PRINCIPLES OF FORMATION AND FOCUSING OF INTENSE BEAMS OF CHARGED PARTICLES

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1. INTRODUCTION

ALMOST all the scientific and technological discoveries of our time are related to the use of electronic devices. Among these are, for example, the charged particle accelerators of nuclear physics, high-frequency amplifier and oscillator tubes, the electron microscope, photomultipliers, electron beam devices, etc.

The basis of every electronic device is a beam of charged particles, in the majority of cases a beam of electrons. In such a beam, as in any statistical ensemble of particles, characteristic internal forces act on the particles. These forces may manifest themselves under various conditions, but we may disregard them entirely and treat the beam as consisting of a number of noninteracting charged particles. This neglect of internal forces in the beam is the procedure in electron microscopy and in most problems of electron optics. With the development of technology, there are more and more applications of long beams of charged particles with high current density. The obtaining and, more important, the focusing of such beams over a sizeable length has many special features which result from the interaction of the charged particles with one another and which are referred to as "space charge effects."

Beams of charged particles in which the internal forces play an important part, and cannot be neglected, are said to be intense. It is obvious that the concept of intense beams is introduced simply as a convention, though it is also very convenient from the practical point of view. Thus the magnitude of the space charge is characterized by the ratio of the current strength I carried by the beam to the $\frac{3}{2}$ power of the voltage (or energy) φ of the beam (the "perveance" of the beam). Computation shows that the influence of space charge can be neglected in electron beams with a perveance less than 10^{-7} amp/v$^{3/2}$.

For obtaining intense beams, one can use ordinary thermionic and oxide cathodes, powerful L-cathodes,[1,2] and also the phenomenon of surface ionization,[3] which consists in the formation of ions during the passage of neutral atoms through heated metallic surfaces. The focusing of beams which have a high density of space charge presents considerable difficulties. One can no longer speak of focusing in the sense in which it is used in electron optics. The term "focusing of an intense beam" should be taken to mean the forming of a current of particles into a beam with a more or less sharp boundary and with, possibly, laminar flow of the particles, as well as the maintenance of a given configuration over a given length and its preservation from the disrupting action of space charge and other perturbations along the path of motion of the particles. The shape of an intense beam and the distribution of particles over its cross section can be found in many cases by the use of a system of electrical probes, and can be recorded on ordinary photographic plates.[4] This is explained by the fact that an intense beam produces strong ionization along its trajectory even at low pressures of residual gas. The subsequent recombination with oppositely charged particles produces a noticeable illumination of the ionized region.

Although the methods of forming and focusing of beams in electronic devices are closely related to the mechanism of interaction of the beam with the controlling fields of the grids, decelerating electrodes and resonators, there are various general requirements on beams which permit us to disregard the interaction mechanism and treat the different focusing systems from a unified point of view.

The theoretical analysis of the behavior of intense beams in electromagnetic fields involves the solution of systems of nonlinear partial differential equations, so it is an extremely complicated mathematical problem. Over the course of the last few years, a large number of approximate solutions have been found, and several successful devices have been found for obtaining and focusing intense beams. An important part in the development of this vital branch of electronics was played by the work of S. A. Boguslavskii and V. R. Bursian on the influence of space charge on the motion of beams of charged particles, of V. S. Lukoshkov on methods for modeling intense beams, the work of Ya. I. Frenkel', S. A. Bobkovskii, M. M. Bredov, and S. M. Braginskii on problems of neutralization of space charge in beams, the work of V. T. Ovcharov and Z. S. Chernov on the theory and on new methods for focusing nonrectilinear flows. Of the work in other countries we should mention the surveys and original work by J. Pierce and H. Ivey, and the papers of P. Kirstein, B. Meltzer, G. Walker and others.

The purpose of the present survey is to collect and summarize the results of papers on the formation and preservation of intense beams of charged particles which have appeared in many different journals and books, and to point out problems which still await solution.

The existing literature surveys[5-12] on the effects of space charge treat the problems of formation and focusing of intense beams only indirectly, and are already obsolete.

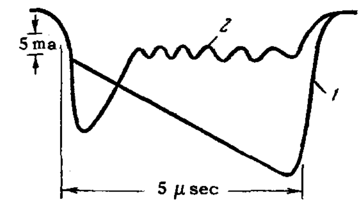
2. FUNDAMENTAL ASSUMPTIONS CONCERNING PROPERTIES OF INTENSE BEAMS

An individual charged particle in a beam is subjected along its path to the action of the following types of electromagnetic fields: 1) external electric and magnetic fields, usually stationary; 2) a certain averaged field, which results from the action on the particle of the other particles in the beam and the residual gases and ions in the system; 3) fluctuating fields which result from collisions of particles or from fluctuations of the current density in the beam.

Usually one assumes that the electromagnetic field strengths satisfy Maxwell's equations, and that the motion of the individual particle is described either by Newton's laws, if the particle velocities are small, or by the Einstein equations if the velocities are relativistic. However, if the fields change by a sizable amount over distances comparable to the de Broglie wavelength, $\lambda = h/p$ (where $h = 6.624 \times 10^{-27}$ erg-sec is Planck's constant, and p the momentum of the particle), we cannot use the concept of a trajectory to describe the motion of the particle, and we must take the particle's wave properties into account. In addition we must include the fact that every charged particle[13] has an intrinsic orbital angular momentum and magnetic moment. Finally, in any electronic vacuum system there are molecules of residual gas which will interact with the particles in the beam, and this interaction is described by the laws of quantum mechanics.

In most designs the vacuum is so high that we can neglect collisions of particles in the beam with molecules of residual gases. The effect of residual gas manifests itself in an additional focusing of beams of particles of the opposite charge[14,15] and in oscillatory behavior of the current density in the beam[16-20] (Fig. 1).

FIG. 1. Current oscillogram along axis of beam. The pressure in the system was 5×10^{-7} mm Hg. As the pressure is increased, the pulse 1 changes to the shape 2 (the pressure at this time was $\sim 10^{-4}$ mm Hg).



This focusing is explained by the fact that particles of the opposite charge which are formed as a result of ionization of the residual gases are captured inside the beam and are kept there by the attractive electrostatic forces. This type of compensation of the space charge is the simplest, and is observed at pressures from 10^{-2} to 10^{-7} mm Hg. However, the processes of ionization and trapping of charged particles are subject to strong fluctuations over the length of the beam.[21-23] In addition, the presence of residual gases in the system speeds the destruction of the electrodes through bombardment by ions of the residual gases.

The de Broglie wavelength λ in the systems which are in use at present is of the order of $10^{-8} - 10^{-9}$ cm, so that we may[24,25] disregard quantum effects. However these effects impose a limit on the resolving power of electron microscopes. A consistent quantum mechanical analysis of the behavior of intense beams of charged particles has not been carried out.

As was pointed out by Meltzer[26-27] and Winwood,[28] the magnetic interaction of particles in a beam can be neglected only for the case of nonrelativistic velocities and for low values of the ratio of width to length of the beam. For example, let us consider the flow of electrons in an infinite plane capacitor. Let us consider a current tube of radius r. According to Ampere's law, around the current tube there is a magnetic field which will have a focusing action on the current, and will change its motion and the charge distribution in it. The magnetic forces can be neglected only[27] if $r/l \ll 6\,(c/v)^2$, where r/l is the ratio of the radius of the beam to its length, and $c = 3 \times 10^{10}$ cm/sec is the velocity of light. For example, if the accelerating voltage is 50 kv, r/l must be less than

30. The main effects associated with magnetic interactions in beams are discussed in references 29 — 31.

Hydrodynamical analogies have received extensive application in the study of intense beams. For example, it is often possible to mark off a set of nonintersecting layers in the flow of charged particles, where the behavior of the current at each point of a particular layer can be characterized at any one time by a single velocity vector. By analogy with hydrodynamics, such flows are called laminar. Usually it is assumed that the individual particle moves, not in a discrete, but in a *continuous, electrically charged medium*. However in experimental work many phenomena are encountered which cannot be described by means of the concept of laminar flow. For example there are the formation of voids in beams, irregular fluctuations of current density along the beam, anomalous distributions of particle velocities, etc. Studies have been made of the conditions for breakdown of the laminar character of the flow in strong magnetic fields[32,33] and the breakdown from resulting nonuniform distribution of velocities over the cross section of the beam,[32,34] fluctuation of charge density,[35,36] various thermal effects,[37–39] etc.

3. TYPES OF FOCUSING. SPECIAL PROBLEMS OF THE THEORY

Intense beams with the desired dimensions and shape can be obtained only by the use of external restricting forces, which must compensate the disruptive forces of the space charge. Such forces may be those of external electrostatic fields, external magnetic fields, or the force fields of particles with the opposite charge which are produced as the intense beam passes through the residual gas in the system. In accordance with this description, we distinguish three types of focusing: electrostatic, magnetic, and gas focusing. There are also combined methods of focusing.[40,41] For example, in M- and O-type carcinotrons, perpendicular electric and magnetic fields are used[42–44] for the focusing of electron beams. However, attempts to produce high efficiency electron guns of this type meet with serious difficulties associated with the changing electromagnetic fields which act at the point of entry of the beam into the focusing system. Moreover, focusing by crossed electric and magnetic fields is closely related to the mechanism of removal of energy from the beam, so that the focusing effect cannot in this case be treated separately from the method for removing the high frequency energy.

Gas focusing methods have a strong influence on electron beams, but with increasing mass of the charged particles these effects decrease rapidly because of the high leakage of electrons from the region of the ion beam. In Sec. 2 we discussed the basic phenomena which accompany the neutralization of the space charge of a beam. An excellent survey of literature concerning this question can be found in reference 45.

The methods of focusing by use of electric and magnetic fields can, in turn, be divided into focusing by inhomogeneous and by uniform fields. In the first case the focusing fields do not change along the beam, while in the second the beam configuration is preserved by having it pass through a set of magnetic or electric lenses, prisms or mirrors.

Attempts to find the shape of the forming and focusing electrodes analytically leads to the problem of solving the Laplace equation, which gives the distribution of the potential outside a certain, in general, open surface, on which the values of the potential are known and where we impose the condition that there be no forces normal to the surface. In mathematical physics such a problem is called the Cauchy problem for elliptic equations. It is known[46] that this problem is not correct in the ordinary sense, i.e., its solution is very sensitive to slight changes in the boundary conditions. For example, if the exact value of the potential is given at two points on the x axis which are separated by the distance a, while the potential is known with an error $\epsilon \sin(\pi x/a)$ at the intermediate points on the x axis, one can show that in the region $y > 0$ the error in the solution will be given by $\epsilon \sin(\pi x/a) \cosh(\pi y/a)$, i.e., the error will increase rapidly with increasing y. With increasing precision in the fixing of the boundary conditions, i.e., with decreasing a, the error will oscillate and increase in absolute value. Such an instability of the solution was found in the work of Brewer,[47] Hechtel,[48] and Berz.[49] In reference 48, for example, the value of the potential at the boundary of the beam was expanded in power series; when three terms of the expansion were used, he obtained a shape for the electrodes which made physical sense, but these electrodes were completely changed in shape when six terms of the series were used.

It can be shown[50,51] that the Cauchy problem has a solution expressed in analytic functions only in restricted cases. Let us consider an ideally conducting plane which is at zero potential, and several isolated regions of accumulation of electric charge. Then as a result of induction a charge distribution appears on the plane, whose density at any point of the plane is determined by the value of the normal derivative of the potential. If now we forget how this distribution of charges on the plane was produced, and find the value of potential at an arbitrary point outside the plane, we are required to solve the Laplace equation with boundary conditions of the Cauchy type. However this solution must surely be incorrect in regions of accumulation of the electric charges, since there we must satisfy the Poisson equation which contains the density of the charges in such regions.

The stability of the plane Cauchy problem for the Laplace equation was demonstrated by Carleman.[52] Estimates characterizing the stability of the spatial Cauchy problem have been obtained by M. M. Lavrent'ev[53,54] for functions which are given over an arbitrary domain with sufficiently smooth boundary, and by E. M.

Landis[55] for second-order elliptic equations with certain restrictions on the coefficients in the equation. Radley[56] has shown how to find the solution of the Cauchy problem by using convergent series.

From the practical point of view, the instability of the solution has its bright side, since even sizable inaccuracies in the shaping of electrodes will produce only small perturbations of the flow along their boundaries. This permits us to apply conclusions which have been drawn for the case of infinite electrodes to actual focusing systems of finite dimensions.

4. FUNDAMENTAL CONSERVATION LAWS IN THE HYDRODYNAMIC THEORY OF INTENSE CURRENTS

According to Lorentz, the equation of motion for an individual particle of the beam has the form

$$\left(\frac{\partial}{\partial t}+(\mathbf{v}\nabla)\right)\frac{\mathbf{v}}{\sqrt{1-(v/c)^2}}=-\eta\{\mathbf{E}+[\mathbf{v},\ \mathbf{B}]\}, \tag{1*}$$

where the electric and magnetic fields **E** and **B** satisfy the Maxwell equations:

$$\left.\begin{aligned}&\operatorname{rot}\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0, && \operatorname{div}\mathbf{E}=\frac{\varrho}{\varepsilon},\\ &\operatorname{rot}\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}=\mu\varrho\mathbf{v}, && \operatorname{div}\mathbf{B}=0,\end{aligned}\right\} \tag{2†}$$

and ρ is the charge density, ϵ and μ are the permeability and dielectric constant of the vacuum, **v** is the velocity vector of the particle, and η is the charge-to-mass ratio of the particle.

In place of the field intensities, we use the vector potential **A** and scalar potential φ, with

$$\mathbf{B}=\operatorname{rot}\mathbf{A},\qquad \mathbf{E}=-\nabla\varphi-\frac{\partial\mathbf{A}}{\partial t}. \tag{3}$$

Just as in hydrodynamics, we can obtain the following fundamental conservation laws.

The law of conservation of charge:

$$\operatorname{div}\varrho\mathbf{v}+\frac{\partial\varrho}{\partial t}=0. \tag{4}$$

In the case of conservative fields, i.e., when **A** and φ are independent of the time, along any trajectory of the charged particles the law of conservation of energy is satisfied:

$$\frac{c^2}{\sqrt{1-(v/c)^2}}-\eta\varphi=\text{const}, \tag{5}$$

or, in the nonrelativistic case,

$$v^2-2\eta\varphi=\text{const}. \tag{6}$$

As Gabor[58] has shown, Lagrange's theory is valid for intense beams: the value of the integral

$$\oint_C(\mathbf{p}-e\mathbf{A})\,d\mathbf{l} \tag{7}$$

along a closed contour C surrounding a current tube is a constant for the given current tube. Obviously if the current tube emerges from a point, the Lagrange invariant is equal to zero. In the absence of magnetic fields and for nonrelativistic velocities, the Thomson theorem, which is well known in hydrodynamics,[59] follows from Lagrange's theorem:

$$\oint_C\mathbf{v}\,d\mathbf{l}=\text{const}. \tag{8}$$

In investigating nonlaminar flows with different statistical distributions of the velocities of the particles in the beam, one uses Liouville's theorem (cf. references 60 and 61). In this case the motion of an individual particle is treated as a motion in phase space, characterized by the coordinate triple x, y, z and the corresponding momentum values p_x, p_y, p_z, where the density of charges in a portion of phase space which moves with the particle does not change with time, i.e., the equation

$$\int dx\,dy\,dz\,dp_x\,dp_y\,dp_z=\text{const} \tag{9}$$

is satisfied along the trajectory of the charged particle.

The fundamental conservation laws which have been enumerated enable us, in many cases, to simplify calculations and to obtain quickly the necessary relations between various parameters of intense flows. Thus the study of one-dimensional flows of charged particles (the computation of plane, cylindrical, and spherical diodes) is based on the laws of conservation of charge and energy, and on Gauss' theorem, which enables us to find the value of the field intensity E. In making electron-optical systems, extensive use is made of Lagrange's theorem and its special case, Busch's theorem (cf. reference 62). The study of the propagation of electromagnetic waves of space charge[63,64] along intense beams in the linear approximation is based on the Maxwell equations and the general laws of motion. The investigation of different models of the statistical distribution of the initial velocities of the particles[60,61] and the finding of the limits to focusing of beams as a function of cathode temperature are based mainly on Liouville's theorem.

5. STATIONARY NONRELATIVISTIC BEAMS IN THE ABSENCE OF MAGNETIC FIELDS

If we set $v^2/c^2 \ll 1$ and $\mathbf{B} = 0$ in (1) and (2), the following equations, which describe the motion of an individual particle in such beams, will be satisfied:

$$\left.\begin{aligned}(\mathbf{v}\nabla)\mathbf{v}&=-\eta\mathbf{E},\\ \operatorname{div}\mathbf{E}&=\frac{\varrho}{\varepsilon},\\ \operatorname{rot}\mathbf{E}&=0.\end{aligned}\right| \tag{10}$$

In order for the system (10) to be complete, we must give the connection between the velocity and the charge density. To do this, we use the charge conservation equation (4), which in this case has the form

$$\operatorname{div}\varrho\mathbf{v}=0. \tag{11}$$

*[v, B] = v × B; vB = v · B.
†rot = curl.

The system (10) and (11) can be solved for the velocity:

$$\left.\begin{aligned} \operatorname{div}\{\mathbf{v}\nabla(\mathbf{v}\nabla)\mathbf{v}\} &= 0, \\ \operatorname{rot}\{(\mathbf{v}\nabla)\mathbf{v}\} &= 0. \end{aligned}\right\} \tag{12}$$

If we neglect the initial velocities of the particles and assume that the potential of the cathode is constant, all the particles in the beam will have the same energy. Such beams are said to be "normal"; the necessary and sufficient condition for such beams[65] is that

$$[\mathbf{v},\ \operatorname{rot}\mathbf{v}] = 0, \quad \text{i.e.,} \quad (\mathbf{v}\nabla)\mathbf{v} = \frac{1}{2}\nabla\mathbf{v}^2. \tag{13}$$

Thus, for normal beams the second of equations (12) is identically satisfied, and the problem reduces to the solution of the single scalar equation

$$\operatorname{div}\{\mathbf{v}\,\Delta v^2\} = 0. \tag{14}$$

Further simplifications of Eq. (14) were made by Spangenberg[66] and Walker[67] by using the Thomson theorem (8). The condition for the existence of laminar flows, in which all the particles in the beam move with the same velocity, is

$$\operatorname{rot}\mathbf{v} = 0, \quad \text{i.e.,} \quad \mathbf{v} = \nabla W, \tag{15}$$

where W is a scalar function which is called the action function (in hydrodynamics it is called the velocity potential). Then Eq. (14) can be replaced by a single nonlinear fourth-order partial differential equation for the determination of the single scalar function W. The other beam parameters are determined from W as follows:

$$\left.\begin{aligned} &\text{the potential } \varphi = \frac{(\nabla W)^2}{2\eta}, \\ &\text{the charge density } = -\frac{\varepsilon}{2\eta}\Delta(\nabla W)^2, \\ &\text{the trajectory } \frac{dx}{\frac{\partial W}{\partial x}} = \frac{dy}{\frac{\partial W}{\partial y}} = \frac{dz}{\frac{\partial W}{\partial z}}. \end{aligned}\right\} \tag{16}$$

To investigate equations (14) and (15), we choose the orthogonal, curvilinear coordinates x_1, x_2, x_3 so that the surface x_1 = const corresponds to W = const. We know[68] that the square of the line element ds^2 in such a coordinate system is expressed in terms of the Lamé coefficients h_1, h_2, h_3 as follows:

$$ds^2 = h_1^2 dx_1^2 + h_2^2 dx_2^2 + h_3^2 dx_3^2. \tag{17}$$

The necessary and sufficient condition for the existence of a one-dimensional laminar flow of charged particles, i.e., for a flow in which *all* the beam parameters are constant on the surface x_1 = const, and where the velocity vector is perpendicular to x_1, is[69–71] the equation

$$\frac{h_2 h_3}{h_1^5} W'[(W')^2]'' + \frac{\partial}{\partial x_1}\left(\frac{h_2 h_3}{h_1^5}\right) W'[(W')^2]' + \frac{h_2 h_3}{h_1}\Delta\frac{(W')^3}{h_1^2} = F(x_2 x_3),$$

where F is an arbitrary function of the variables x_2 and x_3, and the prime denotes differentiation with respect to x_1.

Consequently such a flow is possible only when either

$$\left.\begin{aligned} &\frac{h_2 h_3}{h_1}\Delta\frac{1}{h_1^2} = F(x_2 x_3), \text{ i.e., } W' = \text{const}, \\ \text{or} \\ &\frac{h_2 h_3}{h_1} = M(x_1)N(x_2 x_3) + \sum_i M_i(x_1)N_i(x_1 x_2 x_3), \end{aligned}\right\} \tag{18}$$

where M, M_i and N_i are arbitrary functions of the corresponding variables.

A similar investigation has been made recently[40] of two-dimensional flows of charged particles in systems with an external magnetic field.

No general methods for solving Eqs. (14) and (15) exist at present. Therefore only special cases are studied. Thus in references 67 and 82 the method of separation of variables was applied to the problem of propagation of a strong flow of charged particles between plane non-parallel electrodes and between two coaxial right cones with a common vertex. In references 83 – 85, the method of separation of variables was applied to systems with cylindrical and spherical symmetry. To find the action function, one sometimes uses conformal mapping methods.[79,86] However, as Meltzer[87] has pointed out, in multielectrode systems the beam parameters cannot be expressed in terms of analytic functions. In such systems it is suggested that one seek a solution of the Hadamard form.[46,54,55]

6. SHAPING OF BEAMS BY MEANS OF PIERCE ELECTRODES

The spreading of beams under the action of space charge forces can be neutralized by external electrostatic fields. A practical method for achieving such a result was first proposed by Pierce.[5,88] This method consists of the following. If, say, in a plane diode which is completely filled with space charge, we look at a portion of the flow of charged particles, then at the boundary of this partial beam the repulsive space charges will be compensated by the volume charge of the other particles. The same result can obviously be achieved if the region of space outside the beam is free of charge, but the electrodes are given such a geometrical shape and their potentials are so chosen that the location of the edges of the beam is unchanged. In general only two electrodes, which are at the potentials of the cathode and anode, respectively, are required for this. Such a system of electrodes is called a "Pierce gun," and the electrodes are called Pierce electrodes.

This problem of compensating the repulsive forces of the space charge at the boundary of the beam can be solved for the general case if one knows the potential distribution over the beam boundary. In mathematical physics, this problem is called the Cauchy problem for elliptic equations, and all the difficulties in solving it were discussed by us in Sec. 3. The required potential distribution over the beam boundary can be found by solving the Poisson equation in the interior of beam, or by experiment. We give below the electrode shapes as found analytically for different types of beams.

a) Ribbon beam:[5,88]

$$\varphi(r, \vartheta) = r^{4/3} \cos \frac{4}{3} \vartheta. \tag{19}$$

The shape of the electrodes is shown in Fig. 2.

b) Wedge beam:[58,89]

$$\varphi(r, \vartheta) = \mathrm{Re}\,\{\omega^{4/3} \sum_{k=0}^{\infty} b_k \omega^k\}, \tag{20}$$

where

$$\omega = \ln \frac{r_k}{r} + i(\vartheta - \vartheta_0).$$

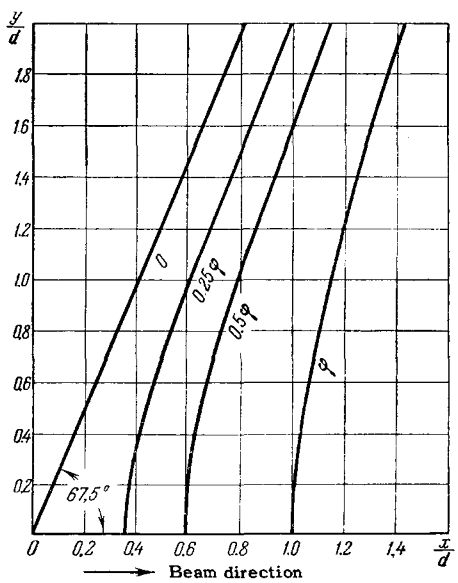


FIG. 2. Equipotential lines of the field required for producing a parallel beam of rectangular cross section. x, y – cartesian coordinates; d – interelectrode distance.

The shape of the electrodes is shown in Fig. 3.

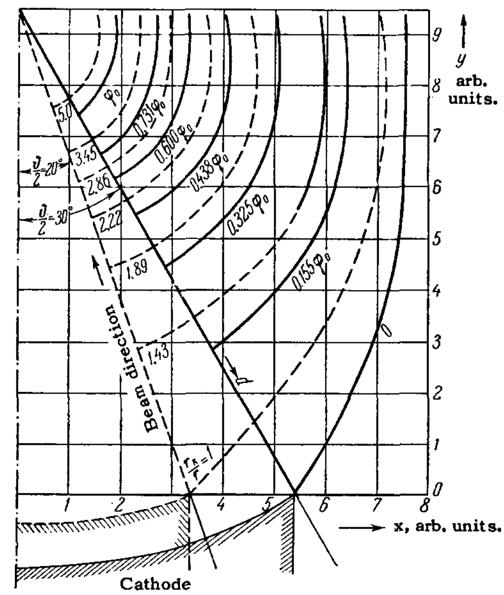


FIG. 3. Equipotential lines of the field required for producing a wedge beam, for different values of ϑ (40 and 60°). d – interelectrode distance; x – distance from cathode to a fixed point in the beam; φ – anode potential; r_k/r – ratio of cathode radius to fixed radius; y – distance from cathode along the beam axis.

c) Cylindrical beam:[90]

$$\varphi(r, z) = Az^{4/3} \sum_{k=0}^{\infty} (-1)^k \left(\frac{a}{z}\right)^{2k} \Phi_{2k}\left(\frac{r}{a}\right), \tag{21}$$

where a is the beam radius. In the region near the edge of the beam

$$\Phi_0 = 1, \quad \Phi_2\left(\frac{r}{a}\right) = \frac{1}{9}\left\{\left(\frac{r}{a}\right)^2 - 2\ln\frac{r}{a} - 1\right\},$$

$$\Phi_4\left(\frac{r}{a}\right) = \frac{10}{1296}\left\{\left(\frac{r}{a}\right)^4 - 8\left(\frac{r}{a}\right)^2\left[\ln\frac{r}{a} - \frac{1}{r}\right] - 4\ln\frac{r}{a} - 5\right\}\ldots,$$

while in the region $R < a^2$,

$$\varphi(r, z) = R^{2/3}\cos\frac{4}{3}\vartheta + \frac{R^{5/3}}{4a}\left\{\sin\frac{\vartheta}{3} - \frac{1}{7}\sin\frac{7}{3}\vartheta\right\} + \ldots \tag{22*}$$

where

$$R = z^2 + (r-a)^2, \quad \vartheta = \mathrm{arctg}\,\frac{r-a}{z}.$$

*arctg = $\tan^{-1}$.

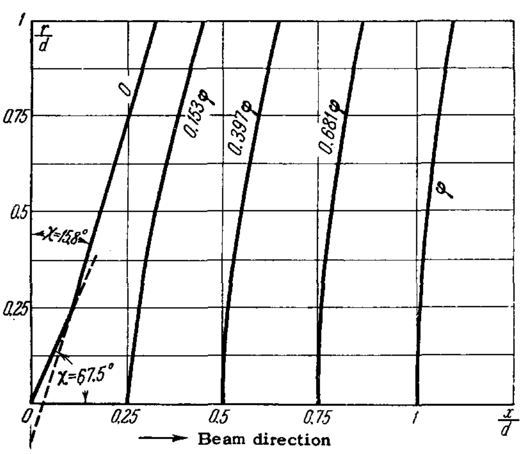


FIG. 4. Equipotential lines of the field required for producing a plane cylindrical beam. χ – angle of inclination of the shaping electrode relative to the beam.

The shape of the electrodes is shown in Fig. 4.

d) Hollow cylindrical beam.[90]

If the inner and outer radii of the beam are equal to a and b respectively, the shape of the equipotentials outside the beam is given by Eqs. (21) and (22), while for the region inside the beam we must replace a by b.

e) Conical beam:[56,91]

$$\varphi(r, \vartheta) = \left(\frac{\sin\vartheta_0}{\sin\vartheta}\right)^{1/2}\omega^{4/3}\left\{\cos\frac{4}{3}\psi - \frac{2}{5}\omega\cos\frac{7}{3}\psi + \frac{3}{7}\omega\frac{7\,\mathrm{ctg}\,\vartheta_0 + \mathrm{ctg}\,\vartheta}{8}\sin\frac{7}{3}\psi + O(\omega^2)\right\}, \tag{23*}$$

where

*ctg = cot.

$$\omega=\left|\ln\frac{r}{r_{\text{K}}}+i(\vartheta-\vartheta_0)\right|, \quad \psi=\operatorname{arctg}\frac{\vartheta-\vartheta_0}{\ln(r_{\text{K}}/r)}.$$

The shape of the electrodes for different opening angles ϑ_0 is shown in Fig. 5.

f) Azimuthal cylindrical beam:[78,79]

$$\varphi(r,\ \vartheta)=-2^{-4/3}a^{-2}\times\left[1.402+\frac{4\cos\left(\vartheta+\frac{2}{3}\pi\right)}{r}-\frac{5}{18}\frac{\cos\left(4\vartheta+\frac{2}{3}\pi\right)}{r^4}+\ldots\right], \tag{24}$$

if $r > 1$, and

$$\varphi(r,\ \vartheta)=2^{-4/3}a^{-2}\left[\frac{2\cos\left(2\vartheta-\frac{2}{3}\pi\right)}{r^2}+1.402+\frac{4}{3}r\cos\left(\vartheta+\frac{2}{3}\pi\right)+\frac{1}{6}r^4\cos\left(4\vartheta+\frac{2}{3}\pi\right)+\ldots\right], \tag{25}$$

if $r < 1$.

The shape of the electrodes is given in Fig. 6, where a is the radius of the limiting trajectory of the beam.

g) Azimuthal spherical beam:[74]

$$\varphi(r,\ \vartheta)=\theta(\vartheta,\ \vartheta_0)\left\{1+2\left(1-\frac{r}{a}\right)+[3-(2\theta^{3/2}\sin\vartheta)^{-1}]\left(1-\frac{r}{a}\right)^2+[4-(\theta^{3/2}\sin\vartheta)^{-1}]\left(1-\frac{r}{a}\right)^3+\ldots,\right. \tag{26}$$

where $\theta(\vartheta,\ \vartheta_0)$ is the function tabulated in reference 74, and a is the limiting trajectory of the beam. The shape of the electrodes for particular angles ϑ_0 is given in reference 74.

By writing the Laplace equation in orthogonal curvilinear coordinates for the small region in front of the cathode, Radley[56] showed that the focusing electrode at zero potential should in general make an angle of

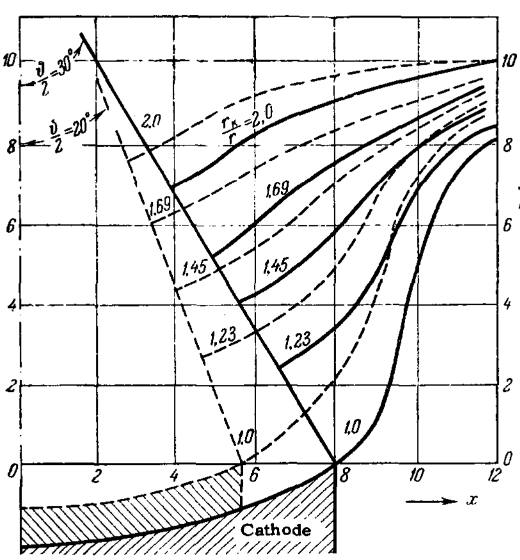


FIG. 5. Shape of equipotential electrodes which produce a converging conical beam with opening angle ϑ (40 and 60°).

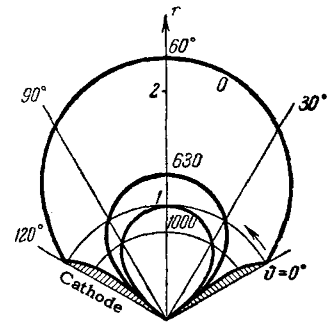


FIG. 6. Polar plot of shape of equipotential electrodes for obtaining an azimuthal cylindrical beam.

67.5° with the boundary of the beam. Changing this angle[92] changes the angle at which the beam leaves the cathode, and also changes the current at the point where the electrode intersects the beam. In Fig. 7 we show the dependence of the angle β which the cathode shaping electrode makes with the vertical (measured clockwise from the vertical) on the angle α between the electrodes and the boundary of the beam.

If we know how to construct the gun using a two-electrode system, we can use[93] this solution to obtain a triode type Pierce electron gun. For example, let us consider a typical system of equipotential lines, found by analytic methods or by various modeling apparatus, and shown schematically in Fig. 8. If we ex-

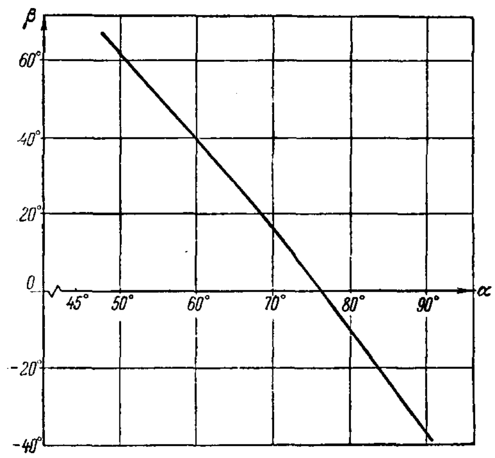


FIG. 7. Dependence of the angle β, which the cathode shaping electrode makes with the vertical, on the angle α between the electrode and the boundary of the rectangular beam. The angle β is measured clockwise from the vertical.

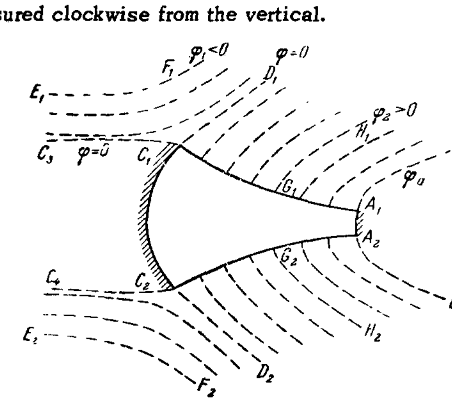


FIG. 8. Shape of equipotential curves of a two-electrode Pierce gun.

tend the zero-potential electrode along the equipotentials C_1D_1 and C_2D_2, and extend the electrode at anode potential along the equipotentials A_1B_1 and A_2B_2, the forces exerted by the space charge in the beam will be compensated by the external field. It is obvious that the boundary of the beam will not change its shape if 1) we place an electrode with the appropriate positive potential φ_2 at the position of any equipotential G_1H_1 and G_2H_2; 2) we replace the electrodes C_1D_1 and C_2D_2 by any other pair of electrodes at zero potential, located along equipotentials C_1C_3 and C_2C_4 and put electrodes with the appropriate negative potential φ_1 at the position of the equipotentials E_1F_1 and E_2F_2. In the first case we get a triode with a positive grid bias, and in the second one with negative grid bias. Thus in the general case one should proceed as follows: first find the position of the zero equipotential in the triode system; then use this equipotential as the cathode for a new beam-shaping system, whose anode is the anode of the triode; by various approximate methods, find the shape of the equipotential curves in the resultant two-electrode system. If we now replace the electrodes which are at zero potential by another pair of electrodes which are placed along the zero potential equipotentials which have been found, and if we put at the position of the corresponding equipotential an electrode at the potential of the grid of the triode, then in the beam-forming system thus obtained the repulsive forces of the space charge will be compensated.

The shapes found for Pierce electrodes are, strictly speaking, valid only for solid, infinitely extended electrodes. However, in practice the Pierce system is used for extracting intense beams through an anode aperture onto the screens of oscillograph tubes, in regions of high frequency fields, to obtain reaction drives,[94] etc. The defocusing of the beam in the anode aperture is found from formulas of electron optics,[62, 95–97] or from the empirical results of Glewe.[98] For the case of converging cylindrical beams, the location and size of the minimum beam cross section beyond the anode opening were found by Pierce.[5] Wedge-shaped beams in systems with a cylindrical cathode were treated[89,99] by a similar method recently. Regions were found with partial focusing (the beam narrows in width, but does not reach the axis) and complete focusing (the beam intersects the axis of the system) of wedge flows in systems with eccentric cylindrical electrodes and in systems with a rod anode (Fig. 9). The shapes of the electrodes needed to form such beams were also studied.

The formulas of electron optics are valid[100,101] only for thin lenses, and are not applicable to beam-shaping systems in which the diameter of the anode opening is comparable with the interelectrode spacing. Copeland[102] tried to replace the anode apertures by thick electrostatic lenses, but this replacement was very crude. If the aperture is large, the space charge forces are most

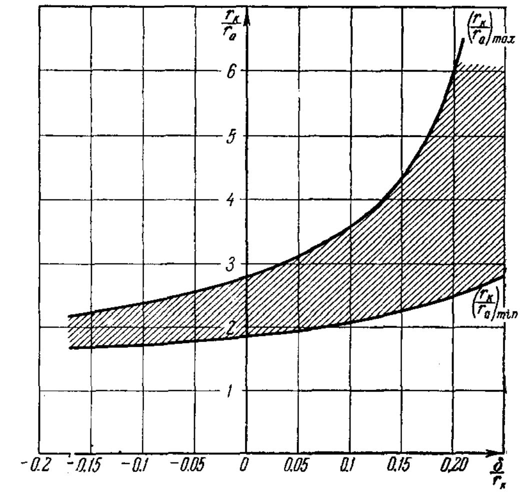


FIG. 9. Region of focusing of wedge-shaped beams in a system of eccentric cylindrical electrodes. r_k and r_a are the radii of cathode and anode, respectively; δ is the distance between the centers of curvature of the electrodes; the sign of δ is taken to be positive if the center of the anode is to the left of the center of the cathode. The region of partial focusing is shaded.

important in the region of the anode, which leads to a violation of the boundary conditions and to defocusing of the beam in front of the anode aperture. It has been suggested[103,104] that one take account of the influence of the finite dimensions of the anode opening by computing the capacity of the beam-shaping system and comparing it with the capacity of the similar system of solid electrodes. In this way one can determine the perveance of any complex beam-shaping system. The most complete enumeration of effects associated with divergence of intense beams in anode openings is given in reference 91 for values of the perveance between 1 and 8×10^{-7} amp/v$^{3/2}$, where it is shown by laborious computations that the focal length is usually 10% less than the values which are obtained from thin lens theory. Extensive graphical material is given in this paper.

Various new designs have been proposed to reduce the effect of the anode aperture on focusing of beams. Thus, in reference 106 an auxiliary anode, whose potential was higher than that of the main anode was used (Fig. 10b) for forming a conical beam with a perveance of 2.2×10^{-6} amp/v$^{3/2}$. In another construction, it is proposed to change the shape of the cathode shaping electrode (Fig. 10c), so that the potential distribution along the boundary of the beam satisfies Langmuir's law[72,73] out to distances equal to 0.6 of the interelectrode separation, and then approaches asymptotically the potential distribution in a cylindrical diode with an anode radius equal to $\frac{2}{3}$ of the true radius.

Figure 11 shows a beam-shaping system with an ellipsoidal cathode.[107] In this case the effect of the spherical aberrations of the anode aperture is balanced by the inhomogeneous distribution of the space charge density over the cross section of the beam. Such a system of electrodes is very sensitive to the choice of sizes of its individual elements. Below we give the

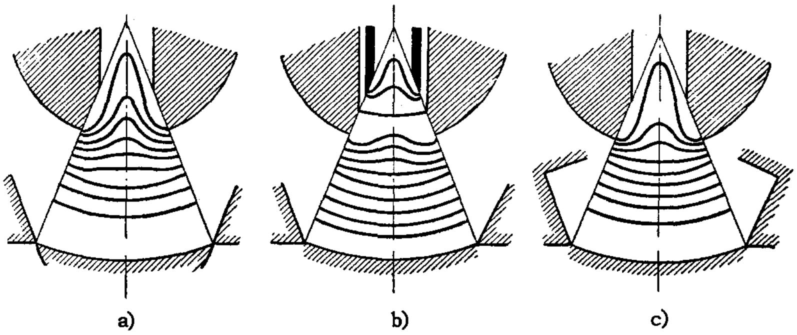


FIG. 10. Effect of anode aperture on shape of conical electron beam. The perveance of the system is $\sim 2.2 \times 10^{-6}$ amp/v$^{3/2}$. a) Langmuir case; b) method of auxiliary anode; c) method of auxiliary cathode shaping electrode.

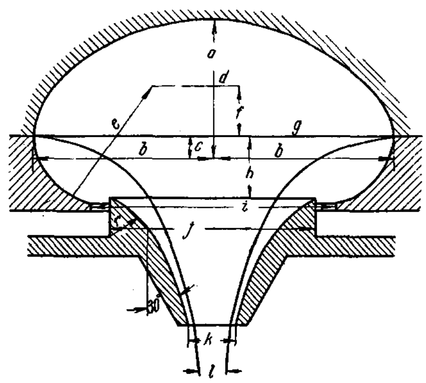


FIG. 11. Electron gun with ellipsoidal cathode.

main dimensions for a focusing system with ellipsoidal cathode, which[108] had a perveance of 3×10^{-6} amp/v$^{3/2}$ and a ratio of the minimum beam diameter to its size at the cathode equal to 75:

$a = 1.0,$	$f = 0.360,$	$j = 1.655,$
$b = 1.3,$	$g = 1.283,$	$k = 0.40,$
$c = 0.152,$	$h = 0.424,$	$l = 0.234,$
$d = 0.292,$	$i = 1.931,$	$r = 0.276.$
$e = 1.055,$		

In 1922 it was already proposed[109] to protect the cathode of electronic vacuum systems from ion bombardment by using hollow beams. With the development of research in the microwave region, another advantage of hollow beams became apparent: the interaction of a beam with the decelerating structure is most effective at the edge of the beam and drops off toward the center. The largest perturbations produced by the anode aperture are also at the axis of the beam, and are thus unimportant for hollow beams. However, the use of hollow beams faces serious difficulties — for stable focusing of such beams it is necessary that the current density in the beam vary inversely as the fourth power of the beam radius.

Another defect of the Pierce system is the need for thermal insulating gaps between the cathode and the shaping electrodes at zero potential. Experiments have shown[110] that the current density at the anode and, consequently, the focusing conditions depend to a large extent on the size and shape of the gaps. There is as yet no theoretical explanation of these phenomena.

7. INTENSE BEAMS OF CHARGED PARTICLES IN MAGNETIC FIELDS

To compensate the electrostatic forces of the space charge one can also use external magnetic fields. Such focusing of beams has been studied carefully only for systems whose cathode is screened from the magnetic field, and for beams with a uniform distribution of particle velocities over the cross section of the beam. Problems of magnetic screening of the cathode were discussed in references 111 – 114, where it was shown that both from theoretical and experimental considerations the optimum position of the magnetic screen should be that for which the longitudinal component of the magnetic field, at the point where the beam is narrowest, is about 70% of its maximum value. If an intense beam is placed in a sufficiently strong longitudinal magnetic field, which may be produced by a solenoid or a permanent magnet, the charged particles will move along the magnetic force lines. Using a longitudinal magnetic field enables us to obtain very stiff focusing, i.e., to retain a given beam configuration in the presence of large perturbations of the motion of the charged particles in the beam. However, this method has some fundamental defects, which include the following: 1) the need for cumbersome and heavy permanent magnets or solenoids, whose magnetization requires additional power: 2) the precision of matching of the individual elements of the focusing system; 3) the requirement of strict homogeneity and laminarity of the intense flow; 4) sensitivity to slight temperature changes, etc. All these deficiencies hinder the application of focusing methods which use a longitudinal magnetic field to present-day microwave equipment.

A further development of the methods of focusing using a magnetic field was the "Brillouin flow."[115-117] Such a flow rotates as a whole in an external magnetic field. A radial force acts on the particles of the beam which, in certain cases, can balance the centrifugal force and the repulsive forces of the space charge. The conditions for obtaining stable Brillouin flow are derived from the general theory of focusing of currents which was developed by V. T. Ovcharov.[118]

For a circular cylindrical beam of radius r, which propagates inside a cylinder of radius R having potential φ_a, these conditions give the result that the maximum current I which can be carried in such a beam is equal to

$$I = I_B\left(1 + 2\ln\frac{R}{r}\right)^{-1}, \tag{27}$$

and that the value of the magnetic field should be taken to be

$$H = H_B\left(1 + 2\ln\frac{R}{r}\right)^{-1/2}, \tag{28}$$

where

$$\left.\begin{aligned} I_B &= \frac{8\pi\varepsilon}{3}\sqrt{\frac{2}{3}\eta}\,\varphi_a^{3/2}, \\ H_B &= \frac{4}{r}\sqrt{\frac{\varphi_a}{3\eta}}. \end{aligned}\right\} \tag{29}$$

Values of I/I_B and H/H_B as a function of r/R are shown in Fig. 12; the focusing of hollow beams in a magnetic field has been treated in references 116, 117, 119 – 121. If the inner and outer radii of the beam are equal to r_1 and r, respectively, and the potential at the outer boundary is φ_a, then the maximum current I and the magnetic field intensity H are equal, respectively, to

$$\left.\begin{aligned} I &= I_B\frac{(r/r_1)^2-1}{(r/r_1)^2+1}, \\ H &= H_B\left(1-(r_1/r)^2\right)^{-1}, \end{aligned}\right\} \tag{30}$$

where I_B and H_B are determined by Eqs. (29).

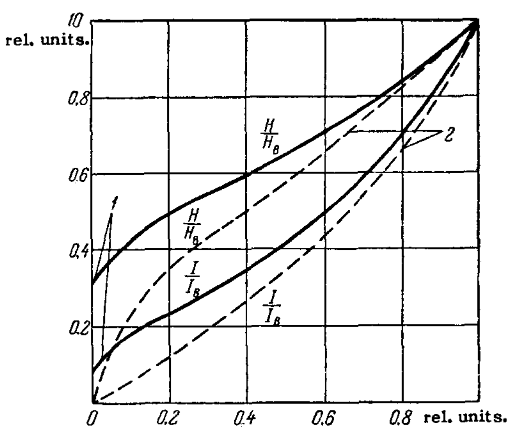


FIG. 12. Focusing of beams in a magnetic field. The solid lines correspond to focusing of cylindrical beams, the dashed lines to focusing of ribbon beams. The abscissa is the ratio of the geometric parameter of the beam to the corresponding quantity for the accelerating electrode: $1 - r_k/R$; $2 - a/d$; the ordinates give the values of I/I_B and H/H_B.

As Hines[122] has shown, inhomogeneous magnetic fields are necessary for Brillouin focusing of converging intense beams. For small angles of convergence ϑ_0, the field can be produced by parabolic pole tips. The maximum value of the current in such a beam is found from expression (29), while the magnetic field must have the following components:

$$H_r = \frac{8}{x\vartheta_0}\left(\frac{\varphi_a}{3\eta}\right)^{1/2}, \quad H_\vartheta = -\frac{4}{x}\frac{\vartheta}{\vartheta_0}\left(\frac{\varphi_a}{3\eta}\right)^{1/2}, \tag{31}$$

where x is the distance from the focus.

The focusing of ribbon beams of width a by a magnetic field of intensity H between plane parallel electrodes separated by a distance d was treated by Pierce[5] and Brillouin.[116] If φ_a is the potential of the electrodes, the maximum value of the current carried by the beam and the corresponding value of the magnetic field H are

$$I = I_B\frac{a}{2d-a}, \quad H = H_B\left(\frac{a}{2d-a}\right)^{1/2}, \tag{32}$$

where the quantities I_B and H_B are given by the equations:

$$I_B = \frac{16\varepsilon}{3a}\left(\frac{2}{3}\eta\right)^{1/2}\varphi_a^{3/2}, \quad H_B = \frac{4}{a}\left(\frac{\varphi_a}{3\eta}\right)^{1/2}. \tag{32'}$$

Graphs of the dependence of I/I_B and H/H_B on a/d are shown by the dashed lines in Fig. 12.

It has not been possible in practice to obtain an ideal Brillouin flow: usually for stable focusing of a beam one requires a magnetic field which is 1.5 – 2 times as strong as that predicted by theory. This is explained by the nonuniformity and nonlaminarity of the flow at its entrance into the focusing system. From the practical point of view, Brillouin flow has made possible only a slight reduction in the value of the magnetic field necessary for focusing, while it has retained and in some cases aggravated the defects of focusing by a longitudinal magnetic field.

Another characteristic feature of the behavior of intense flows in magnetic fields is their marked turbulence.[75] Thus, for certain values of the current, depending on the geometry of the beam, the flow becomes unstable.[123–125] It was found[32–36,126,127] that with increasing magnetic field the beam changed its shape, forming periodic arrow-shaped pieces, or broke up into several vortices moving in the interelectrode space.

8. METHODS OF CENTRIFUGAL AND PERIODIC FOCUSING

We require of any focusing system (and especially of the focusing system of high-frequency electronic apparatus) that it have, on the one hand, reliable and long-time operation, and on the other hand that it be compact and lightweight. The electrostatic focusing systems which we have considered are very cumbersome, and have a short life under conditions of high current density. Magnetic focusing systems can guarantee steady focusing of intense beams with high current density, but the dimensions and weight of such systems become very large. Thus magnetic focusing methods are applied at present exclusively in laboratory models of electronic apparatus. Thus one of the most pressing problems of present day electronics is to find new and effective methods of electrostatic focusing and to reduce the weight and size of magnetic focusing systems. This problem is partially solved

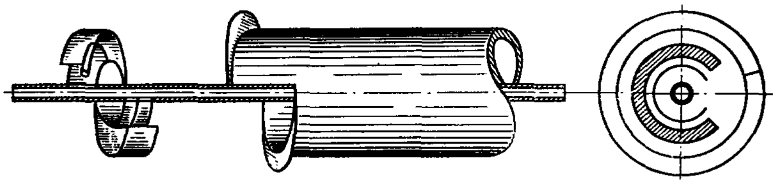


FIG. 13. Electron-optical system with centrifugal-electrostatic focusing.

by the use of centrifugal-electrostatic and periodic focusing.

The space charge of a beam is subject to strong irregular fluctuations, which must be taken into account in constructing focusing systems. One therefore deliberately introduces into the focusing system some additional defocusing forces, so that in this background the repulsive forces of the space charge are a small contribution. Such systems are more reliable and more flexible. The additional defocusing force is usually the centrifugal force resulting from rotation of the beam in a homogeneous or periodic external field. If the external field is a homogeneous electrostatic field, this method is called centrifugal-electrostatic focusing.[128,129] It is based on the stability of the motion of a charged particle in the radial field of a cylindrical capacitor.[129] In this case, the electrode system consists of an electron gun and two coaxial cylinders (Fig. 13) between which a voltage is applied with the inner cylinder at the higher potential. Because of the helical cuts on the cathode and anode, the electron is given some angular momentum and, having entered the field of the cylindrical capacitor at some angle to the axis, begins a motion along a spiral trajectory. Devices in which such a focusing method is used are called "spiratrons." A computation of the stability of beams in such systems[28] shows that it is possible to send sizable currents through the system. Experimentally, beams have been obtained with a current of several tens of milliamperes and coefficients of current transmission of 90 — 98%. Centrifugal-electrostatic focusing can be applied[131] to backward-wave tubes.

The advantages of centrifugal-electrostatic focusing are: 1) the absence of extrenal cumbersome focusing systems and any additional expenditure of power for focusing: 2) the possibility of dispensing with exact adjustment of the system; 3) simplicity and reliability of control of current and beam shape; 4) high efficiency of interaction of the beam with the high frequency fields of the decelerating structure: 5) increased length of service of emitting electrodes, their protection against bombardment by ions of residual gases, etc. A defect of such focusing systems is the need to add auxiliary electrodes in the decelerating structure, which sometimes cannot be done.

Another method of centrifugal focusing is the Harris-Crumly[132,133] method for focusing hollow beams. In this case the hollow beam is formed in a magnetically screened gun, and then brought into the field of a cylindrical condenser. The initial twisting of the electrons into the spiral trajectories is accomplished in a transition region in which the hollow beam passes through a radial magnetic field. The defect of such a system is the necessity for introducing a cumbersome magnetic circuit into the vacuum system, and selecting carefully the magnitude and configuration of the focusing electric and magnetic fields in the transition region.

In addition to focusing methods using homogeneous electric and magnetic fields, one also uses for focusing purposes fields which vary periodically with position. In this case the beam alternately converges and diverges as it passes through the system of magnetic or electric lenses, prisms or mirrors. The stability of the beam is determined by the stability of the oscillatory motion which is performed by the particles of the beam as they pass through the system of alternating fields. Characteristically, such an oscillatory process results in a series of stable and unstable bands.

Periodic electrostatic focusing[134-140,167,213] is achieved by compensating the defocusing forces of the space charge by the radial component of an external electric field which is periodic along the beam. The simplest electrode system consists of a series of circular disks with holes, having alternately higher and lower potentials. The beam of charged particles moves along the axis of the system, and periodically narrows and spreads out. The conditions for stability of solid and hollow beams in such focusing systems were discussed in references 140 — 142. Compared with focusing by homogeneous fields, periodic electrostatic focusing enables one to obtain an almost parallel beam, and the length of such a beam can be large, since the periodic field can be extended easily by adding electrostatic lenses. The main defect of such systems is that the resultant focusing is weak, and the beam is destroyed by small perturbations. A way of getting higher rigidity has been proposed by Chang, by using biperiodic systems,[143] and systems with a rotating beam.[144]

In the first case the beam is focused by the action of two external electrostatic fields which change direction rapidly. A bifilar helix is an example of such a focusing system. The usual methods of focusing by means of periodic fields are based on the periodic action on the beam of external electric forces, where the magnitude and period of these forces varies slowly along the length of the beam. Because of this, any perturbations which arise in the space charge of the beam may not be compensated by the slowly rising focusing forces. The way out of this situation is to reduce the period of the focusing electric fields. However, shortening the period of the focusing system leads to overlapping interaction of the fields in neighboring lenses. Such a mutual influence of neighboring electrostatic lenses can be avoided by adding a sharp drop in the electric field at the end of each of the focusing sections. This last effect is achieved in a system using a bispiral, as a result of the interaction between the electric fields of the two helices. A system with a bispiral may have a very small period and an exponentially decreasing (or increasing) electric focusing field. According to Chang's calculations, a system with a bispiral will guarantee stable focusing of electron beams with perveances up to 10^{-5} amp/v$^{3/2}$. Experimentally, focusing has been obtained for beams with a perveance of 2×10^{-6} amp/v$^{3/2}$ and a total current of 4 ma, for a current transmission coefficient of 97%.

The possibility of obtaining higher rigidity in rotating-beam systems amounts essentially to introducing additional defocusing forces into the system, so that the repulsive forces of the space charge play an unimportant part in the balance of forces. The initial rotation of the beam is usually started by means of a radial magnetic field located in the region of the entrance of the beam into the focusing system. The centrifugal force which comes from this turning of the beam is balanced by the focusing forces of the periodic field of the bispiral. Even though the theory shows that there are important advantages of such a system with stiffer focusing over the usual periodic systems, the values of beam current achieved in experimental models (about 1 ma) are practically equal to the limiting current for focusing by the usual periodic fields. In addition, the combination of an electrostatic focusing method with magnetic turning of the beam is illogical, and the construction is complicated.

An original type of periodic electrostatic focusing was proposed by Kompfner.[145,146] This focusing has been called "slalom" focusing, and is shown schematically in Fig. 14. In this case the beam moves through a system of successive electrostatic prisms, and performs a wave-like motion as it turns past the focusing rods. In each cell the motion of the beam is similar to the motion in the deflecting field of a cylindrical condenser. The focusing rods may be the elements of a delay line of a microwave component like the traveling wave tube. The problem of stability of a "slalom" beam was discussed in reference 147. An experimental backward-wave oscillator, using this type of focusing, was able to achieve continuous tuning over the range 3300 — 4700 Mc/sec, with a coefficient of current transmission of 97%. A positive feature of "slalom" focusing is the high efficiency of interaction of the beam with the electromagnetic field of the decelerating system, since in such a system the beam goes completely around the delay line. A disadvantage of this type of focusing is the need to have a monochromatic beam and precise adjustment of the periodic system.

In the work of Hogg,[148] another type of biperiodic electrostatic focusing is proposed, which the author calls a "double staircase." The system consists of two parallel long rectangular sheets which are connected electrically. The sheets have transverse slots, and the slots in one are opposite the cross ties in the other. At a certain distance from these sheets there are two other solid sheets, which are at the same focusing potential, negative with respect to the potential of the inner sheets. The beam passes alternately past the slots and is deflected in the focusing field. The maximum transmission in devices with this type of focusing reached 70% for a beam of 750 volts, a focusing voltage of 300 volts, and a beam angle of 21° at the entrance to the system.

Methods of periodic magnetic focusing are used very often in practice.[106,139,149-152,211-212] The reason for this is that one can send much greater currents through such systems than through systems with electrostatic focusing. The methods of periodic magnetic focusing are based on the collecting action of a sequence of short or long magnetic lenses. Such focus-

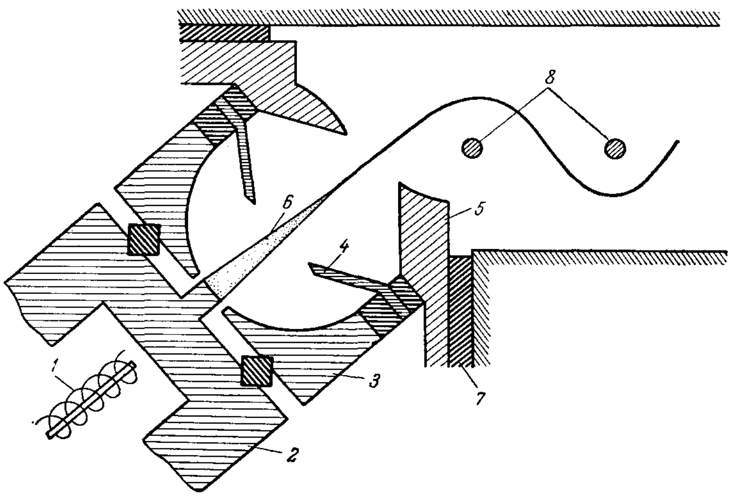


FIG. 14. Electron-optical system with "slalom" focusing. 1 – heater; 2 – cathode; 3 – focusing electrode; 4 – first anode; 5 – second anode; 6 – beam; 7 – insulator; 8 – focusing rod.

ing is characterized by the occurrence of stable and unstable regions,[153,154] and depends on how the beam is brought into the magnetic field, i.e., on how much the cathode of the system is screened from the magnetic field.[155] The beam stability for periodic magnetic focusing depends on two parameters: one of these is related to the space charge density in the beam, and is equal to $a = 1540\,K(L/d)^2$ (where K is the perveance of the beam, d is the initial diameter of the beam, and L is the period of the magnetic field), and the second is related to the magnetic field, $b = 5.58 \times 10^{-4}\,\overline{H^2}L^2/\varphi$ (where $\overline{H^2}$ is the mean square of the magnetic field strength in gauss, and φ is the beam voltage in volts). If b is small, the amplitude and period of oscillation of the circulating beam is quite large, but in the absence of disturbances the beam retains its original diameter. With increasing b the period and the amplitude of oscillation decrease, and when $b = a$ the current becomes approximately parallel. If $b > a$, the space charge can be neglected, the oscillation amplitude increases and destroys the laminarity of the flow. The paths of the electrons cross the axis, but the electrons still do not leave the beam, even though the oscillations may be large. For $b \approx 0.7$, the flow becomes unstable and the beam rapidly loses electrons, while for $b > 0.7$ new bands of stability and instability of the beam appear. In these stability regions, the electron orbits cross the axis often.

In comparison with focusing by homogeneous fields, periodic magnetic focusing has a whole series of advantages, some of which should be mentioned: 1) increase in the limiting perveance of the beam, and 2) reduction in weight and size of focusing system. For example, in a low power traveling wave tube, the value of the perveance was approximately equal to 1.5×10^{-7} amp/v$^{3/2}$. The limiting value of the perveance[5] for which focusing is possible in a tube or spiral of diameter D and length l is equal to $3.9 \times 10^{-5}\,(D/l)^2$. If the beam is focused by n magnetic lenses whose separations are equal to $L = l/n$, the limiting perveance of such a system will be proportional to the square of the number of focusing lenses. Thus the typical value for the perveance of a traveling wave tube can be obtained for $n = 2$ and $l/D = 32$.

Furthermore, suppose there is a permanent magnet, which produces a homogeneous field over a certain length, and we want to increase the size of the region with this magnetic field strength by a factor of n. To do this we would have to multiply all the dimensions of the magnet by n, whereas in the case of periodic focusing it would be sufficient to multiply only the length by n. Consequently the weight of the periodic system will be n times smaller than the weight of the corresponding homogeneous system. Experiments have shown[156] that a periodic system weighing 1.5 kg was required for a 100-watt traveling wave tube, whereas a homogeneous magnet system would have weighed 25 kg.

Disadvantages of periodic magnetic focusing systems are: 1) more complicated control of the magnetic system: 2) the presence of regions in which the current is in an unstable state: 3) high sensitivity to temperature changes; 4) the necessity for special magnetic materials, and various others. Thus, for example, for values of the perveance exceeding 2×10^{-6}, the lenses must be placed at such a distance from one another that the field of a lens influences the field in neighboring lenses, the zone of stability of the beam becomes very much smaller, and control of the focusing system becomes complicated. An essential requirement on the flow is that it must be homogeneous and laminar at its entrance into the focusing system. However in practice it is difficult to satisfy this requirement, so that the values of the focusing fields usually exceed the computed values by 50 – 100%.

By means of quadrupole magnetic fields, one can introduce a centrifugal defocusing force, and thus increase the stiffness of focusing. Such focusing has been called "meander" focusing (and is shown schematically in Fig. 15). By using quadrupole magnets we make the field have opposite directions on the two sides of the axis. An electron which enters the system in a plane perpendicular to the magnetic field will move along a zigzag curve which periodically crosses the axis. Such a beam has been used for excitation of high frequency oscillations in a resonator. In this case the first instability zone begins[139] when the magnetic field parameter $b = 0.44$.

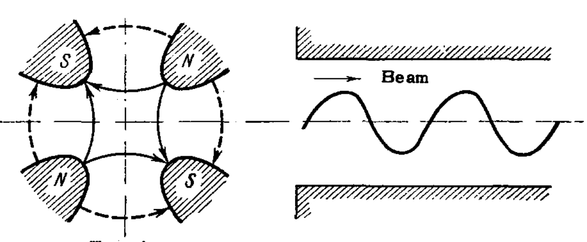


FIG. 15. Schematic of "meander" focusing of a beam.

9. APPROXIMATE METHODS OF COMPUTING INTENSE BEAMS

As we have seen, the general equations of motion of charged particles in stationary fields, when we include the effects of space charge, are too complicated for general application. If, however, we limit ourselves to treating the paths of particles which move near the axis of the system and make small angles with it (the so-called paraxial rays), the differential equations take on a relatively simple form. The methods of paraxial optics have found wide application in electronics for the study of complex electrostatic and magnetic lenses, and have been applied successfully in the work of Sturrock and Kirstein[158,159] to the study of space charge effects in curvilinear beams. Waters[38] has applied the method of paraxial optics to the study of periodic focusing of beams.

Let us consider briefly the basic idea of the method. Suppose that we know the distribution of potential along the trajectory S of a charged particle. In the paraxial region in the vicinity of the curve S we may regard all trajectories of charged particles as parallel to S. We choose an orthogonal curvilinear coordinate system with its x axis along S and its y axis along the normal to the trajectory. In this coordinate system we represent the action function as a series

$$W = \sum_{k=0} y^k w_k(x)$$

and substitute W in Eqs. (14) and (15). Collecting the coefficients of the zeroth power of the variable y, we obtain the fundamental equation of the paraxial ray

$$\frac{dz}{dx} = -\frac{2w_2 z}{w_0'}, \tag{33}$$

where $z = R\rho_0 w_0'$, where R is the distance from a particular point on S to the axis of the system. The coefficients of higher powers of y give equations describing corrections to the paraxial trajectories.

If the boundary of the beam is given by the equation $y = \pm y_1(x)$, then by using $\frac{dy_1}{dx} = \frac{2w_2 y_1}{w_1}$ we get the important relation $y_1 z = \text{const}$. Since the convergence of the beam is determined by the ratio $y_1(0)/y_1(x) = z(x)/z(0)$, it can be found by solving the one paraxial equation (33).

Reference 159 gives a comparison of the results of exact and paraxial calculations of trajectories of hollow beams of charged particles in systems with a ring cathode. The results of the comparison are given in Fig. 16, where the solid lines are the trajectories found from the exact solution[83,84] of the differential equations, while the dashed lines are the paraxial trajectories. Thus, in computing focusing systems, there is no need to give the curve S and the potential distribution along it analytically; a numerical assignment of these data along the curve is sufficient, except for a small region near the cathode.

Comparatively recently, Stuart and Meltzer[160] have proposed the use of perturbation theory methods for studying stationary intense beams of charged particles. If one assumes that the beam parameters differ only a little from the values which can be found easily by analytic methods, corrections to these values can be found by solving ordinary differential equations. In solving the space charge equations, Kirstein[80] used a more general method, which includes not only the perturbations of the solutions of the differential equations but also perturbations of the space metric. By this method he found the solution of the space charge equations in a system with a toroidal cathode, by using the corresponding solution for a circular cylindrical beam.

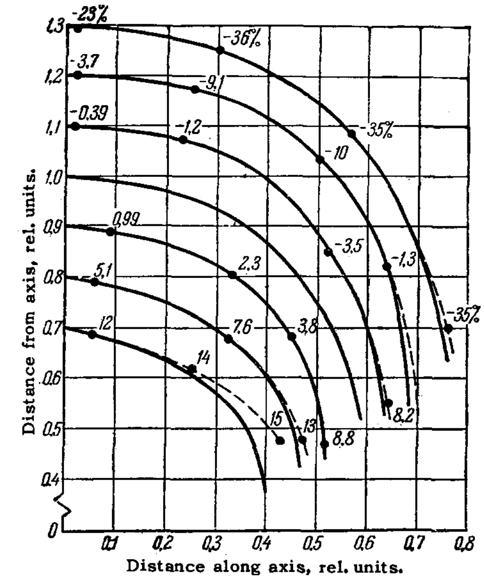


FIG. 16. Shape of trajectories of charged particles in a hollow cylindrical beam, obtained by the methods of paraxial optics (dashed curves) and by exact calculation (solid curves). The numbers along the trajectories give the percentage error in the determination of the potential. Numbers along the trajectories (in %) give the relative errors of the potential values along the particular trajectory.

Among the other approximate methods, we should mention the method of the equivalent diode,[86,161] which is based on the assumption that the shapes of the equipotential surfaces in systems with a cylindrical or spherical cathode are the same in the presence of space charge as in its absence, but differ only in the values of the potential at each surface in the two cases. In this method the anode of a complex configuration is replaced by a fictitious electrode with such a geometry that it forms, together with the cathode, a diode with straight-line trajectories for the motion of the particles, and the dimensions of the anode are chosen so that the equivalent diode has the same capacity as the real system. To determine the capacity of the whole system, one uses the standard methods of conformal mapping. Ivey[162] has tabulated 27 cases of computation by this method of complex configurations of electrodes with cylindrical and spherical external and internal cathodes.

10. THE INFLUENCE OF RELATIVISTIC EFFECTS AND INITIAL VELOCITIES

Relativistic effects manifest themselves in the increase of mass of the charged particles with increasing velocity, and thus in the reduction of the beam current, and also in the increasing magnetic interaction of the individual particles in the beam. As was shown in references 168 — 170, the potential distribution for relativistic velocities of the beam particles in systems with plane, cylindrical, and spherical symmetry differs considerably from the familiar Langmuir solution.

With increasing accelerating voltage,[168–170] the magnetic field produced by the beam current increases, and this in turn has a focusing action on the beam. The effect of this is that the beam can carry a current which is $[1 + (\eta\varphi/2c^2)]^{3/2}$ times greater than the value obtained when relativistic effects are omitted. Thus for a voltage of 50 kv the increase in electron current is 1.07, and it reaches 1.82 for a voltage of 500 kv. Relativistic effects result in an increase of the focal lengths of electrostatic and magnetic lenses.

It should be noted that in general, as the velocities of charged particles approach the velocity of light, their interactions with one another (i.e., space charge effects) decrease to zero. However the interaction cannot change sign,[171] so the beam cannot become self-focusing. Obviously the change of the potential distribution over the boundary of the beam and the increase in magnetic interaction of the particles leads to a change of the previously found shapes of Pierce electrodes.

In the case of magnetic focusing of intense beams, relativistic effects cause a decrease in the value of the Brillouin field H_B by a factor[120] $[1 + (\eta\varphi_a/2c^2)]^{-1/4}$.

The effect of initial velocities of the particles on the parameters of intense beams are related mainly to the fact that the charged particles leave the cathode along directions which are not perpendicular to its surface. This results in an inhomogeneous distribution of particle velocities over the cross section of the beam.[60,61,172–174] In Brillouin flow, for example, the effect of the thermal velocities of the particles is that part of the beam current is carried outside the computed limiting radius. Pierce and Walker[5,57] found the dependence of the fraction of the total current carried beyond the radius r on the value of the ratio r/r_B and the parameter $\mu = 1.76 \times 10^{-8}(r_B/r_K)^2(I/T\sqrt{\varphi})$, where r_B is the Brillouin radius, r_K is the cathode radius, T is the cathode temperature (°K), I is the total beam current in amperes, and φ is the anode potential in volts. This dependence is shown in Fig. 17.

The initial velocities of the particles affect the quality of the focusing system. In addition to distortion of images, the thermal velocities limit the current which can be carried by the beam, which causes an increase in the diameter of the beam. The solution of the space charge equations for a given distribution of particle velocities at the cathode is quite complicated, and gives quite involved results, among which we should mention those of Van der Ziel[175] and Langmuir[176] for plane electrodes, and the solution of this problem in systems with cylindrical and spherical symmetry;[176–180] there has also been a study of intense beams with inhomogeneous distributions of the particle velocities by the density matrix method.[181] The investigations of Meltzer[61] on different types of statistical distributions of initial velocities in an intense beam are interesting.

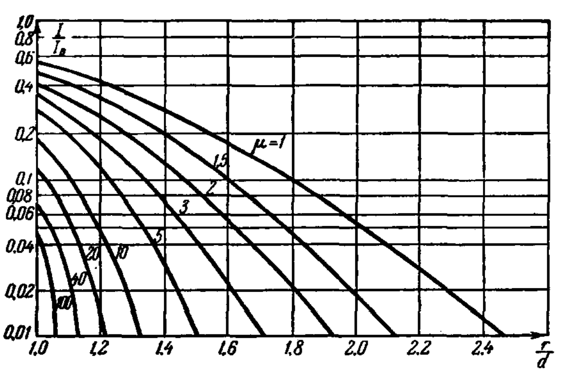


FIG. 17. Effect of initial particle velocities on focusing of cylindrical beams in a magnetic field.

In practice the initial velocities are usually taken into account by Pierce's method:[5] one first finds the condition for focusing of the beam in the absence of space charge, but including the initial velocities of the particles; then one compares this with the results obtained when the initial velocities are neglected, but the space charge effects are included. At high anode voltages, the minimum dimensions of beams and the aberrations of focusing systems are determined mainly by the action of the space charge, whereas they are determined by the thermal velocities of the particles[5,99] when the voltage is low.

11. APPLICATION OF ANALOG COMPUTER SYSTEMS AND METHODS FOR MODELING INTENSE BEAMS

The rapid development of radar technique during the Second World War is closely connected with the use of self-consistent field methods for space charge problems. The self-consistent field method, which has been applied extensively in quantum physics[13] and in solid state physics,[182] consists of the following. From physical considerations we know some initial potential distribution, with which we compute numerically the trajectories of the charged particles and the density distribution of the charge in this field. From the values found for the charge density, we find the solution of the Poisson equation for the required boundary conditions. With this solution of the Poisson equation we repeat the procedure until the difference in the distribution functions for two successive cycles is less than the required accuracy. It turns out[183] that this method of successive approximations converges very slowly. The use of the method is largely limited by the speed of computing and the size of the memory of present-day digital computing machines.

The "electrical analog"[76,184–189] methods have much faster convergence. Here, the actual electrode system is replaced by a set of resistances which are connected electrically. The voltage drops in the resistors are chosen to correspond to the potentials at which the electrodes of the system are kept. The potential at an arbitrary point of the electrical network will correspond to the voltage drop in the interelectrode space, and can be measured easily. The space charge is modeled in such a system by supplying a current to the network; the point of supply and the size of the current are found by successive approximations, subject to the boundary conditions of the problem.

Among other modeling methods, the most widely used are the electrolytic tank methods: 1) the method of profiling the tank bottom,[190–192] 2) the method of current input elements,[100,194–196] and 3) the method of the three-dimensional electrolyte.[197] In the first method the bottom of the tank is made of rubber, so that one can easily change the depth $h(x, y)$ of electrolyte in the tank. Since the conductivity of a homogeneous electrolyte is proportional to the depth of the electrolyte, in the stationary case it follows from Ohm's law that $\operatorname{div}(\sigma\nabla\varphi) = 0$, and consequently,

$$\Delta\varphi + (\nabla\varphi)\nabla\ln h = 0. \tag{34}$$

We denote by d/ds the derivative along a force line of the electric field, and introduce the absolute value of the field strength $|\nabla\varphi|$; the right side of (34) can be written as

$$\Delta\varphi = \frac{d\ln h}{ds}|\nabla\varphi|.$$

It is obvious that to solve the Poisson equation by the electrolytic tank method, one must, in addition to producing the appropriate boundary conditions in the tank, give the bottom of the tank such a shape that the expression

$$\frac{d\ln h}{ds}|\nabla\varphi| = -\frac{\varrho}{\varepsilon} \tag{35}$$

is valid.

For practical computations it is more convenient to write Eq. (35) in integral form:

$$\ln h = -\int_{s_0}^{s}\frac{\varrho\,ds}{\varepsilon|\nabla\varphi|}. \tag{36}$$

If one has a sufficiently dense family of force lines marked in the tank, which cover the region under investigation, one can by computing the integral along each of them find the distribution $h(x, y)$ over the whole region of the tank bottom which is of interest. It is obvious that the equation $\Delta\varphi = -\rho/\epsilon$ can be solved by this method only by using successive approximations, since in calculating the distribution $h(x, y)$ over the tank bottom we need to know $|\nabla\varphi|$ and the distribution of lines of force, for which we may take the corresponding values found in the tank in the preceding approximation. This method requires very laborious computations and converges slowly.

In the method using current input elements, the space charge is modeled by little conducting rods connected to external electrical elements. By repeated reflection in the inclined tank bottom and in the surface of the electrolyte, one produces a model of a discrete cloud of space charge. In order to avoid jumps in the potential at the current supply elements, they are made in the form of wires which barely emerge from the bottom of the tank, or are simply holes in the bottom of the tank which are filled with electrolyte,[198] while the shape of the cathode is modeled by a plate electrode whose plane makes an angle of $3\pi/8$ with the surface of the electrolyte.

In the method of the three-dimensional electrolyte the modeling is done with an electrolyte having anisotropic conductivity, which makes possible a considerable increase in the rate of convergence of the successive approximations.

We should also mention the rubber sheet method[77,199–202] and the conducting paper method.[203,204] In the first, the space charge is modeled by a distribution of loads which, at each point of the membrane, is proportional to the density of space charge, and the potentials of the external electrodes are modeled by additional loads along definite lines in the sheet. Using heavy steel balls and stroboscopic illumination, one determines the trajectories and velocities of the particles in zeroth approximation, from which one computes a new distribution of the volume charge, and thus achieves the required accuracy by successive approximations. For the solution of three-dimensional space charge problems by this method, it is suggested[201] that one reduce the air pressure below the membrane.

If one adds a small amount of carbon black and graphite to a mass of paper pulp, one can obtain a whole range of specific resistances of the paper sheets, from tens of ohms to tens of megohms per sheet. The modeling of an electric field by regions of different conductivity can be accomplished using such paper, either by perforating it or by pasting pieces from different types of paper, in accordance with the given relative conductivities. To change the resistance, one can also tint and retouch the paper. Electrodes for the conducting paper are prepared from metal foil, or are deposited on the paper by using silver paint, a suspension which has a high conductivity. Such suspensions are used for making printed circuits, and are a mixture of metallic silver, bakelite lacquer and alcohol. For probes one uses a hard graphite pencil or a sharp metal needle. The actual process of modeling the space charge is that described above using current input elements, which consist of a set of needles appropriately arranged on the paper. It is possible to use modeling systems with variable current inputs to various points on the paper,[203] and also systems with fixed values of the current but with varying numbers of modeling elements.[214] In practice one usually uses isotropic conducting paper. However as the technology of preparing the paper develops, applications are being found also for anisotropically conducting paper. A defect of the conducting paper is its quite high negative temperature coefficient of resistance.

In systems of modeling the volume charge of beams, there is extensive application of methods for computing particle trajectories which take account of their interaction with one another.[81,152,205-209]

Reference 210 describes an original electrolytic tank with an arrangement for automatically constructing the trajectories of charged particles in electric and magnetic fields having axial symmetry. The precision of drawing the trajectories was 1%.

12. COMPARISON OF DIFFERENT FOCUSING METHODS

In comparing magnetic and electrostatic focusing systems, we should point out that, in principle, electrostatic systems have an obvious superiority over magnetic systems, in that their weight, size, and power cost is much less. However the electrostatic methods which have been developed are not yet brought to perfection and cannot supplant magnetic focusing in the most important UHF equipment. Thus for the practical focusing of an intense beam with a current density of 1000 — 10 000 amp/cm^2 by the methods known at present, it would require either a magnetic field of tens of thousands of gauss or an electrostatic field of millions of volts per cm, which has not yet been produced. Thus the development of new methods of focusing, and in particular electrostatic methods, is one of the main problems of electronics.

As for the comparison of focusing methods using homogeneous or periodic fields, the latter require strict matching of geometrical dimensions and equality of amplitudes of the field in each period of the system. To solve this problem is more difficult than to produce a homogeneous focusing field along the whole length of the beam, though the latter case requires considerably higher weight of equipment. Therefore, in laboratory models, and sometimes in practical constructions, the use of homogeneous fields for focusing is preferred.

A long stable beam of charged particles can be produced by compensating all the radial forces acting on the particles of the beam. However in actuality the particles of the beam are subjected to different perturbations at their entrance into the focusing system and during their passage through it. Thus focusing imposes the requirement of stability under random perturbations of the intense beam. In other words, the law of variation of the focusing and defocusing forces in the system must be such that for a random displacement of the charged particle from its equilibrium trajectory, the required restoring force will develop. The faster this force increases with change in radius of the beam, the higher the stiffness of the focusing, other things being equal. In Fig. 18 we show a graph of the change in the restoring forces for different focusing methods (where, for convenience, the defocusing force is assumed always to have the same value at the equilibrium point). The rigidity of the focusing method is characterized not only by the rate of rise but also by the magnitude of the restoring force for random displacements of the charged particles from the equilibrium radius. As we see from the graph, the rate of rise of the restoring force is almost the same for Brillouin flow and for the method of centrifugal focusing; it is largest for the system using periodic electrostatic focusing and for magnetically limited beams. Nevertheless the actual value of the restoring force is very small in the case of periodic focusing. Therefore in this case, even though rapidly increasing forces begin to act on a beam which leaves the equilibrium trajectory, the magnitude of these forces is of the same order as the space charge forces. The result is that periodic focusing has very low rigidity. As we have seen in Sec. 8, to increase the rigidity of such a system the distribution of charge density over the cross section of the beam must be nonuniform. The rigidity of focusing systems can be increased by introducing into the system additional defocusing forces which produce a background in which the action of the space charge and its random fluctuations appear smoothed out. Therefore centrifugal methods of focusing are always more rigid, and such systems are more reliable and easily controlled.

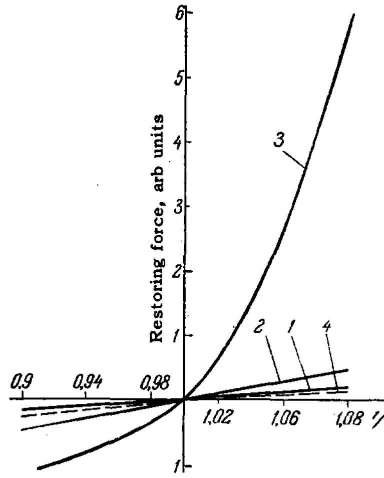


FIG. 18. Dependence of restoring force on displacement of charged particle from the equilibrium trajectory in systems with various types of focusing. 1 – Brillouin flow; 2 – intense flow in a longitudinal magnetic field; 3 – system with periodic electrostatic focusing and rotating hollow beam; 4 – system with centrifugal electrostatic focusing.

The principles of formation and focusing of intense beams discussed here do not, of course, exhaust the whole multitude of processes which occur in intense flows of charged particles. We have considered only the most important methods of focusing, which have been applied successfully in electronics; this may help the development of new, more perfect focusing systems.

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