

*THE APPLICATION OF THE METHODS OF QUANTUM FIELD THEORY
IN STATISTICAL PHYSICS*

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INTRODUCTION

IN recent years the quantum-mechanical theory of many-particle systems has received much attention. Important advances in this field have been the result of the successful application of well developed methods of quantum field theory to the study of systems made up of large numbers of interacting particles. Since in a brief survey it is impossible to include all of the various approaches that have been developed, we shall present mainly (as the most promising method) the method of Green's functions,¹ which in its application to many-body problems has been most intensively developed in papers by Soviet authors,²⁻¹¹ and also in some other papers.^{12,13} The advantage of the Green's-function method lies not only in a clear formulation of the problems and in the existence of flexible methods for solving them, but also in the possibility of extending the results to the case of temperatures other than absolute zero. In this connection there have recently been many papers devoted to the study of important problems of statistical physics on the basis of the Green's-function method.¹⁴⁻³⁵

The present review gives a more or less detailed exposition of this method as applied to statistics. The basis adopted is the widely used method of Matsubara,¹⁴ which has been significantly improved in recent papers by Soviet authors.^{15,16-20}

In the first section we present a new thermodynamic perturbation theory based on the methods of quantum field theory, and at the same time explain the main ways to apply the mathematical apparatus of quantum field theory for the purposes of statistical physics. The exposition is given in considerable detail. The contents of this section convincingly show the advantages of the new thermodynamic perturbation theory. Whereas in the old thermodynamic perturbation theory mathematical complications made it impossible to advance beyond the first (one or two) approximations in

the interaction between particles, in the new formulation the use of the diagram technique makes the construction of the perturbation-theory series so intuitively clear that it is possible to carry out a selective summation of an infinite number of terms of the series. Such a selective summation leads to physical approximations that go beyond the framework of the perturbation theory itself.

The second section is devoted to calculations without the use of thermodynamic Green's functions. Here particular attention is given to methods for working with Green's functions. The size of the article does not allow a really full treatment of the applications to specific problems. The exposition needed can, however, be found in the original literature to which references are made. This remark also applies to the third section, which is devoted to time and temperature dependent Green's functions.

In the fourth section we examine the application of the principles here expounded to a certain class of problems in kinetics. The case of the deceleration of a particle in passing through a plasma is used as an illustration. The method is so general, however, that it can be extended without difficulty to other similar problems, namely: bremsstrahlung and pair production in the passage of particles through plasmas, plasma radiation, stopping of electrons in metals, and so on.

1. THERMODYNAMIC PERTURBATION THEORY

The perturbation-theory series. Let us consider a quasi-closed statistical system, which in general can consist of several types of fermions and bosons. For simplicity in the writing, however, we shall suppose that the system contains only one type of fermions and one type of bosons, since the extension of the results to the case of many-component systems presents no difficulties. We regard the particles—the fermions

and bosons—as quanta of fermion and boson fields, so that in the Schrödinger representation (Matsubara¹⁴) the Hamiltonian of the statistical (for example, electron-phonon) system is of the following form:

$$H = H_0 + H_1, \quad (1.1a)$$

$$H_0 = \sum_{\mathbf{p}, \mathbf{r}} \epsilon_{\mathbf{p}} a_{\mathbf{p}\mathbf{r}}^* a_{\mathbf{p}\mathbf{r}} + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^* b_{\mathbf{k}}, \quad (1.1b)$$

$$H_1 = g \sum_{\mathbf{p}, \mathbf{r}, \mathbf{k}} \left(\frac{\omega_{\mathbf{k}}}{2V} \right)^{1/2} (a_{\mathbf{p}+\mathbf{k}, \mathbf{r}}^* a_{\mathbf{p}\mathbf{r}} b_{\mathbf{k}} + a_{\mathbf{p}-\mathbf{k}, \mathbf{r}}^* a_{\mathbf{p}\mathbf{r}} b_{\mathbf{k}}^*), \quad (1.1c)$$

where H_0 is the Hamiltonian of the noninteracting electron and phonon fields; H_1 is the operator for the interaction of these fields; $a_{\mathbf{p}\mathbf{r}}$ and $a_{\mathbf{p}\mathbf{r}}^*$ are the respective operators for absorption and production of an electron with momentum \mathbf{p} , polarization \mathbf{r} , and energy $\epsilon_{\mathbf{p}} = \mathbf{p}^2/2m$; $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^*$ are analogous operators for a phonon with momentum \mathbf{k} and energy $\omega_{\mathbf{k}} = \hbar |\mathbf{k}|s$; V is the volume of the statistical system; and the coupling constant g is given by the expression

$$g = \left(\frac{VC^2}{N'Ms^2} \right)^{1/2},$$

in which M and N' are respectively the mass and total number of lattice ions, C is the ordinary constant for the interaction of electron and lattice, and s is the speed of sound. The operators for production and absorption of particles satisfy the commutation relations:

$$\begin{aligned} [a_{\mathbf{p}\mathbf{r}}, a_{\mathbf{p}'\mathbf{r}'}^*]_{\pm} &\equiv a_{\mathbf{p}\mathbf{r}} a_{\mathbf{p}'\mathbf{r}'}^* + a_{\mathbf{p}'\mathbf{r}'}^* a_{\mathbf{p}\mathbf{r}} = \delta_{\mathbf{r}\mathbf{r}'} \delta_{\mathbf{p}\mathbf{p}'}, [a_{\mathbf{p}\mathbf{r}}, a_{\mathbf{p}'\mathbf{r}'}]_{\pm} \\ &= [a_{\mathbf{p}\mathbf{r}}^*, a_{\mathbf{p}'\mathbf{r}'}^*]_{\pm} = 0, \\ [b_{\mathbf{k}}, b_{\mathbf{k}'}^*]_{\pm} &\equiv b_{\mathbf{k}} b_{\mathbf{k}'}^* - b_{\mathbf{k}'}^* b_{\mathbf{k}} = \delta_{\mathbf{k}\mathbf{k}'}, [b_{\mathbf{k}}, b_{\mathbf{k}'}]_{\pm} = [b_{\mathbf{k}}^*, b_{\mathbf{k}'}^*]_{\pm} = 0. \end{aligned} \quad (1.2)$$

By means of the operators for the free electron field $\psi(\mathbf{x})$ and the free phonon field $\varphi(\mathbf{x})$,

$$\psi_{\alpha}(\mathbf{x}) = V^{-1/2} \sum_{\mathbf{p}, \mathbf{r}} a_{\mathbf{p}\mathbf{r}} u_{\alpha}^{\mathbf{r}} e^{i\mathbf{p}\mathbf{x}/\hbar}, \quad \psi_{\alpha}^*(\mathbf{x}) = V^{-1/2} \sum_{\mathbf{p}, \mathbf{r}} a_{\mathbf{p}\mathbf{r}}^* u_{\alpha}^{\mathbf{r}} e^{-i\mathbf{p}\mathbf{x}/\hbar}, \quad (1.3a)$$

$$\varphi(\mathbf{x}) = \sum_{\mathbf{k}} \left(\frac{\omega_{\mathbf{k}}}{2V} \right)^{1/2} (b_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}/\hbar} + b_{\mathbf{k}}^* e^{-i\mathbf{k}\mathbf{x}/\hbar}), \quad (1.3b)$$

where $u^{\mathbf{r}}$, $\mathbf{r} = 1, 2$, are spin wave functions of the electron, corresponding to the two different spin projections, the interaction operator H_1 can conveniently be written in the following form:

$$H_1 = g \int \psi_{\alpha}^*(\mathbf{x}) \varphi(\mathbf{x}) \psi_{\alpha}(\mathbf{x}) d^3x. \quad (1.4)$$

All of these operators act on the Schrödinger function Φ_{nN} , which for the case in which the system considered is completely closed satisfies the Schrödinger equation

$$i\hbar \frac{\partial \Phi_{nN}}{\partial t} = H \Phi_{nN}, \quad (1.5)$$

where the index n numbers the energy levels of the system consisting of phonons and N electrons. It follows from the form of H_1 that the operator for the total number of electrons

$$N = \sum_{\mathbf{p}, \mathbf{r}} a_{\mathbf{p}\mathbf{r}}^* a_{\mathbf{p}\mathbf{r}} \quad (1.6)$$

commutes with the Hamiltonian H (in what follows the operator for the total number of particles and its eigenvalues will be denoted by the same symbol N). Consequently, in addition to the total energy of a closed system, the number of electrons is also a conserved quantity, and therefore each state Φ_{nN} of the system is characterized not only by the energy quantum number n but also by an indication of the number N of electrons. Furthermore, the energy levels E_{nN} of the system are also different for different values of N . If the system under consideration is quasi-closed, both the energy and the total number of electrons will fluctuate around their average values. To describe the behavior of such a system in thermodynamic equilibrium one uses a Gibbs distribution with variable number of particles, according to which the probability for the system to contain N particles and furthermore be in a state with energy E_{nN} is given by the expression (cf., e.g., Landau and Lifshitz³⁶)

$$W_{nN} = e^{(\Omega + \mu N - E_{nN})\beta}, \quad (1.7)$$

where $\beta = 1/kT$, Ω is the thermodynamic potential of the system, and μ is the chemical potential of the electrons.

For the determination of the thermodynamic characteristics of the system it is sufficient to calculate the statistical sum

$$Z = \sum_{nN} e^{(\mu N - E_{nN})\beta} = \sum_{nN} \Phi_{nN}^* e^{(\mu N - H)\beta} \Phi_{nN} = \text{Sp} e^{(\mu N - H)\beta}, \quad (1.8)$$

where the operators N and H that appear under the trace sign are defined in Eq. (1.6) and Eqs. (1.1a)–(1.1c). For example, the thermodynamic potential Ω of the system is given by

$$\Omega = -\beta^{-1} \ln \text{Sp} e^{(\mu N - H)\beta}. \quad (1.9)$$

If F is an operator that refers to the entire system, then the statistical average value of this operator over the ensemble of interacting particles, $[F]_{\text{av}}$, is given by

$$[F]_{\text{av}} = \sum_{nN} W_{nN} \Phi_{nN}^* F \Phi_{nN} = \text{Sp} (e^{(\mu N - H)\beta} F) / \text{Sp} e^{(\mu N - H)\beta}. \quad (1.10)$$

It is easy to see that the density matrix,

$$\varrho = e^{(\mu N - H)\beta}, \quad (1.11)$$

which appears in the expressions (1.8)–(1.10), satisfies the following equation:

$$\frac{\partial \varrho}{\partial \beta} = (\mu N - H) \varrho. \quad (1.12)$$

Following the work of Matsubara,¹⁴ let us represent the density matrix (1.11) in the form

$$\varrho = e^{(\mu N - H_0)\beta} S(\beta), \quad (1.13)$$

from which it can be seen that the operator $S(\beta)$ can be defined as the solution of the equation

$$-\frac{\partial S}{\partial \beta} = H_1(\beta) S \quad (1.14)$$

with the initial condition $S(0) = 1$, and with the definition

$$H_1(\tau) = e^{-(\mu N - H_0)\tau} H_1 e^{(\mu N - H_0)\tau}. \quad (1.15)$$

The transformation (1.15) from the Schrödinger operator H_1 to the operator $H_1(\tau)$, which depends on τ , is a transition to a peculiar "interaction representation." The dependence of operators on τ given by Eq. (1.15) differs from that used in the well known paper of Matsubara¹⁴ by the presence of the chemical potential μ . As will be seen later, the presence of the parameter μ in Eq. (1.15) greatly facilitates the calculations and enables us to obtain a considerable extension of the Matsubara method.¹⁴ The dependence of the operators on μ as shown in Eq. (1.15) was first introduced independently by Abrikosov, Gor'kov, and Dzyaloshinskii¹⁵ and by Fradkin.^{16,18,19}

As is well known (cf., e.g., the book by Akhiezer and Berestetskii³⁷), the solution of (1.14) with the initial condition $S(0) = 1$ can be written in the form of the series

$$S(\beta) = T e^{-\int_0^\beta H_1(\tau) d\tau} \\ \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \dots \int_0^\beta d\tau_n T [H_1(\tau_1) H_1(\tau_2) \dots H_1(\tau_n)], \quad (1.16)$$

in which, in accordance with Eqs. (1.15) and (1.4), the operator $H_1(\tau)$ is given by

$$H_1(\tau) = g \int \psi_a^\dagger(x) \varphi(x) \psi_a(x) d^3x, \quad (1.17)$$

where the variable x denotes the combination of \mathbf{x} and τ , and the dependence of the operators $\psi^\dagger(x)$, $\psi(x)$, and $\varphi(x)$ on the variable τ is the same as that of $H_1(\tau)$ shown in Eq. (1.15); for example,

$$\psi^\dagger(x) = e^{-(\mu N - H_0)\tau} \psi^\dagger(\mathbf{x}) e^{(\mu N - H_0)\tau}, \quad (1.18)$$

that is, the field operators in Eq. (1.17) are written in the "interaction representation." The symbol T placed before any operators $A(\tau_1) B(\tau_2) \dots F(\tau_n)$ indicates the T -product of these operators, which by definition is of the form (Wick³⁸)

$$T[A(\tau_1) B(\tau_2) \dots F(\tau_n)] = \delta_p B(\tau_2) F(\tau_n) \dots A(\tau_1), \quad (1.19)$$

where $B(\tau_2) F(\tau_n) \dots A(\tau_1)$ is the same set of operators as $A(\tau_1) B(\tau_2) \dots F(\tau_n)$, but arranged so that the numerical value of τ in the operators increases from right to left. $\delta_p = +1$ or -1 , depending on the parity of the number of interchanges of fermion operators that must be made.

If we use the following definition of the statistical average value of an arbitrary operator F in the absence of interaction between the particles ($H_1 = 0$):

$$\langle F \rangle = \text{Sp}(e^{(\mu N - H_0)\beta} F) / \text{Sp}(e^{(\mu N - H_0)\beta}), \quad (1.20)$$

then by using Eqs. (1.11), (1.13), (1.16), and (1.17) we can put the partition function in the form of an infinite

series in which each term is the statistical average of a T -product of field operators over the ensemble of noninteracting particles

$$Z = Z_0 \langle S(\beta) \rangle, \quad (1.21)$$

$$\langle S(\beta) \rangle = \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \int d^4x_1 \int d^4x_2 \dots \int d^4x_n \langle T [(\psi^\dagger(x_1) \varphi(x_1) \psi(x_1)) \dots (\psi^\dagger(x_n) \varphi(x_n) \psi(x_n))] \rangle, \quad (1.22)$$

where $d^4x = d^3x d\tau$ and the integral over \mathbf{x} is taken over the volume of the system, while the integral over the variable τ is taken over the interval from 0 to β .

Similarly we have for the thermodynamic potential

$$\Omega = \Omega_0 - \beta^{-1} \ln \langle S(\beta) \rangle, \quad (1.23)$$

where the logarithm of $\langle S(\beta) \rangle$ can also be represented as a series (see below).

The index 0 is always used to designate quantities that refer to the system of noninteracting particles. If the convergence is rapid, the series (1.21) – (1.23) can serve as the basis for the calculation of all the thermodynamic characteristics of the system; this is the main content of the new thermodynamic perturbation theory (cf. also an interesting version of the new thermodynamic perturbation theory in papers by Hugenholz³⁹ and Ch'en Ch'un-Hsien⁴⁰).

Rules for calculating the terms of the series. The series (1.22) for the operator $S(\beta)$ has a great formal resemblance to the series for the S matrix in quantum field theory, for which there is a well developed mathematical apparatus. Unlike the case of quantum field theory, however, in Eq. (1.22) we must calculate the average values of the T -products of free-field operators not over the ground state (vacuum), but over states of the system which contain arbitrarily large numbers of particles (statistical average). For this reason we must reexamine the well known propositions of the mathematical apparatus of quantum field theory as applied to the calculation of statistical averages of T -products of operators.

Since the statistical averages in Eq. (1.22) are taken over the system of noninteracting particles, the free-field operators that appear in Eq. (1.22) have the following forms:

$$\psi(x) = V^{-1/2} \sum_{\mathbf{p}, r} a_{\mathbf{p}, r} e^{i\mathbf{p}\mathbf{x} \cdot \mathbf{h} - (\epsilon_{\mathbf{p}} - \mu)\tau}, \\ \psi^\dagger(x) = V^{-1/2} \sum_{\mathbf{p}, r} a_{\mathbf{p}, r}^\dagger e^{-i\mathbf{p}\mathbf{x} \cdot \mathbf{h} + (\epsilon_{\mathbf{p}} - \mu)\tau}, \quad (1.24)$$

$$\varphi(x) = \sum_{\mathbf{k}} \left(\frac{\omega_{\mathbf{k}}}{2V} \right)^{1/2} (b_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x} \cdot \mathbf{h} - \omega_{\mathbf{k}}\tau} + b_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\mathbf{x} \cdot \mathbf{h} - \omega_{\mathbf{k}}\tau}). \quad (1.25)$$

Let us put ψ^\dagger , ψ , and φ in the form of sums of two parts

$$\psi(x) = v_1(x) + v_2^*(x), \quad \psi^\dagger(x) = v_1^*(x) + v_2(x), \quad (1.26)$$

$$\varphi(x) = \varphi_-(x) + \varphi_+(x), \quad (1.27)$$

where

$$v_1(x) = V^{-1/2} \sum_{p,r} (1 - q_{pr}) a_{pr} u^r e^{ipx/h - (\epsilon_p - \mu)\tau},$$

$$v_1^+(x) = V^{-1/2} \sum_{p,r} (1 - q_{pr}^*) a_{pr}^* u^r e^{-ipx/h + (\epsilon_p - \mu)\tau}, \quad (1.28)$$

$$\varphi_-(x) = \sum_{\mathbf{k}} \left(\frac{\omega_{\mathbf{k}}}{2V} \right)^{1/2} [(1 - f_{1\mathbf{k}}) b_{\mathbf{k}} e^{i\mathbf{k}x/h - \omega_{\mathbf{k}}\tau} + f_{2\mathbf{k}} b_{\mathbf{k}}^* e^{-i\mathbf{k}x/h + \omega_{\mathbf{k}}\tau}], \quad (1.29)$$

and the operators $v_2(x)$, $v_2^+(x)$ and $\varphi_+(x)$ are found from Eqs. (1.26) – (1.29) by simple subtraction. According to the work of Wick³⁸ we define the \mathcal{N} -product of operators $v_1(x) v_2(x) v_1^+(x) v_2^+(x) \dots \varphi_-(x) \varphi_+(x) \dots$ as a product in which all the v_1^+ are to the left of all v_1 , v_2 , and v_2^+ , all v_2^+ are to the left of all v_1 and v_2 , all v_2 are to the left of all v_1 , and, finally, all φ_+ are to the left of all φ_- , so that:

$$\mathcal{N}(v_1(x) v_2(x) v_1^+(x) v_2^+(x) \dots \varphi_-(x) \varphi_+(x) \dots)$$

$$= \delta_p v_1^+(x) v_1^+(x') \dots v_2^+(x) v_2^+(x') \dots v_2(x) v_2(x')$$

$$\dots v_1(x) v_1(x') \dots \varphi_+(x) \varphi_+(x') \dots \varphi_-(x) \varphi_-(x') \dots, \quad (1.30)$$

where the sign symbol δ_p is equal to ± 1 , depending on the parity of the number of interchanges of fermion operators. The T- or \mathcal{N} -product of a sum of operators is equal to the sum of the T- or \mathcal{N} -products of the individual terms, so that for the working out of an expression of the form

$$\mathcal{N}(\psi\psi^+ \dots \varphi \dots) \text{ or } T(\psi\psi^+ \dots \varphi \dots) \quad (1.31)$$

one must use Eqs. (1.26), (1.27), and (1.30). With this definition of the \mathcal{N} -product the fermion field operators (1.24) anticommute under the sign of the \mathcal{N} -product, and the boson operators (1.25) commute. The difference of the T- and \mathcal{N} -products of two operators A and B of the types (1.24) – (1.27) is called the contraction \underline{AB}

$$T(AB) - \mathcal{N}(AB) = \underline{AB}. \quad (1.32)$$

As is well known, the T- and \mathcal{N} -products (1.19) and (1.30) – (1.31) of operators ABCDE...F obey Wick's algebraic theorem,³⁸ according to which the T-product of a set of operators is equal to the sum of all the \mathcal{N} -products of the same operators with all possible contractions:

$$T(ABCDE \dots F) = \mathcal{N}(ABCDE \dots F)$$

$$+ \mathcal{N}(\underline{AB}CDE \dots F) + \dots + \mathcal{N}(\underline{ABCDE} \dots F)$$

$$+ \dots + \mathcal{N}(\underline{ABCDE} \dots \underline{F}). \quad (1.33)$$

Since we are concerned only with the statistical average of a T-product of a set of operators of the types (1.24) – (1.25) over a system of noninteracting particles, we shall try to choose the constants q_{pr} , $f_{1\mathbf{k}}$, and $f_{2\mathbf{k}}$ in Eqs. (1.28) and (1.29) in such a way that all statistical averages of \mathcal{N} -products will satisfy the equation

$$\langle \mathcal{N}(\psi\psi^+ \dots \varphi \dots) \rangle = 0. \quad (1.34)$$

Then Wick's theorem (1.33) for free-field operators ABCDE...F of Eqs. (1.24) – (1.25) will read simply:

$$\langle T(ABCDE \dots F) \rangle = \underline{ABCDE} \dots \underline{F} + \dots, \quad (1.35)$$

where the right member is the sum of products of the operators with all possible contractions. The expression (1.35) is different from zero only if there is an even number of boson operators φ under the sign of the T-product, and if the fermion operators ψ and ψ^+ occur in pairs. This is a consequence of the fact that for arbitrary values of q_{pr} , $f_{1\mathbf{k}}$, and $f_{2\mathbf{k}}$ there are only two nonvanishing contractions of the operators (1.24) – (1.25):

$$\underline{\psi_{\alpha}(x) \psi_{\beta}^+(x')} \equiv \mathcal{G}_{0\alpha\beta}(x - x')$$

$$= \begin{cases} V^{-1} \sum_{p,r} (1 - |q_{pr}|^2) u_{\alpha}^r u_{\beta}^{r*} e^{ip(x-x')/h - (\epsilon_p - \mu)(\tau - \tau')}, & \tau > \tau' \\ -V^{-1} \sum_{p,r} |q_{pr}|^2 u_{\alpha}^r u_{\beta}^{r*} e^{ip(x-x')/h - (\epsilon_p - \mu)(\tau - \tau')}, & \tau < \tau', \end{cases} \quad (1.36)$$

$$\underline{\varphi(x) \varphi(x')} \equiv \mathcal{D}_0(x - x')$$

$$= \begin{cases} \sum_{\mathbf{k}} (\omega_{\mathbf{k}}/2V) [(1 - f_{1\mathbf{k}})(1 - f_{2\mathbf{k}}) e^{i\mathbf{k}(x-x')/h - \omega_{\mathbf{k}}(\tau - \tau')} - \\ - f_{1\mathbf{k}} f_{2\mathbf{k}} e^{-i\mathbf{k}(x-x')/h + \omega_{\mathbf{k}}(\tau - \tau')}], & \tau > \tau', \\ \sum_{\mathbf{k}} (\omega_{\mathbf{k}}/2V) [(1 - f_{1\mathbf{k}} f_{2\mathbf{k}}) e^{i\mathbf{k}(x'-x)/h - \omega_{\mathbf{k}}(\tau' - \tau)} + \\ + (f_{1\mathbf{k}} f_{2\mathbf{k}} - f_{1\mathbf{k}} - f_{2\mathbf{k}}) e^{-i\mathbf{k}(x'-x)/h + \omega_{\mathbf{k}}(\tau' - \tau)}], & \tau' > \tau. \end{cases} \quad (1.37)$$

On the other hand, by using Eqs. (1.24) and (1.25) one can easily verify that

$$\langle T(\psi_{\alpha}(x) \psi_{\beta}^+(x')) \rangle$$

$$= \begin{cases} V^{-1} \sum_{p,r} (1 - \langle a_{pr}^* a_{pr} \rangle) u_{\alpha}^r u_{\beta}^{r*} e^{ip(x-x')/h - (\epsilon_p - \mu)(\tau - \tau')}, & \tau > \tau', \\ -V^{-1} \sum_{p,r} \langle a_{pr}^* a_{pr} \rangle u_{\alpha}^r u_{\beta}^{r*} e^{-ip(x-x')/h - (\epsilon_p - \mu)(\tau - \tau')}, & \tau < \tau', \end{cases} \quad (1.38)$$

$$\langle T(\varphi(x) \varphi(x')) \rangle = \sum_{\mathbf{k}} \frac{\omega_{\mathbf{k}}}{2V} [(1 + \langle b_{\mathbf{k}}^* b_{\mathbf{k}} \rangle) e^{i\mathbf{k}(x-x')/h - \omega_{\mathbf{k}}(\tau - \tau')} + \langle b_{\mathbf{k}}^* b_{\mathbf{k}} \rangle e^{-i\mathbf{k}(x-x')/h + \omega_{\mathbf{k}}(\tau - \tau')}], \quad (1.39)$$

where $\langle a_{pr}^* a_{pr} \rangle$ and $\langle b_{\mathbf{k}}^* b_{\mathbf{k}} \rangle$ are the statistical averages of the numbers of electrons with the energy ϵ_p and polarization r and of phonons with the energy $\omega_{\mathbf{k}}$, in the absence of interaction between the particles. In order to calculate $\langle a_{pr}^* a_{pr} \rangle$ and $\langle b_{\mathbf{k}}^* b_{\mathbf{k}} \rangle$, let us consider

$$Z_0 = \text{Sp } e^{(\mu N - H_0) \beta} \quad (1.40)$$

and the statistical operator W_0 of the system of noninteracting particles⁴¹

$$W_0 = e^{(\Omega_0 + \mu N - H_0) \beta} = Z_0^{-1} e^{(\mu N - H_0) \beta}, \quad (1.41)$$

where

$$\mu N - H_0 = \sum_{p,r} (\mu - \epsilon_p) a_{pr}^* a_{pr} + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^* b_{\mathbf{k}}. \quad (1.42)$$

Since the different terms of the sums in Eq. (1.42) commute with each other, the statistical operator

(1.41) can be represented as a product of statistical operators referring to each individual state of an electron or phonon (to the individual field oscillators):

$$W_0 = \prod_{p,r} Z_{pr}^{-1} e^{(\mu - \epsilon_p) \beta \hat{n}_{pr}} \prod_{\mathbf{k}} Z_{\mathbf{k}}^{-1} e^{-\omega_{\mathbf{k}} \beta \hat{n}_{\mathbf{k}}} \equiv \prod_{p,r} W_{pr} \prod_{\mathbf{k}} W_{\mathbf{k}}, \quad (1.43)$$

where

$$\hat{n}_{pr} = a_{pr}^\dagger a_{pr}, \quad \hat{n}_{\mathbf{k}} = b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, \quad Z_0 = \prod_{p,r} Z_{pr} \prod_{\mathbf{k}} Z_{\mathbf{k}}. \quad (1.44)$$

Using the normalization of the statistical operators, $\text{Sp } W_{pr} = 1$, $\text{Sp } W_{\mathbf{k}} = 1$, and also the diagonal values of the operators \hat{n}_{pr} and $\hat{n}_{\mathbf{k}}$, which are respectively 0, 1 and 0, 1, 2, ..., ∞ , we easily find that

$$Z_{pr} = 1 + e^{(\mu - \epsilon_p) \beta}, \quad Z_{\mathbf{k}} = (1 - e^{-\omega_{\mathbf{k}} \beta})^{-1}. \quad (1.45)$$

From Eqs. (1.45) and (1.44) we get, by the way, the well known expression for the thermodynamic potential Ω_0 of the electron-phonon system in the absence of interaction between the particles

$$\Omega_0 = \frac{V}{(2\pi\hbar)^3 \beta} \int \ln(1 - e^{-\omega_{\mathbf{k}} \beta}) d^3k - \frac{2V}{(2\pi\hbar)^3 \beta} \int \ln(1 + e^{(\mu - \epsilon_p) \beta}) d^3p, \quad (1.46)$$

where in passing from the sum to the integral we have used the relation

$$\sum_{p,r} = \frac{2V d^3p}{(2\pi\hbar)^3}. \quad (1.46a)$$

Knowledge of the statistical operators W_{pr} and $W_{\mathbf{k}}$ for the individual particle states enables us to calculate the distribution functions n_{pr} and $n_{\mathbf{k}}$ of noninteracting electrons and phonons:

$$n_{pr} = \langle a_{pr}^\dagger a_{pr} \rangle = (1 + e^{(\mu - \epsilon_p) \beta})^{-1} \text{Sp } e^{(\mu - \epsilon_p) \beta \hat{n}_{pr}} \hat{n}_{pr} = (e^{(\epsilon_p - \mu) \beta} + 1)^{-1},$$

$$n_{\mathbf{k}} = \langle b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \rangle = (1 - e^{-\omega_{\mathbf{k}} \beta}) \text{Sp } e^{-\omega_{\mathbf{k}} \beta \hat{n}_{\mathbf{k}}} \hat{n}_{\mathbf{k}} = (e^{\omega_{\mathbf{k}} \beta} - 1)^{-1}. \quad (1.47)$$

The average number of electrons with a given energy ϵ_p , Eq. (1.47), does not depend on the spin state of the electron. Therefore we shall hereafter omit the index r on the distribution function (1.47): $n_{pr} \equiv n_p$.

Thus if we take for the constants q_{pr} , $f_{1\mathbf{k}}$, and $f_{2\mathbf{k}}$ the values

$$|q_{pr}|^2 = (e^{(\epsilon_p - \mu) \beta} + 1)^{-1}, \quad (1.48)$$

$$f_{1\mathbf{k}} = (1 + e^{\omega_{\mathbf{k}} \beta/2})^{-1}, \quad f_{2\mathbf{k}} = (1 - e^{\omega_{\mathbf{k}} \beta/2})^{-1}, \quad (1.49)$$

then according to Eqs. (1.36) – (1.37) and (1.38) – (1.39) (Matsubara¹⁴) we have

$$\langle \mathcal{N}'(\psi(x)\psi^*(x')) \rangle = 0, \quad \langle \mathcal{N}'(\varphi(x)\varphi(x')) \rangle = 0, \quad (1.50)$$

and the contraction of a product is equal to the statistical average of the T-product of the given operators.

If for $\psi(x)$ and $\psi^*(x)$ we insert their series expansions (1.24), then in virtue of the statistical independence of the individual states of the particles, Eq. (1.43), the theorem (1.50) will also be true for each term of the sum so obtained; for example,

$$\langle \mathcal{N}'(a_{pr}^\dagger a_{pr}) \rangle = 0, \quad (1.51)$$

where as before the \mathcal{N}' -product is understood in the sense of Eq. (1.30), and the parts of v_1 , v_1^\dagger , v_2 , and v_2^\dagger are played by their individual terms α_{1pr} , α_{1pr}^\dagger , α_{2pr} , and α_{2pr}^\dagger , which are given by (Thouless⁴²)

$$\alpha_{1pr} = (1 - q_{pr}) a_{pr}, \quad \alpha_{2pr} = q_{pr}^\dagger a_{pr}^\dagger,$$

$$a_{pr} = \alpha_{1pr} + \alpha_{2pr}, \quad a_{pr}^\dagger = \alpha_{1pr}^\dagger + \alpha_{2pr}^\dagger. \quad (1.52)$$

All unessential factors are omitted in Eqs. (1.51) and (1.52).

An \mathcal{N}' -product of a larger number of factors $\psi\psi^\dagger$ is expanded into a sum of \mathcal{N}' -products of operators $a_{pr} a_{pr}^\dagger, a_{pr}^\dagger a_{pr}, \dots$. If all the fermion operators a_{pr} and a_{pr}^\dagger refer to different states of the electron, then the average of such an \mathcal{N}' -product is zero by Eq. (1.51). If there are any two identical operators among the a_{pr} (or among the a_{pr}^\dagger), then because of their anticommutation the \mathcal{N}' -product is zero. Consequently, when Eqs. (1.50) and (1.51) hold, the average of an \mathcal{N}' -product containing an arbitrary number of fermion operators $\psi(x)$ and $\psi^\dagger(x)$ is equal to zero. This last assertion also holds for boson operators $\varphi(x)$ (Thouless⁴²). To see this, according to Eq. (1.35) we have only to show that the average of the T-product of any number of boson operators (1.25) is identically equal to the sum of all products of these operators with all possible contractions, where by a contraction we mean the expression given by Eqs. (1.37) and (1.49).

The statistical operator W_0 , Eq. (1.43), which determines the average value, is factored. Therefore we can carry out the proof of the theorem for each individual boson state. The expression $\langle T(\varphi_{\mathbf{k}}(x)\varphi_{\mathbf{k}}(x')\dots) \rangle$, where $\varphi_{\mathbf{k}}(x)$ is the term of the sum (1.25), $\varphi(x) = \sum_{\mathbf{k}} \varphi_{\mathbf{k}}(x)$, that refers to the boson state with the energy $\omega_{\mathbf{k}}$, breaks up into a sum of terms containing $\langle b_{\mathbf{k}}^{\dagger l} b_{\mathbf{k}}^m \rangle$. Because of the orthogonality of the wave functions Φ_{nN} with respect to which one takes the average, the only nonvanishing terms will be those that contain equal numbers of production operators $b_{\mathbf{k}}^\dagger$ and absorption operators $b_{\mathbf{k}}$ ($l = m$).

According to Eq. (1.2) the occupation-number operators $n_{\mathbf{k}} = b_{\mathbf{k}}^\dagger b_{\mathbf{k}}$ satisfy the following relations (Bloch and DeDominicis⁴¹):

$$\hat{n}_{\mathbf{k}} b_{\mathbf{k}} = b_{\mathbf{k}} (\hat{n}_{\mathbf{k}} - 1),$$

$$\hat{n}_{\mathbf{k}} b_{\mathbf{k}}^2 = b_{\mathbf{k}}^2 (\hat{n}_{\mathbf{k}} - 2),$$

$$\dots$$

$$\hat{n}_{\mathbf{k}} b_{\mathbf{k}}^m = b_{\mathbf{k}}^m (\hat{n}_{\mathbf{k}} - m), \quad (1.53)$$

from which we have

$$b_{\mathbf{k}}^{\dagger(m)} b_{\mathbf{k}}^m = b_{\mathbf{k}}^{\dagger(m-1)} \hat{n}_{\mathbf{k}} b_{\mathbf{k}}^{(m-1)} = b_{\mathbf{k}}^{\dagger(m-1)} b_{\mathbf{k}}^{(m-1)} (\hat{n}_{\mathbf{k}} - m + 1) = b_{\mathbf{k}}^{\dagger(m-2)} b_{\mathbf{k}}^{m-2}$$

$$\times (\hat{n}_{\mathbf{k}} - m + 2) (\hat{n}_{\mathbf{k}} - m + 1) = \hat{n}_{\mathbf{k}} (\hat{n}_{\mathbf{k}} - 1) \dots (\hat{n}_{\mathbf{k}} - m + 1). \quad (1.54)$$

Averaging Eq. (1.54) by means of the statistical operator $W_{\mathbf{k}}$ of Eqs. (1.43) – (1.45),

$$W_{\mathbf{k}} = (1 - e^{-\omega_{\mathbf{k}}\beta}) e^{-\omega_{\mathbf{k}}\beta\hat{n}_{\mathbf{k}}} \equiv (1 - Z) Z^{\hat{n}_{\mathbf{k}}}, \quad (1.55)$$

we get

$$\begin{aligned} \langle b_{\mathbf{k}}^{(m)} b_{\mathbf{k}}^m \rangle &= (1 - Z) \sum_{n=0}^{\infty} n(n-1) \dots (n-m+1) Z^n \\ &= (1 - Z) Z^m \left(\frac{d}{dZ} \right)^m \sum_{n=0}^{\infty} Z^n = (1 - Z) Z^m \left(\frac{d}{dZ} \right)^m \frac{1}{1-Z} \\ &= m! \left(\frac{Z}{1-Z} \right)^m \equiv m! \langle b_{\mathbf{k}}^+ b_{\mathbf{k}} \rangle^m, \end{aligned} \quad (1.56)$$

i.e., in the averaging of $\langle b_{\mathbf{k}}^{+m} b_{\mathbf{k}}^m \rangle$ the first factor $b_{\mathbf{k}}^+$ can be combined in a pair with any of the m factors $b_{\mathbf{k}}$, the second $b_{\mathbf{k}}^+$ with any of the $m-1$ remaining factors $b_{\mathbf{k}}$, and so on, making $m!$ combinations in all.

The result (1.56) applied to $\langle T(\varphi_{\mathbf{k}}(x) \varphi_{\mathbf{k}}(x') \varphi_{\mathbf{k}}(x'') \dots) \rangle$ means that the average of the T-product of any number of boson operators $\varphi_{\mathbf{k}}(x)$, $\varphi_{\mathbf{k}}(x')$, $\varphi_{\mathbf{k}}(x'')$, ... is equal to the sum of all products of these operators with all possible contractions; as in Eqs. (1.37), (1.49), this sum is given by⁴²

$$\begin{aligned} \varphi_{\mathbf{k}}(x) \varphi_{\mathbf{k}}(x') &= \frac{\omega_{\mathbf{k}}}{2V} [(1 + \langle b_{\mathbf{k}}^+ b_{\mathbf{k}} \rangle) e^{i\mathbf{k}(x-x')/h - \omega_{\mathbf{k}}|\tau-\tau'|} \\ &+ \langle b_{\mathbf{k}}^+ b_{\mathbf{k}} \rangle e^{-i\mathbf{k}(x-x')/h + \omega_{\mathbf{k}}|\tau-\tau'|}]. \end{aligned} \quad (1.57)$$

The diagram technique. The individual terms of the perturbation-theory series (1.22) are averages of T-products of free-field operators. Since the average of a T-product is different from zero only for an even number of boson operators $\varphi(x)$, the sum (1.22) contains terms with even values of the summation index. In the expansion of a T-product into a sum of products with all possible contractions we must not contract operators $\psi(x)$ and $\psi^+(x)$ that have the same variable of integration, because the term with such a contraction is zero.¹⁴

$$\begin{aligned} &\int \underbrace{(\psi^+(x) \varphi(x) \psi(x)) \dots (\psi^+(x') \varphi(x') \psi(x')) \dots}_{\dots} d^4x d^4x' \\ &= \int \mathcal{G}_0(x-x) \mathcal{D}_0(x-x') \dots \underbrace{(\psi^+(x') \psi(x'))}_{\dots} d^4x d^4x' \\ &= \mathcal{G}_0(0) \int \mathcal{D}_0(x) d^4x \int \dots \underbrace{(\psi^+(x') \psi(x'))}_{\dots} d^4x', \end{aligned} \quad (1.58)$$

where the integral of the function $\mathcal{D}_0(x)$ over all the space of \mathbf{x} is zero, since the sum (1.37) does not contain the term with phonon momentum $\mathbf{k} = 0$.

It is convenient to represent an individual term in the average of the T-product (1.22) by a graph according to the following rule (the analogous method of constructing graphs in quantum electrodynamics is described in detail, for example, in the book of Akhiezer and Berestetskii,³⁷ to which reference is made). To each variable of integration x_i there corresponds a point, a vertex of the graph. Since the variable x_i is

contained in three operators $\psi^+(x_i) \varphi(x_i) \psi(x_i)$, three lines come together at a vertex – two solid lines, corresponding to the operators $\psi^+(x_i)$ and $\psi(x_i)$, and one dotted line, corresponding to $\varphi(x_i)$.

The operators ψ^+ , ψ , and φ occur only in the form of contractions. Therefore the solid and dotted lines will begin and end at vertices of the graph; to the contraction $\mathcal{G}_0(x-x')$ of fermion operators there corresponds in the graph a solid line going from the vertex x' to the vertex x , and to the contraction $\mathcal{D}_0(x-x')$ of boson operators there corresponds a dotted line connecting the vertices x and x' . The operators $\psi^+(x')$ and $\psi(x)$ that are involved in $\mathcal{G}_0(x-x')$ contain only production and absorption operators, respectively. Therefore it is said that a solid line going from a vertex x' to a vertex x describes the motion of a virtual electron produced at the point x' and absorbed at x , and $\mathcal{G}_0(x-x')$ is called the propagation function of the electron. Analogously, $\mathcal{D}_0(x-x')$ is called the propagation function of the phonon. As has been noted earlier, the graphs consist of even numbers of vertices. For example, there is only one term of order g^2 in the series (1.22),

$$-\frac{g^2}{2} \int d^4x_1 d^4x_2 \mathcal{G}_0(x_1-x_2) \mathcal{D}_0(x_1-x_2) \mathcal{G}_0(x_2-x_1), \quad (1.59)$$

to which there corresponds a graph with two vertices (Fig. 1). Conversely, from the form of the graph one can write down the corresponding term of the expansion of the T-product, remembering that to each graph with n vertices there corresponds a factor $(-1)^{n+l} g^n/n!$, where l is the number of closed electron loops contained in the graph in question. The additional factor $(-1)^l$ comes from the anticommutation of the Fermi operators under the sign of the T-product. In order to get all graphs with n vertices, one

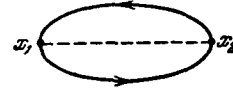


FIG. 1

must connect n points x_1, x_2, \dots, x_n (the vertices) by lines in all possible ways such that two solid lines and one dotted line meet at each vertex. Then to each such graph there will correspond a definite term of the expansion of the T-product (1.22), proportional to g^n . In writing down the terms from the forms of the graphs one must take account of the fact that several terms can be numerically equal, differing only by interchanges of the variables of integration. For example, the graphs of different structures with four vertices are shown in Fig. 2. The four-vertex graphs not shown are obtained from the graphs of Fig. 2 by giving different designations to the vertices. One thus gets, for example, six different graphs of type c) and three of type d); to such sets of graphs there correspond numerically equal terms in the expansion of the T-product.

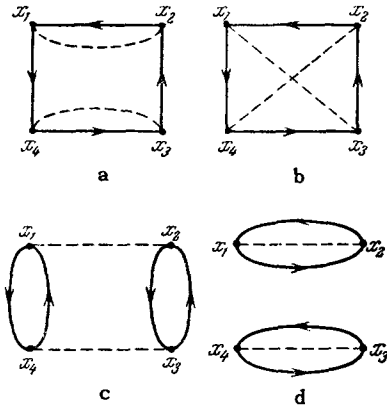


FIG. 2

Thus $\langle S(\beta) \rangle$ is the sum of all possible graphs with closed solid lines. Since to each line of a graph there corresponds a function of a difference of coordinates, the expression corresponding to an individual connected closed graph is proportional to the volume V . In particular, the sum $\langle S(\beta) \rangle$ contains terms that each correspond to several separate connected closed graphs, for example, the graphs d) of Fig. 2. Each such term in the sum $\langle S(\beta) \rangle$ is proportional to V^m , where m is the number of separate connected graphs.

On the other hand, the thermodynamic potential (1.23) must be proportional to the first power of the volume V . It follows that the sum $\langle S(\beta) \rangle$ must be of the form $\exp L$, where L is the sum of all connected closed graphs. In fact, if we denote by L_S the contribution from one connected closed graph of type s , then a term of the sum $\langle S(\beta) \rangle$ that breaks up into several types of connected graphs is obviously equal to $\prod_s (l_s!)^{-1} (L_S)^{l_s}$, where l_s is the number of graphs of a given type. Then the entire infinite sum $\langle S(\beta) \rangle$ can be written in the following way (Vedenov and Larkin⁴³):

$$\langle S(\beta) \rangle = \sum_{l_s=0}^{\infty} \prod_s \frac{1}{l_s!} (L_S)^{l_s} = \prod_s e^{L_S} = e^{\Sigma L_S},$$

where ΣL_S is the sum of all connected closed graphs. For the thermodynamic potential (1.23) we have $\Omega = \Omega_0 - \beta^{-1} \Sigma L_S$.

Besides simplicity and the well known "automatic" character of the calculations, this method has the advantage that by means of intuitively significant graphs

$$\mathcal{G}_0(\mathbf{x}_1 - \mathbf{x}_2, \tau_1 - \tau_2) \equiv \langle T(\Psi(\mathbf{x}_1) \Psi^+(\mathbf{x}_2)) \rangle = \begin{cases} Z_0^{-1} \text{Sp} e^{(\mu N - H_0)\beta} e^{-(\mu N - H_0)(\tau_1 - \tau_2)} \Psi(\mathbf{x}_1) e^{(\mu N - H_0)(\tau_1 - \tau_2)} \Psi^+(\mathbf{x}_2), & \tau_1 > \tau_2, \\ -Z_0^{-1} \text{Sp} e^{(\mu N - H_0)\beta} e^{-(\mu N - H_0)(\tau_2 - \tau_1)} \Psi^+(\mathbf{x}_2) e^{(\mu N - H_0)(\tau_2 - \tau_1)} \Psi(\mathbf{x}_1), & \tau_2 > \tau_1, \\ Z_0^{-1} \text{Sp} e^{(\mu N - H_0)\beta} e^{-(\mu N - H_0)(\tau_2 - \tau_1 + \beta)} \Psi^+(\mathbf{x}_2) e^{(\mu N - H_0)(\tau_2 - \tau_1 + \beta)} \Psi(\mathbf{x}_1), & \tau_1 > \tau_2, \\ -Z_0^{-1} \text{Sp} e^{(\mu N - H_0)\beta} e^{-(\mu N - H_0)(\tau_1 - \tau_2 + \beta)} \Psi(\mathbf{x}_1) e^{(\mu N - H_0)(\tau_1 - \tau_2 + \beta)} \Psi^+(\mathbf{x}_2), & \tau_2 > \tau_1. \end{cases}$$

From this we have for $\tau_1 - \tau_2 = \tau < 0$

$$\mathcal{G}_0(\mathbf{x}_1 - \mathbf{x}_2, \tau) = -\mathcal{G}_0(\mathbf{x}_1 - \mathbf{x}_2, \tau + \beta), \quad (1.61)$$

one can get a deeper understanding of the structure of the higher approximations. Given the possibility of writing down any term of the perturbation-theory series, one can sum an infinite set of terms of a definite class, and thus get a result that is different from ordinary calculations by perturbation theory (an expansion in terms of a different parameter, which is especially valuable in cases in which perturbation theory is inapplicable). Moreover, in the diagram technique one can very simply get estimates of the orders of magnitude of omitted graphs, and thus can easily establish the range of applicability and the accuracy of a selective summation. A remarkable example of this is a paper by Vedenov and Larkin,⁴³ in which a selective summation of an infinite number of terms in the perturbation-theory series is used to find the free energy of a completely ionized gas [with the Hamiltonian (1.71a) - (1.71c), in which $v(\mathbf{x} - \mathbf{x}')$ is the Coulomb potential for the interaction of a pair of particles] in the form of an expansion in the density n . In addition to the Debye term, the authors succeeded in getting the next two terms of the expansion, proportional to $n^2 \ln n$ and to n^2 .

The change to the p representation. The technique for calculating the terms of the series (1.22) is greatly facilitated by going from the coordinate representation to the momentum representation of the functions \mathcal{G}_0 and \mathcal{D}_0 . In doing so it must be kept in mind that the contractions $\mathcal{G}_0(\mathbf{x}_1 - \mathbf{x}_2, \tau_1 - \tau_2)$ and $\mathcal{D}_0(\mathbf{x}_1 - \mathbf{x}_2, \tau_1 - \tau_2)$ are functions of the differences $\mathbf{x}_1 - \mathbf{x}_2$ and $\tau_1 - \tau_2$, and that as functions of $\tau_1 - \tau_2$ they are defined only in the range from $-\beta$ to β . Let us define the functions $\mathcal{G}_0(\mathbf{x}, \tau)$ and $\mathcal{D}_0(\mathbf{x}, \tau)$ on the entire τ axis by continuing them periodically. Then we can expand $\mathcal{G}_0(\mathbf{x}, \tau)$ and $\mathcal{D}_0(\mathbf{x}, \tau)$ in Fourier series with respect to the variable τ , and in Fourier integrals with respect to the space coordinates \mathbf{x} ; for example:

$$\mathcal{G}_0(\mathbf{x}, \tau) = \frac{1}{(2\pi\hbar)^3 \beta} \sum_n \int \mathcal{G}_0(\mathbf{p}, \omega_n) e^{i(\mathbf{p}\mathbf{x}/\hbar - \omega_n\tau)} d^3p, \\ \mathcal{G}_0(\mathbf{p}, \omega_n) = \frac{1}{2} \int_{-\beta}^{\beta} d\tau \int d^3x \mathcal{G}_0(\mathbf{x}, \tau) e^{-i(\mathbf{p}\mathbf{x}/\hbar - \omega_n\tau)}. \quad (1.60)$$

The change to the p representation is much simplified owing to an important property of the functions \mathcal{G}_0 and \mathcal{D}_0 . According to the definition (1.32) of a contraction of operators, and also Eqs. (1.50), (1.20), and (1.18), we have for the function \mathcal{G}_0 :

where the minus sign of $\mathcal{G}_0(\mathbf{x}_1 - \mathbf{x}_2, \tau + \beta)$ is due to the anticommutation of the fermion operators. An analogous relation also holds for the contraction of the boson operators for $\tau < 0$

$$\mathcal{D}_0(\mathbf{x}, \tau) = \mathcal{D}_0(\mathbf{x}, \tau + \beta). \quad (1.62)$$

If we continue the functions $\mathcal{G}_0(\mathbf{x}, \tau)$ and $\mathcal{D}_0(\mathbf{x}, \tau)$ of the variable τ periodically along the entire τ axis, then the relations (1.61) and (1.62) will be satisfied for all τ (Abrikosov, Gor'kov, and Dzyaloshinskii¹⁵).

As we have shown by the diagram technique, each vertex of a graph joins an even number of fermion

lines. Therefore all of the integrals $\int_0^\beta \dots d\tau$ that

occur in Eq. (1.22) can be replaced by $\frac{1}{2} \int_{-\beta}^\beta \dots d\tau$. In

fact, by making a change of variables and using Eqs. (1.61) and (1.62), we get

$$\begin{aligned} & \int_0^\beta \mathcal{G}_0(x_1 - x) \mathcal{D}_0(x_3 - x) \mathcal{G}_0(x - x_2) d\tau \\ &= \int_{-\beta}^0 \mathcal{G}_0(x_1 - x) \mathcal{D}_0(x_3 - x, \tau_3 - \tau + \beta) \mathcal{G}_0(x - x_2) d\tau \\ &= \int_{-\beta}^0 \mathcal{G}_0(x_1 - x) \mathcal{D}_0(x_3 - x) \mathcal{G}_0(x - x_2) d\tau. \end{aligned} \quad (1.63)$$

This fact allows us to carry out the Fourier transformation easily in all of the terms of the series (1.22).

It also follows from the relations (1.61) and (1.62) that in the Fourier expansion (1.60) of the fermion contraction $\mathcal{G}_0(\mathbf{x}, \tau)$ the only nonvanishing components are those with $\omega_n = (2n + 1)\pi/\beta$, and in the expansion of the boson contraction $\mathcal{D}_0(\mathbf{x}, \tau)$ there are only components with $\omega_n = 2n\pi/\beta$, where $n = 0, \pm 1, \pm 2, \dots$. According to Eqs. (1.38), (1.39), and (1.47) the explicit forms of $\mathcal{G}_0(\mathbf{p}, \omega_n)$ and $\mathcal{D}_0(\mathbf{k}, \omega_n)$ are as follows:

$$\mathcal{G}_{0\alpha\beta}(\mathbf{p}, \omega_n) = -(i\omega_n + \mu - \varepsilon_p)^{-1} \delta_{\alpha\beta}, \quad \omega_n = (2n + 1)\pi/\beta, \quad (1.64)$$

$$\mathcal{D}_0(\mathbf{k}, \omega_n) = \frac{\omega_k^2}{\omega_k^2 + \omega_n^2}, \quad \omega_n = 2n\pi/\beta. \quad (1.65)$$

In the p representation a propagation function $\mathcal{G}_0(\mathbf{p}, \omega_n)$ corresponds to each solid line of a graph, and a function $\mathcal{D}_0(\mathbf{k}, \omega_n)$ to each dotted line. To each vertex of a graph there corresponds a factor $\delta(\Sigma\mathbf{p})\delta_{0\Sigma\omega_n}$, where $\Sigma\mathbf{p}$ and $\Sigma\omega_n$ are sums of the components of the four-dimensional momenta of the solid and dotted lines that come together at the vertex in question, and $\delta_{0\Sigma\omega_n}$ is the Kronecker δ symbol,

$$\delta_{0\Sigma\omega_n} = \begin{cases} 1 & \text{for } \sum \omega_n = 0, \\ 0 & \text{for } \sum \omega_n \neq 0. \end{cases}$$

We can agree to include in the sums $\Sigma\mathbf{p}$ and $\Sigma\omega_n$ the momenta of the particles that are produced at the vertex with the plus sign, and those of particles absorbed

at the vertex with the minus sign. There is to be an integration over all the three-dimensional momenta of the solid and dotted lines, and a summation over all the fourth components of the momenta. To an individual graph with n vertices there corresponds a factor

$$(-1)^{n+l} g^n [(2\pi\hbar)^3 \beta]^{n-m}/n!,$$

where l is the number of closed loops of fermion lines and m is the number of solid and dotted lines contained in the given graph. For example, the expression $Z^{(1)}$ that corresponds to the graph of Fig. 1 has the following form in the p representation:

$$\begin{aligned} Z^{(1)} = & -\frac{g^2 V}{(2\pi\hbar)^6} \beta \sum_{\omega, \omega_1} \int \frac{\omega_k^2 d^3 p_1 d^3 k}{(i\omega_1 + \mu - \varepsilon_{p_1}) [i(\omega_1 + \omega) + \mu - \varepsilon_{p_1 + k}] (\omega_k^2 + \omega^2)}, \\ & \omega_1 = (2n + 1)\pi/\beta, \quad \omega = 2m\pi/\beta, \end{aligned} \quad (1.66)$$

where we have used

$$\delta(0) = \frac{1}{(2\pi\hbar)^3} \int e^{i\mathbf{p}\mathbf{x}/\hbar} d^3 x \Big|_{\mathbf{p}=0} = \frac{V}{(2\pi\hbar)^3}, \quad (1.67)$$

since the integration over the space coordinate \mathbf{x} is always over the volume V of the statistical system under consideration. The summation over ω and ω_1 in Eq. (1.66) leads to the following result:

$$Z^{(1)} = \frac{2g^2 V \beta}{(2\pi\hbar)^6} \int \frac{\{(n_p - n_{p+k})n_k + (1 - n_{p+k})n_p\} \omega_k}{\varepsilon_{p+k} - \varepsilon_p + \omega_k} d^3 p d^3 k, \quad (1.68)$$

where n_p and n_k are the respective distribution functions of the electrons and phonons, Eq. (1.47). To terms of order g^2 the partition function of Eq. (1.21) is

$$Z = Z_0(1 + Z^{(1)}), \quad (1.69)$$

and the thermodynamic potential Ω of the system is

$$\Omega = -\beta^{-1} \ln Z = \Omega_0 - \beta^{-1} Z^{(1)}, \quad (1.70)$$

where $-\beta^{-1} Z^{(1)}$ is the first correction to the thermodynamic potential $\Omega_0 = -\beta^{-1} \ln Z_0$ [Eq. (1.46)] of the noninteracting particles, caused by the interaction.

Generalization to various cases of interaction of the particles. The apparatus of the thermodynamic perturbation theory has been illustrated by the example of the electron-phonon system. It is easy to see, however, that the apparatus is suitable for the study of any statistical system composed of fermions (or bosons) that interact in pairs through a potential $v = v(\mathbf{x} - \mathbf{x}')$.

Let us consider a statistical system of identical particles of a single type, which interact in pairs with the potential energy $v(\mathbf{x} - \mathbf{x}')$. Extension of the results to a multicomponent system presents no difficulty. We regard an individual particle as a quantum of the ψ -field, which is a function of the coordinates and satisfies a definite equation. In the second-quantization scheme the Hamiltonian of the statistical system (Matsubara¹⁴) is written

$$H = H_0 + H_1, \quad (1.71a)$$

$$H_0 = \int \psi^*(x) \left(-\frac{\hbar^2 \nabla^2}{2m} \right) \psi(x) d^3x, \quad (1.71b)$$

$$H_1 = \frac{1}{2} \int v(x-x') \psi^*(x) (\psi^*(x') \psi(x')) \psi(x) d^3x d^3x', \quad (1.71c)$$

where for a system of fermions

$$\psi_\alpha(x) = V^{-1/2} \sum_{p,r} a_{p,r} u_\alpha^r e^{ipx/\hbar}, \quad \psi_\alpha^*(x) = V^{-1/2} \sum_{p,r} a_{p,r}^* u_\alpha^{r*} e^{-ipx/\hbar},$$

$$[a_{p,r}, a_{p',r'}^*]_+ = \delta_{pp'} \delta_{rr'}, \quad [a_{p,r}, a_{p',r'}]_+ = [a_{p,r}^*, a_{p',r'}^*]_+ = 0, \quad (1.72)$$

and for a system of bosons

$$\psi(x) = V^{-1/2} \sum_k b_k e^{ikx/\hbar}, \quad \psi^*(x) = V^{-1/2} \sum_k b_k^* e^{-ikx/\hbar},$$

$$[b_k, b_{k'}]_- = \delta_{kk'}, \quad [b_k, b_{k'}]_- = [b_k^*, b_{k'}^*]_- = 0. \quad (1.73)$$

With an interaction of the type (1.71c) the total number of particles

$$N = \int \psi^*(x) \psi(x) d^3x \quad (1.74)$$

is conserved along with the total energy (1.71a), if the system is closed. To describe a quasi-closed system with a variable number of particles we again use the Gibbs distribution (1.7), in which the parameter μ now means the chemical potential of the particles (1.74) of the system. Just as in Eq. (1.7), the parameter μ is chosen so as to obtain the correct average number \bar{N} of particles in the system,

$$\bar{N} = \text{Sp} N e^{(\mu N - H)/\beta} / \text{Sp} e^{(\mu N - H)/\beta}.$$

As before, the partition function of the system is of the form of Eqs. (1.21), (1.16),

$$Z = Z_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \dots$$

$$\times \int_0^\beta d\tau_n \langle T [H_1(\tau_1) H_1(\tau_2) \dots H_1(\tau_n)] \rangle, \quad (1.75)$$

where

$$H_1(\tau) = \frac{1}{2} \int v(x-x') \psi^*(x, \tau) (\psi^*(x', \tau)$$

$$\times \psi(x', \tau)) \psi(x, \tau) d^3x d^3x', \quad (1.76)$$

and the dependence of $\psi(x)$ and $\psi^*(x)$ on the variable τ is the same as in Eq. (1.18) ("interaction representation").

To use the rules of calculation developed previously, we rewrite the expression (1.76) in the form

$$\int_0^\beta H_1(\tau) d\tau = \frac{1}{2} \int U(x-x') T [(\psi^*(x) \psi(x)) (\psi^*(x') \psi(x'))] d^4x d^4x'$$

$$- \frac{1}{2} v(0) \int \psi^*(x) \psi(x) d^4x, \quad (1.77)$$

$$U(x-x') = v(x-x') \delta(\tau-\tau'), \quad (1.78)$$

where

$$\int_0^\beta F(\tau-\tau') \delta(\tau-\tau') d\tau' = \frac{1}{2} [F(+0) + F(-0)],$$

$$\delta(\tau+\beta) = \delta(\tau). \quad (1.79)$$

As the $\delta(\tau)$ of Eq. (1.79) we can take, for example,

$$\delta(\tau) = \beta^{-1} \sum_{n=-\infty}^{\infty} \exp(i2n\pi\tau/\beta). \quad (1.80)$$

Noting that for arbitrary operators A, B, ... we have the relation

$$T[T(A), T(B), \dots] = T[A, B, \dots],$$

let us rewrite the partition function (1.75) in the convenient form

$$Z = Z_0 \left\langle T \left[\exp \left\{ -\frac{1}{2} \int (\psi^*(x) \psi(x)) \right. \right. \right.$$

$$\times U(x-x') (\psi^*(x') \psi(x')) d^4x d^4x'$$

$$\left. \left. + \frac{1}{2} v(0) \int \psi^*(x) \psi(x) d^4x \right\} \right] \right\rangle, \quad (1.81)$$

to which all of the rules of calculation previously established can be applied. In the case of Fermi-Dirac statistics the contraction of two operators is given by

$$\underline{\psi_\alpha(x) \psi_\beta^*(x')} \equiv \mathcal{G}_{0\alpha\beta}(x-x')$$

$$= \begin{cases} V^{-1} \sum_p (1-n_p) \delta_{\alpha\beta} e^{ip(x-x')/\hbar - (\epsilon_p - \mu)(\tau - \tau')}, & \tau > \tau', \\ -V^{-1} \sum_p n_p \delta_{\alpha\beta} e^{ip(x-x')/\hbar - (\epsilon_p - \mu)(\tau - \tau')}, & \tau < \tau', \end{cases} \quad (1.82)$$

$$n_p = (e^{(\epsilon_p - \mu)\beta} + 1)^{-1},$$

and for Bose-Einstein statistics

$$\underline{\psi(x) \psi^*(x')} \equiv \mathcal{G}_0(x-x')$$

$$= \begin{cases} V^{-1} \sum_p (1+n_p) e^{ip(x-x')/\hbar - (\epsilon_p - \mu)(\tau - \tau')}, & \tau > \tau', \\ V^{-1} \sum_p n_p e^{ip(x-x')/\hbar - (\epsilon_p - \mu)(\tau - \tau')}, & \tau < \tau', \end{cases} \quad (1.83)$$

$$n_p = (e^{(\epsilon_p - \mu)\beta} - 1)^{-1}.$$

A difference from the case of the electron-phonon system is that in expanding the T-product of Eq. (1.81) we must include among the possible contractions contractions that contain $\psi(x)$ and $\psi^*(x)$ with the same argument. To each such contraction there corresponds in the graph a solid-line loop that begins and ends at the same point. To a dotted line in the graph there corresponds the factor $U(x-x')$. Besides vertices at which three lines (two solid and one dotted) meet, the graphs now have vertices at which only two solid lines meet. The appearance of these is due to the expression $\int \psi^*(x) \psi(x) d^4x$ in Eq. (1.81). The graphs that correspond to terms of the series (1.81) proportional to the first power of the potential v are shown in Fig. 3. Corresponding to these graphs we have as the expression Z_1 (for fermions)

$$Z_1 = -\frac{1}{2} v(0) \int \mathcal{G}_{0\alpha\alpha}(0) d^4x$$

$$- \frac{1}{2} \int U(x-x') \mathcal{G}_{0\alpha\alpha}(0) \mathcal{G}_{0\beta\beta}(0) d^4x d^4x'$$

$$+ \frac{1}{2} \int U(x-x') \mathcal{G}_{0\alpha\beta}(x-x') \mathcal{G}_{0\beta\alpha}(x'-x) d^4x d^4x'. \quad (1.84)$$

Using the relations

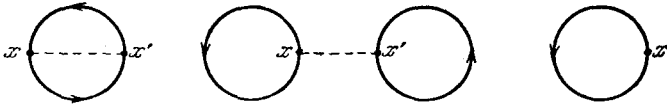


FIG. 3

$$\mathcal{G}_{0\alpha\beta}(0) = -V^{-1} \sum_{\mathbf{p}} n_{\mathbf{p}} \delta_{\alpha\beta}, \quad v(0) = V^{-1} \sum_{\mathbf{p}} v_{\mathbf{p}},$$

$$\int v(\mathbf{x} - \mathbf{x}') d^3x = v_0,$$

$$\begin{aligned} \mathcal{G}_{0\alpha\beta}(x-x') \mathcal{G}_{0\beta\alpha}(x'-x) \\ = -2V^{-2} \sum_{\mathbf{p}} \sum_{\mathbf{p}'} n_{\mathbf{p}} (1-n_{\mathbf{p}'}) e^{i(\mathbf{p}-\mathbf{p}')(\mathbf{x}-\mathbf{x}')/\hbar - (e_{\mathbf{p}'} - e_{\mathbf{p}})|\tau - \tau'|}, \end{aligned}$$

where $v_{\mathbf{p}}$ is the Fourier component of $v(\mathbf{x})$, we transform Eq. (1.84) to the form (Matsubara¹⁴)

$$\begin{aligned} Z_1 &= V^{-1} \beta \sum_{\mathbf{p}} \sum_{\mathbf{p}'} \{v_{\mathbf{p}} n_{\mathbf{p}'} - 2v_0 n_{\mathbf{p}} n_{\mathbf{p}'} - v_{\mathbf{p}'-\mathbf{p}} n_{\mathbf{p}} (1-n_{\mathbf{p}'})\} \\ &= -V^{-1} \beta \sum_{\mathbf{p}} \sum_{\mathbf{p}'} (2v_0 - v_{\mathbf{p}'-\mathbf{p}}) n_{\mathbf{p}} n_{\mathbf{p}'}, \end{aligned} \quad (1.85)$$

from which it can be seen that the presence of the expression

$$v(0) \int \psi^+(x) \psi(x) d^4x$$

in Eq. (1.81) means the subtraction of a proper-energy term in the final result. In the case of Bose-Einstein statistics there will be a plus sign in the round brackets of Eq. (1.85), the term v_0 is not multiplied by 2, and the entire expression is divided by 2.

The calculation of the terms of the series (1.81) can also be done in the \mathbf{p} representation. In fact, by using a general property of the propagation functions (1.82), (1.83),

$$\mathcal{G}_0(\mathbf{x}, \tau) = \mp \mathcal{G}_0(\mathbf{x}, \tau + \beta) \quad (1.86)$$

(the upper sign is for fermions, the lower for bosons), and the fact that an even number of fermion lines meet at each vertex, and that $U(\mathbf{x}, \tau + \beta) = U(\mathbf{x}, \tau)$, one can easily verify that the equation

$$\int_0^{\beta} \dots d\tau = \frac{1}{2} \int_{-\beta}^{\beta} \dots d\tau$$

holds for each of the variables τ in all the terms of the series (1.81). Then the expansion in Fourier series can be carried out in each term of the series (1.81), if the term does not reduce to a mere constant.

In the \mathbf{p} representation there corresponds to each solid line a factor $\mathcal{G}_{0\alpha\beta}(\mathbf{p}, \omega_n)$, Eq. (1.64), for fermions, and a factor

$$\mathcal{G}_0(\mathbf{p}, \omega_n) = -(i\omega_n + \mu - \varepsilon_{\mathbf{p}})^{-1}, \quad \omega_n = 2n\pi/\beta \quad (1.87)$$

for bosons. The factor that corresponds to a dotted line is

$$U(\mathbf{k}, \omega_n) = v_{\mathbf{k}}, \quad v_{\mathbf{k}} = \int v(\mathbf{x}) e^{-i\mathbf{k}\mathbf{x}/\hbar} d^3x. \quad (1.88)$$

The further remarks on the diagram technique are the same as in the case of the electron-phonon system.

In the generalization to different forms of pair interaction of the particles it has been tacitly assumed that there is no superconductivity in the Fermi system considered, and that the Bose system is at a temperature above that of the Bose condensation, since our proof of the very important theorem of Eqs. (1.34) and (1.35), which is the basis of this thermodynamic perturbation theory, does not extend to such cases. A thermodynamic perturbation theory for Fermi systems with superconductivity must be constructed on the basis of an extension of the work of Gor'kov,⁷ and such a theory for Bose systems below the point of the Bose condensation must be based on an extension of the work of Belyaev.⁶

2. THERMODYNAMIC GREEN'S FUNCTIONS

The system of coupled equations. In the preceding section we have expounded the new thermodynamic perturbation theory based on the methods of quantum field theory. The application of the methods of quantum field theory to statistical problems can be greatly extended if we use Green's functions. Just as in quantum field theory, in statistical physics the Green's-function method makes it possible to obtain physical approximations which differ from the expansions of the perturbation theory, being as a rule the result of the summation of an infinite set of definite terms of the perturbation-theory series. The use of Green's functions has been especially fruitful in the study of the thermodynamic properties of a system, in the calculation of the total energy, and also in calculating the energy spectrum of weakly excited states of a statistical system.

Let us first consider the so-called thermodynamic Green's functions of a statistical system, which we define in the following way:

$$\mathcal{G}_1(x, x') = \langle T[\psi(x) \psi^*(x') S] \rangle / \langle S \rangle, \quad (2.1)$$

$$\mathcal{G}_2(x_1 x_2, x'_1 x'_2) = \langle T[\psi(x_1) \psi(x_2) \psi^*(x'_1) \psi^*(x'_2) S] \rangle / \langle S \rangle, \quad (2.2)$$

$$\begin{aligned} \dots \\ \mathcal{G}_n(x_1 \dots x_n, x'_1 \dots x'_n) \\ = \langle T[\psi(x_1) \dots \psi(x_n) \psi^*(x'_1) \dots \psi^*(x'_n) S] \rangle / \langle S \rangle, \end{aligned} \quad (2.3)$$

and which we call respectively the one-particle function, the two-particle function, and so on. In the case of the electron-phonon system the ψ operators that appear in the electron thermodynamic Green's functions, Eqs. (2.1) – (2.3), are defined in Eqs. (1.18) and (1.3), and the function $S = S(\beta)$ is given in Eqs. (1.16) and (1.17); the phonon Green's functions are defined in analogy with Eqs. (2.1) – (2.3). For example,

$$\mathcal{L}_1(x, x') = \langle T[\varphi(x) \varphi(x') S] \rangle / \langle S \rangle \quad (2.4)$$

is the one-particle phonon thermodynamic Green's function.

In the case of a statistical system with particles interacting by pairs, Eqs. (1.71a) – (1.71c), we must

take for the function $S = S(\beta)$ in Eqs. (2.1) – (2.3) the function given by Eqs. (1.16) and (1.77).

In working with Green's functions it is convenient to use the following well known theorem connecting the T and N products of operators, which is proved in detail in the papers of Anderson⁴⁴ and Matsubara.¹⁴ Let $F(\psi^+, \psi, \varphi)$ be a functional (regarding functional differentiation, see Appendix 1) which can be expanded in a functional series in the operators ψ^+ , ψ , and φ (for definiteness we shall think of an electron-phonon system); then

$$T[F(\psi^+, \psi, \varphi)] = \mathcal{N} [e^{\Delta} e^{\Sigma} F(\psi^+, \psi, \varphi) e^{-\Delta} e^{-\Sigma}], \quad (2.5)$$

where

$$\Sigma = \int d^4x d^4y \mathcal{G}_{0\alpha\beta}(x-y) \frac{\delta}{\delta\psi_{\beta}^+(y)} \frac{\delta}{\delta\psi_{\alpha}(x)}, \quad (2.6)$$

$$\Delta = \frac{1}{2} \int d^4x d^4y \mathcal{D}_0(x-y) \frac{\delta}{\delta\varphi(x)} \frac{\delta}{\delta\varphi(y)}, \quad (2.7)$$

$$\left[\frac{\delta}{\delta\psi_{\alpha}(x)}, \psi_{\beta}(y) \right]_{+} = \left[\frac{\delta}{\delta\psi_{\alpha}^+(x)}, \psi_{\beta}^+(y) \right]_{+} = \delta_{\alpha\beta} \delta(x-y),$$

$$\left[\frac{\delta}{\delta\psi_{\alpha}(x)}, \psi_{\beta}^+(y) \right]_{+} = \left[\frac{\delta}{\delta\psi_{\alpha}^+(x)}, \psi_{\beta}(y) \right]_{+} = 0, \quad (2.8)$$

$$\left[\frac{\delta}{\delta\psi_{\alpha}(x)}, \frac{\delta}{\delta\psi_{\beta}(y)} \right]_{+} = \left[\frac{\delta}{\delta\psi_{\alpha}^+(x)}, \frac{\delta}{\delta\psi_{\beta}^+(y)} \right]_{+}$$

$$= \left[\frac{\delta}{\delta\psi_{\alpha}(x)}, \frac{\delta}{\delta\psi_{\beta}(y)} \right]_{+} = 0,$$

$$\left[\frac{\delta}{\delta\varphi(x)}, \varphi(y) \right]_{-} = \delta(x-y), \quad \left[\frac{\delta}{\delta\varphi(x)}, \frac{\delta}{\delta\varphi(y)} \right]_{-} = 0. \quad (2.9)$$

When applied to the thermodynamic Green's functions (2.1) – (2.4) this theorem gives, for example,

$$\mathcal{G}_n(x_1 \dots x_n, x'_1 \dots x'_n) = \langle \mathcal{N} [e^{\Delta} e^{\Sigma} \psi(x_1) \dots \psi(x_n) \psi^+(x'_1) \dots \psi^+(x'_n) \sigma] \rangle / \langle \mathcal{S} \rangle, \quad (2.10)$$

$$\sigma = \exp \left(-g \int \psi^+(x) \varphi(x) \psi(x) d^4x \right). \quad (2.11)$$

Here the operator $\exp(-\Delta) \exp(-\Sigma)$ is replaced by unity, since the wave functions of the statistical system, with respect to which the averaging in Eq. (2.10) is done, are not functionals of the operators ψ^+ , ψ , and φ , and consequently all terms of the series $\exp(-\Delta) \exp(-\Sigma)$ that contain functional derivatives are zero.

It is easy to obtain the system of coupled equations for the thermodynamic Green's functions (2.1) – (2.3) if we use the commutation relations

$$e^{\Sigma} \psi_{\alpha}(x) = \psi_{\alpha}(x) e^{\Sigma} + \int d^4y \mathcal{G}_{0\alpha\beta}(x-y) \frac{\delta}{\delta\psi_{\beta}^+(y)} e^{\Sigma},$$

$$e^{\Delta} \varphi(x) = \varphi(x) e^{\Delta} + \int d^4y \mathcal{D}_0(x-y) \frac{\delta}{\delta\varphi(y)} e^{\Delta}, \quad (2.12)$$

which can be verified by direct calculation. In fact, let us commute the factor $\exp \Sigma$ with $\psi(x_1)$ in Eq. (2.10), remembering that terms in which operators ψ (or ψ^+) and φ stand to the left of $\exp \Sigma$ and $\exp \Delta$, respectively, are zero in accordance with the theorem about averages of \mathcal{N} -products. As the result of this commutation we get a functional-derivative operation $\delta/\delta\psi^+$, and on taking the derivative accord-

ing to the rules (2.8) we get among other things, a term with the operator φ , according to the formula

$$\left[\frac{\delta}{\delta\psi_{\beta}^+(y)} \right] \sigma = -g \varphi(y) \psi_{\beta}(y) \sigma. \quad (2.13)$$

It is easy to get rid of the $\varphi(y)$, however, by commuting the operator $\exp \Delta$ with $\varphi(y)$ by means of Eq. (2.12). As the result we get the following infinite system of coupled integral equations:

$$\mathcal{G}_n(x_1 \dots x_n, x'_1 \dots x'_n) = \sum_{s=1}^n (-1)^{n+s} \mathcal{G}_0(x_1 - x'_s)$$

$$\times \mathcal{G}_{n-1}(x_2 \dots x_n, x'_1 \dots x'_{s-1} x'_{s+1} \dots x'_n) - g^2 \int d^4y d^4z \mathcal{G}_0$$

$$\times (x_1 - y) \mathcal{D}_0(y - z) \mathcal{G}_{n+1}(y x_2 \dots x_n z, z x'_1 \dots x'_n),$$

$$n = 1, 2, \dots, \infty. \quad (2.14)$$

Here for brevity the spinor indices in the \mathcal{G} functions are included in the coordinate variables, so that integration with respect to a coordinate also implies summation over the corresponding spinor index. This system of coupled equations (2.14) for the Green's functions is very similar to the integral equations connecting molecular distribution functions of different orders, which have been found by Born and Green⁴⁵ and by Kirkwood⁴⁶ in classical statistical mechanics (in this connection see papers by Bogolyubov⁴⁷ and by Ch'en Ch'un-Hsien²⁸). Since it is impossible to find the exact solution of the system of equations (2.14), one tries to find various approximate solutions. Thus in studying the infinite system of equations (2.14) one usually breaks it off by assigning a definite form to the Green's function of some fixed order, and finds the solution of the "broken off" closed system of equations. For example, if one prescribes a definite form of the three-particle Green's function $\mathcal{G}_3(x_1 x_2 x_3, x'_1 x'_2 x'_3)$, then the one- and two-particles Green's functions are found from the equations

$$\mathcal{G}_1(x, x') = \mathcal{G}_0(x - x')$$

$$- g^2 \int d^4y d^4z \mathcal{G}_0(x - y) \mathcal{D}_0(y - z) \mathcal{G}_2(yz, zx'), \quad (2.15)$$

$$\mathcal{G}_2(x_1 x_2, x'_1 x'_2) = \mathcal{G}_0(x_1 - x'_2) \mathcal{G}_1(x_2, x'_1) - \mathcal{G}_0(x_1 - x'_1) \mathcal{G}_1(x_2, x'_2)$$

$$- g^2 \int d^4y d^4z \mathcal{G}_0(x_1 - y) \mathcal{D}_0(y - z) \mathcal{G}_3(yx_2z, zx'_1 x'_2). \quad (2.16)$$

In writing the equations for the Green's functions one can take into account the interaction between the electrons by introducing a self-consistent field in the operator for the interaction energy, Eq. (1.4), and consequently also in the operator S , Eq. (1.16). In this way Matsubara¹⁴ has determined the energy spectrum of the electrons in a metal. His results agreed with those of Bardeen,⁴⁸ which were obtained in an entirely different way.

Equations for the thermodynamic Green's function. In applications it is most convenient to use equations that contain only the one-particle thermodynamic Green's functions. The shortest way to obtain such

equations is to use the Schwinger technique of variational derivatives (Schwinger¹). As an illustration let us consider a statistical system of electrons and positrons interacting with each other through the electromagnetic field (a plasma). For generality we shall take the relativistic problem. The Hamiltonian of the system in the Schrödinger representation is written as follows:*

$$H = H_0 + H_1, \quad (2.17a)$$

$$H_0 = \sum_{\mathbf{p}} \epsilon_{\mathbf{p}} \left(\sum_{r=1}^2 a_{\mathbf{p}r}^{\dagger} a_{\mathbf{p}r} + \sum_{r=3}^4 b_{\mathbf{p}r}^{\dagger} b_{\mathbf{p}r} \right) + \sum_{\mathbf{k}, \lambda=1}^2 \omega_{\mathbf{k}} c_{\mathbf{k}\lambda}^{\dagger} c_{\mathbf{k}\lambda}, \quad (2.17b)$$

$$H_1 = - \int j(\mathbf{x}) A(\mathbf{x}) d^3x, \quad (2.17c)$$

$$j_{\nu}(\mathbf{x}) = ieN (\bar{\psi}(\mathbf{x}) \gamma_{\nu} \psi(\mathbf{x})) = (ie/2) \gamma_{\nu\alpha\beta} [\bar{\psi}_{\alpha}(\mathbf{x}) \psi_{\beta}(\mathbf{x}) - \psi_{\beta}(\mathbf{x}) \bar{\psi}_{\alpha}(\mathbf{x})], \quad (2.17d)$$

$$\left. \begin{aligned} \psi(\mathbf{x}) &= V^{-1/2} \left(\sum_{\mathbf{p}, r=1}^2 a_{\mathbf{p}r} u^r(\mathbf{p}) e^{i\mathbf{p}\mathbf{x}} + \sum_{\mathbf{p}, r=3}^4 b_{\mathbf{p}r}^{\dagger} u^r(-\mathbf{p}) e^{-i\mathbf{p}\mathbf{x}} \right), \\ \bar{\psi}(\mathbf{x}) &= V^{-1/2} \left(\sum_{\mathbf{p}, r=1}^2 a_{\mathbf{p}r}^{\dagger} \bar{u}^r(\mathbf{p}) e^{-i\mathbf{p}\mathbf{x}} + \sum_{\mathbf{p}, r=3}^4 b_{\mathbf{p}r} \bar{u}^r(-\mathbf{p}) e^{-i\mathbf{p}\mathbf{x}} \right), \\ A_{\nu}(\mathbf{x}) &= \sum_{\mathbf{k}, \lambda=1}^2 (2\omega_{\mathbf{k}} V)^{-1/2} (c_{\mathbf{k}\lambda} e^{i\mathbf{k}\mathbf{x}} + c_{\mathbf{k}\lambda}^{\dagger} e^{-i\mathbf{k}\mathbf{x}}) l_{\nu}^{\lambda}. \end{aligned} \right\} \quad (2.18)$$

Here H_0 is the Hamiltonian of the free electron-positron and photon fields, and H_1 is the Hamiltonian for the interaction between them; $u^r(\mathbf{p})$ with $r = 1, 2$ is a solution of the Dirac equation

$$(i\hat{p} + m) u^r(\mathbf{p}) = 0 \quad (2.19)$$

for positive energy, $\epsilon_{\mathbf{p}} = (\mathbf{p}^2 + m^2)^{1/2} = -ip_4$, and for $r = 3, 4$ it is a solution for the negative energy $-(\mathbf{p}^2 + m^2)^{1/2}$; $\bar{u}^r = u^{r*} \gamma_4$; A_{ν} is the four-dimensional vector potential of the electromagnetic field; $a_{\mathbf{p}r}$ ($b_{\mathbf{p}r}$) and $a_{\mathbf{p}r}^{\dagger}$ ($b_{\mathbf{p}r}^{\dagger}$) are the operators for absorption and production of an electron (positron) with momentum \mathbf{p} , polarization r , and energy $\epsilon_{\mathbf{p}} = (\mathbf{p}^2 + m^2)^{1/2}$, and $c_{\mathbf{k}\lambda}$ and $c_{\mathbf{k}\lambda}^{\dagger}$ are the analogous operators for a photon with momentum \mathbf{k} , polarization vector l^{λ} and energy $\omega_{\mathbf{k}} = |\mathbf{k}|$. The operators for changes of the numbers of particles satisfy the commutation relations:

$$[a_{\mathbf{p}r}, a_{\mathbf{p}'r'}^{\dagger}]_{-} = [b_{\mathbf{p}r}, b_{\mathbf{p}'r'}^{\dagger}]_{-} = \delta_{rr'} \delta_{\mathbf{p}\mathbf{p}'}, \quad [c_{\mathbf{k}\lambda}, c_{\mathbf{k}'\lambda'}^{\dagger}]_{-} = \delta_{\lambda\lambda'} \delta_{\mathbf{k}\mathbf{k}'}. \quad (2.20)$$

In applications to statistics it is a very important fact that besides conservation of the energy (2.17a) in a closed system we also have conservation of the total charge Q ,

$$Q = \frac{e}{2} \int \text{Sp} [\psi^*(\mathbf{x}) \psi(\mathbf{x}) - \psi(\mathbf{x}) \psi^*(\mathbf{x})] d^3x = e \left(\sum_{\mathbf{p}, r=1}^2 a_{\mathbf{p}r}^{\dagger} a_{\mathbf{p}r} - \sum_{\mathbf{p}, r=3}^4 b_{\mathbf{p}r}^{\dagger} b_{\mathbf{p}r} \right), \quad (2.21)$$

*In what follows we use a system of units in which $\hbar = c = 1$, $e^2/4\pi\hbar c = 1/137$, and adopt the following rule of summation over vector indices: $\mathbf{p}\mathbf{q} = p_{\nu} q_{\nu} = p_1 q_1 + p_2 q_2 + p_3 q_3 + p_4 q_4$. Furthermore $\hat{q} = q_{\nu} \gamma_{\nu}$, where γ_4 and $\gamma_{1,2,3} = -i\alpha_{1,2,3}$ are the usual Dirac matrices. N denotes the N-product in the sense of quantum electrodynamics.³⁷

or, what is the same thing, of the difference between the total number of electrons N^- and the total number of positrons N^+ ,

$$N^- = \sum_{\mathbf{p}, r=1}^2 a_{\mathbf{p}r}^{\dagger} a_{\mathbf{p}r}, \quad N^+ = \sum_{\mathbf{p}, r=3}^4 b_{\mathbf{p}r}^{\dagger} b_{\mathbf{p}r}. \quad (2.22)$$

Therefore it is convenient to describe a quasi-closed statistical system of electrons and positrons by means of a Gibbs distribution with a variable number of particles, in which the probability $W_{n, N^- - N^+}$ for the system to have a given difference $N^- - N^+$ between the numbers of electrons and positrons and to be in the state with the energy $E_{n, N^- - N^+}$ is

$$W_{n, N^- - N^+} = e^{(\Omega + \mu(N^- - N^+) - E_{n, N^- - N^+})\beta}, \quad (2.23)$$

where μ and $-\mu$ are the respective chemical potentials of electrons and positrons. Repeating the arguments of the preceding section, we can easily find the partition function Z for the system,

$$Z = \text{Sp} e^{(\mu(N^- - N^+) - H_0)\beta} = Z_0 \langle \mathcal{S}(\beta) \rangle, \quad (2.24)$$

$$\begin{aligned} \mathcal{S}(\beta) &= T e^{\int j(\mathbf{x}) A(\mathbf{x}) d^4x} \\ &= \sum_{n=0}^{\infty} \frac{(ie)^n}{n!} \int d^4x_1 \dots \int d^4x_n T [N(\bar{\psi}(x_1) \hat{A}(x_1) \psi(x_1)) \\ &\dots N(\bar{\psi}(x_n) \hat{A}(x_n) \psi(x_n))], \end{aligned} \quad (2.25)$$

where, as in the case of Eqs. (1.16) and (1.75), the field operators are written in a special "interaction representation," so that the dependence of the operators $\bar{\psi}(\mathbf{x})$, $\psi(\mathbf{x})$, and $A(\mathbf{x})$ on the variable τ is given by a single law for all operators, for example

$$\psi(x) = e^{-(\mu(N^- - N^+) - H_0)\tau} \psi(x) e^{(\mu(N^- - N^+) - H_0)\tau}. \quad (2.26)$$

By using Eq. (2.25) one can find the electronic and photonic thermodynamic Green's functions, and also other thermodynamic characteristics of the system, by the methods of perturbation theory, but with the modification that as the contractions of the electron-positron operators, $\underline{\psi(x) \bar{\psi}(x')} \equiv \mathcal{S}_0(x - x')$, and of the photon-field operators, $\underline{A_{\mu}(x) A_{\nu}(x')} \equiv \mathcal{D}_{0\mu\nu}(x - x')$, one must use the following expressions:

$$\mathcal{S}_0(x - x') = \begin{cases} \frac{\hat{p} - m}{(2\pi)^3} \int \frac{d^3p}{2\epsilon_{\mathbf{p}}} [(n_{\mathbf{p}}^- - 1) e^{i\mathbf{p}(x-x') - (\epsilon_{\mathbf{p}} - \mu)(\tau - \tau')} \\ + n_{\mathbf{p}}^+ e^{-i\mathbf{p}(x-x') + (\epsilon_{\mathbf{p}} + \mu)(\tau - \tau')}] , \tau > \tau', \\ \frac{\hat{p} - m}{(2\pi)^3} \int \frac{d^3p}{2\epsilon_{\mathbf{p}}} [(n_{\mathbf{p}}^+ - 1) e^{-i\mathbf{p}(x-x') + (\epsilon_{\mathbf{p}} + \mu)(\tau - \tau')} \\ + n_{\mathbf{p}}^- e^{i\mathbf{p}(x-x') - (\epsilon_{\mathbf{p}} - \mu)(\tau - \tau')}] , \tau < \tau', \end{cases} \quad (2.27)$$

$$\begin{aligned} \hat{p} &= \gamma_4 \left(\frac{\partial}{\partial \tau} - \mu \right) + \boldsymbol{\gamma} \nabla \equiv \gamma_{\nu} \frac{\partial}{\partial x_{\nu}} - \mu \gamma_4, \\ \mathcal{D}_{0\mu\nu}(x - x') &= \frac{\delta_{\mu\nu}}{(2\pi)^3} \int \frac{d^3k}{2\omega_{\mathbf{k}}} [(1 + n_{\mathbf{k}}) e^{i\mathbf{k}(x-x') - \omega_{\mathbf{k}}|\tau - \tau'|} \\ + n_{\mathbf{k}} e^{-i\mathbf{k}(x-x') + \omega_{\mathbf{k}}|\tau - \tau'|}], \end{aligned} \quad (2.28)$$

where \bar{n}_p , n_p^+ , and n_k are the respective distribution functions for electrons, positrons, and photons,

$$n_p^\mp = (e^{(\epsilon_p \mp \mu)\beta} + 1)^{-1}, \quad n_k = (e^{\omega_k \beta} - 1)^{-1}. \quad (2.30)$$

More valuable calculations, however, are made without perturbation theory, on the basis of the solution of the equations for the Green's functions. To derive these equations we note that by means of the relations

$$\begin{aligned} T[\psi(x) \dots \psi(x') S] &= ST[\tilde{\psi}(x) \dots \tilde{\psi}(x')], \\ T[A(x) \dots A(x') S] &= ST[\tilde{A}(x) \dots \tilde{A}(x')] \end{aligned} \quad (2.31)$$

the thermodynamic Green's functions can also be written in a different form. For example, for the one-particle thermodynamic Green's functions (we shall hereafter omit the index "1") we get

$$\begin{aligned} \mathcal{G}_{\alpha\beta}(x, x') &= \text{Sp} \{ e^{(\Omega + \mu(N^- - N^+) - H)\beta} T(\tilde{\psi}_\alpha(x) \tilde{\psi}_\beta(x')) \} \\ &\equiv [T(\tilde{\psi}_\alpha(x) \tilde{\psi}_\beta(x'))]_{\text{av}}, \\ \mathcal{D}_{\mu\nu}(x, x') &= \text{Sp} \{ e^{(\Omega + \mu(N^- - N^+) - H)\beta} T(\tilde{A}_\mu(x) \tilde{A}_\nu(x')) \} \\ &\equiv [T(\tilde{A}_\mu(x) \tilde{A}_\nu(x'))]_{\text{av}}, \end{aligned} \quad (2.32)$$

where the tilde \sim denotes operators in the Heisenberg representation,

$$\tilde{\psi}(x) = e^{-(\mu(N^- - N^+) - H)\tau} \psi(x) e^{(\mu(N^- - N^+) - H)\tau}. \quad (2.33)$$

By using the commutation relations of the operators $\psi(\mathbf{x})$, $\bar{\psi}(\mathbf{x})$, and $A(\mathbf{x})$ one can verify without difficulty that the Heisenberg field operators satisfy the following equations:

$$(\hat{p} - ie\hat{A}(x) + m)\tilde{\psi}(x) = 0, \quad (2.34)$$

$$\left(\nabla^2 + \frac{\partial^2}{\partial \tau^2}\right)\tilde{A}(x) = -ieN(\tilde{\psi}(x)\gamma\tilde{\psi}(x)), \quad (2.35)$$

which coincide with the corresponding equations of quantum electrodynamics if we make the replacements $\tau \rightarrow it$, $\mu \rightarrow 0$.

In order to get the equations for the thermodynamic Green's functions (2.32) we formally introduce an auxiliary external-current function $J(\mathbf{x})$, where $J(\mathbf{x})$ is an unquantized function. Then in the interaction operator (2.17c) there is an additional term $-\int J(\mathbf{x})A(\mathbf{x})d^3x$, and the Heisenberg operators of the electron-positron and photon fields will satisfy the same equations (2.34), (2.35), except that the external current $J(\mathbf{x})$ is added to the electron-positron current in Eq. (2.35). Let us consider the Green's function in the presence of the external field; we define it by the formulas (2.1), (2.4), and (2.32), in which

$$S(\beta) = T e^{\int (j(x) + J(x))A(x) d^4x}. \quad (2.36)$$

In this case the Green's functions are functionals of the external field $J(\mathbf{x})$.

We can set up equations for these Green's functions, and for $J = 0$ the solution of these equations will give the thermodynamic Green's functions of the real statistical system without the external field. For this

purpose we consider $T[\tilde{\psi}(\mathbf{x}, \tau)\tilde{\psi}(\mathbf{x}', \tau')]$ as a function of τ . For $\tau = \tau'$ this function has a discontinuity equal to $\beta\delta(\mathbf{x} - \mathbf{x}')$. Consequently,

$$\frac{\partial}{\partial \tau} T[\tilde{\psi}(x)\tilde{\psi}(x')] = T\left[\frac{\partial \tilde{\psi}(x)}{\partial \tau}\tilde{\psi}(x')\right] + \beta\delta(x - x')\delta(\tau - \tau'). \quad (2.37)$$

Performing an averaging $[\dots]_{\text{av}}$ of both members of Eq. (2.37) and using Eqs. (2.34), (2.1), and (2.36), we easily find

$$\left(\hat{p} - ie[\hat{A}(x)]_{\text{av}} + m - ie\gamma_\nu \frac{\delta}{\delta J_\nu(x)}\right)\mathcal{G}(x, x') = \delta(x - x'). \quad (2.38)$$

Similarly, averaging the equation for $\tilde{A}(x)$ and then taking the variational derivative with respect to the current, $\delta/\delta J_\nu(x')$, on both sides of this equation and using Eq. (2.31), we get

$$\left(\nabla^2 + \frac{\partial^2}{\partial \tau^2}\right)\mathcal{D}_{\mu\nu}(x, x') = -\delta_{\mu\nu}\delta(x - x') + ie\gamma_{\mu\alpha\beta} \frac{\delta \mathcal{G}_{\beta\alpha}(x, x')}{\delta J_\nu(x')}, \quad (2.39)$$

where we have used the notation

$$\frac{\delta[\tilde{A}_\mu(x)]_{\text{av}}}{\delta J_\nu(x')} = [T(\tilde{A}_\mu(x)\tilde{A}_\nu(x'))]_{\text{av}}$$

$$-[\tilde{A}_\mu(x)]_{\text{av}}[\tilde{A}_\nu(x')]_{\text{av}} \equiv \mathcal{D}_{\mu\nu}(x, x')$$

which for $J = 0$ agrees with the definition of the thermodynamic Green's function, Eqs. (2.32), (2.4), since $[\tilde{A}(\mathbf{x})]_{\text{av}}|_{J=0} = 0$.

The system of equations (2.38), (2.39) is usually written in a different form by expressing the variational derivatives in terms of a mass operator M , a polarization operator $\Pi_{\mu\nu}$, and a vertex operator Γ_ν , by the formulas:

$$\begin{aligned} -ie\gamma_\nu \frac{\delta \mathcal{G}(x, x')}{\delta J_\nu(x')} &= e^2 \int \gamma_\nu \mathcal{G}(x, y) \Gamma_\mu(yz, y') \mathcal{D}_{\mu\nu}(y', x) \mathcal{G}(z, x') d^4y d^4y' d^4z \\ &\equiv \int M(x, z) \mathcal{G}(z, x') d^4z, \end{aligned} \quad (2.40)$$

$$\begin{aligned} -ie\gamma_{\mu\alpha\beta} \frac{\delta \mathcal{G}_{\beta\alpha}(x, x')}{\delta J_\nu(x')} &= e^2 \text{Sp} \int \gamma_\mu \mathcal{G}(x, y) \Gamma_\nu(yz, y') \mathcal{G}(z, x) \\ &\times \mathcal{D}_{\nu\alpha}(y', x') d^4y d^4y' d^4z \equiv \int \Pi_{\mu\nu}(x, y) \mathcal{D}_{\nu\alpha}(y, x') d^4y, \end{aligned} \quad (2.41)$$

$$\Gamma_\nu(yz, y') = i \frac{\delta \mathcal{G}^{-1}(y, z)}{\delta e[\tilde{A}_\nu(y')]_{\text{av}}} = \gamma_\nu \delta(y - z) \delta(z - y') + i \frac{\delta M(y, z)}{\delta e[\tilde{A}_\nu(y')]_{\text{av}}}. \quad (2.42)$$

When the external field is turned off, $J = 0$, the Green's functions that appear in the relations (2.38) – (2.42) coincide with the thermodynamic Green's functions (2.32), which because of the homogeneity of the statistical system under consideration are functions of the differences of coordinates, $\mathcal{G}(x, x') \equiv \mathcal{G}(x - x')$, $\mathcal{D}(x, x') \equiv \mathcal{D}(x - x')$ [one can easily verify this by direct calculation, by carrying through the arguments

that lead to Eq. (3.4)]. Consequently, the mass, polarization, and vertex operators are also functions of the differences of coordinates,

$$M(x, x') \equiv M(x - x'), \quad \Pi(x, x') \equiv \Pi(x - x'), \\ \Gamma(xy, z) \equiv \Gamma(x - y, y - z),$$

and as functions of the difference of the fourth coordinate, $\tau - \tau'$, all of these functions are defined in the range from $-\beta$ to β . We can, however, define them on the entire τ axis by continuing them periodically. By repeating the proof of Eq. (1.61) for the functions \mathcal{G} and \mathcal{D} one easily finds that also in the general case $H_1 \neq 0$ these thermodynamic Green's functions have the properties (1.61) and (1.62) (Abrikosov, Gor'kov, and Dzyaloshinskii¹⁵). According to the definition (2.40) - (2.42), the mass and polarization operators have these same properties:

$$M(x, \tau) = -M(x, \tau + \beta), \quad \Pi(x, \tau) = \Pi(x, \tau + \beta). \quad (2.43)$$

In the change to the p representation this fact allows

us to replace all integrals $\int_0^\beta \dots d\tau$ contained in the relations (2.38) - (2.42) by integrals $\frac{1}{2} \int_{-\beta}^\beta \dots d\tau$. After

this the Fourier transformation (1.60) is easily carried out in all of the expressions (2.38) - (2.42). As the result one gets the following equations in the p representation for the thermodynamic Green's functions (2.32) (Fradkin^{18,20}):

$$[ip\gamma - (ip_4 + \mu)\gamma_4 + m + M(p)] \mathcal{G}(p) = 1, \quad (2.44)$$

$$[k^2 - \Pi(k)] \mathcal{D}(k) = 1, \quad (2.45)$$

$$M(p) = \frac{e^2}{(2\pi)^3 \beta} \sum_{k_4} \int \gamma \mathcal{G}(p+k) \Gamma(p+k, k) \mathcal{D}(k) d^3k, \quad (2.46)$$

$$\Pi(k) = \frac{e^2}{(2\pi)^3 \beta} \sum_{p_4} \int \text{Sp} \gamma \mathcal{G}(p+k) \Gamma(p+k, k) \mathcal{G}(p) d^3p, \\ \Gamma(p, p') = \gamma + \Lambda(p, p'), \\ p_4 \equiv \omega_n = (2n+1)\pi/\beta, \quad k_4 \equiv \omega_m = 2m\pi/\beta. \quad (2.47)$$

Here $\Lambda(p, p')$ is defined as a series in e^2 which contains the set of all vertex-part graphs except the simple vertex (point).

Let us rewrite Eqs. (2.44) and (2.45) in the form

$$\mathcal{G}(p) = \mathcal{G}_0(p) - \mathcal{G}_0(p) M(p) \mathcal{G}(p), \quad (2.48)$$

$$\mathcal{D}(k) = \mathcal{D}_0(k) + \mathcal{D}_0(k) \Pi(k) \mathcal{D}(k), \quad (2.49)$$

where the zeroth-approximation ($e^2 = 0$) thermodynamic Green's functions

$$\mathcal{G}_0(p) = [ip\gamma - (ip_4 + \mu)\gamma_4 + m]^{-1}, \quad \mathcal{D}_0(k) = k^{-2} \quad (2.50)$$

are the same as the contractions of operators, Eqs. (2.27) and (2.29), written in the p representation. If we solve Eqs. (2.48) and (2.49) by the iteration method, then we get the perturbation-theory series for \mathcal{G} and \mathcal{D} . A better physical approximation, however, is found by expanding the kernels of Eqs. (2.48) and (2.49) in

series in e^2 or in another parameter of the thermodynamic problem, and then solving exactly the equation with an approximate kernel. For example, if in Eq. (2.49) we keep only the first nonvanishing approximation $\Pi^1(k)$ to the kernel $\Pi(k)$ [this approximation is obtained if in Eq. (2.47) we replace all functions by their zeroth approximations], we find

$$\mathcal{D}_{\mu\nu}(k) = k^{-2} [1 - k^{-2} \Pi^1(k)]_{\mu\nu}^{-1}, \quad (2.51)$$

where $[1 - k^{-2} \Pi^1(k)]_{\mu\nu}^{-1}$ is the matrix that is the reciprocal of the matrix $\delta_{\mu\nu} - k^{-2} \Pi_{\mu\nu}^1(k)$. In the language of perturbation theory, the solution (2.51) is the result of the summation of an infinite number of terms of a definite class,

$$\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_0 \Pi^1 \mathcal{D}_0 + \mathcal{D}_0 \Pi^1 \mathcal{D}_0 \Pi^1 \mathcal{D}_0 + \dots, \quad (2.52)$$

to which there correspond the graphs of Fig. 4.

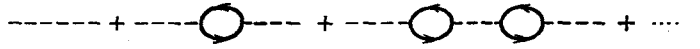


FIG. 4

In the concrete use of the solution of Eqs. (2.48) - (2.49) infinities may appear which are associated with the renormalization of the mass and charge of the particle. These infinities, however, are easily removed by the same recipes as exist in electrodynamics. This question has been treated by a number of authors in connection with the application to statistics, for example by Fradkin¹⁹ and by Akhiezer and Peletminskii.²³ The latter authors have calculated the thermodynamic potential of a gas of electrons, positrons, and photons to and including terms proportional to $e^4 (\ln e)^2$. The divergences that arose were removed by renormalization of the charge and mass of the electron and redefinition of the vacuum level.

In the nonrelativistic approximation, in which the mean energy of the thermal motion of a particle is much smaller than its rest mass, $\beta^{-1} \ll m$, we must make the replacement $\epsilon_p \rightarrow m + (p^2/2m)$, $\mu \rightarrow m + \mu'$, $\gamma_4 \rightarrow \delta_{\alpha\beta}$ (α and β run through the values 1, 2) and regard all quantities as small in comparison with m ; then

$$\mathcal{G}_0(p) = \frac{-ip\gamma + (ip_4 + \mu)\gamma_4 + m}{(p_4 - i\mu)^2 + p^2 + m^2} \rightarrow \frac{-\delta_{\alpha\beta}}{ip_4 + \mu' - \frac{p^2}{2m}}, \quad (2.53)$$

which agrees with Eq. (1.64). Similarly, the expression (2.27) goes over in the nonrelativistic case into Eqs. (1.38), (1.47), since the positron distribution function (2.30) becomes identically zero. Neglecting terms of the type of a retarded interaction, we get for the function $\mathcal{D}_0(k)$ in the nonrelativistic case

$$\mathcal{D}_{0\mu\nu}(k) = \frac{\delta_{\mu\nu}}{k^2 + k_4^2} \rightarrow \frac{1}{k^2}, \quad (2.54)$$

and in the polarization operator (2.47) we must set $\gamma_{1,2,3} = 0$, $\gamma_4 = \delta_{\alpha\beta}$, and replace \mathcal{G}_0 by the nonrelativistic value (2.53). Remembering these points, one can write down without difficulty the equation for the

thermodynamic function $\mathcal{G}(p)$ (and also that for the function \mathcal{L}) that describes a system of nonrelativistic particles interacting through the Coulomb field [see later argument, Eqs. (2.66) – (2.71)].

When we go to the absolute zero of temperature $\beta \rightarrow \infty$ in all of the relations (2.44) – (2.49) and the sums over discrete frequencies p_4 and k_4 are replaced by integrals,

$$\frac{1}{\beta} \sum_{p_4} \dots \rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \dots d^4p. \quad (2.55)$$

In nonrelativistic formulas (for example, in the use of perturbation theory) one may encounter integrals of several factors of the form (2.53). In such integrals one can change the variable of integration, $p'_4 = ip_4$, and then turn the path of integration back to the real axis. Then the passage around the poles will be as shown in Fig. 5 (Abrikosov, Gor'kov, and Dzyaloshinskii¹⁵).

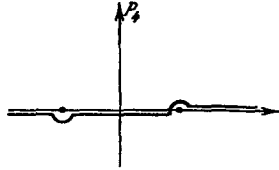


FIG. 5

In the case of the electron-phonon system, Eqs. (1.1a) – (1.1c), the equations for the electron and phonon thermodynamic Green's functions \mathcal{G} and \mathcal{L} of Eqs. (2.1) and (2.4) will also be of the form of Eqs. (2.48) – (2.49) and (2.46) – (2.47), in which, however, one must put $e \rightarrow g$, $\gamma \rightarrow 1$ and reverse the signs + and - in the right members of Eqs. (2.48) – (2.49). Furthermore the zeroth-order thermodynamic Green's functions \mathcal{G}_0 and \mathcal{L}_0 must now be taken to mean the expressions (1.64) and (1.65).

In the case of statistical systems of particles interacting in pairs, Eqs. (1.71a) – (1.71c), the equations for the Green's functions have been studied in papers by Abrikosov, Gor'kov, and Dzyaloshinskii¹⁵ and by Fradkin.¹⁹

Connection between the thermodynamic Green's functions and the thermodynamic functions of the system. We shall find some general relations by considering as an example a statistical system consisting of a gas of electrons, positrons, and photons, Eqs. (2.17a) – (2.17c). The extension of the results to other systems can be made without difficulty.

Let us formally differentiate the thermodynamic potential of the system, $\Omega = -\beta^{-1} \ln Z$, with respect to the charge e ; then according to Eqs. (2.24) – (2.25) and (2.40) – (2.41) we get

$$\begin{aligned} \frac{\partial \Omega}{\partial e} &= i\beta^{-1} \int d^4x \langle T [N(\psi(x)\hat{A}(x)\bar{\psi}(x))S] \rangle / \langle S \rangle \\ &= -\beta^{-1} e^{-1} \int \Pi_{\mu\nu}(x, y) \mathcal{L}_{\nu\mu}(y, x) d^4y \\ &= -\beta^{-1} e^{-1} \int M(x, Z) \mathcal{G}(Z, x) d^4Z. \end{aligned} \quad (2.56)$$

Integrating Eq. (2.56) with respect to the charge and remembering Eq. (2.48), we can express Ω in terms of the electron-positron thermodynamic Green's function

$$\Omega = \Omega_0 + \frac{V}{(2\pi)^3 \beta} \int_0^e \frac{de}{e} \sum_{p_4} \int d^3p [\mathcal{G}(p) - \mathcal{G}_0(p)]_{\alpha\beta} \mathcal{G}_0^{-1}{}_{\beta\alpha}(p), \quad (2.57)$$

where Ω_0 is the thermodynamic potential for the case of no interaction between the particles ($e^2 = 0$). If the system consists of particles that interact in pairs through a potential $v(\mathbf{x} - \mathbf{x}')$, Eq. (1.71c), then instead of the charge e one introduces an auxiliary parameter Λ through the formula $v(\mathbf{x} - \mathbf{x}') \rightarrow \Lambda v(\mathbf{x} - \mathbf{x}')$, and integrates from 0 to 1 with respect to this parameter. In this case there is an additional factor $\frac{1}{2}$ before the integral in Eq. (2.57).

It is often advantageous to express Ω in terms of the polarization operator. To do this we again carry out a Fourier transformation in Eq. (2.56), use Eq. (2.49), and then integrate with respect to the charge; then

$$\Omega = \Omega_0 - \frac{V}{(2\pi)^3 \beta} \int_0^e \frac{de}{e} \sum_{k_4} \int d^3k \frac{1}{k^2} \Pi_{\mu\nu} \left(1 - \frac{1}{k^2} \Pi\right)_{\nu\mu}^{-1}. \quad (2.58)$$

For applications it is quite sufficient to solve Eq. (2.49) with the kernel Π^1 , proportional to e^2 ; then the relation (2.58) can be rewritten (Akhiezer and Peletminskii²³):

$$\begin{aligned} \Omega &= \Omega_0 + \frac{V}{2(2\pi)^3 \beta} \sum_{k_4} \int d^3k \int_0^e d \left(1 - \frac{1}{k^2} \Pi^1\right)_{\mu\nu} \left(1 - \frac{1}{k^2} \Pi^1\right)_{\nu\mu}^{-1} \\ &= \Omega_0 + \frac{V}{2(2\pi)^3 \beta} \sum_{k_4} \int d^3k \ln \det \left| \delta_{\mu\nu} - \frac{1}{k^2} \Pi^1_{\mu\nu} \right|, \end{aligned} \quad (2.59)$$

where we have kept only the terms proportional to e^2 and also terms of the form $e^4 f(e^2)$, where $f(0) = 0$. A study of the photon polarization operator $\Pi^1_{\mu\nu}(k)$ in the relativistic region has been made in papers by Fradkin²⁰ and by Akhiezer and Peletminskii.²³

When the thermodynamic potential is known it is not hard to determine all the other thermodynamic characteristics of the system (cf. Landau and Lifshitz³⁶).

As is well known, the mean energy E of a statistical system is determined by the relation

$$E - \mu \bar{N} = \sum_{n, N} (E_{nN} - \mu N) e^{(\Omega + \mu N - E_{nN})\beta} = \Omega - \beta^{-1} \frac{\partial \Omega}{\partial \beta^{-1}}, \quad (2.60)$$

where \bar{N} is the mean number of particles in the system (or the difference of the mean numbers of electrons and positrons, $N^- - N^+$). When one neglects the fluctuation of the total number of particles in the system it is easy to express the change of the energy of the system caused by the interaction of the particles, $\Delta E = E - E_0$, in terms of the change of the potential, $\Delta \Omega = \Omega - \Omega_0$ (the index "0" always denotes the value for no interaction between the particles). According to Eq. (2.60) we have at the absolute zero of temperature

$$E - \mu \bar{N} = \Omega = \Omega_0 + \Delta\Omega, \quad (2.61)$$

where $\Omega_0 = E_0 - \mu \bar{N}_0$. By using the expression

$$\bar{N}^- - \bar{N}^+ = \frac{2V}{(2\pi)^3} \int (n_p^- - n_p^+) d^3p$$

we can eliminate the chemical potential from $\Delta\Omega$ in Eq. (1.61) and express it in terms of $\bar{N}^- - \bar{N}^+$; then the energy correction ΔE caused by the interaction of the particles (at the absolute zero of temperature) can be written in the form (Fradkin¹⁷)

$$\begin{aligned} \Delta E &= \frac{V}{(2\pi)^4} \int_0^e \frac{de}{e} \int d^4p [\mathcal{G}(p) - \mathcal{G}_0(p)]_{\alpha\beta} \mathcal{G}_{\beta\alpha}^{-1}(p) \\ &= -\frac{V}{(2\pi)^4} \int_0^e \frac{de}{e} \int d^4k \frac{1}{k^2} \Pi_{\mu\nu} \left(1 - \frac{1}{k^2} \Pi\right)_{\nu\mu}^{-1}. \end{aligned} \quad (2.62)$$

Putting $(\mathcal{G} - \mathcal{G}_0) \mathcal{G}_0^{-1} = M\mathcal{G}$ in the form of an infinite series in e^2 , we get the well known expansion of the energy (2.62) in terms of connected diagrams (for more details on this point see the paper by Klein and Prange¹²).

By applying Eqs. (2.62) and (2.59) to a gas of electrons, positrons, and photons, Akhiezer and Peletminskii²³ have calculated the thermodynamic potential, and have used it to compute the energy correction ΔE caused by the interaction in this system. Their results in part overlap the well known results of Gell-Mann and Brueckner,⁴⁹ Vedenov,⁵⁰ and Fradkin.^{18,19} Because the result is cumbersome in form, we present here only the limiting values ΔE_{nr} for the nonrelativistic case and ΔE_{er} for the extreme relativistic case (at low temperatures):

$$\begin{aligned} \Delta E_{nr} &= -(3\pi^2)^{4/3} (2\pi)^{-4} V n^{4/3} e^2 \\ &+ (1 - \ln 2) (2\pi)^{-4} V m n e^4 \ln(e^2 m n^{-1/3}), \end{aligned} \quad (2.63)$$

$$\begin{aligned} \Delta E_{er} &= (3\pi^2)^{4/3} 2^{-1} (2\pi)^{-4} V n^{4/3} e^2 \\ &+ (3\pi^2)^{4/3} 2^{-1} (2\pi)^{-8} V n^{4/3} e^4 \ln e^2, \end{aligned} \quad (2.64)$$

where the first terms, proportional to e^2 , are the values of the exchange energy, and the second terms are the values of the correlation energy, which is of higher order in e^2 ; $n = (\bar{N}^- - \bar{N}^+)/V$, and m is the mass of the real electron.

In particular, by setting $\mu = 0$ in the general formulas for a system of electrons, positrons, and photons, these authors²³ found the correction to the energy of black-body radiation caused by the interaction of the particles; in the extreme relativistic limit $\beta^{-1} \gg m$ the result becomes quite simple, namely

$$\Delta E = \frac{25}{264\pi^2} E_{id} \left(e^2 - \frac{8e^3}{5V^{3\pi}} \right), \quad (2.65)$$

where $E_{id} = 33\pi^2 V / 180\beta^4$ is the energy of the ideal gas of electrons, positrons, and photons for $\beta^{-1} \gg m$.

System of particles interacting by Coulomb's law.

Let us first consider the case of a multicomponent

system of various types of fermions interacting through the electromagnetic field in the relativistic region.

Each particular type of fermions λ is described by a Hamiltonian H_λ , which after the subtraction of $\sum_{\mathbf{k}, \lambda=1}^2 \omega_{\mathbf{k}} c_{\mathbf{k}\lambda}^\dagger c_{\mathbf{k}\lambda}$ is exactly of the form of Eqs. (2.17a)

– (2.17c), except that the fermion operators carry the additional index λ , which designates the given type of fermions. The total Hamiltonian of such a multicomponent system is equal to the sum of the Hamiltonians H_λ of all the separate types of fermions plus the Hamiltonian of the free electromagnetic field. By repetition of the previous arguments, it is not hard to write down the partition function and construct the thermodynamic perturbation theory for a multicomponent system. The most interesting procedure, however, is to study the system by means of Green's functions. Previously, in writing down the equations for the Green's functions we started from the equations for the field operators in the Heisenberg representation, Eqs. (2.34) and (2.35). In the case of a multicomponent system each individual operator $\tilde{\psi}_\lambda$ of the fermion field of type λ also satisfies Eq. (2.34) with the chemical potential $\mu = \mu_\lambda$ and the mass $m = m_\lambda$, and the right member of (2.25) for the electromagnetic field operator $A(x)$ will contain the sum of the currents from all the types of fermions. Therefore the equation for the thermodynamic Green's function \mathcal{G}_λ of the fermion of type λ will have the previous form (2.44), and the equation for the photon thermodynamic Green's function \mathcal{D} will contain a polarization operator equal to the sum of polarization operators calculated from Eq. (2.47) with \mathcal{G} replaced by \mathcal{G}_λ . In the nonrelativistic approximation these equations can describe a system of ions and electrons interacting by Coulomb's law. By the results noted in Eqs. (2.53) and (2.54), the equations that describe a multicomponent system of nonrelativistic fermions interacting by Coulomb's law are of the following forms:

$$\mathcal{G}_\lambda(p) = \mathcal{G}_{0\lambda}(p) - \mathcal{G}_{0\lambda}(p) M_\lambda(p) \mathcal{G}_\lambda(p), \quad (2.66)$$

$$\mathcal{D}(k) = \mathcal{D}_0(k) + \mathcal{D}_0(k) \Pi(k) \mathcal{D}(k), \quad (2.67)$$

$$M_\lambda(p) = \frac{Z_\lambda^2 e^2}{(2\pi)^3 \beta} \sum_{k_4} \int \mathcal{G}_\lambda(p+k) \Gamma(p+k, k) \mathcal{D}(k) d^3k, \quad (2.68)$$

$$\Pi(k) = \sum_{\lambda} \frac{Z_\lambda^2 e^2}{(2\pi)^3 \beta} \sum_{p_4} \int \text{Sp} \mathcal{G}_\lambda(p+k) \Gamma(p+k, k) \mathcal{G}_\lambda(p) d^3p, \quad (2.69)$$

$$\Gamma(p, p') = 1 + \Lambda(p, p'), \quad (2.70)$$

where $p_4 = (2n+1)\pi/\beta$, $k_4 = 2m\pi/\beta$ ($m, n = 0, \pm 1, \pm 2, \dots$); $Z_\lambda e$ is the charge of a fermion of type λ (for the electron $Z = -1$); the zeroth-order Green's functions are

$$\mathcal{G}_{0\lambda}(p) = \frac{-\delta_{\alpha\beta}}{i p_4 + \mu_\lambda - \frac{p^2}{2m_\lambda}}, \quad \mathcal{D}_0(k) = \frac{1}{k^2}, \quad (2.71)$$

where μ_λ and m_λ are respectively the chemical po-

tential and mass of the fermion of type λ , and the spinor indices α and β run through the values 1, 2 for fermions of spin $\frac{1}{2}$. With Eq. (2.66) written as it is, we could dispense altogether with the spin of the fermion. Then \mathcal{G} would not depend on the spinor indices, and in Eq. (2.71) $\delta_{\alpha\beta} \rightarrow 1$. We could then take account of the spin of the fermion when we write the number of states of a particle with a given momentum, Eq. (1.46a); this procedure is especially convenient when the spin of a fermion is larger than $\frac{1}{2}$.

Let us find the thermodynamic potential by using the solution of the equation (2.67) with the kernel Π in the first nonvanishing approximation in e^2 , which in this case is

$$\begin{aligned} \Pi^1(k) &= \sum_{\lambda} \frac{2Z_{\lambda}^2 e^2}{(2\pi)^3} \int \frac{(n_{p+k}^{\lambda} - n_p^{\lambda}) d^3p}{\varepsilon_{p+k}^{\lambda} - \varepsilon_p^{\lambda} - ik_4} \\ &= \sum_{\lambda} \frac{2Z_{\lambda}^2 e^2}{(2\pi)^3} \int \frac{\omega_{p\lambda} (n_{p+k}^{\lambda} - n_p^{\lambda})}{\omega_{p\lambda}^2 + k_4^2} d^3p, \end{aligned} \quad (2.72)$$

$$\varepsilon_p^{\lambda} = \frac{p^2}{2m_{\lambda}}, \quad \omega_{p\lambda} = \varepsilon_{p+k}^{\lambda} - \varepsilon_p^{\lambda}, \quad n_p^{\lambda} = [e^{(\varepsilon_p^{\lambda} - \mu_{\lambda})/\beta} + 1]^{-1}. \quad (2.73)$$

According to Eq. (2.59) the thermodynamic potential of the multicomponent system of fermions is of the form

$$\Omega = \Omega_0 + \frac{V}{2(2\pi)^3\beta} \sum_{k_4} \int d^3k \ln \left(1 - \frac{1}{k^2} \Pi^1(k) \right). \quad (2.74)$$

If we separate out from $\Delta\Omega = \Omega - \Omega_0$ as given by Eq. (2.74) the self-consistent part proportional to e^2 , the remaining correlation part $\Delta\Omega_{\text{cor}}$ of the thermodynamic potential can be written in the following way (Fradkin^{18,19}):

$$\Delta\Omega_{\text{cor}} = \frac{V}{2(2\pi)^3\beta} \sum_{k_4} \int d^3k \left[\ln \left(1 - \frac{1}{k^2} \Pi^1(k) \right) + \frac{1}{k^2} \Pi^1(k) \right]. \quad (2.75)$$

Let us see what $\Delta\Omega_{\text{cor}}$ corresponds to in the classical limit $\hbar = 0$, $n^{2/3} \hbar^2 \beta m^{-1} \ll 1$ (high temperatures and small densities n). For this purpose we rewrite Eq. (2.75) in the form

$$\Delta\Omega_{\text{cor}} = - \frac{V}{(2\pi)^3\beta} \int_0^e \frac{de}{e} \sum_{k_4} \int \frac{[\Pi^1(k, k_4)]^2 d^3k}{[k^2 - \Pi^1(k, k_4)] k^2}, \quad (2.76)$$

from which it can be seen that small values of k^2 are the main region for the integration. At high temperatures the function $\Pi^1(k, k_4)$ falls off exponentially as k^2 increases, and for small k^2 has different behaviors, depending on the value of k_4 , namely: for $k^2 \rightarrow 0$ we have $\Pi^1(k, k_4 \neq 0) \rightarrow 0$, whereas $\Pi^1(k, 0) \rightarrow \Pi^1(0, 0) \neq 0$, where

$$\begin{aligned} \Pi^1(0, 0) &= \sum_{\lambda} \frac{2Z_{\lambda}^2 e^2}{(2\pi)^3} \int \frac{\partial n_p^{\lambda}}{\partial \varepsilon_p^{\lambda}} d^3p = - \sum_{\lambda} \frac{2Z_{\lambda}^2 e^2 \beta}{(2\pi)^3} \int n_p^{\lambda} d^3p \\ &= - e^2 \beta \sum_{\lambda} Z_{\lambda} n_{\lambda} \equiv - \kappa^2. \end{aligned} \quad (2.77)$$

Here κ is the reciprocal of the Debye radius, and n_{λ} is the density of fermions of type λ

$$n_{\lambda} = \frac{2}{(2\pi)^3} \int n_p^{\lambda} d^3p. \quad (2.78)$$

Thus in the entire sum over k_4 in Eq. (2.76) the largest contribution comes from the term with $k_4 = 0$. Assuming that the screening is weak, $e^2 \beta n^{1/3} \ll 1$, we use in the denominator (and also in the numerator) the value of the function $\Pi^1(k, 0)$ at $k = 0$: $\Pi^1(k, 0) = \Pi^1(0, 0)$. The integral over the variable k is then easily calculated, and we arrive at the well known result of the Debye-Hückel theory⁶³

$$\begin{aligned} \Delta\Omega_{\text{cor}} &= - \frac{V}{(2\pi)^3\beta} \int_0^e \frac{de}{e} \frac{4\pi\kappa^4}{e} \int_0^{\infty} \frac{dk}{k^2 + \kappa^2} \\ &= - \frac{2V\pi V}{3} \left(\frac{e^2}{4\pi} \beta^{1/3} \sum_{\lambda} Z_{\lambda}^2 n_{\lambda} \right)^{3/2}. \end{aligned} \quad (2.79)$$

The present method, however, also allows us to determine without especial difficulty all of the subsequent corrections to the thermodynamic potential in terms of the small parameter $e^2 \beta n^{1/3} \ll 1$ of the Debye-Hückel theory and the parameter for classical theory to apply, $n^{2/3} \hbar^2 \beta m^{-1} \ll 1$. For example, for a mixture of two types of ions at high temperatures, $\beta \rightarrow 0$, and equal densities, Fradkin¹⁹ has given an expansion of the thermodynamic potential per unit volume which contains in addition to the Debye term (2.79) other terms proportional to β and $\beta^{3/2}$.

In applying the present theory to solutions of ions one can formally include effects of a medium which weakens the Coulomb interaction of the ions by the substitution $Z_{\lambda} \rightarrow Z_{\lambda} \epsilon^{-1/2}$, where ϵ is the dielectric constant of the solvent, $\epsilon = \text{const}$. In this connection we mention papers by Vedenov,^{50,51} and also papers by Dzyaloshinskii and Pitaevskii²¹ and by Dzyaloshinskii, Lifshitz, and Pitaevskii.²² In the latter papers^{21,22} the method of Green's functions is extended to the case of an absorbing medium with a complex dielectric constant.

The formula (2.74) is also valid for the opposite limiting case of low temperatures and large densities of the components of the mixture, for which the small parameter is $m e^2 / \hbar^2 n^{1/3}$ (the Gell-Mann-Brueckner approximation). Setting $\beta \rightarrow \infty$ in Eq. (2.75), we find $\Delta\Omega_{\text{cor}}$ for the electron gas $m_{\lambda} = m$, $Z_{\lambda} = -1$ at absolute zero temperature in nonrelativistic approximation. After this, by means of Eqs. (2.62), we get the following expression for the correlation energy E_{cor} of an electron gas with $\beta = 0$ (Fradkin^{18,19}):

$$\begin{aligned} E_{\text{cor}} &= \frac{V}{2(2\pi)^3} \int d^3k \left[\ln \left(1 - \frac{2e^2}{k^2 (2\pi)^3} \int \frac{\omega_p (n_{p+k} - n_p)}{\omega_p^2 + k_4^2} d^3p \right) \right. \\ &\quad \left. + \frac{2e^2}{k^2 (2\pi)^3} \int \frac{\omega_p (n_{p+k} - n_p)}{\omega_p^2 + k_4^2} d^3p \right], \\ \varepsilon_p &= \frac{p^2}{2m}, \quad \omega_p = \varepsilon_{p+k} - \varepsilon_p, \quad n_p = \begin{cases} 0 & \text{for } \varepsilon_p > \mu, \\ 1 & \text{for } \varepsilon_p < \mu. \end{cases} \end{aligned} \quad (2.80)$$

If we expand the E_{cor} of Eq. (2.80) in powers of the small parameter $m e^2 / \hbar^2 n^{1/3} \ll 1$, we get the well known

results of Gell-Mann and Brueckner,⁴⁹ Sawada,⁵² and Sawada and others.⁵³

3. TIME-DEPENDENT GREEN'S FUNCTIONS

The analytical properties of time-dependent Green's functions. As has been noted earlier, the use of thermodynamic Green's functions is extremely helpful in the study of the thermodynamic properties of a system. For other purposes, however, such as the study of the weak excitations (quasi-particles) of a system, and also the study of various kinetic phenomena, for example the passage of particles through matter, the absorption or scattering of light and sound, and so on, it is necessary to know temperature Green's functions which depend on the time t , or, as they are sometimes called, time-dependent Green's functions

$$G(x, x') = -i \text{Sp} \{ e^{(\Omega + \mu N - H)\beta} T(\tilde{\psi}(x)\tilde{\psi}^+(x')) \} \\ \equiv -i [T(\tilde{\psi}(x)\tilde{\psi}^+(x'))]_{av}, \quad (3.1)$$

where the sign \sim denotes operators in the Heisenberg representation

$$\tilde{\psi}(x) = e^{i(H - \mu N)t} \psi(x) e^{-i(H - \mu N)t}, \quad (3.2)$$

and N is the operator for the total number of particles (which is conserved along with the total energy of a closed system) or the difference of the operators for the total numbers of electrons and positrons. Equation (3.1) defines the one-particle time-dependent Green's function both for fermions and for bosons. Many-particle time-dependent Green's functions are defined in analogy with Eqs. (2.2) and (2.3).

Comparing the definitions of the thermodynamic Green's function $\mathcal{G}(x, x')$, Eqs. (2.1) – (2.4) and (2.32) – (2.33), and of the time-dependent Green's function $G(x, x')$, Eq. (3.1), we see that for homogeneous systems both functions depend on the difference $x - x'$ and that the change from \mathcal{G} to G in coordinate space and in the range of τ from $-\beta$ to β is accomplished by replacing τ by t and multiplying \mathcal{G} by the factor $-i$. It is useful, however, to know the relation between the Fourier components of these functions, and for this purpose we shall examine the Fourier expansion of the time-dependent Green's function (Landau⁵).

The matrix element of the operator (3.2) is obviously of the form

$$\tilde{\psi}_{nm}(x) = \psi_{nm}(0) e^{i(\omega_{nm}t - \mathbf{p}_{nm}x)}, \\ \omega_{nm} = E_n - E_m + \mu, \quad \mathbf{p}_{nm} = \mathbf{P}_n - \mathbf{P}_m, \quad (3.3)$$

where n and m denote states of a closed system in which the total energy E , the total number of particles N , and the total momentum \mathbf{P} have definite values, and for a self-adjoint operator ψ^+ we get $(\psi^+)_{nm} = (\psi^+)_{mn}$. Using Eq. (3.3), we rewrite the Green's function (3.1) in the form

$$G(x_1 - x_2) = -i \sum_n e^{(\Omega + \mu N_n - E_n)\beta} [T(\psi(x_1)\psi^+(x_2))]_{nn} \\ = \begin{cases} -i \sum_{nm} e^{(\Omega + \mu N_n - E_n)\beta} |\psi_{nm}(0)|^2 e^{i(\omega_{nm}t - \mathbf{p}_{nm}x)}, & t > 0, \\ \pm i \sum_{nm} e^{(\Omega + \mu N_n - E_n)\beta} |\psi_{mn}(0)|^2 e^{i(\omega_{mn}t - \mathbf{p}_{mn}x)}, & t < 0, \end{cases} \quad (3.4)$$

where the upper sign is for fermions and the lower for bosons, and $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$, $t = t_1 - t_2$. In the double sum with $t < 0$ it is convenient to interchange the summation indices, $m \rightarrow n$, $n \rightarrow m$, and use the fact that the matrix element ψ_{nm} is different from zero only for $N_m = N_n + 1$. Then the expression under the summation sign for $t < 0$ will differ from that for $t > 0$ only by the factor $\exp \omega_{nm}\beta$. Defining the Fourier transformation by

$$G(\mathbf{x}, t) = \frac{1}{(2\pi)^4} \int G(\mathbf{p}, \omega) e^{i(\mathbf{p}x - \omega t)}, \quad (3.5)$$

we get from Eq. (3.4) the following Fourier expansion with respect to the space variables:

$$G(\mathbf{p}, t) = \begin{cases} i \int_{-\infty}^{\infty} g(\mathbf{p}, E) e^{-iEt} dE, & t > 0, \\ \mp i \int_{-\infty}^{\infty} g(\mathbf{p}, E) e^{-E\beta} e^{-iEt} dE, & t < 0, \end{cases} \quad (3.6)$$

$$g(\mathbf{p}, E) \\ = -(2\pi)^3 \sum_{nm} e^{(\Omega + \mu N_n - E_n)\beta} |\psi_{nm}(0)|^2 \delta(\mathbf{p} + \mathbf{p}_{nm}) \delta(E + \omega_{nm}). \quad (3.7)$$

In going to the Fourier component with respect to the variable t we must use the formula

$$\int_0^{\infty} e^{iat} dt = \pi \delta(a) + \frac{i}{a}. \quad (3.8)$$

We then have finally

$$G(\mathbf{p}, \omega) = \int \frac{g(\mathbf{p}, E)}{E - \omega} (1 \pm e^{-E\beta}) dE + i\pi g(\mathbf{p}, \omega) (1 \mp e^{-\omega\beta}) \\ \equiv G'(\mathbf{p}, \omega) + iG''(\mathbf{p}, \omega). \quad (3.9)$$

A comparison of the two terms in Eq. (3.9) leads to the conclusion that there is a definite relation between the real part G' of the time-dependent Green's function and the imaginary part G'' (Landau⁵): for Fermi statistics

$$G'(\mathbf{p}, \omega) = \frac{1}{\pi} \int \coth \frac{E\beta}{2} \frac{G''(\mathbf{p}, E)}{E - \omega} dE, \quad (3.10)$$

and for Bose statistics

$$G'(\mathbf{p}, \omega) = \frac{1}{\pi} \int \tanh \frac{E\beta}{2} \frac{G''(\mathbf{p}, E)}{E - \omega} dE. \quad (3.11)$$

Thus the Fourier transform $G(\mathbf{p}, \omega)$ of the time-dependent Green's function is not an analytic function of the variable ω . The following two functions are

analytic in the upper half of the plane of ω :

$$G'(\mathbf{p}, \omega) + i \coth \frac{\omega\beta}{2} G''(\mathbf{p}, \omega), \quad (3.12)$$

$$G'(\mathbf{p}, \omega) + i \tanh \frac{\omega\beta}{2} G''(\mathbf{p}, \omega), \quad (3.13)$$

as becomes obvious if we represent each of these functions as a contour integral, for example:

$$G'(\mathbf{p}, \omega) + i \coth \frac{\omega\beta}{2} G''(\mathbf{p}, \omega) = \frac{1}{\pi} \int_{\mathcal{C}} \coth \frac{\omega'\beta}{2} \frac{G''(\mathbf{p}, \omega')}{\omega' - \omega} d\omega'. \quad (3.14)$$

It turns out that the functions (3.12) and (3.13) are the Fourier transforms of the so-called retarded function $G^R(x_1 - x_2)$ (Bogolyubov and Tyablikov,²⁵ Abrikosov, Gor'kov, and Dzyaloshinskiĭ¹⁵)

$$G^R(x_1 - x_2) = \begin{cases} -i [\tilde{\psi}(x_1) \tilde{\psi}^*(x_2) \pm \tilde{\psi}^*(x_2) \tilde{\psi}(x_1)]_{av}, & t_1 > t_2, \\ 0, & t_1 < t_2, \end{cases} \quad (3.15)$$

where the upper sign is for Fermi statistics [for which the Fourier transform is Eq. (3.12)] and the lower sign is for Bose statistics [for which the Fourier transform of G^R is given by Eq. (3.13)]. Using Eqs. (3.12) – (3.14), we can represent the Fourier component $G^R(\mathbf{p}, \omega)$ of the retarded function (3.15) by an expansion of the Lehmann type⁵⁴

$$G^R(\mathbf{p}, \omega) = \int_{-\infty}^{\infty} \frac{\varrho(\mathbf{p}, E)}{E - \omega - i\delta} dE, \quad (3.16)$$

where

$$\varrho(\mathbf{p}, E) = g(\mathbf{p}, E) (1 \pm e^{-E\beta}) \quad (3.17)$$

is a real function, whose integral $\int_{-\infty}^{\infty} \rho(\mathbf{p}, E) dE$ is finite.

It is easy to obtain the same sort of Lehmann expansion for the thermodynamic Green's function $\mathcal{G}(x_1 - x_2)$ of Eq. (2.32). In fact, by repeating the arguments that led to Eq. (3.6) we get for $G(\mathbf{p}, \tau)$

$$\mathcal{G}(\mathbf{p}, \tau) = \begin{cases} - \int_{-\infty}^{\infty} g(\mathbf{p}, E) e^{-E\tau} dE, & \tau > 0, \\ \pm \int_{-\infty}^{\infty} g(\mathbf{p}, E) e^{-E(\beta+\tau)} dE, & t < 0, \end{cases} \quad (3.18)$$

where the upper sign is for fermions, the lower for bosons, and $g(\mathbf{p}, E)$ is given by Eq. (3.7). Then making the Fourier transformation (1.60) we have

$$\mathcal{G}(\mathbf{p}, \omega_n) = - \int_{-\infty}^{\infty} \frac{\varrho(\mathbf{p}, E)}{E - i\omega_n} dE, \quad (3.19)$$

where $\rho(\mathbf{p}, E)$ is given by Eq. (3.17). Comparing Eqs. (3.16) and (3.19), we find (Abrikosov, Gor'kov, and Dzyaloshinskiĭ¹⁵)

$$\mathcal{G}(\mathbf{p}, \omega_n) = -G^R(\mathbf{p}, i\omega_n), \quad \omega_n > 0. \quad (3.20)$$

If we consider the integral (3.19) formally as a func-

tion of the complex variable $i\omega_n$, it defines a function that is analytic in the upper half-plane. According to Eq. (3.20), this function coincides with $G^R(\mathbf{p}, \omega)$ at the infinite set of points $i\omega_n$ ($\omega_n > 0$), which have a point of condensation. From the theorem of analytic continuation we conclude that $-G^R(\mathbf{p}, \omega)$ is the analytic continuation of the function $G(\mathbf{p}, -i(i\omega_n))$ to the upper half-plane of the complex variable:

$$-G^R(\mathbf{p}, \omega) = \mathcal{G}(\mathbf{p}, -i\omega). \quad (3.21)$$

In addition to the spectral resolution (3.5) – (3.21) of the one-particle Green's functions (which has also been treated in papers by Gor'kov,⁷ Martin and Schwinger,^{13,35} Fradkin,^{18,19} Bonch-Bruevich,²⁶ and Kogan²⁷) it can be helpful in some problems (for example, in the theory of electric conductivity, cf. Bonch-Bruevich³⁰) to have the spectral representations of many-time temperature Green's functions. A paper by Bonch-Bruevich²⁴ is devoted to this problem.

Determination of the energy spectrum of a system.

The use of quantum-field-theory methods has been particularly fruitful in the study of the weak excitations of a system consisting of a large number of interacting particles. As is well known, the appearance of elementary excitations in such a system can be interpreted as the appearance of quasi-particles. A set of elementary excitations forms a gas of quasi-particles, which can be described very conveniently by the apparatus of Green's functions. As has been noted in papers by Galitskiĭ and Migdal³ and by Galitskiĭ,⁴ the description of a system of an enormous number of particles by the method of elementary excitations (quasi-particles) is exact only in the case of an ideal gas. When there is interaction between the particles the weakly excited states are not stationary states of the system, and this leads to damping of the elementary excitations.

For concreteness let us consider the electron-phonon system of Eqs. (1.1a) – (1.1c). We define the electron and phonon single-particle Green's functions $G(x - x')$ and $D(x - x')$ by Eqs. (3.1) and (3.2) and

$$D(x - x') = -i [\tilde{\varphi}(x) \tilde{\varphi}(x')]_{av}. \quad (3.22)$$

The equations for these Green's functions are derived by means of the previously described technique of variational derivatives (cf., e.g., papers by Fradkin,^{16,18} Bonch-Bruevich,⁵⁵ and Kogan²⁷) or by summing infinite sets of graphs of definite classes. An example of such a summation is shown in Fig. 4, which leads to Eq. (2.49) with Π^1 as an approximate kernel for Π . Without going into the rather simple calculations, we present the equations for G and D in the p representation:

$$G(p) = G_0(p) - G_0(p) M(p) G(p), \quad (3.23)$$

$$D(k) = D_0(k) + D_0(k) \Pi(k) D(k), \quad (3.24)$$

$$M(p) = \frac{ig^2}{(2\pi)^4} \int G(p+k) \Gamma(p+k, k) D(k) d^4k, \quad (3.25)$$

$$\Pi(k) = \frac{ig^2}{(2\pi)^4} \int \text{Sp} G(p+k) \Gamma(p+k, k) G(p) d^4p, \quad (3.26)$$

$$\Gamma(p, p') = 1 + \Lambda(p, p'), \quad (3.27)$$

where $G_0(p)$ and $D_0(p)$ are the Green's functions of the electron and phonon in the absence of interaction:

$$G_0(p, \omega) = \frac{1-n_p}{\omega - \epsilon_p + \mu + i\delta} + \frac{n_p}{\omega - \epsilon_p + \mu - i\delta}, \quad (3.28)$$

$$D_0(k, \epsilon) = \frac{\omega_k}{2} \left(\frac{1+n_k}{\epsilon - \omega_k + i\delta} - \frac{n_k}{\epsilon - \omega_k - i\delta} + \frac{n_k}{\epsilon + \omega_k + i\delta} - \frac{1+n_k}{\epsilon + \omega_k - i\delta} \right), \quad (3.29)$$

and $\Lambda(p, p')$ is the set of all vertex-part graphs except the simple vertex (point).

There is a simple connection between the one-particle Green's function and the spectrum of elementary excitations, or quasi-particles (Galitskiĭ and Migdal,³ Galitskiĭ,⁴ Bonch-Bruевич,⁵⁵ Fradkin,^{18,19} and Kogan²⁷). The energies and dampings of the quasi-particles are determined by the poles of the analytic continuation of the one-particle time-dependent Green's function as function of the fourth component of the four-dimensional momentum. There can, however, also be excitations in a system of a large number of interacting particles that are not of a one-particle character. The energy spectrum of such excitations is determined in terms of the two-particle Green's function (Galitskiĭ and Migdal³). When in $G^{-1}(p, \omega) = \omega + \mu - \epsilon_p + M(p, \omega)$ we set the variable ω equal to $E_p - \mu - i\gamma_p$ we find the following equation for the determination of the poles of the analytic continuation of the electron Green's function of Eq. (3.23):

$$E_p - i\gamma_p - \epsilon_p + M(p, E_p - \mu - i\gamma_p) = 0. \quad (3.30)$$

A solution of this equation gives the energy E_p and the damping γ_p of a quasi-particle: $E_p < \mu$ corresponds to holes in the Fermi distribution, and $E_p > \mu$ corresponds to quasi-particles above the Fermi sphere. If the damping is small, $\gamma_p/E_p \ll 1$, then we get from Eq. (3.30) the approximate relations

$$E_p - \epsilon_p + \text{Re} M(p, E_p - \mu) = 0, \quad (3.31)$$

$$\gamma_p = \text{Im} M(p, E_p - \mu) / \left(1 + \left(\frac{\partial \text{Re} M}{\partial \omega} \right)_{\omega=E_p-\mu} \right). \quad (3.32)$$

Similarly, the poles of the analytic continuation of the phonon Green's function $D(k, \epsilon)$ of Eq. (3.24) give the energies and dampings of the phonon excitations. In this way Migdal⁸ has found the energy spectrum of the electrons and the dispersion of the lattice vibrations in a normal metal at absolute zero temperature without assumptions about smallness of the interaction between electrons and phonons.

As an illustration we shall determine the energy spectrum of the electrons for a temperature different from absolute zero in the case of a superconductor.

The energy spectrum of a superconductor. As is well known (cf., e.g., the review by Abrikosov and Khalatnikov⁵⁶) in a small range of energies in the neighborhood of the Fermi sphere a gas of fermions with a direct interaction between the particles (of the nature of a weak attraction) shows the property of superconductivity. In this connection let us consider a quasi-closed system of electrons, Eqs. (1.71a) – (1.72), with a direct four-fermion interaction $v(\mathbf{x} - \mathbf{x}') = g\delta(\mathbf{x} - \mathbf{x}')$, where g is a small coupling constant that is different from zero only in a narrow range of energies ϵ_p near the Fermi surface, $\epsilon_F - \kappa \leq \epsilon_p \leq \epsilon_F + \kappa$ (Gor'kov⁷). The electron-field operators in the Heisenberg representation, $\tilde{\psi}(\mathbf{x})$ and $\tilde{\psi}^+(\mathbf{x})$, Eq. (3.2), satisfy the equations

$$\left(i \frac{\partial}{\partial t} + \mu + \frac{\nabla^2}{2m} \right) \tilde{\psi}(\mathbf{x}) - g(\tilde{\psi}^+(\mathbf{x}) \tilde{\psi}(\mathbf{x})) \tilde{\psi}(\mathbf{x}) = 0, \quad (3.33)$$

$$\left(i \frac{\partial}{\partial t} - \mu - \frac{\nabla^2}{2m} \right) \tilde{\psi}^+(\mathbf{x}) + g\tilde{\psi}^+(\mathbf{x})(\tilde{\psi}^+(\mathbf{x}) \tilde{\psi}(\mathbf{x})) = 0. \quad (3.34)$$

Therefore for the electron time-dependent Green's function $G(\mathbf{x}, \mathbf{x}') \equiv G(\mathbf{x} - \mathbf{x}')$ of Eq. (3.1) we have

$$\left(i \frac{\partial}{\partial t} + \mu + \frac{\nabla^2}{2m} \right) G_{\alpha\beta}(\mathbf{x} - \mathbf{x}') + ig[T(\tilde{\psi}_\sigma^+(\mathbf{x}) \tilde{\psi}_\sigma(\mathbf{x}) \tilde{\psi}_\alpha(\mathbf{x}) \tilde{\psi}_\beta^+(\mathbf{x}'))]_{\text{av}} = \delta_{\alpha\beta} \delta(\mathbf{x} - \mathbf{x}'). \quad (3.35)$$

Let us now use a physical idea of Cooper,⁵⁷ according to which the weak attraction near the Fermi surface between two electrons with opposite momenta and spins leads to the formation of a bound state of the pair of electrons with a negative binding energy. Furthermore the interaction between the electrons will be taken into account to the extent that it leads to the formation of a stationary pair. Then the average of the T-product of operators in Eq. (3.35) can be written in the following way (Gor'kov⁷):

$$[T(\tilde{\psi}_\sigma^+(\mathbf{x}) \tilde{\psi}_\sigma(\mathbf{x}) \tilde{\psi}_\alpha(\mathbf{x}) \tilde{\psi}_\beta^+(\mathbf{x}'))]_{\text{av}} = -\hat{F}_{\alpha\sigma}(0) \hat{F}_{\sigma\beta}^*(\mathbf{x} - \mathbf{x}'), \quad (3.36)$$

where

$$\hat{F}_{\alpha\beta}(\mathbf{x} - \mathbf{x}') = [N | \tilde{\psi}_\alpha(\mathbf{x}) \tilde{\psi}_\beta(\mathbf{x}') | N + 2]_{\text{av}},$$

$$\hat{F}_{\alpha\beta}^*(\mathbf{x} - \mathbf{x}') = [N + 2 | \tilde{\psi}_\alpha^+(\mathbf{x}) \tilde{\psi}_\beta^+(\mathbf{x}') | N]_{\text{av}}. \quad (3.37)$$

From Eqs. (3.34) and (3.36) it is not hard to write down the equation for the function $\hat{F}^*(\mathbf{x} - \mathbf{x}')$

$$\left(i \frac{\partial}{\partial t} - \mu - \frac{\nabla^2}{2m} \right) \hat{F}^*(\mathbf{x} - \mathbf{x}') + ig\hat{F}^*(0) G(\mathbf{x} - \mathbf{x}') = 0. \quad (3.38)$$

To determine the energy spectrum of the system it is necessary to find the electron Green's function $G(p, \omega)$ in the p representation. By carrying out the Fourier transformation (3.5) in Eqs. (3.35), (3.36), and (3.38), we get

$$(\omega - \xi_p) G(p, \omega) - ig\hat{F}^*(0) \hat{F}^*(p, \omega) = 1,$$

$$(\omega + \xi_p) \hat{F}^*(p, \omega) + ig\hat{F}^*(0) G(p, \omega) = 0, \quad (3.39)$$

where

$$\xi_p = \frac{p^2}{2m} - \mu.$$

It follows from the definition (3.37) that $\hat{F}^*(0)$ and

$\hat{F}^+(0)$ are of the forms

$$\hat{F}^+(0) = J \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv J\hat{I}, \quad \hat{F}(0) = -J\hat{I}, \quad (3.40a)$$

where J can be regarded as a real constant, since in the equations (3.39) one can always split off a phase factor $\exp i\alpha$ from J and include it in the unknown functions $G(\mathbf{p}, \omega)$ and $\hat{F}^+(\mathbf{p}, \omega)$. According to Eq. (3.39) the function $\hat{F}^+(\mathbf{p}, \omega)$ is also proportional to the matrix \hat{I}

$$\hat{F}^+(\mathbf{p}, \omega) = \hat{I}F^+(\mathbf{p}, \omega), \quad (3.40b)$$

which can thus be got rid of in Eq. (3.39), since $\hat{I}^2 = -1$. After this we rewrite the system of equations (3.39) in the form

$$(\omega^2 - \xi_p^2 - g^2 J^2) F^+(\mathbf{p}, \omega) = -igJ, \quad (3.41)$$

$$(\omega - \xi_p) G(\mathbf{p}, \omega) = 1 + igJF^+(\mathbf{p}, \omega). \quad (3.42)$$

The Green's function $G(\mathbf{p}, \omega)$ is defined as a particular solution of Eq. (3.42), but in finding $F^+(\mathbf{p}, \omega)$ we must take into account both a particular solution of Eq. (3.41) and also the general solution of the corresponding homogeneous equation, which is $A(\mathbf{p}, \beta) [\delta(\omega + E_p) + \delta(\omega - E_p)]$, where $E_p = (\xi_p^2 + \Delta^2)^{1/2}$, $\Delta^2 = g^2 J^2$, and $A(\mathbf{p}, \beta)$ is an arbitrary real function of the momentum and the temperature. Therefore

$$F^+(\mathbf{p}, \omega) = \frac{-igJ}{(\omega + E_p - i\delta)(\omega - E_p + i\delta)} + A(\mathbf{p}, \beta) [\delta(\omega + E_p) + \delta(\omega - E_p)],$$

$$G(\mathbf{p}, \omega) = \frac{\omega + \xi_p}{(\omega + E_p - i\delta)(\omega - E_p + i\delta)} + \frac{igJ}{\omega - \xi_p} A(\mathbf{p}, \beta) [\delta(\omega + E_p) + \delta(\omega - E_p)].$$

Thus the imaginary part of the Green's function $G(\mathbf{p}, \omega)$ is not completely determined by Eqs. (3.41) and (3.42). Let us choose the arbitrary function $A(\mathbf{p}, \beta)$ so that the real and imaginary parts of $G(\mathbf{p}, \omega)$ shall satisfy the well known relation (3.10). Simple calculations lead to the following result:

$$A(\mathbf{p}, \beta) = -\pi \Delta n_p / E_p, \quad n_p = (e^{E_p \beta} + 1)^{-1}, \quad (3.43)$$

and the Green's function takes the final form:

$$G(\mathbf{p}, \omega) = u_p^2 (\omega - E_p + i\delta)^{-1} + v_p^2 (\omega + E_p - i\delta)^{-1} + 2\pi i n_p [u_p^2 \delta(\omega - E_p) - v_p^2 \delta(\omega + E_p)],$$

$$u_p^2 = \frac{1}{2} \left(1 + \frac{\xi_p}{E_p} \right), \quad v_p^2 = \frac{1}{2} \left(1 - \frac{\xi_p}{E_p} \right). \quad (3.44)$$

The spectrum of the elementary excitations is determined by the positive pole of the function $G(\mathbf{p}, \omega)$, which gives

$$E_p = \sqrt{\xi_p^2 + \Delta^2}, \quad (3.45)$$

where Δ is a function of the temperature, which is found from the conditions (3.40a) and (3.40b)

$$J = \hat{F}^+(0) = \frac{1}{(2\pi)^4} \int F^+(\mathbf{p}, \omega) d^3p d\omega$$

$$= -\frac{Jg}{2(2\pi)^3} \int \frac{(1-2n_p)}{\sqrt{\xi_p^2 + \Delta^2}} d^3p \quad (3.46)$$

or

$$1 = -\frac{g}{2(2\pi)^3} \int \frac{\tanh \frac{E_p \beta}{2}}{\sqrt{\xi_p^2 + \Delta^2}} d^3p, \quad |\xi_p| < \kappa. \quad (3.47)$$

At absolute zero temperature, $\beta \rightarrow \infty$, $\tanh(E_p \beta / 2) = 1$, for small negative g (attraction) Eq. (3.47) has the well known solution

$$\Delta_0 = 2\kappa e^{-1/\kappa}, \quad \kappa = p_F |g| m / 2\pi^2, \quad (3.48)$$

and near absolute zero temperature the solution of Eq. (3.47) is of the form (cf., e.g., the review of Abrikosov and Khalatnikov⁵⁶)

$$\Delta = \Delta_0 - \sqrt{\frac{2\pi \Delta_0}{\beta}} e^{-\Delta_0 \beta}. \quad (3.49)$$

As can be seen from Eq. (3.45), the excited states of the system are separated from the ground state by a gap of width Δ , which leads to the phenomenon of superconductivity (Bardeen and others,⁵⁸ Bogolyubov,⁵⁹ Gor'kov,⁷ Abrikosov and Khalatnikov⁵⁶).

Another interesting feature of the Green's-function method is that it enables us to construct a gauge-invariant quantum-field-theory technique in the theory of superconductivity. By this method Abrikosov, Gor'kov, and Khalatnikov³² have investigated superconductors in high-frequency electromagnetic fields, and in particular have calculated the frequency and temperature dependences of the impedance of a massive superconductor. The further development of this technique has been pursued in a series of papers by Gor'kov³³ on the behavior of superconductors in magnetic fields, and also in a paper by Abrikosov and Gor'kov³⁴ on superconducting alloys in constant magnetic fields.

Recently Migdal⁶⁰ has developed a method for treating superfluidity and calculating the moments of inertia of nuclei that is based on the application of Green's functions to systems of finite dimensions that consist of interacting Fermi particles. In subsequent papers this method of Migdal has been successfully applied to the study of pair correlation in nuclei with an odd number of particles (Grin' and others⁶¹) and to the excitation of collective states of nuclei in the scattering of charged particles (Drozdov⁶²).

The connection between time-dependent Green's functions and the thermodynamic characteristics of a system. Given the time-dependent Green's function, it is not hard to determine all the thermodynamic quantities of a system. To do so we need only find the number of particles n as a function of the temperature $1/\beta$ and the chemical potential μ . The particle density n as a function of β and μ is expressed in terms of the time dependent Green's function

$G(x-x')$ of Eq. (3.1) in the following way:

$$G(0)|_{t'=t+0} = \pm i \text{Sp } e^{(\Omega+\mu N-H)\beta} \tilde{\psi}^+(x) \tilde{\psi}(x) = \pm i \frac{\bar{N}}{V} = \pm i n, \quad (3.50)$$

where the upper sign is for Fermi statistics and the lower is for Bose statistics. Using the expansion

$$\psi(x) = V^{-1/2} \sum_{\mathbf{p}} \alpha_{\mathbf{p}} e^{i\mathbf{p}x}, \quad \psi^+(x) = V^{-1/2} \sum_{\mathbf{p}} \alpha_{\mathbf{p}}^{\dagger} e^{-i\mathbf{p}x}, \quad (3.51)$$

where $\alpha_{\mathbf{p}}^{\dagger}$ and $\alpha_{\mathbf{p}}$ are the operators for production and absorption of a particle, we rewrite Eq. (3.50) in the form

$$G(0)|_{t'=t+0} = \pm i V^{-1} \sum_{\mathbf{p}} \text{Sp } e^{(\Omega+\mu N-H)\beta} \alpha_{\mathbf{p}}^{\dagger} \alpha_{\mathbf{p}} = \pm i V^{-1} \sum_{\mathbf{p}} n_{\mathbf{p}} \\ = \pm i \frac{1}{(2\pi)^3} \int n_{\mathbf{p}} d^3 p. \quad (3.52)$$

On the other hand

$$G(0)|_{t'=t+0} = \frac{1}{(2\pi)^4} \int G(\mathbf{p}, \omega) e^{i\omega t} d^3 p d\omega \Big|_{t=|t| \rightarrow 0}. \quad (3.53)$$

On comparing Eqs. (3.52) and (3.53) we find that the distribution function $n_{\mathbf{p}}$ of the particles at an arbitrary temperature is connected with the Green's function in the following way (Migdal²):

$$n_{\mathbf{p}} = \mp i \frac{1}{2\pi} \int G(\mathbf{p}, \omega) e^{i\omega t} d\omega \Big|_{t=|t| \rightarrow 0}. \quad (3.54)$$

We shall now express the mean energy in terms of the Green's function for the case of a system of particles interacting by pairs, Eqs. (1.71a) – (1.71c). By definition the mean energy E of the system is given by

$$E = \int d^3 x \left[\psi^+(x) \frac{-\nabla^2}{2m} \psi(x) \right]_{\text{av}} \\ + \frac{1}{2} \int d^3 x d^3 x' v(x-x') [\psi^+(x) \psi^+(x') \psi(x') \psi(x)]_{\text{av}} \\ = \mp i \int d^3 x \frac{-\nabla^2}{2m} G(xt, x't') \Big|_{t' \rightarrow t+0}^{x' \rightarrow x} \\ - \frac{1}{2} \int d^3 x d^3 x' v(x-x') G_2(x't, xt; x't', x't') \Big|_{t' \rightarrow t+0}, \quad (3.55)$$

where $G(xt, x't')$ and $G_2(x_1 t_1, x_2 t_2; x_1' t_1', x_2' t_2')$ are the one-particle time dependent Green's function of Eq. (3.1) and the two-particle function

$$G_2(x_1, x_2; x_1', x_2') = (-i^2) [T(\tilde{\psi}(x_1) \tilde{\psi}(x_2) \tilde{\psi}^+(x_1') \tilde{\psi}^+(x_2'))]_{\text{av}} \quad (3.56)$$

As before, the upper sign is for fermions and the lower for bosons. Let us introduce the mass operator $M(x, y)$, which is defined by the formula

$$-i \int d^4 y M(x, y) G(y, x') \\ = \int d^3 x'' v(x-x'') G_2(x''t, xt; x't', x''t). \quad (3.57)$$

The mass operator $M(x, y) \equiv M(x-y)$ is usually represented as a power series in the interaction (cf., e.g., the review of Klein and Prange¹²). Performing a Fourier transformation in Eq. (3.55) and using Eq. (3.57), we finally get

$$E = \frac{\mp i V}{(2\pi)^4} \int \left[\epsilon_{\mathbf{p}} \mp \frac{1}{2} M(\mathbf{p}, \omega) \right] G(\mathbf{p}, \omega) d^3 p d\omega, \quad (3.58)$$

where $\epsilon_{\mathbf{p}} = \mathbf{p}^2/2m$ and the integral over ω is taken in the same sense as in Eq. (3.54). For absolute zero temperature the formula (3.58) was first obtained in a paper by Galitskiĭ and Migdal^{3,4} for the case of fermions and in a paper by Belyaev⁶ for bosons. The latter author also included the energy of the particles in the Bose condensed phase, which does not appear explicitly in Eq. (3.58). In the papers cited, the energies of the ground states of nonideal Fermi and Bose gases are calculated on the basis of Eq. (3.58).

4. SOME APPLICATIONS TO SPECIFIC PROBLEMS

The scattering and stopping of charged particles passing through matter, and the ionization and radiation that accompany these phenomena, have been a central object of interest to physicists from the very beginning of the development of present ideas about atoms, nuclei, and elementary particles. Accordingly it is natural to apply to these phenomena the new methods, which give the clearest and most accurate solutions of these problems.

Let us consider as a whole a neutral system of electrons and ions that are in thermal equilibrium. In passing through this system an external charged particle will lose energy mainly through collisions with electrons and with the lighter particles. In their application to this problem the Green's-function method and the diagram technique were first developed by Larkin.³¹ From the beginning, however, he confined himself to the nonrelativistic case. For the sake of generality we shall expound the method in a form that can be applied for arbitrary speeds of the particles. In the present version the method can easily be extended to other problems (bremsstrahlung and pair production in the passage of particles through plasmas, radiation from a plasma, stopping of electrons in metals, and so on).

To calculate the stopping power of an electron-positron plasma we write down in the Schrödinger representation the Hamiltonian of the system of electrons and positrons and the external particle passing through it, with all of these particles interacting through the electromagnetic field,

$$H = H_e + H_{\gamma} + H_1 + H'_0 + H'_1, \quad (4.1)$$

where H_e and H_{γ} are the Hamiltonians of the free electron-positron field and the free photon field and H_1 is the Hamiltonian for their interaction, Eqs. (2.17a) – (2.18), H'_0 is the free-field Hamiltonian of the external particle (a fermion), and H'_1 is the Hamiltonian for its interaction with the photon field. H'_0 and H'_1 have the same structures as the corresponding operators for the electron-positron field.

The S matrix that describes the quantum-mechanical transitions of the plasma particles and the exter-

nal particle satisfies the equation

$$i \frac{\partial S}{\partial t} = (H_e + H_\nu + H_1 + H'_0 + H'_1) S. \quad (4.2)$$

Let us use the transformation

$$S = e^{-i(H_e + H_\nu + H_1 + H'_0)t} \mathcal{S} \quad (4.3)$$

and go over to a different representation, in which the field operators $\psi'(x)$ of the external particle are written in the interaction representation

$$\psi'(x, t) = e^{iH'_0 t} \psi'(x) e^{-iH'_0 t}, \quad (4.4)$$

and the operators $\psi(x)$ of the electron-positron field and $A(x)$ of the photon field are in the Heisenberg representation

$$\begin{aligned} \psi(x, t) &= e^{i(H_e + H_\nu + H_1)t} \psi(x) e^{-i(H_e + H_\nu + H_1)t}, \\ A(x, t) &= e^{i(H_e + H_\nu + H_1)t} A(x) e^{-i(H_e + H_\nu + H_1)t}. \end{aligned} \quad (4.5)$$

The operator $A(x, t)$ then satisfies the equation

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2} \right) A(x) = -ieN(\bar{\psi}(x)\gamma\psi(x)), \quad (4.6)$$

which has as its solution

$$A(x) = A^0(x) + e \int D^0(x-x') N(\bar{\psi}(x')\gamma\psi(x')) d^4x', \quad (4.7)$$

where $A^0(x)$ is the free photon field and $D^0(x-x')$ is the zeroth-order propagation function of a photon in quantum electrodynamics.

In accordance with the sense of the problem, this representation presupposes that the interaction between the external particle and the plasma particles is turned on at the time $t = -\infty$ and off at $t = \infty$, while the interaction between the plasma particles is always turned on.

In the present representation the \mathcal{S} matrix is defined in the following way:

$$\begin{aligned} i \frac{\partial \mathcal{S}}{\partial t} &= H'_1 \mathcal{S}, \quad \mathcal{S} = T e^{-i \int H'_1(x) d^4x}, \\ H'_1(x) &= -ie'N(\bar{\psi}'(x)\hat{A}(x)\psi'(x)), \end{aligned} \quad (4.8)$$

where e' is the charge of the external particle. The \mathcal{S} matrix (4.8) describes the scattering of the external particle by the electron-positron plasma taken as a whole. The expression (4.8) can equally well be applied to the phenomena of radiative scattering, pair production, and so on.

We shall assume further that the external particle is a fast particle ($ee'/\hbar v \ll 1$), so that its interaction with the electromagnetic field produced by the plasma can be treated by perturbation theory. Then the element of the \mathcal{S} matrix that describes scattering in which the external particle makes a transition from a state with momentum p and polarization r to a state with momentum p' and polarization r' , and the plasma goes from state n to state m , is given, in view of Eq. (4.7), by the following formula:

$$\begin{aligned} \mathcal{S}_{mp'r', npr} &= -ee' \left(mp'r' \left| \int \bar{\psi}'(x) \gamma_\nu \psi'(x) \right. \right. \\ &\quad \times D^0(x-x') \bar{\psi}(x') \gamma_\nu \psi(x') d^4x d^4x' \left. \left. \right| npr \right) \\ &= \frac{(2\pi)^4 i e e'}{V(\omega^2 - q^2)} (\bar{u}'^{r'} \gamma_\nu u^{r'}) \gamma_{\nu\alpha\beta} (\bar{\psi}_\alpha(0) \psi_\beta(0))_{mn} \\ &\quad \times \delta(\mathbf{q} - \mathbf{p}_{mn}) \delta(\omega - \omega_{mn}), \\ \mathbf{q} &= \mathbf{p} - \mathbf{p}', \quad \omega = \varepsilon_p - \varepsilon_{p-q}, \quad \mathbf{p}_{mn} = \mathbf{p}_m - \mathbf{p}_n, \\ \omega_{mn} &= E_m - E_n, \quad \varepsilon_p = \sqrt{p^2 + M^2}, \end{aligned} \quad (4.9)$$

where \mathbf{q} and ω are respectively the momentum and the energy transferred to the plasma in the scattering of the external particle with mass M and charge e' .

The probability dW of this process, averaged over initial and summed over final spin states of the external particle and also averaged statistically over the Gibbs distribution (2.23), is determined as follows:

$$dW = \frac{(ee')^2}{(2\pi)^2 (\omega^2 - q^2)^2} T_{\nu\mu} \gamma_{\mu\alpha\beta} \gamma_{\nu\alpha'\beta'} \Phi_{\alpha\beta\alpha'\beta'}(\mathbf{q}, \omega) d^3q, \quad (4.10)$$

$$T_{\nu\mu} = (\delta_{\nu\mu} (pp' + M^2) - p_\nu p'_\mu - p'_\nu p_\mu) / 2\varepsilon_p \varepsilon_{p'}. \quad (4.11)$$

$$\begin{aligned} \Phi_{\alpha\beta\alpha'\beta'}(\mathbf{q}, \omega) &= (2\pi)^3 \sum_{nm} e^{i(\Omega + \mu N_n - E_n)\beta} (\bar{\psi}_\alpha(0) \psi_\beta(0))_{nm} \\ &\quad \times (\bar{\psi}_{\alpha'}(0) \psi_{\beta'}(0))_{mn} \delta(\mathbf{q} - \mathbf{p}_{mn}) \delta(\omega - \omega_{mn}), \end{aligned} \quad (4.12)$$

where N is the difference between the total numbers of electrons and positrons, and $\Phi_{\alpha\beta\alpha'\beta'}(\mathbf{q}, \omega)$ can be expressed in terms of the two-particle thermodynamic Green's function (2.2) of the plasma, with the arguments set equal in pairs, $\mathcal{G}_2(x_1 x_1, x_2' x_2') \equiv K(x_1 - x_1')$. To bring out the connection, we follow the method of Landau,⁵ Eqs. (3.4) - (3.11), and make a spectral resolution of the function $K(x)$. We get

$$K_{\alpha\beta\alpha'\beta'}(\mathbf{q}, \tau) = \begin{cases} \int_{-\infty}^{\infty} \Phi_{\alpha\beta\alpha'\beta'}(\mathbf{q}, \omega) e^{-\omega\tau} d\omega, & \tau > 0, \\ \int_{-\infty}^{\infty} \Phi_{\alpha\beta\alpha'\beta'}(\mathbf{q}, \omega) e^{-\omega(\beta+\tau)} d\omega, & \tau < 0, \end{cases} \quad (4.13)$$

where $-\beta \leq \tau \leq \beta$.

After being periodically continued on the entire τ axis the function (4.13) satisfies the condition $K(\mathbf{q}, \tau) = K(\mathbf{q}, \tau + \beta)$ for arbitrary τ . Furthermore, according to Eq. (1.60), we have

$$K(\mathbf{q}, \omega_n) = \int_{-\infty}^{\infty} \frac{\Phi(\mathbf{q}, \omega) (1 - e^{-\omega\beta})}{\omega - i\omega_n} d\omega, \quad \omega_n = 2n\pi/\beta. \quad (4.14)$$

As a function of the variable $i\omega_n$ the integral (4.14) defines a function $\mathcal{K}(\mathbf{q}, \omega)$ which is analytic in the upper half-plane,

$$\begin{aligned} \mathcal{K}(\mathbf{q}, \omega) &= \int_{-\infty}^{\infty} \frac{\Phi(\mathbf{q}, \omega') (1 - e^{-\omega'\beta})}{\omega' - \omega} d\omega' \\ &= \int_{-\infty}^{\infty} \frac{\Phi(\mathbf{q}, \omega') (1 - e^{-\omega'\beta})}{\omega' - \omega} d\omega' + i\pi \Phi(\mathbf{q}, \omega) (1 - e^{-\omega\beta}), \end{aligned} \quad (4.15)$$

and is the analytic continuation of the function $K(\mathbf{q}, -i(\omega_n))$ into the upper half-plane [cf. also Eqs. (3.20), (3.21)]

$$\mathcal{K}(\mathbf{q}, \omega) = K(\mathbf{q}, -i\omega). \quad (4.16)$$

Since $T_{\nu\mu}\gamma_{\mu\alpha\beta}\gamma_{\nu\alpha'}\beta'\Phi_{\alpha\beta\alpha'\beta'}(\mathbf{q}, \omega)$ is a real function of ω on the real axis, we get from Eqs. (4.15) and (4.16)

$$T_{\nu\mu}\gamma_{\mu\alpha\beta}\gamma_{\nu\alpha'}\beta'\Phi_{\alpha\beta\alpha'\beta'}(\mathbf{q}, \omega) = \frac{\text{Im}(T_{\nu\mu}\gamma_{\mu\alpha\beta}\gamma_{\nu\alpha'}\beta'K_{\alpha\beta\alpha'\beta'}(\mathbf{q}, -i\omega))}{\pi(1-e^{-\omega\beta})}. \quad (4.17)$$

Thus the problem reduces to that of finding the function

$$\gamma_{\mu\alpha\beta}\gamma_{\nu\alpha'}\beta'K_{\alpha\beta\alpha'\beta'}(\mathbf{q}, \omega_n),$$

since by the formal interchange $\omega_n \rightarrow i\omega_n$ we are enabled to determine $\Phi(\mathbf{q}, \omega)$ at once from Eq. (4.17). Elementary calculations lead to the following relation:

$$\begin{aligned} e^2\gamma_{\mu\alpha\beta}\gamma_{\nu\alpha'}\beta' \int \mathcal{L}_\nu(x-x')K_{\alpha\beta\alpha'\beta'}(x-x')d^4x' \\ = - \int \Pi_{\mu\nu'}(x-x')\mathcal{L}_{\nu'\nu}(x'-x)d^4x', \end{aligned} \quad (4.18)$$

or in the \mathbf{p} representation

$$\gamma_{\mu\alpha\beta}\gamma_{\nu\alpha'}\beta'K_{\alpha\beta\alpha'\beta'}(\mathbf{q}, \omega_n) = - \frac{\Pi_{\mu\nu'}(\mathbf{q}, \omega_n)\mathcal{Z}_{\nu'\nu}(\mathbf{q}, \omega_n)}{e^2\mathcal{Z}_0(\mathbf{q}, \omega_n)}, \quad (4.19)$$

where the polarization operator and the thermodynamic Green's functions are defined in Eqs. (2.44) – (2.52). We finally get for the probability dW of Eq. (4.10) the following expression:

$$\begin{aligned} dW = \frac{e'^2 \text{Im}[T_{\nu\mu}\mathcal{Z}^{-1}_{0\nu\nu'}(\mathbf{q}, -i\omega)\Pi_{\nu'\mu'}(\mathbf{q}, -i\omega)\mathcal{Z}_{\mu'\nu}(\mathbf{q}, -i\omega)]}{4\pi^3(\omega^2 - q^2)^2(e^{-\omega\beta} - 1)} d^3q \\ \equiv W_q d^3q, \end{aligned} \quad (4.20)$$

and the energy lost per unit time by the particle passing through the plasma is given by

$$-\frac{d\varepsilon_p}{dt} = \int (\varepsilon_p - \varepsilon_{p-q}) W_q d^3q. \quad (4.21)$$

Using Eq. (2.51) and neglecting terms of higher order in e^2 [cf. Eq. (2.59)], we get for W_q for arbitrary speed of the external particle and arbitrary temperature of the plasma

$$W_q = \frac{e'^2}{4\pi^3(\omega^2 - q^2)^2(e^{-\omega\beta} - 1)} \text{Im} \frac{T_{\nu\mu}\Pi_{\nu'\mu}(\mathbf{q}, -i\omega)}{1 + k^{-2}\Pi_{\nu'\nu'}(\mathbf{q}, -i\omega)}, \quad (4.22)$$

where the term $k^{-2}\Pi^1 \sim e^2$ has been kept in the denominator in order to avoid an infrared divergence in the calculation of the energy loss.

Let us study in more detail the simple case of a nonrelativistic particle being stopped in a nonrelativistic plasma. Here the expression for the energy loss to the electron gas coincides with the result of Larkin³¹

$$-\frac{d\varepsilon_p}{dt} = \frac{e'^2}{2\pi^2} \int_0^\infty dq \int_{-1}^1 dx \frac{\omega}{e^{-\omega\beta} - 1} \text{Im} \frac{\Pi(\mathbf{q}, -i\omega)}{q^2 - \Pi(\mathbf{q}, -i\omega)}, \quad (4.23)$$

where $\mathbf{x} = \mathbf{v} \cdot \mathbf{q}/vq$, $\omega = \varepsilon_p - \varepsilon_{p-q} = vqx - q^2/2M$, and

v is the speed of the particle moving through the plasma.

Let us take for the polarization operator Π the first nonvanishing approximation (2.72),

$$\Pi(\mathbf{q}, -i\omega) = \frac{2e^2}{(2\pi)^3} \int \frac{n_{\mathbf{p}+q/2} - n_{\mathbf{p}-q/2}}{pq/m - \omega} d^3p, \quad (4.24)$$

where $n_{\mathbf{p}}$ is the distribution function of the electrons in the nonrelativistic region, Eq. (1.47). At high temperature $n_{\mathbf{p}} = \exp(\mu - \varepsilon_{\mathbf{p}})\beta$, $\varepsilon_{\mathbf{p}} = p^2/2m$, and therefore the imaginary part of Π is

$$\text{Im} \Pi(\mathbf{q}, -i\omega) = n \frac{V\sqrt{2\pi m\beta}e^2}{2q} (e^{-\omega\beta} - 1) e^{-\frac{m\beta}{2}(\frac{\omega}{q} - \frac{q}{2m})^2}, \quad (4.25)$$

where n is the density of electrons in the plasma.

Suppose the external particle moves with a speed v much larger than the mean thermal speed of the electrons, $v \gg (\beta m)^{-1/2}$. We break up the integral over q in Eq. (4.23) into two ranges $q \geq q'$ and $q \leq q'$, with a value of q' determined from the condition $m/\beta \gg q'^2 \gg \kappa^2$, where $\kappa^2 = -\Pi(0, 0) = \beta n e^2$ is the square of the reciprocal Debye radius (Larkin³¹). Since $\Pi(\mathbf{q}, -i\omega)$ does not exceed κ in order of magnitude, in integrating over the first range we can neglect the term Π in comparison with q^2 ; then

$$\begin{aligned} \left(-\frac{d\varepsilon_p}{dt}\right)_1 = \frac{e'^2 e^2 n \sqrt{2\pi m\beta}}{4\pi^2} \\ \times \int_{q'}^\infty dq \int_{-1}^1 dx \frac{vx - q/2M}{q^2} e^{-\frac{m\beta}{2}(vx - q\frac{M+m}{2mM})^2}. \end{aligned} \quad (4.26)$$

If in Eq. (4.26) we neglect terms of the order $v^{-1}(\beta m)^{-1/2}$, then

$$\left(-\frac{d\varepsilon_p}{dt}\right)_1 = \frac{ne^2 e'^2}{4\pi m v} \ln \frac{2mMv}{q'(M+m)}. \quad (4.27)$$

In the second range $q^2 \leq q'^2 \ll m/\beta$, and therefore in ω we can neglect the term $q^2/2M$ in comparison with mvx . This enables us to change from integration over x to integration over $\omega = vqx$, and

$$\left(-\frac{d\varepsilon_p}{dt}\right)_2 = \frac{e'^2}{2\pi^2} \int_0^{q'} \frac{dq}{vq} \text{Im} \int_{-vq}^{vq} \frac{\omega d\omega}{e^{-\beta\omega} - 1} \frac{\Pi(\mathbf{q}, -i\omega)}{q^2 - \Pi(\mathbf{q}, -i\omega)}. \quad (4.28)$$

The maximum value of $\omega\beta$ in Eq. (4.28) is $vq\beta$. Let us choose q' so that the quantity $\omega\beta$ does not exceed unity, i.e., $q' = 1/v\beta$. Then q' still satisfies our former conditions $q'^2 \ll m/\beta$ if the speed of the external particle is large compared with the thermal velocity. With this choice of q' the integrand in Eq. (4.28) is analytic inside a circle of radius vq . Let us displace the path to the upper arc; then $q^2 \ll m\beta\omega^2$, and consequently we can use for $\Pi(\mathbf{q}, -i\omega)$ the expression

$$\Pi(\mathbf{q}, -i\omega) = \frac{e^2 n}{m} \frac{q^2}{\omega^2} + O\left(\frac{q^2}{m\beta\omega^2}\right). \quad (4.29)$$

The integral along the upper arc then takes the form

$$\int_{-vq}^{vq} \frac{d\omega}{e^{-\omega\beta} - 1} \frac{\omega}{\omega^2 - \omega_0^2}, \quad (4.30)$$

where $\omega_0 = (ne^2/m)^{1/2}$ is the Langmuir frequency. If we bring the path of integration back down to the real axis, the integral (4.30) is found to be zero for $vq < \omega_0$, and for $vq > \omega_0$ it is equal to half the sum of the residues at the points $\pm\omega_0$, which gives for the integral (4.30) the value $\pi ne^2/2m$. Finally, the integral (4.23) over the second range $q \leq q'$ is given by

$$\left(-\frac{d\epsilon_p}{dt}\right)_2 = \frac{ne^2 e'^2}{4\pi m v} \ln \frac{q'v}{\omega_0}. \quad (4.31)$$

In the nonrelativistic region the total energy loss in the slowing down of the particle is given by the sum of Eqs. (4.27) and (4.31):

$$-\frac{d\epsilon_p}{dt} = \frac{ne^2 e'^2}{4\pi m v} \ln \frac{2Mm^{3/2} v}{(M+m)\sqrt{ne^2}}. \quad (4.32)$$

The expression (4.32) gives the total energy loss that comes from two-particle collisions in which the momentum transferred is much larger than the reciprocal of the Debye radius, and also from the excitation of plasma waves. One of these kinds of loss can be separated from the other only with logarithmic accuracy. The energy loss by excitation of plasma waves is determined by the pole of the integrand in Eq. (4.28). This pole also gives the spectrum of the plasma oscillations.

In reference 31 the formula (4.23) is also used as the basis for a detailed study of other limiting cases of the stopping of a nonrelativistic particle, for example the cases of low temperature, of small speed of the external particle in comparison with the mean thermal velocity of the plasma electrons, and so on.

APPENDIX 1

FUNCTIONAL DIFFERENTIATION

Let there be given a class of functions $y(x)$ defined on the interval $[a, b]$. Then $I(y)$ is a functional of the functions $y(x)$ if to each function of the given class there corresponds a certain numerical value of $I(y)$. A general form of a functional $I(y)$ can be written as follows:

$$I(y) = \int_a^b F(x, y, \dot{y}) dx, \quad \dot{y} = \frac{dy}{dx}. \quad (1)$$

For example, the length of a curved line is a functional of its shape; in this case $F(x, y, \dot{y}) = (1 + \dot{y}^2)^{1/2}$, where $y(x)$ is the given curve.

Together with the function $y(x)$ let us consider an infinitesimally different function $\bar{y}(x)$, which coincides with $y(x)$ at the ends of the interval. The difference $\bar{y}(x) - y(x) \equiv \delta y(x)$ is called the variation of the function $y(x)$ at the point x . Thus the variation $\delta y(x)$

is the increment of the function $y(x)$ caused by a change of the form of the function itself. Similarly, the variation δI of the functional is the main linear part of the increment of the functional $I(y)$ caused by a change of the form of the function $y(x)$:

$$\delta I \equiv I(y + \delta y) - I(y) = \int_a^b \left(F_y - \frac{d}{dx} F_{\dot{y}} \right) \delta y(x) dx, \quad (2)$$

where F_y and $F_{\dot{y}}$ are the partial derivatives with respect to y and \dot{y} .

The concept of the functional can be regarded as a generalization of a function of many variables. In fact, let us replace the integral (1) by the integral sum

$$I(y) = \sum_{h=1}^{\infty} F(x_h, y(x_h), \dot{y}(x_h)) \Delta x_h. \quad (3)$$

Then $I(y)$ will be a function of the "variables" $y(x_1), y(x_2), \dots$, since for fixed values of x_1, x_2, \dots a change of the form of the function $y(x)$ leads to changes of the numerical values of the entire set $y(x_1), y(x_2), \dots$. This analogy suggests how we can introduce the concept of the variational (functional) derivative. Whereas the total differential df of a function of many variables $f(x_1, x_2, \dots, x_n)$ is equal to the sum of the partial derivatives multiplied by the increments of the variables,

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i, \quad (4)$$

the variation δI of the functional $I(y)$ is equal to the integral of the variational derivative $\frac{\delta I}{\delta y(x)}$ (taken at the point x) multiplied by the increment $\delta y(x)$ of the function:

$$\delta I = \int_a^b \frac{\delta I}{\delta y(x)} \delta y(x) dx. \quad (5)$$

By comparing (5) and (2) we find that the variational derivative of $I(y)$ with respect to the function y taken at the point x is by definition equal to

$$\frac{\delta I}{\delta y(x)} = F_y(x, y, \dot{y}) - \frac{d}{dx} F_{\dot{y}}(x, y, \dot{y}). \quad (6)$$

The concept of the variational derivative can also be approached in a somewhat different way. To each form of the function $y(x)$ there corresponds the area bounded by the axis Ox and the curve $y(x)$. To a change of the form of the function $y(x)$ there corresponds a change of the area by the amount

$$\sigma = \int_a^b \delta y(x) dx. \quad (7)$$

Suppose the change of form of the function $y(x)$ has occurred only in an infinitely small neighborhood of the point x_0 , for example $\delta y(x) = \epsilon \delta(x - x_0)$. Then in generalizing the operation of partial differentiation to the case of functionals it is natural to give the name

of the variational derivative at the point x_0 to the limit $\lim \{ [I(y + \delta y) - I(y)] / \sigma \}$ as the change of area σ contracts to the point x_0 :

$$\frac{\delta I}{\delta y(x_0)} = \lim_{\sigma \rightarrow 0} \frac{I(y + \delta y) - I(y)}{\sigma} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_a^b \left(F_y - \frac{d}{dx} F'_y \right) \varepsilon \delta(x - x_0) dx$$

$$= \left(F_y - \frac{d}{dx} F'_y \right)_{x=x_0}. \quad (8)$$

We can regard the function $y(x)$ at a fixed argument $x = x'$ as a special case of a functional, $I(y) \equiv y(x')$:

$$y(x') = \int_a^b \delta(x' - x) y(x) dx. \quad (9)$$

Then according to Eqs. (5) and (2) we have

$$\frac{\delta y(x')}{\delta y(x'')} = \delta(x' - x''). \quad (10)$$

If I is a functional of a function $B(x)$ and $B(x)$ for fixed x is a functional of the function $b(y)$, then I is also a functional of $b(y)$, and the variation of the functional I caused by a change of the form of the function $b(y)$,

$$\delta I = \int \frac{\delta I}{\delta b(y)} \delta b(y) dy, \quad (11)$$

can also be written in the form

$$\delta I = \int \frac{\delta I}{\delta B(x)} \delta B(x) = \int \frac{\delta I}{\delta B(x)} \frac{\delta B(x)}{\delta b(y)} dx \delta b(y) dy, \quad (12)$$

where we have used the relation

$$\delta B(x) = \int \frac{\delta B(x)}{\delta b(y)} \delta b(y) dy. \quad (13)$$

From a comparison of (11) and (12) we find

$$\frac{\delta I}{\delta b(y)} = \int \frac{\delta I}{\delta B(x)} \frac{\delta B(x)}{\delta b(y)} dx. \quad (14)$$

If $I(\Phi)$ is a functional of operators $\Phi(x)$, we define the variational derivative in the following way:

$$\delta I = \int \delta \psi(x) \frac{\delta I}{\delta \psi(x)} dx. \quad (15)$$

Let us consider two functionals I_1 and I_2 of the operators $\Phi(x)$. Suppose $[\delta \Phi, I_1]_{\pm} = 0$. Then the variational derivative of $(I_1 I_2)$,

$$\delta(I_1 I_2) = \int \delta \Phi(x) \frac{\delta(I_1 I_2)}{\delta \Phi(x)} dx \quad (16)$$

can also be written in the following way:

$$\delta(I_1 I_2) = I_1(\Phi + \delta \Phi) I_2(\Phi + \delta \Phi) - I_1(\Phi) I_2(\Phi) = (\delta I_1) I_2 + I_1 \delta I_2$$

$$= \int \delta \Phi(x) \left(\frac{\delta I_1}{\delta \Phi(x)} I_2 \mp I_1 \frac{\delta I_2}{\delta \Phi(x)} \right) dx. \quad (17)$$

From a comparison of (16) and (17) we have

$$\frac{\delta}{\delta \Phi(x)} I_1 I_2 = \frac{\delta I_1}{\delta \Phi(x)} I_2 \mp I_1 \frac{\delta I_2}{\delta \Phi(x)}, \quad (18)$$

or

$$\left[\frac{\delta}{\delta \Phi(x)}, I_1 \right]_{\pm} = \frac{\delta I_1}{\delta \Phi(x)}. \quad (19)$$

This last relation is to be understood in the sense that the result of acting on an arbitrary functional with the left and right members of Eq. (19) is the same expression. Setting $\Phi = \psi(x)$, $I_1 = \psi(y)$, or $\Phi = \psi(x)$, $I_1 = \psi^*(y)$, or $\Phi = \varphi(x)$, $I_1 = \varphi(y)$, and so on, we get the formulas (2.8) and (2.9).

APPENDIX 2

THE SUMMATION OF SERIES ENCOUNTERED IN PROBLEMS OF STATISTICAL PHYSICS BY THE USE OF METHODS OF QUANTUM FIELD THEORY

In using quantum-field-theory methods for the purposes of statistical physics one frequently has to sum series over discrete values of the fourth component of the four-dimensional momentum vector. It turns out that one can establish certain general rules and formulas for such summations, which are extremely useful in applications.

For example, by performing the Fourier transformation (1.60) on the zeroth-order phonon thermodynamic Green's function $\mathcal{Z}_0(\mathbf{x}, \tau)$, Eqs. (1.37) and (1.49), and using Eq. (1.65) we get for $\mathcal{Z}_0(\mathbf{p}, \tau)$

$$\frac{1}{\beta} \sum_{\omega_n} \frac{\omega_k^2}{\omega_k^2 + \omega_n^2} e^{-i\omega_n \tau} = \frac{\omega_k}{2} [(1 + n_k) e^{-\omega_k |\tau|} + n_k e^{\omega_k |\tau|}]. \quad (I)$$

If we take the limit $\tau \rightarrow 0$ in both members of Eq. (I), we get

$$\frac{\omega_k}{\beta} \sum_{\omega_n} \frac{1}{\omega_k^2 + \omega_n^2} = \frac{1}{2} + n_k, \quad (II)$$

where $\omega_n = 2n\pi/\beta$, n_k is defined in Eq. (1.47), and ω_k is an arbitrary positive quantity $\omega_k > 0$ of the dimensions of energy.

Similarly, performing the Fourier transformation on the zeroth-order electron thermodynamic Green's function $\mathcal{Z}_0(\mathbf{x}, \tau)$, Eqs. (1.36) and (1.48), and using Eq. (1.64), we get for $\mathcal{Z}_0(\mathbf{p}, \tau)$

$$\frac{1}{\beta} \sum_{p_4} \frac{-\delta_{\alpha\beta}}{i p_4 + \mu - \epsilon_p} e^{-i p_4 \tau} = \begin{cases} (1 - n_p) \delta_{\alpha\beta} e^{-(\epsilon_p - \mu)\tau}, & \tau > 0, \\ -n_p \delta_{\alpha\beta} e^{-(\epsilon_p - \mu)\tau}, & \tau < 0. \end{cases} \quad (III)$$

Since $\mathcal{Z}_0(\mathbf{p}, \tau)$ has a discontinuity at the point $\tau = 0$, for $\tau \rightarrow 0$ we must use in Eq. (III) the half-sum $[\mathcal{Z}_0(\mathbf{p}, \tau + 0) + \mathcal{Z}_0(\mathbf{p}, \tau - 0)]/2$, which is equivalent to multiplying both members of Eq. (III) by $\delta(\tau)$ and then integrating over τ . The result is

$$\frac{\epsilon_p - \mu}{\beta} \sum_{p_4} \frac{1}{p_4^2 + (\epsilon_p - \mu)^2} = \frac{1}{2} - n_p, \quad (IV)$$

where $p_4 = (2n + 1)\pi/\beta$, $\epsilon_p > 0$, and n_p is defined in Eq. (1.47).

We also note that an analogous treatment of the boson Green's function of Eq. (1.83) and (1.87) leads to the following extension of Eq. (II) to the case $\mu \neq 0$:

$$\frac{\epsilon_p - \mu}{\beta} \sum_{p_4} \frac{1}{p_4^2 + (\epsilon_p - \mu)^2} = \frac{1}{2} + n_p, \quad (V)$$

where $p_4 = 2n\pi/\beta$, $\epsilon_p > 0$, and n_p is given by Eq. (1.83).

Repeating these arguments for the relativistic electron thermodynamic Green's function, Eq. (2.27), we get

$$\frac{1}{2\beta} \sum_{p_4} \frac{-i\mathbf{p}\mathbf{Y} + (ip_4 + \mu)\mathbf{Y}_4 + m}{(p_4 - i\mu)^2 + \epsilon_p^2} = \frac{1}{2\epsilon_p} \left[\left(\frac{1}{2} - n_p^- \right) (m + \mathbf{Y}_4 \epsilon_p - i\mathbf{Y}\mathbf{p}) + \left(\frac{1}{2} - n_p^+ \right) (m - \mathbf{Y}_4 \epsilon_p - i\mathbf{Y}\mathbf{p}) \right]. \quad (VI)$$

Let us take the trace of both members of Eq. (VI):

$$\frac{2\epsilon_p}{\beta} \sum_{p_4} \frac{1}{(p_4 - i\mu)^2 + \epsilon_p^2} = 1 - n_p^- - n_p^+,$$

where $p_4 = (2n + 1)\pi/\beta$, $\epsilon_p > 0$, and n_p^- and n_p^+ are defined in Eq. (2.30).

This device can also be extended without much difficulty to other cases of summation. The results are general formulas of more complicated forms [cf., e.g., Eqs. (1.66) and (1.68)].

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