total number of particles. Since the number of spin waves is not fixed (since spin waves can be created and destroyed), the chemical potential of the spin waves is equal to zero. Therefore the potential Ω coincides with the free energy F:

$$\Omega = F.$$

The heat capacity C_S and the magnetic moment of the body $\overline{\mathfrak{M}}_Z = M(T, H)V$ are related to the free energy by the formulas

$$M(T, H) = M - \frac{1}{V}\frac{\partial F}{\partial H}, \quad C_s = -T\frac{\partial^2 F}{\partial T^2},$$

and consequently,

$$M(T, H) = M - \frac{1}{V}\frac{\partial \Omega}{\partial H}, \quad C_s = -T\frac{\partial^2 \Omega}{\partial T^2}. \qquad (8.6)$$

Using formula (8.5) and the expression (2.11) for $\epsilon_{\mathbf{k}}$, and changing in (8.5) from summation to integration over the wave vector $\mathbf{k}$, in accordance with the formula

$$\sum_{\mathbf{k}} f(\mathbf{k}) = \frac{V}{(2\pi)^3}\int f(\mathbf{k})\,d\mathbf{k} \qquad (8.5')$$

we find, after integration by parts, the following expression for Ω:

$$\Omega = -\frac{V}{6\pi^2}\int_0^{\pi/2}\sin\theta\,d\theta\int_{\varepsilon_0}^{\infty}\frac{k^3(\varepsilon,\theta)\,d\varepsilon}{e^{\varepsilon/T}-1}, \qquad (8.6')$$

where $k(\epsilon, \theta)$ is the magnitude of the wave vector of the spin wave expressed in terms of its energy ϵ and the angle θ, and

$$\varepsilon_0 = \sqrt{(\mu H^{(e)})^2 + 4\pi\mu^2 M_0 H^{(e)}\sin^2\theta}, \quad H^{(e)} = H + \beta M_0$$

(Here and in the following we assume that the magnetic field is along the direction of easiest magnetization, $H \equiv H_0$.)

Since we are considering the low-temperature region, the upper limit in the integral (8.6') can be taken equal to infinity.

It is not possible to carry out the integration in (8.6') in general form. We shall therefore consider only some of the more interesting limiting cases.

If the temperature is sufficiently high, $T \gg 2\pi\mu M_0$, $\mu H^{(e)}$, then (F. Bloch[1])

$$M(T,H) = M - \frac{\zeta\left(\frac{3}{2}\right)}{8\pi^{3/2}}\frac{\mu}{a^3}\left(\frac{T}{\theta_C}\right)^{3/2},$$

$$C_s = \frac{15}{32}\frac{\zeta\left(\frac{5}{2}\right)}{\pi^{3/2}}\frac{1}{a^3}\left(\frac{T}{\theta_C}\right)^{3/2}, \qquad (8.7)$$

where $\zeta(x)$ is the Riemann ζ function (the specific heat and magnetic moment are taken for unit volume).

In the temperature region $\mu H^{(e)} \ll T \ll 2\pi\mu M_0$, we have[3,44]

$$M(T,H) = M - \frac{\pi}{48\sqrt{2}}\frac{\mu}{a^3}\left(\frac{T}{2\pi\mu M_0}\right)^{1/2}\left(\frac{T}{\theta_C}\right)^{3/2} \approx$$
$$\approx M - 0.05\,\frac{\mu}{a^3}\left(\frac{T}{2\pi\mu M_0}\right)^{1/2}\left(\frac{T}{\theta_C}\right)^{3/2},$$
$$C_s = \frac{147}{64}\frac{\zeta\left(\frac{7}{2}\right)}{\pi^{3/2}a^3}\frac{T}{2\pi\mu M_0}\left(\frac{T}{\theta_C}\right)^{3/2} \approx 0.5\,\frac{1}{a^3}\frac{T}{2\pi\mu M_0}\left(\frac{T}{\theta_C}\right)^{3/2}. \qquad (8.8)$$

These formulae are valid if the ferromagnet has low anisotropy energy, i.e., if $\beta \ll 2\pi$.

If $\mu H^{(e)} \gg T$, $2\pi\mu M_0$, then[44]

$$M(T,H) = M - \frac{1}{8\pi^{3/2}}\frac{\mu}{a^3}\left(\frac{T}{\theta_C}\right)^{3/2}e^{-\frac{\mu H^{(e)}}{T}},$$

$$C_s = \frac{1}{8\pi^{3/2}a^3}\left(\frac{\mu H^{(e)}}{\theta_C}\right)^{3/2}\left(\frac{\mu H^{(e)}}{T}\right)^{1/2}e^{-\frac{\mu H^{(e)}}{T}}. \qquad (8.9)$$

If, finally, $T \ll \mu H^{(e)}$, $2\pi\mu M_0$, then

$$M(T,H) = M - \frac{1}{16\pi^{3/2}}\frac{\mu}{a^3}\frac{T}{2\pi\mu M_0}\left(\frac{T}{\theta_C}\right)^{3/2}e^{-\frac{\mu H^{(e)}}{T}},$$

$$C_s = \frac{1}{16\pi^{3/2}a^3}\left(\frac{T}{\theta_C}\right)^{1/2}\frac{(\mu H^{(e)})^2}{2\pi\mu M_0\theta_C}e^{-\frac{\mu H^{(e)}}{T}}. \qquad (8.10)$$

We see that, in the case of high fields and very low temperatures, the spin heat capacity and the deviation of the magnetic moment from saturation are exponentially small. The same behavior with temperature occurs in the absence of field if $T \ll \beta\mu M_0$. This phenomenon is related to the presence of an activation energy for the spin wave.

Thus at very low temperature the contribution of the spin waves to the heat capacity is exponentially small.

Let us see at what temperatures the contribution of the spin waves to the heat capacity becomes reasonably large. For this purpose let us compare formulas (8.7) and (8.8) with the familiar expression for the phonon specific heat

$$C_l = \frac{2\pi^2}{5}\frac{1}{a^3}\left(\frac{T}{\theta_D}\right)^3, \qquad (8.11)$$

where θ_D is the Debye temperature, defined as

$$\frac{3}{\theta_D^3} = \frac{2}{\theta_l^3} + \frac{1}{\theta_l^3}, \quad \theta_l = \frac{hs_l}{a}, \quad \theta_t = \frac{hs_t}{a}$$

(s_l and s_t are the velocities of longitudinal and transverse sound vibrations). It is easy to verify that the spin specific heat exceeds the phonon specific heat, or is of the same order of magnitude, in the temperature interval

$$\beta\mu M_0 \ll T \ll \theta_D^2/\theta_C.$$

The lower limit here is several hundreds of degrees for materials with low anisotropy energy ($\beta \ll 1$), and is approximately equal to a degree for uniaxial crystals whose anisotropy energy is large ($\beta \sim 10$); the upper limit is of order $10 - 100°$K.

We note that the behavior of the specific heat of a ferromagnet, and consequently the behavior of its entropy as a function of magnetic field, must result in a magnetocaloric effect of the same type as in the case of ordinary paramagnets.

Formulas (8.7) — (8.10) show that the magnetic moment of a ferromagnet varies with temperature according to the same law as does the spin specific heat, only in the temperature range $T \gg \mu H^{(e)}$, $2\pi\mu M_0$. In other temperature regions these quantities have a different temperature variation.

The temperature dependence of the magnetic moment and specific heat has been studied experimentally,[45] and a law $C_S \sim M - M(T,H) \sim T^{3/2}$, which is in accord with the theory of spin waves, has been observed over a relatively wide range of temperatures. No experiments have been carried out at high magnetic fields and at very low temperatures.

9. Thermal and Magnetic Properties of Antiferromagnets.

The thermal and magnetic properties of antiferromagnets can be studied in the same way as for the case of ferromagnets. Using the expressions (7.13)

and (7.17) for the energy spectrum of the antiferromagnet, we must determine, in accordance with formula (8.5), the thermodynamic potential Ω and then, from formulas (8.5), find the specific heat and magnetic moment of the antiferromagnet.

First we shall give the expressions for the specific heat of an antiferromagnet in a few limiting cases. If there is no magnetic field $\mathbf{H}_0$, then[34]

$$C_s \cong \begin{cases} \left(\frac{2}{\pi^3}\right)^{1/2} \frac{1}{a^3} \left(\frac{\mu H_a}{\theta_C}\right)^3 \left(\frac{\mu H_a}{T}\right)^{1/2} e^{-\frac{\mu H_a}{T}} & (T \ll \mu H_a), \\ \frac{8\pi^2}{15} \frac{1}{a^3} \left(\frac{T}{\theta_C}\right)^3 & (\theta \gg T \gg \mu H_a). \end{cases} \quad (9.1)$$

If $\mathbf{H}_0$ is different from zero and is along the preferred axis of the antiferromagnet, the specific heat is given by the following formulas:[38]

$$C_s \cong \begin{cases} \left(\frac{2}{\pi^3}\right)^{1/2} \frac{1}{a^3} \frac{\mu^3 H_a (H_a - H_0)^2}{\theta_C^3} \left(\frac{\mu H_a}{T}\right)^{1/2} e^{-\frac{\mu(H_a - H_0)}{T}}, \\ \qquad \mu H_a > \mu H_0 \gg T, \\ \frac{4\pi^2}{15} \frac{1}{a^3} \left(\frac{T}{\theta_C}\right)^3, \ \mu H_0 \gg \mu H_a \gg T, \ \mu H_0 \gg T \gg \mu H_a, H_e \gg H_0, \\ \frac{8\pi^2}{15} \frac{1}{a^3} \left(\frac{T}{\theta_C}\right)^3, \quad \theta_C \gg T \gg \mu H_a > \mu H_0. \end{cases} \quad (9.2)$$

If the external magnetic field is oriented perpendicular to the preferred axis, then at sufficiently low temperatures the specific heat falls off exponentially with decreasing temperature:

$$C_s = \begin{cases} \left(\frac{2}{\pi^3}\right)^{1/2} \frac{1}{a^3} \left(\frac{\mu H_a}{\theta_C}\right)^3 \left(\frac{\mu H_a}{T}\right)^{1/2} e^{-\frac{\mu H_a}{T}}, & \mu H_a \gg \mu H_0 \gg T, \\ \frac{1}{2} \left(\frac{2}{\pi^3}\right)^{1/2} \frac{1}{a^3} \left(\frac{\mu H_a}{\theta_C}\right)^3 \left(\frac{\mu H_a}{T}\right)^{1/2} e^{-\frac{\mu H_a}{T}}, \theta_C \gg \mu H_0 \gg \mu H_a \gg T, \\ \frac{1}{4\pi^{3/2}} \frac{1}{a^3} (\mu H_0)^{3/2} \left(\frac{1}{\theta_C^{3/2}} + \frac{1}{\theta_{C_1}^{3/2}}\right) \left(\frac{\mu H_0}{T}\right)^{1/2} e^{-\frac{\mu H_0}{T}}, \mu H_0 \gg \mu H_e \gg T. \end{cases} \quad (9.3)$$

At comparatively high temperatures, the specific heat has a power law dependence on the temperature:

$$C_s = \begin{cases} \frac{8\pi^2}{15} \frac{1}{a^3} \left(\frac{T}{\theta_C}\right)^3, & \theta_C \gg T \gg \mu H_0 > \mu H_a, \\ \frac{4\pi^2}{15} \frac{1}{a^3} \left(\frac{T}{\theta_C}\right)^3, & \theta_C \gg \mu H_0 \gg T \gg \mu H_a, \\ \frac{15}{32} \frac{\zeta(5/2)}{\pi^{3/2}} \frac{1}{a^3} \left(\frac{T}{\theta_C}\right)^{3/2}, & H_0 > H_e, T \gg \mu (H_0 - H_{1e}). \end{cases} \quad (9.4)$$

We see that the spin specific heat of antiferromagnets is, over a wide range of temperatures ($\mu H_a \ll T \ll \theta_C$), proportional to the third power of the temperature, i.e., it behaves in the same way as the phonon specific heat.

Formulas (9.2) show that if $\mathbf{H}_0 \parallel \mathbf{n}$, the specific heat of antiferromagnets depends differently on temperature according as $H_0 > H_a$ and $H_0 < H_a$. If $\mathbf{H}_0 \perp \mathbf{n}$, a singularity in the temperature behavior of the specific heat occurs for $H_0 = H_e$. Near these values of magnetic field there are also singularities in the other thermodynamic quantities. These singularities are related to the change in the structure of the ground state of the antiferromagnet (cf. Sec. 7).

If $\mathbf{H}_0 \parallel \mathbf{n}$ and $H_0 \lesssim H_a$, then, as we see from formulas (7.13), the spin wave branch ϵ_1 has for $H_0 \approx H_a$ a very small activation energy, and that this activation-energy increases with decreasing field $\mathbf{H}_0$. Therefore, an adiabatic switching-on of the field $\mathbf{H}_0$ should lead to an increase in temperature of the antiferromagnet. The change in the temperature of the body can be determined by equating the entropies before and after switching-on the magnetic field

$$S(T_i, H) = S(T_f, 0),$$

where S is the total entropy of the body including the entropies of spin waves and phonons, T_i and T_f are the initial and final temperatures of the body. If $T_i \ll \mu H_a$, $\frac{\theta_D^2}{\theta_C^2} \mu H_a$, the final temperature will be equal to

$$T_f = T_i \frac{\theta_D}{\theta_C} \left(\frac{\mu H_a}{T_i}\right)^{1/2}. \quad (9.5)$$

We note that the heating process can occur in two stages: first, the spin system is heated to a temperature of the order of $\mu H_a / \ln \frac{\mu H_a}{T}$, and then a common value of the temperature of spins and lattice is established, given by formula (9.5). Such a picture will hold if the relaxation time within the spin system is very much less than the time for equalizing the temperatures of the phonons and the spin waves.

Since the magnetic structure of an antiferromagnet depends essentially on the orientation of the magnetic field, a change in orientation of the field without changing its value can also lead to a magnetocaloric effect.[38]

Let us now consider the magnetic properties of antiferromagnets. Since the magnetic moments of the sublattices compensate one another in the absence of a magnetic field, the resulting magnetic moment for not too strong fields is proportional to the magnetic field. Therefore, the magnetic properties of antiferromagnets are conveniently described by the magnetic susceptibility tensor χ_{ik}.

We shall give the formulas for the longitudinal and transverse components of this tensor with respect to the preferred axis of the antiferromagnet, for various limiting cases:[34]

$$\chi_{\parallel} = \begin{cases} \left(\frac{2}{\pi^3}\right)^{1/2} \frac{\mu^2}{a^3 \theta_C} \left(\frac{\mu H_a}{\theta_C}\right)^{3/2} \left(\frac{T}{\theta_C}\right)^{1/2} e^{-\frac{\mu H_a}{T}}, & T \ll \mu H_a, \\ \frac{2\mu^2}{3a^3 \theta_C} \left(\frac{T}{\theta_C}\right)^2, & \theta_C \gg T \gg \mu H_a; \end{cases} \quad (9.6)$$

$$\chi_{\perp} = \begin{cases} \frac{1}{\beta} - \frac{1}{2} \left(\frac{2}{\pi^3}\right)^{1/2} \frac{\mu^2}{a^3 \theta_C} \left(\frac{\mu H_a}{\theta_C}\right)^{1/2} \left(\frac{T}{\theta_C}\right)^{3/2} e^{-\frac{\mu H_a}{T}}, & T \ll \mu H_a, \\ \frac{1}{\beta} - \frac{1}{6} \frac{\mu^2}{a^3 \theta_C} \left(\frac{T}{\theta_C}\right)^2, & \mu H_a \ll T \ll \theta_C. \end{cases} \quad (9.7)$$

These formulas are valid for $H_0 \ll H_a$.

If the external field is sufficiently large, the magnetic moment of the antiferromagnet will not be proportional to the magnetic field. As we have stated, for sufficiently high values of magnetic field there is a change in the ground state of the antiferromagnet, and the value of the magnetic moment depends essentially on the orientation of the magnetic field and the

relative orientation of the magnetic moments.

Using formulas (7.19) and (7.20) for the energy spectrum of antiferromagnets having weak ferromagnetism, we can calculate the temperature dependence of the magnetic moment and the magnetic susceptibility of such bodies.

In the case of an antiferromagnet with weak transverse ferromagnetism, we have:[40,42]

$$M(T) = 2M_0 \frac{H_d}{H_e}\left\{1 - \frac{1}{24}\frac{\mu^2}{a^3\theta_C}\frac{H_e}{M_0}\left(\frac{T}{\theta_C}\right)^2\right\},$$

$$\chi_\perp(T) = 2\frac{M_0}{H_e}\left\{1 - \frac{1}{24}\frac{\mu^2}{a^3\theta_C}\frac{H_e}{M_0}\left(\frac{T}{\theta_C}\right)^2\right\}. \qquad (9.8)$$

For the case of an antiferromagnet with weak longitudinal ferromagnetism,[42]

$$M(T) = \nu M_0\left\{1 - \frac{1}{12}\frac{\mu^2 H_e}{a^3\theta_C M_0}\left(\frac{T}{\theta_C}\right)^2\right\},$$

$$\chi_\parallel(T) = \frac{1}{3}\frac{\mu^2}{a^3\theta_C}\left(\frac{T}{\theta_C}\right)^2. \qquad (9.9)$$

It seems that as yet there are no experiments from which one might determine the spin specific heat of antiferromagnets.

The temperature dependence of the magnetic moment and magnetic susceptibility has been studied,[40] under conditions which are suitable for a comparison of the experimental results with theory, on antiferromagnets with weak ferromagnetism ($MnCO_3$). It appears that the dependence of the magnetic moment on temperature is described satisfactorily by the first of the formulas (9.8), while the magnetic susceptibility depends on the temperature much more weakly than is required by the second formula of (9.8).

(To be concluded in the next issue.)

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