ELECTRODYNAMICS OF MEDIA WITH SPATIAL DISPERSION

A. A. RUKHADZE and V. P. SILIN

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INTRODUCTION

THE phenomenological electrodynamics of material media, which was formulated by Maxwell, did not take into account in its initial form the dependence of the dielectric permittivity and magnetic permeability on the frequency of the electromagnetic field. Therefore, the experimentally observed dependence of the index of refraction on the frequency of light (dispersion, or, more precisely, frequency dispersion), which leads in Maxwell's theory to a dispersion of the permittivity and permeability, was for a while a phenomenon apparently in contradiction with the theory. However, further advances in physics particularly molecular theory and the Lorentz electron theory, made it possible to disclose the microscopic reasons for the dispersion phenomenon. Finally, quantum mechanics, in a definite sense, brought to a culmination the microscopic theory of dispersion.

On the other hand, even at the end of the preceding century it became clear that the phenomenon of natural optical activity can be explained by taking account of the dependence of the permittivity not only on the frequency, but also on the wave vector. Such a point of view was completely confirmed by Born's microscopic theory, which showed that the dependence of the permittivity on the wave vector (referred to in what follows as spatial dispersion of the permittivity*) in naturally optically-active (gyrotropic) media corresponds to taking into account small quantities of the order of the ratio of the dimensions of molecules to the wavelength of electromagnetic radiation. In such a case we may speak of weak spatial dispersion.

In recent years, mainly because of the very rapid development of plasma physics and also the investigation of the electromagnetic properties of metals at low temperatures by radio-frequency methods, it has become clear that, in addition to frequency dispersion of the permittivity, the spatial dispersion possible in many cases is by no means small. In numerous theoretical papers concerning the fundamentals of plasma physics and physics of metals, in addition to specific electromagnetic properties of the particular media, one also investigates as a rule general questions of the electrodynamics of media with spatial dispersion. The present survey is a systematic presentation of the electrodynamics of media with spatial dispersion.

*The term "spatial dispersion" seems to have first been used in reference 3.

1. THE EQUATIONS OF THE ELECTROMAGNETIC FIELD

The electrodynamics of material media differs from the electrodynamics of the vacuum primarily in that under the action of external fields or external field sources there are induced in the medium densities of charge and current. One then has the following field equations for the electric field intensity **E** and the magnetic induction **B**:

$$\operatorname{div}\mathbf{E}=4\pi(\varrho+\varrho_0),\quad \operatorname{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},$$
$$\operatorname{rot}\mathbf{B}=\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}+\frac{4\pi}{c}(\mathbf{j}+\mathbf{j}_0),\quad \operatorname{div}\mathbf{B}=0. \qquad (1.1)^*$$

Here ρ_0 and $\mathbf{j}_0$ are the charge density and current density of the external sources of the field, while ρ and $\mathbf{j}$ are the corresponding densities induced in the medium. For the induced charges and currents one has the continuity equation

$$\frac{\partial\varrho}{\partial t}+\operatorname{div}\mathbf{j}=0. \qquad (1.2)$$

The physical meaning of the electric field **E** and magnetic induction **B** which appear in the field equations (1.1) is determined by the expression for the force **F** acting on a point test charge e moving in the medium with velocity **v**:

$$\mathbf{F}=e\left(\mathbf{E}+\frac{1}{c}[\mathbf{v},\ \mathbf{B}]\right). \qquad (\mathrm{I})\dagger$$

In order that the system of field equations be closed, one needs the so-called material equations, which relate the density of induced currents to the electric field intensity and the magnetic induction. Such relations essentially also determine the electromagnetic properties of the material media. However, the expression of the material equations and, consequently, the corresponding systems of field equations are not unique.

Usually the equations of the electromagnetic field in a medium are written in the form

$$\operatorname{div}\mathbf{D}=4\pi\varrho_0,\qquad \operatorname{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},$$
$$\operatorname{rot}\mathbf{H}=\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}+\frac{4\pi}{c}\mathbf{j}_0,\qquad \operatorname{div}\mathbf{B}=0. \qquad (1.3)$$

Here the magnetic field strength **H** and the electric induction **D** are related in the following way with the density of induced currents:

$$\mathbf{j}=\frac{\partial\mathbf{P}}{\partial t}+c\operatorname{rot}\mathbf{M}, \qquad (1.4)$$
$$\mathbf{H}=\mathbf{B}-4\pi\mathbf{M}, \qquad (1.5)$$

*rot = curl.
†[**v**, **B**] = **v** × **B**.

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}. \tag{1.6}$$

The vector **M** is called the magnetization and **P** the polarization vector of the medium. Substituting (1.4) in (1.1) and using (1.5) in (1.6), we obtain (1.3).

Already the fact that, in place of a single vector quantity characterizing the medium ($\mathbf{j}$), there are two vector quantities in (1.3), shows a definite arbitrariness in the choice of the way of writing the field equations, which is determined, obviously, by considerations of convenience. We note that, from the polarization current $\partial\mathbf{P}/\partial t$ one often separates off the conduction current $\partial\mathbf{P}/\partial t \rightarrow \partial\mathbf{P}'/\partial t + \mathbf{j}_{\text{cond}}$. Such a splitting has a real significance only for slowly varying fields. Thus, for the case of a constant field, setting $\partial\mathbf{P}'/\partial t = 0$, we have $\mathbf{j} = \mathbf{j}_{\text{cond}} + c \operatorname{curl} \mathbf{M}$. The density of the conduction current is defined so that its integral over a surface intersecting the medium is equal to the total current. Therefore the introduction of the magnetization **M** is justified by the possibility of the existence of a non-zero current density also in the case where the total current through any surface is equal to zero.*

However, in the case of varying fields, the splitting of the induced current into parts and also the associating with it of the displacement current $(1/4\pi)(\partial\mathbf{E}/\partial t)$ is quite difficult to justify. On the other hand, by means of the relation

$$\mathbf{D}'(\mathbf{r},\ t) = \mathbf{E}(\mathbf{r},\ t) + 4\pi \int_{-\infty}^{t} dt'\mathbf{j}(\mathbf{r},\ t') \tag{1.7}$$

one can introduce a quantity $\mathbf{D}'$ which enables one to combine the densities of induced charges and current with the displacement current in the field equations (1.1). Then the field equations take the form (cf. reference 2, Sec. 83)

$$\operatorname{div}\mathbf{D}' = 4\pi\varrho_0, \qquad \operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},$$
$$\operatorname{rot}\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{j}_0, \qquad \operatorname{div}\mathbf{B} = 0. \tag{II}$$

Naturally the system of equations (II) must be supplemented by a material equation giving an explicit expression for the quantity $\mathbf{D}'$, which we shall from now on call the electric induction.†

Before discussing possible material equations, we make the following remark: as has been shown, the formulation of the field equations may vary. In particular, the field equations (II) are formulated so that one does not use the concept of magnetic field strength in them. Such a form of the field equations is preferable for the treatment of rapidly varying phenomena. However, considering the fact that often electrodynamics is formulated on the basis of Eq. (1.3), which, in many cases, is entirely justified, and also for the purpose of establishing a connection with other approaches, although we shall give preference to Eq. (II), we shall also use (1.3). In particular, going on to a discussion of material equations supplementing the field equations, we first consider material equations for the case of (1.3).

In the following we shall restrict ourselves to the case of linear electrodynamics. Then the material equations are linear relations. In the case of constant fields, the material equations corresponding to (1.3) are written in the form

$$D_i = \varepsilon_{ij}E_j, \qquad B_i = \mu_{ij}H_j.$$

Here the quantities ϵ_{ij} and μ_{ij}, which are called respectively the permittivity tensor and the permeability tensor, are determined by the specific properties of the medium and, in this sense, characterize its electromagnetic properties. Such material equations are suitable only for sufficiently slowly varying fields. In the case of rapidly varying fields, i.e., fields changing rapidly compared with characteristic relaxation times in the medium, or compared with the period of characteristic normal vibrations of the medium, the situation is more complicated. Then the state of the medium is dependent not only on the field at the given time t, but also on its values at preceding times. This can be understood if we consider, for example, the fact that a relaxation process beginning in a medium under the action of a field appearing at moment t and going to zero during a time much smaller than the relaxation time will, nevertheless, persist after the disappearance of the field. Therefore, in the case of high-frequency fields, one should, as is usually done, use material equations of the form[2]

$$D_i(t) = \int_{-\infty}^{t} dt'\hat{\varepsilon}_{ij}(t-t')\,E_j(t'), \qquad B_i(t) = \int_{-\infty}^{t} dt'\hat{\mu}_{ij}(t-t')\,H_j(t'). \tag{1.8}$$

Relations (1.8) take account of the influence of the prior history on the electromagnetic properties of the medium. Here one usually speaks of temporal, or frequency, dispersion.

In the case of fields which change sharply in space, one must obviously also take account of the influence of the field at remote points on the electromagnetic properties of the medium at a particular point in space. In fact, for example, because of transport processes the state at a definite point in the medium will be determined not only by the value of the field at that point, but also by the field over whole regions of the medium from which the influence of the field is transmitted as a result of convection of matter. Therefore, in place of the material equations (1.8), one must use spatially non-local relations, taking account not only of temporal

*To avoid misunderstanding we state that, in contrast to the definition (1.6) of the induction **D**, which we are using (and which is also used by many authors), the electric induction is often understood in the literature to be the quantity $\mathbf{E} + 4\pi\mathbf{P}'$ (cf., for example, reference 4).

†We use term electric induction for the quantity appearing in Eq. (1.3) and defined by relation (1.6). We hope that this will not give rise to confusion. In each case where there may be some question we shall, in using the term "electric induction," write either **D** or **D'** respectively.

dispersion, but also of spatial dispersion. For a homogeneous, isotropic, and non-gyrotropic medium* such relations can be written in the following form:

$$\mathbf{D}(\mathbf{r}, t) = \int_{-\infty}^{t} dt' \int d\mathbf{r}' \hat{\varepsilon}(t-t', \mathbf{r}-\mathbf{r}') \mathbf{E}(\mathbf{r}', t'),$$

$$\mathbf{B}(\mathbf{r}, t) = \int_{-\infty}^{t} dt' \int d\mathbf{r}' \hat{\mu}(t-t', \mathbf{r}-\mathbf{r}') \mathbf{H}(\mathbf{r}', t'). \quad (1.9)$$

Thus, the electromagnetic properties of such a medium are determined by two functions depending on $\mathbf{r}$ and t.

We have here intentionally not written the relations generalizing the material equations (1.8) to anisotropic media. The point is that, if one made a direct generalization of relations (1.8), there would appear a large number of functions (two tensors depending on $\mathbf{r}$, $\mathbf{r}'$ and $t-t'$) which would be used for describing the electromagnetic properties of the medium. On the other hand, in the field equations (II), there appears only one vector quantity $\mathbf{D}'$ characterizing the properties of the medium. The material equation supplementing the field equations (II) and including both temporal and spatial dispersions can, in the case of linear electrodynamics, be written in the form:

$$D_i'(\mathbf{r}, t) = \int_{-\infty}^{t} dt' \int d\mathbf{r}' \hat{\varepsilon}_{ij}(t-t', \mathbf{r}, \mathbf{r}') E_j(\mathbf{r}', t). \quad \text{(III)}$$

The dependence of the kernel of the integral on the right side of this relation on the difference $t-t'$ is caused by the uniformity of the problem with respect to the time.† In the case of a spatially homogeneous medium, the dependence is also only on the difference of the coordinates.

It can be shown that the material equation (III) is not general, since it includes only the dependence of the electric induction $\mathbf{D}'$ on the electric field $\mathbf{E}$. In principle, one might speak of a dependence of $\mathbf{D}'$ also on the magnetic induction $\mathbf{B}$. However, it is easy to see that in the latter case, by using the field equation curl $\mathbf{E} = -(1/c)(\partial \mathbf{B}/\partial t)$ one could always eliminate $\mathbf{B}$ and write the material equation in the form (III).

The material equation (III) supplements the field equations (II). If, however, we start from the field equations (1.1), then we should use a material equation relating the current density j induced in the medium with the electric field strength:

$$j_i(\mathbf{r}, t) = \int_{-\infty}^{t} dt' \int d\mathbf{r}' \hat{\sigma}_{ij}(t-t', \mathbf{r}, \mathbf{r}') E_j(\mathbf{r}', t'). \quad (1.10)$$

By means of relations (1.7) and (1.11), it is easy to establish the following connection between the quantities $\hat{\epsilon}_{ij}(t, \mathbf{r}, \mathbf{r}')$ and $\hat{\sigma}_{ij}(t, \mathbf{r}, \mathbf{r}')$:

$$\hat{\varepsilon}_{ij}(t, \mathbf{r}, \mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')\,\delta(t)\delta_{ij} + 4\pi \int_{-\infty}^{t} dt' \hat{\sigma}_{ij}(t, \mathbf{r}, \mathbf{r}'). \quad (1.11)$$

Naturally, both forms of the material equation (III) and (1.10), as well as the field equations (II) and (1.1), are completely equivalent. From these relations it follows that the electromagnetic properties of an anisotropic medium are determined by a single tensor depending on $\mathbf{r}$, $\mathbf{r}'$ and $t-t'$. It is therefore clear that the two tensors which appear when one directly generalizes formulas (1.8) to the case of spatial dispersion cannot be independent. This fact points out the uselessness of introducing the magnetic field strength, in addition to the electric induction, for an anisotropic medium when one wants to include spatial dispersion.

The system of field equations must also be supplemented by boundary conditions. Let us formulate the boundary conditions at a uniform surface of separation of media. Here, as a result of the equation, div $\mathbf{B} = 0$, we obtain a condition of continuity of the normal component of magnetic induction $B_{1n} = B_{2n}$. The normal to the surface of separation of the media is assumed to be directed from the first medium toward the second. Furthermore from the equation curl $\mathbf{E} = -(1/c)\times(\partial\mathbf{B}/\partial t)$ we obtain a condition of continuity of the component of the electric field which is tangential to the surface of separation, $E_{1t} = E_{2t}$. These boundary conditions are a consequence of the field equations, in which the medium properties do not appear, and therefore can be used both for the system of field equations (II), as well as for (1.3).

The complication of the boundary problems, when one takes account of spatial dispersion, manifests itself in the conditions related to the material equations. A formal integration of the equation curl $\mathbf{B} = (1/c)\times(\partial\mathbf{E}/\partial t) + (4\pi/c)(\mathbf{j} + \mathbf{j}_0)$ gives the boundary condition $\mathbf{n}\times(\mathbf{B}_2 - \mathbf{B}_1) = (4\pi/c)(\mathbf{i}' + \mathbf{i}_0)$, where n is the normal to the boundary surface, $\mathbf{i}_0$ is the density of current of the external surface sources, and $\mathbf{i}'$ is the density of surface currents. The last quantity is defined by the equation

$$\mathbf{i}' = -\int_1^2 dl\left(\frac{1}{4\pi}\frac{\partial \mathbf{E}}{\partial t} + \mathbf{j}\right) = -\frac{1}{4\pi}\int_1^2 dl\frac{\partial \mathbf{D}'}{\partial t}.$$

Integration is taken over an infinitesimally small depth of the surface layer.

In the case of the field equations (1.3), where we use expression (1.4) for the density of the induced current, we have

$$\mathbf{i}' = -\frac{1}{4\pi}\int_1^2 dl\,\frac{\partial \mathbf{D}}{\partial t} + c[\mathbf{n}, \mathbf{M}_2 - \mathbf{M}_1].$$

Therefore, for the tangential components of the magnetic field, the boundary conditions can be written in the form

*Concerning the material equation for an isotropic and gyrotropic medium, cf. Sec. 7.

†We note that, if under the action of certain external causes [not related to the influence of the electromagnetic field taken into account in (III)] the medium changes its properties in the course of time, then we can no longer speak of a homogeneity of the problem in time, and therefore can no longer use the dependence on time difference in $\varepsilon_{ij}(\mathbf{r}, \mathbf{r}', t, t')$.

$$[\mathbf{n},\ \mathbf{H}_2-\mathbf{H}_1]=\frac{4\pi}{c}(\mathbf{i}+\mathbf{i}_0),$$

where

$$\mathbf{i}=-\frac{1}{4\pi}\int_2^1 dl\frac{\partial\mathbf{D}}{\partial t}.$$

Thus, both the tangential components of the magnetic induction, as well as the tangential components of the magnetic field intensity, even in the absence of external sources of surface currents, may have discontinuities at the surface of separation of two media.

The last boundary condition, which is a consequence of the equation div $\mathbf{E} = 4\pi(\rho+\rho_0)$ has, for the case of Eq. (II), the form $D'_{2n}-D'_{1n}=4\pi(\sigma'+\sigma_0)$. Here σ_0 is the surface density of charge of the external sources, while

$$\sigma'=\frac{1}{4\pi}\int_1^2 dl\,\mathrm{div}\,[\mathbf{n},\ [\mathbf{D},\ \mathbf{n}]].$$

An analogous boundary condition can also be written for the field equation (1.3).

It should also be remarked that, in order to be able to solve the system of field equations, one must also have an expression for the induced surface density of current (and also of charge). In other words, one must have surface material equations.* Thus, summing up, we may say that the field equations (II), supplemented by the material equations (III) (and also by surface material equations), when one takes account of the boundary conditions

$$B_{1n}=B_{2n},\qquad D'_{2n}-D'_{1n}=4\pi(\sigma_0+\sigma'),$$
$$\mathbf{E}_{1t}=\mathbf{E}_{2t},\quad [\mathbf{n},\ \mathbf{B}_2-\mathbf{B}_1]=\frac{4\pi}{c}(\mathbf{i}_0+\mathbf{i}')\qquad\text{(IV)}$$

together with the conditions at infinity, enable us to determine uniquely the electromagnetic field in any part of space.

2. THE COMPLEX PERMITTIVITY TENSOR

The electromagnetic field in the medium can be represented by means of a Fourier expansion as a set of monochromatic components whose time dependence is determined by the functions $e^{i\omega t}$. Such an expansion is also called a spectral resolution. For a monochromatic electromagnetic field with frequency ω, the material equation (III) takes the following form

$$D'_i(\mathbf{r})=\int d\mathbf{r}'\varepsilon_{ij}(\omega,\ \mathbf{r},\ \mathbf{r}')E_j(\mathbf{r}'),\qquad(2.1)$$

where

$$\varepsilon_{ij}(\omega,\ \mathbf{r},\ \mathbf{r}')=\int_0^\infty dte^{i\omega t}\hat{\varepsilon}_{ij}(t,\ \mathbf{r},\ \mathbf{r}').\qquad(2.2)$$

The tensor $\hat{\epsilon}_{ij}(t,\mathbf{r},\mathbf{r}')$ which establishes a linear relation between the real quantities $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{D}(\mathbf{r},t)$ is obviously a real function. The quantity $\epsilon_{ij}(\omega,\mathbf{r},\mathbf{r}')$, defined by the relation (2.2), may generally be complex.

*Examples of this type of surface material equations were considered in reference 5. (Cf. also reference 6, Sec. 18.)

We denote the real and imaginary parts of $\epsilon_{ij}(\omega,\mathbf{r},\mathbf{r}')$ respectively by $\epsilon'_{ij}(\omega,\mathbf{r},\mathbf{r}')$ and $\epsilon''_{ij}(\omega,\mathbf{r},\mathbf{r}')$, i.e.,

$$\varepsilon_{ij}(\omega,\ \mathbf{r},\ \mathbf{r}')=\varepsilon'_{ij}(\omega,\ \mathbf{r},\ \mathbf{r}')+i\varepsilon''_{ij}(\omega,\ \mathbf{r},\ \mathbf{r}').$$

From relation (2.2) it follows that

$$\varepsilon_{ij}(-\omega,\ \mathbf{r},\ \mathbf{r}')=\varepsilon^*_{ij}(\omega,\ \mathbf{r},\ \mathbf{r}'),\ \varepsilon'_{ij}(\omega,\ \mathbf{r},\ \mathbf{r}')=\varepsilon'_{ij}(-\omega,\ \mathbf{r},\ \mathbf{r}'),$$
$$\varepsilon''_{ij}(\omega,\ \mathbf{r},\ \mathbf{r}')=-\varepsilon''_{ij}(-\omega,\ \mathbf{r},\ \mathbf{r}').\qquad(2.3)$$

If the medium is unbounded in space and homogeneous, the kernel of the integral equation (III) is a function of the difference of coordinates;

$$D'_i(\mathbf{r},\ t)=\int_{-\infty}^t dt'\int d\mathbf{r}'\hat{\varepsilon}_{ij}(t-t',\ \mathbf{r}-\mathbf{r}')E_j(\mathbf{r}',\ t).\qquad(2.4)$$

In this case it is convenient to expand the electromagnetic field in a Fourier integral, representing it as a set of plane monochromatic waves whose dependence on coordinates and time is given by the functions $e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$. For such waves the relation (2.4) takes the following form:

$$D'_i=\varepsilon_{ij}(\omega,\ \mathbf{k})E_j,\qquad(2.5)$$

where

$$\varepsilon_{ij}(\omega,\ \mathbf{k})=\int_0^\infty dte^{i\omega t}\int d\mathbf{r}e^{-i\mathbf{k}\mathbf{r}}\hat{\varepsilon}_{ij}(t,\ \mathbf{r})=\int d\mathbf{r}e^{-i\mathbf{k}\mathbf{r}}\varepsilon_{ij}(\omega,\ \mathbf{r}).\qquad(2.6)$$

In the following we shall call the quantity $\epsilon_{ij}(\omega,\mathbf{k})$ the tensor of the complex dielectric permittivity of the medium. It should be emphasized that the dielectric tensor (2.6) can be introduced only for unbounded and spatially homogeneous media, for which the material equation has the form (2.4). The dependence of the tensor $\epsilon_{ij}(\omega,\mathbf{k})$ on the frequency of the field is determined by the frequency dispersion, while the dependence on the wave vector $\mathbf{k}$, determined by the non-locality of the material equation (2.4), characterizes the spatial dispersion.

The quantity $\epsilon_{ij}(\omega,\mathbf{k})$ is, generally speaking, a complex function of the real variables ω and $\mathbf{k}$. From the relation (2.6), taking account of the fact that the function $\epsilon_{ij}(t,\mathbf{r})$ is real, we obtain the following relations for the real part $\epsilon'_{ij}(\omega,\mathbf{k})$ and the imaginary part $\epsilon''_{ij}(\omega,\mathbf{k})$ of the dielectric permittivity tensor:

$$\varepsilon'_{ij}(-\omega,\ -\mathbf{k})=\varepsilon'_{ij}(\omega,\ \mathbf{k}),\ \varepsilon''_{ij}(-\omega,\ -\mathbf{k})=-\varepsilon''_{ij}(\omega,\ \mathbf{k}),$$
$$\varepsilon^*_{ij}(\omega,\ \mathbf{k})=\varepsilon_{ij}(-\omega,\ -\mathbf{k}).\qquad(2.7)$$

For an unbounded and spatially homogeneous medium, one can introduce one more quantity characterizing the electromagnetic properties of the medium. To do this we expand the electromagnetic field in a Fourier integral in the coordinates, representing it as a superposition of fields whose dependence on the coordinates is given by the factor $e^{i\mathbf{k}\cdot\mathbf{r}}$. For such fields the material equation (2.4) takes the form

$$D'_i(t)=\int_{-\infty}^t dt'\varepsilon_{ij}(t-t',\mathbf{k})E_j(t'),\qquad(2.8)$$

where

$$\varepsilon_{ij}(t-t', \mathbf{k}) = \int d\mathbf{r} e^{-i\mathbf{k}\mathbf{r}} \hat{\varepsilon}_{ij}(t-t', \mathbf{r}). \quad (2.9)$$

The tensor $\epsilon_{ij}(t,\mathbf{k})$, like $\epsilon_{ij}(\omega,\mathbf{k})$, is a complex quantity whose real and imaginary parts have the following properties:

$$\varepsilon'_{ij}(t, \mathbf{k}) = \varepsilon'_{ij}(t, -\mathbf{k}), \quad \varepsilon''_{ij}(t, \mathbf{k}) = -\varepsilon''_{ij}(t, -\mathbf{k}). \quad (2.10)$$

The dependence of $\epsilon_{ij}(\omega,\mathbf{k})$ on wave vector leads to the result that even in an isotropic and non-gyrotropic medium the dielectric permittivity retains essentially a tensor form. In fact, in such a medium because of the dependence of $\epsilon_{ij}(\omega,\mathbf{k})$ on the vector $\mathbf{k}$, in addition to the tensor δ_{ij}, one can also form the tensor $k_i k_j$. Therefore, the dielectric tensor in an isotropic and non-gyrotropic medium can be represented in the following form:[7]

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \varepsilon^{\mathrm{tr}}(\omega, \mathbf{k}) + \frac{k_i k_j}{k^2} \varepsilon^{\mathrm{l}}(\omega, \mathbf{k}). \quad (2.11)$$

The coefficient on the right side of formula (2.11), which multiplies the transverse tensor, we shall call the transverse dielectric permittivity $[\epsilon^{\mathrm{tr}}(\omega,\mathbf{k})]$, and the corresponding coefficient of the longitudinal tensor the longitudinal dielectric permittivity $[\epsilon^{\mathrm{l}}(\omega,\mathbf{k})]$ of the isotropic medium. These quantities are complex functions of frequency and wave vector. Introducing the real and imaginary parts of the longitudinal and transverse dielectric permittivities, we obtain from relation (2.7):

$$\begin{aligned} &\varepsilon^{\mathrm{tr}\prime}(\omega, k) = \varepsilon^{\mathrm{tr}\prime}(-\omega, k), \qquad \varepsilon^{\mathrm{tr}\prime\prime}(\omega, k) = -\varepsilon^{\mathrm{tr}\prime\prime}(-\omega, k), \\ &\varepsilon^{\mathrm{l}\prime}(\omega, k) = \varepsilon^{\mathrm{l}\prime}(-\omega, k), \qquad \varepsilon^{\mathrm{l}\prime\prime}(\omega, k) = -\varepsilon^{\mathrm{l}\prime\prime}(-\omega, k). \end{aligned} \quad (2.12)$$

We have already spoken of the two forms of the field equations in a medium. It is useful now to consider the relation of the material equations (1.9) and (III) for a spatially homogeneous isotropic and non-gyrotropic medium, and also the connection of the Maxwell equations (1.3) and (II). However, before doing this, we consider some consequences which follow from the material equations (1.9).

For a monochromatic electromagnetic field with frequency ω we have from equations (1.9)

$$\mathbf{D}(\mathbf{r}) = \int d\mathbf{r}' \varepsilon(\omega, \mathbf{r}, \mathbf{r}') \mathbf{E}(\mathbf{r}'), \quad \mathbf{B}(\mathbf{r}) = \int d\mathbf{r}' \mu(\omega, \mathbf{r}, \mathbf{r}') \mathbf{H}(\mathbf{r}'), \quad (2.13)$$

where

$$\varepsilon(\omega, \mathbf{r}, \mathbf{r}') = \int_0^\infty dt e^{i\omega t} \hat{\varepsilon}(t, \mathbf{r}, \mathbf{r}'), \quad \mu(\omega, \mathbf{r}, \mathbf{r}') = \int_0^\infty dt e^{i\omega t} \hat{\mu}(t, \mathbf{r}, \mathbf{r}') \quad (2.14)$$

Because of the reality of the fields $\mathbf{D}(\mathbf{r},t)$, $\mathbf{E}(\mathbf{r},t)$, $\mathbf{B}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$, obviously the functions $\hat{\epsilon}(t, \mathbf{r}, \mathbf{r}')$ and $\hat{\mu}(t, \mathbf{r}, \mathbf{r}')$ also are real. Then, from expressions (2.14), in the same way as for (2.3), it follows that the real parts of the quantities $\epsilon(\omega, \mathbf{r}, \mathbf{r}')$ and $\mu(\omega, \mathbf{r}, \mathbf{r}')$ are even functions of the frequency, while their imaginary parts are odd functions of the frequency.

In the case of a spatially homogeneous and unbounded medium, the dependence of $\hat{\epsilon}(t, \mathbf{r}, \mathbf{r}')$ and $\hat{\mu}(t, \mathbf{r}, \mathbf{r}')$ on coordinates is a difference dependence. Expanding the field in plane monochromatic waves $e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$, we obtain from the material equations (1.9) the following relation between the quantities $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{B}$, $\mathbf{H}$ in such a wave:

$$\mathbf{D} = \varepsilon(\omega, k)\mathbf{E}, \quad \mathbf{B} = \mu(\omega, k)\mathbf{H}, \quad (2.15)$$

where

$$\begin{aligned} \varepsilon(\omega, k) &= \int_0^\infty dt e^{i\omega t} \int d\mathbf{r} e^{i\mathbf{k}\mathbf{r}} \hat{\varepsilon}(t, \mathbf{r}) = \int d\mathbf{r} e^{-i\mathbf{k}\mathbf{r}} \varepsilon(\omega, \mathbf{r}), \\ \mu(\omega, k) &= \int_0^\infty dt e^{i\omega t} \int d\mathbf{r} e^{i\mathbf{k}\mathbf{r}} \hat{\mu}(t, \mathbf{r}) = \int d\mathbf{r} e^{-i\mathbf{k}\mathbf{r}} \mu(\omega, \mathbf{r}). \end{aligned} \quad (2.16)$$

The quantities $\epsilon(\omega,\mathbf{k})$ and $\mu(\omega,\mathbf{k})$ in the following will be called respectively the permittivity and permeability of the isotropic medium. From the relations (2.16) we obtain the following formulas, characterizing the properties of the real and imaginary parts of these quantities, analogously to (2.12):

$$\begin{aligned} &\varepsilon'(-\omega, k) = \varepsilon'(\omega, k), \qquad \varepsilon''(-\omega, k) = -\varepsilon''(\omega, k), \\ &\mu'(-\omega, k) = \mu'(\omega, k), \qquad \mu''(-\omega, k) = -\mu''(\omega, k). \end{aligned} \quad (2.17)$$

We now turn to the setting up of the relation we are interested in finding between the field equations (1.3) and (II). Equations (1.3), for fields depending on time and coordinates as $e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$, take the following form:

$$\begin{aligned} &i\mathbf{k}\mathbf{E}\varepsilon(\omega, k) = 4\pi\varrho_0(\omega, \mathbf{k}), \qquad [\mathbf{k}, \mathbf{E}] = \frac{\omega}{c}\mathbf{B}, \\ &\frac{i}{\mu(\omega, k)}[\mathbf{k}, \mathbf{B}] = -\frac{i\omega}{c}\varepsilon(\omega, k)\mathbf{E} + \frac{4\pi}{c}\mathbf{j}_0(\omega, \mathbf{k}), \quad \mathbf{k}\mathbf{B} = 0, \end{aligned} \quad (2.18)$$

where $\rho_0(\omega,\mathbf{k})$ and $\mathbf{j}_0(\omega,\mathbf{k})$ are the Fourier components of the charge density and current density of the external sources of the field.

In the same way, the field equation (II) in this case, when we include expression (2.11) for the dielectric tensor, takes the form:

$$\left.\begin{aligned} &i\mathbf{k}\mathbf{E}\varepsilon^{\mathrm{l}}(\omega, k) = 4\pi\varrho_0(\omega, \mathbf{k}), \quad [\mathbf{k}, \mathbf{E}] = \frac{\omega}{c}\mathbf{B}, \quad \mathbf{k}\mathbf{B} = 0, \\ &i[\mathbf{k}, \mathbf{B}]_i = -\frac{i\omega}{c}\left\{\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\varepsilon^{\mathrm{tr}}(\omega, k) + \frac{k_i k_j}{k^2}\varepsilon^{\mathrm{l}}(\omega, k)\right\} E_j \\ &\qquad + \frac{4\pi}{c} j_{0i}(\omega, \mathbf{k}). \end{aligned}\right\} \quad (2.19)$$

From a comparison of (2.18) and (2.19) it immediately follows that

$$\varepsilon^{\mathrm{l}}(\omega, k) = \varepsilon(\omega, k). \quad (2.20)$$

To establish the relation of the quantities $\mu(\omega,\mathrm{k})$ with the quantities $\epsilon^{\mathrm{l}}(\omega,\mathrm{k})$ and $\epsilon^{\mathrm{tr}}(\omega,\mathrm{k})$ we proceed as follows. We subtract from the equation of the system (2.18) which contains on its right side the current density of external field sources $\mathbf{j}_0(\omega,\mathbf{k})$ the corresponding equation of system (2.19). Making use of relation (2.20) and eliminating the magnetic induction $\mathbf{B}$

from the difference by using the field equation

$$\mathbf{B}=\frac{c}{\omega}[\mathbf{k},\ \mathbf{E}] \tag{2.21}$$

we obtain

$$\frac{\omega}{c}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)(\varepsilon^l-\varepsilon^{tr})E_j=\left(1-\frac{1}{\mu}\right)\frac{c}{\omega}[\mathbf{k},\ [\mathbf{k},\ \mathbf{E}]]_i.$$

From this we then have the following relation:[7]

$$1-\frac{1}{\mu(\omega,\ k)}=\frac{\omega^2}{c^2k^2}[\varepsilon^{tr}(\omega,\ k)-\varepsilon^l(\omega,\ k)], \tag{2.22}$$

which enables us to express the magnetic permeability of the isotropic medium $\mu(\omega, k)$ in terms of the longitudinal and transverse dielectric permeabilities.

In the case of an isotropic and non-gyrotropic medium, the material equation (2.5) can also be written in a somewhat different form, whose use is in many cases more convenient for establishing the relation of the field equations in the forms (1.3) and (II). According to expression (2.11), the material equation (2.5) for an isotropic, non-gyrotropic medium takes the form

$$\mathbf{D}'=\varepsilon^{tr}(\omega,\ k)\mathbf{E}-[\varepsilon^{tr}(\omega,\ k)-\varepsilon^l(\omega,\ k)]\frac{\mathbf{k}(\mathbf{kE})}{k^2}. \tag{2.23}$$

Using the field equation (2.21) and also relations (2.20) and (2.22), the material equation (2.23) can be transformed to

$$\mathbf{D}'=\varepsilon(\omega,\ k)\mathbf{E}-\frac{c}{\omega}\frac{4\pi\chi(\omega,\ k)}{\mu(\omega,\ k)}[\mathbf{k},\ \mathbf{B}], \tag{2.24}$$

where we have introduced the notation

$$\chi(\omega,\ k)=-\frac{1}{4\pi}(\mu(\omega,\ k)-1). \tag{2.25}$$

We shall call the quantity $\chi(\omega, k)$ the magnetic susceptibility of the isotropic medium.

Naturally, both forms of the material equations for the field (2.23) and (2.24), as well as (2.15), are completely equivalent. In each of them there appear two functions of the frequency and wave vector which describe the electromagnetic properties of the isotropic and non-gyrotropic medium.

In concluding this section , we introduce the concept of a complex conductivity tensor of the medium. As already mentioned above, in place of the field equations (II) and the material equations (III) one can use the field equations (1.1) and the material equation (1.10). The tensor $\hat{\sigma}_{ij}(t, \mathbf{r}, \mathbf{r}')$ relating the current density $\mathbf{j}$ induced in the medium to the electric field intensity has properties analogous to the properties of $\hat{\epsilon}_{ij}(t, \mathbf{r}, \mathbf{r}')$. In particular, in the case of a spatially unbounded and uniform medium the tensor $\hat{\sigma}_{ij}(t, \mathbf{r}, \mathbf{r}')$ is a function of the difference of the coordinates. Then for fields depending on the time and coordinates as $e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$ we get

$$j_i=\sigma_{ij}(\omega,\ \mathbf{k})E_j, \tag{2.26}$$

where

$$\sigma_{ij}(\omega,\ \mathbf{k})=\int_0^\infty dt e^{i\omega t}\int d\mathbf{r} e^{-i\mathbf{kr}}\hat{\sigma}_{ij}(t,\ \mathbf{r}). \tag{2.27}$$

We shall call the quantity $\sigma_{ij}(\omega, \mathbf{k})$ the complex conductivity tensor of the medium. From relations (1.11), (2.6) and (2.27) we obtain the relation between the dielectric tensor and the conductivity tensor:*

$$\varepsilon_{ij}(\omega,\ \mathbf{k})=\delta_{ij}+4\pi^2\sigma_{ij}(\omega,\ \mathbf{k})\delta_+(\omega). \tag{2.28}$$

Using (2.28), one can easily obtain relations characterizing the properties of the conductivity tensor $\sigma_{ij}(\omega, \mathbf{k})$ when there are corresponding relations for the dielectric permittivity tensor [cf., for example, Eq. (2.7)].

For an isotropic and non-gyrotropic medium, in analogy to formula (2.11), we can write

$$\sigma_{ij}(\omega,\ \mathbf{k})=\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\sigma^{tr}(\omega,\ k)+\frac{k_ik_j}{k^2}\sigma^l(\omega,\ k), \tag{2.29}$$

where σ^{tr} and σ^l are respectively the transverse and longitudinal conductivity of the medium. Then from formulas (2.11), (2.28), and (2.29) we have

$$\varepsilon^{tr,\,l}(\omega,\ k)=1+4\pi^2\sigma^{tr,\,l}(\omega,\ k)\delta_+(\omega). \tag{2.30}$$

3. DISPERSION OF THE DIELECTRIC PERMITTIVITY TENSOR

In the preceding section we have introduced the concept of a dielectric permeability tensor of the medium $\epsilon_{ij}(\omega, \mathbf{k})$, taking into account both frequency and spatial dispersion. Here we consider the behavior of $\epsilon_{ij}(\omega, \mathbf{k})$ for small values of ω and $\mathbf{k}$ and show the relation of this tensor to certain quantities characterizing the electromagnetic properties of the medium.

An electromagnetic field which is variable in time may also be variable in space. In the case of fields which change rapidly in space, one must take into account the effect of field at distant points on the electromagnetic properties at a particular point in space; i.e., we must take into account spatial dispersion. However, if the electromagnetic field varies sufficiently smoothly over space, one can restrict consideration to frequency dispersion and neglect spatial dispersion. On the other hand, it is also possible to have a case where a field which is non-uniform in space can be considered to be static, and consequently one can neglect the frequency dispersion. The electromagnetic properties of the medium in these two limiting cases are described by the limiting expressions for the tensor $\epsilon_{ij}(\omega, \mathbf{k})$ corresponding to $\mathbf{k}/\omega \to 0$ and $\omega/\mathbf{k} \to 0$ respectively. Let us consider these limits on the ex-

*The function $\delta_+(\omega)$ is defined as follows:

$$\delta_+(\omega)=\frac{1}{2\pi}\int_0^\infty d\tau e^{i\omega\tau}=\frac{1}{2}\delta(\omega)+P\frac{i}{2\pi\omega},$$

where the symbol P means that the singularity at $\omega = 0$ is to be understood in the sense of the principal value.

ample of an isotropic and non-gyrotropic medium. In this case, the dielectric tensor (2.11) contains two functions $\epsilon^{tr}(\omega, k)$ and $\epsilon^{l}(\omega, k)$. Let us begin with a consideration of the longitudinal dielectric permittivity $\epsilon^{l}(\omega, k)$.

In the static limit, i.e., as $\omega \to 0$, the external sources of the field may, generally speaking, produce an inhomogeneous electric field in the medium. Then the electric field in the medium will be derivable from a potential, i.e.,

$$\mathbf{E} = -\operatorname{grad}\Phi, \tag{3.1}$$

where Φ is the scalar potential of the field.

We expand the functions $\mathbf{E}$ and Φ in three-dimensional Fourier integrals in the coordinates $\mathbf{r}$:

$$\mathbf{E}(\mathbf{r}) = \int d\mathbf{k} e^{i\mathbf{k}\mathbf{r}}\mathbf{E}, \qquad \Phi(\mathbf{r}) = \int d\mathbf{k} e^{i\mathbf{k}\mathbf{r}}\Phi_{\mathbf{k}}. \tag{3.2}$$

Then according to (3.1) we have

$$\mathbf{E} = -i\mathbf{k}\Phi_{\mathbf{k}}.$$

Substituting this expression in the field equations (2.19), we obtain the following equation for the scalar potential $\Phi_{\mathbf{k}}$:

$$k^2\varepsilon^{l}(0, k)\Phi_{\mathbf{k}} = 4\pi\varrho_0(0, \mathbf{k}). \tag{3.3}$$

In the case where the source of the field in the medium is a point charge at rest, the charge density is equal to

$$\varrho_0(\mathbf{r}) = e\delta(\mathbf{r} - \mathbf{r}_0).$$

Therefore the electrostatic potential of a point charge in an isotropic medium, according to (3.3), has the form:

$$\Phi(\mathbf{r}) = \frac{4\pi e}{(2\pi)^3}\int d\mathbf{k}\,\frac{e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}_0)}}{k^2\varepsilon^{l}(0, \mathbf{k})}. \tag{3.4}$$

The difference of the function $\epsilon^{l}(0, k)$ from unity has the consequence that the field of a point charge in a medium differs from the Coulomb field. In particular, if, for example,

$$\varepsilon^{l}(0, k) = 1 + \frac{1}{r_{scr}^2 k^2}, \tag{3.5}$$

then we obtain from (3.4) the expression

$$\Phi(\mathbf{r}) = \frac{e}{|\mathbf{r} - \mathbf{r}_0|}e^{-\frac{|\mathbf{r}-\mathbf{r}_0|}{r_{scr}}}. \tag{3.6}$$

Such a potential corresponds to the Debye screening of the field of the point charge in the medium (cf. reference 8, Sec. 74). The Debye screening results in a weakening of the field at large distances from the charge and is caused by the fact that the integrand in formula (3.4) in the case where $\epsilon^{l}(0, k)$ has the form (3.5) remains finite for $k = 0$. Here it is important that the second term in formula (3.5) is positive. Thus, in order to have Debye screening of the electrostatic field in an isotropic medium, it is sufficient that the static dielectric permittivity $\epsilon^{l}(0, k)$ for $k = 0$ have a singularity of the type $1/k^2$ and remain positive. The quantity r_{scr}^{-2} defined by the relation

$$r_{scr}^{-2} = \lim_{k\to 0}\lim_{\omega/k\to 0} k^2[\varepsilon^{l}(\omega, k) - 1], \tag{3.7}$$

characterizes the distance over which the weakening of the static field of the charge in the medium occurs.

In the opposite limiting case, i.e., for $k/\omega \to 0$, the quantity

$$\varepsilon^{l}(\omega, 0) = \lim_{k/\omega\to 0}\varepsilon^{l}(\omega, k) \tag{3.8}$$

is the usual dielectric permittivity of the medium, taking into account only frequency dispersion. It should be remarked that in this limit, corresponding to the neglect of spatial dispersion, the dielectric permittivity tensor of an isotropic medium should have the form

$$\varepsilon(\omega)\delta_{ij}.$$

This follows from the fact that, in an isotropic medium, when we neglect spatial dispersion, we can form only a single second-rank tensor δ_{ij}. Starting from this, we conclude that*

$$\varepsilon^{l}(\omega, 0) = \varepsilon^{tr}(\omega, 0) = \varepsilon(\omega). \tag{3.9}$$

Obviously, in the limit $\omega \to 0$ the quantity $\epsilon(\omega)$ cannot give rise to any screening of the static field in the medium, primarily because it not only does not have a singularity $\sim 1/k^2$, but in general does not depend on the wave vector $\mathbf{k}$.

From our presentation it is clear that there can, in general, exist two different limits for the longitudinal dielectric permittivity $\epsilon^{l}(\omega, k)$ for $\omega = 0$ and $k = 0$. In those cases where these limits exist, we shall use for them the following notation:

$$\varepsilon^{l}_{\omega}(0, 0) = \lim_{\omega\to 0}\lim_{k/\omega\to 0}\varepsilon^{l}(\omega, k),$$
$$\varepsilon^{l}_{k}(0, 0) = \lim_{k\to 0}\lim_{\omega/k\to 0}\varepsilon^{l}(\omega, k). \tag{3.10}$$

Thus, the point $\omega = 0$, $k = 0$ can, in certain cases, be an essentially singular point for the longitudinal dielectric permittivity $\epsilon^{l}(\omega, k)$.

It should be remarked that the dielectric permittivity $\epsilon(\omega)$, taking into account only frequency dispersion, may have a singularity in the region of small ω. For conductors, for example, at low frequencies (cf., for example, reference 2, Sec. 62)

$$\varepsilon(\omega) = \frac{4\pi i\sigma_0}{\omega}, \tag{3.11}$$

where σ_0 is the static conductivity of the conductor. For dielectrics in the region of low frequencies, $\epsilon(\omega)$ has no singularity:

$$\varepsilon(\omega) = \varepsilon_0, \tag{3.12}$$

where ϵ_0 is the static dielectric constant.

*For the case of an isotropic and non-gyrotropic medium, the spatial dispersion of the dielectric permittivity was treated in reference 9. However, it was assumed there that Eq. (3.9) also holds for $k \neq 0$.

We now consider the limiting transitions to the cases $\omega = 0$ and $k = 0$ for the transverse dielectric permittivity of the medium $\epsilon^{tr}(\omega, k)$. First of all, we note that the electromagnetic field equations (1.3) allow, in addition to the electrostatic field, the possible existence in the medium of a constant magnetic field produced by external sources. It was not by chance that we used the field equations in the form (1.3). The point is that the field equations (II) are inconvenient for describing a constant magnetic field in the medium. In fact, let us consider the case where the electric field intensity **E** in the medium is equal to zero, while the magnetic induction $\mathbf{B} \neq 0$. The material equation (2.23) is then obviously not applicable, since it does not contain the magnetic induction **B** explicitly. The material equation written in the form (2.24) is also not very suitable for describing a constant field in the medium. For the case of a constant magnetic field, it is more convenient to use the Maxwell equations (1.3) and the corresponding material equations (2.15).

We investigate the conditions under which we may speak of a constant field in the medium. In the case of slow change of magnetic field with time, there appears in the medium a weak, variable electric field determined by the equation

$$\operatorname{rot} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}.$$

We note that it is also possible to have, in general, a constant electric field which is not small. The order of magnitude of the strength of the variable electric field **E** can be found by evaluating both sides of the above equation. If ω is a characteristic frequency, and $1/k$ a characteristic dimension of the inhomogeneity of the magnetic field in the medium, then, according to this equation, $E \sim (\omega/ck) B$. On the other hand, from the field equations (1.1) and (1.4) we have

$$\operatorname{rot} \mathbf{B} = \frac{1}{c}\frac{\partial}{\partial t}(\mathbf{E} + 4\pi\mathbf{P}') + 4\pi \operatorname{rot} \mathbf{M} + \frac{4\pi}{c}\mathbf{j}_{\text{cond}}. \qquad (3.13)$$

From this we see that the magnetic field in the medium can be regarded as constant if, in this equation, we can neglect the term $(1/c)(\partial/\partial t)(\mathbf{E} + 4\pi\mathbf{P}')$. For dielectrics, because of the absence of a conductivity, the quantity $\mathbf{E} + 4\pi\mathbf{P}'$ coincides with the electric induction **D**, and therefore this term in (3.13) for small ω is of order $i\epsilon_0(\omega/c)\mathbf{E}$ where ϵ_0 is the static dielectric constant of the dielectric. For conductors, the quantity $\mathbf{E} + 4\pi\mathbf{P}'$ differs from the induction **D** by a term due to the conduction current. Since the singularity of the dielectric permittivity of a conductor in the region of low frequencies is, as we pointed out above in (3.11), related to the conductivity, we conclude that $(1/c)\times(\partial/\partial t)(\mathbf{E} + 4\pi\mathbf{P}') \sim (i\omega/c)\epsilon_0\mathbf{E}$, where the quantity ϵ_0 remains finite for $\omega \to 0$.* Thus for such media, the term pointed out above in (3.13) can be neglected if $\sqrt{\epsilon_0}\omega/ck \ll 1$. The transition to a constant magnetic field means a transition in the material equations to the limit $\omega/k \to 0$, and consequently the static magnetic permeability should be defined from the relation (2.22) as follows:

$$\mu_k(0, 0) = \lim_{k\to 0}\lim_{\omega/k\to 0}\left\{1 - \frac{\omega^2}{c^2k^2}(\varepsilon^{tr} - \varepsilon^{l})\right\}^{-1}. \qquad (3.14)$$

From this formula it follows that the deviation of the static magnetic permeability from unity (or, what is the same thing, the deviation of the static magnetic susceptibility from zero) means that the right side of relation (2.22) is different from zero in the limit of constant field, i.e., for $\omega/k \to 0$.

In the opposite limiting case for $k/\omega \to 0$, we may speak of weak spatial dispersion and expand the quantities $\epsilon^{l}(\omega, k)$ and $\epsilon^{tr}(\omega, k)$ in series in powers of k/ω:

$$\varepsilon^{l}(\omega, k) \approx \varepsilon(\omega) + \alpha(\omega)\frac{c^2k^2}{\omega^2}, \quad \varepsilon^{tr}(\omega, k) \approx \varepsilon(\omega) + \beta(\omega)\frac{c^2k^2}{\omega}. \qquad (3.15)$$

Then from relation (2.22) we have

$$\mu_\omega(\omega, 0) = \lim_{k/\omega\to 0}\mu(\omega, k) = [1 + \alpha(\omega) - \beta(\omega)]^{-1}. \qquad (3.16)$$

This quantity does not depend on the wave vector **k**, and in the limit of $\omega = 0$ it does not at all coincide with the static magnetic permeability defined by relation (3.14).

It is especially important to emphasize that, when we speak of the frequency dispersion of the magnetic permeability corresponding to the material equations (1.8), we are not talking about the quantity $\mu_\omega(\omega, 0) = \lim_{k/\omega\to 0}\mu(\omega, k)$ but of the quantity $\mu(\omega, k)$ in the neighborhood of the point $\omega/k = 0$. Under specific conditions, in the neighborhood of this point the quantity $\mu(\omega, k)$ may be essentially dependent on the frequency of the field and not dependent on the wave vector **k**. In the language of the quantities $\epsilon^{l}(\omega, k)$ and $\epsilon^{tr}(\omega, k)$, this means the following: first of all, in order for the static magnetic permeability of a medium to be different from unity, or, as one says, in order for the medium to have magnetic properties, it is necessary that the expression $\epsilon^{tr} - \epsilon^{l}$ in the vicinity of the point $\omega/k = 0$ have a singularity of the type k^2/ω^2. For nonmagnetic media this expression does

*We note that the quantity ε_0 does not coincide with the static dielectric permittivity of the conductor, since $\mathbf{E} + 4\pi\mathbf{P}'$ differs from the induction **D**. In the low frequency region, for a conductor,

$$\varepsilon(\omega) \approx \varepsilon_0 + \frac{4\pi i\sigma_0}{\omega}.$$

In the literature, however, the quantity ε_0 is frequently called the dielectric constant of the conductor. The second term in this expression is related to the conduction current.

not have such a singularity in the neighborhood of the point $\omega/k = 0$, and therefore the static magnetic permeability of such media is $\mu_k(0, 0) = 1$. Secondly, if in the expansion of the function $\epsilon^{tr} - \epsilon^l$ in series in powers of ω/k the coefficient of the term k^2/ω^2 is independent of $\mathbf{k}$, but depends on frequency ω, then the quantity $\mu(\omega, k)$, defined by the relation (2.22), will be a function only of frequency in the neighborhood of the point $\omega/k = 0$. It is just this quantity, and not the quantity $\mu_\omega(\omega, 0)$ defined by formula (3.17), which has the significance of a magnetic permeability of the medium when we take account of the frequency dispersion. In the following, in order not to confuse this quantity with $\mu_\omega(\omega, 0)$, we shall denote it by $\mu_k(\omega)$. From all we have said we conclude that the frequency dispersion of the magnetic permeability of a medium $\mu_k(\omega)$ has a meaning only in a restricted region of frequencies in the neighborhood of the point $\omega/k = 0$, where $1/k$ is a characteristic dimension of the inhomogeneities of the field in the medium. In the limit of $\omega = 0$, this quantity naturally coincides with the static magnetic permeability (3.14). It should be mentioned that the possible existence of two limits $\mu_\omega(0, 0)$ and $\mu_k(0, 0)$ is a consequence of the fact that the point $\mathbf{k} = 0$ and $\omega = 0$ can, generally speaking, be an essential singularity for the dielectric permittivity tensor $\epsilon_{ij}(\omega, k)$.

A completely analogous situation holds in the case of anisotropic media. Thus, in the limit of $\omega/k \to 0$, the scalar potential of the field produced by a point charge in an anisotropic medium is determined by the following expression:

$$\Phi(\mathbf{r}) = \frac{4\pi e}{(2\pi)^3} \int d\mathbf{k} \, \frac{e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}_0)}}{k_i \varepsilon_{ij}(0, \mathbf{k}) k_j}. \qquad (3.17)$$

If the function $k_i \epsilon_{ij}(0, \mathbf{k}) k_j$ remains finite for $\mathbf{k} \to 0$, then, as in the case of an isotropic medium, one can have a screening of the electrostatic field of the charge at large distances in an anisotropic medium also. Here the character of the screening of the field in the anisotropic medium will, in general, depend on direction.

In the opposite limiting case, when $k/\omega \to 0$, the quantity $\epsilon_{ij}(\omega, 0)$ is a dielectric permittivity tensor which takes into account only the frequency dispersion and completely neglects spatial dispersion. The function $\epsilon_{ij}(\omega, 0)$ can, in general, have a singularity in the region of low frequencies ω. For conducting media, for example, the function $\epsilon_{ij}(\omega, 0)$ for small ω has a singularity of the type $1/\omega$, while for non-conducting media it remains finite.

From the above, it follows that the point $\omega = 0$, $\mathbf{k} = 0$, just as in the case of an isotropic medium, may be an essential singularity for the dielectric permittivity tensor, and there may exist two different limits of the tensor $\epsilon_{ij}(\omega, \mathbf{k})$ for $\omega = 0$ and $\mathbf{k} = 0$. A manifestation of this fact is the existence of anisotropic media with non-zero magnetic susceptibility. All the arguments presented above concerning the possibility of regarding the magnetic field in an isotropic medium as constant remain valid also for an anisotropic medium; i.e., the magnetic field in the medium can be regarded as constant only in the neighborhood of the point $\omega/k = 0$, where ω is a characteristic frequency and $1/k$ is a characteristic size of the field inhomogeneity. In the neighborhood of this point, the dielectric permittivity tensor can have a singularity $\sim (k/\omega)^2$ which corresponds to the presence in the medium of a non-zero magnetic susceptibility.

4. ENERGY OF THE ELECTROMAGNETIC FIELD IN THE MEDIUM

External sources of the field, which produce an electromagnetic field, change the energy of the medium. The change in energy of the medium is actually determined by the interaction of the electromagnetic field with the sources of the field. Such an interaction energy is determined by the work done by the field on the external sources. We note that the work done during a time dt by the electric field $\mathbf{E}(\mathbf{r}, t)$ on the external currents in the volume $d\mathbf{r}$, characterized by a current density $\mathbf{j}_0(\mathbf{r}, t)$, is equal to

$$\mathbf{E}(\mathbf{r}, t) \, \mathbf{j}_0(\mathbf{r}, t) \, d\mathbf{r} \, dt. \qquad (4.1)$$

Therefore, the total work done by the field over all space, during the time of action of the external sources up to the time t, is given by an integral of this expression,

$$A = \int_{-\infty}^{t} dt' \int d\mathbf{r} \, \mathbf{E}(\mathbf{r}, t') \, \mathbf{j}_0(\mathbf{r}, t'). \qquad (4.2)$$

According to the law of conservation of energy, the work done by the field must be compensated by the change in energy of the electromagnetic field which we denote by W. The rate of change of the field energy is given by the relation

$$\frac{dW}{dt} = -\frac{dA}{dt} = -\int d\mathbf{r} \, \mathbf{E}(\mathbf{r}, t) \, \mathbf{j}_0(\mathbf{r}, t). \qquad (4.3)$$

By using the field equations (1.3), we can eliminate the current density of the external sources from the right side of relation (4.3). We then get

$$\frac{dW}{dt} = \int d\mathbf{r} \left\{ \frac{1}{4\pi} \left(\mathbf{H} \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \frac{\partial \mathbf{D}}{\partial t} \right) + \frac{c}{4\pi} \operatorname{div} [\mathbf{E}, \mathbf{H}] \right\}. \qquad (4.4)$$

On the other hand, to eliminate the current density of the external sources one can use the system of field equations (II). In this case

$$\frac{dW}{dt} = \int d\mathbf{r} \left\{ \frac{1}{4\pi} \left(\mathbf{B} \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \frac{\partial \mathbf{D}}{\partial t} \right) + \frac{c}{4\pi} \operatorname{div} [\mathbf{E}, \mathbf{B}] \right\}. \qquad (4.5)$$

For an unbounded material medium (which is the only case we shall consider here), the field at infinity can be assumed to vanish. Therefore the surface integrals to which the expressions in formulas (4.4) and (4.5) reduce, since they contain divergences also can be assumed to vanish. If, inside the material medium, the magnetic induction **B** and the magnetic field intensity change continuously, then the rate of change of the energy in the unbounded medium is given by formulas (4.4) and (4.5), in which we may drop the terms containing divergences. In particular,

$$\frac{dW}{dt}=\frac{1}{4\pi}\int \partial\mathbf{r}\left\{\mathbf{B}\frac{\partial\mathbf{B}}{\partial t}+\mathbf{E}\frac{\partial\mathbf{D}'}{\partial t}\right\}. \quad (4.6)$$

By using formula (4.6) we can get an expression for the amount of heat liberated in the medium. Let us consider a monochromatic field whose time dependence is $e^{-i\omega t}$. Averaging the expression (4.6) over the time, we get the average energy accumulating in the medium, or, what is the same thing, the amount of heat liberated in the medium per unit time. Because of the fact that the electric field is a real quantity, it can be represented in the form

$$\mathbf{E}(\mathbf{r},\ t)=\mathbf{E}(\mathbf{r},\ \omega)\,e^{-i\omega t}+\mathbf{E}^*(\mathbf{r},\ \omega)\,e^{i\omega t}. \quad (4.7)$$

We can similarly represent the magnetic and electric induction. Substituting such expressions in formula (4.6) and taking the time average, we get

$$Q=\frac{i\omega}{4\pi}\int d\mathbf{r}\,\{\mathbf{E}(\mathbf{r},\omega)\,\mathbf{D}'^*(\mathbf{r},\ \omega)-\mathbf{E}^*(\mathbf{r},\ \omega)\,\mathbf{D}'(\mathbf{r},\ \omega)\}. \quad (4.8)$$

Since for a monochromatic field the material equation (III) takes the form (2.1) and, because, in addition, relation (2.3) is valid, formula (4.8) can be written as follows:

$$Q=\frac{i\omega}{4\pi}\int d\mathbf{r}\,d\mathbf{r}'\,\{\varepsilon_{ij}^*(\omega,\ \mathbf{r},\ \mathbf{r}')-\varepsilon_{ji}(\omega,\ \mathbf{r},\ \mathbf{r}')\}\,E_i(\mathbf{r},\ \omega)\,E_j^*(\mathbf{r}',\ \omega). \quad (4.9)$$

The right side of this relation determines the energy accumulating per unit time in the medium (or, correspondingly, given out by the medium*) as a result of the appearance in the medium of a monochromatic electromagnetic field due to external sources.

In a homogeneous medium with weak absorption, in treating the propagation of plane electromagnetic waves $e^{-i\omega t+i\mathbf{k}\cdot\mathbf{r}}$, one can approximately treat $\mathbf{k}$ as a constant. Then, using formula (4.9), one can obtain the following expression for the heat liberated per unit time per unit volume of the body:

$$\frac{Q}{V}=\frac{i\omega}{4\pi}\{\varepsilon_{ij}^*(\omega,\ \mathbf{k})-\varepsilon_{ji}(\omega,\ \mathbf{k})\}\,E_j^*E_i. \quad (4.10)$$

Under conditions where there is practically no heat liberated, one can neglect the dissipation and consider the medium as nonabsorptive. Then we can assume that the equality

$$\varepsilon_{ij}^*(\omega,\ \mathbf{r},\ \mathbf{r}')=\varepsilon_{ji}(\omega,\ \mathbf{r}',\ \mathbf{r}) \quad (4.11)$$

is satisfied. In the case of a homogeneous medium, we then have

$$\varepsilon_{ij}^*(\omega,\ \mathbf{k})=\varepsilon_{ji}(\omega,\ \mathbf{k}). \quad (4.11')$$

*The latter is possible when the medium is not in a state of thermodynamic equilibrium. In order not to complicate the presentation, we shall speak of an accumulation of energy.

Thus, for a nonabsorbing medium the dielectric permittivity tensor is Hermitian. For an isotropic and non-gyrotropic medium, we should use the expression (2.11) as the dielectric permittivity tensor. Then formula (4.10) becomes

$$Q=\frac{i\omega}{4\pi k^2}\{[\varepsilon^{l*}(\omega,\ k)-\varepsilon^{l}(\omega,\ \mathbf{k})]\,|(\mathbf{kE})|^2+[\varepsilon^{tr*}(\omega,\ k)-\varepsilon^{tr}(\omega,\ \mathbf{k})]\,|[\mathbf{k},\ \mathbf{E}]|^2\}$$
$$=\frac{\omega}{2\pi k^2}\{\varepsilon^{l''}(\omega,\ \mathbf{k})\,|(\mathbf{kE})|^2+\varepsilon^{tr''}(\omega,\ \mathbf{k})\,|[\mathbf{k},\ \mathbf{E}]|^2\}. \quad (4.12)$$

The first term on the right side of formula (4.12) determines the absorption of the longitudinal field in the medium, and the second the absorption of the transverse field. In this sense, we may speak of longitudinal and transverse losses in the medium.

For material media which are in a state of thermodynamic equilibrium as a result of external actions the entropy increases and therefore heat is liberated. In this case Q is positive and, from the expression (4.12), we find

$$\varepsilon^{l''}>0 \text{ and } \varepsilon^{tr''}>0. \quad (4.13)$$

We note that, by using relations (2.15), (2.20), and (2.22), the right side of formula (4.12) can be written as

$$Q=\frac{i\omega}{4\pi}\{(\varepsilon^*-\varepsilon)'\mathbf{EE}^*+(\mu^*-\mu)\,\mathbf{HH}^*\}=\frac{\omega}{2\pi}\{\varepsilon''\,|\mathbf{E}|^2+\mu''\,|\mathbf{H}|^2\}. \quad (4.14)$$

This expression can also be obtained directly by taking a time average of formula (4.4).

We note that from the inequalities (4.13) it generally does not follow that the imaginary part of the magnetic permeability μ'' is positive. Such a condition does not follow in the general case even from formula (4.14), since the transverse electric field $\mathbf{E}^{tr}$ and the magnetic field are related to one another. This last point means that, in order for the expression (4.14) to be positive, it is not necessary that the condition $\mu''>0$ be satisfied.

It is of interest to consider the case of almost monochromatic fields and to determine the rate of systematic change in energy corresponding to this case. In fact, when we usually speak of a field consisting of a superposition of monochromatic fields with frequencies close to some value ω, this means that in the Fourier expansion

$$\mathbf{E}(\mathbf{r},\ t)=\int_{-\infty}^{+\infty} d\omega' e^{-i\omega' t}\mathbf{E}(\mathbf{r},\ \omega')$$

the quantity $\mathbf{E}(\mathbf{r},\omega')$ as a function of ω' has a sharp maximum in the neighborhood of the points $\omega'=\pm\omega$. The two frequency values $(\pm\omega)$ appear because of the fact that the field $\mathbf{E}(\mathbf{r},t)$ is real, so that $\mathbf{E}^*(\mathbf{r},\omega)=\mathbf{E}(\mathbf{r},-\omega)$. All this permits us to represent an almost monochromatic field in the form

$$\mathbf{E}(\mathbf{r},\ t)=\mathbf{E}_0(\mathbf{r},\ t)\,e^{-i\omega t}+\mathbf{E}_0^*(\mathbf{r},\ t)\,e^{i\omega t}, \quad (4.15)$$

where $\mathbf{E}_0(\mathbf{r}, t)$ as a function of time changes very little over a time equal to the period $2\pi/\omega$. It is obvious that

$$\mathbf{E}_0(\mathbf{r}, t) = \int_0^\infty d\omega' e^{i(\omega-\omega')t}\,\mathbf{E}(\mathbf{r}, \omega'),$$

$$\mathbf{E}_0^*(\mathbf{r}, t) = \int_{-\infty}^0 d\omega' e^{-i(\omega+\omega')t}\,\mathbf{E}^*(\mathbf{r}, \omega').$$

In the integrands of such integrals, because of the fact that $\mathbf{E}(\mathbf{r}, \omega)$ is a function with a sharp maximum, we can make an expansion in powers of $\omega' \mp \omega$. This in particular permits us to write the following approximate relations:

$$\frac{\partial}{\partial t}\mathbf{E}_0(\mathbf{r}, t) \approx i\int_0^\infty d\omega'\,(\omega-\omega')\,\mathbf{E}(\mathbf{r}, \omega'),$$

$$\frac{\partial}{\partial t}\mathbf{E}_0^*(\mathbf{r}, t) \approx -i\int_{-\infty}^0 d\omega'\,(\omega+\omega')\,\mathbf{E}^*(\mathbf{r}, \omega').$$

The occurrence on the right sides of these expressions of the quantities $\omega \pm \omega'$ has the result that, for a sufficiently narrow distribution of field in frequency, the function $\mathbf{E}_0$ actually is a slowly varying function. There is an analogous situation for the electric and magnetic inductions.

We now obtain the approximate expression for the time derivative of the electric induction of an almost monochromatic field and pose the problem of expressing it in terms of the slowly varying function $\mathbf{E}_0(\mathbf{r}, t)$. According to formula (2.1) we have

$$\frac{\partial D_i'(\mathbf{r}, t)}{\partial t} = -i\int d\omega' e^{-i\omega' t}\int d\mathbf{r}'\,\varepsilon_{ij}(\omega', \mathbf{r}, \mathbf{r}')\,E_j(\mathbf{r}, \omega').$$

Because of the fact that, in our case of an almost monochromatic field, the main contribution to the ω' integrations comes only from the frequency regions near to $\pm\omega$, we can, in the integrands on the right side of this relation, make an expansion in powers of $\omega' \pm \omega$. Retaining only the first two terms of such an expansion, and also making use of the expressions obtained above for $\partial\mathbf{E}_0/\partial t$ and $\partial\mathbf{E}_0^*/\partial t$, we find

$$\frac{\partial D_i'(\mathbf{r}, t)}{\partial t} \approx -i\omega e^{-i\omega t}\int d\mathbf{r}'\,\varepsilon_{ij}(\omega, \mathbf{r}, \mathbf{r}')\,E_{0j}(\mathbf{r}', t) + i\omega e^{i\omega t}\int d\mathbf{r}'\varepsilon_{ij}^*(\omega, \mathbf{r}, \mathbf{r}')\,E_{0j}^*(\mathbf{r}', t) + e^{-i\omega t}\int d\mathbf{r}'\,\frac{\partial E_{0j}(\mathbf{r}', t)}{\partial t} \times \frac{\partial}{\partial\omega}[\omega\varepsilon_{ij}(\omega, \mathbf{r}, \mathbf{r}')] - e^{i\omega t}\int d\mathbf{r}'\,\frac{\partial E_{0j}^*(\mathbf{r}^*, t)}{\partial t}\,\frac{\partial}{\partial\omega}[\omega\varepsilon_{ij}^*(\omega, \mathbf{r}, \mathbf{r}')].$$

This expression, as well as formula (4.15) and the corresponding expression for the magnetic induction, enable us to determine the quantity of interest to us, the rate of systematic change of energy of an electromagnetic field in the medium. In fact, substituting these expressions in formula (4.6) and averaging over the period $2\pi/\omega$, we obtain for the rate of systematic change of electromagnetic energy the following expression:

$$\left(\frac{dW}{dt}\right)_{\text{av}} = \frac{1}{4\pi}\int d\mathbf{r}\,d\mathbf{r}'\left\{E_{0i}^*(\mathbf{r}, t)\,\frac{\partial E_{0j}(\mathbf{r}', t)}{\partial t}\,\frac{\partial}{\partial\omega}[\omega\varepsilon_{ij}(\omega, \mathbf{r}, \mathbf{r}')] + E_{0j}(\mathbf{r}', t)\,\frac{\partial E_{0i}^*(\mathbf{r}, t)}{\partial t}\,\frac{\partial}{\partial\omega}[\omega\varepsilon_{ji}^*(\omega, \mathbf{r}', \mathbf{r})]\right\} + \frac{1}{4\pi}\int d\mathbf{r}\,\frac{\partial}{\partial t}(\mathbf{B}_0^*\mathbf{B}_0) + Q, \tag{4.16}$$

where Q is the heat liberated per unit time and is given by formula (4.9).

Formula (4.16) enables us to give a quantitative criterion for a nonabsorbing medium. From Eq. (4.11) we can give the conditions when the expression for the heat liberated, Q, is much smaller than the sum of all the remaining terms on the right side of formula (4.16). In this case, taking $\epsilon_{ij}(\omega, \mathbf{r}, \mathbf{r}') \approx \epsilon_{ji}^*(\omega, \mathbf{r}, \mathbf{r}')$, we get from formula (4.16),

$$\left(\frac{dW}{dt}\right)_{\text{av}} = \frac{dU}{dt} = \frac{d}{dt}\left\{\int \frac{d\mathbf{r}}{4\pi}\,\mathbf{B}_0^*\mathbf{B}_0 + \int \frac{d\mathbf{r}, d\mathbf{r}'}{4\pi}\,\mathbf{E}_{0i}^*(\mathbf{r}, t)\,\mathbf{E}_{0j}(\mathbf{r}', t)\,\frac{\partial}{\partial\omega}[\omega\varepsilon_{ij}(\omega, \mathbf{r}, \mathbf{r}')]\right\}. \tag{4.17}$$

It is important that for a nonabsorbing medium the rate of systematic change of energy of the electromagnetic field is given, according to expression (4.17), by a total time derivative. Therefore the quantity U can be considered as the average energy of the electromagnetic field of the medium.

For plane waves whose dependence on space coordinates has the form $e^{i\mathbf{k}\cdot\mathbf{r}}$, we obtain from (4.17)

$$U = \frac{1}{4\pi}\int d\mathbf{r}\left\{\mathbf{B}_0^*\mathbf{B}_0 + E_{0i}^*E_{0j}\,\frac{\partial}{\partial\omega}[\omega\varepsilon_{ij}(\omega, \mathbf{k})]\right\}. \tag{4.18}$$

In the case of an isotropic and non-gyrotropic medium, according to relation (2.11), we can write the right-hand side of formula (4.18) as

$$\frac{1}{4\pi}\int d\mathbf{r}\left\{\mathbf{B}_0^*\mathbf{B}_0 + |\mathbf{E}_0^l|^2\frac{\partial}{\partial\omega}(\omega\varepsilon^l) + |\mathbf{E}^{tr}|^2\frac{\partial}{\partial\omega}(\omega\varepsilon^{tr})\right\} = \frac{1}{4\pi}\int d\mathbf{r}\left\{|\mathbf{E}_0^l|^2\frac{\partial}{\partial\omega}(\omega\varepsilon^l) + |\mathbf{E}_0^{tr}|^2\frac{\partial}{\partial\omega}\left(\omega\left[\varepsilon^{tr} - \frac{c^2k^2}{\omega^2}\right]\right)\right\}. \tag{4.19}$$

Here $\mathbf{E}_0^l$ is the longitudinal (parallel to the vector $\mathbf{k}$) component of the electric field, and $\mathbf{E}_0^{tr}$ is the transverse component ($\operatorname{div}\mathbf{E}_0^{tr} = 0$).

From the condition that expression (4.19) be positive, there follow in particular the inequalities

$$\frac{\partial}{\partial\omega}(\omega\varepsilon^l) \geqslant 0, \qquad \frac{\partial}{\partial\omega}\left\{\omega\left[\varepsilon^{tr} - \frac{c^2k^2}{\omega^2}\right]\right\} \geqslant 0, \tag{4.20}$$

which for $\mathbf{k} = 0$ coincide with one another and go over into the well-known inequality (cf. reference 2, Sec. 64)

$$\frac{d}{d\omega}(\omega\varepsilon(\omega)) \geqslant 0.$$

By using expression (4.19) and also relations (2.15), (2.20), and (2.22), it is not difficult to show that for an isotropic and non-gyrotropic medium formula (4.18) goes over into the form

$$U=\frac{1}{4\pi}\int d\mathbf{r}\left\{|\mathbf{E}_0|^2\frac{\partial}{\partial\omega}(\omega\varepsilon)+|\mathbf{H}_0|^2\frac{\partial}{\partial\omega}(\omega\mu)\right\}. \quad (4.21)$$

This last expression is very convenient to use in the case of nonabsorbing isotropic media, when one considers only frequency dispersion of the dielectric permittivity.[2]

In conclusion, we emphasize that one may speak about an average energy of the electromagnetic field, U, as defined by formulas (4.18), (4.19), and (4.21), only under the condition that the absorption in the medium be negligibly small.

5. ELECTROMAGNETIC WAVES IN A MEDIUM

In the absence of external sources, electromagnetic fields are still possible in vacuum. Such fields are called electromagnetic waves. In particular, they may be plane monochromatic waves whose time and coordinate dependence is $e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$, where ω and $\mathbf{k}$ are real quantities.

In the absence of absorption, it is also possible to have such waves in a medium. On the other hand, in absorbing media the situation is different. If a certain electromagnetic field appears in such a medium at some initial time as a result of the action of external forces, then in the succeeding moments after the completion of the action of the external forces, because of the presence of dissipative processes, the field in the medium will in general be damped. In particular, it is possible to have damped oscillations of the field, i.e., electromagnetic waves which are damped in time.

In this section we shall consider the question of the dependence of the electromagnetic field on time in an unbounded, homogeneous medium in which, at the initial time t = 0, the external sources produce an electromagnetic field, while in the succeeding time the external sources of the field do not act.

To solve such an initial value problem it is not enough to know the magnetic induction $\mathbf{B}(\mathbf{r},0)$, the electric field $\mathbf{E}(\mathbf{r},0)$, and the electric induction $\mathbf{D}'(\mathbf{r},0)$ at the initial time t = 0. In fact, on the right hand side of the material equation (III), which in our case of a homogeneous, unbounded medium has the form (2.4), there enter the values of the electric field both for $t > 0$ as well as for preceding times. It therefore must be clear that to solve the initial value problem one actually needs to know the prior history of the field in the medium.

The electric induction can be represented as a sum of two terms[7] $\mathbf{D}'=\mathbf{D}^{(0)}+\mathbf{D}^{(1)}$, where

$$D_i^{(0)}=\int_{-\infty}^{0}dt'\int d\mathbf{r}'\hat{\varepsilon}_{ij}(t-t',\mathbf{r}-\mathbf{r}')E_j(\mathbf{r}',t'), \quad (5.1)$$

$$D_i^{(1)}=\int_{0}^{t}dt'\int d\mathbf{r}'\hat{\varepsilon}_{ij}(t-t',\mathbf{r}-\mathbf{r}')E_j(\mathbf{r}',t'). \quad (5.2)$$

The quantity $\mathbf{D}^{(1)}$ depends only on the value of the field for $t > 0$; on the other hand, $\mathbf{D}^{(0)}$ depends on the prior history and therefore this quantity must be assigned in the initial value problem. The physical significance of the need for assigning such a quantity, which is a function of the time, is that we thus take account of processes of relaxation and transport accomplished by the particles of the medium and beginning at the time t = 0.*

Thus, for solving the initial value problem in which we are interested we shall assume that we are given $\mathbf{B}(\mathbf{r},0)$ and $\mathbf{D}^{(0)}(\mathbf{r},t)$. We note that $\mathbf{D}'(\mathbf{r},0)=\mathbf{D}^{(0)}(\mathbf{r},0)$. Assuming that these quantities are known, we make use of the Fourier transformation[10-12] to obtain the solution of the field equations†

$$\mathbf{E}(\mathbf{r},t)=\frac{1}{(2\pi)^4}\int d\mathbf{k}\int_{-\infty+i\sigma}^{\infty+i\sigma}d\omega e^{-i\omega t}\mathbf{E}(\mathbf{k},\omega) \quad (t\geqslant 0),$$

$$\mathbf{E}(\mathbf{k},\omega)=\int d\mathbf{r}\int_0^\infty dt\, e^{i\omega t-i\mathbf{k}\mathbf{r}}\mathbf{E}(\mathbf{r},t) \quad (\operatorname{Im}\omega=\sigma>0). \quad (5.3)$$

With respect to the time, we use a one-sided Fourier transform corresponding to the fact that the equation of the field without sources in the initial value problem is valid only for $t > 0$. Formulas analogous to (5.3) are also to be understood as holding for the electric and magnetic induction. Then, from the field equations, we have

$$\frac{\omega}{c}\mathbf{B}(\mathbf{k},\omega)-[\mathbf{k},\mathbf{E}(\mathbf{k},\omega)]=\frac{i}{c}\mathbf{B}(\mathbf{k},t=0), \quad \mathbf{k}\mathbf{B}=0,$$

$$\frac{\omega}{c}\mathbf{D}'(\mathbf{k},\omega)+[\mathbf{k}\mathbf{B}(\mathbf{k},\omega)]=\frac{i}{c}\mathbf{D}'(\mathbf{k},t=0), \quad \mathbf{k}\mathbf{D}'=0.$$

Here

$$\mathbf{D}'(\mathbf{k},t=0)=\int d\mathbf{r}\,e^{-i\mathbf{k}\mathbf{r}}\mathbf{D}'(\mathbf{r},t=0);$$

An analogous relation determines $\mathbf{B}(\mathbf{k},t=0)$. Eliminating the magnetic induction and also representing the electric induction in the form of a sum $D_i'=D_i^{(0)}+D_i^{(1)}$, we get

$$\omega^2\mathbf{D}^{(1)}(\mathbf{k},\omega)+c^2[\mathbf{k},[\mathbf{k},\mathbf{E}(\mathbf{k},\omega)]]$$
$$=\omega\mathbf{D}'(\mathbf{k},t=0)+ic[\mathbf{k}\mathbf{B}(\mathbf{k},t=0)]-\omega^2\mathbf{D}^{(0)}(\mathbf{k},\omega), \quad (5.4)$$

$$\mathbf{k}\mathbf{D}^{(1)}(\mathbf{k},\omega)=-\mathbf{k}\mathbf{D}^{(0)}(\mathbf{k},\omega). \quad (5.5)$$

Supplementing this system by a material equation relating the quantities $\mathbf{D}^{(1)}(\mathbf{k},\omega)$ and $\mathbf{E}(\mathbf{k},\omega)$ and having the following form according to formula (5.2):

$$D_i^{(1)}(\mathbf{k},\omega)=\varepsilon_{ij}(\omega,\mathbf{k})E_j(\mathbf{k},\omega), \quad (5.6)$$

*Naturally, if we should consider the initial value problem not just for the field equations, but also for the equations of motion of the particles of the medium, then the knowledge of the prior history would not be necessary. However, then, in addition to the initial values of the field, we would have to also assign the initial states of the particles of the medium.

†On the assumption that $\mathbf{E}(\mathbf{r},t)$ increases with time no faster than $e^{\sigma t}$, we can assert according to formula (5.3) that $\mathbf{E}(\mathbf{k},\omega)$ as a function of the complex variable ω has no singularities in the complex plane of this variable above the line $\operatorname{Im}(\omega)=\sigma$. The same also applies to the function $\varepsilon_{ij}(\omega,\mathbf{k})$ defined by relation (2.6).

we get a system of linear, algebraic equations for the determination of the quantity $\mathbf{E}(\mathbf{k}, \omega)$:

$$k_i \varepsilon_{ij}(\omega, \mathbf{k}) E_j(\mathbf{k}, \omega) = -\mathbf{k}\mathbf{D}^{(0)}(\mathbf{k}, \omega), \quad (5.4')$$

$$(\omega^2 \varepsilon_{ij}(\omega, \mathbf{k}) - c^2 k^2 \delta_{ij} + c^2 k_i k_j) E_j(\mathbf{k}, \omega) = -\omega^2 D_i^{(0)}(\mathbf{k}, \omega) + i\omega D_i'(\mathbf{k}, t=0) + ic[\mathbf{k}, \mathbf{B}(\mathbf{k}, t=0)]. \quad (5.5')$$

Let us first consider the case of an isotropic, nongyrotropic medium, where expression (2.11) holds for the dielectric permittivity tensor. In this case, the system of equations (5.4′) and (5.5′) breaks up into independent equations for the longitudinal field $\mathbf{E}^l$ (parallel to the vector $\mathbf{k}$) and the transverse field $\mathbf{E}^{tr}$:

$$\varepsilon^l(\omega, \mathbf{k}) \mathbf{E}^l(\mathbf{k}, \omega) = -\mathbf{D}^{(0)l}(\mathbf{k}, \omega),$$

$$\{\omega^2 \varepsilon^{tr}(\omega, \mathbf{k}) - c^2 k^2\} \mathbf{E}^{tr}(\mathbf{k}, \omega) = -\omega^2 \mathbf{D}^{(0)tr}(\mathbf{k}, \omega) + i\omega \mathbf{D}'(\mathbf{k}, t=0) + ic[\mathbf{k}, \mathbf{B}(\mathbf{k}, t=0)].$$

Here $\mathbf{D}^{(0)l}$ and $\mathbf{D}^{(0)tr}$ are the longitudinal and transverse components of the vector $\mathbf{D}^{(0)}$. From this we have immediately

$$\mathbf{E}^l(\mathbf{k}, \omega) = -\frac{\mathbf{D}^{(0)l}(\mathbf{k}, \omega)}{\varepsilon^l(k, \omega)}, \quad (5.7)$$

$$\mathbf{E}^{tr}(\mathbf{k}, \omega) = \frac{-\omega^2 \mathbf{D}^{(0)tr}(\mathbf{k}, \omega) + i\omega \mathbf{D}'(\mathbf{k}, t=0) + ic[\mathbf{k}, \mathbf{B}]}{\omega^2 \varepsilon^{tr}(\omega, k) - c^2 k^2}. \quad (5.8)$$

According to formulas (5.7) and (5.8), the expressions for the longitudinal and transverse fields can be represented as

$$\mathbf{E}^l(\mathbf{r}, t) = \int_0^\infty dt' \int d\mathbf{r}' G_+^l(\mathbf{r} - \mathbf{r}', t - t') \frac{\partial^2}{\partial \mathbf{r}'^2} \mathbf{D}^{(0)}(\mathbf{r}', t'), \quad (5.7')$$

$$\mathbf{E}^{tr}(\mathbf{r}, t) = \int_0^\infty dt' \int d\mathbf{r}' \left\{ \mathbf{D}'(\mathbf{r}', t=0) \frac{\partial}{\partial t} G_+^{tr}(\mathbf{r} - \mathbf{r}', t - t') + G_+^{tr}(\mathbf{r} - \mathbf{r}', t - t') \left[\frac{\partial^2 \mathbf{D}^{(0)}(\mathbf{r}', t')}{\partial t'^2} - c \operatorname{rot} \mathbf{B}(\mathbf{r}', t=0) \right] \right\}, \quad (5.8')$$

where the longitudinal and transverse retarded Green's functions have the respective forms[7]

$$G_+^l(\mathbf{r}, t) = \frac{1}{(2\pi)^4} \int_{-\infty+i\sigma}^{+\infty+i\sigma} d\omega e^{-i\omega t} \int d\mathbf{k} \frac{e^{i\mathbf{k}\mathbf{r}}}{k^2 \varepsilon^l(\omega, k)}, \quad (5.9)$$

$$G_+^{tr}(\mathbf{r}, t) = \frac{1}{(2\pi)^4} \int_{-\infty+i\sigma}^{+\infty+i\sigma} d\omega e^{-i\omega t} \int d\mathbf{k} \frac{e^{i\mathbf{k}\mathbf{r}}}{\omega^2 \varepsilon^{tr}(\omega, k) - c^2 k^2}. \quad (5.9')$$

It is clear that to treat the dependence of the field on the time it is necessary to investigate the form of the retarded Green's functions.

In expressions (5.9) and (5.9′) it is useful to shift the contour of integration over ω into the lower half-plane of the complex variable.* Here the integral along a line parallel to the real axis and lying infinitely far in the lower half-plane gives a zero contribution. The finite contribution comes from circling the poles of the integrand and also going around the cuts in the plane of the complex variable which arise as a result of the presence of branch points.

Let us consider the contribution to the integral (5.9) arising from poles of the integrand which are associated with the vanishing of the longitudinal dielectric permittivity

$$\varepsilon^l(\omega, k) = 0. \quad (5.10)$$

In this case the integral over the contour around such a pole corresponding to the residue of the integrand (5.9) gives a time dependence

$$e^{-i\omega' t + \omega'' t},$$

where $\omega = \omega' + i\omega''$ is the solution of (5.10), determining the dependence of the frequency ω' and the logarithmic decrement $\gamma = -\omega''$ for the wave with wave vector $\mathbf{k}$.* For a given wave vector, there may, in general, be different roots of (5.10). After sufficiently long times only those vibrations among these will be important which are damped most slowly, i.e., which have the smallest logarithmic decrement, which corresponds to the root of (5.10) which is closest to the real axis.

The contribution to the integral (5.9) resulting from branches of the longitudinal dielectric permittivity and associated with the integration along the boundary lines of the cut in the plane of the complex variable ω does not give a purely exponential dependence.[12] We may say that in this case a continuous spectrum of frequencies corresponds to a definite wave vector $\mathbf{k}$. Of especial interest is the case when, near the line of the cut on the neighboring sheets of the complex variable ω, the analytic continuation of the function $\epsilon^l(\omega, \mathbf{k})$ has a zero. In this case, for not too small times, the principal time dependence associated with the integration along the boundary line of the cut will be purely exponential with the complex frequency determined by Eq. (5.10) for the analytic continuation of the longitudinal dielectric permittivity on the neighboring sheets.

An analogous treatment for transverse waves using expression (5.8) obviously leads to the condition

$$\omega^2 \varepsilon^{tr}(\omega, k) - c^2 k^2 = 0, \quad (5.11)$$

which determines the frequencies and logarithmic decrements of the transverse oscillations of the field.

In the case of an anisotropic, homogeneous medium, the solution of the system of linear equations (5.4′) and (5.5′) is proportional to $\Delta^{-1}(\mathbf{k}, \omega)$, where

$$\Delta(\mathbf{k}, \omega) = \left| k^2 \delta_{ij} - k_i k_j - \frac{\omega^2}{c^2} \varepsilon_{ij}(\omega, \mathbf{k}) \right| \quad (5.12)$$

*It should be remarked that in shifting the contour of integration in (5.9) we must make an analytic continuation of the integrand into the lower half-plane of the complex variable ω. The functions $\mathbf{D}^{(0)l}(\mathbf{k}, \omega)$ and $\varepsilon^l(\omega, \mathbf{k})$ as functions of ω are defined by means of the one-sided Fourier transform, according to formulas (2.6) and (5.3), and are analytic everywhere in the upper half-plane of the complex variable ω ($\operatorname{Im}\omega \gtrsim \sigma \geq 0$), beyond, possibly, a band of finite width σ around the real axis.

*ω'' is negative if the pole lies in the lower half-plane. But if the pole is located in the upper half-plane ($\sigma \geq \omega'' > 0$), then we should speak of a logarithmic increment, which can be the case only when the medium is in a non-equilibrium state.

is the determinant of this system of linear equations. Therefore the branch points and zeros of such a determinant determine the dependence of the field on time resulting from the properties of the medium and not caused by the initial state. In particular, in place of the dispersion equations (5.10) and (5.11), giving the dependence of frequency and logarithmic decrement on wave vector, for the case of an anisotropic medium we have

$$\left| k^2\delta_{ij} - k_i k_j - \frac{\omega^2}{c^2}\varepsilon_{ij}(\omega, \mathbf{k}) \right| = 0. \tag{5.13}$$

It is not difficult to see that in the case of an isotropic medium with the expression for the complex dielectric permittivity tensor $\epsilon_{ij}(\omega, \mathbf{k})$ given by formula (2.11), the determinant $\Delta(\mathbf{k}, \omega)$ splits into a product of two factors, as a result of which Eq. (5.13) reduces to Eqs. (5.10) and (5.11).

6. PLANE MONOCHROMATIC WAVES IN A MEDIUM

As we have already pointed out above, in a material medium in the absence of absorption, just as in vacuum, it is possible to have electromagnetic waves of the form

$$e^{i\mathbf{kr} - i\omega t}. \tag{6.1}$$

In a vacuum the frequency ω and wave vector $\mathbf{k}$ are real quantities, and the waves (6.1) are plane. In considering the propagation of the electromagnetic waves (6.1) in material media, it is necessary in the general case to introduce complex values of ω and $\mathbf{k}$. In the preceding section we considered the problem of propagation of electromagnetic waves in a medium, these waves being the result of an arbitrary initial perturbation in the medium. Then, assigning the real wave vector $\mathbf{k}$, we found Eqs. (5.10), (5.11), and (5.13), which made it possible to determine the complex frequency $\omega = \omega' + i\omega''$, whose real part represents the vibration frequency and whose imaginary part is the logarithmic decrement (or logarithmic increment) of the amplitude of the wave with time. However, it is also sensible to consider a different formulation of the problem in which one chooses a real frequency; i.e., one considers propagation in the medium of a monochromatic wave with a fixed frequency ω. We are then required to determine the complex wave vector $\mathbf{k}$. Here we shall consider the propagation of waves of the type (6.1) for such a formulation of the problem in an unbounded homogeneous medium. Equations (II) and (III) describing the electromagnetic field in the medium can then be brought to the following system of homogeneous algebraic equations for the electric field intensity $\mathbf{E}$;

$$\left\{ k^2\delta_{ij} - k_i k_j - \frac{\omega^2}{c^2}\varepsilon_{ij}(\omega, \mathbf{k}) \right\} E_j = 0. \tag{6.2}$$

The condition of compatibility of this system is

$$\left| k^2\delta_{ij} - k_i k_j - \frac{\omega^2}{c^2}\varepsilon_{ij}(\omega, \mathbf{k}) \right| = 0 \tag{6.3}$$

which is the dispersion equation for the electromagnetic waves in the medium. It determines the dispersion law in implicit form; i.e., it determines the dependence of the wave vector on frequency.

For a vacuum, $\epsilon_{ij}(\omega, \mathbf{k}) = \delta_{ij}$, and therefore we find from (6.3) that $k^2 = \omega^2/c^2$. In the case of a material medium, Eq. (6.3) for real ω may also have complex solutions $\mathbf{k} = \mathbf{k}' + i\mathbf{k}''$. It should be noted that complex solutions $\mathbf{k}$ are not necessarily associated with complex values for the dielectric permittivity tensor. In fact, for an isotropic medium, for example, when we neglect spatial dispersion, $\epsilon_{ij}(\omega, \mathbf{k}) = \epsilon(\omega)\delta_{ij}$; therefore, we have from Eq. (6.3), $k^2 = (\omega^2/c^2)\epsilon(\omega)$. For $\epsilon(\omega) < 0$ the roots of this equation are pure imaginary, even though there is no absorption in the medium.

In the general case of complex $\mathbf{k}$, the wave (6.1) may be called "plane" only in some conventional sense. From the coordinate dependence of the field, $e^{i\mathbf{k}\cdot\mathbf{r}} = e^{i\mathbf{k}'\cdot\mathbf{r} - \mathbf{k}''\cdot\mathbf{r}}$, it follows that the planes of constant phase of the wave are planes perpendicular to the vector $\mathbf{k}'$, while the amplitude of the wave is constant on planes perpendicular to the vector $\mathbf{k}''$, along whose direction there is a damping (or rise in amplitude) of the wave. Therefore such waves are called inhomogeneous plane waves, as distinct from the homogeneous plane waves in which the surfaces of constant field value coincide with the surfaces of constant wave amplitude. One can have homogeneous plane waves when the quantity $\mathbf{k}$ is real, as occurs, for example, in vacuum; or when $\mathbf{k}'$ and $\mathbf{k}''$ are parallel to one another. Media in which there correspond to real values of ω real values of $\mathbf{k}$ (more precisely, negligibly small imaginary parts of $\mathbf{k}$) are said to be transparent for the given frequencies.

In the case of homogeneous plane waves, the dispersion equation (6.3) enables us to determine the value of the wave vector $\mathbf{k}$ for each given direction of propagation of the wave. But in those same cases, where in a given problem inhomogeneous plane waves are important, usually in addition to the frequency one also knows two real components of the wave vector. Then the dispersion equation (6.3) determines the third, complex component of $\mathbf{k}$. Such a situation holds, for example, in the problem of reflection and refraction of a plane monochromatic wave for oblique incidence on a plane boundary between vacuum and a medium. The tangential components of the wave vector of the refracted wave are then equal to the tangential components of the wave vector of the incident wave and are real; the normal components are determined from the dispersion equation (6.3) and are complex in general.

After the general comments, let us consider the most important special case of propagation of electromagnetic waves of the type (6.1) in material media.

An especially simple and easily pictured case is the propagation of monochromatic waves in a transparent medium. If the medium is isotropic and non-gyrotropic, then the dispersion equation (6.6) breaks up into two equations. The first of them,

$$\varepsilon^{l}(\omega, k) = 0 \tag{6.4}$$

determines the wave vector of longitudinal waves in the medium. The second equation

$$k^2 - \frac{\omega^2}{c^2}\varepsilon^{tr}(\omega, k) = 0 \tag{6.5}$$

is the dispersion equation for the transverse electromagnetic waves.

Using relations (2.20) and (2.22), the dispersion equations (6.4) and (6.5) can also be written in the following form:

$$\varepsilon(\omega, k) = 0, \tag{6.4'}$$

$$k^2 - \frac{\omega^2}{c^2}\varepsilon(\omega, k)\mu(\omega, k) = 0. \tag{6.5'}$$

This form of writing of the dispersion equations for the longitudinal and transverse waves corresponds to describing the propagation of electromagnetic waves in an isotropic and non-gyrotropic medium by means of the field equations (1.3) and (1.9).

In a transparent medium we can define a vector $\mathbf{n}$ by means of the relation

$$\mathbf{k} = \frac{\omega}{c}\mathbf{n}. \tag{6.6}$$

The quantity $n(\omega)$, which is called the index of refraction of the medium, characterizes the difference between the phase velocity of the waves propagating in a given direction and the velocity of light in vacuum.

Dispersion equations for the longitudinal and transverse waves in an isotropic and non-gyrotropic transparent medium can be written in the following form, by using the quantities $\mathbf{n}$ and ω:

For longitudinal waves:

$$\varepsilon^{l}\left(\omega, \frac{\omega}{c}\mathbf{n}\right) = 0 \quad \text{or} \quad \varepsilon\left(\omega, \frac{\omega}{c}\mathbf{n}\right) = 0, \tag{6.7}$$

For transverse waves:

$$n^2 = \varepsilon^{tr}\left(\omega, \frac{\omega}{c}\mathbf{n}\right) \quad \text{or} \quad n^2 = \varepsilon\left(\omega, \frac{\omega}{c}\mathbf{n}\right)\mu\left(\omega, \frac{\omega}{c}\mathbf{n}\right). \tag{6.8}$$

In an isotropic medium the index of refraction of the wave does not depend on the direction of propagation. This leads to the result that, in an isotropic medium, both the phase and the group velocity can, generally, be directed opposite to the direction of propagation of the wave. In this case we say that the electromagnetic waves have negative group velocity.

When one neglects spatial dispersion (i.e., in the limit of $k/\omega \to 0$), the dispersion equation for longitudinal waves [cf. formula (3.9)]

$$\varepsilon(\omega) = 0 \tag{6.9}$$

determines the discrete frequencies of electromagnetic oscillations of the medium. In this case, the longitudinal waves have zero group velocity and arbitrary phase velocity. When we include spatial dispersion of the dielectric permittivity, as we see from Eqs. (6.4) and (6.7), the frequency of the longitudinal waves becomes a function of the wave vector, and thus the group velocity is different from zero. In this sense, longitudinal waves in a medium, when one takes account of spatial dispersion, become an acceptable branch of the normal waves.

The dispersion equation for the transverse waves (6.5) in an isotropic transparent medium, when we neglect spatial dispersion, has the form

$$n^2 = \varepsilon(\omega). \tag{6.10}$$

Since the dielectric permittivity $\epsilon(\omega)$ is a single-valued function of frequency, we say that in an isotropic medium, when we neglect spatial dispersion, there can propagate only one transverse wave with a given frequency ω. (Here, it is understood, of course, that one can have waves with two different types of polarization.) When we include spatial dispersion, Eq. (6.8) for transverse waves has, in general, several solutions $n_l^2(\omega)$ (possibly even an infinite number). Consequently, there may propagate in the medium several transverse waves with the same frequency, but with different indices of refraction $n_l(\omega)$.

In the case of an anisotropic transparent medium, the dispersion equation for electromagnetic waves (6.3) can be written, using the quantities $\mathbf{n}$ and ω, in the following form:

$$\left| n^2\delta_{ij} - n_i n_j - \varepsilon_{ij}\left(\omega, \frac{\omega}{c}\mathbf{n}\right)\right| = 0. \tag{6.11}$$

The subdivision of electromagnetic waves into longitudinal and transverse waves is in general not possible in the case of an anisotropic medium. The index of refraction of the wave $n(\omega)$ then depends on its direction of propagation. Therefore, the direction of the group velocity of the wave in an anisotropic medium does not coincide with its direction of propagation, as is the case for an isotropic medium.

Under the conditions where one neglects spatial dispersion, the dispersion equation for the electromagnetic waves in a transparent anisotropic medium

$$| n^2\delta_{ij} - n_i n_j - \varepsilon(\omega) | = 0 \tag{6.12}$$

in the space (n_x, n_y, n_z) determines a certain surface of fourth order — the "wave-vector surface." For each given direction $\mathbf{n}$, this equation is a quadratic equation in n^2. Therefore, in each direction in the anisotropic medium there can, in general, propagate two waves with the same frequency ω. When one includes spatial dispersion, the picture becomes considerably more complicated. The dispersion equation (6.11) in the general case is a surface of higher order than (6.12), and therefore one can have more than two waves propagating in each direction in the medium.

So far we have considered the propagation of plane monochromatic waves (6.1) in transparent media, where the wave vector of the wave $\mathbf{k}$ is a real quantity. However, as we have already remarked above, in treating the problem of propagation of monochromatic waves in material media, one must also introduce complex wave vectors $\mathbf{k} = \mathbf{k}' + i\mathbf{k}''$. Then we can distinguish a large class of homogeneous plane waves for which $\mathbf{k}'$ and $\mathbf{k}''$ are parallel to one another. Included in this type of wave are, for example, electromagnetic waves in an isotropic absorbing medium. The problems of propagation of homogeneous plane waves in absorbing media formally are not different from the corresponding problems in transparent media. The same dispersion equations (6.3) (for anisotropic media) or (6.7) and (6.8) (for isotropic media) determine the value of the complex wave vector in each given direction of propagation of the wave. One must keep in mind that when the imaginary part of the wave vector $\mathbf{k}''$ is large, the concept of a wave loses its meaning; since its amplitude changes considerably over a distance of the order of the wave length $\lambda\!\!\!^{-} = 2\pi/k'$, we have an electromagnetic wave which is essentially exponentially damped in space. For large values of $\mathbf{k}''$ the concept of direction of propagation of the wave also becomes meaningless. In an absorbing, non-transparent medium, in addition to the wave vector, the index of refraction as determined by (6.6) becomes complex, $n = n' + in''$. Here the quantity n' is called the index of refraction, while $\kappa = n''$ is the absorption coefficient of the medium.

As a simple example, let us consider the transverse electromagnetic waves in an isotropic absorbing medium, neglecting spatial dispersion. From (6.8) we then obtain[2]

$$n' = \frac{1}{\sqrt{2}}\sqrt{\varepsilon' + \sqrt{\varepsilon'^2 + \varepsilon''^2}}, \quad n'' = \frac{1}{\sqrt{2}}\sqrt{-\varepsilon' + \sqrt{\varepsilon'^2 + \varepsilon''^2}}. \tag{6.13}$$

From this we see that the quantity $n(\omega)$ can be complex even in the case of a real dielectric permittivity for the medium, i.e., in the absence of absorption in the medium. In particular, for $\epsilon' < 0$ and $\epsilon'' = 0$ we have, from expression (6.13), $n' = 0$, $n'' = \sqrt{|\epsilon'|}$. For conductors in the low-frequency range where formula (3.11) is valid, we find from the expression (6.13) that n' and n'' coincide in value and are equal to $n' = n'' = \sqrt{2\pi\sigma_0/\omega}$. The most general class of waves (6.1) in material media are inhomogeneous plane waves in which the real part $\mathbf{k}'$ and the imaginary part $\mathbf{k}''$ of the wave vector are not parallel to one another. Inhomogeneous plane waves appear essentially, for example, in problems of reflection and refraction of plane waves at a plane boundary of separation between two homogeneous media. In such problems one usually knows two real components of the wave vector $\mathbf{k}$, and from the dispersion equations (6.3), (6.4), and (6.5) one determines the third complex component as a function of the frequency of the wave and the two known components of the wave vector.

7. PROPAGATION OF ELECTROMAGNETIC WAVES IN MEDIA WITH WEAK SPATIAL DISPERSION

In treating electromagnetic waves in unbounded, spatially homogeneous media, we have used the material equation

$$D_i' = \varepsilon_{ij}(\omega, \mathbf{k}) E_j. \tag{7.1}$$

We have not restricted ourselves to any explicit functional dependence of the dielectric permittivity tensor $\epsilon_{ij}(\omega, \mathbf{k})$ on wave vector $\mathbf{k}$. If the electromagnetic field varies sufficiently smoothly in space, the tensor $\epsilon_{ij}(\omega, \mathbf{k})$ can be expanded in series in powers of $\mathbf{k}$. Limiting ourselves to the first terms in the expansion, we write[13] (cf. also references 4, 6, and 14)

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \varepsilon_{ij}(\omega) + i\gamma_{ijl}(\omega) n_l + \alpha_{ijlm}(\omega) n_l n_m, \tag{7.2}$$

where $\mathbf{n} = (c/\omega)\mathbf{k}$. In this case of a field varying slowly over space, the coefficients γ_{ijl} and α_{ijlm} are small, and the expansion (7.2) is a series in powers of a small parameter.* In such cases we may speak of weak spatial dispersion.

The propagation of plane electromagnetic waves in material media, when we include weak spatial dispersion, obviously can be investigated more completely than was done in the preceding section. In addition, in the propagation of electromagnetic waves in such media there appear certain characteristic effects which are absent when we neglect spatial dispersion.

It follows from expression (6.2) that the effects of weak spatial dispersion can become important for small values of the components of the tensor $\epsilon_{ij}(\omega)$, which is the dielectric permittivity tensor of the medium when one includes only frequency dispersion. In fact, then in the expansion (7.2), the second and third terms are important, these being the terms associated with spatial dispersion.

The expansion (7.2), however, is not always sufficient for describing the effects of weak spatial dispersion in a medium. The point is that, if all the components of the tensor $\epsilon_{ij}(\omega)$ are large, then we can restrict ourselves in expression (7.2) to just the first term. At the same time, the components of the tensor $\epsilon_{ij}^{-1}(\omega)$ may be small, and in the expansion[13-15]

$$\varepsilon_{ij}^{-1}(\omega, \mathbf{k}) = \varepsilon_{ij}^{-1}(\omega) + i g_{ijl}(\omega) n_l + \beta_{ijlm}(\omega) n_l n_m \tag{7.3}$$

the second and third terms, which take account of the spatial dispersion, will be important. Therefore, to describe the effects of weak spatial dispersion in a

*This parameter depends on the electromagnetic properties of the medium. In a plasma, for longitudinal waves, the small parameter for describing the spatial dispersion is the ratio $r_D/\lambda\!\!\!^{-}$, where r_D is the Debye radius and $\lambda\!\!\!^{-}$ is the wave length of the longitudinal field. For transverse waves in a plasma, this parameter is $\sim v/c$, where v is the thermal velocity of the particles. For crystalline media and neutral gases, a parameter of this sort is the ratio $a/\lambda\!\!\!^{-}$, where a is the lattice constant or the radius of the molecules of the gas, respectively.

medium we shall use, in addition to the expansion (7.2), the expansion (7.3).

Before proceeding to consider electromagnetic waves in a medium including weak spatial dispersion, let us make some comments concerning the symmetry of the coefficients γ_{ijl} and α_{ijlm}. From the symmetry properties of the dielectric permittivity tensor (cf. Sec. 9)

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \varepsilon_{ji}(\omega, -\mathbf{k}) \tag{7.4}$$

it follows immediately that $\gamma_{ijl} = -\gamma_{jil}$ and $\alpha_{ijlm} = \alpha_{jilm}$. The tensor α_{ijlm} is also symmetric in the indices l and m. The coefficients g_{ijl} and β_{ijlm} obviously have the same symmetry properties as the γ_{ijl} and the α_{ijlm}. In the absence of absorption in the medium, $\epsilon^*_{ij}(\omega, \mathbf{k}) = \epsilon_{ji}(\omega, \mathbf{k})$; therefore in such a medium the tensors γ_{ijl} and α_{ijlm} are real. The further simplification of these quantities is already associated with specific symmetries of the medium. In the following we shall restrict ourselves to treating weakly absorbing media. Therefore the coefficients in the expansion (7.2) and (7.3) will be assumed to be real throughout the following discussion.

In the expansions (7.2) and (7.3), in addition to the linear terms in $\mathbf{k}$, we also will keep quadratic terms. In the majority of cases the expansion of the tensor $\epsilon_{ij}(\omega, \mathbf{k})$ in powers of the wave vector does not contain odd powers of $\mathbf{k}$. The point is that if the individual molecules which constitute the medium have centers of symmetry, or if in the case of a crystalline medium the elementary cell of the crystal has a center of symmetry, then the dielectric tensor of such a medium has the following property:

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \varepsilon_{ij}(\omega, -\mathbf{k}).$$

In this case, the expansion of the tensor $\epsilon_{ij}(\omega, \mathbf{k})$ in powers of the wave vector obviously contains only even powers of $\mathbf{k}$. Such media are said to be non-gyrotropic, or optically inactive. Media which do not have this symmetry property are said to be gyrotropic. In particular, isotropic media may be gyrotropic. An example of such a medium is a solution of cane sugar. For gyrotropic media we may limit ourselves to the first two terms in the expansions (7.2) and (7.3).

We now consider the propagation of electromagnetic waves in media when we include weak spatial dispersion. Let us begin with the treatment of isotropic, gyrotropic media. In an isotropic medium (and also in a crystal with cubic symmetry), the symmetric second rank tensor $\epsilon_{ij}(\omega)$ reduces to a scalar, while the antisymmetric tensor of second rank $\gamma_{ijl}n_l$ reduces to a pseudoscalar. Introducing the notation $\gamma_{ijl} = \gamma e_{ijl}$ and $g_{ijl} = -g e_{ijl}$, where e_{ijl} is the completely antisymmetric unit tensor of third rank, the expansions (7.2) and (7.3) can be written as follows:*

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \varepsilon(\omega)\,\delta_{ij} + i\gamma(\omega)\,e_{ijl}\,n_l, \tag{7.5}$$

$$\varepsilon^{-1}_{ij}(\omega, \mathbf{k}) = \varepsilon^{-1}(\omega)\,\delta_{ij} - ig(\omega)\,e_{ijl}n_l. \tag{7.6}$$

The expression (7.5) should be used to account for weak spatial dispersion, as we have already remarked above, for small values of $\epsilon(\omega)$, whereas for large values of $\epsilon(\omega)$ one should use expression (7.6). When account is taken of frequency dispersion, the dielectric permittivity $\epsilon(\omega)$ is generally a non-monotonic function of frequency. In treating electromagnetic waves near absorption bands of the medium, one frequently uses the following interpolation formula (cf., for example, reference 16, Sec. 149):

$$\varepsilon = \varepsilon' + i\varepsilon'' = \varepsilon_0 - \frac{\omega_0^2}{\omega^2 - \omega_j^2 - i\omega\nu}, \tag{7.7}$$

where ω_0, ω_j, and ν characterize the properties of the medium. For $\nu = 0$ the dielectric permittivity (7.7) is real, i.e., there is no absorption in the medium.

From expressions (7.5) and (7.6) we obtain for longitudinal waves the equation

$$\varepsilon(\omega) = 0, \tag{7.8}$$

which coincides with the dispersion equation for longitudinal waves in an isotropic and non-gyrotropic medium under the condition that one neglects spatial dispersion.†

As regards transverse electromagnetic waves, the expressions (7.5) and (7.6) lead for them to different dispersion equations. Since for the transverse waves, according to Maxwell's equations, $\mathbf{D} = n^2\mathbf{E}$, we obtain from the expansion (7.5) the following dispersion equation:

$$[n^2 - \varepsilon(\omega)]^2 = \gamma^2(\omega)\,n^2. \tag{7.9}$$

Because of the smallness of the quantity γ^2, one can write approximate solutions of (7.9) in the form

$$n^2_{\pm} \approx \varepsilon(\omega) \pm \gamma(\omega)\sqrt{\varepsilon(\omega)}. \tag{7.10}$$

To the two solutions of (7.9) there correspond the

*For an isotropic, gyrotropic medium, the dielectric tensor, including arbitrary spatial dispersion, can be represented as follows:

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\varepsilon^{\mathrm{tr}}(\omega, \mathbf{k}) + \frac{k_i k_j}{k^2}\varepsilon^{l}(\omega, \mathbf{k}) + i\varphi(\omega, \mathbf{k})\,l_{ijl}k_l.$$

In the case of weak dispersion, this expression goes over into (7.5), where $\varphi(\omega, 0) = \gamma(\omega)\,c/\omega$, and into (7.6) with $\varphi(\omega, 0) = \varepsilon^2(\omega)\,g(\omega)\,c/\omega$.

†We should remark that the dispersion equation for longitudinal electromagnetic waves in an isotropic, gyrotropic medium, when one includes spatial dispersion,

$$\varepsilon^l(\omega, \mathbf{k}) = 0$$

also does not differ from the corresponding equation in an isotropic and non-gyrotropic medium.

following two relations between the components of the vector **E** (or **D**):

$$E_x = \pm iE_y$$

(The vector **k** is assumed to be along the z axis.) This means that two waves corresponding to the two solutions of (7.9), have different polarizations: namely, the wave in which $E_x = iE_y$ is right circularly polarized, while the wave in which $E_x = -iE_y$ is left circularly polarized. In such a medium there is a rotation of the plane of polarization of the electromagnetic wave.

If the frequency of a transverse wave is close to one of the natural frequencies of the medium, then, according to (7.5), the dispersion equation has the form[13]

$$g^2n^6 - \left(\frac{n^2}{\varepsilon} - 1\right)^2 = 0. \quad (7.11)$$

Making use of the fact that the quantity g^2 is small, we find the following solutions of this equation*

$$n_{1,2}^2 \cong \varepsilon(\omega)\left[1 \pm g(\omega)\,\varepsilon^{3/2}(\omega)\right], \qquad n_3^2 \cong \frac{1}{\varepsilon^2(\omega)\,g^2(\omega)}, \quad (7.12)$$

corresponding to three transverse waves. It is easy to show that the waves with indices of refraction n_1^2 and n_2^2 have right and left circular polarizations respectively.

Summarizing the above statements, we conclude that in an isotropic, gyrotropic medium, far from absorption bands, two transverse waves propagate, while near to an absorption band, according to (7.13), we can have three transverse waves propagating with the same frequency but with different indices of refraction. In Fig. 1 we show the curves of $n_{1,2,3}^2(\omega)$ near an absorption band for $\epsilon(\omega)$ taken in the form (7.7), where the absorption is neglected ($\nu = 0$). In constructing the graph we have used the following values: $g^2 = 10^{-5}$, $\epsilon_0 = 1$, $\omega_0/\omega_j = 1$. The dashed curves in Fig. 1 are

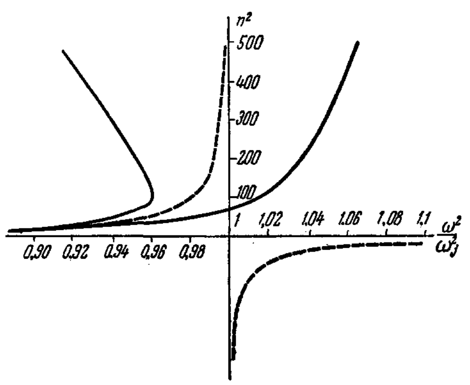

FIG. 1

*Measurements of the quantity g for different media do not exist at present. We may assume, however, that the quantity g is of the same order as γ, which in the optical region of the spectrum for various materials is of the order of $\gamma \sim 10^{-3} - 10^{-4}$ (cf. reference 16, Ch. XXIX).

the limiting curves (7.7). We note that multiple roots of (7.12) correspond to

$$\varepsilon_m^2 = \frac{2^{2/3}}{3g^{2/3}}, \quad n_m^2 = \left(\frac{2}{g}\right)^{2/3}, \quad n^2 = \frac{1}{4}\left(\frac{2}{g}\right)^{2/3},$$

i.e., $\omega^2/\omega_j^2 \approx 0.96$, $n_m^2 \approx 70$, and $n^2 \approx 18$. In the optical frequency region this corresponds to $\Delta\omega \sim 2 \times 10^{-2}\,\omega_j \sim 6 - 12 \times 10^{13}$ sec^{-1}, or $\Delta\lambda\!\!\!^- \sim 80 - 150$ A. These estimates show that the range in which three transverse waves exist in the medium is quite far from the center of the absorption line (natural frequency of the medium). The absorption is then still negligibly small, which makes possible the experimental observation of such waves.

In the case of a non-gyrotropic medium, as we have already pointed out above, the expansion of the tensor $\epsilon_{ij}(\omega, \mathbf{k})$ begins with quadratic terms in **k**. If the medium is also isotropic, the expansions (7.2) and (7.3) take the form

$$\varepsilon_{ij}(\omega, \mathbf{k}) = [\varepsilon(\omega) - \alpha_1(\omega)\,n^2]\,\delta_{ij} - \alpha_2(\omega)\,n_i n_j, \quad (7.13)$$

$$\varepsilon_{ij}^{-1}(\omega, \mathbf{k}) = [\varepsilon^{-1}(\omega) + \beta_1(\omega)\,n^2]\,\delta_{ij} + \beta_2(\omega)\,n_i n_j. \quad (7.14)$$

In writing these expressions we have used the fact that in the case of isotropic media the tensors α_{ijlm} and β_{ijlm} reduce to tensors of second rank with two independent components.

The expression (7.14) leads to the following dispersion equation for longitudinal waves in the medium:

$$n_{\parallel}^2 = \frac{\varepsilon(\omega)}{\alpha_1(\omega) + \alpha_2(\omega)}. \quad (7.15)$$

This equation differs qualitatively from the dispersion equation for longitudinal waves obtained neglecting spatial dispersion (6.9). The difference is that to Eq. (6.9) there correspond only vibrations with discrete frequencies, and consequently longitudinal waves with zero group velocity, whereas the waves defined by Eq. (7.15) have a non-zero group velocity.

For transverse waves we obtain from expression (7.14) the dispersion equation

$$n_{\perp}^2 = \frac{\varepsilon(\omega)}{1 + \alpha_1(\omega)}, \quad (7.16)$$

which, because of the smallness of the quantity $\alpha_1(\omega)$, is practically not different from (6.10), corresponding to neglect of spatial dispersion.

The situation is different for large values of $\epsilon(\omega)$ when, to take account of weak spatial dispersion, we must use expression (7.15). Then the dispersion equation for transverse waves takes the form[13,15]

$$\beta_1 n^4 + \frac{n^2}{\varepsilon} - 1 = 0, \quad (7.17)$$

for which we have the following solutions:

$$n_{1,2}^2 = -\frac{1}{2\varepsilon\beta_1} \pm \sqrt{\left(\frac{1}{2\varepsilon\beta_1}\right)^2 + \frac{1}{\beta_1}}. \quad (7.18)$$

Thus, including weak spatial dispersion in isotropic media leads to a qualitatively new phenomenon near to an absorption band, namely to the appearance of new transverse waves. In Figs. 2 and 3 we show the curves of $n^2_{1,2}(\omega)$ near an absorption band for the case of a real function $\epsilon(\omega)$, i.e., for $\nu = 0$ in formula (7.7). We have used the following values: $\epsilon_0 = 1$, $-\omega_0/\omega_j = 1$ and $|\beta_1| = 10^{-5}$, where $\beta_1 > 0$ in Fig. 2, and $\beta_1 < 0$ in Fig. 3. Curve (7.8) is shown as the dashed curve in both cases.

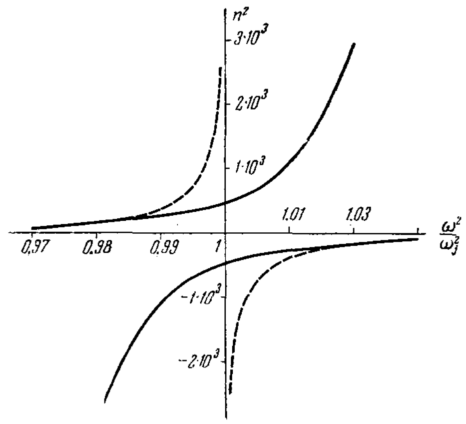

FIG. 2

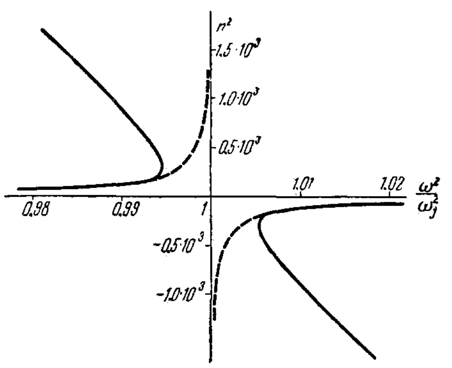

FIG. 3

For $\beta_1 > 0$ one of the roots of (7.18) is negative, and therefore the corresponding wave cannot propagate in the medium. In the case of $\beta_1 < 0$ one can have a propagation of both waves. In this case the multiple roots correspond to the values

$$\varepsilon_m^2 = \frac{1}{4|\beta_1|}, \quad n_m^2 = \frac{1}{\sqrt{|\beta_1|}},$$

i.e., $n_m^2 \approx 300$, and $\omega^2/\omega_j^2 \approx 0.994$. In the optical frequency range this corresponds to $\Delta\omega \sim 3 \times 10^{-3}\,\omega_j \sim 1-2 \times 10^{13}\ \mathrm{sec}^{-1}$, or $\Delta\lambda \sim 10-20\,\mathrm{A}$. This is so close to the natural frequency of the medium that the absorption is then extremely important, as a result of which these waves are very difficult to observe. In fact, for $\nu \neq 0$ we have $n^2 = (n' + in'')^2$, where the absorption coefficient n'' for a frequency corresponding to the multiple root is equal to $n'' \cong \epsilon''|\beta_1|^{1/4} \cong 0.5 \times 10^4\,\nu/\omega_j$. For $\nu/\omega_j \sim 10^{-6}$ we have $n'' \sim 5 \times 10^{-3}$. Since the intensity of the radiation is damped according to the law $e^{-2(\omega/c)n''z} = e^{-\mu z}$, we get $\mu \approx 3 \times 10^3\ \mathrm{cm}^{-1}$, i.e., the intensity of the radiation decreases by a factor of e over a distance $\sim 3 \times 10^{-4}$ cm (where $\lambda \sim 2 \times 10^{-6}$ cm). Far from the center of the absorption line where we can neglect the damping, the index of refraction of one of the waves is so large that the condition for applicability of the expansion (7.14) is violated. From the estimates given it follows that observation of both waves in an isotropic and non-gyrotropic medium is possible only in films of thickness $< 10^{-4}$ cm.

In conclusion we consider very briefly the propagation of electromagnetic waves in media with different crystal structure, when one takes account of weak spatial dispersion. For simplicity we restrict ourselves to considering only non-gyrotropic media. Including weak spatial dispersion naturally reduces the symmetry of the dielectric permittivity tensor of the medium compared to the symmetry which it had when we neglected the dispersion.

In crystalline media with cubic symmetry, the dielectric permittivity $\epsilon_{ij}(\omega)$, when we include only frequency dispersion, is similar to the permeability for an isotropic medium. But when we include weak spatial dispersion, there appears a weak optical anisotropy of cubic crystals, associated with the fact that the tensor α_{ijlm} (and also β_{ijlm}) in a cubic crystal has three indpendent components. The non-zero components of the tensor α_{ijlm} are then[13]

$$\alpha_1 = \alpha_{xxxx} = \alpha_{yyyy} = \alpha_{zzzz}, \quad \alpha_2 = \alpha_{xyxy} = \alpha_{xzxz} = \alpha_{yzyz},$$
$$\alpha_3 = \alpha_{xxyy} = \alpha_{yyxx} = \alpha_{zzxx} = \alpha_{yyzz} = \alpha_{xxzz} = \alpha_{zzyy}.$$

To include a weak anisotropy of cubic crystals in treating transverse waves, it is sufficient to substitute in the small terms of the expression (7.2) and (7.3) the zeroth order values of the index of refraction $n_0^2 = \epsilon(\omega)$, which correspond to neglecting spatial dispersion. We should, however, remember that such a replacement is valid only far from those frequencies for which $\epsilon(\omega)$ is close to zero or infinity. Then the dielectric permittivity tensor $\epsilon_{ij}(\omega, \mathbf{k})$; according to (7.2), depends on the direction of propagation of the wave, which corresponds to a medium with optical anisotropy that manifests itself, for example, in double refraction of a cubic crystal. In ranges of frequency for which $\epsilon(\omega) \to 0$ or $\epsilon(\omega) \to \infty$, there should appear in a cubic crystal the same characteristic effects as in an isotropic medium, but complicated somewhat by the weak anisotropy of the cubic crystal.

Crystals with other symmetries can be treated in a similar fashion. Depending on the symmetry of the crystals, the tensors α_{ijlm} and β_{ijlm} simplify in different ways. For example, in rhombic crystals they have 12 independent components, in tetragonal crystals 7, etc. It should be remarked that in cubic crystals, as in isotropic media, spatial dispersion is much stronger when gyrotropy is present.

8. ENERGY LOSS OF FAST ELECTRONS IN A MEDIUM

A fast charged particle moving in a medium excites electromagnetic waves in it. In an absorbing medium these waves are damped rapidly, which essentially corresponds to a transfer of energy of the particle to the medium via the excitation of electromagnetic waves in it. Therefore a fast charged particle loses part of its energy as it moves in a medium. We shall assume that the energy of the excited electromagnetic waves is small compared with the energy of the particle, and that the change of velocity of the particle in the medium is negligible. The theory of the energy loss of fast charged particles in a medium was developed in papers by Tamm, Frank, and Fermi.[17-20] The generalization of this theory to the case of spatial dispersion of the dielectric permittivity was given in references 7 and 21 – 33.

The energy loss of a moving particle is obviously determined by the work done on the particle by the damping force resulting from the electromagnetic field which it produces in the medium. The work of this force, as determined by expression (I), is given per unit length of the path in the medium by

$$W = \frac{\mathbf{vF}}{v} = \frac{e(\mathbf{vE})}{v}. \tag{8.1}$$

In this formula we should substitute the electric field intensity $\mathbf{E}(\mathbf{r}, t)$ at the point at which the charge is located. We shall consider the motion of the fast charged particle in a spatially homogeneous and unbounded medium. We shall represent the electromagnetic field produced by the particle by means of its Fourier expansion in a sum of waves of the form $e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$. Going over in the field equations (II) to Fourier components and eliminating the magnetic induction $\mathbf{B}$, we obtain the following equation for determining the Fourier components of the electric field intensity $\mathbf{E}$:

$$\left\{k^2\delta_{ij} - k_ik_j - \frac{\omega^2}{c^2}\varepsilon_{ij}(\omega, \mathbf{k})\right\}E_j = \frac{4\pi i\omega}{c^2} j_{0j}(\omega, \mathbf{k}), \tag{8.2}$$

where $\mathbf{j}_0(\omega, \mathbf{k})$ are the Fourier components of the current density of the external sources of the field.

In an isotropic and non-gyrotropic medium, where the dielectric permittivity tensor has the form (2.11), we find from Eq. (8.2) the following expressions for the Fourier components of the electric field intensity $\mathbf{E}$:

$$E_i = \frac{4\pi i\omega}{k^2}\left\{\frac{k^2\left(\delta_{ij} - \frac{k_ik_j}{k^2}\right)}{c^2\left[k^2 - \frac{\omega^2}{c^2}\varepsilon^{tr}(\omega, k)\right]} - \frac{k_ik_j}{\omega^2\varepsilon^{l}(\omega, k)}\right\} j_{0j}(\omega, \mathbf{k}). \tag{8.3}$$

The intensity of the electric field in the medium at any point at the time t is then determined by means of the Fourier transformation formula

$$\mathbf{E}(\mathbf{r}, t) = \int_{-\infty}^{+\infty} dt e^{-i\omega t}\int d\mathbf{k}\, e^{i\mathbf{kr}}\mathbf{E}(\omega, \mathbf{k}). \tag{8.4}$$

Formulas (8.3) and (8.4) enable us to find the electromagnetic field in an isotropic and non-gyrotropic medium produced by an arbitrary field source with current density $\mathbf{j}_0(\mathbf{r}, t)$. For the case where the field source is a point charge moving with velocity $\mathbf{v}$,

$$\mathbf{j}_0(\mathbf{r}, t) = e\mathbf{v}\delta(\mathbf{r} - \mathbf{v}t). \tag{8.5}$$

Then from formulas (8.3) and (8.4) we get

$$\mathbf{E}(\mathbf{r}, t) = -\frac{4\pi ie}{(2\pi)^3}\int d\mathbf{k}\frac{e^{i\mathbf{k}(\mathbf{r}-\mathbf{v}t)}}{k^2}\left\{\frac{\mathbf{k}}{\varepsilon^{l}} - \frac{k^2(\mathbf{kv})\left(\mathbf{v} - \frac{(\mathbf{kv})\mathbf{k}}{k^2}\right)}{c^2\left[k^2 - \frac{\omega^2}{c^2}\varepsilon^{tr}\right]}\right\}. \tag{8.6}$$

Taking the value of the electromagnetic field intensity at the point where the charge is located, i.e., at the point $\mathbf{r} = \mathbf{v}t$, we find, using formula (8.1), for the energy loss of the particle per unit length of path in the medium,

$$W = \frac{ie^2}{2\pi^2 v}\int d\mathbf{k}\frac{(\mathbf{kv})}{k^2}\left\{\frac{1}{\varepsilon^{l}(\mathbf{kv}, k)} - \frac{k^2\left[v^2 - \frac{(\mathbf{kv})^2}{k^2}\right]}{c^2\left[k^2 - \frac{\omega^2}{c^2}\varepsilon^{tr}(\mathbf{kv}, k)\right]}\right\}. \tag{8.7}$$

Introducing the notation $\omega = \mathbf{k}\cdot\mathbf{v}$, $q^2 = k^2 - (\mathbf{k}\cdot\mathbf{v})^2/v^2$, we write formula (8.7) in the following form:

$$W = \frac{ie^2}{\pi v^2}\int_{-\infty}^{+\infty}\omega\, d\omega\int_0^{\infty}\frac{q\,dq}{q^2 + \frac{\omega^2}{v^2}}\left\{\frac{1}{\varepsilon^{l}\left(\omega, \sqrt{q^2 + \frac{\omega^2}{v^2}}\right)} - \frac{v^2}{c^2}\frac{q^2}{q^2 + \omega^2\left[\frac{1}{v^2} - \frac{1}{c^2}\varepsilon^{tr}\left(\omega, \sqrt{q^2 + \frac{\omega^2}{v^2}}\right)\right]}\right\}. \tag{8.8}$$

Since the real and imaginary parts of the longitudinal and transverse dielectric permittivities are even functions of frequency, while the imaginary parts are odd, we have from formula (8.8)

$$W = W^{l} + W^{tr}, \tag{8.9}$$

where

$$W^{l} = -\frac{2e^2}{\pi v^2}\int_0^{\infty}\omega\, d\omega\int_0^{\infty}\frac{q\,dq}{q^2 + \frac{\omega^2}{v^2}}\operatorname{Im}\frac{1}{\varepsilon^{l}\left(\omega, \sqrt{q^2 + \frac{\omega^2}{v^2}}\right)}, \tag{8.10}$$

$$W^{tr} = \frac{2e^2}{\pi c^2}\int_0^{\infty}\omega\, d\omega\int_0^{\infty}\frac{q^3\,dq}{q^2 + \frac{\omega^2}{v^2}} \times \operatorname{Im}\frac{1}{q^2 + \omega^2\left[\frac{1}{v^2} - \frac{1}{c^2}\varepsilon^{tr}\left(\omega, \sqrt{q^2 + \frac{\omega^2}{v^2}}\right)\right]}. \tag{8.11}$$

At first glance it may seem that the important contributions to the energy loss of the charged particle in the medium come only from those regions of the arguments ω and $\mathbf{k}$ in which there is considerable absorption. This is not so, however. In the expression (8.8) there is also contained a considerable contribution from regions in which the imaginary parts of ϵ^{l} and ϵ^{tr} are negligibly small. The point is that in such regions the denominators of the first and second terms in the curly brackets of (8.8) may generally pass through zero, and the integrand may then have a pole. In Sec. 4 it was shown that for media which are in thermodynamic equilibrium, $\mathrm{Im}\,\epsilon^l \geq 0$ and $\mathrm{Im}\,\epsilon^{tr} \geq 0$. Taking this into account and also using the relation

$$\lim_{\delta\to+0}\frac{1}{x+i\delta}=P\frac{1}{x}-i\pi\delta(x),$$

we find the following contributions to W^l and W^{tr} from those regions of the variables ω and $\mathbf{k}$ in which there is no absorption:

$$\Delta W^l=\frac{2e^2}{v^2}\int\omega\,d\omega\int\frac{q\,dq}{q^2+\frac{\omega^2}{v^2}}\,\delta\left[\varepsilon^l\left(\omega,\sqrt{q^2+\frac{\omega^2}{v^2}}\right)\right], \quad (8.12)$$

$$\Delta W^{tr}=\frac{2e^2}{c^2}\int\omega\,d\omega\times\int\frac{q^3\,dq}{q^2+\frac{\omega^2}{v^2}}\,\delta\left[q^2+\omega^2\left(\frac{1}{v^2}-\frac{1}{c^2}\,\varepsilon^{tr}\left(\omega,\sqrt{q^2+\frac{\omega^2}{v^2}}\right)\right)\right]. \quad (8.13)$$

The integrations with respect to ω and q in these expressions extend over the regions of weak absorption in the medium. From formulas (8.12) and (8.13) it is clear that in regions of weak absorption the energy losses are determined by those values of the variables ω and $\mathbf{k}$ at which the arguments of the δ functions appearing in the formulas are equal to zero. According to (6.4) and (6.5) these values of ω and $\mathbf{k}$ correspond to longitudinal and transverse electromagnetic waves in the isotropic and non-gyrotropic medium.

We have written the expression for the energy loss of the particle in the medium as a sum of two terms W^l and W^{tr}. The first term W^l in (8.9) represents the energy loss of a nonrelativistic electron in the medium, and is caused by radiation of longitudinal electromagnetic waves; the second term W^{tr} represents the energy loss of the electron to excitation of transverse electromagnetic waves in the medium. Frequently in the literature the energy losses of a particle corresponding to the quantity W^l are called polarization losses, or Bohr losses, while the losses associated with the term W^{tr} in (8.9) are called Cerenkov losses. It should be emphasized that such a division, in a certain sense, is purely a convention, since both the first and the second terms in (8.9) apply to energy losses of the charged particle to excitation of longitudinal and transverse waves respectively in the medium. In an anisotropic medium, in which the division of electromagnetic waves into longitudinal and transverse is not possible in general, the concept of a division of losses into polarization and Cerenkov losses loses its meaning.

In the case of the motion of a charge particle in an anisotropic medium, we obtain from formulas (8.1), (8.2), and (8.4) the following expression for the energy loss of the particle per unit length of path:[30]

$$W=\frac{ie^2}{2\pi^2vc^2}\int d\mathbf{k}\,(\mathbf{kv})\left[(v_i\alpha_{ij}^{-1}v_j)+\frac{(v_i\alpha_{ij}^{-1}k_j)(k_i\alpha_{ij}^{-1}v_j)}{1-k_i\alpha_{ij}^{-1}k_j}\right], \quad (8.14)$$

where

$$\alpha_{ij}=k^2\delta_{ij}-\frac{(\mathbf{kv})^2}{c^2}\,\varepsilon_{ij}(\omega,\mathbf{k}).$$

For a non-absorbing medium the energy losses of a particle in the medium will be determined by the poles of the integrand in (8.14). These poles coincide with the roots of (6.3), which is the dispersion equation for electromagnetic waves in the anisotropic medium.

Formula (8.14) simplifies considerably in the case where the medium can be assumed to satisfy the inequality $(v^2/c^2)\,\epsilon_{ij}(\omega,\mathbf{k}) \ll 1$ for all values of the arguments ω and $\mathbf{k}$. Then formula (8.14) takes the following form:

$$W=-\frac{ie^2}{2\pi^2v}\int\frac{(\mathbf{kv})\,d\mathbf{k}}{k_i\varepsilon_{ij}(\mathbf{kv},\mathbf{k})\,k_j}. \quad (8.15)$$

Since expression (8.15) is obtained from (8.14) by a formal transition to the limit $c\to\infty$, one says that it determines the total nonrelativistic energy loss of a charged particle in an anisotropic medium.

From the above it follows that the division of the losses into longitudinal and transverse, as is the case for an isotropic medium, becomes meaningless for the case of an anisotropic medium. We may, however, speak of the nonrelativistic losses of the fast particle, as given by expression (8.15), and the total losses of the particle, as given by formula (8.14).

Let us now explain what characteristic features for energy loss in isotropic and non-gyrotropic media result from inclusion of spatial dispersion. Let us consider each term in formula (8.9) individually.

As we have already stated above, the quantity W^l represents the energy loss of a charged particle due to radiation of longitudinal waves in the medium. Suppose that a particle with momentum $\mathbf{p}$, as a result of interaction with the medium, radiates a longitudinal electromagnetic wave with frequency ω and wave vector $\mathbf{k}$, and is then scattered through an angle $\vartheta \ll 1$. In the language of quantum mechanics, such a wave may be called a longitudinal quantum with energy $\hbar\omega$ and momentum $\hbar\mathbf{k}$. From the laws of conservation of energy and momentum, we have

$$\hbar^2k^2=p^2\vartheta^2+\frac{\hbar^2\omega^2}{v^2}=(\Delta p)^2+p^2\vartheta^2. \quad (8.16)$$

From this, using the notations $k^2 = q^2 + \omega^2/v^2$, we have $q = p\vartheta/\hbar$. It should be remarked that such a quantum mechanical treatment applies only in regions of transparency of the medium, when the imaginary parts of ω and $\mathbf{k}$ are small. From formula (8.10) we find the following expression for the probability of scattering of a fast particle through an angle $\vartheta \ll 1$ with emission of a longitudinal quantum of frequency ω, per unit time of motion in the medium:

$$\frac{v\,dW^l}{\hbar\,\omega\,d\omega\,\vartheta\,d\vartheta}=-\frac{2e^2}{\pi\hbar v}\,\frac{1}{\vartheta^2+\left(\frac{\hbar\omega}{pv}\right)^2}\,\mathrm{Im}\,\frac{1}{\varepsilon^l\left(\omega,\sqrt{\left(\frac{p\vartheta}{\hbar}\right)^2+\frac{\omega^2}{v^2}}\right)}. \quad (8.17)$$

Using the relation $\omega = \mathbf{k}\cdot\mathbf{v}$, and also formula (8.16), it is easy to obtain from the expression (8.17) the probability of emission per unit time of a longitudinal quantum with wave vector **k** by a fast electron moving in the medium with velocity **v**. For non-absorbing media this probability is equal to

$$\frac{v\,dW^l}{\hbar\omega\,d\mathbf{k}}=\frac{e^2}{2\pi\hbar}\frac{\delta\left[\varepsilon^l(\mathbf{kv},\mathbf{k})\right]}{k^2}. \tag{8.18}$$

When we neglect spatial dispersion, formula (8.17) becomes the following:[2]

$$\frac{v\,dW^l}{\hbar\omega\,d\omega\vartheta\,d\vartheta}=-\frac{2e^2}{\pi\hbar v}\frac{1}{\vartheta^2+\left(\frac{\hbar\omega}{pv}\right)^2}\operatorname{Im}\frac{1}{\varepsilon(\omega)}, \tag{8.19}$$

where $\epsilon(\omega)=\epsilon^l(\omega,0)$. Comparing formulas (8.17) and (8.19) we conclude that including spatial dispersion changes the angular dependence of the probability of scattering of a fast particle in a medium with emission of a longitudinal quantum. In certain cases, for not too small angles of scattering, the difference between (8.17) and (8.19) may be very important.

Formula (8.18) for the probability of emission of a longitudinal quantum, when we neglect spatial dispersion, takes the form

$$\frac{v\,dW^l}{\hbar\omega\,d\mathbf{k}}=\frac{e^2}{2\pi\hbar}\frac{\delta\left[\varepsilon^l(\mathbf{kv})\right]}{k^2}. \tag{8.20}$$

Finally, we give one more form for the expression for the longitudinal (polarization) loss (8.10) which is useful when the spatial dispersion can be neglected. Expressing the dielectric permittivity $\epsilon(\omega)$ in terms of the index of refraction n' and the absorption coefficient κ,

$$\varepsilon(\omega)=(n'+i\varkappa)^2=n^2(\omega)$$

and integrating with respect to q, we get from formula (8.10)

$$W^l=\frac{2e^2}{\pi v^2}\int_0^\infty\omega\,d\omega\frac{2n'\varkappa}{(n'^2+\varkappa^2)^2}\ln\frac{q_0 v}{\omega}. \tag{8.21}$$

The upper limit of integration over q, the quantity q_0, is determined from the condition that one can neglect spatial dispersion of the longitudinal dielectric permittivity in expression (8.10).

In a completely analogous fashion we obtain from formula (8.11) the expression for the probability of scattering of a fast particle per unit time with emission of a transverse quantum of frequency ω:

$$\frac{v\,dW^l}{\hbar\omega\,d\omega\vartheta\,d\vartheta}=\frac{2e^2p^2}{\pi\hbar^3c^2}\frac{\vartheta^2}{\vartheta^2+\left(\frac{\hbar\omega}{pv}\right)^2}\times\operatorname{Im}\frac{1}{\left(\frac{p\vartheta}{\hbar}\right)^2+\omega^2\left[\frac{1}{v^2}-\frac{1}{c^2}\varepsilon^{tr}\left(\omega,\sqrt{\left(\frac{p\vartheta}{\hbar}\right)^2+\frac{\omega^2}{v^2}}\right)\right]}. \tag{8.22}$$

Just as in (8.17), the angular dependence in expression (8.22) differs from the angular dependence of the corresponding expression obtained when we neglect spatial dispersion. However, unlike the case of (8.17), the inclusion of spatial dispersion in (8.22) for a nonrelativistic particle leads to a weak effect, since, as one sees from formula (8.22), the changes refer only to the small term of order $(v^2/c^2)\epsilon^{tr}$. Besides, the contribution of expression (8.22) to the total probability of scattering of a particle in a medium through an angle $\vartheta \ll 1$ with radiation of electromagnetic waves is a small quantity of order v^2/c^2. For a relativistic particle, the effect of spatial dispersion may become significant. The inclusion of spatial dispersion has an important effect on the spectral and angular distribution of transverse radiation of a particle. In order to convince oneself of this, let us consider the case of a non-absorbing medium. Then

$$W^{tr}=-\frac{2e^2}{c^2}\int_0^\infty\omega\,d\omega\times\int_0^\infty\frac{q^3\,dq}{q^2+\frac{\omega^2}{v^2}}\delta\left(q^2+\omega^2\left[\frac{1}{v^2}-\frac{1}{c^2}\varepsilon^{tr}\left(\omega,\sqrt{q^2+\frac{\omega^2}{v^2}}\right)\right]\right). \tag{8.23}$$

Neglecting spatial dispersion in this expression, we have

$$W^{tr}=-\frac{2e^2}{c^2}\int_0^\infty\omega\,d\omega\int_0^\infty\frac{q^3\,dq}{q^2+\frac{\omega^2}{v^2}}\delta\left(q^2+\omega^2\left[\frac{1}{v^2}-\frac{1}{c^2}\varepsilon(\omega)\right]\right), \tag{8.24}$$

where $\epsilon(\omega)=\epsilon^{tr}(\omega,0)$.

From expression (8.24) it follows that there is Cerenkov radiation with frequency ω only if the condition

$$v\geqslant\frac{c}{\sqrt{\varepsilon(\omega)}}=\frac{c}{n(\omega)},$$

is satisfied, where $n(\omega)$ is the index of refraction of the radiated transverse wave. After integration over q, we obtain from (8.24) the final formula

$$W^{tr}=\frac{e^2}{c^2}\int_0^\infty\omega\,d\omega\left(1-\frac{c^2}{v^2n^2(\omega)}\right), \tag{8.25}$$

which determines the total intensity of the Cerenkov radiation.

Introducing the angle θ between the direction of motion of the charged particle and the direction of propagation of the Cerenkov radiation of frequency ω, and noting that for a Cerenkov wave $\omega=\mathbf{k}\cdot\mathbf{v}=kv\cos\theta=n(\omega/c)v\cos\theta$, we find that this radiation is distributed over the surface of a cone with opening angle

$$\cos\theta=\frac{c}{vn(\omega)}. \tag{8.26}$$

When we include spatial dispersion, we obtain from formula (8.23) the following condition for radiation of a Cerenkov wave with frequency ω:

$$v>\frac{c}{n_i(\omega)},$$

where $n_i(\omega)$ is one of the roots of equation (6.8), which is the dispersion equation for transverse electromagnetic waves of the medium. Here the radiation is distributed over the surface of a cone with opening angle θ_i:

$$\cos\theta_i=\frac{c}{vn_i(\omega)}. \tag{8.27}$$

Since equation (6.8) has in general several roots, the Cerenkov radiation with frequency ω may be distributed over the surfaces of several cones; whereas when we neglect spatial dispersion, all of the radiation is distributed over the surface of a single cone.[13,30]

9. FLUCTUATIONS OF THE ELECTROMAGNETIC FIELD

Fluctuation oscillations of the density in a material medium give rise to local spontaneous, or, as one says, stochastic currents $\mathbf{j}_{st}$, which produce a fluctuating electromagnetic field. In place of the stochastic currents it is more convenient to introduce a stochastic induction $\mathbf{K}$, which is related to $\mathbf{j}_{st}$ by the relation

$$\mathbf{j}_{st}=\frac{1}{4\pi}\frac{\partial \mathbf{K}}{\partial t}. \quad (9.1)$$

These quantities are also called "external" to emphasize the fact that in the Maxwell equations for the fluctuation field they play the role of external sources of the field.

The theory of fluctuations of the electromagnetic field in material media, when one includes only frequency dispersion of the dielectric permittivity, has been developed in detail in references 2 and 34—38. The theory of electromagnetic fluctuations in media, when one takes account of spatial dispersion, is at present the subject of only a small number of papers.[6,39–43] In this section we shall explain briefly the results of these papers.

Suppose that there appears in a medium a stochastic current with frequency ω. This current, regarded as an external source, gives rise to fluctuating electromagnetic fields in the medium. Following the same arguments given in reference 2 (Sec. 90), we get the following expression for the correlation of the stochastic currents $\mathbf{j}_{st}$:[40]

$$(\mathbf{j}_{st\,i}(\mathbf{r})\,\mathbf{j}_{st\,j}(\mathbf{r}'))_\omega=-\frac{\hbar\omega}{4\pi}\left[\sigma_{ij}(\omega,\mathbf{r},\mathbf{r}')+\sigma^*_{ji}(\omega,\mathbf{r}',\mathbf{r})\right]\operatorname{cth}\frac{\hbar\omega}{2\varkappa T}. \quad (9.2)^*$$

Using relations (1.11) and (9.1), we obtain from expression (9.2) the correlation formula for the fluctuations of the stochastic induction in the medium[6]

$$(K_i(\mathbf{r})\,K_j(\mathbf{r}'))_\omega=i\hbar\operatorname{cth}\frac{\hbar\omega}{2\varkappa T}\left[\varepsilon^*_{ji}(\omega,\mathbf{r}',\mathbf{r})-\varepsilon_{ij}(\omega,\mathbf{r},\mathbf{r}')\right]. \quad (9.3)$$

It should be noted that non-locality of the correlations of the random currents and random induction is caused by spatial dispersion. When the spatial dispersion can be neglected

$$\varepsilon_{ij}(\omega,\mathbf{r},\mathbf{r}')=\varepsilon_{ij}(\omega)\,\delta(\mathbf{r}-\mathbf{r}'),\quad \sigma_{ij}(\omega,\mathbf{r},\mathbf{r}')=\sigma_{ij}(\omega)\,\delta(\mathbf{r}-\mathbf{r}').$$

Formulas (9.2) and (9.3) in this case take their usual form[2]

$$(j_{st\,i}(\mathbf{r})\,j_{st\,j}(\mathbf{r}'))_\omega=-\frac{\hbar\omega}{4\pi}\operatorname{cth}\frac{\hbar\omega}{2\varkappa T}\left(\sigma_{ij}(\omega)+\sigma^*_{ji}(\omega)\right)\delta(\mathbf{r}-\mathbf{r}'), \quad (9.4)$$

$$(K_i(\mathbf{r})\,K_j(\mathbf{r}'))_\omega=i\hbar\operatorname{cth}\frac{\hbar\omega}{2\varkappa T}\left(\varepsilon^*_{ji}(\omega)-\varepsilon_{ij}(\omega)\right)\delta(\mathbf{r}-\mathbf{r}'). \quad (9.5)$$

Let us now consider the case of a spatially homogeneous, isotropic, and non-gyrotropic medium. Using a Fourier expansion in the coordinates and remembering relation (2.11), we obtain from formulas (9.2) and (9.3)

$$(j_{st\,i}(\mathbf{k})\,j_{st\,j}(\mathbf{k}'))_\omega=-\frac{\hbar\omega}{(2\pi)^4}\operatorname{cth}\frac{\hbar\omega}{2\varkappa T}\left\{\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\operatorname{Re}\sigma^{tr}(\omega,k)+\frac{k_ik_j}{k^2}\operatorname{Re}\sigma^{l}(\omega,k)\right\}\delta(\mathbf{k}+\mathbf{k}'), \quad (9.6)$$

$$(K_i(\mathbf{k})\,K_j(\mathbf{k}'))_\omega=\frac{2\hbar}{(2\pi)^3}\operatorname{cth}\frac{\hbar\omega}{2\varkappa T}\left\{\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\operatorname{Im}\varepsilon^{tr}(\omega,k)+\frac{k_ik_j}{k^2}\operatorname{Im}\varepsilon^{l}(\omega,k)\right\}\delta(\mathbf{k}+\mathbf{k}'). \quad (9.7)$$

In order to determine the correlation of the fluctuations of the electric field in the medium, it is necessary to solve Maxwell's equations in which the stochastic inductions play the part of external sources. For an isotropic and non-gyrotropic medium, we have

$$\left\{k^2\delta_{ij}-k_ik_j-\frac{\omega^2}{c^2}\left[\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\varepsilon^{tr}+\frac{k_ik_j}{k^2}\varepsilon^{l}\right]\right\}E_j(\mathbf{k})=\frac{\omega^2}{c^2}K_i(\mathbf{k}). \quad (9.8)$$

From the solution of this equation, taking account of formula (9.7), we get[42]

$$(E_i(\mathbf{k})\,E_j(\mathbf{k}'))_\omega=\frac{2\hbar}{(2\pi)^3}\operatorname{cth}\frac{\hbar\omega}{2\varkappa T}\,\psi_{ij}(\omega,\mathbf{k})\,\delta(\mathbf{k}+\mathbf{k}'), \quad (9.9)$$

where*

$$\psi_{ij}(\omega,\mathbf{k})=\frac{k_ik_j}{k^2}\frac{\operatorname{Im}\varepsilon^{l}}{|\varepsilon^{l}|^2}+\frac{\omega^4}{c^4}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\frac{\operatorname{Im}\varepsilon^{tr}}{\left|k^2-\frac{\omega^2}{c^2}\varepsilon^{tr}\right|^2}. \quad (9.10)$$

Finally, using the inverse Fourier transformation we find the spatial correlation of the electric field in an isotropic and non-gyrotropic medium:

$$(E_i(\mathbf{r})\,E_j(\mathbf{r}'))_\omega=2\hbar\operatorname{cth}\frac{\hbar\omega}{2\varkappa T}\,\Phi_{ij}(\omega,\mathbf{r}-\mathbf{r}'), \quad (9.11)$$

where

$$\Phi_{ij}(\omega,\mathbf{R})=\frac{1}{(2\pi)^3}\int d\mathbf{k}\,e^{i\mathbf{k}\mathbf{R}}\psi_{ij}(\omega,\mathbf{k}). \quad (9.12)$$

We note that the first term on the right of (9.9) corresponds to the correlation of the longitudinal field, while the second term is caused by the transverse field. From this the correlation formula for the longitudinal field can be written in the form[42]

$$(\mathbf{E}_{\parallel}(\mathbf{r})\,\mathbf{E}_{\parallel}(\mathbf{r}'))_\omega=2\hbar\operatorname{cth}\frac{\hbar\omega}{2\varkappa T}\int\frac{d\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}\frac{\operatorname{Im}\varepsilon^{l}}{|\varepsilon^{l}|^2}. \quad (9.13)$$

Similarly, we can write a correlation formula for the transverse electromagnetic field. Contracting on the indices i and j in formula (9.11) we get

$$(\mathbf{E}(\mathbf{r})\,\mathbf{E}(\mathbf{r}'))_\omega=2\hbar\operatorname{cth}\frac{\hbar\omega}{2\varkappa T}\int\frac{d\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\mathbf{R}}\left\{\frac{\operatorname{Im}\varepsilon^{l}}{|\varepsilon^{l}|^2}+2\frac{\omega^4}{c^4}\frac{\operatorname{Im}\varepsilon^{tr}}{\left|k^2-\frac{\omega^2}{c^2}\varepsilon^{tr}\right|^2}\right\}, \quad (9.14)$$

*cth = coth.

*We note that the inequality (4.13) follows from (9.9) and (9.10).

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. If in this expression we neglect the spatial dispersion and make use of relation (3.9), we get the familiar form[2]

$$(\mathbf{E}(\mathbf{r})\mathbf{E}(\mathbf{r}'))_\omega = 2\hbar \operatorname{cth}\frac{\hbar\omega}{2\varkappa T}\left\{\frac{\operatorname{Im}\varepsilon(\omega)}{|\varepsilon(\omega)|^2}\delta(\mathbf{R}) - \frac{1}{4\pi}\frac{i\omega^2}{Rc^2}\left[e^{-\frac{\omega}{c}\sqrt{-\varepsilon(\omega)}R} - e^{-\frac{\omega}{c}\sqrt{-\varepsilon^*(\omega)}R}\right]\right\}. \quad (9.15)$$

One of the peculiarities of this formula is the presence of the δ function in front of the term proportional to the imaginary part of the dielectric permittivity $\epsilon(\omega)$, which leads to an infinitely large fluctuation of the longitudinal field in absorbing media. The expression (9.14) does not contain a singularity of the form $\delta(\mathbf{R})$. The second peculiar feature of formula (9.15) is the presence of a divergence of the fluctuations of the longitudinal field for $\epsilon(\omega) = 0$, i.e., when both the real and imaginary part of the dielectric permittivity are equal to zero. The physical reason for this divergence is quite simple.[42] The point is that the condition $\epsilon(\omega) = 0$ is the condition for having longitudinal oscillations where the frequency of the oscillation is independent of wave vector. Consequently, to one frequency of the longitudinal oscillations there corresponds an infinite number of waves with arbitrary wave vectors. This last point means that the fluctuation longitudinal field with the frequency of the longitudinal oscillations corresponds to a thermal excitation of an infinite number of degrees of freedom, which results in a singularity in formula (9.15) at a frequency equal to the frequency of the longitudinal vibrations. It is easily understood that when we include spatial dispersion, in which case the frequency of longitudinal waves becomes a function of the wave vector, the singularity described above cannot occur. In fact, from formula (9.11), we see that the correlation of fluctuations of the longitudinal field has no singularity for $\epsilon^l(\omega, k) = 0$. This is caused by the fact that when we include spatial dispersion, longitudinal waves, like transverse waves, become a legitimate branch of the normal modes in the medium and give rise to effects similar to those resulting from transverse electromagnetic waves.

In the case of a non-absorbing medium, it is easy to obtain from expression (9.9) the following formulas for the fluctuations of the longitudinal and transverse fields in the medium:

$$(\mathbf{E}^l(\mathbf{k}))^2 = \frac{\hbar}{(2\pi)^2}\operatorname{cth}\frac{\hbar\omega}{2\varkappa T}\delta[\varepsilon^l(\omega, k)],$$
$$(\mathbf{E}^{tr}(\mathbf{k}))^2 = \frac{2\hbar}{(2\pi)^2}\operatorname{cth}\frac{\hbar\omega}{2\varkappa T}\delta\left[\varepsilon^{tr}(\omega, k) - \frac{c^2k^2}{\omega^2}\right]. \quad (9.16)$$

Substituting these expressions in formula (4.19), which determines the density of energy of the electromagnetic field in a non-absorbing medium, we obtain[43]

$$\frac{dW^l}{d\omega} = \frac{\hbar\omega^3}{4\pi^2c^3}\operatorname{cth}\frac{\hbar\omega}{2\varkappa T}\sum_i (n_\parallel^i(\omega))^2\frac{d}{d\omega}(\omega n_\parallel^i(\omega)),$$
$$\frac{dW^{tr}}{d\omega} = \frac{\hbar\omega^3}{2\pi^2c^2}\operatorname{cth}\frac{\hbar\omega}{2\varkappa T}\sum_i (n_\perp^i(\omega))^2\frac{d}{d\omega}(\omega n_\perp^i(\omega)), \quad (9.17)$$

where $n_\parallel^i$ and $n_\perp^i$ are the indices of refraction for longitudinal and transverse waves, i.e., for the solutions of (6.7) and (6.8) respectively.

In conclusion, we consider the question of the symmetry of the dielectric permittivity tensor of a medium. We use the property of temporal symmetry of the fluctuations of the electric field

$$(E_i(\mathbf{r})E_j(\mathbf{r}'))_\omega = (E_j(\mathbf{r}')E_i(\mathbf{r}))_\omega. \quad (9.18)$$

A completely analogous symmetry property applies to the fluctuations of the random induction $\mathbf{K}$. A consequence of this, according to (9.5), is the identity

$$\varepsilon_{ij}(\omega, \mathbf{r}, \mathbf{r}') - \varepsilon_{ji}^*(\omega, \mathbf{r}', \mathbf{r}) = \varepsilon_{ij}(\omega, \mathbf{r}', \mathbf{r}) - \varepsilon_{ij}^*(\omega, \mathbf{r}, \mathbf{r}'). \quad (9.19)$$

From this we have

$$\varepsilon_{ij}'(\omega, \mathbf{r}, \mathbf{r}') = \varepsilon_{ji}'(\omega, \mathbf{r}', \mathbf{r}), \quad (9.20)$$

where $\epsilon_{ij}'(\omega, \mathbf{r}, \mathbf{r}')$ is the real part of the tensor $\epsilon_{ij}(\omega, \mathbf{r}, \mathbf{r}')$. But the real and imaginary parts of the dielectric tensor of a medium which is in an equilibrium state are related to one another by the linear integral relations — the Kramers-Kronig formulas. In fact, $\epsilon_{ij}(\omega, \mathbf{r}, \mathbf{r}')$ as a function of ω, defined by means of the one-sided Fourier transformation (2.2), is analytic everywhere in the upper half-plane of the complex variable ω, except possibly for a band of finite width, $\operatorname{Im}\omega \geq \sigma \geq 0$. For media which are in an equilibrium state, $\sigma = +0$ (cf. Sec. 5). Therefore for such media

$$\varepsilon_{ij}(\omega, \mathbf{r}, \mathbf{r}') - \delta_{ij} = \frac{1}{\pi i}\int_{-\infty}^{+\infty} P\frac{\varepsilon_{ij}(\omega', \mathbf{r}, \mathbf{r}') - \delta_{ij}}{\omega' - \omega}d\omega', \quad (9.21)$$

where P means that the integral is to be understood in the sense of a principal value. Separating the real and imaginary parts in (9.21), we obtain the familiar Kramers-Kronig formulas*

*For an isotropic medium the Kramers-Kronig relations (9.21) hold both for the longitudinal, $\varepsilon^l(\omega, k)$, and for the transverse, $\varepsilon^{tr}(\omega, k)$, dielectric permittivities. Therefore according to (2.22) we get for the magnetic permeability

$$1 - \left[\frac{1}{\mu(\omega, k)}\right]' = \frac{1}{\pi}\int_{-\infty}^{+\infty} P\frac{d\omega'}{\omega' - \omega}\cdot\frac{\mu''(\omega', k)}{|\mu(\omega', k)|^2},$$

$$\frac{\mu''(\omega, k)}{|\mu(\omega, k)|^2} = -\frac{1}{\pi}\int_{-\infty}^{+\infty} P\frac{d\omega'}{\omega' - \omega}\left\{1 - \left[\frac{1}{\mu(\omega', k)}\right]'\right\}.$$

If the function $\mu''(\omega, k)$ does not have a singularity for $\omega = 0$, then from these formulae we have in particular the following relation for the static magnetic permeability of the isotropic medium:

$$\left[\frac{1}{\mu(\omega, k)}\right]' = 1 - \frac{2}{\pi}\int_0^\infty \frac{d\omega'}{\omega'}\frac{\mu''(\omega', k)}{|\mu(\omega', k)|^2}.$$

In Sec. 4 we have already mentioned that the quantity $\mu''(\omega, k)$, unlike $\varepsilon^{l''}(\omega, k)$ and $\varepsilon^{tr''}(\omega, k)$, may be either positive or negative. In fact, the inequality $\mu''(\omega, k) > 0$ [or $\mu''(\omega, k) < 0$] would mean the impossibility of existence of diamagnetic (or paramagnetic) media.

$$\varepsilon'_{ij}(\omega, \mathbf{r}, \mathbf{r}') - \delta_{ij} = \frac{1}{\pi} \int_{-\infty}^{+\infty} P \frac{\varepsilon''_{ij}(\omega', \mathbf{r}, \mathbf{r}')}{\omega' - \omega} d\omega',$$

$$\varepsilon''_{ij}(\omega, \mathbf{r}, \mathbf{r}') = -\frac{1}{\pi} \int_{-\infty}^{+\infty} P \frac{\varepsilon'_{ij}(\omega', \mathbf{r}, \mathbf{r}') - \delta_{ij}}{\omega' - \omega} d\omega'. \quad (9.22)$$

From these relations it follows that the imaginary part $\epsilon''_{ij}(\omega, \mathbf{r}, \mathbf{r}')$ as well as the real part $\epsilon'_{ij}(\omega, \mathbf{r}, \mathbf{r}')$, has the symmetry property (9.20). We thus arrive at the final result

$$\varepsilon_{ij}(\omega, \mathbf{r}, \mathbf{r}') = \varepsilon_{ji}(\omega, \mathbf{r}', \mathbf{r}). \quad (9.23)$$

In the case of an unbounded and spatially homogeneous medium, we obtain from this the following symmetry property of the dielectric permittivity tensor:[44]

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \varepsilon_{ji}(\omega, -\mathbf{k}). \quad (9.24)$$

The form of the relations (9.23) and (9.24) is changed somewhat if there is a constant magnetic field $\mathbf{B}_0$ in the medium produced by external sources. In this case, when we reverse the sign of the time, we must also make the change $\mathbf{B}_0 \to -\mathbf{B}_0$. Therefore, in place of the relations (9.23) we obtain

$$\varepsilon_{ij}(\omega, \mathbf{k}, \mathbf{B}_0) = \varepsilon_{ji}(\omega, -\mathbf{k}, \mathbf{B}_0). \quad (9.25)$$

In the case of an unbounded and spatially homogeneous medium, we have

$$\varepsilon_{ij}(\omega, \mathbf{r}, \mathbf{r}', \mathbf{B}_0) = \varepsilon_{ji}(\omega, \mathbf{r}', \mathbf{r}, -\mathbf{B}_0). \quad (9.26)$$

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