STABILITY OF PLASMA

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CONTENTS

I. INTRODUCTION

PLASMA stability, to which the present review is devoted, is no less important, both theoretically and practically, in plasma physics than the corresponding question of stability in ordinary hydrodynamics, and its solution is probably more difficult.

One of the most important applications of the theory of plasma stability is in the problem of controllable thermonuclear fusion. Even to produce and to heat a plasma under laboratory conditions, we must be able to insulate it, and this is possible only with the aid of external or self-magnetic fields. These fields must decrease sufficiently the diffusion of heat and of particles from the region occupied by the plasma. Such a plasma state can in a certain sense be called metastable.

However, very stringent conditions must be satisfied if the metastable state is to exist a sufficiently long time. It is essential here to prevent the excitation of both macroscopic degrees of freedom (which would lead to the escape of the plasma as a whole from the occupied region), and microscopic ones, since the fields produced by the build-up of oscillattions increase the diffusion of the particles sharply.

In addition, many heating methods are usually connected with excitation of individual degrees of freedom of a plasma. For example, heating of a plasma situated in a stationary magnetic field by means of a field of frequency close to the cyclotron frequency of the ions or electrons (called heating by cyclotron resonance) increases only the energy of rotation of the particle in the Larmor orbits in the field; heating with current increases the kinetic energy of the electrons, etc. Instability may set in in this case, too, and a considerable portion of the energy may go to the field of the oscillations. The field, in turn, again increases the diffusion of the particles.

Only a theory of laminar plasma motion exists at present, and the limits of its applicability can be evaluated by stability theory. Moreover, stability theory is an essential basis for the development of a theory for turbulent plasma.

Directly related to stability theory is dissipation of energy in the plasma during time intervals considerably shorter than the pair-collision times, i.e., the question of the so-called "collisionless" dissipation in plasma. The mechanisms of this dissipation, which are connected with the exchange of energy between different degrees of freedom, must also be known before a theory can be developed for turbulent plasma.

The authors of this article did not aim at a detailed analysis of the mathematical formalism of stability theory; we tried to report the principal physical results of stability theory, derived from well-drawn physical considerations. In the few cases when, in our opinion, the formal approach attains the purpose more rapidly, we do not engage in qualitative considerations.

The choice of the material and its arrangement in the review were aimed at establishing, where possible, *the internal connections* between the physical causes of various types of plasma instability. In the introductory chapter we discuss the formulation of the problem and explain briefly the various approaches to the stability problem (including the choice of approximations) and methods of describing the plasma.

Chapter II is devoted to the so-called aperiodic type of instability (where the departure from equilibrium increases monotonically with time). Instabilities of this type characterize essentially static plasma equilibrium configurations that are contained by the pressure of the magnetic field. This applies, for example, to stability of a plasma in a gravitational field (the Kruskal-Schwartzschild problem) and to the closely related problem of stability of a plasma in a "barrel-shaped" magnetic field (Longmire, Rosenbluth), to the stability of a pinch, i.e., a current-carrying plasma column, etc. We consider next the stability of a plasma pinch with allowance for rotation, and the stability of a plasma contained by the pressure of a high-frequency electromagnetic field. In addition, we consider several types of aperiodic instability in a rarefied plasma with nonmaxwellian velocity distribution in a magnetic field.

The second part of Chapter II is devoted to the influence of dissipative effects on aperiodic plasma instability, viz., stability of a pinch with allowance for finite conductivity, and stability of rotation of a real plasma.

Oscillating plasma instabilities and the conditions under which they are produced are described in Chapter III (in these cases the system oscillates with a *continuously increasing amplitude after departing from* the initial state). The problem discussed there concerns so-called "beam" instability of a rarefied plasma and of instability of nonmaxwellian plasma in general. Criteria are derived for the stability under the launching of various types of oscillations.

Certain oscillating instabilities are characteristic of plasma currents in a magnetic field. This question is considered in the magnetohydrodynamic approximation in Sec. 10 of Chapter III. In conclusion, we consider the conditions under which periodic convection arises in an unevenly heated plasma placed in a magnetic field, as well as the question of the instability of the positive column of a gas discharge in a magnetic field.

The last chapter, IV, is devoted to a discussion of nonlinear effects in the theory of plasma instability. The so-called quasilinear method is developed and applied both to problems in hydrodynamics of plasma (convection), and to problems involved in the build-up of oscillations in a rarefied plasma. These effects are particularly important for an estimate of the influence of the instability on transport processes in the plasma.

Individual mathematical derivations for various chapters are contained in the appendices.

1. Linear and Nonlinear Stability Theory

The stability of any system is investigated by the perturbation method. If the initial perturbation of the stationary state of the system increases with time, the state is unstable under a perturbation of this type.

The question of instability was investigated most thoroughly only as applied to small perturbations. The deviations from the initial stationary state are considered here to be so small that the equations for these deviations can be linearized, i.e., expanded in powers of the perturbation amplitude and all terms of order higher than the first neglected. The present review is devoted essentially to linear stability theory.

In view of the linearity of the equations of the theory, it is natural to employ the method of Fourier expansion in terms of time and of the space coordinates in which the system is homogeneous. The problem reduces in this case to an investigation of the behavior of the individual Fourier component of some physical quantity. Where such an analysis leads to misunderstanding, it becomes necessary to turn to a current solution of the problem with initial conditions taken into account. Usually, however, one can seek a solution of the system of equations simply in the form

$$F(\mathbf{r},\ t) = F(\mathbf{r})\, e^{-i\omega t},$$

where $\omega = \omega_r + i\omega_i$ is the complex frequency and F is the deviation of any physical quantity from its stationary value.

The linear theory of stability does not distinguish in principle between metastable and stable states, i.e., it is not applicable in the case when two stationary states are separated by a barrier. The latter problem is already part of the nonlinear stability theory. With the exception of several particular cases pertaining to ordinary hydrodynamics, there is no such "subcritical" nonlinear theory of stability.

Great progress was made in the development of "supercritical" stability theory. This theory deals with the effect of perturbations on the average "background"* and with the development of small perturbations against this background. Either energy considerations or the usual methods of perturbation theory are employed. The physical meaning of this approximation is that distortion of the average background

*We use the term "background" to describe the stationary state. In this case the stationary state is the one averaged over many periods of small oscillations.

by the perturbation decreases the energy transfer to the perturbations. A balance is established at some finite perturbation amplitude between the perturbation energy flux and the flux of the energy dissipated in the perturbations. This establishes the stationary amplitude of the perturbations (see reference 1 for a general formulation of the problem).

This method is valid when the critical stability conditions are exceeded slightly. If λ is some parameter (the critical value of which is λ_c), then the motion consists of a stationary component plus oscillations whose amplitude is proportional to the square root of $\lambda - \lambda_c$. Under these conditions each mode develops independently and the interaction between modes can be neglected. Further deviation from the critical state causes interaction and energy exchange to set in between the different modes.

The question of the conditions that lead to turbulence in the system is beyond the scope of the present review.

2. Oscillating and Aperiodic Instabilities

Physically, it is meaningful to distinguish between two kinds of instability:

a) aperiodic departure, in which the deviation from equilibrium position increases monotonically with time, and

b) oscillations whose amplitude increases with time.

Mathematically the condition under which the departure is aperiodic is $\mathrm{Re}\,\omega = 0$, so that all the quantities that characterize the deviation from equilibrium have the form

$$F(\mathbf{r})\,e^{\omega_i t},$$

where ω_i is a real number. A criterion for such an instability is best formulated in energy terms. The system is unstable if there exist perturbations that cause its potential energy to decrease.

An instability is called oscillating if $\mathrm{Re}\,\omega \neq 0$, i.e., oscillating perturbations that build up in time exist in the system. Such an instability can be due to the following causes.

In many cases resonance between a group of plasma particles and the perturbation wave can set in in a coordinate system moving with a certain velocity. Such a phenomenon, which leads to the development of oscillating instability, will be called phase resonance.

An oscillating instability can also be produced as a result of competition between relaxation processes, such as heat conduction and the diffusion of the magnetic field in convection.

Oscillating instabilities differ in character. If small perturbations are initiated in some small region of space and increase in that region without limit as $t \to \infty$ we call such an instability absolute. On the other hand, if these perturbations both increase and move out of the system, this will be called a drift instability.

3. Methods of Describing a Plasma

Before we proceed to report the results accumulated in the extensive literature on plasma stability, let us describe briefly the initial mathematical formalism used to describe a plasma in problems involving the stability of various states of a plasma.

The plasma, being an aggregate of an electron gas and ion gases (several sorts of ions may exist in the plasma),* is frequently described in terms of a different distribution function $f(\mathbf{r}, \mathbf{v}, t)$ for each sort of charge. These distribution functions can in principle be determined by solving a system of Boltzmann equations, in which account is taken not only of particle pair collisions, but also of the action of electric and magnetic fields on the plasma particles. The fields themselves are related in turn, by the Maxwell equations, with the spatial current densities $e\int \mathbf{f}\mathbf{v}\,d\mathbf{v}$ and the charge density $e\int f\,d\mathbf{v}$.

To obtain reasonable physical results, one usually resorts to simplified mathematical models, which permit the use, under certain assumptions, of equations simpler than Boltzmann's.

As in an ordinary gas, we can use the gas dynamic approximation if we consider space scales L considerably greater than the average mean free path L_{av} of the ions (electrons) prior to collision.

Formally, the system of equations of magnetic gas dynamics consists of the equations for the lower moments (up to the third, inclusive) of the ion and electron distribution functions and the Maxwell equations for self-consistent fields, the plasma being assumed quasineutral. The latter assumption, which is valid for scales considerably greater than the Debye radius, is known to be satisfied in the gasdynamic case.

In magnetohydrodynamic stability investigations the plasma can be regarded as an ideal liquid, i.e., we can discard the dissipative terms from the initial magnetohydrodynamic equations. This approximation is meaningful if the processes of interest to us have a duration much shorter than the diffusion time of the field

$$\tau_m = \frac{4\pi\sigma L^2}{c^2},$$

where L is the characteristic dimension and σ the conductivity of the plasma, shorter than the characteristic time of "velocity diffusion"

$$\tau_V = \frac{L^2}{\nu},$$

where ν is the kinematic viscosity, and shorter than the characteristic time of temperature diffusion

*We confine ourselves here from the outset to the "gas" approximation, i.e., we assume the plasma to be an almost perfect gas, which is correct if the Debye radius of the ions (electrons) exceeds appreciably the average distance between particles. However, many results concerning stability, particularly those pertaining to the magnotohydrodynamic approximation, have a much wider range of application.

$$\tau_T = \frac{L^2}{\chi},$$

where χ is the coefficient of temperature conductivity of the plasma. The corresponding dimensionless parameters are called respectively the magnetic Reynolds number

$$\mathrm{Re}_m = \frac{\tau_m}{\tau} = \frac{4\pi\sigma L^2}{c^2\tau},$$

where τ is the time scale, the hydrodynamic Reynolds number

$$\mathrm{Re}_g = \frac{\tau_g}{\tau} = \frac{L^2}{\nu\tau}$$

and the Peclet number

$$\mathrm{Pe} = \frac{\tau_T}{\tau} = \frac{L^2}{\chi\tau}.$$

Thus, in order for the approximation of an "ideal" plasma to be valid, it is necessary to have

$$\mathrm{Re}_m \gg 1, \quad \mathrm{Re}_g \gg 1, \quad \mathrm{Pe} \gg 1.$$

It should be noted that these conditions are sometimes found to be insufficient, for when $R \gg 1$ there exists a whole class of phenomena which are not described by the ideal-plasma model, phenomena connected with the so-called paradox of zero and vanishing dissipation (see Secs. 5a and 10). The ideal magnetohydrodynamic system of equations obtained in this approximation has the form

$$\frac{\partial \varrho}{\partial t} + \operatorname{div} \varrho \mathbf{v} = 0, \qquad (3.1)$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\mathbf{v} = -\frac{1}{\varrho}\nabla p + \frac{1}{4\pi\varrho c}[\operatorname{rot}\mathbf{H}, \mathbf{H}], \qquad (3.2)^*$$

$$\frac{\partial \mathbf{H}}{\partial t} = \operatorname{rot}[\mathbf{v}, \mathbf{H}], \qquad (3.3)$$

$$p = p(\varrho). \qquad (3.4)$$

In this system (3.1) is the continuity equation for the density ρ, (3.2) is the equation of motion describing the variation of the average velocity $\mathbf{v}$ of a plasma element under the influence of a force connected with the pressure drop p and a ponderomotive force $\frac{1}{c}\mathbf{j}\times\mathbf{H} = \frac{1}{4\pi}\operatorname{curl}\mathbf{H}\times\mathbf{H}$, Eq. (3.3) states that the force lines of the magnetic field are "glued" to an ideally conducting plasma, and (3.4) is the equation of state.

The ponderomotive force is conveniently written in the form of the sum

$$\frac{1}{4\pi}[\mathbf{H}, \operatorname{rot}\mathbf{H}] = \nabla\frac{\mathbf{H}^2}{8\pi} - \frac{1}{4\pi}(\mathbf{H},\nabla)\mathbf{H}.$$

The first (potential) term is the gradient of the "magnetic pressure." The second term yields the projection

$$\mathbf{n}\frac{1}{4\pi}(\mathbf{H},\nabla)\mathbf{H} = \frac{\mathbf{H}^2}{4\pi R},$$

where $\mathbf{n}$ is the normal to the force line and R is the radius of curvature of the latter, i.e., the second term is analogous in form to the elastic force that arises when a stretched string is bent. It is therefore called the "tension of the magnetic force lines." This descriptive language will be frequently used henceforth.

*Rot = curl. $[\mathbf{v},\mathbf{H}] = \mathbf{v}\times\mathbf{H}$, $(\mathbf{v},\nabla) = \mathbf{v}\cdot\nabla$.

The introduction of the dissipative effects complicates the system (3.1) — (3.4). Thus, a term

$$\eta\Delta\mathbf{v} + \left(\frac{\eta}{3} + \zeta\right)\operatorname{grad}\operatorname{div}\mathbf{v},$$

due to the influence of viscosity (η and ζ are viscosity coefficients) appears in the right half of (3.2), while the right half of (3.3) contains the term that takes into account the electric resistance of the plasma

$$\frac{c^2}{4\pi\sigma}\Delta\mathbf{H},$$

where σ is the electric conductivity of the plasma and c is the velocity of light in vacuum. The pressure no longer obeys the adiabatic law, and we must use in lieu of (3.4) the two equations $p = p(\rho, T)$ and

$$\frac{\partial}{\partial t}\left\{\varrho\left(\frac{v^2}{2} + c_v T\right) + \frac{H^2}{8\pi}\right\}$$
$$= -\operatorname{div}\left\{\varrho\mathbf{v}\left(\frac{\mathbf{v}^2}{2} + c_p T\right) + \frac{c}{4\pi}[\mathbf{E}, \mathbf{H}] - (\mathbf{v}\hat{\sigma}') - \varkappa\nabla T\right\},$$

where

$$\sigma'_{ik} = \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right) + \zeta\delta_{ik}\frac{\partial v_l}{\partial x_l},$$

κ is the coefficient of heat conduction. The quantities σ, ν, and κ are scalars only when the mean free path is much less than the average Larmor radii of the ions (electrons). In a strong magnetic field, the Larmor radii $r_H \sim vmc/eH$ (v is the average thermal velocity) may become small and our equation is made complicated by the anisotropy of the transport coefficients. A strong magnetic field, for example, decreases the heat flow transverse to the force lines by a factor l^2_{av}/r^2_H.*

If the characteristic spatial scales of interest to us are, to the contrary, much smaller than the mean free path, we can use the Boltzmann kinetic equation without collision integrals to describe the plasma, for in this limiting case each ion and electron of the plasma moves in its own trajectory under the influence of the electric and magnetic fields, which depend in turn on the joint motion of all the electrons and ions

$$\left.\begin{aligned} &\frac{\partial f}{\partial t} + (\mathbf{v}\nabla)f + \left\{\frac{e\mathbf{E}}{m} + \frac{e}{mc}[\mathbf{v},\mathbf{H}]\right\}\frac{\partial f}{\partial \mathbf{v}} = 0,\\ &\operatorname{div}\mathbf{E} = 4\pi e\left(\int f_i\,d\mathbf{v} - \int f_e\,d\mathbf{v}\right), \quad \operatorname{div}\mathbf{H} = 0,\\ &\operatorname{rot}\mathbf{H} = \frac{4\pi}{c}e\left(\int f_i\mathbf{v}\,d\mathbf{v} - \int f_e\mathbf{v}\,d\mathbf{v}\right) + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}, \quad \operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}.\end{aligned}\right\} \qquad (3.\mathrm{I})$$

*It must be noted that, unlike in stability of mechanical equilibrium, allowance for the dissipative terms leads not only to a change in the rate of build-up of the perturbations, but also to a change in the stability condition itself (see Sec. 5 for the reason).

In this form, the system (3.I) permits a practical investigation of stability only in the simplest cases of "homogeneous" or almost "homogeneous" background, i.e., in such idealized equilibrium states, in which the unperturbed distribution of the quantities that describe the plasma depends very weakly on the spatial coordinates. In practice this means that we consider perturbation wavelengths considerably shorter than the characteristic scales of the unperturbed spatial distribution.

The system (3.I), which is based on kinetic equations without account of collisions, can be greatly simplified if the characteristic spatial scales considerably exceed the average Larmor radii of the ions (electrons), and the time scales are much greater than the period of Larmor revolution. In this case the trajectory of each charge in the plasma is a superposition of a slow drift transverse to the force lines, motion along the lines, and rapid Larmor rotation about the magnetic field. Averaging over the rapid rotations, we can obtain in this approximation (called the "drift" approximation)[2–4] simple equations of motion for the center of the Larmor circle of the charge

$$\frac{d\mathbf{r}}{dt} = v_{\|}\frac{\mathbf{H}}{H} + c\frac{[\mathbf{E},\mathbf{H}]}{H^2} + \frac{c}{eH^2}[\mathbf{F},\ \mathbf{H}].$$

Here $v_{\|}$ is the velocity along the force line, while the second term describes the electric drift.

The last term describes the drift under the influence of a force

$$\mathbf{F} = -\mu\nabla H - M(\boldsymbol{\varepsilon}_0\nabla)\boldsymbol{\varepsilon}_0 v_{\|}^2 - \frac{d}{dt}c\frac{[\mathbf{E},\mathbf{H}]}{H^2}M,$$

where $\mu\nabla H$ is the force acting on a particle with magnetic moment $\mu = Mv_{\perp}^2/2H$ in an inhomogeneous magnetic field, and $-M(\epsilon_0\cdot\nabla)\epsilon_0 v_{\|}^2 - \frac{d}{dt}c\frac{\mathbf{E}\times\mathbf{H}}{H^2}M$ is the inertia force, where M is the particle mass. The magnetic moment is conserved in the drift approximation. The equation of motion along the force line has the form

$$M\frac{dv_{\|}}{dt} = -\mu\frac{(\nabla H,\ \mathbf{H})}{H} + e\frac{(\mathbf{E},\ \mathbf{H})}{H}.$$

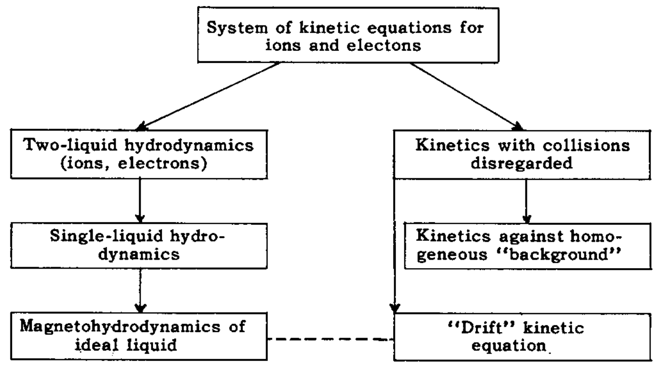
In the drift approximation it is possible to replace the distribution function $f(\mathbf{v}, \mathbf{r}, t)$ by the function $f_{dr}(v_{\|}, \mu, \mathbf{r}, t)$, with one less independent variable. The kinetic equation for $f_{dr}(v_{\|}, \mu, \mathbf{r}, t)$ has obviously the form of a continuity equation in the space $v_{\|}, \mu, \mathbf{r}$:

$$\frac{\partial f_{dr}}{\partial t} + \mathrm{div}_{\mathbf{r}}\left(\frac{d\mathbf{r}}{dt},\ f_{dr}\right) + \frac{\partial}{\partial v_{\|}}\frac{\partial v_{\|}}{\partial t}f_{dr} = 0. \qquad (3.II)$$

The drift equation (3.II), together with Maxwell's equations for the fields **E** and **H**, has a unique feature: when there is no dependence on the spatial coordinate along the force line, the equations for the moments of f_{dr} have the same form as the magnetohydrodynamic equations, apart from the adiabatic exponent γ, which is equal to 2 in the drift approximation. Actually, $p_{\perp} \sim nmv_{\perp}^2 \sim n\mu H \sim n^2$, since μ is a constant and $H \sim n$ is a result of the "gluing" of the force lines.

Thus, even in the absence of pair collisions the equations of magnetohydrodynamics are formally valid for motion transverse to the force lines.[5]

The different mathematical models used for describing the plasma are compared in the following scheme, which enables us to trace the "genealogy" of various approximate methods and their interrelationships.



The fact that a similar mathematical formalism is used, in final analysis, to describe a plasma under the most contradictory limiting cases, $L \gg l_{av}$ and $L \ll l_{av}$, enables us frequently to deduce the stability or instability of a rarefied plasma ($L \ll l_{av}$) simply from the results of the magnetohydrodynamic investigation ($L \gg l_{av}$).

II. APERIODIC PLASMA INSTABILITY

4. Ideal Plasma

4a. The energy principle.[6–8] A remarkable property of the system of linearized equations for small perturbations of an ideal plasma in the magnetohydrodynamic approximation is that they are self adjoint (this takes place when the unperturbed plasma is at rest). It follows from (3.1) — (3.4) that a small displacement of the plasma from the equilibrium position obeys the equation of motion

$$\varrho\ddot{\xi} = \mathbf{E}\{\xi\}, \qquad (4a.1)$$

where

$$\mathbf{F}\{\xi\} = \nabla\{\gamma p\,\mathrm{div}\,\xi + \xi\nabla p\} + \frac{1}{c}[\mathbf{j}\,\mathrm{rot}\,[\xi\mathbf{B}]] - \frac{1}{4\pi}[\mathbf{B},\ \mathrm{rot}\,\mathrm{rot}\,[\xi\mathbf{B}]] + [\mathrm{div}\,p\xi]\nabla\varphi.$$

Here p is the pressure, ρ the density, **B** the intensity of the magnetic field, φ the potential of the external forces, **j** the current density in equilibrium, and $\boldsymbol{\xi}$ the displacement from the equilibrium position. The self adjointness of (4a.1) follows from the fact that the force depends only on the displacement ξ, and not on its derivatives with respect to the time.

Obviously the expression

$$K\{\xi,\ \xi\} = \frac{1}{2}\int dV\varrho\,|\dot{\xi}|^2$$

corresponds to the kinetic energy, while

$$\delta W\{\xi, \xi\} = -\frac{1}{2}\int dV\, \xi \mathbf{F}\{\xi\}$$

corresponds to the potential energy. If we seek solutions in the form of normal oscillations $\exp(-i\omega_n t)$, it follows from (4a.1) that

$$-\omega_n^2 \varrho \xi_n = \mathbf{F}\{\xi_n\}. \quad (4a.2)$$

From the self adjointness of (4a.1) it follows that ω_n^2 is real. Therefore in an ideal plasma at rest only aperiodic instability can exist in the magnetohydrodynamic approximation. Because of the self adjointness of (4a.2), ξ_n could be chosen to be orthonormalized and used to make up a complete system, in which any displacement satisfying the boundary conditions on the plasma surface can be expanded. Therefore for any displacement $\xi = \Sigma a_n \xi_n$ we have $\delta W = \frac{1}{2}\Sigma a_n^2 \omega_n^2$. δW can become negative if and only if there exists at least one imaginary frequency $\omega_n^2 < 0$. Consequently, the solution of the stability problem reduces to a determination of the sign of δW under arbitrary displacements satisfying the boundary conditions, particularly the continuity of the total pressure on the boundary.

This last limitation can be lifted by introducing the generalized energy principle. For this purpose it is necessary, with account of the correct boundary conditions, to express δW as the sum of a volume term (over the volume of the plasma) δW_F, a surface term δW_S, and a vacuum term δW_V:

$$\delta W = \delta W_F + \delta W_S + \delta W_V, \quad (4a.3)$$

where

$$\delta W_F = \frac{1}{2}\int_F dV\Big\{\frac{1}{4\pi}|\operatorname{rot}[\xi\mathbf{B}]|^2 - \frac{1}{c}\mathbf{j}[\operatorname{rot}[\xi\mathbf{B}], \xi]$$

$$+ \gamma p(\operatorname{div}\xi)^2 + (\operatorname{div}\xi)(\xi\nabla p) - (\xi\nabla\varphi)\operatorname{div}(p\xi)\Big\}, \quad (4a.3a)$$

$$\delta W_S = \frac{1}{2}\int_S d\sigma(\mathbf{n}\xi)^2\,\mathbf{n}\langle\nabla(p + B^2 8\pi)\rangle, \quad (4a.3b)$$

$$\delta W_V = \frac{1}{8\pi}\int_V dV H^2. \quad (4a.3c)$$

The first integral is over the volume occupied by the plasma, the second over the boundary between the plasma and the vacuum (n is the normal to the surface and $\langle f\rangle$ denotes the jump in f on going through the surface), and the third over the vacuum (**H** is the field in the vacuum). It is assumed that the force lines do not cross the plasma boundary. It can be shown[8] that the necessary and sufficient condition for instability is the existence of $\boldsymbol{\xi}$, **H**, and **E** which satisfy only the electrodynamic conditions on the separation boundary and make the potential energy (4a.3) negative.

The self-adjoint equation (4a.1) can be derived from the variational principle for

$$\omega^2 = \frac{\delta W\{\xi, \xi\}}{K\{\xi, \xi\}}, \quad \Delta\omega^2 = 0.$$

Using the variational principle, we can determine not only the stability conditions, but also the increments; however, the energy principle, which does not require the normalization condition $K\{\boldsymbol{\xi}, \boldsymbol{\xi}\} = 1$, is much simpler to use.

With the aid of the energy principle we can readily establish several comparison theorems. We give here the following example: If system II differs from system I in that the part occupied in system I by the vacuum is occupied in system II by a plasma of zero pressure, then instability of system II implies instability of system I. To prove this we note that whereas in the region of system I occupied by the vacuum the contribution to δW is due to the term

$$\delta W_V = \frac{1}{8\pi}\int dV\cdot\mathbf{H}^2,$$

where **H** is the field perturbation in the vacuum, in system II we have

$$\delta W_F = \frac{1}{8\pi}\int |\operatorname{rot}[\xi, \mathbf{B}]|^2\, dV.$$

if $\boldsymbol{\xi}$ and H are chosen such that $\delta W_{II} < 0$, then by choosing $\boldsymbol{\xi}_I = \boldsymbol{\xi}_{II}$ and $\mathbf{H}_I = \mathbf{H}_{II}$ everywhere outside the region of the vacuum of the system I, and $H_I = \operatorname{curl}[\boldsymbol{\xi}\times\mathbf{B}]$ in this region, we find that $\delta W_I < 0$. This choice is possible, since $\mathbf{H}_I$ satisfies the electrodynamic conditions on the separation boundary.

Using the energy principle in the form (4a.3) it is easy to show, for example, that a sufficient condition for the stability of a sharp plasma boundary with no internal magnetic field is the inequality $\partial H^2/\partial n > 0$, i.e., for stability, the magnetic pressure must increase everywhere with increasing distance from the plasma boundary.

From energy considerations it follows in the general case that the most dangerous are deformations which do not increase the energy of the magnetic field, i.e., those in which the magnetic force lines are neither "stretched" nor bent. Such deformations are "flutes" along the force lines of the magnetic field, in which $\boldsymbol{\xi}\perp\mathbf{B}_0$.

This last circumstance causes the energy principle, which has been formally derived from linearized equations of the hydrodynamic approximation, to have a sensible meaning also for a rarefied plasma in the drift approximation. Actually, for the most stable perturbations ($\boldsymbol{\xi}\perp\mathbf{B}_0$) the motion is transverse to the force lines, where again hydrodynamics is applicable, but now with non-isotropic pressure. It can be shown that the stability criterion obtained from the variational principle in the drift approximation[9] is somewhat less rigid than the corresponding magnetohydrodynamic criterion.

4b. Stability of the plasma boundary. When it comes to applications, one of the most interesting problems (for example, in controllable fusion) is the stability of a plasma confined by a magnetic field, i.e., the question of the stability of the magnetic insulation of a plasma. This problem is simplest to analyze by means of a simple example, first investigated by Kruskal and Schwarzschild.[10] Consider a layer of plasma bordering on a vacuum. The boundary is maintained in equilibrium by the pressure of the magnetic field $H^2/8\pi$. The ends of the plasma are in contact with ideally conducting plates perpendicular to the magnetic field and spaced a distance L apart (Fig. 1).[48]

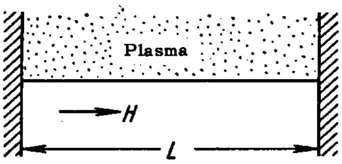


FIG. 1

The fields inside and outside the plasma are parallel. A force $\mathbf{f} = \rho \mathbf{g}$ is perpendicular to these fields. The fields inside and outside the plasma are B_0 and H_0, respectively.

If the plasma boundary is displaced vertically by δz, and if the disturbance has a length on the order of l transverse to the field and, naturally, L along the field, then the pressure on the portion of the boundary that has deviated the most is increased by an amount equal to the weight of a plasma column of height δz:

$$\delta p = \varrho g \delta z. \tag{4b.1}$$

The distortion of the magnetic field produces a quasi-elastic force. If this force is greater than the change in pressure, the equilibrium is stable.

In the vacuum this force is connected with the change of the magnetic pressure $\delta H^2/8\pi = H_0 \delta H_{||}/4\pi$, where $\delta H_{||}$ is the component of the magnetic field disturbances along H_0. The field near the boundary of an ideally conducting plasma remains parallel to the plasma. Therefore, $\delta H_\perp \sim H_0 \delta z/L$. Since the motion is quasi stationary $(v/c \to 0)$, we have outside the plasma $\mathbf{H} = \nabla\varphi$ and $\nabla^2\varphi = 0$. Therefore when $l \ll L$ the characteristic dimension of the disturbance of the field in a direction perpendicular to the boundary is l. Consequently, $\delta H_{||} \sim \delta H_\perp l/L$ and $\delta H^2/8\pi \sim H_0^2 l \delta z/L^2$. Inside the plasma the field is "frozen in" and the volume force reduces to the Maxwellian tension $B^2/4\pi R$, and R is the radius of curvature of the force line; since $R \sim L^2/\delta z$, the quasi elastic volume force is equal to $B_0^2 \delta z/L^2$. Therefore, if the condition

$$\frac{B_0^2 + H_0^2}{L^2} > \frac{g\varrho}{l} \tag{4b.2}$$

is satisfied, the equilibrium is stable.

For any field, perturbations exist with values of l so small as to make the sharp boundary of the plasma unstable. Obviously, such perturbations diffuse the boundary to a width

$$l_b \sim g\varrho L^2/B^2.$$

Actually, when an element of inhomogeneous plasma is displaced, a quasi-elastic force $B^2 \delta z/L^2$ and an Archimedean force $g\delta\rho$ (where $\delta\rho$ is the difference in the densities of the displaced element and surrounding plasma) set in. But

$$\delta\varrho = \nabla\varrho_0 \cdot \delta z, \tag{4b.3}$$

so that the stability condition is

$$\frac{B^2}{L^2} > g\nabla\varrho_0 \sim \frac{g\varrho}{l_b} \tag{4b.4}$$

i.e., a diffuse boundary of the plasma, of width l_b is actually stable.

If a dense plasma occupies not the entire region between the plates, but only a part of length L_1, and a rare plasma is located between it and the plates, it is necessary to replace ρ in (4b.4) by an "effective" density $\rho^* = \rho L_1/L$. The exact necessary and sufficient condition for stability has the form

$$\frac{\pi B_0^2}{4L_1 L\varrho} > \frac{g}{l}. \tag{4b.5}$$

Although we have started out with the hydrodynamic picture, the results are qualitatively valid for a rare plasma, too.

It is easy to see that in the absence of a stabilizing force connected with the secured ends, i.e., as $L \to \infty$, the instability which is described above hydrodynamically will develop with an increment* $\omega_i \sim \sqrt{g/l}$, for the change in pressure (4b.1) leads in this case to an acceleration

$$\omega_i^2 \delta z \sim \delta\ddot{z} = \frac{\delta p}{\varrho l} = \frac{g}{l}\,\delta z. \tag{4b.5a}$$

Any force acting perpendicular to the magnetic field and independent of the sign of the charge, will give rise to the above-described instability; the conducting ends, as in the case of a gravitational field, exert a stabilizing influence.

Such a force may be, first, the centrifugal force connected with the motion of the particles along the curved force lines; it is necessary here to replace in (4b.4) $\mathbf{g}$ by $\mathbf{R}v_{||}^2/R^2$, where $\mathbf{R}$ is the radius of curvature of the force line. Secondly, this force can be connected with the drift in the inhomogeneous magnetic field (see Sec. 3). In this case $g \to \frac{1}{2}\mathbf{R}v_\perp^2/R^2$. Adding the two effects, we obtain

$$g \to \frac{\mathbf{R}}{R^2}(v_{||}^2 + v_\perp^2/2) \sim \frac{\mathbf{R}}{R^2}\frac{p_{||} + p_\perp}{\varrho}. \tag{4b.6}$$

It is seen therefore that a convex plasma boundary is unstable.[11] On the other hand, if conducting plates are present on the ends of the system, it is essential for stabilization, as can be seen from (4b.5) and (4b.6), to have

*L. A. Artsimovich has called our attention to the fact that the system remains unstable because of the finite conductivity of the end material, particularly of the rare secondary plasma through which contact is made with the ends (but the increments decrease).

$$B^2 > \frac{4\bar{L}^2\varrho}{l_b} \frac{v_\parallel^2 + v_\perp^2/2}{R}, \qquad (4b.7)$$

where $\bar{L}$ is the distance between plates, averaged with allowance for the variation of the plasma density along the force lines, R is the radius of curvature of the force line, and l_b is the thickness of the diffuse boundary.

If a plasma cylinder is located in crossed electric and magnetic fields, and consequently rotates with a velocity v = cE/H, then instability may result from the centrifugal force. In this case

$$g_{eff} \sim \frac{v^2}{r} \sim \frac{c^2E^2}{rH^2}, \qquad (4b.8)$$

where r is the radius of the plasma boundary. This instability is discussed in Secs. 4d, 5a, and 10c, and the physical causes of plasma rotation are treated in Sec. 13. It follows from (4b.8) that

$$\omega_i \sim \sqrt{\frac{g_{eff}}{l}} \sim \frac{cE}{H\sqrt{rl}}. \qquad (4b.9)$$

If the plasma borders on an inhomogeneous magnetic field in vacuum, then, as already mentioned in the preceding section, its stability depends on whether the magnetic field in the vacuum increases or decreases with increasing distance from the plasma.

If a certain perturbation in the form of a "tongue" appears on the boundary and penetrates through the field flux lines, we can determine its future fate by employing already known reasoning. The pressure on the end of a slowly-moving tongue is equal to the pressure of the plasma, i.e., $H_b^2/8\pi$, where H_b is the intensity of the magnetic field on the boundary. The surrounding field pressure is $H^2/8\pi$, where H is the intensity of the magnetic field at a distance δz from the boundary, i.e.,

$$\delta p = \frac{H^2(\delta z) - H^2(0)}{8\pi} = \frac{H_b}{4\pi}\frac{dH}{dz}\delta z,$$

and from (4b.5a)

$$\omega_i \sim \sqrt{H\nabla H/4\pi\varrho l}. \qquad (4b.10)$$

For a constant magnetic field this derivation is merely a duplicate of the derivation of the criterion (4b.6) but in the macroscopic language of magnetohydrodynamics. However, it remains meaningful also for high frequency fields (see Sec. 4g). A stability criterion, with allowance for the fixed ends of the force lines, can be readily obtained in analogy with (4b.7).

A plasma can be made unstable not only by forces not connected with the sign of the charge but also by an electric field. Let us consider, for example, a plasma near a conducting boundary. If an electric field E exists in the vacuum, and a surface charge

$$\sigma = \frac{E}{4\pi},$$

exists on the boundary, then the displaced element is acted on by the force

$$\delta p = \frac{E}{4\pi}\delta E.$$

Therefore

$$\omega_i^e \sim \frac{E}{\sqrt{4\pi\varrho l d}},$$

where d is the distance from the plasma to the conducting plane.

However, the question of the stability of the boundary is not completely limited to hydrodynamic stability theory. The currents flowing over the boundary and the sharp gradients cause a strong deviation from equilibrium. We shall therefore return to this problem when we investigate the microscopic instability of the plasma.

To conclude this section we note that if the plasma is accelerated, then the analog of "**g**" is the acceleration "**a**" of the boundary. The perturbation increment in this case is $\omega_i \sim \sqrt{a/l}$. An instability of this type is observed in experiments on the contraction of a plasma by an axial magnetic field ("θ pinch").

We have confined ourselves to an investigation of a dense plasma, in which the dielectric constant is $\epsilon = 1 + 4\pi NMc^2/B^2 \gg 1$ (N is the particle density and M the mass of the plasma ions). It has therefore been tacitly assumed in all the formulas that $4\pi NMc^2/B^2 \gg 1$, and unity was neglected in the expression for ϵ. The surface charge σ, developed on the curved surface by various particle drifts in the force field, produces in the plasma an electric field $E' \sim 4\pi\sigma/\epsilon$. Consequently, when $\epsilon \sim 1$, the electric field E' decreases considerably and with it the rate of growth of the instability:

$$\omega_i' \sim \omega_i \sqrt{\frac{4\pi NMc^2}{B^2}}.$$

4c. Convective instability. Unlike the preceding case, in which the plasma boundary was considered, let us examine the stability of the internal part of a plasma with closed force lines.

We confine ourselves, for the sake of clarity, to an axially symmetrical plasma configuration. Let the field in the plasma have only an azimuthal component B_0, i.e., let current flow along the axis of the plasma, and let the current density, the intensity of the magnetic field, the pressure, and the plasma density depend only on the distance to a certain axis.

Because of the tension in the bent force lines of the magnetic field, each force tube (Fig. 2) tends to contract towards the axis. Opposing this is the gradient of the magnetic and gas pressures. In the stationary

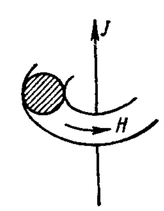


FIG. 2

state both forces are balanced. If a small radial displacement of the tube towards the axis causes the tension in the tube to increase more rapidly than the pressure gradient, the equilibrium is obviously unstable, and vice versa.

To investigate the stability of the plasma let us calculate the total force acting on the displaced tube. For this purpose we consider a thin force tube at a distance r_0 away from the axis. The gradient of the total pressure is balanced by the tension of the magnetic force lines of the tube (see Sec. 3):

$$\frac{\partial}{\partial r}(p + B_0^2/8\pi) = -B_0^2/4\pi r_0. \qquad (4c.1)$$

Let us find the forces acting on a tube which is displaced a distance δr. Such a displacement changes the field in the tube B_T, in view of the conservation of the total magnetic flux through its section S_T, by an amount

$$\frac{\delta B_T}{B_T} = -\frac{\delta S_T}{S_T} = -\frac{\delta V_T}{V_T} + \frac{\delta r_T}{r_T}, \qquad (4c.2)$$

where V_T is the volume of the tube. But in adiabatic motion $\delta V_T/V_T = -\delta p/\gamma p$. Therefore

$$\frac{\delta B_T}{B_T} = \frac{1}{\gamma}\frac{\delta p}{p} + \frac{\delta r}{r}. \qquad (4c.3)$$

On the other hand, the total pressure in the tube should be equal to the external pressure at the point $r + \delta r$:

$$p_T + B_T^2/8\pi = p_0(r+\delta r) + B_0^2(r+\delta r)/8\pi. \qquad (4c.4)$$

The left and right halves of (4c.4) are respectively equal to

$$p_0(r) + B_0^2(r)/8\pi + \delta p + B_0\delta B_T/4\pi$$

and

$$p_0(r) + B_0^2(r)/8\pi + \frac{dp_0}{dr}\delta r + \frac{B_0}{4\pi}\frac{dB_0}{dr}\delta r.$$

Therefore

$$\delta p + \frac{B_0\delta B_T}{4\pi} = (p_0' + B_0B_0'/4\pi)\,\delta r. \qquad (4c.5)$$

Substituting δB_T from (4c.3), we obtain

$$\frac{\delta B_T}{B_0} = \frac{1}{\gamma p_0}\frac{p' + B_0(B_0' - B_0/r)/4\pi}{1 + B_0^2/4\pi\gamma p}\delta r + \frac{\delta r}{r}. \qquad (4c.6)$$

The change in the tension in the magnetic tube is

$$\delta\frac{B_T^2}{4\pi r} = \frac{B_0\delta B_T}{2\pi r} - \frac{B_0^2}{4\pi r^2}\delta r. \qquad (4c.7)$$

Since the gradient of the total pressure at the point $r + \delta r$ balances the tension of all the tubes contained there, the total gradient changes after displacement by

$$\frac{B_0^2(r+\delta r)}{4\pi(r+\delta r)} - \frac{B_0^2(r)}{4\pi r} = \frac{B_0B_0'}{2\pi r}\delta r - \frac{B_0^2}{4\pi r^2}\delta r. \qquad (4c.8)$$

The total force acting on the tube is

$$\delta F = \delta\frac{B_T^2}{4\pi r} - \delta\frac{B_0^2}{4\pi r} = \frac{B_0^2}{2\pi r}\left(\frac{\delta B_T}{B_0} - \frac{B_0'}{B_0}\delta r\right). \qquad (4c.9)$$

From (4c.6) and (4c.9) we get

$$\delta F = -\frac{B_0^2}{2\pi r}\left\{\frac{1}{\gamma p_0}\frac{p_0' + B_0(B_0' - B_0/r)/4\pi}{1 + B_0^2/4\pi\gamma p_0} + \frac{1}{r} - \frac{B_0'}{B_0}\right\}\delta r. \qquad (4c.10)$$

If $\delta F/\delta r > 0$, the tube accelerates and the equilibrium is unstable. Therefore the condition for stability against rearrangement of the force tubes has the form

$$\delta F/\delta r < 0. \qquad (4c.11)$$

From (4c.1) it follows that

$$p_0' = -\frac{B_0}{4\pi}(B_0' + B_0/r),$$

so that the plasma is stable if

$$-\frac{B_0^2/2\pi r\gamma p_e}{1 + B_0^2/4\pi\gamma p_0} + \frac{1}{r} - \frac{B_0'}{B_0} > 0,$$

or, in a different form,

$$\frac{B_0'}{B_0} - \frac{1}{r} + \frac{B_0^2}{4\pi\gamma p_0}\left(\frac{B_0'}{B_0} + \frac{1}{r}\right) < 0. \qquad (4c.12)$$

For stability in an incompressible liquid ($\gamma \to \infty$) it is necessary and sufficient that the field increase not faster than the distance from the axis. For stability of a rarefied plasma (when $B^2/8\pi p = 1/\beta \gg 1$), however, it is necessary that the field away from the axis decrease more rapidly than $1/r$. This last condition coincides with the condition obtained in reference 12. Actually, let us introduce the quantity $U = -\int dl/B_0 = -2\pi r/B_0$. It is easy to see that (4c.12) assumes the form

$$\nabla U \nabla p < \gamma p\frac{(\nabla U)^2}{|U|}. \qquad (4c.13)$$

This condition is valid[11,12] for closed force lines of any shape.

If $\beta > 2\gamma/3$, the most dangerous perturbations of the equilibrium configurations are those with no axial symmetry — "kinks." The stability condition for them has the form

$$-\frac{d\ln p}{d\ln r} < \frac{1}{\beta}. \qquad (4c.14)$$

On the other hand, the condition (4c.12) has the form

$$-\frac{d\ln p}{d\ln r} < \frac{4\gamma}{2+\gamma\beta}. \qquad (4c.15)$$

It follows from this, actually, that when $\beta > 2\gamma/3$ the stability is determined by the condition (4c.14).

4d. Stability of a cylindrical pinch. The stability of a plasma pinch has been the subject of numerous researches. The very first photographs of a plasma pinch compressed by its own current have shown that it is unstable against "neck" (sausage) and flexure (kink) deformations.

The instability of a pinch with longitudinal current flowing over the surface is theoretically evident from the fact that the magnetic field diminishes everywhere from the plasma boundary towards the outside.

Various authors[13-15] have proposed, almost simultaneously, to stabilize such a pinch with a strong magnetic field directed along the pinch axis. In this case the deformations of the pinch actually perform work to increase the energy of this magnetic field, thereby causing the stabilizing effect, which is most clearly pronounced for perturbations with a large wave vector along the pinch axis. In long-wave perturbations, to the contrary, the changes in the longitudinal magnetic field are small and the instability against perturbations with wavelengths considerably greater than the radius of the pinch still remains. The condition of pinch stability against "kinks" and "sausages" is best visualized as follows.

We consider first kink perturbations (Fig. 3). We assume that the pinch, of radius a, contains a frozen-in axial field B, while outside the pinch there is the azimuthal field H of the current flowing over its surface. If the pinch is bent (length of the flexure $\sim \lambda$),

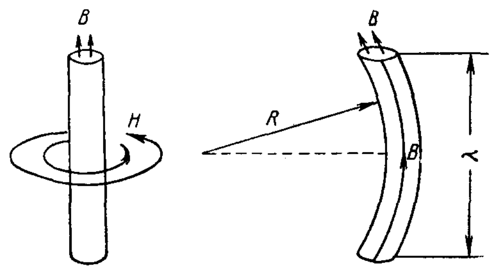


FIG. 3. Kink instability of a pinch.

then the force lines of the azimuthal field become denser on the outside than on the inside. The internal part of the pinch (turned towards the center of curvature) is therefore under greater magnetic pressure. On the other hand, the bending of the force lines of the frozen-in field produce a force in the opposite direction (quasi-elastic force).

The force produced by the azimuthal field on a unit of pinch length can be calculated in the following manner. We isolate around the pinch a cylindrical volume of radius $\sim \lambda$, bounded by planes passing through the center of curvature. Since the force lines of the azimuthal field lie in these planes, the total force acting in the direction of the displacement consists of the corresponding component of the magnetic pressure on the ends

$$2\int_a^\lambda \frac{H^2}{8\pi}\cdot 2\pi r\,dr\alpha,$$

(where the angle of inclination is $\alpha = \lambda/2R$ and R is the radius of curvature) and the pressure on the side surfaces, which can be neglected. At a distance $\sim \lambda$, the field perturbation vanishes.

The force on a unit pinch length, exerted by the field perturbation due to the bending, is therefore

$$\frac{1}{R}\int_a^\lambda \frac{H^2}{8\pi}\cdot 2\pi r\,dr = \frac{H_0^2}{4R}\ln\frac{\lambda}{a}\cdot a^2,$$

where $H_0 = H(a)$ – the field on the surface of the pinch.

The quasi-elastic volume force is

$$-\frac{B^2}{4\pi R}\pi a^2 = -\frac{B^2}{4R}a^2,$$

so that the total force is

$$\delta F = \frac{a^2}{4R}\left(H_0^2\ln\frac{\lambda}{a} - B^2\right).$$

From this we get the known stability condition

$$\frac{B^2}{H^2} > \ln\frac{\lambda}{a}.$$

Since it follows from the equilibrium condition

$$p + \frac{B^2}{8\pi} = \frac{H^2}{8\pi}$$

that $B^2 < H^2$, it is clear that the pinch cannot be stabilized against long-wave perturbations by a strong internal longitudinal field.

If a longitudinal magnetic field exists inside and outside the pinch with the axial current, then the total field is helical. The pinch, bending along the helical force lines in such a field, can penetrate between the force lines of the field without bending them. Such an instability will occur if the perturbation of the pinch surface is helical and if the pitch λ of this helix is equal to or greater than the pitch $2\pi a H_z/H_\varphi$ of the force line on the surface of the pinch. Consequently, the pinch will be stable against helical perturbations of wavelength

$$\lambda < 2\pi a\frac{H_z}{H_\varphi}.$$

If the wavelength of the perturbation is bounded from above by the dimensions of the system (for example, the length $2\pi R$ of a toroidal pinch), then instability sets in when the current exceeds the critical value

$$I_{cr} \sim c\frac{a^2}{R}H_z,$$

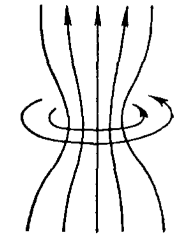
sometimes called the Shafranov-Kruskal current.

Thus, in both cases a maximum wavelength exists for the perturbation that can be stabilized by a magnetic field.

The condition of stability against "sausages" (Fig. 4) can be obtained in the following manner. Let the radius of the pinch change by δa. Then, owing to the conservation of the flux, the field inside the pinch is changed by

$$\delta B = -B_0\frac{2\delta a}{a}.$$

FIG. 4. Sausage instability of a pinch.



On the other hand, the azimuthal field H outside the pinch is

$$H = \frac{2I}{ca},$$

where I is the total current. Therefore $\delta H = -H_{\varphi}\delta a/a$. The total change in the difference of magnetic pressures inside and outside the pinch is consequently equal to

$$\delta p_m = -\frac{B^2}{4\pi}\frac{2\delta a}{a} + \frac{H^2}{4\pi}\frac{\delta a}{a},$$

so that the stability condition has the form

$$B^2 > \frac{H^2}{2}.$$

Thus, a sufficiently strong magnetic field will suppress the "sausages" but cannot stabilize a pinch against long-wave "kinks."

However, long waves can be stabilized by surrounding the plasma pinch with a conducting coaxial sheath. The displacement of the pinch should induce in the sheath currents which interact with the pinch and tend to return it to the initial position. A combination of both stabilization methods found use in many well-known experimental installations for the production of high-temperature plasma ("Zeta," "Columbus," "Alpha," "Tocomac," etc.).

If we forego the idealized picture of the "surface current," then the stability criteria will depend essentially on the profile of current distribution (and consequently on the magnetic field, plasma pressure, etc.) over the cross section of the pinch.

In Sec. 4c we derived the condition that the profile of the distribution of these quantities must satisfy in order for the pinch to be stable (in the absence of a longitudinal field!). Physically, however, such a profile cannot be realized, since it contains a singularity in the current density on the axis.

The idealized picture of distributed longitudinal and azimuthal fields in a fast discharge corresponds more or less to reality only in a short time interval. The finite conductivity of the plasma causes the fields to become "intermixed" and a helical field is produced in the discharge. On the other hand, in installations designed for prolonged confinement of the plasma it is necessary, as we have already seen, to employ a helical field in order to maintain stability (Stellarator, Tocomac). Thus it becomes necessary to investigate the stable equilibrium of the plasma in a helical field.

The usual interchange instability is impossible in a helical field, since the pitch of the force line is different on each magnetic surface, the tubes become "entangled" upon radical displacement, and a quasi elastic force is produced. In each given layer, however, the tube can move if its displacement is uniform along the force lines. With increasing deviation from its own layer, the quasi elastic force increases, and consequently a surface instability localized near a certain magnetic surface can occur.

In order for the perturbation not to bend the force line, its wave vector must be perpendicular to the line everywhere. If we seek a perturbation in the form $\exp(im\varphi + ikz)$, then this condition has the form

$$\mathbf{kB} = \frac{m}{r}B_{\varphi} + kB_z = 0, \tag{4d.1}$$

where $\mathbf{B}$ is the unperturbed helical field.

Let us assume that this condition is satisfied for given m and k at a certain radius r_0. If the scale of the perturbation is $\lambda \sim 1/k \ll r_0$, then the equation of motion can be expanded in powers of λ/r_0 about r_0, which is determined from the condition (4d.1). Let us find the conditions under which slow displacements of the tubes are possible, i.e., let us find the instability limit. The tube equilibrium condition has the form

$$\nabla\Phi + (\mathbf{B}\nabla)\mathbf{B}/4\pi = 0, \quad \Phi = p + B^2/8\pi, \tag{4d.2}$$

the condition on $\mathbf{B}$ being $\operatorname{div}\mathbf{B} = 0$.

Linearizing these equations we obtain

$$\left.\begin{aligned} i\mathbf{kB}b_r - \frac{2B_{\varphi}}{r}b_{\varphi} - \Phi_1' &= 0,\\ i\mathbf{kB}b_{\varphi} + (B_{\varphi}' + B_{\varphi}/r)\,b_r - \frac{im}{r}\Phi_1 &= 0,\\ i\mathbf{kB}b_z - ik\Phi_1 &= 0,\\ \frac{1}{r}\frac{d}{dr}(zb_r) + i\mathbf{kb} &= 0, \end{aligned}\right\} \tag{4d.3}$$

where $\mathbf{b}$ and Φ_1 are the perturbations of the field and of the total pressure.

We introduce the variable $x = r/r_0 - 1$ and expand (4d.1) in powers of λ/r_0: $\mathbf{k}\cdot\mathbf{B} = Sx$, where

$$S = r_0(\mathbf{kB})'. \tag{4d.4}$$

From the system (4d.3) we obtain for b_r the following differential equation

$$b_r'' + \{-k^2 + \varkappa^2/x^2\}\,b_r = 0, \tag{4d.5}$$

where

$$\varkappa^2 = 2B_{\varphi}(B_{\varphi}' + B_{\varphi}/r_0)\,k^2r_0/S^2 = -\frac{8\pi p'/rB_z^2}{(\mu'/\mu)^2},$$

and $\mu = B_{\varphi}/rB_z$ is the torsion (i.e., a quantity reciprocal to the radius of torsion) of the force line. Generally speaking, (4d.5) does not hold near $x = 0$, and for a correct description of this region it is necessary to take either inertia or dissipation into account. However, a formal solution of (4d.5) is

$$b_r = b^0\sqrt{x}\,K_{i\nu}(kx),$$

where $\nu = \sqrt{\kappa^2 - 1/4}$. When $\nu^2 > 0$ the funtion has, as is well known, an infinite number of zeros near $x = 0$. The solutions that are damped at infinity can therefore be joined to any solution near $x = 0$. When $\nu^2 < 0$ this cannot be done. Therefore the critical value of κ^2 is $1/4$. An exact account of the inertia confirms this result.

Thus, the condition for the instability of a plasma in a helical field has the form $\kappa^2 < 1/4$, i.e.,

$$\frac{r}{4}\left(\frac{\mu'}{\mu}\right)^2 + \frac{8\pi p'}{B_z^2} < 0. \tag{4d.6}$$

This condition was first established by Suydam.[16] In the general case it follows from (4d.6) that the greater the relative change in the torsion of the force lines μ'/μ in the radial direction, the greater the pressure gradient that can be contained stably by the magnetic field. The necessary and sufficient condition for the stability can be obtained from the energy principle (see Sec. 4a):

> if the pinch is stable for m = 1, $-\infty < k < \infty$ and for $k \to 0$, m = 0, it is also stable for all m and k.

Even with the aid of this theorem, however, the investigation of the stability of the pinch against nonlocal perturbation is a rather complicated matter. Results were obtained only for some special types of field distributions.[50]

4e. Stability of rotating non-uniform plasma in a magnetic field. In a whole series of devices such as "Homopolar," magnetic plasma condensers or traps in which the rotation of the plasma improves the confinements, we deal with a non-uniform plasma that rotates in an axial magnetic field (Fig. 5).

In systems of the "Homopolar" type the plasma borders on its ends with insulators (Fig. 6). Therefore, generally speaking, interchange instability can develop. The plasma can move transversely to the field without disturbing it. However this type of instability can be produced only under rather specific conditions. Since this instability is no longer aperiodic, we shall consider it in greater detail in Chapter IV. As will follow from Chapter IV, if the plasma density is not decreased during displacement away from the axis, then the rotation is unstable only when the flow contains a layer in which the velocity curl has an extremum. If there is no such layer, only perturbations with axial symmetry are dangerous.

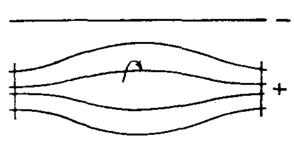


FIG. 5. Trap with rotating plasma.

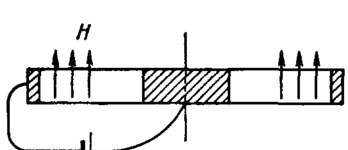


FIG. 6. Homopolar.

We have thus obtained the condition of stability against perturbations that bend the magnetic force lines.

Let us consider a magnetic force line, the ends of which are secured at two points a distance 2L apart, and whose central part is displaced a distance δr from the equilibrium position (Fig. 7). Each tube rotates with an angular velocity $\Omega(r)$. It is clear that in the case of a sufficiently slow displacement of the tube, its angular velocity remains the same in any arbitrarily weak field B, for otherwise the tube would be "wound up" by the rotation and the magnetic field would increase by an arbitrarily large amount (Fig. 8).

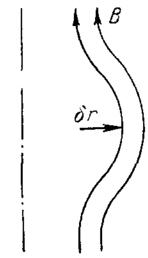


FIG. 7

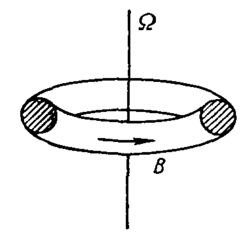


FIG. 8

As before, we confine ourselves to the case of small β. The forces acting radially on the tube are made up of the following:

a) Tension in the magnetic force lines in the tube:

$$-\frac{B^2}{4\pi R} = -\frac{B^2}{2\pi L^2}\,\delta r, \tag{4e.1}$$

the minus sign means that this force tends to return the tube.

b) Centrifugal force connected with the rotation of the tube:

$$\Omega^2\varrho r + \delta(\Omega_T^2\varrho r) = \Omega^2\varrho r + \Omega^2\varrho\delta r, \tag{4e.2}$$

we can neglect the change of the density in the tube when β is small;

c) Gradient of total pressure at the point $r + \delta r$

$$\nabla\left(p + \frac{B^2}{8\pi}\right) = [\Omega^2 r\varrho]_{r+\delta r} = \Omega^2\varrho r + \Omega^2\varrho' r\delta r + (\Omega^2)'\,\varrho r\,\delta r + \Omega^2\varrho\delta r. \tag{4e.3}$$

From (4e.2) and (4e.3) we obtain

$$\frac{\delta F_m}{\delta r} = -\left\{\frac{B^2}{2\pi L^2} + r\frac{d\varrho\Omega^2}{dr}\right\}, \tag{4e.4}$$

so that the stability condition has the form

$$\frac{B^2}{2\pi L^2} > -r\frac{d\varrho\Omega^2}{dr}. \tag{4e.5}$$

The difference between this problem and the analogous one of stability in the absence of a field is that in the latter the angular momentum of the liquid element is conserved,[1] whereas in the presence of a field it is the angular velocity that is conserved. The velocity is conserved no matter how weak the field, so long as dissipation processes do not come into play, i.e., so long as the plasma does not become "unglued" from the force lines of the magnetic field. For details regarding an incompressible liquid, see reference 17.

If an azimuthal current flows through the rotating plasma, then the centrifugal effect is superimposed on the magnetic convective instability. In an inviscid fluid the angular momentum of the toroidal magnetic force tube is conserved as the tube moves uniformly in a radial direction (expansion or contraction), i.e., $M = \rho\Omega r^2 = \text{const}$.

It follows from (4c.5) that the change of the magnetic field in the tube is given by

$$\frac{\delta B}{B} = \frac{1}{\gamma p}\,\frac{p' + B(B' - B/r)/4\pi}{1 + B^2/4\pi\gamma p}\,\delta r + \frac{\delta r}{r},$$

and the change in pressure is

$$\delta p = \frac{p' + B(B' - B/r)/4\pi}{1 + B^2/4\pi\gamma p}.$$

On the other hand, if the motion of the tube is adiabatic

$$\frac{\delta \varrho}{\varrho} = \frac{1}{\gamma}\,\frac{\delta p}{p}.$$

Therefore the total force acting on the displaced tube, equal to the sum of the variations in the centrifugal force, in the tension of the magnetic force lines, and in the gradient of the total pressure,

$$\delta F = \delta F_c - \delta F_m - \delta\nabla(p + B^2/8\pi)$$
$$= \left\{ -\frac{(\Omega r)^2 + B^2/2\pi\varrho}{c_T^2 r}\,\frac{p' + B(B' - B/r)/4\pi}{1 + B^2/4\pi\gamma p} - \frac{p'\Omega^2 r\gamma}{c_T^2} + \frac{B(B' - B/r)}{2\pi r} - \frac{2\varrho\Omega\,(\Omega r^2)}{r} \right\}.$$

The first two terms are due to the compressibility of the tube, the third to the change in the tension of the magnetic force lines in the tube, and the last is connected with the effect of the centrifugal force.

The sufficient condition for stability is $\delta F/\delta r < 0$ at each point of the rotating plasma.

4f. Aperiodic instability of nonmaxwellian plasma. A curious type of aperiodic instability can occur in a uniform plasma located in a constant magnetic field, if the plasma is not in thermodynamic equilibrium, i.e., if the particle velocity distribution is nonmaxwellian.[18-20] This instability develops within a time much shorter than the time of pair collisions, and we shall therefore exclude particle collisions from further consideration.

Let us consider perturbations with wave lengths λ much greater than the average Larmor radii of the electrons and ions (and with a characteristic time of variation considerably greater than the period of revolution of the particles in the magnetic field). Here we can use in the calculations the drift approximation, within the framework of which the plasma is considered as an aggregate of quasi particles ("Larmor circles") with conserved magnetic moment $\mu = \epsilon_\perp/H$. The particle distribution is described by a function $f(\mu, v_\parallel)$, where $v_\parallel$ is the velocity along the constant magnetic field and $\epsilon_\perp$ is the rotation energy.

Instability can occur in our case for waves propagating at an arbitrary angle to the electromagnetic field. For the sake of simplicity we consider two limiting cases: a wave propagating strictly along the magnetic field (Alfven wave),* and a magnetoacoustic wave propagating almost perpendicularly to the magnetic field.

1. Instability of Alfven wave. As is well known, Alfven waves can be visualized as oscillations of "elastic filaments" — the force lines of the magnetic field. To determine the instability conditions, let us consider the forces arising when a force line is bent (Fig. 9). Since the particles are "attached" to the force line, motion along the curved portion of the force line gives rise to a centrifugal force

$$F_c = \int f\,\frac{mv_\parallel^2}{R}\,dv_\parallel\,d\mu,$$

which tends to increase the curvature.

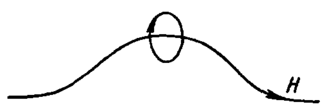


FIG. 9

Since, in addition, each "quasi particle" has a magnetic moment $-\mu\epsilon_0$, oriented opposite to the magnetic field $\epsilon_0 H$, a particle in an inhomogeneous magnetic field will be acted upon by a force due to the presence of the magnetizing current $j_\mu = c\,\text{curl}\int \mu f\,dv_\parallel\,d\mu$

$$F_\mu = \frac{[j_\mu, H]}{c} = \left[\text{rot}\int \mu f(\mu, v_\parallel)\,dv_\parallel\,d\mu,\ H\right]. \quad (4f.1)$$

This force, together with the "tension" force of the magnetic field lines

$$F_H = \frac{[j, H]}{c} = \frac{1}{4\pi}[\text{rot}\,H,\ H] \quad (4f.2)$$

tends to return the force line to the equilibrium position.

If $F_c > F_\mu + F_H$, the system deviates from equilibrium position, i.e., instability sets in. In this perturbation, the only non-vanishing wave-vector component is $k_\parallel$. Substituting $\nabla = \{0, 0, ik\}$ in (4f.1) and (4f.2), we obtain the following instability condition

$$p_\parallel - p_\perp > \frac{H^2}{8\pi}, \quad (4f.3)$$

where

*Actually, if the instability criterion which we are about to derive is satisfied, the perturbation does not behave like a wave (this manifests itself formally in the fact that ω^2 becomes negative). Nonetheless, we shall speak for the sake of brevity of "instability on an Alfven wave," bearing in mind that as the anisotropy is decreased this type of perturbation is gradually converted into an Alfven wave.

$$p_{\|} = \int mv_{\|}^2 f\, dv_{\|}\, d\mu, \qquad p_{\perp} = \int \mu H f\, dv_{\|}\, d\mu.$$

When condition (4f.3) is satisfied, the force takes the system continuously away from the equilibrium condition, and therefore the instability is aperiodic and varies as $\exp(\gamma t)$.

The increment γ can be readily obtained by equating the sum of the forces $F_C - F_\mu - F_H$ to the product of the mass of the plasma element by the acceleration

$$\dot{v} = \frac{d}{dt}\frac{cE}{H_0}.$$

Since it follows from Maxwell's equations that $E = H\gamma/ck$, the acceleration is $\dot{v} = \gamma^2 H/kH_0$. Substituting the values of the forces F, we obtain

$$-\gamma^2 = 2k^2\left(\frac{H^2}{8\pi} + p_\perp - p_\|\right)/\varrho_0.$$

2. "Instability on magnetoacoustic waves" propagating almost perpendicular to the magnetic field. We obtain the stability limit by equating the total force F, acting on the plasma element under an almost-transverse perturbation ($k_\perp \gg k_\|$) to the force

$$F_\perp = -\nabla_\perp p - \frac{[\mathbf{H}_0, \operatorname{rot}\mathbf{H}]_\perp}{4\pi} = 0. \qquad (4f.4)$$

Here p is the correction to the transverse pressure, equal to

$$\int mv_\perp^2 f_1\, d\mathbf{v}, \qquad (4f.5)$$

where f_1 is the correction to the distribution function, and H is the correction to the magnetic field. The linearized kinetic equation yields for the correction f_1, which in the drift approximation is a function of $\mu = mv_\perp^2/2H$ and $v_\|$, the following expression ($\omega = 0$ on the stability boundary)

$$f_1 = f_0\frac{H}{H_0} + \frac{\mu H}{m}\frac{\partial f_0}{v_\|\partial v_\|}\,^* .$$

Substituting f_1 (4f.2) and recognizing that in an almost transverse wave

$$[\mathbf{H}_0, \operatorname{rot}\mathbf{H}]_\perp = ik_\perp H_0 H,$$

we obtain from (4f.1) the stability limit

$$1 + \frac{p_\perp^0}{H_0^2/8\pi} + 4\pi\int \mu^2 \frac{\partial f_0}{\partial\left(\frac{mv_\|^2}{2}\right)}\, d\mu\, dv_\| = 0, \qquad (4f.6)$$

where $p_\perp^0 = \int \mu H f_0\, dv$ is the equilibrium value of the transverse pressure.

For an equilibrium Maxwellian distribution function, which is known to be stable, the last two terms in (4f.3) cancel each other; consequently, for a plasma to be stable against perturbations of this type, the left half of (4f.3) should be positive. In the particular case of a "anisotropic Maxwellian" distribution funtion $T_\perp > T_\|$, the stability criterion has the form

$$1 + \frac{n_0 T_\perp (1 - T_\perp/T_\|)}{H_0^2/8\pi} > 0. \qquad (4f.7)$$

We see from formulas (4f.3) — (4f.7) that the instability types considered in this section arise in a low-pressure plasma (i.e., $p \ll H^2/8\pi$) only in the case of sufficiently large anisotropy. Thus, for example, according to (4e.7), $(T_\perp - T_\|)/T_\|$ should reach a value greater than $H^2/8\pi p_\perp$, in order for an aperiodic instability to be able to develop in the plasma. Under these conditions the oscillator instability, which will be considered later on in Sec. 7, should set in much earlier.

4g. Stability of a plasma contained by the pressure of a high-frequency electromagnetic field. Distinct types of instability are inherent in configurations produced by high-frequency (hf) containment, i.e., in a plasma insulated by the pressure of an alternating electromagnetic field. The feasibility of such a containment is based on the fact that the alternating magnetic field cannot penetrate into a conductor (such as a plasma at frequencies $\omega < \omega_0$) and thereby produces a pressure drop $\overline{H}^2/8\pi$ at the boundaries of the conductor.

It is simplest to analyze the stability against perturbations that vary little over a single cycle of the containing hf field. In this case we can average over the fast oscillations of the hf field and introduce the average pressure $\overline{H}^2/8\pi$.

The matter becomes even simpler if the hf field is quasi-stationary, i.e., the displacement currents can be neglected. Actually, the magnetic field is then related with the current density in the same manner as in the "static" case, all the results of the energy approach (see Sec. 4a) are also valid in the magnetohydrodynamic treatment of the plasma. Such a system will be most stable if $\overline{H}^2/8\pi$ increases from the plasma outward. Unlike the "static" case, it is much easier* in principle to obtain by hf containment a magnetic-mirror geometry that ensures stability.

In order for the hf field to be able to contain the plasma stably against the action of an external force (gravity, centrifugal force, etc.) it is necessary that no plasma "tongues" be able to penetrate into the spaces between the force lines of the field. To avoid this, it is possible to "rotate" the polarization of the waves so as to "smooth" the tongues. This, however, can cause dragging of the plasma and further destabilization.

If the displacement current and the associated wave effects are appreciable in the containing hf field, (i.e., if the wavelength of the containing hf field is comparable with the characteristic mirror dimension), the following "resonant" effect can appear.

*For simplicity we disregard here the variation in the distribution function, due to the longitudinal electric field. This is valid if the electrons are "cold" and therefore annihilate the longitudinal field.

*On the other hand, the practical production of a hf magnetic field of sufficient amplitude calls for tremendous power to supply the hf apparatus.

We consider for simplicity a plane plasma boundary, contained by the pressure of an electromagnetic wave, (say a standing wave), excited in a layer of thickness l, bounded on the one side by the plasma surface and on the other by some conducting surface (Fig. 10). Let the amplitudes E^0 and H^0 of the hf

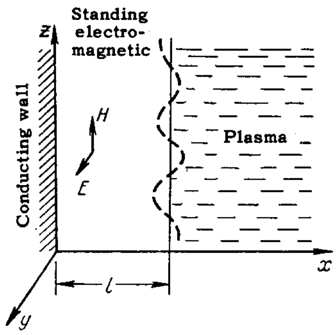


FIG. 10

fields E and H have an equilibrium distribution in space

$$\left.\begin{aligned} E_y^0 &= E^0 \sin \frac{\omega}{c} x, \\ H_z^0 &= H^0 \cos \frac{\omega}{c} x, \end{aligned}\right\} \qquad (4g.1)$$

where

$$\frac{\omega}{c} = \frac{\pi}{e}, \qquad E^0 = -H^0.$$

A wave-like perturbation [~ exp (ikz)] of the plasma boundary causes a change in the spatial distribution of the hf field, and thereby a change in the distribution of its pressure $\overline{H}^2/8\pi$ along the plasma boundary. If the pressure on the crests of the plasma boundary has increased, the system is stable (it tends to return to the initial state). In the opposite case, the plasma will depart from the equilibrium position (aperiodic instability will set in). Maxwell's equation for the corrections to the standing-wave amplitude have the form

$$H_z^1 = -\frac{c}{\omega}\frac{dE_y^1}{dx}, \qquad \frac{d^2E_y^1}{dx^2} + \left(\frac{\omega^2}{c^2} - k^2\right)E_y^1 = 0, \qquad (4g.2)$$

and the boundary conditions call for the vanishing of the tangential component of E on both boundaries, at $x = 0$ and $x = 1 + \xi$, where ξ is a small displacement of the plasma boundary. The first condition yields simply $E_y^1|_{x=0} = 0$, while the second, accurate to first-order quantities, is

$$E_y^1|_{x=l} + \xi \left.\frac{dE_y^0}{dx}\right|_{x=l} = 0. \qquad (4g.3)$$

Equations (4g.2) and (4g.3) have a solution satisfying the boundary conditions, namely,

$$E_y^1 = -\frac{\omega}{c}E^0\xi\frac{\sin \alpha x}{\sin \alpha l}, \quad H_z^1 = E^0\xi\alpha\frac{\cos \alpha x}{\sin \alpha l},$$

where

$$\alpha = \sqrt{\omega^2/c^2 - k^2}.$$

The correction to the pressure on the plasma boundary is proportional to $H^0H^1 \sim -\xi\frac{(H^0)^2}{4\pi}\alpha \cot \alpha l$. For stability is essential that the pressure increase where ξ is negative, i.e., where the plasma boundary is convex. Consequently, the plasma is stable if*

$$\operatorname{ctg} \sqrt{\omega^2/c^2 - k^2}\cdot l > 0$$

or[21]

$$k^2 > \frac{3}{4}\frac{\pi^2}{l^2}.$$

We see therefore that instability sets in when the surface perturbations have long wavelengths $2\pi/k$, comparable with the length of the electromagnetic standing waves. No such instability occurs if the dimensions of the plasma configuration are much less than c/ω.

This instability can be interpreted qualitatively as geometric resonance, which occurs when the wavelength of the perturbation is comparable with the wavelength of the containing electromagnetic wave.

We have considered for simplicity only the simplest deformation of the plasma boundary [~ exp (ikz)], a wave along the direction of the hf magnetic field. An analysis of more general deformations, of the form exp [i ($k_z z + k_y y$)], hardly changes the stability criterion, viz., the system is stable if

$$k_z^2 > \frac{\pi^2}{l^2}.$$

5. Aperiodic Plasma Instability with Account of Dissipation

Dissipative processes — viscosity, conductivity, and heat conduction — can in principle convert the aperiodic instability into oscillations with increasing amplitude, for the problem is no longer self adjoint. In some cases, however, it can be shown that if the finite dissipation is taken into account the instability region is determined by condition $\omega = 0$, i.e., the system departs from equilibrium aperiodically. In this case, in analogy with an ideal plasma, we can introduce a variational stability criterion and use it for numerical computations.

In the present section we consider problems involving rotation and magnetic convection in a plasma, with allowance for dissipative forces.

Aperiodic thermal convection in a plasma is considered in detail in Sec. 13, where we determine the conditions for the occurrence of convection and of "supercritical" convection.

5a. Stability of plasma rotation. In this section we continue the study of the stability of a rotating plasma against axially-symmetrical perturbations. For simplicity we confine ourselves to an incompressible liquid with finite conductivity, the only case investigated in detail to date.

*ctg = cot.

We have seen in Sec. 4d that the influence of the magnetic field on the rotating stability of an ideal plasma lies in the fact that the angular momentum ceases to be conserved under displacements of small parts of the plasma, but the angular velocity of these parts is conserved. The change in the stability limit for a uniform ideally conducting plasma between two rotating cylinders that bound the plasma (angular velocities Ω_1 and Ω_2, radii R_1 and R_2; $R_2 > R_1$) is shown in Fig. 11 for the case of a weak magnetic field. Allowing for the magnetic quasi elastic force, the stability condition is

$$\frac{B^2}{L^2} + r\frac{d\Omega^2\varrho}{dr} > 0, \tag{5a.1}$$

i.e., the stability region is somewhat broader than shown in Fig. 11.

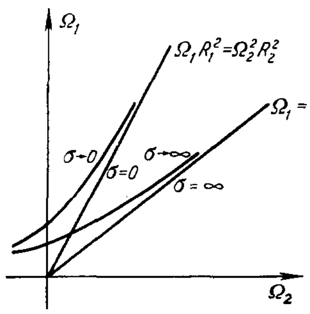


FIG. 11

As the conductivity decreases, the stability region increases. This is brought about by damping due to Joule heating of the plasma and by the "ungluing" of the plasma from the magnetic force lines; this causes the instability boundary to shift towards the Rayleigh boundary ($\Omega_1R_1^2 = \Omega_2R_2^2$).

At a certain finite conductivity, the stabilizing effect of the magnetic field begins to manifest itself — Joule losses begin to exceed the influence of "magnetic untwisting." Let us consider this effect for a poorly conducting liquid, $\frac{B}{\sqrt{4\pi\rho}}\,\frac{R_2 - R_1}{c^2}\,4\pi\sigma < 1$. As is well known, the motion of the plasma disturbs the magnetic field little in this case. The effect of the field reduces therefore to the appearance of a retarding force

$$\frac{jH}{c} \sim \sigma\frac{vH}{c}\frac{H}{c} = \frac{\sigma v H^2}{c^2}. \tag{5a.2}$$

Let us find the conditions for the existence (i.e., the stability boundary) of stationary motion maintained in a plasma by the gradient of the centrifugal force.

We have seen (Sec. 4e) that when a plasma element is rapidly displaced (compared with the equalization velocity) it is acted upon by the difference between the change in the total pressure gradient and the centrifugal force

$$\delta F = -\frac{2\varrho\Omega}{r}(\Omega r^2)'\,\delta r, \tag{5a.3}$$

where Ω is the angular velocity of rotation of the layer from which the element begins its motion, r is the distance of this layer from the axis of rotation, and ρ is the density.

The displacement time can be estimated by equating this force to the electromagnetic retarding force and to the force of viscous friction

$$-\frac{2\varrho\Omega}{r}(\Omega r^2)'\,\delta r = \frac{\sigma H^2}{c^2}\frac{\delta r}{\delta t} + \frac{\eta}{\lambda^2}\frac{\delta r}{\delta t}, \tag{5a.4}$$

where λ is the characteristic dimension of the perturbations, hence

$$\delta t_1 \cong \frac{\frac{\sigma H^2}{c^2} + \frac{\eta}{\lambda^2}}{-\frac{2\varrho\Omega}{r}(\Omega r^2)'}. \tag{5a.4'}$$

On the other hand, this time should not be sufficient for the rotational velocity of the element to become equal to the velocity of the surrounding plasma. The rate of equalization is determined from the approximate relation

$$\varrho\frac{\delta v}{\delta t_2} \cong \frac{\sigma H^2}{c^2}\delta v + \frac{\eta}{\lambda^2}\delta v. \tag{5a.5}$$

Equating δt_1 and δt_2, we obtain the stability boundary

$$-\frac{2\Omega}{r}(\Omega r^2)' \sim \left(\frac{\nu}{\lambda^2} + \frac{\sigma H^2}{c^2\varrho}\right)^2, \tag{5a.6}$$

or

$$\mathsf{T}^{cr} = -\frac{2\Omega(\Omega r^2)'(R_2 - R_1)^4}{r\nu^2} \sim \frac{(R_2 - R_1)^4}{\lambda^4}\left(1 + \frac{\sigma H^2\lambda^2}{c^2\varrho\nu}\right)^2, \tag{5a.7}$$

where T is the Taylor number.

For solid cylinders $T^{cr} \to T_0^{cr} = 1700$ as $H \to 0$. Putting

$$\frac{\sigma H^2}{c^2\varrho\nu}(R_2 - R_1)^2 = \mathbf{H}^2,$$

where $\mathbf{H}$ is the Hartmann number, we see that

$$\frac{\mathsf{T}^{cr}}{\mathsf{T}_0^{cr}} \sim \left(1 + \mathbf{H}^2\frac{\lambda^2}{(R_2 - R_1)^2}\right)^2\frac{\lambda_0^4}{\lambda^4}. \tag{5a.8}$$

The length λ involved in all these expressions is some average dimension of the perturbation

$$\lambda \approx (R_2 - R_1)A,$$

and the exact value of the numerical factor A depends on the boundary conditions.

In a strong field ($\mathbf{H} \gg 1$) the stability limit is determined by the dimensionless parameter

$$\Lambda = -\frac{2\varrho^2\Omega(\Omega r^2)'c^4}{\sigma^2H^4r}, \quad \Lambda^{cr} \approx 1. \tag{5a.9}$$

Figure 12 shows the dependence of T^{cr} on $\mathbf{H}$. The initial portion of the curve was determined numerically in reference 22. When $T > T_{cr}$ the expression is unstable.

5b. Pinch stability. Considering that the longitudinal current flows in a real, dense plasma (we regard a plasma as dense if the mean free path is much less than the dimensions of the vessel), an account must be taken of the finite conductivity and viscosity of the plasma.

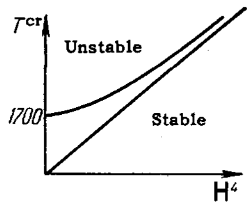


FIG. 12

Let us examine qualitatively two types of phenomena, which are influenced by the finite conductivity of the plasma: a) stability of the discharge column as a whole, b) internal (magnetic convective) stability of the pinch.

It is clear that if the finite conductivity is taken into account, all the previous conclusions regarding the stability of the pinch with respect to "sausages" and "kinks" remain in force, since the current always remains inside the pinch. But once a longitudinal magnetic field is applied, the picture is substantially changed.

Let us consider, for example, a screw-like bending of the entire pinch as a whole. Whereas in the case of ideal conductivity the force lines are dragged by the pinch, in the case of poor plasma conductivity the distortion of the external field can be neglected. Therefore the field exerts a radial force on the φ component of the current. It is clear that of the two possible helices, one will cause instability, pressing of the pinch against the wall (Figs. 13 – 14).

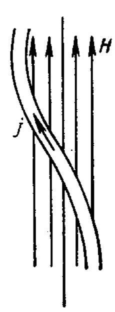


FIG. 13

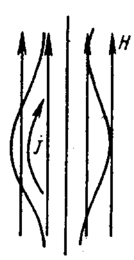


FIG. 14

In sausage-like perturbations the main effect is the occurrence of a moment in the z direction, due to the force $j_r H_z/c$. This moment causes rotation of the pinch as a whole, and a centrifugal force is produced. If the pinch is located in vacuum, the force has a highly destabilizing effect – it "splashes" the pinch. On the other hand, if the pinch is surrounded by an atmosphere of cold (and dense) gas, then the centrifugal effect, like in a centrifuge, will stabilize the pinch. The pinch will be in a certain "dynamically stable" state, rotating at a velocity such that the rise in the pressure gradient in the cold "jacket," due to the centrifugal force, offsets the drop in the magnetic pressure, and the work done by the current in the external field offsets the work of the friction forces.

The internal stability of the pinch is connected with the extent of the damping due to viscosity and due to finite conductivity. Actually, let us consider the convective instability investigated in Sec. 4d, with allowance for viscosity and finite conductivity of the plasma. A tube of magnetic force lines, accelerated by the difference between the gradient of the total pressure and the self-magnetic tension

$$\delta F = -\frac{B_0^2}{2\pi r_0}\left\{\frac{1}{\gamma p_0}\frac{p_0' + \frac{B_0}{4\pi}(B_0' - B_0/r)}{1 + B_0^2/4\pi\gamma p_0} + \frac{1}{r_0} - \frac{B_0'}{B_0}\right\}\delta r \qquad (5b.1)$$

[see (4c.10)] has, with allowance for the finite viscosity, a velocity

$$v \cong \frac{\delta r}{\delta t} \cong \frac{\delta F}{\eta}\lambda^2, \qquad (5b.2)$$

where λ is the dimension of the tube.

If the fields inside and outside the tube cannot be equalized within this time, i.e.,

$$\delta t < \frac{4\pi\sigma\lambda^2}{c^2}, \qquad (5b.3)$$

then convection develops. The critical conditions for the occurrence of magnetic convection are obtained by substituting δt from (5b.2) in (5b.3):

$$\mathrm{T}_m = \frac{\delta F}{\delta r}\frac{4\pi\sigma d^4}{\eta c^2}$$
$$= -\frac{2B_0^2\sigma}{\eta c^2 r}\left\{\frac{1}{\gamma p}\frac{p' + \frac{B_0}{4\pi}(B_0 - B_0/r)}{1 + B_0^2/4\pi\gamma p} + \frac{1}{r} - \frac{B_0'}{B_0}\right\} d^4 > \mathrm{T}_m^{cr} \sim \frac{d^4}{\lambda^4}. \qquad (5b.4)$$

The exact value of the critical magnetic Taylor number T_m^{cr} depends on the boundary conditions, and amounts to 1700 for convection between the isolating walls in a thin cylindrical layer of radius r and thickness d, and to 657 in the case of convection in a layer with free surfaces.

For a rarefied plasma ($\beta = 8\pi p/B^2 \ll 1$) this condition yields

$$\mathrm{T}_m = \frac{8\pi p'\sigma}{\eta c^2 r} d^4 \sim \frac{8\pi\sigma d^3}{c^2 r\nu}\frac{p}{\varrho} \sim \frac{6}{\gamma}\frac{4\pi e^2}{mc^2} n d^2 \frac{d}{r}\sqrt{\frac{M}{m}}\left(\frac{l_s}{r_{\lambda i}}\right)^2.$$

It is assumed that the plasma has "magnetized" viscosity. This quantity is usually very large, so that the convection is strongly "transcritical," i.e., with developed turbulence.

On the other hand, in a cold and dense plasma ($l_s \ll r_{\lambda i}$), of the type produced in plasmotrons and in stabilized arcs, convection may not set in.

If a longitudinal magnetic field exists inside the pinch, new effects may appear, allowance for which, however, is practically impossible in the general case. The opposite limiting case, when the longitudinal magnetic field is so strong that the azimuthal magnetic field can be neglected along with the plasma pressure (compared with $H_z^2/8\pi$), admits nonetheless of a simple analysis.[46] Let us assume that an electric field E_0 is applied in equilibrium along z and produces a current $j_0 = \sigma_0 E_0$, with σ_0 variable along the x coordinate (for simplicity we assume flat rather than cylindrical geometry). We consider perturbations of the form $\exp\{i(k_y y + k_z z) + i\omega t\}$, which develops at frequencies ω such that the distortion of the magnetic field can be neglected:

$$\text{rot}\,\mathbf{E} \approx 0, \quad \text{rot}\,\mathbf{H} \approx 0 \tag{5b.5}$$

(this means that the phase velocity ω/k of the perturbations should be much less than the characteristic velocity $H/\sqrt{4\pi\rho}$ connected with the magnitude of the magnetic field).

The x component of the velocity of the perturbed current-carrying tube will be

$$v_x = c\frac{E_y}{H_0}. \tag{5b.6}$$

The perturbation of the electric field is determined from the condition

$$j = \sigma E_0 + \sigma_0 E_z = 0. \tag{5b.7}$$

If we neglect the heat conduction, the change in conductivity σ will be due only to the motion of the plasma

$$\sigma = \frac{d\sigma_0}{dx}\xi_x, \tag{5b.8}$$

where ξ_x is the displacement of the current tube. Combining (5b.6), (5b.7), and (5b.8) and recognizing that $E_y = k_y E_z/k_z$ from (5b.5), we obtain the connection between the velocity v and the displacement ξ_x

$$v_x = c\frac{k_y}{k_z}\frac{E_0}{H_0}\frac{d\ln\sigma_0}{dx}\xi_x. \tag{5b.9}$$

Choosing the proper sign of k_y/k_z, we can construct perturbations which rise exponentially in time $e^{\nu t}$ (v_x is directed along the displacement ξ_x). The increment of such an instability will be

$$i\omega = \nu = \frac{v_x}{\xi_x} = c\frac{k_y}{k_z}\frac{E_0}{H_0}\frac{d\ln\sigma_0}{dx}. \tag{5b.10}$$

In this idealized analysis, any equilibrium in which the conductivity σ_0, meaning also the plasma temperature (as well as the density, if the plasma is not fully ionized), vary in space should be unstable. In fact, an account of the equalizing action of the heat conduction (for example, along the force lines) would yield instead of (5b.10) the following formula for the increment

$$\nu = -k_z^2\chi + c\frac{k_y}{k_z}\frac{E_0}{H_0}\frac{d\ln\sigma_0}{dx}. \tag{5b.11}$$

In connection with the foregoing instability, it is interesting to note that when the plasma is heated by the Joule heat created by the current flowing through it, the tendency towards spontaneous rise in the inhomogeneity in the spatial distribution of the temperature may become significant. This is caused by the fact that the conductivity σ is greater in a local volume with high temperature, and consequently the release of Joule heat σE^2 exceeds the average level. Thus, a unique "thermal" instability sets in.

III. OSCILLATING PLASMA INSTABILITY

Proceeding to investigate the oscillating instability of a plasma, we shall consider in Sec. 6 the instability of beams, and in Sec. 7 the microscopic instability of a "nonmaxwellian" plasma, due to phase resonance between the perturbation waves and individual groups of particles. In Secs. 8 and 9 we shall discuss the microscopic instability of a non-uniform plasma, due to phase resonance between the perturbation waves and the particle drift. Section 10 is devoted to instability of a plasma flowing in a magnetic field, due to phase resonance between the perturbation waves and individual layers of the flow, while Sec. 11 covers "periodic convection" in a plasma, i.e., instability of a heavy plasma, heated from below, subject to the condition $4\pi\sigma\chi/c^2 > 1$, i.e., when the magnetic field is equalized more slowly than the temperature. The occurrence of instability of the type of Alfven oscillations with increasing amplitude is connected in this last case with the competition between two different diffusion processes.

6. Instability of Beams in a Plasma

Because the mean free path of the particles in the plasma may in many cases be much greater than the dimensions of the vessel, groups of particles with different average velocities — beams — can exist in the same place in the plasma. These beams can be introduced into the plasma artificially (electron beams in electron-beam amplifiers, ion beams in the case of injection in a trap) or may be produced in the plasma by external fields (for example, "runaway" electrons). In many cases the plasma itself can be regarded as several interpenetrating electron and ion beams. In this section we are interested essentially in almost monoenergetic beams (the spread in particle velocities is much less than the average velocity). The instability of diffuse beams will be considered in Sec. 6a.

An interesting common property of almost-monoenergetic beams is the intense energy loss, which cannot be explained from the point of view of pair collisions.[23] Apparently this experimental fact is in many cases in good agreement with the theory of instability of beams.

We cannot cover in this review all the results of this theory,[23–25] and confine ourselves therefore to a study of infinite homogeneous beams, disregarding effects connected with finite transverse beam dimensions. In addition, we consider only quasi neutral beams. As is well known, almost-monoenergetic beams are well described by the hydrodynamic approximation, and only this approximation will be used in the present section.

In Sec. 6a we consider the instability of two beams, in Sec. 6b we analyze briefly the character of this instability, and in Sec. 6c we study the instability of two ion beams in a plasma with hot electrons. Finally, in Sec. 6d we consider the effect of the magnetic field on beam instability.

6a. Instability of two beams. We consider two particle beams with charges e_1 and e_2, masses m_1 and m_2, densities N_1 and N_2, and velocity spreads c_1 and c_2. The first beam is at rest, and the second moves at a velocity V.

The equation relating the complex frequency ω with the wave number k (derived in Appendix I) has the form

$$F\left(\frac{\omega}{k}\right)=\frac{\omega_1^2}{\left(\frac{\omega}{k}\right)^2-c_1^2}+\frac{\omega_2^2}{\left(\frac{\omega}{k}-V\right)^2-c_2^2}=k^2, \quad (6a.1)$$

where

$$\omega_1^2=\frac{4\pi e_1^2N_1}{m_1} \text{ and } \omega_2^2=\frac{4\pi e_2^2N_2}{m_2}.$$

Figure 15 shows the left half of (6a.1) as a function of ω/k for two cases: a) $c_1 + c_2 < V$, b) $c_1 + c_2 > V$.

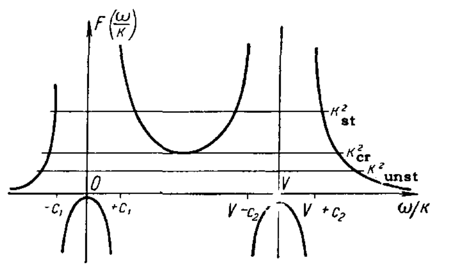


FIG. 15

If the line k^2 crosses $F(\omega/k)$ at four points, then (6a.1) has four real roots and the perturbations are purely periodic, i.e., the beams are stable against perturbations with this wave number.

In Fig. 15, the line k_{st}^2 corresponds to non-rising perturbations and k_{unst}^2 corresponds to perturbations whose amplitudes increase in time. The line k_{cr}^2 separates the stability and instability regions.

It follows from Fig. 16 that if $c_1 + c_2 > V$, the beams are stable.

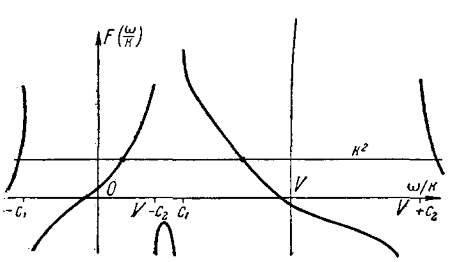


FIG. 16

When $c_1 + c_2 < V$, instability is brought about only by perturbations of sufficiently long wavelength. For monoenergetic beams it follows from (6a.1) that

$$k_{cr}^2=\frac{\omega_{p1}^2}{V^2}\left\{1+\left(\frac{\omega_{2p}}{\omega_{1p}}\right)^{2/3}\right\}^3. \quad (6a.2)$$

Of the four possible waves, two do not build up. These two waves are analogous to the ordinary Langmuir oscillations. The other two waves do not build up if they are sufficiently long.

Let us consider in greater detail the mechanism of "electrostatic" instability (as the instability against perturbations such as longitudinal plasma oscillations is sometimes called). By way of an example we choose greatly different beams — an electron beam moving through an ion gas at rest (or, perfectly analogously, a dense beam moving through a rarefied one). We assume that a long-wave perturbation ($k \ll \omega_2/V$, where ω_2 is the plasma frequency of the electrons) has been produced in the plasma at the initial instant of time. Since we are not interested in fast oscillations, this initial perturbation should be quasi neutral. We assume, however, that it has a small excess negative charge. Then ions will begin to gather in the field of this charge. On the other hand, the moving electrons slow down in it and their density in a region of negative space charge also increases. Therefore the initial quasi neutral perturbation builds up. The characteristic growth time of such a perturbation can be obtained from the following considerations.

From the conservation of the energy of the moving electrons it follows that

$$Vv_e-\frac{e}{m}\varphi=\text{const}$$

(the constant can be set equal to zero). Here v_e are the perturbed electron velocities and φ is the potential. From the conservation of the electron current it follows that

$$n_e=-\frac{N_ev_e}{V_0}=-\frac{N_ee}{mV^2}\varphi,$$

where n_e is the perturbation of the electron density. The Poisson equation then gives the following expression for the ion-density perturbation

$$n_i=\frac{1}{4\pi e}\left\{-\frac{\partial^2\varphi}{\partial x^2}-\frac{4\pi e^2N}{mV^2}\varphi\right\}. \quad (6a.3)$$

It follows therefore that for sufficiently long waves, as we have seen earlier, both the electron density and the ion density increase in the negative space charge region. From the equation of motion of ions in the field φ and from the continuity equation it follows that

$$\frac{\partial v_i}{\partial t}=-\frac{e}{M}\frac{\partial\varphi}{\partial x},$$

$$\frac{\partial n_i}{\partial t}=-N_i\frac{\partial v_i}{\partial x}, \text{ i.e., } \frac{\partial^2 n_i}{\partial t^2}=\frac{N_ie}{M}\frac{\partial^2\varphi}{\partial x^2}.$$

From (6a.4) and (6a.3) we obtain for a spatially sinusoidal perturbation

$$\frac{\partial^2 n_i}{\partial t^2}=-\frac{4\pi e^2N_i}{M}\frac{k^2}{k^2-\frac{4\pi e^2N}{mV^2}}n_i.$$

The characteristic growth time is

$$t_{gr}=\left\{\frac{\frac{4\pi e^2N}{mV^2}-k^2}{\frac{4\pi e^2N}{M}k^2}\right\}^{1/2}\sim\frac{1}{kV}\left(\frac{M}{m}\right)^{1/2}. \quad (6a.5)$$

The closer the wavelength is to "resonant" ($k_{res} = \omega_e/V$), the faster the perturbations grow.

The shortest growth time — growth at resonance — is obtained from (6a.1) by putting $k = \omega_e/V$ and neglecting ω/ω_e compared with unity:

$$t_{gr} \sim \frac{1}{\omega_i} = \left(\frac{2M}{m}\right)^{1/3} \frac{2}{\omega_e \sqrt{3}}. \quad (6a.6)$$

Here ω_i is the imaginary part of ω (increment).

In weak beams, the time (6a.6) is much shorter than that obtained from (6a.5). Therefore the velocity spread in the beam reduces the increment considerably even before the instability has completely vanished. It can be shown that when $c_e/V \sim (m/M)^{1/3}$ the mechanism described in the next section is more important to the growth of beam oscillations.

6b. Absolute and drift instability of beams. As already mentioned in the introduction, it is expedient to differentiate between absolute and drift instability. The system is absolutely unstable if a small perturbation, occurring at some instant of time and in a limited region of space, increases without limit as $t \to \infty$ in the same region. On the other hand, if the perturbation is transported and grows, but diminishes with time at the initial point, this is drift instability. Thus, the very formulation of the problem is meaningful only within the framework of the linear approximation; and only with respect to one particular type, namely perturbations that differ from zero at the initial instant of time only in a limited region of space.

Absolutely unstable systems can be used only as generators. On the other hand, systems in which drift instability exists, can be used as amplifiers.

To answer this question, we must solve the initial-value problem for the system of equations that describes small beam perturbations. A beam in a plasma at rest and beams moving in one direction are not absolutely unstable.

Beams moving in opposite directions are absolutely unstable. The physical reasons for this lie in the existence of feedback between the source of perturbation and the perturbation, which grows as it propagates.

6c. Stability of ion beams in a plasma. Let us consider an ion-electron plasma containing an ion beam. For simplicity we assume that the ion temperature is much lower than the electron temperature.

We introduce the following notation: ω_{p_1} — Langmuir frequency of the plasma ions (first beam), ω_{p_2} — Langmuir frequency of the beam ions (second beam), ω_e — Langmuir frequency of the electrons, V — velocity of ions of the second beam, k — wave number of the perturbations, $c_e = \sqrt{\gamma T/m}$ — thermal velocity of the electrons.

The dispersion equation, which is derived in Appendix 1, has the form

$$F\left(\frac{\omega}{k}\right) = \frac{\omega_{p1}^2}{(\omega/k)^2} + \frac{\omega_{p2}^2}{(\omega/k - V)^2} + \frac{\omega_e^2}{(\omega/k)^2 - c_e^2} = k^2. \quad (6c.1)$$

Assuming $V < c_e$, we obtain the dependence of the left half on ω/k, as shown in Fig. 17. Obviously the roots 1, 2, 3, and 4, obtained where the curves cross the line k^2 (which is parallel to the abscissa axis) are purely periodic perturbations, which do not lead to instability.

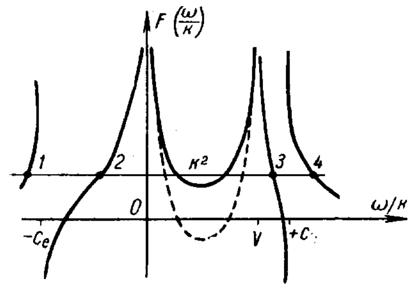


FIG. 17

The stability or instability of the beams depends on whether this straight branch intersects $F(\omega/k)$ when $0 < \omega/k < V$. For this purpose it is sufficient to know whether $F(\omega/k)$ has zeros in the range 0 — V. Thus, the beams are unstable if $F = 0$ has two real roots, and are stable if the number of roots is four. Let us consider the particular case $V \ll c_e$. We can then neglect in (6c.1) ω/k compared with c_e, and get

$$\frac{\omega_{p1}^2}{(\omega/k)^2} + \frac{\omega_{p2}^2}{(\omega/k - V)^2} = \frac{\omega_e^2}{c_e^2}. \quad (6c.2)$$

This equation is exactly equivalent to (6a.1). We therefore obtain from (6a.2) the stability condition

$$\frac{V^2}{c_e^2} > \frac{\omega_{p2}^2}{\omega_e^2} \left\{1 + \left(\frac{\omega_{p1}}{\omega_{p2}}\right)^{2/3}\right\}^3.$$

On the other hand, our analysis is valid only for $V^2 < c_e^2$. If this condition is violated, electrostatic instability occurs on the electronic branch. Therefore the condition for stability against excitation of the foregoing perturbations has the form

$$1 > \frac{V^2}{c_e^2} > \frac{\omega_{p2}^2}{\omega_{e1}^2} \left\{1 + \left(\frac{\omega_{p1}}{\omega_{p2}}\right)^{2/3}\right\}^3. \quad (6c.3)$$

We have considered here only one particular type of perturbation — oscillations in the direction of the beam velocity. If "oblique" perturbations ($k \nparallel V$) are taken into account, ion beams are always unstable. However, if there exists in the plasma a sufficiently strong magnetic field parallel to the beam velocity, then the criterion (6c.3) remains in force, i.e., sufficiently fast ion beams are stable. This is due to the fact that the magnetic field "suppresses" the oscillations perpendicular to **H**, and therefore the wave vector component $k_\perp$ drops out of the equation.

6d. Effect of magnetic field on beam instability. The presence of a magnetic field perpendicular to the direction of motion of the beam can appreciably change the results of the preceding section.

The magnetic field influences the motion of the ions in the wave whenever $\lambda \gtrsim r_{\lambda i}$. Here λ is the wavelength $r_{\lambda i}$ is the Larmor radius of the ion, and for the "resonant" wavelength $\lambda = 2\pi/k = 2\pi V/\omega_{pi}$. On the other hand, $r_{\lambda i} \sim v_{Ti}/\omega_{Hi}$, where v_{Ti} is the thermal velocity and ω_{Hi} is the gyrofrequency of the ions.

The field therefore affects the perturbed motion of ions only if

$$V/v_{Ti} > \frac{\omega_{pi}}{\omega_{Hi}} = \sqrt{4\pi NMc^2/H^2}\,. \tag{6d.1}$$

This last quantity is very large for a dense plasma. Consequently the field affects the motion of the ions in such a plasma only when very narrow beams are perturbed.

Let us consider a beam in a sufficiently weak field, i.e., we assume that $V/v_{Ti} < \sqrt{4\pi NMc^2/H^2}$. In this case the field influences only the electrons. If the "resonant" wavelength of the perturbations is greater than the Larmor radius of the electrons, this influence is appreciable. Condition (6d.1) is then replaced by

$$\frac{V}{c_e} > \frac{\omega_{pi}}{\omega_{He}}, \text{ i.e., } \frac{V}{v_{Ti}} > \left\{\frac{T_e}{T_i}\frac{4\pi Nmc^2}{H^2}\right\}^{1/2}. \tag{6d.2}$$

In the opposite case, $\dfrac{V}{v_{Ti}} > \left\{\dfrac{T_e}{T_i}\dfrac{Nmc^2\cdot 4\pi}{H^2}\right\}$, the field does not influence the perturbations at all. But if

$$\left\{\frac{4\pi NMc^2}{H^2}\right\}^{1/2} > \frac{V}{v_{Ti}} > \left\{\frac{T_e}{T_i}\frac{4\pi Nmc^2}{H^2}\right\}^{1/2}, \tag{6d.3}$$

then the electrons are "magnetized," i.e., they drift under the influence of the electric field of the perturbation wave.

The perturbation of the x component of the electron velocity (the x axis is directed, as everywhere in this section, along the velocity of the ion beam) is thus, according to Sec. 3,

$$v_{ex} \sim -\frac{c^2m}{eH^2}\frac{\partial E_x}{\partial t}. \tag{6d.4}$$

From the continuity equation for the electrons it follows that

$$\frac{\partial n_e}{\partial t} = -N_e\frac{\partial v_{ex}}{\partial x} = \frac{c^2mN_e}{e^2H^2}\frac{\partial^2 E_x}{\partial t\,\partial x}, \tag{6d.5}$$

i.e.,

$$n_e \cong \frac{c^2mN_e}{eH^2}\frac{\partial E_x}{\partial x}.$$

The electric field in the longitudinal wave is determined by the expression

$$\frac{\partial E_x}{\partial x} = -4\pi en_e + 4\pi en_i = -\frac{4\pi mN}{H^2}c^2\frac{\partial E_x}{\partial x} + 4\pi en_i.$$

Substituting n_i from Appendix I and neglecting the left half, we obtain

$$\frac{4\pi Nmc^2}{H^2}k^2 = \frac{\omega_{i1}^2}{(\omega/k)^2 - v_{Ti1}^2} + \frac{\omega_{i2}^2}{(\omega/k - V)^2 - v_{Ti2}^2}. \tag{6d.6}$$

Qualitatively this equation corresponds fully to (6a.1). Therefore, as before, the stability condition has the form $V > v_{T_1} + v_{T_2}$ and the kinetic mechanism of the instability becomes more appreciable when

$$\frac{v_{Ti}}{V} > \left(\frac{\omega_{i1}}{\omega_{i2}}\right)^{2/3} \text{ for } \frac{\omega_{i1}}{\omega_{i2}} \ll 1.$$

The resonant wavelength is now

$$k_{\text{res}} = \frac{\omega_{i2}\omega_H}{\omega_e V} \cong \frac{\sqrt{\omega_{Hi}\omega_{He}}}{V}\ (\omega_{i2} \gg \omega_{i1}),$$

i.e., resonance takes place at the "geometric mean" frequency.

Formula (6d.6) is no longer valid when the beam velocity V, being considerably greater than the thermal velocity of the ions, approaches the Alfven velocity $H/\sqrt{4\pi\rho}$. At greater velocities, as shown by calculation, the beam is unstable against oblique perturbations $k \nparallel V$.

If the charged-particle beams move along the field, then Cerenkov radiation of these waves sets in when the relative beam velocity exceeds the phase velocities of the waves. In addition, a strong interaction takes place between the beam particles and the waves when the wave frequency, changing as a result of the Doppler effect, coincides with the overtones of the Larmor frequency of rotation of the particles in the longitudinal field. This interaction will be considered in detail qualitatively in Sec. 7.

7. Microscopic Instability of "Nonmaxwellian" Plasma

In Sec. 4f we considered the aperiodic instability of a rarefied plasma with nonmaxwellian particle-velocity distribution. Generally speaking, the deviation of the particle distribution from equilibrium (Maxwellian) distribution can lead to build up of waves in the plasma, i.e., to the appearance of oscillating instability. The criterion for the occurrence of such an instability, i.e., the condition under which the imaginary part ω_i of the frequency $\omega = \omega_r + i\omega_i$ reverses sign, can be readily obtained by examining the balance of energy exchange between the plasma particles and any plasma wave produced by the fluctuations. At very small ω_i ($\omega_i \ll \omega_r$) the wave with given ω and corresponding wave vector k is almost periodic. The average energy of the plasma ions (or electrons) oscillating in the periodic field of the wave will not change. The only exception are those particles whose velocity satisfies the condition of resonance with the wave. In the absence of a magnetic field, the only particles at resonance in the unperturbed plasma are those whose velocity is close to the phase velocity ω/k of the wave (the resonance condition is $\omega - \mathbf{k}\cdot\mathbf{v} = 0$). In the presence of a constant external magnetic field, the wave will also interact effectively with the particles for which, in their own coordinate system, the Doppler effect causes the wave frequency $\omega' = \omega - k_{\parallel}v_{\parallel}$ to be close to the cyclotron frequency $\omega_H = eH/mc$ (or to one of its harmonics in ω_H): $\omega - k_{\|}v_{\|} - n\omega_H = 0$, $n = \pm 1, \pm 2, \ldots$.

Particles whose velocity component $v_{\|}$ along the magnetic field satisfies this condition will be continuously accelerated by the field of the wave, just as ions are accelerated in the cyclotron.

Let us examine some specific conditions for the build up of different waves in a plasma.

1. In the absence of a constant magnetic field, the waves that can propagate in a homogeneous plasma are either purely transverse or purely longitudinal. We do not consider transverse waves, since their phase velocity exceeds the velocity of light ($\epsilon = 1 - \omega_p^2/\omega^2$). The phase velocity of longitudinal Langmuir electron oscillations has a lower limit on the order of the thermal velocity of the electrons (the corresponding minimum wavelength is on the order of the Debye radius) and increases with increasing wavelength. Let us consider a Langmuir wave of frequency ω (and phase velocity ω/k); in the coordinate system moving with a velocity ω/k relative to the laboratory system, the profile of variation of the electrostatic potential is a stationary sine wave with amplitude φ_0, an alternation of the potential "wells" and "crests" for the electrons. Electrons with a velocity sufficiently different from ω/k will move freely in this periodic field, and their average energy remains constant. The electrons whose velocity v differs from ω/k by an amount less than $\sqrt{2e\varphi_0/m}$, will be reflected from the potential "crests." These electrons can be divided into two groups with velocities greater than and smaller than ω/k, respectively. The electrons of the first group catch up with the potential "crests," are reflected and thereby give up energy to the waves. The electrons of the second group are "whiplashed" by the wave and acquire energy from it. The amplitude of the wave will increase if on the whole energy is transferred from the electrons to the wave. This takes place if the number of electrons in the first group is greater than in the second group, i.e., if

$$\left(\frac{\partial f_0}{\partial v}\right)_{v=\frac{\omega}{k}} > 0.$$

To satisfy this condition it is necessary that the electron velocity distribution function have at least one additional maximum in the region above thermal velocity. On the other hand, if $\partial f_0/\partial v < 0$, $\omega_i < 0$ everywhere, i.e., the wave attenuates (Landau damping). The value of ω_i near the instability region ($\omega_i \ll \omega$) can be obtained (apart from a numerical factor) from a simple examination of the energy exchange between the wave and the resonant particles. As is well known,

$$\omega_i \equiv \gamma = \frac{1}{2\mathscr{E}} \frac{d\mathscr{E}}{dt},$$

where $\mathscr{E}$ is the energy density in the wave, which in our case is equal to the sum of the energy of the electric field and the kinetic energy $E_0^2/8\pi + (m/2)\sum_i v_i^2$ (E_0 is the amplitude of the electric field and v_i is the amplitude of the velocity of the i-th electron in the wave).

For Langmuir oscillations of frequency ω close to ω_0, the kinetic energy is equal to the energy of the electric field, so that

$$\mathscr{E} = E_0^2/4\pi = k^2\varphi_0^2/4\pi.$$

The rate of change of the energy density, $d\mathscr{E}/dt$, consists of the rate of change of energy delivered per unit time to the wave by the electrons of the first group

$$\dot{\mathscr{E}}_1 = n\frac{m}{2} \int_{\omega/k}^{\omega/k+\sqrt{2e\varphi_0/m}} \left[\frac{v^2}{2} - \frac{(2\omega/k - v)^2}{2}\right] \frac{-\frac{\omega}{k}+v}{\lambda} f_0(v)\,dv,$$

and the energy received from the wave by the electrons of the second group

$$\dot{\mathscr{E}}_2 = n\frac{m}{2} \int_{\omega/k-\sqrt{2e\varphi_0/m}}^{\omega/k} \left[\frac{v^2}{2} - \frac{(2\omega/k - v)^2}{2}\right] \frac{v-\omega/k}{\lambda} f_0(v)\,dv.$$

For a wave of low amplitude, $e\varphi_0 \ll m(\omega/k)^2$, the integrals are readily calculated by expanding $f_0(v)$ near $v = \omega/k$. Here

$$\frac{d\mathscr{E}}{dt} = \dot{\mathscr{E}}_1 + \dot{\mathscr{E}}_2 \cong \varphi_0^2\omega_0^3 \left(\frac{df_0}{dv}\right)_{\omega/k}$$

and

$$\gamma \sim \frac{\omega_0^3}{k^2}\left(\frac{df_0}{dv}\right)_{v=\omega/k}$$

2. In the presence of a constant magnetic field, there are many modes of plasma oscillation. At frequencies $\omega \ll \omega_{Hi}$ both Alfven and magnetoacoustic waves can propagate. We consider first the build-up of a magnetoacoustic wave, propagated at an angle to the magnetic field. When $\omega \ll \omega_{Hi}$ the magnetic moments μ of the electrons and ions are conserved. In the inhomogeneous magnetic field of the wave, these moments are acted upon by a force $-\mu\nabla H$. A wave with specified ω and k produces a periodic pattern of condensation and rarefaction of magnetic force lines, the pattern moving at the phase velocity ω/k of the wave. Under the influence of the force $-\mu\nabla H$, the particles moving along **H** will be reflected from the regions of denser force lines, if the component of the particle velocity in the direction of the magnetic field is not too different from the corresponding projection of the phase velocity of the wave

$$v_{\|} \sim \omega/k_{\|}.$$

Arguments analogous to those given in Sec. 1 lead to almost the same instability criterion

$$\int \mu \left(\frac{\partial f_0}{\partial v_{\|}}\right)_{\omega/k_{\|}} d\mu > 0.$$

The phase velocity of magnetoacoustic waves, as is well known, is of the same order of magnitude as

$H/\sqrt{4\pi\rho}$; in a sufficiently strong magnetic field, this velocity greatly exceeds the thermal velocity of the particles, so that instability against build-up of magnetoacoustic waves presents no danger in many of the typical experimental devices produced for the containment of plasma.

3. Let us examine the build-up produced by cyclotron resonance. In the first two sections we have considered the build-up of different types of oscillations; in either case, the instability is due to particles moving with a velocity close to the phase velocity of the wave.

We now consider the appearance of instability due to a group of particles which are in cyclotron resonance with the wave ($\omega - k_{\|}v_{\|} = n\omega_H$). It should be noted that in the case of waves propagating at an angle to the magnetic field, when the electric field of the wave has components both longitudinal and transverse relative to the constant magnetic field, the stability criterion for the build-up of the waves is determined by both mechanisms. In order to investigate the role of cyclotron resonance in pure form, we shall consider the simplest type of wave, propagating along a constant magnetic field ($k_\perp = 0$), with transverse polarization. Such a wave will interact effectively with particles of velocity

$$v_{\|} = (\omega - \omega_H)/k.$$

To derive the criterion of plasma stability against build-up of transverse waves with $k_\perp = 0$, let us estimate the work performed by the electric field of the wave on the plasma particles

$$\frac{d\mathcal{E}}{dt} \sim \overline{\mathbf{v}\mathbf{E}} \sim \bar{v}E,$$

where

$$\bar{v} = \int f_1 v\, dv,$$

and f_1 is the correction to the unperturbed distribution function f_0, due to the action of the wave on the particles. This correction is linear in the field of the wave and proportional to the quantity

$$\left(\mathbf{E} + \frac{[\mathbf{v},\ \mathbf{H}]}{c}\right)\frac{\partial f_0}{\partial \mathbf{v}}. \qquad (7.1)$$

Here $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields of the wave, connected by the relation

$$\mathbf{H} = \frac{c}{\omega}[\mathbf{k},\ \mathbf{E}].$$

Since the vector $\mathbf{E}$ of our transverse wave is perpendicular to the constant magnetic field $\mathbf{H}_0$, expression (7.1) can be rewritten

$$-Ev_\perp\left[\left(1 - \frac{k_{\|}v_{\|}}{\omega}\right)\frac{\partial f_0}{\partial \varepsilon_\perp} + \frac{k}{\omega}\frac{\partial f_0}{\partial (mv_{\|})}\right] \qquad (7.2)$$

[under the assumption that $f_0 = f_0\ (\epsilon_\perp,\ v_{\|})$, where $\epsilon_\perp = mv_\perp^2/2$].

Inasmuch as the particles effectively interacting with the wave have a velocity $v_{\|}$ satisfying the condition $v_{\|} = (\omega - \omega_H)/k$, we must put $v_{\|} = (\omega - \omega_H)/k$ in (7.2). Integrating (7.2) with respect to $\epsilon_\perp$, we then obtain the condition that the particles must satisfy if they are to give up energy to the field of the wave, i.e., the following instability takes place

$$\int \varepsilon_\perp \left[\left(1 - \frac{k_{\|}v_{\|}}{\omega}\right)\frac{\partial f_0}{\partial \varepsilon_\perp} + \frac{k}{\omega}\frac{\partial f_0}{\partial (mv_{\|})}\right]_{v_{\|} = \frac{\omega - \omega_H}{k}} d\varepsilon_\perp > 0. \qquad (7.3)$$

Let us consider, for example, an "anisotropic Maxwellian" distribution with different temperatures $T_{\|}$ and $T_\perp$:

$$f_0 \sim e^{-\frac{\varepsilon_\perp}{T_\perp} - \frac{mv_{\|}^2}{2T_{\|}}};$$

in this case the instability criterion (7.3) has the form

$$\frac{T_\perp}{T_{\|}} + \frac{\omega_H}{\omega}(1 - T_\perp/T_{\|}) < 0. \qquad (7.4)$$

For an isotropic plasma, $T_\perp = T_{\|}$ and the second term vanishes; if the degree of anisotropy is small (i.e., if $|1 - T_\perp/T_{\|}| \ll 1$), then instability can occur only for waves with frequencies ω considerably lower than the cyclotron frequency ω_H.

As is well known, transverse waves propagating along the magnetic field H_0 are circularly polarized; the direction of rotation of the vector of polarization is determined in our formulas by the sign of the frequency ω. Therefore, depending on the sign of $1 - T_\perp/T_{\|}$, instability can occur no matter what the sign of the anisotropy $1 - T_\perp/T_{\|}$, whether right-hand or left-hand polarized waves are excited.

It follows from (7.4) that instability takes place even at very low temperature anisotropy $|T_\perp - T_{\|}|/T$; in this case, however, the increment ω_i is exponentially small. Actually, the frequency ω_i is proportional to

$$f_0\left(v_{\|}\Big|_{\frac{\omega-\omega_H}{k}}\right) \sim \exp\left[-\frac{m}{2}\frac{(\omega - \omega_H)^2}{Tk^2}\right].$$

When $\omega \ll \omega_H$ we have $k^2 = \omega^2/c_H^2$, where $c_H^2 = H^2/4\pi\rho$; since according to (7.4) instability takes place when $\omega \leq \omega_H(T_{\|} - T_\perp)/T$, the increment ω_i for the most "dangerous" waves will be proportional to

$$\exp\left\{-\frac{m}{2}\frac{c_H^2}{T}\left(\frac{T}{T_{\|} - T_\perp}\right)^2\right\}.$$

It should be noted that the instability just considered may not occur in a real situation with low anisotropy, for the "cut-off" Maxwellian distribution (which apparently obtains in the experiment) does not contain the particles with large longitudinal velocities $v_{\|} \sim \sqrt{T/M}\ T/|T_{\|} - T_\perp|$, which are responsible for the build-up of the oscillations. In practice, therefore, the wave build-up due to such an instability can be noticeable only if the degree of anisotropy is sufficiently large and increases with the ratio of the magnetic pressure to the plasma pressure. Let, for example, $T_\perp > T_\parallel$. Let us estimate at what degree of anisotropy does the exponential factor $\exp\left\{-\frac{m}{2}\frac{(\omega-\omega_H)^2}{T_\parallel k^2}\right\}$, which is contained in the expression for the increment, become of the order of unity

$$\frac{m(\omega-\omega_H)^2}{T_\parallel k^2} \sim 1. \qquad (7.5)$$

The square of the wave vector k^2 can be expressed in terms of ω with the aid of the following well known dispersion relations for "cold" plasma:

$$N^2 = \frac{k^2c^2}{\omega^2} = \frac{\omega_{0i}^2}{\omega_{Hi}(\omega_{Hi}-\omega)},$$

if we deal with a wave having a polarization vector that rotates in the same direction as the ions rotate in the magnetic field. Substituting $k^2 = \omega_{0i}^2\,\omega^2/c^2\omega_{Hi}(\omega_{Hi}-\omega)$ in (7.5) and taking account of the fact that for instability, according to (7.4), we must have $\omega \lesssim \omega_{Hi}(1 - T_{\parallel i}/T_{\perp i})$, we obtain $T_{\perp i}/T_{\parallel i} \gtrsim \sqrt{H^2/8\pi n T_{\perp i}}$. Under this condition, the instability increment becomes noticeable. An analogous condition takes place also for a wave polarized in the direction of rotation of the electrons: $T_{\perp e}/T_{\parallel e} \gtrsim \sqrt{H^2/8\pi n T_{\perp e}}$. It is clear that, other conditions being equal, the electron mode should have a greater increment. Returning to the results of Sec. 4f, where we derived the conditions for aperiodic instability of a nonisotropic plasma, we note that when $H^2/8\pi nT$ is large the oscillating instability should occur at much lower anisotropy.

We have considered here essentially the simplest case of a transversely polarized wave propagating along a constant magnetic field. In an oblique wave ($\mathbf{k} \nparallel \mathbf{H}_0$) the situation under which instability sets in is more complicated, since it is also necessary to take into account the particles that are in multiple cyclotron resonance with the wave $\omega \pm n\omega_H + k_\parallel v_\parallel = 0$, where n is any integer. A similar problem was considered for highly idealized "background" distribution functions in reference 27. The instability due to anisotropy on the "ionic sound" branch was considered in reference 28. Ionic sound, as is well known (see Sec. 8) can exist only in a strongly nonisothermal plasma, $T_e \gg T_i$. It was shown accordingly in reference 28 that the corresponding instability should occur (for example, if $T_{e\parallel} > 5T_{i\parallel}$) when $T_{i\perp} > 2T_{i\perp}$.

8. Build-up of Oscillations in a Plasma in the Presence of Relative Motion of Ions and Electrons

A plasma can also be unstable when both the electrons and the ions have Maxwellian distributions, but move with a certain velocity u relative to each other, i.e., when current flows. It is natural to expect instability of the beam type to appear here (see Sec. 6) if, for example, we regard the electrons moving relative to the ions as a "beam." Instability should occur when u goes through a certain critical value, on the order of the phase velocity of the corresponding wave.

Let us consider, for simplicity, a case when there is no magnetic field (all the arguments are readily extended to the case when there exists a constant magnetic field, but the oscillations are along the force lines). The lowest phase velocity $v_{ph} \sim \sqrt{(T_e + T_i)/M}$ is possessed by the ionic longitudinal oscillations (ionic sound). However, if ω/k is not too much greater than the average thermal velocity of the ions $\sqrt{T_i/M}$, these oscillations attenuate rapidly (practically within several cycles), and transfer their energy to the ions, which move with a velocity on the order of the phase velocity of the wave. For these oscillations really to exist it is necessary that the condition $ZT_e \gg T_i$ be satisfied. In practice this condition is frequently satisfied even for a plasma with $Z \sim 1$ (Ze is the ion charge).

Let us determine the criterion for the instability of the plasma against build-up of ionic "sound" oscillations, by considering the interaction between the particles and the potential "crests," as was done in Sec. 7.

Assume that, in the coordinate system in which the average velocity of the ions is zero, the ion and electron distribution functions f_i and f_e have the form

$$f_i = (2\pi T_i/M)^{-1/2}\exp\{-Mv^2/2T_i\},$$
$$f_e = (2\pi T_e/m)^{-1/2}\exp\{-m(v-u)^2/2T_e\},$$

f_i and f_e are the complete distribution functions, integrated over the transverse velocities. The balance of energy exchanged between the plasma particles and the wave is determined by the energy transferred per unit time to the wave from the electrons

$$\frac{d\mathscr{E}_e}{dt} \sim \varphi_0^2\omega_0^2\omega\left(\frac{df_e}{dv}\right)_{v=\omega/k} \qquad (8.1)$$

and by the energy transferred from the wave to the ions

$$\frac{d\mathscr{E}_i}{dt} \sim \varphi_0^2\Omega_0^2\omega\left(\frac{df_i}{dv}\right)_{v=\omega/k}. \qquad (8.2)$$

The instability condition for the build-up of ionic sound assumes the form

$$\frac{d\mathscr{E}}{dt} = \frac{d\mathscr{E}_e}{dt} - \frac{d\mathscr{E}_i}{dt} > 0,$$

i.e.,

$$\frac{u-\omega/k}{\omega/k} > \sqrt{\frac{M}{m}}\left(\frac{T_e}{T_i}\right)^{3/2}\exp\left\{-\frac{1}{2}\left(\frac{T_e}{T_i}+3\right)\right\},\ Z=1. \qquad (8.3)$$

It is seen from (8.3) that for instability it is necessary that u be greater than ω/k; with increasing T_e/T_i, the excess of u over the phase velocity ω/k, necessary for the build-up of the ionic sound, decreases.

For clarity, we present a table showing the dependence of the quantity $y = \left(u_c - \frac{\omega}{k}\right)\Big/\frac{\omega}{k}$ (where u_c

x	8	9	10	11	13
y	5	4	3	2	1

is the relative velocity necessary for the occurrence of instability) on the ratio $x = T_e/T_i$ for a plasma with $Z = 1$ and $M/m = 3600$.

With the aid of (8.1) and (8.2) we can readily determine the increment $\gamma = \dot{\mathscr{E}}/2\mathscr{E}$, by dividing the rate of energy transfer from the particles to the wave by the energy density in the wave

$$\mathscr{E} \sim \frac{n}{M} \frac{e^2\varphi_0}{(\omega/k)^2},$$

$$\omega_i \sim \omega \left\{ \frac{\left(u - \frac{\omega}{k}\right)}{\omega/k} \frac{\sqrt{m}}{T_e^{3/2}} - \frac{\sqrt{M}}{T_i^{3/2}} e^{-\frac{1}{2}\left(\frac{T_e}{T_i}+3\right)} \right\} \frac{T_e^{3/2}}{\sqrt{M}}. \tag{8.4}$$

The ion-oscillation frequency spectrum has an upper limit on the order of the ionic Langmuir frequency $\Omega_i = \sqrt{4\pi ne^2/m}$. The phase velocity ω/k remains approximately equal to $\sqrt{ZT_e + T_i/M}$ in the entire region of the spectrum. Consequently, the increment ω_i increases in proportion to the frequency, and the most likely to be excited are the short waves with wavelength on the order of several times the ionic Debye radius.

9. Microscopic Instabilities of an Inhomogeneous Plasma

We have seen in Sec. 6 that the presence of a "beam" passing through a plasma can lead to instability. A special form of such "beam" instability can occur in a non-uniform plasma in the presence of a current $\mathbf{j}_0 = (e/4\pi)\,\mathrm{curl}\,\mathbf{H}_0 \neq 0$. The electrons and ions participating in the production of the current $\mathbf{j}_0$ comprise in some respect a "beam."

1. This effect can become particular pronounced in the thin boundary layer between the region of the magnetic field and the plasma current incident on this region (the Chapman-Ferraro problem). The electrons entering the region of the magnetic field are "reflected" from it, and penetrate only a distance on the order of the electronic Larmor radius r_H. On the other hand, the ions are hardly influenced by the magnetic field and are contained by the electrostatic forces.

In such a boundary layer, of thickness $\delta \sim r_H$, there should flow an electric current that ensures equality of the plasma and magnetic-field pressures

$$p = \frac{H^2}{8\pi}, \quad j \sim \frac{c}{4\pi}\frac{H}{\delta}.$$

The relative velocity of the ions and electrons carrying the current should be of the following order of magnitude inside this layer

$$v_0 = \frac{j}{en} \sim \frac{c}{4\pi}\frac{H}{en\,\delta} \sim \sqrt{\frac{p}{nm}}.$$

Thus, the relative velocity of the electrons inside the Chapman-Ferraro layer is on the order of the average thermal velocity and the "beam" instability described in Sec. 6 is possible.

2. In the case of small inhomogeneities (i.e., at low current densities) the velocity of the "beam" will be quite small and it can interact only with the slowest of the waves propagating in the plasma.

Let us consider the simplest equilibrium of a plasma confined by the pressure of a magnetic field with a constant direction everywhere, say along the z axis (Fig. 18).

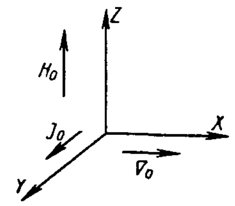


FIG. 18

Assume that in equilibrium all the quantities depend only on x. The current $\mathbf{j}_0$ is transverse to the z and x axes and parallel to the y axis. We consider the interaction between the particles and sound waves propagating transverse to $\mathbf{H}_0$ in the direction of the y axis. The particles effectively interacting with the wave will be those whose drift velocity v_{dr} is close to the phase velocity of the wave ω/k. Inasmuch as such electrons (ions) always have the same phase relative to the wave, they will be acted upon by a constant force. Under the influence of the y component of this force, the particles will drift with constant velocity $cF_y \times \mathbf{H}_0/eH_0^2$ along x. Electrons (ions) which will fall into the weakest magnetic field will lose energy, becoming adiabatically demagnetized (since the condition $\mu = mv_\perp^2/2H = \mathrm{const}$ implies $mv_\perp^2 \sim H$). To the contrary, the charges entering into regions when the field is stronger will acquire energy. The first tendency predominates if the region with the stronger field contains more electrons (ions) of the required energy (v_{dr} depends on $v_\perp^2 m/2$, namely $v_{dr} = c\frac{E_0}{H_0} + \frac{c}{eH_0^2}\frac{dH_0}{dx}\frac{mv_\perp^2}{2}$), i.e., when

$$H_0' \frac{\partial}{\partial x}\left(\frac{f_0}{H_0}\right)_{v_{dr}=\omega/k} < 0. \tag{9.1}$$

In this case the plasma energy as a whole will decrease and accordingly the wave energy will increase (i.e., instability takes place). For a Maxwellian distribution function and a weakly inhomogeneous plasma, (9.1) is equivalent to the condition[29,30]

$$\frac{d\ln T}{d\ln H} > 1.$$

However, at small inhomogeneities $H_0'/H_0 \ll 1/r_{Hi}$, the average particle drift velocities are very small compared with the thermal velocities. Only particles of energy much greater than the average thermal energy can resonate with the wave. The increment, which is proportional to the fraction of such particles, will consequently be exponentially small.

In principle, owing to the occurrence of anomalous dispersion at frequencies close to multiple cyclotron frequencies, waves can be found propagating transverse to the magnetic field with as small a phase velocity as desired. Such waves could interact effectively with particles carrying electric current, even in a weakly non-uniform plasma. This phenomenon can, however, hardly lead to a "universal" instability of any non-uniform plasma, inasmuch as anomalous dispersion is always accompanied by an anomalous absorption (at multiple cyclotron frequencies), thus ensuring a stability "margin."

3. The spontaneous growth of ionic sound oscillations, considered in Sec. 8, can occur under certain conditions in a non-uniform plasma. The reason for such an instability can be visualized as follows. It is well known that in a uniform plasma located in a strong magnetic field, the frequency of the ionic sound waves is independent of the wave-vector component transverse to the magnetic field if $\omega \ll eH/Mc$ (the magnetic field suppresses the transverse motions). On the other hand if the unperturbed plasma is not uniform, for example if it has a temperature that varies transversely to the force lines, then the transverse motions in the wave will be accompanied by heat flow ($\mathbf{v}$ grad T_0). An ionic sound wave can be imagined as an alternation of regions of compression (increase in temperature) and rarefaction (reduction in temperature), moving in space with a velocity $\sim \sqrt{T/M}$. Depending on the sign of the ratio of the components of the wave vector $k_{||}/k_{\perp}$ (parallel to and perpendicular to the magnetic field), the transverse influx of heat will take place, either in the compression region, or in the rarefaction region, owing to the inhomogeneity of the initial temperature. It is found that in the former case the ionic sound is unstable if the transverse motion is sufficiently large ($k_{\perp} \gg k_{||}$). The instability criterion (see Appendix IV) has the form

$$\frac{k_{\perp}}{k_{||}} > \frac{a}{r_{Hi}}.$$

Here r_{Hi} is the average Larmor radius of the ions, and a is the distance over which the temperature varies appreciably.

10. Stability of Plasma Flow in a Magnetic Field

The magnetic field affects the stability of a moving plasma in two ways. First, it changes the spatial distribution of the velocity of flow, and the stability depends greatly, as is well known, on the velocity profile. Second, the field also has a direct effect on the perturbations, i.e., on the stability. If the field does not have time to diffuse out of the flow perturbations within the inertia time, i.e., if $Re_m = 4\pi\sigma Lv/c^2 \gg 1$, then the direct effect of the field on the perturbation is in the appearance of the quasi elastic force connected with the deformation of the magnetic force lines. In this case the extent of the effect is naturally determined by the ratio of the magnetic energy density to the hydrodynamic energy density, i.e., by the dimensionless number $A = \sqrt{\frac{H^2/8\pi}{\rho v^2/2}} = \sqrt{\frac{H^2}{4\pi\rho v^2}}$, the so-called Alfven number. On the other hand, if the field is not perturbed, i.e., $Re_m < 1$, then its effect on the perturbations reduces to the appearance of an electromagnetic retardation force

$$f_m = \frac{jH}{c} \sim \frac{\sigma vH}{c}\frac{H}{c}.$$

The ratio of the density of this force to the density of the inertia forces yields a parameter (the Stuart number) which determines the extent of the effect of the field on a poorly conducting liquid:

$$\mathrm{St} = \frac{\sigma vH^2}{c^2}\frac{L}{\varrho v^2} = \frac{\sigma H^2 L}{c^2 \varrho v},$$

where L is the characteristic dimension of the system.

A distinguishing feature of the action of the magnetic field is that it does not affect the short-wave perturbations, since $Re_m^{(p)} \to 0$ and $St^{(p)} \to 0$ when $L^{(p)} \to 0$, where $L^{(p)}$ is the characteristic dimension of the perturbations.

Let us proceed now to an investigation of the stability of some specific flows. We consider first a flow in which the velocity distribution does not change when the magnetic field is applied. This includes plane laminar flow parallel to the field.

10a. Stability of plane flow. 1. The simplest example of plane flow in a longitudinal field is a separate small jet in a liquid at rest (Fig. 19). Let us consider long-wave perturbations in the jet. We can neglect here the electric resistance of the plasma and the tension

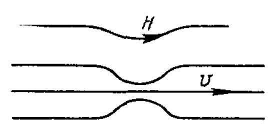


FIG. 19

of the curved force lines $(\mathbf{H}\cdot\nabla)\mathbf{H}$, and confine ourselves to an analysis of the potential terms. The pressure distribution and the field outside the jet are also little perturbed. In quasi-stationary bending

$$\frac{\varrho v^2}{2} + p = w = \text{const}.$$

For simplicity we confine ourselves to an incompressible liquid. Then $\rho v^2/2 = (\rho v_0^2/2)(\Sigma_0^2/\Sigma^2)$, where Σ is the cross section of the tube. The transverse total pressure will be in this case

$$\Phi = w + \frac{H_0^2}{8\pi}\Sigma_0^2/\Sigma^2 - \frac{\varrho v_0^2}{2}\Sigma_0^2/\Sigma^2 = \Phi_0 + \frac{\Sigma_0^2}{\Sigma^2}\left(\frac{H_0^2}{8\pi} - \frac{\varrho v_0^2}{2}\right).$$

The stability condition is that when the cross section increases the pressure inside the tube decreases, i.e., $\partial\Phi/\partial\Sigma < 0$. But

$$\frac{\partial\Phi}{\partial\Sigma} = -\frac{2\Sigma_0^2}{\Sigma_0^3}\left(\frac{H_0^2}{8\pi} - \frac{\varrho v^2}{2}\right),$$

so that the stability condition is $A > 1$.

For short-wave surface perturbations ($\lambda \ll \sqrt{\Sigma_0}$) we arrive at the problem of stability of a tangential discontinuity; it is easily seen that here the quasi elastic force, the perturbation of the external magnetic pressure, and the perturbation of the internal magnetic pressure are all of the same order of magnitude. As shown in reference 31, the stability condition for $Re_m \gg 1$ has the form

$$(H_1^2 + H_2^2)/4\pi \gg v_0^2\, \varrho_1\varrho_2/(\varrho_1 + \varrho_2),$$

where $H_{1,2}$ and $\rho_{1,2}$ are the field and density on the two sides of the tangential discontinuity. At equal densities $\rho_1 = \rho_2$ and equal fields $H_1 = H_2$, this condition becomes $A > 1/2$. As shown in reference 31, the compressibility affects this result very little.

For still shorter waves, $Re_m \to 0$, and the field no longer stabilizes them for any A. The question of whether such waves are dangerous is connected with the real structure of the layer, determined by the viscosity.

2. Let us examine the influence of the field on the stability of plane longitudinal flows with continuous velocity distribution. The velocity profile of a flow of this kind is determined by the viscous and inertia forces (Fig. 20).

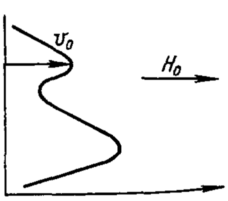


FIG. 20. Laminar longitudinal flow.

As is well known,[35] the stability of such a flow in the absence of a field is connected with the presence of a point at which $v_0''(z) = 0$, i.e.,

$$\nabla \operatorname{rot}_y \mathbf{v}_0 = 0.$$

This condition, first derived by Rayleigh, admits of a simple interpretation. If the vortex tube goes outside the layer in which it originally moved, acceleration is produced in a direction opposite to the motion of the tube, proportional to

$$\int\int [v'(x, y)]^2 \nabla \operatorname{rot}_y \mathbf{v}_0\, dx\, dy,$$

and therefore the tube returns to the original layer.

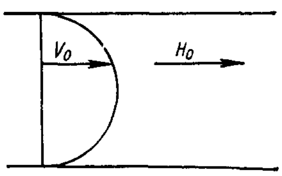
On the other hand, if somewhere in the flow we have $\nabla \operatorname{curl}_y \mathbf{v}_0 = 0$, nothing hinders the motion of the tube near this layer. On the other side of the layer, however, the acceleration reverses sign and the tube continues to move further. This causes the initial perturbation to "work loose," and the energy is transferred from the main flow to the perturbations.

The magnetic field should exert a stabilizing action on such inertially-unstable flows when $A \sim 1$, if $Re_m \gg 1$, and when $St \sim 1$, if $Re_m \ll 1$.

If the curl of the velocity does not have an extremum anywhere inside the flow, then the instability of flow can be connected only with the effect of viscosity, as was demonstrated for the first time by Heisenberg.[32] Let us consider the interaction between an individual perturbation wave and the flow. It is obvious that at large Reynolds numbers the viscosity can manifest itself only in the layer in which the phase velocity of the perturbation is equal to the velocity of flow. It is precisely near this layer that energy exchange takes place between the perturbation and the flow. The elements of the liquid can move gradually from one side of the layer to the other and become untwisted by the flow. The instability has an oscillating character, and we see that it is connected with phase resonance.

The stability investigated in greatest detail is that of plane Poiseuille flow between infinite parallel plates. If $Re_m \gg 1$, then phase resonance takes place between the flow and the Alfven waves. It is shown in reference 33 that the critical Reynolds number increases with increasing $A^2 = H^2/4\pi\rho_0 v_0^2$ and when $A = 0.1$ the Poiseuille flow becomes fully stable (Fig. 21).

FIG. 21. Poiseuille flow in a longitudinal field.



With decreasing conductivity, the field necessary to stabilize the flow increases, as shown in reference 34. As $Re_m \to 0$ (poor conductivity) the effect of the field, as has already been shown, reduces to an electromagnetic retarding force

$$jH/c \sim \sigma H^2 v^2/c^2.$$

The magnitude of this force is independent of the dimensions of the perturbation. As is well known from ordinary hydrodynamics, the fastest to grow are perturbations with wavelengths on the order of transverse dimensions of the stream. For shorter waves, the stabilizing role of the viscosity increases, and for longer ones, the destabilizing role decreases. The curves that separate the stability region from the instability region on the plane (k, Re_g), ($k = 2\pi/\lambda$ and λ is the wavelength of the perturbations) are shown in Fig. 22. They are similar to the equal-increment curves obtained in reference 35. The critical Reynolds number increases slowly with increasing Stuart number, but when $St \to 0.1$, the instability region of the stream contracts to a point.

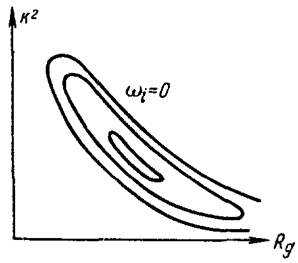


FIG. 22

The dependence of the critical Alfven number $A = \sqrt{H^2/4\pi\rho v_0^2}$ on the magnetic Reynolds number is illustrated in Fig. 23.

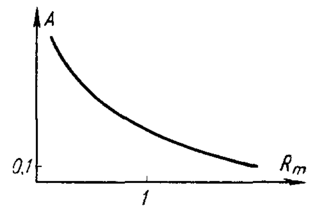


FIG. 23

$$\text{As } Re_m \to 0 \quad A^2 \to 0.1/Re_m,$$
$$\text{as } Re_m \to \infty \quad A \to 0.1.$$

Thus, in the case of poor conductivity Re_m is so to speak a measure of the action of a field on the flow.

10b. Stability of flows perpendicular to the field. We know that if the velocity of the liquid is perpendicular to the direction of the magnetic field, a Hartmann boundary layer forms near the wall perpendicular to the field. Its thickness is determined by the equality of the viscous and magnetic forces:

$$\frac{\eta v}{L_H^2 \varrho} \simeq \frac{\sigma H^2 v}{\varrho c^2},$$

i.e.,

$$L_H = \left(\frac{\eta c^2}{\sigma H^2}\right)^{1/2}.$$

On the other hand, if the wall is parallel to the field, then the thickness of the boundary layer is $\sim L/H^{1/2}$, where $H = (\sigma H^2 L^2/\eta c^2)^{1/2}$ is the Hartmann number, and L is the characteristic dimension of the flow.

In the first boundary layer, naturally, viscous forces predominate over all internal processes. Therefore the influence of the field on the flow is confined to its influence on the velocity profile. The critical Reynolds number is

$$Re_g^{cr} = vL_H/\nu = (vL/\nu)(1/H) \simeq 50\,000,$$

i.e., the Reynold number, calculated from the characteristic dimension of the flow is, on the order of

$$Re_g^{cr} \sim 50\,000 H.$$

The number 50,000 is the critical Reynolds number for an exponential boundary layer, known from ordinary hydrodynamics.

In the second type of boundary layer

$$Re_g^{cr} = vL/\nu \sqrt{H} \simeq 50\,000.$$

Thus, a magnetic field perpendicular to the flow increases in most cases the stability of the flow by concentrating the gradients near the walls. This effect is particularly appreciable in flow of the diffusor type, where reversal of flow is possible in the absence of the field. It should be noted, however, that the field can destabilize the flow. Actually, if in the absence of a field Re_g lies in the region on the left of the stability region of Fig. 22 (case of small gradients and high viscosity), then the field can destabilize the flow by increasing the velocity gradient, if Re_g falls in the instability region.

10c. Stability of a rotating plasma. In Sec. 4e we considered the stability of a rotating plasma against axially-symmetrical perturbations. Such perturbations bend the magnetic force lines and the stability depends on the ratio of the quasi elastic force produced by the perturbed field, to the force due to the faster decrease in the stationary pressure gradient compared with the centrifugal force acting on the perturbed force tubes.

In a large magnetic field, the quasi elastic force suppresses the instability. The field, however, does not influence "flute-like" perturbations, which are homogeneous along the axis of rotation (Fig. 24).

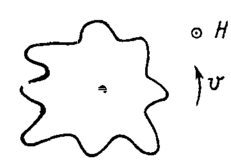


FIG. 24. Flute instability of rotating plasma.

If the density of the plasma decreases away from the axis of rotation, an instability of the type investigated in Section 4b, that of a plasma boundary in a gravitational field, is produced. Therefore such a rotation is absolutely unstable, as shown in Appendix I for $\beta = 8\pi p/H^2 \ll 1$.

If the plasma rotates like a solid, then the first to come out are two "tongues" on opposite sides.

If, however, the speed of rotation changes more rapidly than the density, then only the small-scale perturbations are unstable. These, however, are effectively suppressed by the viscosity (they may also be forbidden by the finite value of the Larmor radius in the rarefied plasma), so that the rotation may prove to be stable.

If the density of the plasma is constant, then the question of the stability of the rotation reduces to the question, already considered in Sec. 10a, of the instability of plane flow. The necessary condition for instability, as we have seen, is the presence in the plasma of a layer with an extremal velocity curl, i.e., a layer in which at some point

$$\frac{d}{dr}\,\mathrm{rot}_z \mathbf{v} = \frac{d}{dr}(v' + v/r) = 0.$$

On the other hand, if the rotation is inertially stable, then a finite viscosity can apparently cause it to become unstable at large Reynolds numbers, as usually occurs in plane flow.

11. "Oscillating Convection" in a Plasma

It was already mentioned in Sec. 5 that if the plasma contains a magnetic field whose diffusion is slower than the heat exchange, i.e.,

$$4\pi\sigma\chi/c^2 > 1, \tag{11.1}$$

where χ *is the coefficient of temperature conductivity*, then, as was shown first in reference 36, in a heavy plasma heated from below, Alfven waves with an amplitude that increases in time (Fig. 25) occur at a certain critical temperature gradient. This is connected with the fact that the plasma element is, as it were,

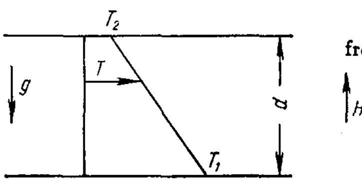


FIG. 25. Plasma layer heated from below in gravitational field.

"tied" to the force line, owing to the quasi elastic force. Under the influence of the Archimedean force the plasma element stretches the force line, and consequently it slows down and its temperature has time to become equal to the ambient temperature. With this, the Archimedean force decreases, the magnetic quasi elastic force again returns the element, which is heated again. The development of such an oscillating instability increases the heat transfer, like the development of *aperiodic convection. The conditions for the occurrence* of a periodic instability in a plasma heated from below, with allowance for viscosity, are:

$$\mathrm{Ra}^{\mathrm{cr}} = \frac{27}{4}\pi^4 \frac{S_1S_2 - \chi\nu c^2/4\pi\sigma}{\chi\nu(\chi+\nu)}, \quad \mathbf{H}^2 \to 0, \tag{11.2}$$

$$\mathrm{Ra}^{\mathrm{cr}} = \pi^2 \frac{(c^2/4\pi\sigma)(\nu + c^2/4\pi\sigma)}{\chi(\chi+\nu)}, \quad \mathbf{H}^2 \to \infty \tag{11.3}$$

for the free boundary.

Here

$$\mathrm{Ra} = \frac{\alpha g(T_1 - T_2)}{\chi\nu} d^3,$$
$$S_1 = \chi + \nu + c^2/4\pi\sigma,$$
$$S_2 = \chi\nu + \nu c^2/4\pi\sigma + \chi c^2/4\pi\sigma.$$

ν is the kinematic viscosity of the plasma, χ is the temperature conductivity, and α is the coefficient of *thermal expansion* (Secs. 4 — 5). *We know that without* a field $\mathrm{Ra}^{\mathrm{cr}} = 27\pi^4/4$ for a free boundary (1700 for a rigid one). We see therefore that the curve $\mathrm{Ra}^{\mathrm{cr}}(\mathrm{H})$ for an oscillating instability in weak fields always lies above the aperiodic $\mathrm{Ra}^{\mathrm{cr}}(\mathrm{H})$ curve.

In strong fields when $\chi > c^2/4\pi\sigma$, as follows from (11.3), the aperiodic instability occurs at lower Rayleigh numbers than the oscillating instability, i.e., at lower temperature gradients.

12. Instability of Positive Column of Gas Discharge in a Magnetic Field[37,38]

By observing the variation of the coefficient of diffusion of an incompletely ionized plasma from the positive column of a gas discharge in a long tube with varying external magnetic field parallel to the discharge axis, Lehnert observed that oscillations are produced in the plasma column when the field is increased above a certain critical value, and that the diffusion of the plasma on the tube walls increases at the same time.

The condition for the occurrence of instability and for the development of oscillations in a plasma column (considered in greater detail in Appendix II) can be derived for very long waves in the following fashion. The most dangerous perturbations are those in which the electron density remains almost constant, dn/dt = 0; these perturbations are cumulative in time. For *these slowly-varying perturbations the total derivative* d/dt is approximately equal to $\mathbf{v}\cdot\nabla$, where $\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$ is the electron velocity in the stationary distribution, made up of the velocity along the tube $\mathbf{v}_{\parallel}$ (the cause of the discharge current) and the drift velocity

$$\mathbf{v}_{\perp} = \frac{c}{eH^2}[\mathbf{HF}]$$

The velocity is due to the electrostatic force $e\nabla\varphi$ and the gradient of the pressure of the electron gas $\nabla(nT_e)$:

$$\mathbf{F} = e\nabla\varphi + T_e\nabla n/n.$$

If the perturbation varies in space as $\exp\{i(kz + m\theta)\}$, then the condition $(\mathbf{v}\cdot\nabla)n = 0$ yields

$$kv_{\parallel} + \frac{m}{r}\frac{c\Phi'}{H} = 0, \tag{12.1}$$

where $\Phi' = T_eN'/eN$ is the radial electric field, and N is the density of the electrons in the stationary distribution. It is obvious that this condition coincides with Eq. (III.6) in the limit of very long waves, as $k \to 0$. But to satisfy this condition for small $v_{\parallel}$, we must already have large values of k, for which diffusion must be taken into account. Allowance for diffusion leads to the appearance in (12.1) of a term proportional to k^4 (see Appendix II); the criterion obtained thereby for the stability of a plasma column has the form

$$v_{\parallel} < v_{\parallel}^{\mathrm{cr}}.$$

The critical electron velocity, accurate to a numerical factor, is

$$v_{\parallel}^{\mathrm{cr}} \sim \frac{D}{a}\sqrt{\frac{|m|}{\Omega\tau}},$$

where D is the coefficient of diffusion, a is the radius of $\Omega = eH/mc$, and τ^{-1} is the frequency of collisions between the electrons and the molecules.

As the magnetic field increases, $v_{\|}^{cr}$ decreases and when

$$v_{\|}^{cr} < v_{\|}$$

oscillations are produced in the plasma column which increase the diffusion flow of the particles towards the walls of the tube.

IV. PROBLEMS IN NONLINEAR STABILITY THEORY

In this chapter we consider a quasi linear approach to the "supercritical" state of the plasma, an approach used below to study the behavior of a plasma near the stability boundary. The quasi linear approach consists of accounting for the reaction of perturbations of finite amplitude on an average background, and of neglecting the interaction between various perturbation modes (i.e., the energy transfer between perturbations with different scales). This approach, naturally, is valid only at low "supercriticality" and reduces essentially to perturbation theory with expansion in terms of the small parameter

$$(\Lambda - \Lambda_{cr})/\Lambda_{cr},$$

where Λ is a dimensionless parameter characterizing the state of the plasma.

In Sec. 13 we consider "supercritical" convection in a plasma, and in Sec. 14 "supercritical" oscillating instability in a nonmaxwellian plasma.

13. Steady-State Convection in a Plasma and "Anomalous Diffusion"

Thomson[36] derived the conditions under which convection arises in a conducting heavy liquid heated from below. He showed that when $4\pi\sigma\chi/c^2 < 1$ (where χ is the temperature conductivity and σ the electric conductivity of the liquid) the convection is aperiodic and the effect of the magnetic field reduces to the action of an electromagnetic retardation force.

The aperiodic convection denotes that steady motion of the liquid — Benard cells — sets in under supercritical conditions (for example, when the temperature gradient exceeds some critical value). Motion of this kind in the absence of a magnetic field was investigated by Stuart[39] and by Gor'kov.[40] A similar study, with allowance for the magnetic field, was made by Nakagawa.[41] In this section we estimate the velocity amplitude and the heat transfer by convective motion when $4\pi\sigma\chi/c^2 < 1$.

We consider for this purpose a simplified convection model, wherein the liquid moves in a thin current tube of dimensions indicated in Fig. 26. It is obvious that the velocities of the liquid along and across the magnetic field are related by

$$v_{\|}/\lambda_{\|} \sim v_{\perp}/\lambda_{\perp}.$$

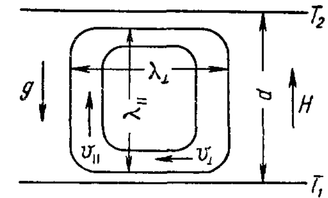


FIG. 26. Model of convection cell.

The work performed by the viscous forces per unit volume in a unit time is

$$\eta|\Delta|v_{\|}^2 + \eta|\Delta|v_{\perp}^2 = \eta|\Delta|v_{\|}^2(1 + \lambda_{\perp}^2/\lambda_{\|}^2),$$

$$|\Delta| = \frac{1}{\lambda_{\|}^2} + \frac{1}{\lambda_{\perp}^2}. \tag{13.1}$$

The work performed by the electromagnetic retardation force is

$$j\frac{H}{c}v_{\perp} \sim \frac{\sigma H^2}{c^2}v_{\perp}^2 \cong \frac{\sigma H^2}{c^2}v_{\|}^2\lambda_{\perp}^2/\lambda_{\|}^2, \tag{13.1a}$$

and that of the Archimedean force is

$$(\varrho - \langle\varrho\rangle)gv_{\|} = \alpha\langle\varrho\rangle(T - \langle T\rangle)gv_{\|}, \tag{13.2}$$

where α is the volume coefficient of expansion of the liquid, and $\langle T\rangle$ is the temperature averaged over all tubes at a given height.

Averaging the equation of heat conduction over all tubes, we get

$$\langle \mathbf{v}_{\|}\nabla(T - \langle T\rangle)\rangle = \chi\Delta\langle T\rangle, \tag{13.3}$$

on the other hand, inside each tube we have

$$\Delta(T - \langle T\rangle) = \frac{1}{\chi}\mathbf{v}_{\|}\nabla\langle T\rangle. \tag{13.4}$$

It follows from (13.3) that

$$\nabla\langle T\rangle = \frac{v_{\|}(T - \langle T\rangle)}{\chi} + \text{const}. \tag{13.5}$$

Substituting (13.5) in (13.4) we obtain

$$\nabla\langle T\rangle = \text{const}\left(1 - \frac{v_{\|}^2}{\chi^2}\frac{1}{|\Delta|}\right). \tag{13.6}$$

The constant is determined from the condition

$$\int_0^d \nabla\langle T\rangle\, dz = T_2 - T_1.$$

Therefore

$$\nabla\langle T\rangle = \frac{T_2 - T_1}{d}\left\{1 + \frac{\langle v_{\|}^2\rangle - v_{\|}^2}{\chi^2|\Delta|}\right\}, \quad \langle v_{\|}^2\rangle \equiv \frac{1}{d}\int_0^d v_{\|}^2\, dz \tag{13.7}$$

and from (13.4)

$$T - \langle T\rangle = -\frac{1}{|\Delta|}\frac{v_{\|}}{\chi}\frac{T_2 - T_1}{d}\left\{1 + \frac{\langle v_{\|}^2\rangle - v_{\|}^2}{\chi^2|\Delta|}\right\}. \tag{13.8}$$

It follows from (13.7) that the gradient of the temperature increases near the walls, where $v_{\|} \to 0$, and molecular transfer is effective, and decreases in the center where the heat is transferred by convection, as shown in Fig. 27. This change in the temperature profile reduces the work of the Archimedean force, i.e., leads to establishment of a finite convection amplitude, and also increases the heat flow.

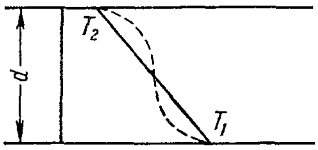


FIG. 27

Actually, near the wall

$$q = -\varkappa\nabla\langle T\rangle = \varkappa\frac{T_1-T_2}{d}\left\{1+\frac{\langle v_{\parallel}^2\rangle}{\chi^2|\Delta|}\right\} \qquad (13.9)$$

where κ is the heat conductivity of the plasma.

Substituting (13.8) and (13.2) and equating the work of the Archimedean force to the work of the viscous force and to the work of electromagnetic retardation, we obtain the square of the limiting convection amplitude

$$\langle v_{\parallel}^2\rangle = \frac{\chi^2}{A}\,\frac{(\lambda_{\perp}/\lambda_{\parallel})^2+1}{(\lambda_{\perp}/\lambda_{\parallel})^2}\left\{\frac{\mathrm{Ra}-\mathrm{Ra}^{\mathrm{cr}}}{\mathrm{Ra}}\right\}\lambda_{\parallel}^{-2}. \qquad (13.10)$$

Here $A = \langle v_{\parallel}^4\rangle/\langle v_{\parallel}^2\rangle^2 - 1$, and $\mathrm{Ra} = -\alpha\rho g(T_2 - T_1)d^3/\chi\eta$ is the Rayleigh number, the critical value of which is

$$\mathrm{Ra}^{\mathrm{cr}} = \frac{\lambda_{\parallel}^{-2}}{|\Delta|^{-1}}d^4\{|\Delta|\lambda_{\parallel}^2(1+\lambda_{\perp}^2/\lambda_{\parallel}^2)+\mathbf{H}^2(\lambda_{\perp}^2/\lambda_{\parallel}^2)(\lambda_{\parallel}^2/d^2)\}, \qquad (13.11)$$

$$\mathbf{H}^2 = \frac{H^2\sigma d^2}{c^2\eta}. \qquad (13.12)$$

The parameter A and the ratio $(d/\lambda_{\parallel})$ are determined by the boundary conditions (generally speaking, $d/\lambda_{\parallel} \sim \pi$).

Substituting (13.10) in (13.9) we obtain the heat flux

$$\frac{q}{q_{\mathrm{cr}}} = \left(1+\frac{1}{A}\right)\frac{\mathrm{Ra}}{\mathrm{Ra}^{\mathrm{cr}}} - \frac{1}{A}, \quad \mathrm{Ra} > \mathrm{Ra}^{\mathrm{cr}}, \qquad (13.13)$$

and

$$\frac{q}{q_{\mathrm{cr}}} = \frac{\mathrm{Ra}}{\mathrm{Ra}^{\mathrm{cr}}}, \quad \mathrm{Ra} < \mathrm{Ra}^{\mathrm{cr}},$$

where q_{cr} is the heat flux in the critical mode.

The heat flux transferred by convection can be written in the form

$$q_{\mathrm{conv}} = \frac{1}{A}\,\frac{\mathrm{Ra}-\mathrm{Ra}^{\mathrm{cr}}}{\mathrm{Ra}^{\mathrm{cr}}}\,q_{\mathrm{cr}}.$$

The form of the Benard cells and the dependence of the critical Rayleigh number $\mathrm{Ra}^{\mathrm{cr}}$ on the magnetic field can be obtained by minimizing the right half of (13.11). From our crude model

$$\lambda_{\perp}/\lambda_{\parallel} \sim \mathbf{H}^{-1/3},$$

i.e., the cells stretch out vertically with increasing magnetic field and

$$\mathrm{Ra}^{\mathrm{cr}} \sim \frac{d^2}{\lambda_{\parallel}^2}\mathbf{H}^2 \sim \pi^2|\mathbf{H}|^2, \quad \mathbf{H} \to \infty.$$

As $\mathbf{H} \to 0$ we have, for convection between solid plates, $\mathrm{Ra}^{\mathrm{cr}} = 1700$.

It follows from (13.13) that as $\mathbf{H} \to \infty$ we have $q/q_{\mathrm{cr}} \sim \mathbf{H}^{-2}$, i.e., the field effectively decreases the convective heat flux.

Exact values of A and $\mathrm{Ra}^{\mathrm{cr}}$ as functions of $\mathbf{H}$ are given in reference 41. At large $\mathbf{H}^2$ the parameter A depends little on the form of the boundary and approaches $1/2$. In this case $\mathrm{Ra}^{\mathrm{cr}} \to \pi^2\mathbf{H}^2$.

With increasing Ra, the above-described convection becomes unstable, new modes appear, the number of free phases increases, and turbulent convection sets in.

We have considered the dynamics of a liquid with an equation of state $\rho = \rho_0\{1 - \alpha(T - T_0)\}$, as is usually done in all papers on convection. It is easy to extend our analysis to include a real plasma. This involves replacing the temperature gradient $(T_2 - T_1)/d$ in the Rayleigh number by the difference between this gradient and the adiabatic temperature gradient, i.e., for a plasma,

$$\mathrm{Ra} = -\left(\nabla T + \frac{g}{c_p}\right)\frac{g\varrho}{T}\,\frac{d^4}{\chi\eta}.$$

Thus our analysis is valid when

$$|\nabla T| > \frac{g}{c_p},$$

where c_p is the specific heat at constant pressure.

We can investigate analogously supercritical flow in a rotating plasma (Taylor vortices), inasmuch as the problems are quite similar, both mathematically and physically. The difference lies in the dual role assumed by the viscosity. On the one hand, the viscosity slows down the convection due to the momentum transfer, and on the other hand it leads to diffusion of the velocity curl, thereby reducing the centrifugal effect. The analog of the Rayleigh number is the Taylor number. The average amplitude of the velocity is, according to (13.10),

$$\sqrt{\langle v^2\rangle} = \mathrm{const}\,(\mathrm{T} - \mathrm{T}^{\mathrm{cr}})^{1/2}.$$

As shown in Sec. 5,

$$\mathrm{T}^{\mathrm{cr}} \sim \mathbf{H}^4 \text{ as } |\mathbf{H}| \to \infty.$$

The change in the form of the Taylor vortices, naturally, coincides with the change in the form of the Benard cells — they stretch along the field. The problem without magnetic field is solved in reference 39.

In the case of magnetic convection, the amplitude in a plasma with closed force lines is limited by the redistribution of the current. The analog of the Rayleigh number is the magnetic Taylor number. The mean square velocity of motion of the magnetic force tubes is

$$\sqrt{\langle v^2\rangle} = \mathrm{const}\cdot(\mathrm{T}_m - \mathrm{T}_m^{\mathrm{cr}})^{1/2}.$$

If the convection is associated with a transfer of particles and heat, such as takes place near an absorbing boundary or a diaphragm, then the time of departure of the particles from the volume in supercritical mode is proportional to the velocity of the tubes and inversely proportional to the dimensions of the system, and not to the square of these dimensions, as would occur in molecular or developed turbulent diffusion. This dependence can permit a clarification of the true picture of diffusion by experiment (Fig. 28).

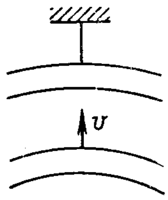


FIG. 28

If the force tubes in the plasma are not closed, then the pattern of supercritical motion is complicated by the appearance of quasi elastic forces, which obviously limit the motion of the tubes. The pattern of development of weakly supercritical instability reduces in this case to the following. Assume that the local condition of Sec. 4d is violated in some place in the plasma:

$$r\left(\frac{\mu'}{\mu}\right)^2 + \frac{32\pi p}{H^2} > 0.$$

Then the surface oscillations in this layer decrease the gradient of the pressure, and the instability shifts to the neighboring layer. Although the instability has a local character in each layer, the diffusion of particles in the entire plasma affected by the developing oscillations is sharply increased.

Closely related with the foregoing problems is that of "anomalous diffusion." We use quotation marks here to note that in all the experiments described below the "anomalous" character of diffusion is apparently connected, at a certain critical value of the field, with the macroscopic instability of the ground state, in which classical diffusion takes place. This phenomenon was observed experimentally in a plasma without current.

In Sec. 12 and in Appendix III we consider the stability of a gas discharge in a long tube in a longitudinal field. At a certain critical field value, as shown in Appendix III, the discharge becomes unstable. The resultant oscillations distort the average background and increase the particle flux across the magnetic field to the wall. In this case the instability of the discharge and the increase in the diffusion are connected in principle with the presence of a longitudinal current.

In many experiments, however, there is no current. In principle, disregarding the method by which the plasma is produced, it can be stated that these devices comprise a longitudinal field, an equipotential wall 1, and the region occupied by the plasma, 2 (Fig. 29). At a certain critical field, depending on the type and pressure of the plasma and of the residual gas in the apparatus, the ordinary diffusion is greatly disturbed and the flux of particles to the wall increases. This phenomenon is explained as follows.

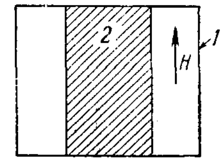


FIG. 29

Owing to the large mobility of the electrons, a potential difference is produced between the plasma and the quter case, on the order of the temperature of the electrons in the plasma. In a radial electric and axial magnetic field the plasma rotates with a certain average angular velocity

$$\langle\Omega\rangle \sim \frac{cE}{\langle r\rangle H} \sim \frac{c\varphi_0}{\langle r^2\rangle H},$$

where φ_0 is the potential of the plasma and $\langle r\rangle$ is the average distance to the axis. Since the density of the plasma decreases away from the axis, the deductions of Sec. 10 show the rotation of the plasma to be unstable. The drift time of the plasma is $\sim 1/\Omega \sim H/\varphi_0 \sim H/T_e$. On the other hand, the time of equalization of the density (connected with the ordinary diffusion) is $\sim H^2$. There is therefore a certain critical field $H_{cr} \sim 1/T_e$, at which instability can develop. When $H > H_{cr}$ the drift of the plasma to the walls is sharply slowed down. Strictly speaking, this process is no longer diffusion. The "tongues" of the rotating plasma simply "wriggle through" between the force lines and go to the wall. The electron temperature T_e and the critical magnetic field H_{cr} depend on the pressure in the chamber and on the gas in it.

14. Quasi Linear Approximation in the Analysis of Oscillating Instabilities in a Rarefied Plasma

In Secs. 4e, 7 and 8 we considered in the linear approximation the periodic instability caused by deviations of the plasma-particle distribution from Maxwellian. If this deviation is small and the increment is much less than the frequency, we can attempt, by taking into account terms of second order of smallness, to analyze approximately the establishment of equilibrium in the system and to estimate the relaxation time.

We assume that the distribution function is the sum of a rapidly oscillating part f^1 and a slowly-varying function f^0. The electric and magnetic fields of the oscillations are then represented as functions with high space and time frequencies but with slowly-varying amplitudes.

We consider for simplicity the case of longitudinal oscillations in a plasma without external magnetic field. The distribution function has the form

$$f = f^0 + f^1,$$

where

$$f^1 = \sum f^1_{\mathbf{k}} e^{i\mathbf{kr} - i\omega_{\mathbf{k}} t}, \quad f^1_{\mathbf{k}} = \frac{(e/m)\,\mathbf{E}_{\mathbf{k}}}{-i\omega_{\mathbf{k}} + i\mathbf{kv}} \nabla_v f^0,$$

and $\mathbf{E_k}$ is the Fourier component of the electric field.

Substituting this value of f^1 in the kinetic equation and averaging over the fast space and time oscillations we obtain an equation for the distribution function of the "background" f^0, which is slowly distorted under the influence of the oscillations:

$$\frac{\partial f^0}{\partial t}+\left\langle\frac{e}{m}E\nabla_v f^1\right\rangle^{(\mathbf{r},t)}=\frac{\partial f^0}{\partial t}+\nabla_v(A\nabla_v f^0)=0, \qquad (14.1)$$

where

$$A(v)=-\frac{e^2}{m^2}\sum_{\mathbf{k}}|\mathbf{E_k}|^2\pi\delta_{\omega_\mathbf{k}-\mathbf{kv}}.$$

In accordance with the arguments of Sec. 7, the oscillations interact effectively with the particles that move in resonance with the phase velocity of the wave

$$\omega=\mathbf{kv}.$$

These particles are indeed responsible for the distortion of the "background."

The amplitude of the electric field of the wave $|\mathbf{E_k}|$ satisfies the following equation (the linear approximation is adequate here)*

$$\frac{d|\mathbf{E_k}|^2}{dt}=2\gamma|\mathbf{E_k}|^2, \qquad (14.2)$$

where γ is the increment, a functional of the "background" f^0,

$$\gamma=\frac{\pi}{2}\frac{\omega_k^3}{k^2}(df^0/dv)_{v=\omega/k}. \qquad (14.3)$$

Accordingly, $\omega_\mathbf{k}$ is a function of $\mathbf{k}$, determined by the dispersion equation of the linear approximation.

To clarify the meaning of the approximation used, we must note that it is essentially analogous to the well known Van der Pol method, in which the motion of an inharmonic oscillator is represented by a superposition of rapidly-oscillating and slowly-varying functions. The equation for the slowly-varying part is obtained by averaging the initial equations, with allowance for the quadratic terms, over the rapidly oscillating function, which is determined in turn from the linear approximation. For this method to be applicable it is necessary not only that the amplitude of the oscillations be small, but also that the oscillations be "fast." It is found that the last condition makes the quasi linear approximation inapplicable to a monochromatic wave no matter how small the amplitude. It is applicable for the analysis of wave packets, the "width" of which is not too small,[49] $\Delta(\omega/k)\gg\sqrt{e\varphi/m}$ (here φ is the amplitude of the potential in the wave). This limitation can be readily understood by changing over to a system of coordinates connected with the resonant particles, $v = \omega/k$. In a monochromatic wave resonance takes place only for a single value of the velocity, and the corresponding frequency vanishes in the moving system of coordinates. It is clear that the condition of "fastness" of oscillations is violated here for the particles responsible for the distortion of the distribution function.

To estimate the relaxation time of a plasma which is unstable in the initial state against build-up of plasma oscillations, we make use of Eqs. (14.1) and (14.2). When a certain time τ has elapsed after the occurrence of the oscillations, the "background" changes in such a way that the oscillations stop increasing (a quasi-stationary distribution is established), so that $\partial f^0/\partial t = 0$.

Integrating (14.1) with respect to time from $t_1 = 0$ to $t_2 \sim \tau$, we obtain, using (14.2), the approximate equality

$$\Delta f^0=\int_{t_1}^{t_2}dt\,\nabla_v(A\nabla_v f^0)\sim\frac{A_0\nabla_v^2 f^0}{\gamma}e^{\gamma\tau}, \qquad (14.4)$$

where $\Delta f^0 = f^0(t_2) - f^0(t_1)$ is the total change in the "background" and A_0 is the value of A at the initial instant.

Relation (14.4) determines the order of magnitude of the relaxation time[42]

$$\tau\sim\gamma^{-1}\ln\frac{\gamma\Delta f^0}{A_0\nabla_v^2 f_0},$$
$$\tau\sim\frac{2}{\pi}\frac{k^2}{\omega_k^3}\frac{1}{(\partial f_0/\partial v)_{v=\omega/k}}\ln\frac{\gamma\Delta f^0}{A_0\nabla_v^2 f_0}. \qquad (14.5)$$

The form of the "background" in the final state (after relaxation) is determined by the equation

$$\nabla_\mathbf{v}(A\nabla_\mathbf{v}f^0)=0.$$

Instability sets in, as is well known (see Sec. 7) in the case when $(\partial f_0/\partial v)_{v=\omega/k} > 0$, i.e., the distribution function has, in addition to the principal maximum, at least one more maximum. The quasi stationary state is established when this additional maximum decreases and turns into a "plateau" (Fig. 30).

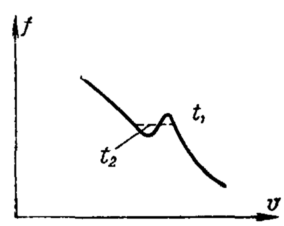


FIG. 30

*It is easy to include the radiation of the oscillations due to fluctuations in the quasi linear system. For this purpose we add to the right half of (14.2) the intensity $I_\mathbf{k}$ of this radiation. In the approximation of a perfect gas, each electron radiates independently and to obtain the total intensity it is therefore necessary to sum the contributions of the individual electrons $I_\mathbf{k} = \int \mathcal{J}_\mathbf{k} f^0(v)\,dv$. The spectral intensity of radiation of plasma waves by a separate electron is $\mathcal{J}_\mathbf{k} = \frac{1}{2\pi}\frac{\omega^2}{k^2}\delta(\omega - \mathbf{kv})$. We obtain finally $I_\mathbf{k} = \frac{1}{2\pi}\frac{\omega^2}{k^2}e^2\int f^0(v_x, v_y, \frac{\omega}{k})dv_x dv_y$. With the aid of this addition we can, for example, derive from (14.2) the stationary fluctuation level of the oscillations, by putting $d/dt\,|\mathbf{E_k}|^2 = 0$. Naturally, such a level exists only in the stability region ($df^0/dv < 0$). As the stability region is approached ($df^0/dv \to 0$), the noise amplitude increases.

15. Developed Instability

Approximate methods, of the type employed in Secs. 13 and 14, make it possible to investigate the "trans-critical mode" only at small deviations from the instability boundary. If the initial unperturbed state of the plasma is far beyond the stability region, the development of instability can bring the plasma to a state in which its motion will be turbulent; an essential feature of this state is the interaction between different scales, or in other words between different k. This interaction leads to a "fragmentation of the scales" and the pumping of energy from larger to smaller scales.

The theory of turbulence is fraught with tremendous difficulties even in the hydrodynamic approximation. Only the homogeneous isotropic case can be considered in practice. Reliable dimensionality considerations can be developed only for local isotropic turbulence (The Kolmogorov-Obukhov relations). It is clear that a magnetic field can disturb the isotropy. Still another difficulty arises in the kinetic analysis and consists of the following. In hydrodynamics the turbulence spectrum is always bounded from above, $k \ll 1/l$ (more accurately, the minimum turbulence scale in the mean free path l, multiplied by $\frac{\langle v\rangle}{v_0}\mathrm{Re}_g^{-1/4}$ where $\langle v\rangle/v_0$ is the ratio of the thermal velocity to the directed velocity). What can we assume for the minimum turbulence scale in a rarefied plasma, when $1/k$ is assumed at the very outset to be much less than l^2? This is a rather serious question, inasmuch as we deal with a dissipation mechanism that limits the "fragmentation" of the scales. In a rarefied plasma this mechanism cannot be ordinary viscosity, since we have already neglected pair collisions. If we resort to the "collisionless" damping mechanisms (similar to those considered in Sec. 7), which again have been investigated only in linearized theory, we obtain several such characteristic lengths: the Debye radius v_T/ω_0, the Larmor radii of the electrons v_{Te}/ω_{He} and of the ions v_{Ti}/ω_{Hi}. (It is known that oscillations of wavelengths approaching these values are anomalously absorbed.)

An additional effect in the damping of small-scale perturbations is the randomizing effect of "intersection of trajectories."[43]

Let us consider in greater detail some characteristic features of the decay of the spectrum of plasma oscillations, caused by interaction between harmonics. If the oscillation amplitude is small, it can serve as the expansion parameter. In the first approximation we have the non-interacting harmonics

$$f_k^{(1)} \sim e^{-i\omega_k t + i\mathbf{k}\mathbf{x}}.$$

In the second approximation (including terms quadratic in the amplitude) the generation of new harmonics is described by an equation of the type

$$Lf^{(2)} \cong \sum_{\mathbf{k}_1\mathbf{k}_2} f_{\mathbf{k}_1}^{(1)} f_{\mathbf{k}_2}^{(1)*} e^{-i(\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2})+i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{x}},$$

where L is a differential operator whose spectrum is $\omega_k = \omega(\mathbf{k})$. The expression in the right half can be considered as a "driving force" acting on the oscillator. If the force is in resonance with the natural vibrations of the oscillator, i.e., $\omega_2 = \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2}$ is a natural frequency corresponding to the wave vector $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$, then this force excites (generates) a new harmonic $\omega_2(\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2)$.

If $\mathbf{k}_1 \parallel \mathbf{k}_2$ (one-dimensional spectrum), this is possible, generally speaking, only for a linear dispersion law $\omega = ck$. In the general case, the possibility of such a decay depends on the form of a function $\omega(\mathbf{k})$ (Fig. 31).

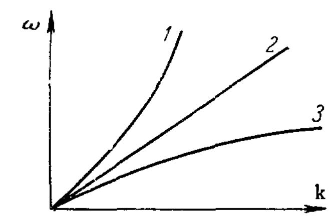


FIG. 31

For the dispersion curve 1 it is obvious that $|\mathbf{k}| < |\mathbf{k}_1| + |\mathbf{k}_2|$, and a triangle can be formed with k_1 and k_2. For curve 3 this cannot be done, and decay is therefore possible only for perturbations of type 1.

Langmuir oscillations, for example (see Sec. 7), belong to type 3, so that second-order decay is impossible for these oscillations. In the third order, the driving force will have the form

$$\sim \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} f_{\mathbf{k}_1} f_{\mathbf{k}_2} f_{\mathbf{k}_3} e^{-(\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}+\omega_{\mathbf{k}_3})+i(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)}.$$

It is easy to see that the system

$$\omega_{\mathbf{k}} = \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} + \omega_{\mathbf{k}_3},$$
$$\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3$$

has a solution for any dispersion law. The decay time of an arbitrary spectrum, generally speaking, is thus inversely proportional to the square of the amplitude. These arguments show[44] that the decay of a spectrum can be regarded as the consequence of collisions between quasi-particles or harmonics, for which the conservation laws $\omega = \Sigma\omega_i$ for the "energy" and $\mathbf{k} = \Sigma\mathbf{k}_i$ for the momentum are satisfied.

The ions and electrons of the plasma will execute "Brownian" motion under the influence of the electric and magnetic field. Knowing the spectrum of these oscillations, we can, in principle, readily obtain the corresponding diffusion coefficients. So far this has been done, unfortunately, only for the uninteresting case of equilibrium thermal vibrations, since the determination of the non-equilibrium spectrum entails great difficulties.

APPENDIX I

STABILITY OF BEAMS IN A PLASMA

We consider several beams moving with velocities V_i and consisting of particles of charge e_i and mass m_i. The particle temperature in the beams is T_i and the particle density is N_i. The Euler equation for small perturbations has the form

$$\frac{\partial v_i}{\partial t}+V_i\frac{\partial v_i}{\partial x}=-\frac{1}{mN_i}\frac{\partial p_i}{\partial x}+\frac{e_i}{m_i}E, \qquad (I.1)$$

the continuity equation is

$$\frac{\partial n_i}{\partial t}+V_i\frac{\partial n_i}{\partial x}+N_i\frac{\partial v_i}{\partial x}=0, \qquad (I.2)$$

and the equation for the longitudinal electric field is

$$\frac{\partial E}{\partial x}=4\pi\sum_i e_i n_i. \qquad (I.3)$$

In these equations v_i, n_i, and p_i are the perturbations of the velocity, density, and pressure of the particles in the i-th beam, and E is the intensity of the electric field of the perturbations. As is well known, for a perfect gas of particles we have in each beam

$$\frac{1}{m_i}p_i=\frac{\gamma_i T_i}{m_i}n_i, \qquad (I.4)$$

where γ_i is some effective adiabatic exponent, determined from the exact kinetic theory.

As usual, we seek the perturbations in the form $\exp(-i\omega t + ikx)$, and obtain the connection between ω and k from (I.3), substituting in it (I.1), (I.2) and (I.4):

$$k^2=\sum\frac{\omega_i^2}{(\omega/k-V_i)^2-c_i^2},$$

where

$$\omega_i^2=\frac{4\pi e_i^2 N_i}{m_i}, \quad c_i^2=\gamma_i\frac{T_i}{m_i}.$$

APPENDIX II

STABILITY OF ROTATING PLASMA

Let us derive the stability conditions for the rotation of the plasma, as given in Sec. 10c.

Electric fields can exist at the ends of the system (because the plasma borders on the vacuum) and consequently the occurrence of "interchange" instability (see Secs. 4b and 4c), i.e., instability of a plasma with $m \neq 0$, is possible. With this, the "tongues" of the plasma wriggle through between the force lines of the magnetic field without bending the latter. We shall find later on the conditions for stability of rotation against perturbations of this type. We confine ourselves to the case of a strong magnetic field ($8\pi p/H^2 \gg 1$). The motion of a plasma in such a field is similar to the motion of an incompressible liquid. From the equations of Chapter II we obtain an equation for the perturbation of the radial component of velocity of rotation

$$\left\{\frac{d^2}{dr^2}+\ln'(\varrho r)\frac{d}{dr}+\frac{\ln'(\Delta+\varrho)\Delta_+}{\frac{\omega r}{m}-V_0}+\frac{V_0^2}{r}\frac{\ln'\varrho}{\left(\frac{\omega r}{m}-V_0\right)^2}-\frac{m^2}{r^2}\right\}v_r=0 \qquad (II.1)$$

with boundary conditions $v_r = 0$ when $r = R_1$ and R_2.

Replacing v_r by $u = \sqrt{\rho}\, r^{3/2} v_r$, we obtain from (II.1)

$$u''+u\left\{-\frac{1}{2}\ln''(\varrho r)-\frac{1}{4}\ln'^2(\varrho r)+\frac{\ln'(\varrho\Delta_+)\Delta_+}{\frac{\omega r}{m}-V_0}+\frac{V_0^2}{r}\frac{\ln'\varrho}{\left(\frac{\omega r}{m}-V_0\right)^2}-\frac{m^2}{r^2}\right\}u=0. \qquad (II.2)$$

In these equations v_r is the radial component of the plasma-velocity perturbation, $\rho(r)$ is the density of the plasma, $V_0(r)$ is the azimuthal velocity of the plasma, $\Delta_+ = V_0'(r) + V_0/r$ is the curl of the plasma velocity, and the prime denotes differentiation with respect to the radius.

The third term in the braces is connected with the usual conservation of the velocity curl in the flow, and the fourth contains in its numerator $(\rho V_0^2/r)\nabla\rho - \nabla p \nabla\rho$, i.e., it is connected with the Taylor instability of a heavy liquid supported by a light one in a field of centrifugal forces.

Let us consider several particular examples.

a) Homogeneous rotation of a plasma. Let the entire plasma rotate with angular velocity ω. From (11.2) it follows that in this case

$$u''_{\xi\xi}+u\left\{-\xi^2+2E-\frac{m^2-\frac{1}{4}}{\xi^2}\right\}=0, \qquad (II.3)$$

where

$$\xi=\frac{r}{\sqrt{2}a}, \quad E=\frac{2(\omega-m\Omega)^2-\omega^2}{(\omega-m\Omega)^2}$$

and it is assumed that $\rho = \rho_0 \exp\{-r^2/2a^2\}$. Equation (II.3) is the Schrödinger equation for a three-dimensional isotropic oscillator with $l = m - \frac{1}{2}$. Therefore local solutions are possible when

$$E=m-\frac{1}{2}+n+\frac{3}{2}=m+n+1, \qquad (II.4)$$

where

$$n=0,\ 1\ \ldots$$

Solving the corresponding dispersion equation, we obtain

$$\omega_i=\frac{m\Omega\sqrt{E-2}}{E-1}, \quad \omega_r=\frac{\Omega m(E-2)}{E-1}. \qquad (II.5)$$

It is obvious that ω_i has a maximum when $E = 3$ and $m = 2$. The rotation is always unstable, and the most likely to wriggle out are two "tongues" on opposite sides. The presence of an internal or external wall merely changes the distribution of the levels in the effective potential well, as shown in Fig. 1.

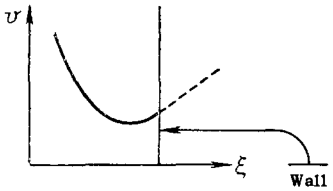


FIG. 1

b) Rotation of a plasma under the influence of an electric field in a cylindrical capacitor. If we apply to the electrodes a potential difference φ_0, then the field between them is

$$E=\frac{\varphi_0}{r\ln\frac{R_2}{R_1}}=\frac{M}{r}. \tag{II.6}$$

It is easy to see that in this case $\Delta_+ = 0$. Multiplying (II.1) by $(rv_r)^*$ and integrating by parts, we obtain from the condition that the real part of the integral vanish

$$\omega_i^2=-m^2\frac{\int\limits_{R_1}^{R_2}\left\{|(rv_r)'|^2+\left[\frac{m^2}{r^2}-\frac{V_0^2\varrho'}{r\varrho}\frac{\left(\frac{\omega_r r}{m}-V_0\right)^2}{|\omega r/m-V_0|^4}\right]|rv_r|^2\right\}rdr\varrho}{\int\limits_{R_1}^{R_2}|rv_r|^2 r^2V_0^2\varrho'\frac{dr}{|\omega r/m-V_0|^4}}, \tag{II.7}$$

where it is assumed that $\omega = \omega_r + i\omega_i$.

It follows from (II.7) that when $\rho' < 0$ the rotation is unstable for all φ_0. This is due to the lack of a stabilizing force. On the other hand, if the speed of rotation (or the electric field in our example) varies for some reason (say space charge) faster than r^{-1}, then a stabilizing force appears. For simplicity we assume that the plasma is contained in a thin gap between cylindrical walls (the gap width λ is much less than the radius r). In this case the dispersion equation has the form

$$\frac{\pi^2}{\lambda^2}=-\frac{1}{2}\ln''(\varrho r)-\frac{1}{4}\ln'^2(\varrho r)+\frac{\ln'(\varrho\Delta_+)\Delta_+}{\frac{\omega r}{m}-V_0}+\frac{V_0^2}{r}\frac{\ln'\varrho}{\left(\frac{\omega r}{m}-V_0\right)^2}-\frac{m^2}{r^2}, \tag{II.8}$$

$$\omega=\frac{m}{r}\left\{V_0+\frac{\Delta_+\ln'(\varrho\Delta_+)}{2A}\pm\sqrt{\frac{[\Delta_+\ln'(\varrho\Delta_+)]^2}{4A^2}+\frac{V_0^2}{r}\frac{\ln'\varrho}{A}}\right\}, \tag{II.9}$$

where r is the average radius of the plasma layer and

$$A=\frac{\pi^2}{\lambda^2}+\frac{m^2}{r^2}+\frac{1}{2}\ln''(\varrho r)+\frac{1}{4}\ln'^2(\varrho r). \tag{II.10}$$

The rotation is unstable if the density falls off away from the axis so much faster than the change in the velocity (regardless of the sign), that

$$-\frac{\varrho'}{\varrho}>\frac{r}{4AV_0^2}\left\{\frac{[\varrho(V_0'+V_0/r)]'}{\varrho}\right\}^2. \tag{II.11}$$

If $\rho'/\rho \sim -1/2$ and $V_0'/V_0 \sim -1/L$, then the rotation is unstable when the following inequality is satisfied

$$\frac{\frac{L}{l}}{\left[\frac{L}{l}+1\right]^2}>\frac{r}{4AL^3}. \tag{II.12}$$

As $m \to \infty$, it follows from (II.10) that ω has an imaginary part if $\rho' < 0$. In this case the rotation of a plasma of decreasing density is unstable. But, if the dissipative processes (or the finite Larmor radius of the ion) limit the dimensions of the perturbations (so that m cannot be very large), then the rotation may prove to be stable, provided the speed of rotation changes more rapidly than the density, as indicated in the text (Sec. 10c).

APPENDIX III

STABILITY OF A POSITIVE COLUMN[38]

We consider in greater detail the occurrence of plasma instability in the positive column of a gas discharge in a long tube, when a sufficiently large constant magnetic field, parallel to the axis of the discharge, is applied.

The initial equations of the problem are the equations for the balance of the number of ions n_i and electrons n_e per unit volume of the plasma (whereupon, by virtue of the quasi-neutrality, $n_i = n_e = n$)

$$\frac{\partial n}{\partial t}+\nabla n\mathbf{v}_i=\alpha n, \tag{III.1a}$$

$$\frac{\partial n}{\partial t}+\nabla n\mathbf{v}_e=\alpha n, \tag{III.1b}$$

where α is the number of charged particles formed per second by one electron.

The velocity of the ions $\mathbf{v}_i$ is determined by their mobility b_i and by the electric field $\mathbf{E} = -\nabla\varphi$

$$\mathbf{v}_i=-b_i\nabla\varphi;$$

In the presence of a strong longitudinal magnetic field H, the electrons drift in a transverse direction with velocity

$$\mathbf{v}_{e\perp}=\frac{c}{eH}[\mathbf{h},\mathbf{F}] \tag{III.2a}$$

and, in addition, move along the magnetic field with velocity

$$\mathbf{v}_{e\|}=b_e\frac{\mathbf{hF}}{e}. \tag{III.2b}$$

The total force $\mathbf{F}$ involved in these formulas and acting on the electron is made up of an electrostatic force $e\nabla\varphi$ and the gradient of the pressure of the electron gas $\nabla(nT_e)$, referred to a single particle:

$$\mathbf{F}=e\nabla\varphi+T_e\frac{\nabla n}{n}. \tag{III.3}$$

We consider the case of vanishingly small ion mobility b_i, when the diffusion to the walls and the rate of ionization tend to zero. We can drop then the αn terms in (III.1) after first linearizing these equations. In addition, we shall consider low-frequency oscillations in a plasma column, and therefore neglect the $\partial n/\partial t$ term in (III.1b). Then, denoting the equilibrium densities and the potential of the plasma by N and Φ, the state of electrons by V, and retaining the symbols n, φ, and v, for the deviations of these quantities from N, Φ, and V, we obtain by linearizing (III.1)

$$\frac{\partial n}{\partial t}-b_i\nabla(N\nabla\varphi+n\nabla\Phi)=0,\quad \nabla(N\mathbf{v}+n\mathbf{V})=0. \tag{III.4}$$

Let us consider a helical perturbation of the density and potential in the form $f(r)\exp(-i\omega t+im\theta+ikz)$ (z is the coordinate along the discharge axis and θ the azimuth) and assume for simplicity $|m|\gg 1$. Substituting the perturbation in (III.4) and using (III.2) and (III.3), we obtain, accurate to terms $\sim 1/m$, two algebraic equations for n and φ:

$$An-B\varphi=0,\quad -i\omega n+C\varphi=0, \tag{III.5a}$$

where

$$A=iku+\frac{im}{r}\frac{b_e}{\Omega\tau}\Phi'+k^2D,\quad B=k^2b_eN+\frac{im}{r}\frac{b_e}{\Omega\tau}N',\quad C=b_i\frac{m^2}{r^2}, \tag{III.5b}$$

$\Omega = eH/mc$, and u is the stationary velocity of the electrons along the discharge axis, while the prime denotes differentiation with respect to the radius.

After obtaining the frequency ω from (III.5a), we now determine the stability boundary of the plasma column from the condition $\operatorname{Im}\omega = 0$

$$\operatorname{Im}\omega=\operatorname{Re}i\omega=\frac{1}{c}\operatorname{Re}AB^*=0.$$

Substituting the values of A and B from (III.5b), we obtain the following equation* for the wave vector k

$$(kr)^4+(kr)\frac{ur}{D}\frac{m}{\Omega\tau}\frac{rN'}{N}+\left(\frac{m}{\Omega\tau}\frac{rN'}{N}\right)^2=0. \tag{III.6}$$

Instability is produced by the perturbation with $k = k_{cr}$ and $m < 0$ if the longitudinal electron velocity exceeds u_{cr}. The values of k_{cr} and u_{cr} can be obtained from (III.6) and from the derivative of (III.6) with respect to k_{cr}. Considering that $rN'/N \sim 1$, we obtain, apart from coefficients of order unity,

$$\frac{u_{cr}r}{D}\sim k_{cr}r\sim\sqrt{|m|/\Omega\tau}. \tag{III.7}$$

If we now express the longitudinal electron velocity u in terms of the electric field E, we can obtain from (III.7) the connection between the critical longitudinal electric field E_{cr} and the magnetic field H_{cr} at which the positive column loses stability

$$\frac{aeE_{cr}}{T_e}=\lambda\left(\frac{|m|}{\Omega_{cr}\tau}\right)^{1/2},\quad \Omega_{cr}=\frac{eH_{cr}}{mc}, \tag{III.8}$$

a is the radius of the column and λ is a numerical factor of order unity.

Instability occurs in the experiment if the magnetic field H exceeds the critical value H_{cr} given by formula (III.8).

*We note that if we neglect the last term of this formula (considering the case of very large $\Omega\tau$), the stability criterion (III.6) coincides with the stability criterion in the presence of a plasma-conductivity gradient (5b.11).

APPENDIX IV

IONIC SOUND IN A NON-UNIFORM PLASMA

We derive the dispersion equation for ionic sound oscillations in a non-uniform plasma. We direct the magnetic field H_0 along the z axis and assume that the plasma pressure is much less than the magnetic pressure, $p_0 \ll H_0^2/8\pi$. Let the equilibrium distribution of the electrons and ions depend on the coordinate κ. For perturbations of the form $\exp i(k_yy+k_zz)$, the corrections to the distribution functions $f_0(v,x)$ will have the following form (assuming that $\omega \ll eH_0/Mc$ — the drift approximation)

$$f_{i,e}=i\frac{1}{\omega+k_zv_z}\left\{c\frac{E_y}{H_0}\frac{\partial f_{0_{i,e}}}{\partial x}+\frac{e_{i,e}}{m_{i,e}}E_z\frac{\partial f_{0_{i,e}}}{\partial v_z}\right\}. \tag{IV.1}$$

The electric field can be considered with sufficient degree of accuracy to be potential, $E = -\operatorname{grad}\varphi$. Then the dispersion equation can be obtained from the condition

$$\operatorname{div}\mathbf{j}\approx ik_zj_z\approx 0,$$
$$j_z=\sum_{i,e}e_{i,e}\int v_zf_{i,e}dv_z. \tag{IV.2}$$

(The transverse components of the current density are suppressed by the magnetic field and do not influence the stability criterion. They must, however, be taken into account when the space dependence of the perturbation — the eigenfunction — is determined.)

The dispersion equation assumes the form

$$\sum_{i,e}\int\frac{v\,dv_z}{\omega+k_zv_z}\left\{\frac{c}{H_0}\frac{k_y}{k_z}\frac{\partial f_{0_{i,e}}}{\partial x}+\frac{e_{i,e}}{m_{i,e}}\frac{\partial f_{0_{i,e}}}{\partial v_z}\right\}=0. \tag{IV.3}$$

For Maxwellian ion and electron distribution functions with x-dependent temperature, we can readily obtain from (IV.3) the simple expression

$$\omega^3-2k_z^2k_y\frac{cT_0\frac{d}{dx}T_0}{eMH_0}\approx 0, \tag{IV.4}$$

which holds when the following condition is satisfied

$$\frac{k_y}{k_z}\gg\frac{eH_0}{c\sqrt{T_0M}\,\frac{d\ln T_0}{dx}}. \tag{IV.5}$$

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