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PROPERTIES OF THE QUANTIZED VORTICES GENERATED BY THE ROTATION OF HELIUM II

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1. INTRODUCTION

1.1. In this review we present an account of the experimental and theoretical research on the hydrodynamics of oscillating solids immersed in rotating helium II. The progress achieved of late in this direction is quite significant with respect to the present day status of the superfluidity problem. A successful solution of this problem has made it possible to advance, within less than a decade, from an aggregate of contradictory facts concerning the rotation of helium II to a deep insight into the physical nature of the observed phenomena and to the development of a quantum theory of vortical motion of a superfluid liquid.

1.2 Supercritical phenomena connected with the rotation of helium II were noted first by Kapitza,[1] who observed the transfer of heat along a capillary within which a glass rod was placed. Rotation of the rod inside the stationary capillary led immediately to a sharp reduction in heat transfer. The properties of rotating helium II were subsequently investigated by Osborne and, independently, by Andronikashvili and Kaverkin,[3] who observed visually the form of the meniscus of the liquid helium contained in a transparent vessel and set in motion by rotation of the latter. The visual observations allowed this investigation to be conducted only at speeds in excess of 6 sec^{-1}.

If, as would follow from the Landau theory (see reference 4, p. 610), the superfluid component were to be at rest in these experiments with only the normal component of helium II participating in the rotation, then the depth of the meniscus would be

$$h=\frac{\varrho_n}{\varrho}\frac{\omega_0^2R^2}{2g}, \tag{1.1}$$

where ρ_n is the density of the normal component, ρ the total density of helium II, ω_0 the angular velocity of rotation, R the radius of the cylindrical vessel, and g the acceleration due to gravity.

In actuality, the depth of the meniscus did not differ, over in the entire range of speeds investigated, from six to 40 sec^{-1}, from that for a classical liquid, and satisfied the expression

$$h=\frac{\omega_0^2R^2}{2g}. \tag{1.2}$$

This means, therefore, that at these angular velocities the helium II moves as a whole. This situation is to be explained on the hypothesis that for sufficiently high rotational velocities the superfluid component is drawn into motion along with the vessel, either by the latter directly, or as a consequence of the passage of the relative velocity of motion of the superfluid and normal (entrained by the beaker) component through a critical value corresponding to the onset of interaction between the two.

One might think that the effect observed is due to the loss by the helium of its superfluid properties. However, such typically superfluid phenomena as the thermomechanical effect, which was investigated in rotating helium II by Andronikashvili and Kaverkin,[3] as well as the propagation of second sound under the same conditions, observed subsequently by Hall and Vinen[5] and by Lane's group,[6] are preserved at all speeds of rotation.

The experiments of Osborne[2] and of Andronikashvili and Kaverkin[3] have demonstrated that to rotating helium II there must be attributed special physical properties. Attempts along these lines were undertaken in many theoretical papers,[7,8,9] but a correct explanation of these experiments was obtained only in conjunction with the hypothesis, proposed by Onsager[10] and independently by Feynman,[11] that quantized vortex filaments can be produced in helium II.

The principal premises of the Feynman theory, as well as the results of the most important experimental investigations, which have confirmed this theory and which in turn have stimulated its further development, will be detailed later on.

1.3. The Onsager-Feynman hypothesis is that in the superfluid component of helium II there can arise vortex filaments, the circulation Γ of which is an integral multiple of $2\pi\hbar/m$ and is expressed by

$$\Gamma = \oint v_{sl}\, dl = n \frac{2\pi\hbar}{m} \qquad (n = 1, 2, \ldots), \qquad (1.3)$$

where v_{sl} is the projection of the superfluid component velocity along the tangent to the contour of integration, while m is the mass of the helium atom.

Each vortex has a definite energy ϵ, amounting to

$$\varepsilon = n^2 \pi \varrho_s \frac{\hbar^2}{m^2} \ln \frac{b}{a_0} \qquad (n = 1, 2, \ldots) \qquad (1.4)$$

per unit length of the vortex. Here a_0 is the radius of the core of the vortex (on the order of interatomic distances) and b is the effective radius of the vortex, i.e., the effective radius of the region in which the superfluid component of helium II executes potential rotation about a given vortex with a velocity inversely proportional to the distance r to its axis:*

$$v_s = \frac{\Gamma}{2\pi r} = n \frac{\hbar}{mr} \qquad (n = 1, 2, \ldots). \qquad (1.5)$$

Since the energy of the vortex filament is proportional to its length ($E = \epsilon l$), the vortex resists the force that tends to stretch it and, as can be readily seen, possesses a tension ϵ, which coincides numerically with its energy per unit length ($\partial E/\partial l = \epsilon$).

Feynman's ideas concerning the possibility of formation of vortex filaments of the type just described in the superfluid component of helium II have contributed to the understanding of previously unexplained peculiarities of critical phenomena. It was ascertained that in the supercritical mode a superfluid liquid is somewhat like a turbulized ideal liquid. By the same token it was shown that the low experimental values obtained for the critical velocity (0.1 — 100 cm/sec) do not by any means contradict the Landau theory,[12] which predicts destruction of superfluidity at velocities on the order of 70 m/sec. At the observed critical velocities, the superfluid component interacts with solids and with the normal component of helium II via the vortex formation which commences at these velocities, and not through loss of its capacity to undergo non-viscous motion. However, these aspects of the Onsager-Feynman theory are beyond the scope of the present survey, and we shall consequently confine ourselves solely to the application of the theory to the rotation of helium II.

1.4. By solving the corresponding variational problem it can be shown that in a liquid situated in a vessel which rotates uniformly with an angular velocity ω_0, the velocity distribution

$$v = \omega_0 r, \qquad (1.6)$$

which corresponds to rotation of the liquid as a whole, is the most desirable from the thermodynamic point of view.

Rotation of this kind is ensured in solids and in viscous liquids by an interatomic or viscous interaction, but it is very difficult to visualize any mechanism which could set into motion of this sort a superfluid liquid, the individual layers of which do not interact with each other.

According to Feynman,[11] the superfluid component of helium II is entrained by the rotating vessel through a specific mechanism, wherein a system of quantized vortex filaments, such as described above, is generated in the liquid. It should be noted that the question of how the widely-known requirements of the classical theorems of Helmholtz, Thomson, and Lagrange, forbidding vortex formation in classical ideal fluids, are evaded, still remains open. Bypassing this problem, however, Feynman[13] provided a quantum-mechanical description of the possibility of vortex formation in a superfluid liquid. On the other hand, he showed,[11] that a specific system of vortex filaments, which will be described presently, will establish in a superfluid liquid a velocity distribution very close to the thermodynamically desirable distribution (1.6).

Feynman proposed that superfluid helium situated in a vessel rotating with an angular velocity ω_0 is permeated by a system of uniformly distributed vortex filaments parallel to the axis of rotation. The number of such vortices per unit cross-sectional area perpendicular to this axis is given by

$$N = \frac{m\omega_0}{n\pi\hbar} \qquad (n = 1, 2, \ldots). \qquad (1.7)$$

The interaction of these vortex filaments with each other causes them to move together with the vessel at a velocity v_L, given by

$$v_L = \omega_0 r. \qquad (1.6')$$

The array of vortices just described produces in the superfluid component of helium II a rather complicated distribution of velocities, as shown schematically in Fig. 1. Examining this figure, one can readily see that the average velocity of the superfluid component $\bar{v}_s$ is

$$\bar{v}_s = \omega_0 r \qquad (1.6'')$$

(like the velocity of the vortex filaments and the velocity of the normal component). The distribution of velocities in the superfluid component rotated by the quantized vortices is found to be the closer to the thermodynamically most desirable distribution, the greater the number of vortex filaments extending through the rotating liquid. In this connection, in using formulas (1.3) — (1.5) and (1.7), we shall henceforth ascribe to the quantum number n its most probable minimum value, n = 1 (see references 11 and 14).

As regards the conditions under which vortex filaments begin to form in helium II in a rotating vessel, these are determined from the condition that the free

*It is interesting to note that when r reaches a value on the order of interatomic distances, v_s increases to about 70 m/sec, whereupon, according to the Landau theory, destruction of superfluidity should occur.

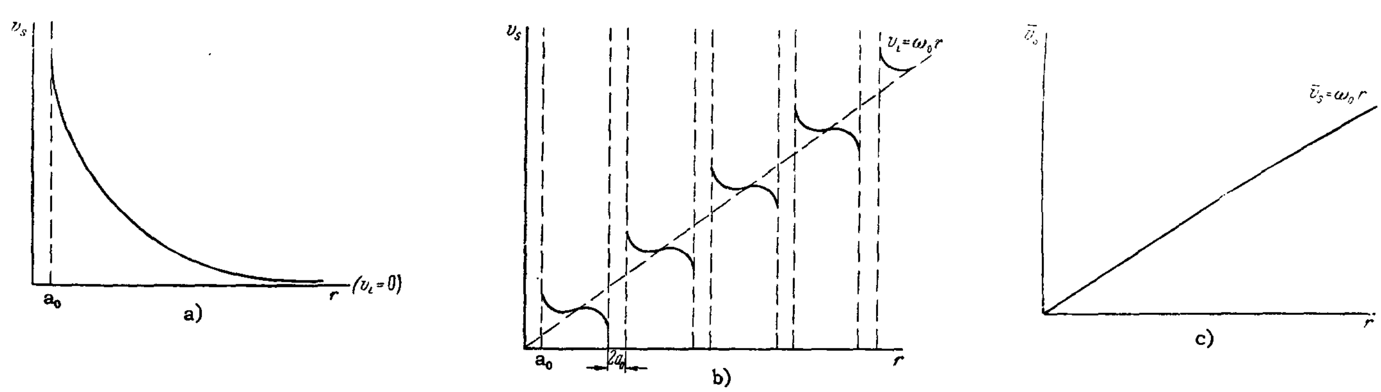

FIG. 1. Distribution of velocities in a superfluid liquid moved by vortices. a) Single (stationary) vortex, --- a_0; b) aggregate of moving vortices; c) result of averaging the velocity distribution shown in the preceding figure.

energy of the liquid must be a minimum. This leads to the following formula[15] for the critical angular velocity for vortex formation $\omega_{0\,cr}$:

$$\omega_{0cr} = \frac{\hbar}{m(R^2 - a^2)} \ln \frac{R}{a_0}, \qquad (1.8)$$

where R is the radius of the cylindrical vessel. Using the last formula, we can readily see that even for relatively small vessels, on the order of one centimeter in radius, the critical velocity is exceedingly small ($\omega_{0\,cr} \sim 10^{-3}$ sec^{-1}). Consequently, the rotation of helium II under normal conditions is always supercritical, and the liquid contains enough vortices to cause rotation of the superfluid component in accordance with (1.6″).

It should be noted that the derivation of (1.6″) is one of the simplest examples of the expedient, widely used in the hydrodynamics of rotating helium II, of averaging-out the details of a complicated pattern of motion. This makes it possible to avoid second-order details, and to describe, using averaged quantities, only the basic aspects of the phenomena under consideration.

1.5. In the preceding section we have shown how potential rotation about individual vortices [formula (1.5)] leads on the average to a vortical rotation [formula (1.6″)] with non-vanishing average curl of the superfluid component velocity $\bar{\omega}$:

$$\bar{\omega} = 2\omega_0. \qquad (1.9)$$

Inasmuch as a noticeable deviation from (1.6″) is observed only in micro-regions surrounding the cores of individual vortices, macroscopic observation of the form of the meniscus naturally produces the impression of rotation of the liquid as a whole.

On the other hand, there is a quite distinct difference between the character of the rotation of the superfluid component and the rotation of a classical liquid under similar conditions. In the case of a classical liquid we deal with a dense (continuous) vortex layer, for which Eq. (1.6) is satisfied with absolute rigor, while formula (1.9) is valid for the curl of the velocity curl $\mathbf{v} = \boldsymbol{\omega}$ at any point of the volume occupied by the liquid, and not for its average value $\bar{\omega}$. In the case of a superfluid liquid, however, the circulation is distributed among the individual quantized vortices, forming a discrete array. The energy of these vortices is localized in individual regions associated with the singularities in the velocity microdistribution of the superfluid component.

It would be natural to expect these singularities in the rotation of the superfluid liquid to be detected by a suitably arranged experiment; the ideas following from Feynman's theory have received brilliant confirmation in a series of experiments described in the survey (references 14 and 16, and especially 17) and by Vinen.[15]

Investigating the propagation of second sound in a direction parallel to the rotational axis of the resonator (i.e., along the direction of the vortex filaments), Hall and Vinen[17] established the virtual absence of any excess attenuation of the second-sound waves due to the rotation, while propagation in the transverse direction led to a considerable increase in attenuation as compared with stationary helium II. Here the excess attenuation, as expected from the Feynman theory, depends on the angular velocity of rotation; i.e., on the vortex density. The authors have also shown that the mutual friction between the normal and superfluid components is due to scattering of the helium II thermal excitation quanta by the vortices, and have determined the values of the coefficients of mutual friction along the three axes of the cylindrical coordinate system. Hall and Vinen were able to derive from their experimental data information on the value of the elementary circulation, which was found to be unity in units of $2\pi\hbar/m$.

A special experiment, aimed at measuring a single quantum of circulation, was undertaken by Vinen,[15] who proved in the most direct fashion that its value is indeed $2\pi\hbar/m$.

Since it is not the purpose of the present article to describe all the investigations connected with vortex production in helium II, the foregoing is fully adequate for an understanding of the nature of the phenomena associated with the oscillation of bodies immersed in rotating helium II, and for relating it to the properties of the Onsager-Feynman vortex system.

2. OSCILLATIONS OF SOLID BODIES IN ROTATING HELIUM II

2.1. In the experiments which we shall now describe, solid bodies oscillating in rotating helium II had the same function as did second sound in the investigations of Hall and Vinen,[17] described above; they played the role of special "probes," with which the peculiar features of vortical rotation of the superfluid component detailed in the preceding chapter were investigated.

2.2. Among these peculiarities the chief is the presence in the rotating helium of a system of vortices, the energy properties of which render each vortex similar to a stretched string (see Sec. 1.3). In this connection, the deformation of a vortex filament, or of a system of such filaments, should lead to the development of elastic forces within the rotating helium II. It is obvious that these elastic forces should manifest themselves most prominently in oscillatory experiments, in which they can cause a reduction in the period of oscillations of a body immersed in rotating helium II. For this purpose it is clearly necessary that the surface of the body intersect the vortex filaments and that their ends be attached (at least partially) to the vibrating surface.

It is appropriate in this connection to consider the conditions under which the ends of the vortex filaments are secured to a solid surface. It is essential to note that although the ends of the vortices are indeed attached to the surface by "suction" with a force ϵ, this does not by any means require that the vortex must follow all of its displacements.* If this displacement is perpendicular to the vortex direction, then in order to follow the surface the vortex would have to become longer, which is not economical, energetically speaking. Therefore the vortices can slip along the surface (if the end of the vortex filament follows the moving surface, the tension of the deflected vortex restores it to its initial position).

Full slippage, however, is possible only for the case of an absolutely smooth surface. For a rough surface, it is especially disadvantageous for the vortices to slip off the projecting regions, which thus localize the ends of the vortex filaments. In this case a surface displacement perpendicular to the vortices causes the vortex filaments to bend and their tension acquires a component tangent to the plane containing the irregularities. It is precisely this component of the vortex tension which causes their elastic properties to manifest themselves in the laws that govern the oscillations of solids of suitable shape immersed in rotating helium II.

2.3. That rotating helium II has the expected elastic properties was directly demonstrated in the experiments of Andronikashvili and Tsakadze,[18] who observed the oscillations of a stack of thin discs, superimposed on a rotation in synchronism with that of the liquid. The distance between discs was uniform and equal to 0.2 mm.

By showing that the rotation of helium II can lead to a reduction in the period of the oscillations of the stack, Andronikashvili and Tsakadze have thus demonstrated the existence of shear elasticity in rotating helium, analogous to that observed in the twisting of a solid body.

2.4. Concurrently with Andronikashvili and Tsakadze, Hall[19] carried out a similar but more detailed investigation of the oscillations of a stack of discs in rotating helium II. Hall's stack was dismountable. By varying the number of discs, he could alter the distance between them from 0.2 to seven millimeters.

Particular attention was paid to the question of the influence of surface roughness, for which purpose the experiments were carried out both with smooth discs and with discs whose surfaces were covered with sand grains of the order of $50\,\mu$ in diameter.* As expected, it was found that smooth and rough surfaces behave differently. For long periods of oscillation (25 sec), however, a smooth surface becomes similar in its properties to a rough one (cf. Sec. 2.6).

Hall expressed the results of his experiments in terms of an effective density ρ' for the superfluid component participating in the motion of the disc, determined from the experiment with the aid of the relation

$$\frac{\rho'}{\rho_s}=\frac{T_\omega^2-T_0^2}{T_2^2-T_1^2}\frac{I_2-I_1}{I_s}, \qquad (2.1)$$

where T_0 is the oscillation period in stationary helium, T_ω is the period in helium rotating with angular velocity ω_0, T_1 and T_2 are the periods corresponding to moments of inertia I_1 and I_2 at room temperature, $(I_1 - I_2)$ is the known moment of inertia, and I_s is the moment of inertia of a solid body with density ρ_s filling the space between the discs.

Hall's experimental results are illustrated in Figs. 2 and 3.

The curve shown in Fig. 2 was obtained under conditions of "rapid" rotation, for which the oscillation frequency Ω is much less than twice the angular velocity of rotation ($\Omega \ll 2\omega_0$). In this case the value of ρ'/ρ_s increases monotonically with increasing product $\sqrt{\omega_0}\,l$ ($2l$ is the distance between discs in the stack) for both smooth and rough discs, and tends asymptotically to $\rho'/\rho_s = 1$ (see Sec. 4.9).

In the opposite limiting case, that of "slow" rotation ($\Omega \gg 2\omega_0$), the period of oscillation of a stack of smooth discs is practically independent of the speed of rotation or of the distance between discs. However, in the case of rough discs, as can be seen from Fig. 3, the rotation

*Formula (1.4) for ϵ can be obtained by using Bernoulli's equation to determine and to integrate over the surface the reduction in pressure due to the rotation of the liquid about the vortex filament [formula (1.5)].

*It should be noted that the stack used by Andronikashvili and Tsakadze was not subjected to similar treatment; however, the discs employed acquired a sufficient roughness during the course of their production (by pressing).

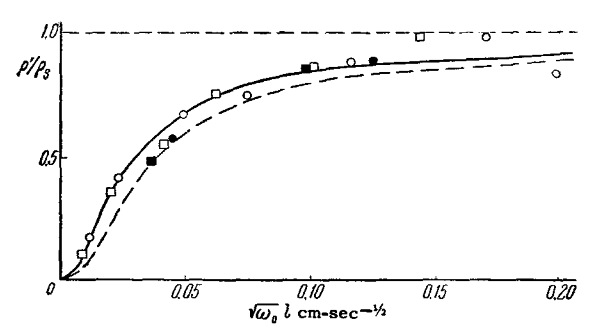

FIG. 2. Dependence of the effective density of the superfluid component on the speed of rotation and on the spacing between discs for the case $\Omega \ll 2\omega_0$ (see also caption to Fig. 3).

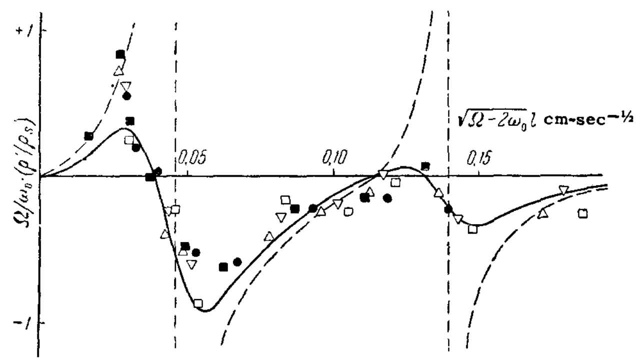

FIG. 3. Dependence of the effective density of the superfluid component on the velocity of rotation and the spacing between discs[19] in the case when $\Omega \gg 2\omega_0$. The dotted (theoretical) curve was constructed by Hall without allowance for slippage (see Sec. 4.9). The solid curve was obtained by suitable choice of the slip coefficient (see Chapters 3 and 4).

exerts quite a strong influence on the period of the oscillations.

If we take $\sqrt{\Omega - 2\omega_0}\, l$ as abscissas and $\frac{\Omega}{\omega_0}\frac{\rho'}{\rho_s}$ as ordinates, then the experimental points fit a curve similar to the dispersion curve associated with the presence of resonance effects. Here, as in the experiments of Andronikashvili and Tsakadze,[18] rotation may cause the period of oscillations not only to increase, but also to decrease.

2.5. Postponing a more detailed discussion of the theoretical problems until Chapter 4, we shall merely make a few remarks necessary for a qualitative understanding of the experimental data, as well as for a clear idea of the goals sought in the formulation of subsequent experiments.

On the basis of the results of his experiments, Hall (in the same paper[19]) developed a theory to explain the phenomenon just described by using the concept of circularly-polarized elastic transverse waves, propagating along the vortex filaments as the disc surfaces oscillate. Under conditions in which an odd number of half-waves fits between neighboring discs, the resultant resonance can lead to infinite peaks in ρ'/ρ_S. Partial slippage of the vortices, however, which takes place even in the case of a rough surface, reduces the resonance values of ρ'/ρ_S to zero (there exists here a fundamental analogy with the theory of anomalous dispersion of light).

As shown by Hall, when $2\omega_0 < \Omega$ the principal role is played by one of the waves generated by the oscillations of the discs in the superfluid component and in the vortices permeating it (see Secs. 4.3 and 4.8). This is the wave whose wave number is given by

$$k_{s0}^{(-)2} = \frac{\Omega - 2\omega_0}{\nu_s} \qquad (2.2)$$

The quantity ν_S in this formula ($\nu_S = \eta_S/\rho_S = \epsilon/\rho_S\Gamma$) is one of the principal parameters of the hydrodynamics of rotating helium (see Chapter 3). It follows from formula (2.2) that the aforementioned resonances are determined by the condition

$$\sqrt{\Omega - 2\omega_0}\, l = (2n-1)\frac{\pi}{2}\sqrt{\nu_s} \qquad (n = 1, 2, \ldots), \qquad (2.3)$$

which enabled Hall to determine, using the data given in Fig. 3, the numerical value of the parameter $\nu_S = (8.5 \pm 1.5) \times 10^{-4}\ \mathrm{cm}^2/\mathrm{sec}$.

Thus, Hall's main premises were essentially in complete agreement with those ideas regarding the elastic properties of the Onsager-Feynman vortex system which were described in the beginning of this chapter, in Section 2.2, and which served as the basis of the experiment by Andronikashvili and Tsakadze. However, without making direct use of the theory which he himself developed, Hall, placing primary emphasis on the concept of an effective density for the superfluid component participating in the oscillations, represented the reduction in the period ($T_\omega < T_0$) as the appearance of negative values of ρ', associated in his opinion with the ability of the superfluid component to move in phase opposition with the motion of the stack.

Without wishing to belittle the significance of Hall's work,[19] which was one of the first and most fundamental investigations of oscillatory phenomena in rotating helium II, we cannot ignore the shortcomings of the concept of effective density. This concept not only hinders the correct understanding of the physical nature of the phenomena under consideration, but, as will be shown in Chapter 4 (see Sec. 4.8), leads to further errors in principle.

The difference between Hall's ideas and those of the Tbilisi group, which insists that the elastic properties of the vortices are of prime importance, was emphasized in the statements made by Andronikashvili in the discussion following Hall's paper at the Fourth All-Union Conference on Low-Temperature Physics (Moscow, 1957). It was noted in this statement that information on the period of the oscillation is not sufficient to establish the complete hydrodynamic picture, but that data on the damping are also necessary.

2.6. In the cryogenic laboratory of Tbilisi University, a systematic investigation was made of the damping of the oscillations of bodies with axial symmetry immersed in rotating helium II.

The first subject of these investigations was a single "heavy" disc; i.e., a disc with sufficiently large moments of inertia so that the changes in its frequency of oscillation caused by interaction with the liquid were insignificant. A disc of this description was suspended from an elastic fiber, placed in a vessel filled with helium II, and set in rotation together with the vessel and a special measuring apparatus, while at the same time it executed axial-torsional oscillations.

The results of the very first measurements of the damping of the oscillations of the disc, carried out in 1958,[20] have disclosed a most unique feature of the dependence of the logarithmic damping decrement upon the speed of rotation; specifically, it was demonstrated that this dependence is characterized by a maximum which occurs at the same speed of rotation for various helium temperatures, and for both rough and smooth surfaces.

Before we proceed to describe in greater detail the data on the damping of the oscillations of the disc in rotating helium II, we must note that this effect was quite unexpected from the point of view of the hydrodynamics of a classical liquid. Actually, Mamaladze and Matinyan[20,21] have shown that for classical viscous liquids the dependence of the damping of the disc's oscillations on the speed of rotation is characterized not by a maximum, but by a minimum at $2\omega_0 = \Omega$, as shown in Fig. 4, in which is plotted the formula[20,21]

$$\delta - \frac{\Omega_0}{\Omega}\delta_0 = \frac{\pi^2 R^4}{2I\Omega}\sqrt{\frac{\eta\rho}{2}}\left(\sqrt{\Omega+2\omega_0}+\sqrt{|\Omega-2\omega_0|}\right) \times\left(1+\frac{2d}{R}+\frac{4\lambda^{(+)}\lambda^{(-)}}{R\,(\lambda^{(+)}+\lambda^{(-)})}\right). \qquad (2.4)$$

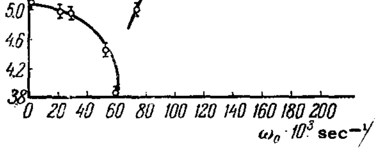

FIG. 4. Dependence of the logarithmic damping decrement of a disc oscillating in water on the speed of rotation.

Here R and I are the radius and the moment of inertia of the disc and ρ and η are the density and dynamic viscosity of the liquid. The last factor is a correction for edge effects. In this factor, d denotes the thickness of the disc and $\lambda^{(\pm)}$ the penetration depth of the two waves generated by the oscillating disc in the rotating liquid:

$$\lambda^{(\pm)} = \sqrt{\frac{2\nu}{|\Omega \pm 2\omega_0|}}\,. \qquad (2.5)$$

Formula (2.4) was confirmed by the experimental data of Mesoed and Tsakadze,[22] obtained for a disc oscillating in rotating water, and also plotted in Fig. 4.

It is easy to see that the presence of a maximum in the damping vs. speed of rotation of the helium II curve cannot be attributed to energy dissipation due to mutual friction between the superfluid component, rotating in the manner described in Sec. 1.4, and the normal component of the helium II, which is directly entrained by the oscillations of the disc. In reality the mutual friction, which increases with increasing number of vortices, can give rise only to a linear increase in damping with increasing speed of rotation. In addition, it was found that the damping due to the mutual friction cannot exceed a value at least one order of magnitude smaller than that obtained.

Thus, it was beyond any doubt even in the first stage of the investigation, while the subsequently established direct connection between the observed phenomena and the elastic properties of the vortices (see Chapter 4, particularly Sec. 4.5) was as yet unclear, that the velocity dependence of the damping observed by Andronikashvili and Tsakadze was uniquely characteristic of rotating helium II and that this phenomenon deserves detailed experimental and theoretical study.

The velocity dependence of the damping of the oscillations of a disc was investigated in 1959 — 1960 over a considerably expanded range of rotational speeds; both smooth and rough discs having various periods of oscillation were investigated. The results of these measurements[20,23,24] are illustrated in Figs. 5, 6, 7, and 8.

Figure 5 shows typical data obtained with a rough disc, the surface of which was covered with sand grains with linear dimensions on the order of 50 μ. As can be seen from this figure, the experimental curve first rises, then reaches a maximum at $2\omega_0/\Omega = 0.21$, and again decreases sharply, thereafter running parallel to the x axis. As $2\omega_0/\Omega$ approaches unity, the curve rises and the behavior of helium II simulates to some extent the behavior of a classical liquid. This can be readily verified by comparing the curve described with Fig. 4, or with the velocity dependence of the damping

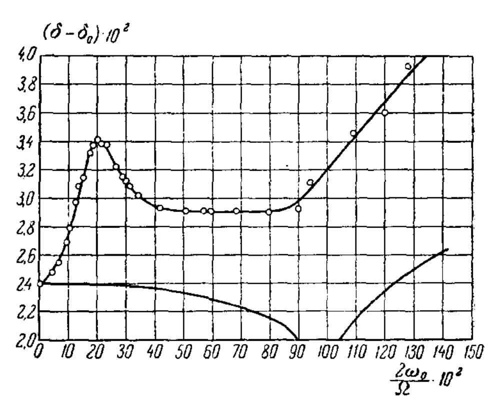

FIG. 5. Dependence of the logarithmic damping decrement of the oscillations of a rough disc on the speed of rotation. The experimental points were obtained with a "heavy" disc at $\Omega = 0.581$ sec^{-1} and a temperature of 1.78°K. The lower curve is calculated from formula (2.4) for $\eta = \eta_n$, $\rho = \rho_n$, and $\nu = \nu_n$.

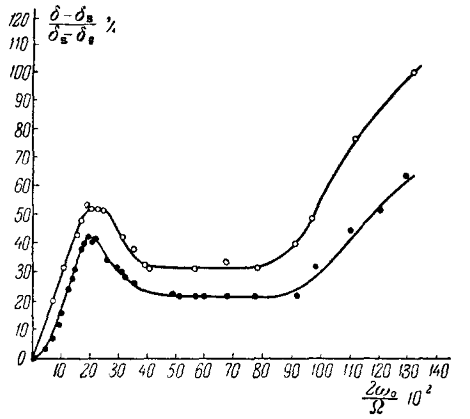

FIG. 6. Dependence of the logarithmic damping decrement of a rough "heavy" disc on the speed of rotation at different frequencies of oscillation. The upper curve corresponds to $\Omega = 0.361$ sec^{-1}, the lower to $\Omega = 0.581$ sec^{-1}. δ_s is the decrement for oscillations in stationary helium (T = 1.78°K).

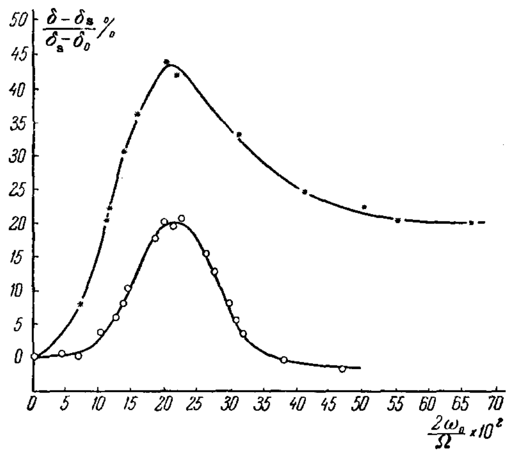

FIG. 7. Dependence of the logarithmic damping decrement of a smooth "heavy" disc on the speed of rotation at different frequencies of oscillation. The upper curve is plotted for $\Omega = 0.368$ sec^{-1}, the lower for $\Omega = 0.551$ sec^{-1} (T = 1.78°K).

due to the viscous properties of the normal component of helium II, calculated from formula (2.4), and shown in Fig. 5.

Figure 6 illustrates the influence of an increase in the period of oscillation on the velocity dependence of the damping. At low speeds of rotation, the damping increases much more rapidly, and the maximum shifts towards smaller values of ω_0 in such a way that its position remains unchanged if plotted in units of $2\omega_0/\Omega$. The same holds for the region where the secondary increase in damping begins.

Figure 7 shows the results obtained with smooth discs. They have the same general character as the data shown in Fig. 6. In particular, the position of the singular points in the graphs along the x axis is precisely the same for both rough and smooth surfaces. It must be noted, however, that for relatively short periods of oscillation (lower curve of Fig. 7), the damping on both sides of the maximum in rotating helium is equal to the damping in the stationary liquid, while the maximum is much lower than in the case of a rough surface. An increase in the period of oscillations (upper curve of Fig. 7 brings the curve obtained for the smooth disc closer to the dependence found for the roughened surface, both in the respect to the qualitative features and quantitative characteristics.

The position of the maximum was found to be almost completely independent of the temperature, which affects only its height. Figure 8 shows how the maximum damping increases with decreasing temperature, reaching at 1.4°K a value on the order of 170% of the damping in stationary helium.

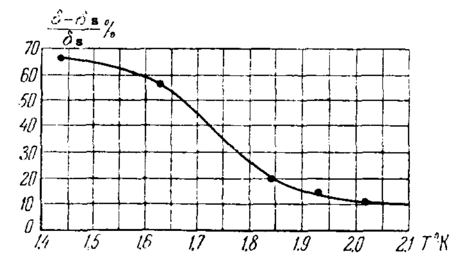

FIG. 8. Temperature dependence of the damping of the oscillations of a disc for $\Omega = 0.581$ sec^{-1} and $\omega_0 = 0.055$ sec^{-1}.

2.7. A few qualitative and quantitative remarks must be made concerning the experimental facts reported in the preceding section, without resorting to an analysis of the hydrodynamic equations, which is postponed to Chapter 4.

A comparison of the experimental curve shown in Fig. 5 with the curve for a normal liquid, shown in the same figure, demonstrates that the measured damping is primarily associated with the interaction between the disc and the rotating superfluid component of the helium II. This is also confirmed by the fact (Fig. 8) that the damping of the oscillations increases with increasing density of the superfluid component ρ_s.

Inasmuch as direct interaction between the solid surface and the superfluid liquid is lacking, while the mutual friction does not provide sufficiently strong coupling (see Sec. 2.6), it is clear that the disc interacts with the superfluid component of the helium II via the vortices attached to its surface. From this point of view, the difference in the amount of damping between rough and smooth discs with similar geometric parameters becomes fully understandable (see Figs. 6 and 7). It is obvious that slippage of the vortices, which should weaken the interaction between the disc and the superfluid component, manifests itself more strongly in the case of smooth surfaces. Moreover, if the period of oscillation is relatively short (lower curve of Fig. 7), virtually complete slippage is observed in the regions on both sides of the maximum; this slippage becomes partial as the period of the oscillations is increased (upper curve of Fig. 7). Indications of partial slippage of the vortices along the rough surface become clearly evident upon comparison of the initial portions of the damping curves plotted for different periods of oscillation (Fig. 6).

Of greatest interest in the treatment of the experimental data on the velocity dependence of the damping of the disc is, naturally, the question of the physical nature of the maximum. In this connection, one must not ignore the fact that the maximum damping is observed at an angular velocity very close to that at which there should occur a phenomenon which can be termed collectivization of the vortex oscillations.[20,23]

According to Thomson, the effective radius R_{osc} of the oscillations of a single vortex filament is given by (see references 19 and 25)

$$R_{osc} \approx \frac{1}{k}, \tag{2.6}$$

where k is the wave number of the wave propagating along the vortex.

It must be emphasized that the effective radius R_{osc} has nothing in common with the effective radius b, which appears in formula (1.4) for the static energy of the vortex, and is on the order of the dimensions of the vessel or of the distance between vortices.* Unlike b, R_{osc} determines the energy of the vortex oscillation over and above the static energy ϵ given by (1.4). The vortex oscillation energy ϵ_{osc} per unit vortex length is

$$\varepsilon_{osc} = \pi \rho_s \frac{\hbar^2}{m^2} \frac{q^2 k^2}{2} \ln \frac{R_{osc}}{a_0}, \tag{2.7}$$

where q is the amplitude of the oscillations (see reference 25).

Using formula (2.6) for the wave which, as already noted in Sec. 2.5, plays a particularly important role in the phenomena that occur in oscillations of discs in a superfluid liquid, and also using formula (2.2), we have

$$R_{osc} = \sqrt{\frac{\nu_s}{\Omega - 2\omega_0}}. \tag{2.8}$$

Let us now compare the effective radius R_{osc} with the distance c between vortices, which, according to (1.7), is

$$c = \sqrt{\frac{\pi\hbar}{m\omega_0}}. \tag{2.9}$$

A comparison of (2.8) and (2.9) shows that at sufficiently small speeds of rotation ω_0, we have $R_{osc} \ll c$. Under these conditions, the oscillations of the vortices can be considered independent of one another. However, at some definite speed $\tilde{\omega}_0$ the effective radius of oscillation of the vortex filaments should become equal to half the distance between vortices, and consequently for $\omega_0 > \tilde{\omega}_0$ the oscillations of the vortices must be regarded as collectivized. Proceeding from the definition of ω_0 and (2.8) and (2.9), we can readily obtain

$$\frac{2\tilde{\omega}_0}{\Omega} = \frac{1}{1 + \frac{2m}{\pi\hbar}\nu_s}. \tag{2.10}$$

Substituting into (2.10) the numerical values of m and ħ, as well as the value $\nu_s = 8.5 \times 10^{-4}$ cm^2/sec as measured by Hall, we obtain $2\tilde{\omega}_0/\Omega = 0.22$, which is actually very close to the position of the maximum in the disc damping vs. speed of rotation curve ($2\omega_0/\Omega = 0.21$).

From the ideas just expressed, we can imagine the following development for the phenomena associated with the damping of a smooth disc, for a relatively short period of oscillation (lower curve of Fig. 8). At small ω_0, where there are few vortices, there is almost complete slippage of the vortex filaments along the smooth surface. An increase in the number of vortices, resulting from an increase in the speed of rotation, causes part of the vortices to become attached (at least partially) to the surface irregularities which are always present to some extent. At this point, an interaction comes into force between the oscillating disc and the Onsager-Feynman vortex system, which leads to additional damping of the oscillations, increasing with increasing $\tilde{\omega}_0$. However, the collectivization of the vortex oscillations, which occurs when $\omega_0 = \tilde{\omega}_0$, favors an increase in the slippage of the ends of the vortex filaments along the solid surface. As a result, the damping again diminishes and virtually complete slippage is observed again at some distance from the maximum. Naturally, the same phenomena are also observed, but to a much lesser degree, in the case of long periods of oscillations and rough surfaces.

2.8. New data, confirming the hypothesis just advanced, were obtained in 1960 by Andronikashvili and Tsakadze[26] from a comparison of the velocity dependence of the damping and oscillation frequency of a "light" disc. The disc which they employed had a sufficiently low moment of inertia that its frequency was noticeably affected by the interaction with the rotating helium II, and possessed a rough surface. The velocity dependence of its damping, shown in Fig. 9a, is quite analogous, as expected, to the curves obtained for "heavy" discs, and is of no interest in itself. However, in comparing it with the curve for velocity dependence of the oscillation frequency shown in Fig. 9b, one must take into consideration the following circumstances.

At low rotational velocities the oscillation frequency Ω increases with increasing ω_0. As shown in Sec. 2.2, this should indeed be the result of the interaction of an oscillating body with the elastic vortex filaments perpendicular to its surface. Furthermore, it stands to reason that in the case of the oscillations of a single disc (but not of a stack of closely spaced discs) in

*A definite analogy exists here with the hydrodynamics of a viscous classical liquid, where the steady motion of solid bodies affects the distribution of velocities in the liquid right out to its boundaries, while the influence of an oscillating body extends only over a layer whose magnitude is determined by the penetration depth of the viscous wave.

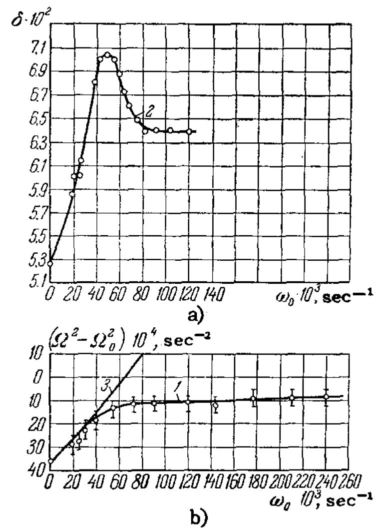

FIG. 9. Velocity dependence of the damping (a) and oscillation frequency (b) of a "light" disc. Curves 1 and 2 are experimental data, obtained at 1.47°K; curve 3 is computed theoretically (see Sec. 4.4).

an unbounded liquid* there can be no resonant effects to produce periodic changes in the oscillation frequency. Therefore the only result of an increase in the angular velocity should have been a monotonic increase in the oscillation frequency, due to the increasing number of vortices. As can be readily seen from Fig. 9, however, the rate of increase in the frequency of oscillation falls off at approximately the same velocities ω_0, at which the maximum damping is observed, and a tendency to saturation soon sets in.

It is obvious that this fact confirms the assumption, stated in Sec. 2.7, that the slippage of the vortices can increase as ω_0 approaches $\tilde{\omega}_0$. In addition, the results obtained with a light disc are clear evidence that the same physical factors govern the velocity dependence of the damping of the disc's oscillations, on one hand, and of the frequency of its oscillations, on the other.

2.9. Let us return to the elastic transverse waves generated by the oscillations of the disc in the Onsager-Feynman vortex system, to which we have already referred several times.

Interesting information converning these waves was obtained in the experiments of Hall[27] (see also reference 14) and of Andronikashvili and Tsakadze[28] in which a single rough disc was made to oscillate under the free surface of rotating helium II; this surface was made either to rise by inflow through the film or to drop by evaporation of the liquid. Thus, in these experiments the vortices whose ends were attached to the oscillating surfaces of the disc were gradually lengthened or shortened, while the opposite ends of the vortices terminated in the free surface of the helium II.

It can easily be seen that the vortex filament must be perpendicular to the free surface at the point of emergence, since otherwise the tension in the vortex would cause it to move until it reached this equilibrium state.

Under such conditions, the lengthening or shortening of the vortex filaments along which the wave described in Sec. 2.5 propagates whould be accompanied by resonance effects similar to those observed with a stack of discs. However, the resonance condition in this case will be that an odd number of quarter wavelengths (for more details see Sec. 4.10) be contained within the distance H from the disc to the free surface. Therefore, we obtain instead of the resonance condition (2.3)

$$\sqrt{\Omega - 2\omega_0} H = (2n-1)\frac{\pi}{2}\sqrt{\nu_s} \qquad (n = 1, 2, \ldots). \qquad (2.11)$$

These resonance phenomena were indeed observed in the aforementioned experiments of Hall and of Andronikashvili and Tsakadze.

In Hall's experiments[27] the period of oscillation of the discs was measured as a function of time, the value of H also varying with time. The oscillatory dependence of the period on the depth of immersion of the disc observed in these experiments, and illustrated in Figs. 10 and 11, yields the value $\nu_s = 9.7 \times 10^{-4}$ cm²/sec.

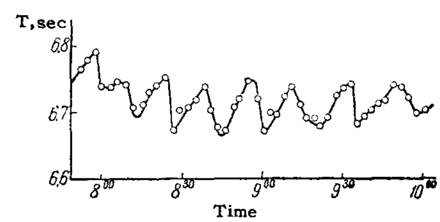

FIG. 10. Variation of the period of oscillation of a single disc with inflow of liquid helium II through the film ($\omega_0 = 0.140$ sec⁻¹, T = 1.3°K).

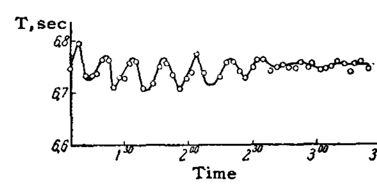

FIG. 11. Variation of the period of oscillation of a single disc with inflow of helium II through the film ($\omega_0 = 0.140$ sec⁻¹, T = 1.6°K).

In an analogous experiment, Andronikashvili and Tsakadze[28] measured the damping of the oscillations.* Their results are illustrated in Figs. 12 and 13. The distance between two resonant points was found to be 0.065 cm. According to these data, the length of the wave propagating along the vortex at $\omega_0 = 0.055$ sec⁻¹ and $\Omega = 0.581$ sec⁻¹ is 0.26 cm (see Sec. 4.3). Evalu-

*The experiments with a light disc, unlike the first experiments with a heavy disc, were undertaken after a theory had been developed for oscillating phenomena in helium II (see Chapter 4, particularly Sec. 4.3). It was therefore well known that the smallness of the penetration depth ensures the "unboundedness" of the liquid under the conditions of the experiment under consideration.

*In this experiment the same disc was used whose damping is illustrated in Fig. 5. The difference in experimental configurations consisted of the following: in determining the dependence of the damping on the velocity, the oscillations were carried out beneath an "infinitely" remote vessel cover (see footnote on p. 8), while in the experiment described here the oscillations were performed under a free surface which approached the disc.

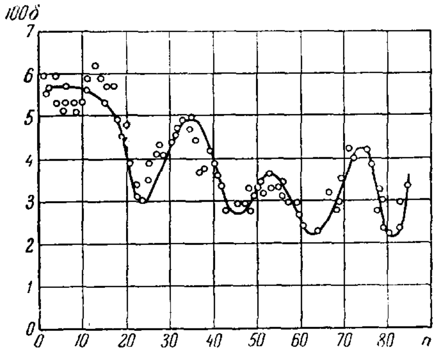

FIG. 12. Variation in the logarithmic damping decrement of the disc with falling level of liquid He II as a result of evaporation. Rate of evaporation -0.5×10^{-2} cm/min ($\Omega = 0.581$ sec^{-1}, $\omega_0 =$ 0.055 sec^{-1}, $T = 1.38$°K); *n is the number of half cycles of oscillation.* The initial portions of the curves in Figs. 12 and 13 correspond to the buildup of the oscillations in the system.

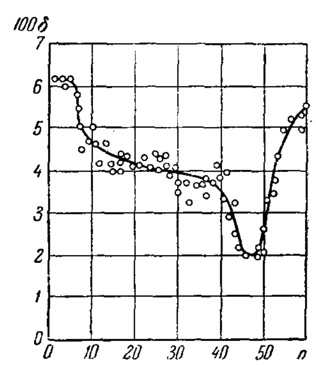

FIG. 13. Variation in the logarithmic damping decrement of the disc with falling level of liquid helium II as a result of evaporation. Rate of evaporation -3.6×10^{-2} cm/min ($\Omega = 0.581$ sec^{-1}, $\omega_0 = 0.055$ sec^{-1}, $T = 1.38$°K).

ation of ν_S with the aid of formula (2.11) yields $\nu_S = 8 \times 10^{-4}$ cm^2/sec.

2.10. It is clear from the foregoing analysis that the above-described manifestations of the elastic vortex properties should not appear if the surface oscillating in the helium II is parallel to the Onsager-Feynman vortex filaments. However, the characteristic features of the rotation of a superfluid liquid should be evident even in this type of experiment, owing to the mutual friction between the superfluid and the normal component of the helium entrained by the oscillating body.

A similar investigation was also carried out in the cryogenic laboratory of the Tbilisi University, where Tsakadze and Chkheidze[29] investigated the velocity dependence of the damping of torsional-axial oscillations of a hollow cylinder, the oscillations being superimposed upon rotation of the cylinder together with the liquid.

It must be noted (see Sec. 4.11) that the damping of the oscillations of a cylinder in a rotating classical liquid should not depend on the velocity of rotation of the latter. This assertion was verified and confirmed in preliminary experiments. Thus, the velocity dependence of the damping of the oscillations of the cylinder was due in its entirety to mutual friction between the superfluid and normal components of the helium II.

From this point of view, the linear increase in damping with increasing velocity of rotation (see Fig. 14) observed by Tsakadze and Chkheidze is completely legitimate; it is the natural result of the fact that the number of vortices increases in proportion to ω_0.

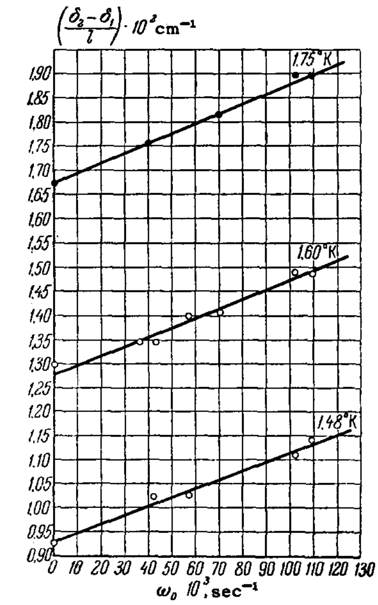

FIG. 14. Dependence of the logarithmic damping decrement of a cylinder on the velocity of rotation. δ_2 and δ_1 – damping decrements at depths of immersion l_2 and l_1, $l = l_2 - l_1$. The straight lines represent the theoretical formula (see Sec. 4.11).

3. HYDRODYNAMICS OF ROTATING HELIUM II

3.1. The hydrodynamic theory for the phenomena considered in this survey was developed concurrently with the experimental investigations described in the preceding chapter. It was developed to fill the need for a quantitative interpretation of the phenomena observed in the very first experiments and, in turn, stimulated the formulation of new experiments. Through this combination of the efforts of theoreticians and experimenters, a unique hydrodynamic theory has been formulated for a rotating quantum liquid, which we shall now proceed to describe, after first explaining why the description of the phenomena occurring in rotating helium II calls for a special hydrodynamic theory.

It is quite clear that the well known equations of the "two-fluid" hydrodynamics of Landau (see, for example, reference 4, Part I, Chapter XVI) should describe rotating helium II as well, just as the Navier-Stokes equation describes the behavior of a classical viscous liquid in principle, no matter what the character of its motion.

However, the characteristics of the rotation of a superfluid liquid make it more advantageous to describe the latter using averaged values of the velocities and the curl of the velocity of the superfluid component.* It has already been pointed out, in Sec. 1.4, that such a description facilitates the separation of

*The averaging is over volume elements containing a sufficiently large number of vortex filaments.

the main features of the investigated phenomena from secondary details. On the other hand, the forces with which the liquid acts on the cores of the vortex filaments take on, through the averaging process, the character of volume forces and therefore enter into the hydrodynamic equations, increasing their complexity. It is clear, however, that such complication is much less burdensome than the need for regarding these forces as surface ones, introducing boundary conditions on the cores of the entire aggregate of vortex filaments; this is precisely what would occur if we were to employ the ordinary hydrodynamic equations for helium II, which describe it without the aid of averaged quantities.

3.2. It follows from the foregoing that the additional terms by which the hydrodynamic equations of rotating helium II differ from the ordinary equations of two-fluid hydrodynamics must take into account the features of vortical rotation of the superfluid already described in Chapter 2. Specifically, they must describe the elastic properties of the vortex filaments (or the equivalent energy relationships), on the one hand, and the mutual friction between the superfluid and normal components of the helium, on the other.

It is easy to see that each unit length of a vortex filament with tension ϵ, being curved, experiences a rectifying force equal to $\epsilon(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is a unit vector tangent to the vortex filament;* $\boldsymbol{\sigma} = \boldsymbol{\omega}/\omega$, $\boldsymbol{\omega} = \text{curl}\,\mathbf{v}_s$. If we recall also that the density N of the vortex filaments equals ω/Γ [see formula (1.7)], it is quite natural to use the following form for the equation of motion of a rotating superfluid obtained by Hall,[19] the derivation of which we shall omit:†

$$\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s, \nabla)\mathbf{v}_s = \nu_s(\boldsymbol{\omega}, \nabla)\frac{\boldsymbol{\omega}}{\omega} + \nabla\Phi. \qquad (3.1)‡$$

Here, the symbol Φ embraces all the terms under the gradient sign, while ν_s is the parameter already introduced in Chapter 2, equal to

$$\frac{\varepsilon}{\varrho_s\Gamma} = \frac{\hbar}{2m}\ln\frac{b}{a_0}.$$

It is obvious that Eq. (3.1) describes the motion of the superfluid component in the absence of a normal component. Strictly speaking, therefore, it is valid only at absolute zero. Under actual conditions, the motion of the superfluid and normal components must be considered simultaneously, since they are coupled by a mutual friction force. The need for taking this force into account was also noted by Hall, but in reference 19 he solved only Eq. (3.1) (together with the continuity equation for the superfluid component).

Recently, in the work of Hall, Bekarevich and Khalatnikov, and Mamaladze and Matinyan, Hall's equation has been incorporated, together with the equation of motion of the normal component and the continuity equations for the superfluid and normal components, into a unified system.[14,30,31] This leads to the following set of equations for the hydrodynamics of rotating helium II:

$$\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s, \nabla)\mathbf{v}_s + \nu_s\left[\boldsymbol{\omega}, \text{rot}\,\frac{\boldsymbol{\omega}}{\omega}\right] = \nabla\Phi + \mathbf{F}_{sn}, \qquad (3.2')*$$

$$\frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n, \nabla)\mathbf{v}_n - \nu_n\Delta\mathbf{v}_n = \nabla\Psi + \mathbf{F}_{ns}, \qquad (3.2'')$$

$$\text{div}\,\mathbf{v}_s = 0, \quad \text{div}\,\mathbf{v}_n = 0. \qquad (3.2''')$$

Here, in Eq. (3.2″), the symbol Ψ incorporates the terms under the gradient sign, $\mathbf{F}_{sn}$ is the mutual friction force acting on unit mass of the superfluid component, and $\mathbf{F}_{ns}$ is the same force acting on unit mass of the normal component ($\mathbf{F}_{ns} = -\rho_s\mathbf{F}_{sn}/\rho_n$).

The expression for the force† $\mathbf{F}_{sn}$ was given by Bekarevich and Khalatnikov.[31] It has the following form:

$$\mathbf{F}_{sn} = \alpha_n\left[\boldsymbol{\omega}, \mathbf{v}_n - \mathbf{v}_s - \nu_s\,\text{rot}\,\frac{\boldsymbol{\omega}}{\omega}\right] + \beta_n\left[\frac{\boldsymbol{\omega}}{\omega}, \left[\boldsymbol{\omega}, \mathbf{v}_n - \mathbf{v}_s - \nu_s\,\text{rot}\,\frac{\boldsymbol{\omega}}{\omega}\right]\right] + \gamma_n\frac{\boldsymbol{\omega}}{\omega}\left(\boldsymbol{\omega}, \mathbf{v}_n - \mathbf{v}_s - \nu_s\,\text{rot}\,\frac{\boldsymbol{\omega}}{\omega}\right). \qquad (3.3)$$

Here α_n and β_n are the coefficients of mutual friction, related to the Hall and Vinen coefficients B and B′, the temperature dependence of which was calculated in references 5, 14, 25, and 32:

$$\alpha_n = -\frac{1}{2}\frac{\varrho_n}{\varrho}B', \quad \beta_n = -\frac{1}{2}\frac{\varrho_n}{\varrho}B. \qquad (3.4)$$

In addition, expression (3.3) contains as well a third mutual-friction coefficient γ_n, which determines the component of the friction force along the vortex filaments.

Introducing the notation

$$\alpha_s = \frac{\varrho_s}{\varrho_n}\alpha_n = -\frac{1}{2}\frac{\varrho_s}{\varrho}B',$$

$$\beta_s = \frac{\varrho_s}{\varrho_n}\beta_n = -\frac{1}{2}\frac{\varrho_s}{\varrho}B, \quad \gamma_s = \frac{\varrho_s}{\varrho_n}\gamma_n, \qquad (3.5)$$

we write for the force $\mathbf{F}_{ns}$ an expression analogous to (3.3):

*The vector $(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{\sigma}$ is directed along the normal, and its magnitude is equal to the curvature of the line whose tangent unit vector is $\boldsymbol{\sigma}$. Later on we shall also use the equation $(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{\sigma} = -\boldsymbol{\sigma}\times\text{curl}\,\boldsymbol{\sigma}$.

†Equation (3.1) differs from the ordinary equation of motion for an ideal liquid only by the term $\nu_s(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{\omega}/\omega$. For simplicity we shall omit throughout the average sign over the symbols $\mathbf{v}_s$, $\mathbf{v}_n$, and $\boldsymbol{\omega}$.

‡$(\mathbf{v}_s, \nabla) = \mathbf{v}_s\cdot\nabla$.

*$[\mathbf{a}, \mathbf{b}] = \mathbf{a}\times\mathbf{b}$; rot = curl; $\Delta = \nabla^2$.

†The derivation of the expression for the force $\mathbf{F}_{sn}$ in Hall's paper[14] contains a computational error. See formula (51) in that paper, where an incorrect transformation results in the inclusion of a term with a different direction in the sum of the vectors perpendicular to $\boldsymbol{\omega}$.

$$\mathbf{F}_{ns} = -\alpha_s\left[\boldsymbol{\omega}, \mathbf{v}_n - \mathbf{v}_s - \nu_s \operatorname{rot}\frac{\boldsymbol{\omega}}{\omega}\right] - \beta_s\left[\frac{\boldsymbol{\omega}}{\omega}, \left[\boldsymbol{\omega}, \mathbf{v}_n - \mathbf{v}_s + \nu_s \operatorname{rot}\frac{\boldsymbol{\omega}}{\omega}\right]\right] - \gamma_s\frac{\boldsymbol{\omega}}{\omega}\left(\boldsymbol{\omega}, \mathbf{v}_n - \mathbf{v}_s - \nu_s \operatorname{rot}\frac{\boldsymbol{\omega}}{\omega}\right). \quad (3.6)$$

Equations (3.2) follow from the complete system of hydrodynamic equations for rotating helium II (derived by Bekarevich and Khalatnikov from conservation laws[31]) under the condition of constant temperature. Consequently, the system (3.2) is quite sufficient for the analysis of the problems discussed in the present survey.

We note also that in writing out formulas (3.2′), (3.3), and (3.6) we have neglected the possible dependence of the parameter ν_s on ω.*

In concluding this section we note that, owing to the peculiarities of the vortical rotation of helium II, the curl of the velocity of the superfluid component becomes one of the quantities determining the state of the liquid, a fact also reflected in the thermodynamic equations. In particular, the energy E per unit volume of helium II, in the rotating reference frame in which the superfluid component is, on the average, at rest, obeys the thermodynamic identity[31]

$$dE = T\,dS + \mu\,d\varrho + (\mathbf{v}_n - \mathbf{v}_s, d\mathbf{p}) + \eta_s d\omega, \quad (3.7)\dagger$$

where S and $\mathbf{p}$ are the entropy and momentum per unit volume, μ is the chemical potential per unit mass, and ρ is the total density of the helium II.

Since the vortex energy per unit volume is $N\epsilon = \omega\epsilon/\Gamma$ [see formulas (1.4), (1.7), and (1.9)], we have

$$\eta_s = \frac{\partial E}{\partial \omega} = \frac{\varepsilon}{\Gamma}, \quad (3.8)$$

from which it follows that the parameter ν_s (which equals $\epsilon/\rho_s\Gamma$) is determined by the relation

$$\nu_s = \varrho_s \eta_s = \varrho_s \frac{\partial E}{\partial \omega}. \quad (3.9)$$

It must be noted, finally, that to Eqs. (3.2) there corresponds the following expression for the momentum flux tensor.

$$\Pi_{ik} = p\delta_{ik} + \varrho_n v_{ni} v_{nk} - \eta_n\left(\frac{\partial v_{ni}}{\partial x_k} + \frac{\partial v_{nk}}{\partial x_i}\right) + \varrho_s v_{si} v_{sk} + \eta_s \omega \delta_{ik} - \eta_s \frac{\omega_i \omega_k}{\omega}. \quad (3.10)$$

The last two terms of this formula are contributed by the singularities of the vortical motion of the superfluid component; p is the pressure.

*As already noted, $\nu_s = (\hbar/2m)\ln(b/a_0)$. If, as is customarily assumed,[11] the effective radius b corresponds in order of magnitude to the distance between the vortices [see formula (2.9)], then ν_s depends on ω_0. However, in view of the smallness of the radius a_0 of the vortex core, which, as established by Feynman,[11] is on the order of 4×10^{-8} cm, or, from Hall's experimental data,[19] 3×10^{-7} cm, the logarithmic dependence of ν_s on ω_0 is found to be very weak.

†$(\mathbf{v}_n - \mathbf{v}_s, d\mathbf{p}) = (\mathbf{v}_n - \mathbf{v}_s)\cdot d\mathbf{p}$

3.3. In solving the system (3.2) for the velocity of the normal component, use is made of the customary boundary conditions, which coincide under the conditions considered here (T = const) with the boundary conditions for the hydrodynamics of a classical viscous liquid. However, the problem of the boundary conditions for the superfluid component velocity requires special consideration.

The occurrence in the first equation of (3.2) of the term $\nu_s \boldsymbol{\omega} \times \operatorname{curl}(\boldsymbol{\omega}/\omega)$, containing second-order space derivatives of $\mathbf{v}_s$, requires the imposition of boundary conditions not only on the component of $\mathbf{v}_s$ perpendicular to the surface, but also on the tangential component. The idea was first advanced by Hall[19] that by establishing a connection between the velocity $\mathbf{v}_s$ of the superfluid component and the velocity of motion $\mathbf{v}_L$ of the vortex filaments, it is possible to obtain the necessary boundary conditions from the requirement that the vortices be secured to the solid surface. The relation between $\mathbf{v}_s$ and $\mathbf{v}_L$ was derived by Hall,[19] and generalized to the case $\mathbf{F}_{sn} \neq 0$ by Mamaladze and Matinyan[30] (see also references 14 and 31):

$$[\mathbf{v}_L, \boldsymbol{\omega}] = \left[\mathbf{v}_s + \nu_s \operatorname{rot}\frac{\boldsymbol{\omega}}{\omega}, \boldsymbol{\omega}\right] + \mathbf{F}_{sn}. \quad (3.11)$$

As regards the conditions for attachment of the vortices to the solid surface, these are determined by assuming that the difference in the tangential components of the vortex velocity $\mathbf{v}_L$ and the surface velocity $\mathbf{v}_\sigma$ is proportional to the tangential component of the force $\epsilon\boldsymbol{\omega}/\omega$ with which a vortex with tension ϵ acts on the solid surface:[30]

$$(\mathbf{v}_L - \mathbf{v}_\sigma)_t = a\left(\frac{\boldsymbol{\omega}}{\omega}\right)_t \quad (3.12)$$

(at the solid surface) where a is the slip coefficient. The value of a = 0 corresponds to complete attachment of the vortices (absolutely rough surface) while $a = \infty$ denotes complete slippage (absolutely smooth surface). Formula (3.12) is equivalent to the analogous formula used earlier by Hall.[19] On the other hand, in the case of small deviations from a velocity distribution of the form $\mathbf{v}_s = \mathbf{v}_n = \mathbf{v}_L = \mathbf{v}_\sigma = \boldsymbol{\omega}_0 \times \mathbf{r}$, this formula is the consequence of a more general equation obtained by Bekarevich and Khalatnikov:[31]

$$\mathbf{v}_L - \mathbf{v}_\sigma = a\left[\frac{\boldsymbol{\omega}}{\omega} \times \left[\mathbf{N}_\sigma \times \frac{\boldsymbol{\omega}}{\omega}\right]\right] \quad (3.13)$$

(at the solid surface), where $\mathbf{N}_\sigma$ is the unit vector normal to the surface.

In the case of a free surface, as already noted in Sec. 2.9, the vortices must be perpendicular to the surface

$$[\boldsymbol{\omega} \times \mathbf{N}_\sigma] = 0 \quad (3.14)$$

(at the free surface).

3.4. Formulas (3.11) and (3.3) can be reconciled only by assuming that the coefficient γ_n of the only term not perpendicular to ω vanishes (in which case, naturally, $\gamma_s = 0$ as well). On the other hand, Eq. (3.11) is quite necessary, since only when it is satisfied can Eq. (3.2′) be cast in the form of a conservation law for the vortices

$$\frac{\partial \mathbf{v}_s}{\partial t} - [\mathbf{v}_L, \boldsymbol{\omega}] = \nabla\left(\Phi - \frac{1}{2} v_s^2\right). \qquad (3.15)$$

Henceforth, therefore, we shall set $\gamma_n = \gamma_s = 0$ when using formulas (3.3) and (3.6).

It should be noted that the expression (3.15) is a less convenient relation than (3.11) with which to establish a connection between $\mathbf{v}_s$ and $\mathbf{v}_L$.

4. THEORY OF SMALL OSCILLATIONS OF BODIES WITH AXIAL SYMMETRY IN ROTATING HELIUM II

4.1. The system of hydrodynamic equations (3.2) has served as the basis for developing a quantitative theory for the phenomena that appear when solid bodies of various shapes are caused to oscillate in rotating helium II.

The solution of the corresponding specific problems is greatly simplified by the fact that the system (3.2) can be linearized, in view of the smallness of the amplitudes of oscillation of the solid bodies (this requirement was in fact satisfied in all the experimental investigations described in Chapter 2). For this purpose, all the velocities $\mathbf{v}_s$, $\mathbf{v}_n$, $\mathbf{v}_L$, and $\mathbf{v}_\sigma$ should be represented in the form of the sum of the fundamental "rotational" terms $\mathbf{v}_0 = \boldsymbol{\omega}_0 \times \mathbf{r}$ and small "oscillatory increments" to these terms:

$$\mathbf{v} = [\boldsymbol{\omega}_0, \mathbf{r}] + \mathbf{u}\exp(i\Omega t), \qquad (4.1')$$

where the quantities $\mathbf{v}$ and $\mathbf{u}$ can have as subscripts s, n, L, or σ. The functions Φ and Ψ are also written in analogous form, as sums

$$\Phi = \Phi_0 + \chi_s \exp(i\Omega t), \qquad (4.1'')$$

$$\Psi = \Psi_0 + \chi_n \exp(i\Omega t). \qquad (4.1''')$$

Here Φ_0 and Ψ_0, like $\mathbf{v}_0$, denote the solutions of the system (3.2) in the absence of oscillations. In substituting Eqs. (4.1) into the system (3.2) we discard terms of order higher than first in $\mathbf{u}$, obtaining as a result a system of linear differential equations in the new variables $\mathbf{u}_s$, $\mathbf{u}_n$, χ_s, and χ_n. We shall not write out this rather cumbersome system, which can be readily derived following the method just described.

Using the same notation, and in the same approximation, we can rewrite the conditions (3.12) or (3.13) for attachment of the vortices in the following form:*

$$u_{Lr} = -\frac{a}{2\omega_0}\frac{du_{s\varphi}}{dz}, \qquad u_{L\varphi} = i\Omega\varphi_0 r + \frac{a}{2\omega_0}\frac{du_{sr}}{dz} \qquad (4.2)$$

(at the solid surface). We take into account here the fact that in all the cases of interest to us the solids execute torsional-axial oscillations; therefore $u_{\sigma r} = 0$ and $u_{\sigma\varphi} = i\Omega\varphi_0 r$, where φ_0 is the amplitude of the oscillations. We also linearize Eqs. (3.11) and (3.15), which are needed to obtain the boundary conditions for the tangential components of the velocity $\mathbf{v}_s$ from the boundary conditions for $\mathbf{v}_L$. These reduce to the relations

$$u_{sr} = \frac{2\omega_0}{i\Omega} u_{L\varphi} + \frac{1}{i\Omega}\frac{\partial \chi_s}{\partial r}, \qquad u_{s\varphi} = -\frac{2\omega_0}{i\Omega} u_{Lr}. \qquad (4.3)$$

4.2. The same general procedure was followed in solving all of the hydrodynamic problems considered in this chapter. It is therefore appropriate to describe the computational procedure in this introductory section, to obviate the need for returning to it later.

First of all, the form of the derived solution to the hydrodynamic equations was selected, as dictated by the symmetry of the problem given and by its boundary conditions. In particular, account was taken in every case of the axial symmetry, as a consequence of which all the physical parameters become independent of the φ coordinate (this situation was also assumed in writing down the equation given in Sec. 4.1). Then, solving the system (3.2), we found the boundary conditions for the problem under consideration.* The values thus determined for $\mathbf{v}_s$ and $\mathbf{v}_n$ (as well as for $\omega = \text{curl}\ \mathbf{v}_s$) were used to calculate the momentum flux tensor Π_{ik} (formula 3.10), with the aid of which the force dF acting on the surface element $d\sigma$ of the oscillating body is found (see reference 4, p. 69):

$$dF_i = \Pi_{ik} d\sigma_k. \qquad (4.4)$$

It is now easy, by integrating over the surface of the solid body, to determine the moment M of the forces acting on the body, which always has the form $M = m\varphi_0 \exp(i\Omega t)$, where φ_0 is the amplitude of the oscillations.

Finally, knowing m, we can determine the frequency Ω and the logarithmic damping decrement δ of the oscillations, using for this purpose the following formulas which result from the moment equation:†

$$\Omega^2 - \Omega_0^2 = -\frac{1}{I}\operatorname{Re}(m), \qquad (4.5)$$

$$\delta - \frac{\Omega_0}{\Omega}\delta_0 = -\frac{\pi}{I\Omega^2}\operatorname{Im}(m). \qquad (4.6)$$

Here Ω_0 and δ_0 are the vacuum values of the frequency and the decrement.

4.3. Let us consider first the oscillations of a disc or a stack of discs in rotating helium II. It is quite obvious that the general solution of the hydrodynamic equations is the same in both cases. Mamaladze and Matinyan have shown[30,33] that the oscillations of the disc generate in the rotating helium II a composite wave, which propagates along the axis of rotation z and is made up of four simple waves. [More accu-

*Henceforth, we shall use cylindrical coordinates (r, φ, z) throughout, with the z axis coinciding with the axis of rotation.

*Beyond this point Hall's computational procedure[19] differs from that described here (see Sec. 4.8).

†Here and below the damping of the oscillations is always considered to be weak: $\delta \ll 1$.

rately, the r and φ components of each of the quantities u_s, u_n, and u_L are represented in the form of a product $rw(z)$, where $w(z)$ is the sum of four exponentials.] The wave numbers of these waves are given by the following formulas* (see reference 33) [all the Im (k) are positive]:

$$k_{1,2}^2 = \frac{1}{2\left(1-\frac{\nu_s}{2\omega_0}q_1^{(+)}\right)}\left\{k_{s0}^{(+)^2}+k_{n0}^{(+)^2}+q_1^{(+)}\left(1-\frac{\nu_s}{2\omega_0}k_{n0}^{(+)^2}\right)+q_2^{(+)}\right.$$
$$\left.\pm\sqrt{\left[k_{s0}^{(+)^2}-k_{n0}^{(+)^2}+q_1^{(+)}\left(1+\frac{\nu_s}{2\omega_0}k_{n0}^{(+)^2}\right)-q_2^{(+)}\right]^2+4q_1^{(+)}q_2^{(+)}\left(1+\frac{\nu_s}{2\omega_0}k_{s0}^{(+)^2}\right)}\right\},$$
$$k_{3,4}^2 = \frac{1}{2\left(1-\frac{\nu_s}{2\omega_0}q_1^{(-)}\right)}\left\{k_{s0}^{(-)^2}+k_{n0}^{(-)^2}+q_1^{(-)}\left(1-\frac{\nu_s}{2\omega_0}k_{n0}^{(-)^2}\right)+q_2^{(-)}\right.$$
$$\left.\pm\sqrt{\left[k_{s0}^{(-)^2}-k_{n0}^{(-)^2}+q_1^{(-)}\left(1+\frac{\nu_s}{2\omega_0}k_{n0}^{(-)^2}\right)-q_2^{(-)}\right]^2+4q_1^{(-)}q_2^{(-)}\left(1+\frac{\nu_s}{2\omega_0}k_{s0}^{(-)^2}\right)}\right\}. \quad (4.7)$$

We use here the following notation:

$$k_{s0}^{(\pm)^2} = \mp\frac{\Omega\pm 2\omega_0}{\nu_s}, \quad k_{n0}^{(\pm)^2} = -i\frac{\Omega\pm 2\omega_0}{\nu_n}, \quad (4.8)$$

$$q_1^{(\pm)} = \mp 2\omega_0 i\frac{\beta_n\mp i\alpha_n}{\nu_s}, \quad q_2^{(\pm)} = 2\omega_0\frac{\beta_s\mp i\alpha_s}{\nu_n}. \quad (4.9)$$

Thus, the quantities q represent in (4.7) the influence of the mutual friction on the character of the wave motion propagating under these conditions in rotating helium II. Setting q = 0, we readily see that the wave numbers k_1 and k_3 are related only to the properties of the superfluid, while k_2 and k_4 are associated with those of the normal component:

$$k_{1,3} = k_{s0}^{(\pm)}, \quad k_{2,4} = k_{n0}^{(\pm)} \quad (\text{for } q_{1,2}^{(\pm)} = 0). \quad (4.10)$$

Using the same approximation, q = 0, the velocity distribution of the superfluid component and the velocity v_L contain only the two exponentials with wave numbers $k_{s0}^{(\pm)}$, while the velocity distribution of the normal component contains two exponentials with wave numbers $k_{n0}^{(\pm)}$). This is natural, since the $k_{s0}^{(\pm)}$ are the wave numbers characterizing the oscillation of discs in a rotating superfluid liquid in the absence of the normal component,[19] while the $k_{n0}^{(\pm)}$ play an analogous role for classical viscous liquids.[21]

With respect to Eqs. (4.10), if the corrections for mutual friction are insignificant (see below), k_1 and k_3 can be regarded as wave numbers associated fundamentally with the motion of the superfluid component, even for q ≠ 0, while k_2 and k_4 can be associated with motion of the normal component of the helium II. The simultaneous presence of all four wave numbers in the distributions of the velocities v_s, v_L, and v_n is due only to mutual friction.

The wave numbers $k_{n0}^{(\pm)}$ describe ordinary viscous waves, characterized by finite wavelengths, penetration depths, and propagation velocities.* In this respect, it is well known that the penetration depths for viscous waves, under the experimental conditions described in Chapter 2, are very small as compared with dimensions on the order of one centimeter.

The waves described by the wave numbers $k_{s0}^{(\pm)}$ are quite different in nature [see (4.8)]. The number $k_{s0}^{(+)}$ is pure imaginary. It therefore describes a wave with a finite penetration depth, comparable in order of magnitude with the penetration depth of viscous waves (the quantity ν_s does not differ very greatly from the viscosity of the normal component ν_n). However, the propagation velocity and wavelength of this wave are infinitely large. The number $k_{s0}^{(-)}$ is of similar character when $\Omega < 2\omega_0$ (rapid rotation). In the opposite case of slow rotation ($\Omega > 2\omega_0$) the number $k_{s0}^{(-)}$ is purely real. Then, with the velocity and wavelength finite, the penetration depth is infinitely large.

Analysis of Eqs. (4.7), as well as analysis of the calculations made by Kiknadze and Tkemaladze on the basis of these formulas using the "Ural" computer, shows that allowance for mutual friction (as represented by the q's) does not significantly alter the character of the wave numbers.† Formulas (4.10) can be assumed to hold true, to a more or less rough approximation, everywhere except in the region $\omega_0 \approx \Omega/2$, where, for example, the quantities k_3 and k_4 are determined completely by the mutual friction force (since in this region $k_{s0}^{(-)} \approx k_{n0}^{(-)} \approx 0$).

However, no matter how small the effect of including the q's in the computation of the wave numbers may be, the influence of the mutual friction is fundamental, in that it reduces the infinite quantities $L_{s0}^{(+)}$ and $\lambda_{s0}^{(-)}$ to the finite values L_4 and λ_3. Specifically, to the first approximation in the mutual coefficient β_n, the penetration depth λ_3 is determined by the equation

$$\lambda_3 \approx -\frac{2\sqrt{(\Omega-2\omega_0)\nu_s}}{\beta_n\Omega} \quad (\beta_n < 0.2\omega_0 < \Omega). \quad (4.11)$$

Under the conditions prevailing in the experiments described in Chapter 2, this formula leads to values which are sufficiently small so that a single disc oscillating in a vessel with dimensions on the order of one

*We recall that the complex wave $k = \sigma + i\tau$ ($\tau > 0$) determines the (phase) velocity of propagation of the wave $-\Omega/\sigma$, the wavelength $L = 2\pi/|\sigma|$, and the penetration depth $\lambda = 1/\tau$.

*These formulas are a generalization of expressions previously obtained by the same authors[30] for the case in which F_{sn} contains terms in curl (ω/ω) (see Sec. 3.2).

†In particular, for $\Omega = 0.581$ sec^{-1} and $\omega_0 = 0.055$ sec^{-1}, the wavelength L_3 is very close at various temperatures to the quantity $L_{s0}^{(-)} = 2\pi\sqrt{\nu_s}/\sqrt{\Omega-2\omega_0}$ and to the experimental value $L_3 = 0.26$ cm, given in Chapter 2.

centimeter may be regarded as immersed in an "unbounded" liquid. At the same time, however, the value of λ_3 considerably exceeds the penetration depths of all of the other waves. This is responsible for the special role, already noted in Chapter 2, which the wave represented by the wave number k_3 ($\approx k_{s0}^{(-)}$) plays in the explanation of the resonance effects observed in the presence of closely-spaced solid surfaces (as in a stack of discs) or in the displacement of the free surface of the rotating helium II when this lies a short distance above the disc.

Moreover, the relatively large value of λ_3 implies that at a certain distance from the oscillating surface of the disc the waves corresponding to the wave number k_3 begin to predominate not only in the motion of the superfluid and the vortex filaments, but also in the motion of the normal component.

We thus arrive at the following description of the wave field generated by a disc oscillating in rotating helium II. In a thin layer near the wall, the motions of the superfluid and normal components of the helium are different. While waves with wave numbers $k_{1,3}$, approximately equal to $k_{s0}^{(\pm)}$, propagate in the superfluid and along the vortex filaments, waves with wave number $k_{2,4}$, nearly equal to $k_{n0}^{(\pm)}$, are observed in the normal component. However, outside of the layer adjacent to the wall, the superfluid, the vortices, and the normal component move in a perfectly similar manner, in accordance with the character of the wave with wave number k_3, which penetrates farthest along the vortex filaments. As a result, it becomes possible to speak of the penetration depth λ_3 as being the penetration depth of the wave in helium II as such, and not in one or another of its components. Such a description becomes all the more reasonable, when, as the number of vortices increases, the viscous penetration depth $(2\nu_n/\Omega)^{1/2}$ reaches half the distance between the vortices, $0.5(\pi\hbar/m\omega_0)^{1/2}$, for in this case there participate in the oscillations, even at some distance from the disc, not only the thin "sleeves" of the normal component surrounding the vortex filaments, but the entire normal fluid as a whole (within the confines of the layer bounded by the penetration depth λ_3).

4.4. Postponing consideration of the more accurate formulas to Sec. 4.6, let us consider the problem of the oscillations of a single disc in rotating helium II, neglecting completely both mutual friction and slippage of the vortices ($q = 0$, $a = 0$). In this approximation it is much easier to analyze the extremely cumbersome expression for the moment **M** of the force acting on the disc [see formula (4.21)]. On the other hand, the main features of the phenomena under consideration are quite clearly evident even in this rough approximation.

As has already been noted several times, in the absence of mutual friction the normal and superfluid components of helium II move independently of one another. Accordingly, the expression for the moment of the force acting on a disc oscillating in rotating helium II breaks up into two independent parts. As a result, the changes in the frequency and damping of its oscillations also appear in the form of a sum of two terms, due respectively to the action of the normal and superfluid components of the helium II on the disc. This is easily seen upon examination of the following formulas:

$$\Omega^2-\Omega_0^2=\frac{\pi R^4\Omega}{2I}\Big[-\sqrt{\frac{\eta_n\varrho_n}{2}}\left(\sqrt{\Omega+2\omega_0}+\sqrt{\Omega-2\omega_0}\right)+\sqrt{\eta_s\varrho_s}\,\frac{2\omega_0}{\Omega}\sqrt{\Omega+2\omega_0}\,\Big], \quad (4.12)$$

$$\delta-\frac{\Omega_0}{\Omega}\delta_0=\frac{\pi^2R^4}{2I\Omega}\Big[\sqrt{\frac{\eta_n\varrho_n}{2}}\left(\sqrt{\Omega+2\omega_0}+\sqrt{\Omega-2\omega_0}\right)+\sqrt{\eta_s\varrho_s}\,\frac{2\omega_0}{\Omega}\sqrt{\Omega-2\omega_0}\,\Big], \quad (4.13)$$

which are valid for $2\omega_0 < \Omega$.* The first of these formulas describes the monotonic increase in the frequency of the oscillations with increasing velocity of rotation ω_0 (see curve 3 of Fig. 9). The second formula describes the dependence of the damping upon the velocity of rotation. The presence of the product $\omega_0\sqrt{\Omega-2\omega_0}$ in this formula insures that this relation goes through a maximum, in qualitative agreement with the experimental data illustrated in Figs. 5, 6, 7, and 9, with the exception that the experimental maximum in damping is observed appreciably to the left ($\omega_0/\Omega \approx 0.1$) of the position expected from formula (4.13) ($\omega_0/\Omega \approx 0.3$).

In the region $2\omega_0 > \Omega$, formulas (4.12) and (4.13) assume the form:

$$\Omega^2-\Omega_0^2=\frac{\pi R^4\Omega}{2I}\Big[-\sqrt{\frac{\eta_n\varrho_n}{2}}\left(\sqrt{\Omega+2\omega_0}-\sqrt{2\omega_0-\Omega}\right)+\sqrt{\eta_s\varrho_s}\,\frac{2\omega_0}{\Omega}\left(\sqrt{\Omega+2\omega_0}+\sqrt{2\omega_0-\Omega}\right)\Big], \quad (4.12')$$

$$\delta-\frac{\Omega_0}{\Omega}\delta_0=\frac{\pi^2R^4}{2I\Omega}\sqrt{\frac{\eta_n\varrho_n}{2}}\left(\sqrt{\Omega+2\omega_0}+\sqrt{2\omega_0-\Omega}\right). \quad (4.13')$$

Formula (4.12′) shows that the monotonic increase of Ω with increasing ω_0 must also persist for rapid rotation, while the quantity $2\omega_0-\Omega$ appears in the second term along with $2\omega_0+\Omega$. In formula (4.13′) the second term, associated with the action of the superfluid component, disappears completely, while the first term describes the increase of δ with increasing ω_0 characteristic of a classical viscous liquid [see formula (2.4) and Fig. 5].

4.5. In expressions (4.8) for the wave numbers $k_{s0}^{(\pm)}$, and particularly in formula (4.13) for the damping, the quantities ν_s and η_s appear as the kinematic and dynamic "viscosities of the superfluid component" (which explains the notation used). If we recall that η_s is the ratio of the tension in the vortex to its circulation, this result may seem somewhat unexpected. We must therefore consider the mechanism of interaction between the oscillating disc and the rotating helium II.

*The first terms of formulas (4.12) and (4.13) are identical with (3.8), (3.9), and (3.10), derived in reference 21 for classical liquids [see also formula (2.4) of the present article].

We note first of all that in accordance with formulas (3.10) and (4.4), the additional terms in the momentum flux tensor due to the rotation of the superfluid component give rise to the following contribution to the φ-component of the force $\mathbf{F}$ with which the liquid acts on the surface of the disc:

$$dF_\varphi = \eta_s \omega_\varphi \, d\sigma. \tag{4.14}$$

(It is recognized here that $N_{0z} = -1$ and that, in an approximation linear in the amplitude of the oscillation, $\omega_z/\omega \approx 1$.) As indicated in reference 30, this force can be interpreted as resulting from the action on the disc of the vortex filaments attached to its surface. Actually, taking Eq. (1.7) into consideration and bearing in mind that the tension vector of the vortex filament is $\epsilon\boldsymbol{\omega}/\omega$, we have* $dF_\varphi = (\omega/\Gamma)(\epsilon\omega_\varphi/\omega)\,d\sigma = \eta_s\omega_\varphi\,d\sigma$.

Thus, in full agreement with the qualitative considerations developed in Chapter 2, the interaction between the oscillating disc and the rotating superfluid liquid is produced by means of vortex filaments attached to the solid surface. However, even from this point of view it is not quite clear how the generation of elastic waves propagating along the vortex filaments leads to dissipation of the energy of the disc. To clarify this problem, Mamaladze[34] undertook the following investigation into the mechanism of interaction between the vortex and the oscillating surface.

Solution of the system (3.2), subject to the appropriate boundary conditions and in the approximation $q = 0$, $a = 0$, shows that there propagate along the vortex filaments two waves with opposite circular polarizations:

$$w_L^{(\pm)} \equiv w_{Lr} \pm i w_{L\varphi} = \mp \Omega\varphi_0 \exp(ik_{s0}^{(\pm)} z) \tag{4.15}$$

(The disc surface corresponds to $z = 0$).

In this connection, the projection of the vortex tension force $\mathbf{F} = \epsilon\boldsymbol{\omega}/\omega$ on the surface of the disc (we shall denote this projection by $\mathbf{F}_{r\varphi}$) can be represented in the form of a sum of two vectors, $\mathbf{F}_{r\varphi}^{(+)}$ and $\mathbf{F}_{r\varphi}^{(-)}$, rotating in opposite directions. This can be readily verified by recalling that $\boldsymbol{\omega} = \operatorname{curl} \mathbf{v}_s$, and by using the relation†

$$w_s^{(\pm)} \equiv w_{sr} \pm i w_{s\varphi} = \mp \frac{2\omega_0}{\Omega} w_L^{(\pm)}.$$

This yields

$$\mathbf{F}_{r\varphi} = \mathbf{F}_{r\varphi}^{(+)} + \mathbf{F}_{r\varphi}^{(-)} = \frac{1}{2}\varphi_0 \varepsilon r (\mathbf{f}^{(+)} + \mathbf{f}^{(-)}), \tag{4.16}$$

where

$$\left.\begin{aligned} f_r^{(-)} &= \sigma_{s0}^{(-)} \cos\Omega t, \\ f_\varphi^{(-)} &= -\sigma_{s0}^{(-)} \sin\Omega t, \end{aligned}\right\} 2\omega_0 < \Omega, \quad \left.\begin{aligned} f_r^{(-)} &= -\tau_{s0}^{(-)} \sin\Omega t, \\ f_\varphi^{(-)} &= -\tau_{s0}^{(-)} \cos\Omega t, \end{aligned}\right\} 2\omega_0 > \Omega,$$
$$f_r^{(+)} = \tau_{s0}^{(+)} \sin\Omega t, \quad f_\varphi^{(+)} = -\tau_{s0}^{(+)} \cos\Omega t. \tag{4.17}$$

In (4.17) a change has been made from complex to real notation. The quantities $\sigma_{s0}^{(\pm)}$ and $\tau_{s0}^{(\pm)}$ are the real and imaginary parts of the complex wave numbers $k_{s0}^{(\pm)}$:

$$\left.\begin{aligned} \sigma_{s0}^{(-)} &= -\sqrt{\frac{\Omega - 2\omega_0}{\nu_s}}, \\ \tau_{s0}^{(-)} &= 0, \end{aligned}\right\} 2\omega_0 < \Omega, \quad \left.\begin{aligned} \sigma_{s0}^{(-)} &= 0, \\ \tau_{s0}^{(-)} &= \sqrt{\frac{2\omega_0 - \Omega}{\nu_s}}, \end{aligned}\right\} 2\omega_0 > \Omega,$$
$$\sigma_{s0}^{(+)} = 0, \quad \tau_{s0}^{(+)} = \sqrt{\frac{\Omega + 2\omega_0}{\nu_s}}. \tag{4.18}$$

It is obvious that the rotation of the vectors $\mathbf{f}^{(+)}$ and $\mathbf{f}^{(-)}$ is due to the $w_L^{(+)}$ and $w_L^{(-)}$ waves, respectively.

Let us now compare the sequence of directions of the vectors $\mathbf{F}_{r\varphi}^{(\pm)}$ (in time) with the sequence of directions of the disc's oscillations, which (also in real notation) is given by the formula

$$\dot{\varphi} - \omega_0 = -\Omega\varphi_0 \sin\Omega t \tag{4.19}$$

[the motion of the disc obeys the law $\varphi = \omega_0 t + \varphi_0 \exp(i\Omega t)$].

Examining Eqs. (4.17) and (4.19) we can readily see that each of the vectors $\mathbf{F}_{r\varphi}^{(-)}$, for $2\omega_0 > \Omega$, and $\mathbf{F}_{r\varphi}^{(+)}$, through each quarter-cycle, alternately accelerates the motion of the disc (as the latter approaches the equlibrium position) and retards it (as the disc moves away from this position). Thus, the action of these components of the tension force causes the vortices to behave as a sort of addition to the elastic suspension of the disc. This gives rise to the increase in the oscillation frequency represented by the second terms in formulas (4.12) and (4.12′). It is clear that an interaction of this kind cannot lead to damping of the oscillations.

A different behavior is exhibited by the vector $\mathbf{F}_{r\varphi}^{(-)}$ when $2\omega_0 < \Omega$. As shown in Fig. 15c, this vector now constantly retards the motion of the disc, and consequently gives rise to the excess damping described by the second term in (4.13). Thus, the dissipation of the oscillation energy of the rotating superfluid liquid is explained by the transfer of the disc's energy to infinity via the vortex filaments along which the $w_L^{(-)}$ wave propagates (when $2\omega_0 < \Omega$).

It is quite clear that were the mutual friction actually to vanish completely, the damping mechanism just described would be possible only in an unbounded liquid. Otherwise the $w_L^{(-)}$ wave, which has an infinite penetration depth, would be reflected from the boundary surface, however remote, and the returning wave of the same amplitude would completely suppress the retarding effect, returning the energy carried away by the traveling wave. However, as already noted in Sec. 4.3, the presence of mutual friction makes the penetra-

*We note that the validity of this result is not confined to the approximation $q = 0$, $a = 0$ used in the present section.

†We use here Eqs. (4.3) taking account of the fact that

$$\frac{\partial \chi_s}{\partial r} = 0, \quad u_{sr} = r w_{sr}(z) \text{ and } u_{s\varphi} = r w_{s\varphi}(z)$$

correspond to the boundary conditions for the problem now under consideration.

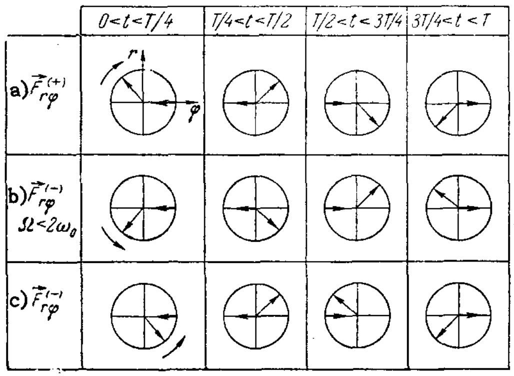

FIG. 15. Schematic representation of the sequence of directions of the components of the vortex tension projected onto the surface of the disc, $\mathbf{F}_{r\varphi}^{(+)}$ and $\mathbf{F}_{r\varphi}^{(-)}$ and the direction of oscillation of the disc. a) During the first quarter cycle the disc approaches the equilibrium position, and the force $\mathbf{F}_{r\varphi}^{(+)}$ accelerates its motion; in the second quarter cycle the disc, after passing through equilibrium position, moves away from it, and the force $\mathbf{F}_{r\varphi}^{(+)}$ begins to retard its motion; in the third and fourth quarter cycles the force $\mathbf{F}_{r\varphi}^{(+)}$ again accelerates the disc towards the equilibrium position and retards the departure from equilibrium. b) The behavior of the force $\mathbf{F}_{r\varphi}^{(-)}$ is similar, in the case of fast rotation of the liquid ($2\omega_0 > \Omega$). c) In the case of slow rotation of the liquid ($\Omega > 2\omega_0$) the force $\mathbf{F}_{r\varphi}^{(-)}$ constantly retards the motion of the disc.

tion depth λ_3 finite [see formula (4.11)]. This means, in turn, that under real conditions the energy of the disc, which is elastically transferred to the vortices, is subsequently dissipated by the mutual friction, which causes attenuation of the $w_L^{(-)}$ wave. Rotation of the superfluid component of the helium II; therefore, will always cause appreciable damping of the oscillations of the disc, provided the nearest solid surface does not come much closer to the disc than the penetration depth λ_3.

Thus, it is the tension of the elastically-deformed vortex filaments, rather than the change in the effective density of the superfluid component (see Sec. 2.5 and also p. 20 which is the true cause of the damping and change in frequency of the oscillations of discs in a rotating superfluid liquid. This is particularly clear in the case of a single disc oscillating in an "infinite" liquid, for which (see p. 16 formulas (4.3) assume the form

$$w_{sr} = \frac{2\omega_0}{i\Omega} w_{L\varphi}, \qquad w_{s\varphi} = -\frac{2\omega_0}{i\Omega} w_{Lr}. \tag{4.20}$$

It follows from these formulas that when the vortices are fully attached to the disc surface ($w_{L\varphi} = i\Omega\varphi_0$, $w_{Lr} = 0$), the superfluid component near the surface moves only radially* ($w_{s\varphi} = 0$, $w_{sr} = 2\omega_0\varphi_0$), i.e., perpendicular to the direction of motion of the disc, and not in or out of phase with the disc. It should also be noted that even if there were a direct interaction between the disc and the superfluid liquid, it could not lead under these conditions to a change in the frequency and damping of the disc.

4.6. Mamaladze and Matinyan[33] have considered the problem of the oscillations of a disc in an unbounded liquid without imposing the limitations $q = 0$ and $a = 0$, introduced in Sec. 4.4 and 4.5. They have shown that the moment M of the force acting on the oscillating disc has the form† $M = m\varphi_0 \exp(i\Omega t)$, where

$$m = \frac{\pi R^4}{\varphi_0}\left[\eta_n\left(\frac{dw_{n\varphi}}{dz}\right)_{z=0} + \eta_s\left(\frac{dw_{sr}}{dz}\right)_{z=0}\right] = \frac{\pi R^4}{2}\left\{\eta_n \frac{A^{(+)}}{(k_1+k_2)\left(1-\frac{\nu_s q_1^{(+)}}{2\omega_0}\right)-\frac{a}{\Omega}\left[k_1k_2\left(1-\frac{\nu_s q_1^{(+)}}{2\omega_0}\right)+k_{s0}^{(+)2}+q_1^{(+)}\right]}\right.$$

$$+\eta_n \frac{A^{(-)}}{(k_3+k_4)\left(1-\frac{\nu_s q_1^{(-)}}{2\omega_0}\right)-\frac{a}{\Omega}\left[k_3k_4\left(1-\frac{\nu_s q_1^{(-)}}{2\omega_0}\right)+k_{s0}^{(-)2}+q_1^{(-)}\right]} + i\eta_s \frac{2\omega_0\left[k_1k_2\left(1-\frac{\nu_s q_1^{(+)}}{2\omega_0}\right)+k_{s0}^{(+)2}+q_1^{(+)}\right]+q_1^{+}\Omega}{(k_1+k_2)\left(1-\frac{\nu_s q_1^{(+)}}{2\omega_0}\right)-\frac{a}{\Omega}\left[k_1k_2\left(1-\frac{\nu_s q_1^{(+)}}{2\omega_0}\right)+k_{s0}^{(+)2}+q_1^{(+)}\right]}$$

$$\left.+i\eta_s \frac{2\omega_0\left[k_3k_4\left(1-\frac{\nu_s q_1^{(-)}}{2\omega_0}\right)+k_{s0}^{(-)2}+q_1^{(-)}\right]-q_1^{(-)}\Omega}{(k_3+k_4)\left(1-\frac{\nu_s q_1^{(-)}}{2\omega_0}\right)-\frac{a}{\Omega}\left[k_3k_4\left(1-\frac{\nu_s q_1^{(-)}}{2\omega_0}\right)+k_{s0}^{(-)2}+q_1^{(-)}\right]}\right\}, \tag{4.21}$$

where

$$A^{(+)} = -\Omega\left[(k_1^2+k_1k_2+k_2^2)\left(1-\frac{\nu_s q_1^{(+)}}{2\omega_0}\right)-k_{s0}^{(+)2}-q_1^{(+)}\right]-2\omega_0 q_2^{(+)}\left(1+\frac{\nu_s k_{s0}^{(+)2}}{2\omega_0}\right)+ak_1k_2(k_1+k_2)\left(1-\frac{\nu_s q_1^{(+)}}{2\omega_0}\right)$$

and

$$A^{(-)} = -\Omega\left[(k_3^2+k_3k_4+k_4^2)\left(1-\frac{\nu_s q_1^{(-)}}{2\omega_0}\right)-k_{s0}^{(-)2}-q_1^{(-)}\right]+2\omega_0 q_2^{(-)}\left(1+\frac{\nu_s k_{s0}^{(-)2}}{2\omega_0}\right)+ak_3k_4(k_3+k_4)\left(1-\frac{\nu_s q_1^{(-)}}{2\omega_0}\right).$$

Examining this formula, we readily see that, as before (cf. Sec. 4.4), it consists of two parts, describing respectively the action on the disc by the superfluid (terms containing η_s) and normal (terms containing η_n) components of the helium II. The mutual friction, however, gives rise to the inclusion in each of these parts of quantities associated with the effects of the other component.

It must also be noted that in the case of full slippage of the vortex filaments, when $a = \infty$, the terms proportional to η_s vanish. This is in full accordance with the mechanism described in Sec. 4.5 for the interaction between the vortices and the oscillating disc. In this

*The radial motion of the superfluid for axial displacements of the vortices is a consequence of the Magnus effect.

†This formula was obtained by the authors of reference 30 for the case in which terms in curl ($\boldsymbol{\omega}/\omega$) are introduced into $\mathbf{F}_{sn}$ (see Sec. 3.2).

case as well, however, the superfluid exerts an appreciable influence on the oscillations of the disc, because of the presence of mutual friction (see Sec. 4.7).

4.7. Formula (4.21) is too cumbersome to be useable without the aid of computer techniques. Figure 16 shows the preliminary results of a calculation of the velocity dependence of the damping of the disc, currently being carried out by Kiknadze and Tkemaladze with the "Ural" computer, using formulas (4.5), (4.6),

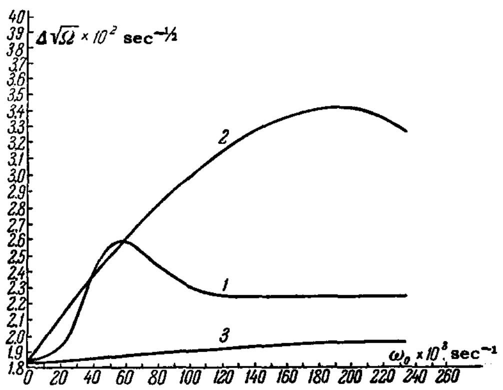

FIG. 16. Velocity dependence of the damping of the disc oscillations ($\Delta = \delta - \delta_0\, \Omega_0/\Omega$, $\Omega = 0.581$ sec^{-1}). 1 – Smoothed curve drawn through the experimental data of Fig. 5; 2 – theoretical curve based on formulas (4.6) and (4.21) for $a = 0$ (absolutely rough disc), $\nu_s = 8.5 \times 10^{-4}$ cm^2/sec, and for values of the mutual friction coefficients corresponding to $T = 1.78$°K (see reference 32); 3 – theoretical curve for $a = \infty$ (absolutely smooth disk).

and (4.21) for values of the parameters corresponding to the experimental conditions illustrated in Fig. 5. The slight rise of the experimental data above the theoretical curve, 2 ($a = 0$), observed for $\omega_0 = 0.04 - 0.06$ sec^{-1}, apparently means that the value $\nu_s = 8.5 \times 10^{-4}$ cm^2/sec, used in the calculations was not sufficiently large.

It is characteristic that the curve 3 ($a = \infty$) differs rather substantially from the lower curve of Fig. 5. This illustrates the effect of the presence of the Onsager-Feynman vortex filaments on the oscillations of the disc, even in the complete absence of any direct connection between the disc and the vortices slipping along its surface.

It can also be seen that curve 2 does not differ greatly from the corresponding curve obtained in reference 30. This means that the terms containing curl (ω/ω) in the expression for $\mathbf{F}_{sn}$ exert a much smaller influence on the form of the velocity dependence of the damping of the disc than on the expression for the penetration depth λ_3 given in Sec. 4.3 (where these terms played the decisive role at small values of ω_0).

Outside the interval $\omega_0 = 0.04 - 0.06$ sec^{-1}, the experimental data lie between the two theoretical curves corresponding to rigid attachment and to complete slippage of the vortices. A set of values was therefore selected for the slip coefficient (again using the "Ural" computer), to provide complete agreement between theory and experiment. The results of these calculations, presented in Fig. 17, are evidently in full agreement with the ideas advanced in Sec. 2.7.

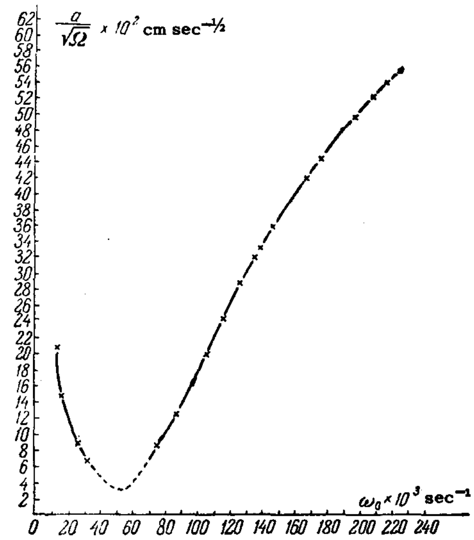

FIG. 17. Velocity dependence of the slippage coefficient. The dependence was chosen such as to make formulas (4.6) and (4.21) result in curve 1 of Fig. 16.

In the region $2\omega_0 \gtrsim \Omega$ (not shown in Figs. 16 and 17), calculation indicates an increase in damping with increasing ω_0 for all values of a. However, the picture in this region is not quite clear as yet, especially when it comes to the behavior of the slip coefficient at $2\omega_0 \approx \Omega$.

4.8. In Chapter 2, particularly in Section 2.5, we compared the concept of the effective density of the superfluid component with the concept of the rotating superfluid liquid as a medium whose behavior is determined by the elastic-plastic properties of the array of Onsager-Feynman vortices permeating it.

In Sec. 4.5 the second concept was corroborated by using the example of a single disc, in which case not only the frequency variation, but also the excess damping was found to be associated with the generation of elastic waves in the vortex filaments.

The critical analysis of the concept of effective density leads to even more characteristic results if the oscillations of a stack of discs are used as an example. (The arguments that follow have been advanced by Mamaladze.)

We note first that the problem of the oscillations of a stack of discs in a rotating superfluid liquid can be solved by the method described in Sec. 4.2. If we use again the approximation $q = 0$, $a = 0$, then the excess damping of the stack vanishes, and the frequency of the disks is given by*

*We write here for the difference $\Omega^2 - \Omega_0^2$ the sum $(\Omega^2 - \Omega_n^2) + (\Omega_n^2 - \Omega_0^2)$, where the second term represents the effect of the normal component of the helium II on the oscillating body, and the first represents the action of the superfluid component (in the approximation $q = 0$ these effects can be separated from each other).

$$\Omega^2 - \Omega_n^2 = -\frac{I_s}{I}\Omega^2 A(\omega_0, \Omega, l). \qquad (4.22)$$

In (4.22), Ω_n stands for the frequency which would be expected were the stack to interact only with the normal component of the helium II, I_s is the moment of inertia of the superfluid trapped in the stack, I the moment of inertia of the stack itself, and the function $A(\omega_0, \Omega, l)$ is given by

$$A(\omega_0, \Omega, l) = -\alpha \frac{Z^{(+)} - Z^{(-)} + \alpha(Z^{(+)} + Z^{(-)} - 2Z^{(+)}Z^{(-)})}{2 + \alpha(Z^{(+)} - Z^{(-)}) - \alpha^2(Z^{(+)} + Z^{(-)})}, \qquad (4.23)$$

where $\alpha = 2\omega_0/\Omega$, $Z^{(\pm)} = \tan(k_{10}^{(\pm)}l)/k_{s0}^{(\pm)}l$, and l is half the distance between neighboring discs. It must be emphasized that (4.22) is derived from perfectly general equations, and while these can be interpreted in terms of the elastic properties of the vortices, they are not associated with any specific preconceived notion of the mechanism for the phenomena under consideration.

We shall now introduce the concept of an effective density for the superfluid component participating in the oscillations of the stack. We shall start with the familiar formula for the frequency of oscillation in terms of the torque of the suspension f and the moment of inertia of the oscillating body I:

$$\Omega^2 = \frac{f}{I}. \qquad (4.24)$$

In accordance with the nature of the effect of the vortex tension on the oscillating body, as established in the preceding section, one might describe the change in the oscillation frequency as due to an effective change in the torque f as well as the moment of inertia I. However, in order to display the contradiction inherent in the effective density concept, we shall, following Hall, take into consideration only changes in I. We can then write down the following set of equations:

$$\left.\begin{aligned} \Omega_0^2 &= \frac{f}{I}, \\ \Omega_{n0}^2 &= \frac{f}{I + \Delta I_{n0}}, \\ \Omega_n^2 &= \frac{f}{I + \Delta I_n}, \\ \Omega^2 &= \frac{f}{I + \Delta I_n + \Delta I_s}, \end{aligned}\right\} \qquad (4.25)$$

where ΔI_{n0} is the moment of inertia of the normal component entrained by the oscillations of the stack in stationary helium II, ΔI_n is the same quantity in the rotating case, ΔI_s is the moment of inertia of the superfluid component, also entrained by the oscillations of the stack as the liquid rotates; the symbols Ω_0, Ω_n, and Ω have already been defined, while Ω_{n0} is the oscillation frequency in stationary helium II.

The relative effective density of the oscillating superfluid component is naturally defined by the expression

$$\left(\frac{\rho'}{\rho_s}\right)_1 = \frac{\Delta I_s}{I_s} \qquad (4.26)$$

[the symbol I_s has been defined in connection with formula (4.22)]. From (4.25) it follows directly that

$$\left(\frac{\rho'}{\rho_s}\right)_1 = \frac{\Omega_n^2 - \Omega^2}{\Omega^2}\frac{\Omega_0^2}{\Omega_n^2}\frac{I}{I_s}. \qquad (4.27)$$

However, if we rewrite Hall's formula (2.1) using the notation of the present section, it will be seen that Hall measured a somewhat different quantity [which we shall denote by $(\rho'/\rho_s)_2$]:

$$\left(\frac{\rho'}{\rho_s}\right)_2 = \frac{\Omega_{n0}^2 - \Omega^2}{\Omega^2}\frac{\Omega_0^2}{\Omega_{n0}^2}\frac{I}{I_s}. \qquad (4.28)$$

At the same time, $\Omega_{n0} \neq \Omega_n$, since the oscillation frequency depends on the velocity of rotation of the liquid even in the case of classical liquids.*

Thus, the determination of the effective density (4.28) cannot be considered correct even from Hall's point of view [the quantity $(\rho'/\rho_s)_2$ represents the ratio $(\Delta I_s + \Delta I_n - \Delta I_{n0})/I_s$, which has no physical meaning].

Of more fundamental significance, however, is the discrepancy between (4.28) and the results of Hall's theoretical calculations, which lead, in the approximation q = 0 and a = 0, to the formula

$$\left(\frac{\rho'}{\rho_s}\right)_3 = A(\omega_0, \Omega, l). \qquad (4.29)$$

On the other hand, with the aid of (4.27) and (4.22) it can easily be shown that the correctly defined effective density $(\rho'/\rho_s)_1$ is equal to

$$\left(\frac{\rho'}{\rho_s}\right)_1 = \frac{\Omega_0^2}{\Omega_n^2}A(\omega_0, \Omega, l). \qquad (4.30)$$

As regards the Hall effective density $(\rho'/\rho_s)_2$, we readily obtain, by comparing (4.27), (4.28), and (4.30)

$$\left(\frac{\rho'}{\rho_s}\right)_2 = \frac{\Omega_{n0}^2 - \Omega^2}{\Omega_n^2 - \Omega^2}\frac{\Omega_0^2}{\Omega_{n0}^2}A(\omega_0, \Omega, l). \qquad (4.31)$$

Thus, all three quantities $(\rho'/\rho_s)_{1,2,3}$ are different, and it becomes clear (a fact which must be especially emphasized) that the results of Hall's measurements of $(\rho'/\rho_s)_2$ were compared by him with the theoretical variation of the entirely different quantity $(\rho'/\rho_s)_3$.

Naturally, at low temperatures (Hall carried out his experiments at T = 1.27° K) we have $\Omega_n \approx \Omega_{n0} \approx \Omega_0$, and consequently Hall's error was not numerically large. It is nonetheless important to inquire into the reasons for the fundamental discordance just demonstrated between the quantities $(\rho'/\rho_s)_1$ and $(\rho'/\rho_s)_3$.

---

*For example, in the case of a single disc Ω_n is defined, according to reference 21, by the formulas

$$\Omega_n^2 - \Omega_0^2 = -\frac{\pi R^4 \Omega}{2I}\sqrt{\frac{\rho_n \eta_n}{2}}\left(\sqrt{\Omega + 2\omega_0} + \sqrt{\Omega - 2\omega_0}\right) \quad \text{for } 2\omega_0 < \Omega$$

and

$$\Omega_n^2 - \Omega_0^2 = -\frac{\pi R^4 \Omega}{2I}\sqrt{\frac{\rho_n \eta_n}{2}}\left(\sqrt{\Omega + 2\omega_0} - \sqrt{2\omega_0 - \Omega}\right) \quad \text{for } 2\omega_0 > \Omega,$$

while Ω_{n0} is the value of Ω_n when $\omega_0 = 0$.

The point is, that in the theoretical part of his paper[19] Hall followed a procedure which differs from that which we have described in Section 4.2. Solving Eq. (3.1), he determined, not the moment of the force (which would permit direct calculation of the oscillation frequency), but rather the angular momentum of the superfluid oscillating between two neighboring discs. This angular momentum is equal to $2\pi R^2 \rho_S \int_0^l v_{S\varphi}\, dz$. He then defined the quantity $(\rho'/\rho_S)_3$ as the ratio of this angular momentum to the angular momentum of the liquid corresponding to its complete entrainment by the oscillating surface (i.e., to the value $2\pi R^2 \rho_S v_{\sigma\varphi} l$):

$$\left(\frac{\varrho'}{\varrho_s}\right)_3 = \frac{1}{l}\int_0^l \frac{v_{s\varphi}}{v_{\sigma\varphi}}\, dz. \qquad (4.32)$$

It might appear that this equality is in full agreement with our definition of ρ'/ρ_S as equal to $\Delta I_S / I_S$. One must not forget, however, that the superfluid does not by any means take part in the oscillations as a solid body. Consequently, the effective moments of inertia, as determined by various means, need not necessarily agree. This is why the effective density $(\rho'/\rho_S)_1$, determined from the frequency of the oscillations, *does not agree with the effective density* $(\rho'/\rho_S)_3$, determined from the angular momentum. In other words, the inconsistency in Hall's concept results from the excessively literal use of the expression "entrainment of the superfluid liquid by the oscillations of the stack of discs."*

4.9. As has already been noted, one can at low temperatures, assume, approximately, that $\left(\frac{\rho'}{\rho_S}\right)_1 \approx \left(\frac{\rho'}{\rho_S}\right)_2 \approx \left(\frac{\rho'}{\rho_S}\right)_3$. Under these same temperature conditions ($T = 1.27°K$), if they are actually realized, one can also regard the approximation $q = 0$ as satisfactory at rotational velocities ω_0 well away from $\Omega/2$. Therefore the agreement noted by Hall between the theory he developed and his own experimental data (see Figs. 2 and 3) is not fortuitous. To obtain complete quantitative agreement between theory and experiment it would only be necessary to forego the approximation $a = 0$. Generalizing formula (4.23) to the case of partial slippage of the vortex filaments along the surfaces of the discs, Hall obtained for the function $A(\omega_0, \Omega, l)$ the following limiting expressions

$$A(\omega_0, \Omega, l) = \frac{\omega_0}{\Omega}\left[\frac{1}{1+\gamma_1^2 \operatorname{tg}^2(k_1 l)}\frac{\operatorname{tg}(k_1 l)}{k_1 l} - \frac{1}{1+\gamma_1^2 \operatorname{th}^2(k_1 l)}\frac{\operatorname{th}(k_1 l)}{k_1 l}\right] \quad \text{for } 2\omega_0 \ll \Omega \qquad (4.33)\dagger$$

and

$$A(\omega_0, \Omega, l) = 1 - \frac{1}{1+\gamma_2^2 \operatorname{th}^2(k_2 l)}\frac{\operatorname{th}(k_2 l)}{k_2 l} \quad \text{for } 2\omega_0 \gg \Omega. \qquad (4.34)$$

Here $k_1 = (\Omega/\nu_S)^{1/2}$, $k_2 = (2\omega_0/\nu_S)^{1/2}$, $\gamma_1 = ak_1/\Omega$ and $\gamma_2 = ak_2/\Omega$. (Inasmuch as the inequality $2\omega_0 \ll \Omega$ was not particularly rigorously satisfied under the conditions prevailing in his experiments, Hall replaced the quantity k_1 in (4.33) by the wave number $k_{S0}^{(-)} = [(\Omega - 2\omega_0)/\nu_S]^{1/2}$. Hall estimated the empirical values of the coefficients γ_1 and γ_2 to be $\gamma_1 = \gamma_2 = 0.5$, which corresponds in order of magnitude to the values of a given in Fig. 17 ($a/\sqrt{\Omega} \sim 10^{-2}$ cm-sec$^{-1/2}$).

Despite his experimental confirmation of formula (4.33) and (4.34) it must be pointed out that Hall performed the limiting transitions to the extreme cases $2\omega_0 \ll \Omega$ and $2\omega_0 \gg \Omega$ with sufficient precision. Without stopping to consider the error introduced in the former case,* which did not lead to disagreement with the experimental data, let us consider the case $2\omega_0 \gg \Omega$. Inasmuch as slippage of the vortices does not play an important role at high angular velocities (see reference 19), one may again use the approximation $q = 0$, $a = 0$. In this approximation Hall obtained

$$A(\omega_0, \Omega, l) = 1 - \frac{\operatorname{th}(k_2 l)}{k_2 l} \quad \text{for } 2\omega_0 \gg \Omega. \qquad (4.35)$$

However, as follows directly from (4.23), it would be more correct to use the following expression, derived by Mamaladze:

$$A(\omega_0, \Omega, l) = \frac{1 - \frac{\operatorname{th}(k_2 l)}{k_2 l}}{1 - \frac{1}{\left(\frac{2\omega_0}{\Omega}\right)^2 \frac{\operatorname{th}(k_2 l)}{k_2 l}}} \quad \text{for } 2\omega_0 \gg \Omega. \qquad (4.35)$$

It is clear from this equation that the quantity $A(\omega_0, \Omega, l)$ cannot in any sense be considered a function of the product $\omega_0^{1/2} l$ alone (see Fig. 2). As $\omega_0 \to \infty$ this function tends to unity, as was noted by Hall; for $l \to \infty$, however, its limiting value is not unity, but zero. Therefore, upon increasing the spacing of the stack for large but fixed velocity of rotation, one should find that the curve shown in Fig. 2 passes through a maximum and tends asymptotically to the x axis.

4.10. We shall now consider the oscillations of a single disc near the free surface of a rotating superfluid liquid. This phenomenon has already been described qualitatively in Sec. 2.9, where it was shown to lead to the appearance of resonance effects.

Let us again use the approximation $q = 0$, $a = 0$. We already know that this approximation does not provide a complete description of the phenomena considered here. In particular, it leads to zero damping

*In the case of oscillations of a single disc in an unbounded liquid $(\rho'/\rho_s)_3 = 0$, i.e., "entrainment" of the superfluid component does not occur [see also formula (4.20) in Section 4.5 and the subsequent discussion]. However, even in this case the effective density of the superfluid, as determined from the oscillation frequency, differs from zero.

†tg = tan; th = tanh.

*It is connected with the fact that the quantities $\alpha Z^{(-)}$ in formula (4.23) are not negligibly small when $\alpha \ll 1$, but $k_{s0}^{(-)} l \approx (2n-1)\pi/2$.

(from q = 0) and to infinite peaks in the velocity dependence of the oscillation frequency (from a = 0). It is fully adequate, however, for the treatment of the experimental data obtained in the present case, which were used only to determine the wavelength.

In order to provide a theoretical background for the results of these experiments, it is therefore sufficient to determine the resonance conditions.

Using the method described in Sec. 4.2, Mamaladze obtained for the moment M_S of the force with which the superfluid component acts on the surface of the oscillating disc an expression of the form

$$M_s = m_s \varphi_0 \exp(i\Omega t),$$

$$m_s = \frac{1}{2}\pi R^4 \eta_s \omega_0 [k_{s0}^{(+)} \operatorname{tg}(k_{s0}^{(+)}H) + k_{s0}^{(-)} \operatorname{tg}(k_{s0}^{(-)}H)]. \quad (4.36)$$

Here H is the distance from the surface of the disc to the free surface, and is assumed to be much greater than the depth of the meniscus or the amplitude of the oscillations of the free surface. It follows from (4.36) that the resonance condition will be $k_{S0}^{(-)} H = (2n-1)\pi/2$ for which H is an integral multiple of one-quarter of the wavelength $L_{S0}^{(-)} = 2\pi[\nu_S/(\Omega - 2\omega_0)]^{1/2}$ [see formula (2.11)].

It is quite clear that when the distance H is a multiple of one-quarter of the wavelength L_3, conditions for resonance are established even for $q \neq 0$ (it is understood that H is much greater than the penetration depths λ_1, λ_2, and λ_4, but is commensurate with λ_3; see Sec. 4.3). Therefore, in order to replace condition (2.11) by a more exact equation, it is sufficient to replace $L_{S0}^{(-)}$ by the wavelength $L_3 = 2\pi/\mathrm{Re}(k_3)$:

$$H = (2n-1)\frac{L_3}{4} = \frac{1}{2}(2n-1)\pi \operatorname{Re}(k_3) \quad (n = 1, 2, \ldots). \quad (4.37)$$

This formula is a generalization of (2.11) for the case in which mutual friction is taken into account.

4.11. Mamaladze and Matinyan[35] have also solved the problem of the torsional oscillations of a cylinder in rotating helium II. They have shown that the oscillating surface generates in the normal component of the helium II a cylindrical wave, the wave number of which is determined by the following equation (Im (k) > 0):

$$k^2 = -\frac{i\Omega}{\nu_n}\left[1 + i\frac{2\omega_0}{\Omega}\beta_s\left(1 - i\frac{2\omega_0}{\Omega}\frac{\beta_n}{1 + i\frac{2\omega_0}{\Omega}\beta_n}\right)\right]. \quad (4.38)$$

Owing to the mutual friction, similar waves are also propagated in the superfluid component of the helium II, and also in the Onsager-Feynman vortex system. These waves, unlike those in the case of an oscillating disk, are not coupled directly to the oscillating surface.

The penetration depth λ of this wave (which propagates radially) is close to the penetration depth of ordinary viscous waves:

$$\lambda = \frac{1}{\operatorname{Im}(k)} \approx \sqrt{\frac{2\nu_n}{\Omega}} \equiv \lambda_0. \quad (4.39)$$

For the velocity dependence of the damping of the oscillations of a hollow, thin-walled ($R_{id} \approx R_{0d}$), "heavy" ($\Omega \approx \Omega_0$) cylinder, the following formula was obtained:

$$\frac{\delta_2 - \delta_1}{l_2 - l_1} = \frac{2\pi^2 R^3}{I}\sqrt{\frac{2\eta_n\varrho_n}{\Omega}}\left(1 - \frac{\omega_0}{\Omega}\beta_s\right)\left(1 - \frac{3\lambda_0}{R}\right), \quad (4.40)$$

where δ_2 and δ_1 are the logarithmic damping decrements for immersion of the cylinder to depths l_2 and l_1 respectively.* (Subtraction of δ_1 from δ_2 automatically excludes the edge effects).

Formula (4.40) shows that the linear increase in damping with increasing velocity of rotation is due entirely to the presence of mutual friction. It follows from this, in particular, that for classical liquids the damping of the torsional oscillations of the cylinder should not depend on the velocity of rotation of the liquid.

The conclusions of this section have all been fully corroborated by the experimental data reported by Tsakadze and Chkheidze.[29]

* * *

Concluding our review of the experimental and theoretical efforts devoted to the investigation of the oscillations of solids in rotating helium II, we can state with assurance that the whole range of observed phenomena (and their physical interpretation) is accommodated by the Onsager-Feynman scheme. It can be regarded as established that the sharp distinction between the laws governing the oscillations of solids in rotating helium II and the corresponding laws for classical liquids is due entirely to the quantization and localization of the circulation characteristic of the rotation of a superfluid liquid.

It must be noted that the investigation of the vortical motion of a superfluid liquid cannot be regarded as complete. This pertains in particular to such problems as the exact quantum-mechanical description of the structure of the vortex filament, the mechanism of vortex formation, the behavior of the vortex near a solid surface, etc. A solution of these problems may modify somewhat the results presented in Chapters 3 and 4.

Nevertheless, it does not appear that the general results obtained in the investigations described here will change significantly. We can consider it as established that rotating helium II is a medium whose unique characteristics are a consequence of the elastic-plastic properties of a system of Onsager-Feynman vortex filaments lying parallel to the axis of rotation. In particular, the presence of such a system explains the unique spatial anisotropy in the viscous properties of rotating helium, which manifests itself in the differing nature

*Formula (4.40) is written out in an approximate form which is linear in the products of $2\omega_0/\Omega$ with the mutual friction coefficients and in the ratio λ_0/R.

of the velocity dependences found for the damping of the oscillations of bodies whose surfaces are, respectively, perpendicular and parallel to the vortex filaments.

Another characteristic feature of rotating helium is the existence of a penetration depth in helium II, but not in its normal component. This makes it possible to introduce the concept of an effective viscosity for helium II which describes this liquid in the case of relatively fast rotation. It must be emphasized that the viscosity thus introduced will also have spatial anisotropy and will be velocity-dependent.

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