

CASCADE THEORY OF SHOWERS*

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I. BASIC EQUATIONS OF THE THEORY

1. Basic Equations of the Theory

IN traversing matter, high-energy*** electrons

*The present article is an abbreviation of an article by the late S. Z. Belen'kiĭ and I. P. Ivanenko written in 1956. New results have been added.

**Deceased.

***High energy particles are those for which the processes of bremsstrahlung radiation and pair production are substantial when the particles traverse matter.

The process of multiplication and the simultaneous splintering of the energy continues until the energy of the majority of the particles is near the critical energy. Then the charged particles begin to lose a large part of their energy by ionization and the cascade begins gradually to die out. The probabilities of these processes taking place with electron-photon radiation in a substance are given by quantum electrodynamics. The theory of showers must determine the probability that in the element of solid angle $(\omega, \omega + d\omega)$ and in the energy interval $(E, E + dE)$ there will appear, at a certain depth, N_1 electrons and N_2 photons (a beam of electrons or photons with given angular and energy distributions is incident on the boundary of a layer of a certain substance at $t = 0$) which pass through a plane perpendicular to the

shower axis* at the distance $(r, r, + dr)$. Mathematically this problem is very complicated and to obtain the more important characteristics of the cascade shower it is sufficient to know only the average number of electrons and photons at a given depth. We therefore examine the average characteristics of the shower.

In all elementary processes at high energies, the angles at which secondary electrons and photons are emitted are extremely small, of the order of $m_e c^2/E$, where E is the energy of the primary particle. Rutherford scattering of the charged particles of a shower is also small, at least in substances with small Z ; therefore a shower develops basically in the direction of motion of the primary particle. This allows us to examine the development of the shower with its depth and in its angular and space distribution as two different problems. At first we determine the function that describes the development of the shower with depth (without considering the increase in the path of the particles because of scattering), and then we use that function to determine the angular and spatial distributions of the particles in the shower.

We begin with the basic equations of the one-dimensional cascade theory which include the processes of pair production by photons, bremsstrahlung, and ionization losses of electrons:

$$\begin{aligned} \frac{\partial P(t, E)}{\partial t} = & 2 \int_E^\infty \Gamma(t, E') W_p(E', E) dE' \\ & + \int_E^\infty P(t, E') W_r(E', E' - E) dE' - \\ & - \int_0^E P(t, E) W_e(E, E') dE' - \beta \frac{\partial P(t, E)}{\partial E}, \end{aligned} \quad (1.1a)$$

$$\begin{aligned} \frac{\partial \Gamma(t, E)}{\partial t} = & \int_E^\infty P(t, E') W_e(E', E) dE' \\ & - \int_0^E \Gamma(t, E) W_p(E, E') dE'. \end{aligned} \quad (1.1b)$$

A detailed derivation of these equations can be found in the monographs of S. Z. Belen'kiĭ² and B. Rossi.³ Here $P(t, E)$ and $\Gamma(t, E)$ are, respectively, functions giving the average number of electrons and photons in the energy interval $(E, E + dE)$ at a depth t in a layer of the substance under

*The axis of the shower is straight, continuing in the direction of motion of the primary particle.

examination; $W_p(E', E)$ is the differential probability that a photon of energy E' will produce in unit path an electron-positron pair with a positron energy E and an electron energy $E' - E$; $W_e(E, E')$ is the differential probability per unit path for the radiation by an electron of energy E of a photon of energy E' ; β is the average ionization loss of charged particles per unit path of the substance.²

Equations (1.1a) and (1.1b) form a system of two linear homogeneous integro-differential equations. They are similar to equations obtained in the theory of diffusion and are therefore often called diffusion equations. They relate the number of electrons and photons at depth t with the number of electrons and photons at depth $t + dt$. The linear integro-differential operators in the right side of (1.1) act only on the variable E , the particle energy at depth t ; the energy of the primary particle or the boundary energy of the spectrum of primary particles E_0 enters into the equation as a parameter. For a given primary energy E_0 it is possible to compute the quantities $P(E_0, t, E)$ and $\Gamma(E_0, t, E)$ only when the functions P and $\Gamma(E_0, t, E)$ are known for energy E' larger than E , since particles with energy E' at depth t are produced by higher energy particles. This physically obvious circumstance corresponds to the fact that the integro-differential operator in the right side of (1.1) acts only on an energy larger than or equal to E .

Sometimes it is more convenient to use the equations with the variable E_0 ; the variable E in this case enters into the equations as a parameter.^{4,5} We can then compute for a certain secondary energy E the functions $P(E_0, t, E)$ and $\Gamma(E_0, t, E)$ from the primary energy E_0 if the functions P and $\Gamma(E_0, t, E)$ are known for all energies E'_0 between E_0 and E , since a particle of energy E_0 can form secondary particles with energy E either directly or through intermediate particles with energy E'_0 lying between E_0 and E . The corresponding integro-differential operator acts on a variable E'_0 , which is less than or equal to E_0 all the way up to $E'_0 = E$.

Equations (1.1) with boundary conditions that give the number and energy spectrum of electrons or photons at the boundary layer $t = 0$ of a substance are the basic equations in the uniform theory of showers. The case of one primary electron of energy E_0 corresponds to the boundary conditions:

$$P(E_0, 0, E) = \delta(E - E_0); \quad \Gamma(E_0, 0, E) = 0.$$

The case of one primary photon of energy E_0 corresponds to the boundary conditions:

$$P(E_0, 0, E) = 0; \quad \Gamma(E_0, 0, E) = \delta(E - E_0), \quad (1.2)$$

where δ is the Dirac delta function. Generally speaking, (1.1) can be employed with arbitrary initial conditions:

$$P(E_0, 0, E) = \varphi_p(E_0, E); \quad \Gamma(E_0, 0, E) = \varphi_\Gamma(E_0, E),$$

where $\varphi_{p,\Gamma}$ are certain arbitrary functions.

2. The Probabilities of the Basic Processes

According to Bethe and Heitler² the probability that an electron with energy E will radiate a photon with energy E' when going through a layer of substance 1 cm thick is:

$$W_e(E, E') dE' = 4naZ^2r_0^2 \frac{dE'}{E'} \left[\left\{ 1 + \left(1 - \frac{E'}{E} \right)^2 \right\} \Phi_1 - \left\{ 1 - \frac{E'}{E} \right\} \Phi_2 \right]. \quad (1.3)$$

Here Z is the charge of the nucleus, $a = e^2/\hbar c$ is the fine-structure constant, $r_0 = e^2/mc^2$ is the classical radius of the electron, and n is the number of atoms per cm^3 of the substance. Formula (1.3) was derived assuming $E \gg mc^2$. The functions Φ_1 and Φ_2 include screening of the nuclear field by the fields of atomic electrons. Formula (1.3) was obtained in Born approximation and therefore it was not considered applicable to heavy elements. A correction for the inaccuracy of the Born approximation was investigated by Bethe and his co-workers,⁶ who found that for the process of bremsstrahlung the correction was small enough to be disregarded and that (1.3) can be used for any Z . L. Landau and Yu. Rumer⁷ showed that the radiation in the atomic electron field can be approximately accounted for in the expression for the probability of radiation by replacing the multiplier Z^2 by $Z(Z+1)$. The theoretical calculation of bremsstrahlung in the field of atomic electrons, not including screening, was done in references 8 to 11; screening was included in the calculations of reference 12. It was found that with an increase in energy the radiation in the electron field approaches radiation in the field of a nucleus with unit charge, but even with $E = mc^2$ the radiation is 15% less than the latter. On the other hand, reference 12 shows that screening has less influence on radiation in the electron field than in the nuclear field. Thus the exchange of Z^2 for $Z(Z+1)$ takes into consideration the radiation in the electron field with sufficient accuracy.

The probability that a photon of energy E' will form a pair in the nuclear field (that is, a positron with energy E and an electron with energy $E' - E$) upon traversing a layer of a substance 1 cm

thick is given by Bethe and Heitler in the following expression:

$$W_p(E', E) dE = 4naZ^2r_0^2 \frac{dE}{E'} \left[\left\{ \left(\frac{E}{E'} \right)^2 + \left(1 - \frac{E}{E'} \right)^2 \right\} \Phi_3 + \frac{E}{E'} \left(1 - \frac{E}{E'} \right) \Phi_4 \right]. \quad (1.4)$$

Formula (1.4) was derived assuming $E' \gg mc^2$. Functions Φ_3 and Φ_4 include screening of the nuclear field by the atomic electron fields. A detailed account and discussion of the W_e and W_p cross sections can be found in reference 13.

As in the case of radiation, pair production in the atomic electron field can be accounted for with sufficient accuracy by changing the multiplier Z^2 to $Z(Z+1)$ in $W_p(E', E)$. The theory of this phenomenon is similar to the theory of bremsstrahlung in the atomic electron field. The theory without screening was developed in references 8, 9, 10, 11, and 14; screening was included in the calculations of reference 12.

The expressions (1.4) were obtained in Born approximation. Deviations from the cross sections in Born approximation were studied in references 15 and 16 at low energies and heavy elements; the authors used exact Dirac wave functions and found out that in lead the exact cross section is 25 percent more at $5 mc^2$ quantum energy and 100 percent more at $3 mc^2$ energy than the cross section in Born approximation. At these energies they obtained the following dependence for correction on atomic number:

$$aZ^2 + bZ^4.$$

Bethe and Maximon¹⁷ accurately calculated the differential cross section for pair production at very high energies without including screening. Bethe and Davies¹⁸ obtained the complete cross section. The correction is 11.8% of the cross section in lead at 88 Mev and 10.0% at 280 Mev. If we include screening, the correction to the cross section can be written approximately as $0.11 (Z/82)^2$.

For high energies, when the screening can be considered complete, the probabilities for the processes of bremsstrahlung and pair production for one unit of radiation length or unit cascade length are written thus:

$$W_e(E, E') dE' = \frac{dE'}{E'} \left[1 + \left(1 - \frac{E'}{E} \right)^2 - \left(\frac{2}{3} - 2b \right) \left(1 - \frac{E'}{E} \right) \right]. \quad (1.5)$$

$$W_p(E', E) dE = \frac{dE}{E'} \left[\left(\frac{E}{E'} \right)^2 - \left(1 - \frac{E}{E'} \right)^2 + \left(\frac{2}{3} - 2b \right) \left(1 - \frac{E}{E'} \right) \frac{E}{E'} \right]. \tag{1.6}$$

The quantity $b = 1/18 \ln(19Z^{1/3})$ is small in comparison with unity and changes little with a change in Z ; therefore b is taken equal to the average value, $b = 0.0135$, with good accuracy.

If we consider radiation and pair production in the atomic electron field and also the correction for the inaccuracy of the Born approximation in heavy elements, we obtain the following expression for a unit "radiation" length:

$$\frac{1}{t_0} = 4naZ(Z+1)r_0^2 L_i^{rad} \left[1 + a \left(\frac{Z}{82} \right)^2 \right]. \tag{1.7}$$

The coefficient "a" for the process of pair production is 0.11. In the case of primary particles at high energies, where the role of electrons and photons in the development of the shower is almost the same, we obtain a good approximation with $a = 0.06$. The expression L_i^{rad} accounts for the effect of complete screening. According to A. Kirpichov and I. Pomeranchuk¹⁹ the values of L_i^{rad} are about 10% greater than the magnitudes obtained from the Thomas-Fermi model.

Table I.

	Z	t-unit in g/cm ²
Hydrogen	1	62
Carbon	6	40
Nitrogen	7	35.4
Oxygen	8	32.1
Aluminum	13	22.7
Argon	18	18
Iron	26	12.6
Lead	29	11.6
Air	82	5.55
	Composition by weight	
	76.4%	
	21.8%	
	1.5%	34.2
Water	Composition by weight	
	11.1%	
	88.9%	33.9

Table I shows the values of shower units in different substances, calculated according to (1.7). The coefficient a was assumed to be 0.06.

At lower energies the ionization and excitation of the atoms of the medium play an essential role in electron energy losses. We shall use the magnitudes of ionization losses averaged according to the "equilibrium" energy spectrum of the electrons.² The magnitude of the ionization loss of a relativistic electron in a unit cascade length β is called the "critical energy."

In the lower energy region the process of pair production for photons is replaced by the Compton effect, i.e., by the elastic scattering of a photon on free electrons. The differential cross section for the Compton effect was calculated by I. E. Tamm²⁰ and by Klein and Nishina.²¹ Assuming $E'/mc^2 \gg 1$, the probability of a photon of energy E' in the path of one radiation unit hitting an electron and retaining an energy E'' is found to be

$$W_{Comp}(E', E'') dE'' = g \left[1 + \left(\frac{E''}{E'} \right)^2 \right] \frac{dE''}{E'E''}, \tag{1.8}$$

where $g = 55 \sum_i n_i Z_i / \sum_i n_i Z_i (Z_i + 1) L_i^{rad}$ Mev.

The quantity g for different substances can be found in reference 2. Integrating (1.8) over E'' from E' to the minimum value of E'' we obtain the effective cross section for photon absorption as a result of the Compton effect:

$$\sigma_{Comp} = \frac{mc^2}{E'} \cdot \frac{108 \sum_i n_i Z_i L_i^{Comp}}{\sum_i n_i Z_i (Z_i + 1) L_i^{rad}}; \tag{1.9}$$

the "Compton logarithm" is $L^{Comp} = \ln(2E'/mc^2) + 1/2$.

We shall use the completely screened forms of the cross sections of basic processes in the light elements, (1.5) and (1.6). Using asymptotic values of the coefficient of bremsstrahlung radiation should not cause large errors for two reasons: first, because in light substances the exact magnitude $W_p(E, E')$, all the way up to energies of the order of one tenth of the critical energy, does not differ from its asymptotic value by more than 20%. In addition, low-energy electrons (of the order of the critical energy and less) expend most of their energy on ionization; losses by radiation play a small role in the correction. Therefore even a rough calculation of electron radiation losses does not cause a large error. Formula (1.6) for $W_p(E', E)$ is a poorer approximation to the accurate cross section of pair production by photons. However, the Compton effect was not included in the basic equations and its influence is considerable at low energies. There is a well known similarity between the process of pair production and the Compton effect as regards the formation of secondary electrons. The energy distribution of secondary Compton electrons up to $E \sim mc^2$ is very similar to the distribution of electrons in pair production. The Compton effect has noticeable probability in the energy region where the basic process for electrons is ionization loss, while radiation losses are small. Therefore it no longer matters whether the photon will spend its energy on the production of electron-positron pairs or on several low-energy Compton electrons. And for the substances with low atomic numbers the total absorption coefficient for photons remains nearly

constant to the very low energies $\sim 5 \times 10^6$ ev. Therefore if we are not interested in the fate of low-energy electrons and if we accept that the process of pair production follows the cross section (1.6) all the way down to the very small energies, then we have thereby approximately included the Compton effect. In the region near the critical energy the deviation of actual energy losses from those calculated according to (1.5) and (1.6) does not exceed in air 8% for the slowing down of electrons of 18% for photon absorption.²²

The formulas given here for the cross sections for bremsstrahlung and pair production are applicable up to arbitrary high energies if we are observing the processes on one isolated atom.²³ However, the picture changes considerably if we observe high-energy radiation and pair production in matter. The effect of multiple scattering of electrons by the atoms of a medium on bremsstrahlung radiation and pair production have been dealt with in detail by E. L. Feinberg in Usp. Fiz. Nauk 58, 193 (1956).

II. SOLUTION OF THE BASIC EQUATIONS OF THE THEORY. THE METHOD OF FUNCTIONAL TRANSFORMATIONS.

3. Solution by the Double Functional Transformation Method

Asymptotic expressions for cross sections for bremsstrahlung radiation by electrons (1.5) and pair production by photons (1.6) are homogeneous functions of the energy of the primary and secondary particles. The basic equations of the cascade theory (1.1a) and (1.1b) are linear in the functions P and Γ and, with the cross sections of the basic processes as given by (1.5) and (1.6) they are homogeneous in the variables E and E' . As a result, the set (1.1) is comparatively easily solved by the method of functional transformations. Using the Laplace-Mellin transformations for energy, L. D. Landau and Yu. B. Rumer⁷ in 1938 solved (1.1) without accounting for ionization losses. In reference 24 Snyder solved (1.1) by using the method of functional transformations and including ionization losses. However, in references 2 and 25, Snyder's method of solution was the subject of criticism. It was shown that his solution did not accurately satisfy the boundary conditions: it described a shower started by a primary electron of energy E_0 and an additional distribution of electrons and photons possessing a noticeable intensity near the critical energy β . Therefore Snyder's solution was accurate only for $E_0/\beta \gg 1$ and $t \gg 1$, where it did

not noticeably differ from a shower started by one primary electron of energy E_0 . In addition, Snyder's solution involved a certain function determined only for integral values of the argument; for non-integral values of the argument the function was defined by graphic interpolation. We note that Snyder found only the depth dependence of the total number of particles with energy greater than zero; he did not solve the problem of the energy spectrum of particles at an arbitrary depth. In 1939 I. E. Tamm and S. Z. Belen'kiĭ,²⁶ having solved the equations integrated over depth with account of the ionization losses, found the "equilibrium" energy spectrum averaged along the entire cascade curve. In 1941, S. Z. Belen'kiĭ,²⁷ by applying the Laplace-Mellin transformation in the variable E and the Laplace transformation in the variable t , reduced the basic equations of the theory to equations in finite differences, which he succeeded in solving. A certain function included in the equation was replaced by an expression which approximated the function well in the region essential for the problem--in the region of the change of variables. After this it was possible to carry out all the calculations and obtain analytical expressions for the dependence of the number of particles in the shower on depth and energy. Investigations showed that the approximation introduced did not involve a significant error in describing the one-dimensional stage of the shower development. Reference 2 gives the most detailed account of this theory.

In 1942, Bhabha and Chakrabarty²⁵ developed the theory including ionization losses and arrived at expressions for the total number of particles different from those obtained by Snyder and by Belen'kiĭ. It was shown in reference 2 that the solutions obtained by Bhabha and Chakrabarty significantly underestimated the number of low-energy particles. Therefore their expressions for the total number of particles are inaccurate and the expressions for energy spectra of particles become correct at particle energies $E > \beta$.

In 1949,²⁸ having improved the method of solution in reference 24, Snyder obtained an accurate solution to the equation in the form of a series strictly satisfying the boundary condition. However, in spite of considerable improvement in the mathematical methods, Snyder did not succeed in finding a complete solution to the problem. He found the dependence of the total number of particles on depth, and the energy spectrum in the region $E/\beta \ll 1$. In 1948 Bhabha and Chakrabarty,²⁹ having improved their method

of solution,²⁵ also obtained an accurate solution to the equations in the form of a series which strictly satisfied the boundary conditions. However, they did not succeed in obtaining physically new results. As earlier, their solution was significantly different from the Belen'kiĭ and Snyder results.

In 1952 Nishimura and Kamata,³⁰ starting from the solutions in references 25 and 29 and replacing the series in it by an integral, which was equal to the series in the region of its convergence and was the analytical continuation outside the region of convergence,* obtained a solution to the equations including ionization losses. Their method of solution was somewhat simpler than that used in references 28 and 29; their solution coincides with Snyder's solution of reference 28.

In 1956, Chakrabarty and Gupta,³¹ using a method of solution analogous to the one proposed in reference 30, obtained an accurate solution to the basic equations in the form of a series. Their series, in contrast to the series in reference 25, converges rapidly and the first term in the series is an accurate solution for the condition $E_0/\beta \gg 1$, as in reference 2 and 28. By applying the double saddle point method in calculating the integrals they succeeded in obtaining an accurate solution of the entire one-dimensional problem, i. e., in finding an analytic expression for the dependence of the number of particles in the shower on depth and on energy. Their results coincide accurately with the results S. Z. Belen'kiĭ obtained from 1941 to 1946.

The Laplace-Mellin transformation in the energy E , applied in the solution of the basic equations (1.1) to the functions $P(t, E)$ and $\Gamma(t, E)$, is:

$$P(t, E) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} E^{-(s+1)} P(t, s) ds,$$

$$P(t, s) = \int_0^\infty E^s P(t, E) dE \quad (2.1)$$

and the Laplace transformation in the variable t is:

$$P(t, s) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{st} P(\lambda, s) d\lambda,$$

$$P(\lambda, s) = \int_0^\infty e^{-\lambda t} P(t, s) dt \quad (2.2)$$

*We note that S. Z. Belen'kiĭ was the first to use a similar device for calculating the energy spectrum of particles with allowance for ionization losses (see reference 2, for example).

(the function $\Gamma(t, E)$ is transformed analogously). The variables s and λ are complex quantities, the contour of integration along s is a straight line parallel of the imaginary axis, with $\text{Re } s > 0$. Transformations (2.1) and (2.2) allow us to go from the variables E and t to the variables s and λ and vice versa. We multiply (1.1) by $E^s e^{-\lambda t}$ and integrate over E and t from zero to infinity:

$$\left. \begin{aligned} \lambda P(\lambda, s) - \phi_p(s) &= \\ -A(s)P(\lambda, s) + B(s)\Gamma(\lambda, s) - \beta s P(\lambda, s-1), & \\ \lambda \Gamma(\lambda, s) - \phi_r(s) &= C(s)P(\lambda, s) - \sigma_0 \Gamma(\lambda, s), \end{aligned} \right\} (2.3)$$

where $\phi_p(s)$ and $\phi_r(s)$ are correspondingly the transformed boundary conditions. The functions $A(s)$, $B(s)$, $C(s)$ and σ_0 can be found in reference 2. The term $\beta s P(\lambda, s-1)$ accounts for ionization losses; its presence considerably complicates the equations, converting them from algebraic equations to equations in finite differences. We eliminate the function $\Gamma(\lambda, s)$ from (2.3)

$$\left. \begin{aligned} P(\lambda, s) \psi(\lambda, s) + \beta s P(\lambda, s-1) &= \varphi(s, \lambda), \\ \psi(\lambda, s) &= \frac{\lambda + A(s) - B(s)C(s)}{\lambda + \sigma_0} = \frac{[\lambda - \lambda_1(s)][\lambda - \lambda_2(s)]}{\lambda + \sigma_0}, \end{aligned} \right\} (2.4)$$

where the functions $\lambda_1(s)$ and $\lambda_2(s)$ are the roots of the equation $\chi(\lambda, s) = 0$. The function $\phi(\lambda, s)$ is defined by the equality

$$\varphi(\lambda, s) = \frac{B(s) - \phi_r(s)}{(\lambda + \sigma_0) + \phi_p(s)},$$

when $|s| \rightarrow \infty$ the function $\chi(\lambda, s) \sim c_1 \ln s$, where c_1 is a certain constant. The function $\Gamma(\lambda, s)$ is defined by the equality

$$\Gamma(\lambda, s) = \frac{C(s)P(\lambda, s) + \phi_r(s)}{\lambda + \sigma_0}.$$

Let $\pi(\lambda, s)$ be the solution of (2.4) without the right-hand side. We look for a solution of (2.4) with the right-hand side in the form

$$P(\lambda, s) = \Phi(\lambda, s) \pi(\lambda, s). \quad (2.5)$$

Substituting (2.5) in (2.4) we obtain the following equation for the function $\Phi(\lambda, s)$:

$$\Phi(\lambda, s) - \Phi(\lambda, s-1) = -\frac{\varphi(\lambda, s)}{\beta s \pi(\lambda, s-1)}.$$

It is not difficult to verify that the last equation has the following solution :³²

$$\Phi(\lambda, s) = \frac{1}{\beta} \sum_{n=0}^{\infty} \frac{\varphi(\lambda, s+n+1)}{(s+n+1)\pi(\lambda, s+n)}.$$

Thus:

$$P(\lambda, s) = \frac{1}{\beta} \sum_{n=0}^{\infty} \frac{\varphi(\lambda, s+n+1)\pi(\lambda, s)}{(s+n+1)\pi(\lambda, s+n)}.$$

From (2.4) without the right-hand side we can find that

$$\frac{\pi(\lambda, s)}{\pi(\lambda, s+n)} = \frac{(-1)^n \psi(\lambda, s+1)\psi(\lambda, s+2)\dots\psi(\lambda, s+n)}{\beta^n (s+1)(s+2)\dots(s+n)}.$$

It is then easy to obtain a final expression for the function $P(\lambda, s)$:

$$P(\lambda, s) = \sum_{n=0}^{\infty} \frac{(-1)^n \varphi(\lambda, s+n+1)\psi(\lambda, s+1)\psi(\lambda, s+2)\dots\psi(\lambda, s+n)}{\beta^n (s+1)(s+2)\dots(s+n)} \quad (2.6)$$

Equation (2.6) gives a formal solution of the problem. The expression for the function $P(\lambda, s)$ admits of a corresponding Mellin transformation for s , and the series (2.6) can be integrated term by term. But as a function of λ the n -th term of the series behaves like λ^n , and an inverse Laplace transformation cannot be directly applied to it (it is necessary that the transformed function vanish like λ^{-1} as $\text{Re}\lambda \rightarrow \infty$). Consequently we must sum the series (2.6) or obtain some other representation for it.

4. S. Z. Belenkiĭ's Method

S. Z. Belen'kiĭ has shown² that another representation of the series (2.6) can be obtained in the following manner.

It is not difficult to show that the most essential role in the theory is played by the variation of the variables s from 0 to 2 and of λ from 2 to -0.6 . In this region the function $\psi(\lambda, s)$ has two essential singularities:

1. It has a simple pole at $\bar{s} = 0 [C(s) \rightarrow \infty]$.

2. It vanishes at values of λ and s determined by the relation $\lambda = \lambda_1(s)$ or $s = s_1(\lambda)$. Consequently in this region of variation, the function $\psi(\lambda, s)$ can be approximated with good accuracy by the following expression

$$\psi_0(\lambda, s) = \frac{f(\lambda)(s-s_1(\lambda))}{s};$$

$$f(\lambda) = -\frac{[s\lambda_1'(s)(\lambda-\lambda_2(s))]_{s=s_1(\lambda)}}{\lambda+\sigma_0} \quad (2.7)$$

This approximation is analogous to but does not coincide with the approximation first used in reference 26. It was shown that approximation (2.7)

leads to a function $P(E, t)$ that coincides practically with the accurate function at high energies when ionization can be ignored. Approximation (2.7) influences those terms of the equation, which account for pair production and bremsstrahlung. Inasmuch as ionization losses are accurately accounted for in (1.1), we can hope that in the present case the application of approximation (2.7) will give good results. Substituting (2.7) in (2.6) we obtain, for boundary condition (1.2):

$$P(\lambda, s) = \frac{E_0}{\beta} E_0^s \sum_{n=0}^{\infty} \frac{\left[-\frac{E_0}{\beta} f(\lambda)\right]^n [\Gamma(s+1)]^2 \Gamma(s+n+1-s_1)}{\Gamma(s+n+1)\Gamma(s+n+2)\Gamma(s+1-s_1)} \quad (2.8)$$

Hence, using (2.1) and (2.2) and integrating over E we obtain the following expression for the function $N(E, t)$ integrated over E :

$$N(E, t) = -\frac{1}{4\pi^2} \int_{C_s} \int_{C_\lambda} ds d\lambda P(\lambda, s) E^{-s} e^{st} \frac{1}{s} \quad (2.9)$$

Series (2.8) is an expansion of the solution in powers of E_0/β . Using the relationship

$$\frac{\Gamma(s)}{\Gamma(s+n+1)} = \frac{1}{n!} \int_0^1 (1-\tau)^n \tau^{s-1} d\tau,$$

we present (2.8) in the form

$$P(\lambda, s) = \frac{s\Gamma(s+1)E_0^{s+1} \int_0^1 \tau^{s-1} F_1 d\tau}{\beta\Gamma(s+1-s_1)} \quad (2.10)$$

where

$$F_1 = \sum_{n=0}^{\infty} \frac{(-Z)^n \Gamma(s+n+1-s_1)}{n! \Gamma(s+n+2)}$$

$$= \frac{1}{2\pi i} \int_C \frac{\Gamma(y+s+1-s_1)\Gamma(-y)}{\Gamma(s+y+2)} Z^y dy,$$

and

$$Z = \frac{E_0}{\beta} (1-\tau) \cdot f(\lambda).$$

Here the contour of integration was chosen so that the poles of the function $\Gamma(y+s+1-s_1)$ lie to the left of the path of integration, and the poles of function $\Gamma(-y)$ to the right of it. Closing the contour of integration C with a semicircle of infinite radius and taking the residues of the integrand at

the points $y=n$, it is not difficult to verify the correctness of (2.10). In addition the contour integral (2.10) serves as an analytic continuation of the series F_1 in the region $\beta/E_0 < 1$. Closing the contour of integration on the left by the semicircle of infinite radius and taking the residues of the integrand at the points $y = s_1 - s - n - 1$, we obtain the expansion F_1 in inverse powers of Z . Limiting ourselves to the first term of the expansion we obtain:

$$F_1 = \frac{\Gamma(s+1-s_1) Z^{-s+s_1-1} \left[1 + O\left(\frac{\beta}{E_0}\right) \right]}{\Gamma(s_1+1)}.$$

We ignore the integral over the closing contour, which, as can be shown, is of the order of $\exp(-E_0/\beta)$. Thus, substituting the expression for $P(\lambda, s)$ in (2.9) and integrating over s and λ (making the substitution $\lambda = \lambda_1(s)$ after the integration over s) we obtain:

$$N(E_0, E, t) = \frac{1}{2\pi i} \int_{\bar{z}-i\infty}^{\bar{z}+i\infty} \frac{1}{s} H_1(s) D(s) G(s, \epsilon) e^{\lambda_1(s)t + ys} ds. \quad (2.11)$$

where

$$G(s, \epsilon) = e^\epsilon \int_{\bar{x}}^{\infty} e^{-x} \left(1 - \frac{\epsilon}{x}\right)^s dx;$$

$$H_1(s) = \frac{\sigma_0 + \lambda_1(s)}{\lambda_1(s) - \lambda_2(s)}; \quad D(s) = \frac{\left[\frac{-s\lambda_1'(s)}{H_1(s)} \right]^s}{\Gamma(s+1)}.$$

We carry out the integration in (2.11) by the saddle point method. As a result we obtain:

$$\{N_p(E_0, E, t)\}^p = \frac{H_1(s) D(s) G(s, \epsilon) e^{ys + \lambda_1(s)t}}{s \sqrt{2\pi\lambda_1''(s) t}}, \quad (2.12)$$

where s is determined from the equation $t = -y/\lambda_1'(s)$, and

$$\epsilon = \frac{E t (\lambda_1(s))}{\beta}, \quad y = \ln(E_0/\beta).$$

Function $G(s, \epsilon)$ describes the energy spectrum of electrons at different depths. For the value $s = 1$ corresponding to the maximum of the shower, $G(s, \epsilon)$ coincides with the expression for the "equilibrium" spectrum of particles provided $\epsilon_0 \gg \epsilon$.²⁶ The function $G(s, \epsilon)$ is listed in table II for different values of the variables s and ϵ .

We note that expression (2.12) is correct under the following conditions: a) the magnitude of y must be more than unity; b) the energy E is of the order of the critical energy β and less. The depth t must be more than unity, because in computing $N(E, t)$ we have ignored the term proportional to $\exp \lambda_2(s)t$ and in the saddle point method we have expanded the function in powers of $1/t$ and ignored the terms of order $1/t$ and higher.

By analogous calculations we obtain the following results for different special cases:

a) Primary electron energy E_0 ;

1. With allowance for ionization losses $\{N_p(E_0, t, E)\}^p$ was determined in (2.12)

$$\{P(E_0, t, E)\}^p = \frac{H_1(s) D(s) (G_1(s, \epsilon) - EG(s, \epsilon)) \exp(ys + \lambda_1(s)t)}{sE \sqrt{2\pi\lambda_1''(s) t}},$$

$$\lambda_1'(s) = -\frac{y}{t}, \quad G_1(s, \epsilon) = e^\epsilon \int_{\bar{x}}^{\infty} e^{-x} \left(1 - \frac{\epsilon}{x}\right)^s (x - \epsilon) dx.$$

If we make use³ of the approximated expression for the function $C(s) = 1/s$, which differs from the accurate expression by only several percent, then we can obtain the following expressions for the photon distribution functions:

$$\{\Gamma(E_0, t, E)\}^p = \frac{H_1'(s) D(s) G(s, \epsilon) \exp(ys + \lambda_1(s)t)}{sE \sqrt{2\pi\lambda_1''(s) t}},$$

$$\{N_\Gamma(E_0, t, E)\}^p = \frac{H_1'(s) D(s) G_2(s, \epsilon) \exp(ys + \lambda_1(s)t)}{s \sqrt{2\pi\lambda_1''(s) t}}.$$

Here

$$G_2(s, \epsilon) = e^\epsilon \int_{\bar{x}}^{\infty} e^{-x} (x - \epsilon)^{s-1} x^{-s} dx,$$

$$H_1'(s) = \frac{1}{\lambda_1(s) - \lambda_2(s)}; \quad \lambda_1'(s) = -\frac{y}{t}.$$

We can write the following relationships between functions:

$$\{P(E_0, t, E)\}^p = \{N_p\}^p \left(-E + \frac{G_1(s, \epsilon)}{G(s, \epsilon)} \right) \frac{1}{E};$$

$$\{\Gamma(E_0, t, E)\}^p = \frac{\{N_p\}^p}{E(\lambda_1(s) + \sigma_0)};$$

$$\{N_\Gamma(E_0, t, E)\}^p = \frac{\{N_p\}^p G_2(s, \epsilon)}{G(s, \epsilon)(\lambda_1(s) + \sigma_0)}.$$

Here throughout $y = \ln(E_0/E)$

2. Without accounting for the ionization losses:

TABLE II.

Function $G(s, \epsilon)$										
$s \backslash \epsilon$	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
0.05	0.967	0.937	0.913	0.891	0.870	0.850	0.832	0.815	0.799	0.784
0.1	0.947	0.901	0.863	0.830	0.798	0.770	0.744	0.720	0.698	0.677
0.2	0.918	0.846	0.792	0.744	0.701	0.663	0.629	0.598	0.569	0.543
0.5	0.860	0.751	0.667	0.597	0.538	0.489	0.446	0.408	0.376	0.346
0.8	0.823	0.692	0.591	0.511	0.447	0.396	0.355	0.313	0.280	0.252
1.2	0.786	0.634	0.522	0.436	0.369	0.315	0.275	0.238	0.207	0.180
1.6	0.758	0.592	0.472	0.384	0.316	0.263	0.221	0.186	0.160	0.138
2.0	0.736	0.559	0.434	0.345	0.277	0.227	0.186	0.153	0.129	0.109
3.0	0.694	0.499	0.367	0.273	0.214	0.173	0.134	0.112	0.095	0.071
4.0	0.662	0.459	0.321	0.232	0.175	0.135	0.102	0.080	0.062	0.048

Function $G_1(s, \epsilon)$										
$s \backslash \epsilon$	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
0.05	0.990	0.982	0.973	0.965	0.956	0.948	0.938	0.929	0.920	0.917
0.1	0.983	0.960	0.940	0.930	0.920	0.906	0.892	0.879	0.865	0.852
0.2	0.968	0.939	0.911	0.885	0.860	0.836	0.815	0.794	0.773	0.751
0.5	0.935	0.876	0.823	0.775	0.731	0.692	0.665	0.620	0.588	0.558
0.8	0.908	0.829	0.759	0.700	0.642	0.604	0.554	0.516	0.479	0.441
1.2	0.880	0.779	0.694	0.621	0.557	0.515	0.468	0.422	0.382	0.341
1.6	0.855	0.740	0.644	0.566	0.494	0.436	0.387	0.343	0.306	0.274
2.0	0.837	0.708	0.599	0.519	0.445	0.388	0.336	0.297	0.262	0.226
3.0	0.800	0.625	0.514	0.433	0.357	0.288	0.259	0.226	0.196	0.152
4.0	0.767	0.596	0.480	0.399	0.301	0.244	0.202	0.162	0.133	0.111

Function $G_2(s, \epsilon)$										
$s \backslash \epsilon$	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
0.05	7.16	4.48	3.50	2.95	2.59	2.30	2.11	1.96	1.83	1.72
0.1	6.51	3.84	2.88	2.35	2.01	1.73	1.57	1.43	1.31	1.22
0.2	5.86	3.23	2.30	1.82	1.49	1.22	1.08	0.968	0.871	0.792
0.5	5.04	2.50	1.63	1.19	0.923	0.745	0.607	0.515	0.447	0.384
0.8	4.65	2.16	1.37	0.930	0.691	0.525	0.427	0.351	0.292	0.244
1.2	4.32	1.89	1.11	0.738	0.525	0.398	0.310	0.247	0.199	0.157
1.6	4.10	1.71	0.966	0.620	0.427	0.304	0.230	0.177	0.140	0.112
2.0	3.93	1.58	0.865	0.547	0.361	0.255	0.185	0.141	0.108	0.084
3.0	3.64	1.37	0.702	0.412	0.262	0.181	0.118	0.091	0.066	0.048
4.0	3.45	1.23	0.604	0.337	0.205	0.132	0.091	0.069	0.048	0.032

Note: The function G_1 can be expressed in terms of the confluent hypergeometric function $W_{\kappa, \mu}(\epsilon)$:

$$G(s, \epsilon) = \epsilon \Gamma\left(\mu + \frac{1}{2} - \kappa\right) e^{-(\mu + \frac{1}{2})} e^{\frac{\epsilon}{2}} W_{\kappa, \mu}(\epsilon);$$

$$G_1(s, \epsilon) = \epsilon^2 \Gamma\left(\mu + \frac{1}{2} - \kappa\right) e^{-(\mu + \frac{1}{2})} e^{\frac{\epsilon}{2}} W_{\kappa, \mu}(\epsilon);$$

$$G_2(s, \epsilon) = \Gamma\left(\mu + \frac{1}{2} - \kappa\right) e^{-(\mu + \frac{1}{2})} e^{\frac{\epsilon}{2}} W_{\kappa, \mu}(\epsilon),$$

where $\mu = 0$ for $G(s, \epsilon)$; $\mu = 1$ for $G_1(s, \epsilon)$; $\mu = \frac{1}{2}$ for $G_2(s, \epsilon)$
 $\kappa = \frac{1}{2} - s$ for $G(s, \epsilon)$; $\kappa = -s - \frac{1}{2}$ for $G_1(s, \epsilon)$; $\kappa = -s$ for $G_2(s, \epsilon)$

For values $\epsilon > 4$ it is easy to calculate G from the asymptotic formulas for $W_{\kappa, \mu}(\epsilon)$.

$$\{N_p\}^p = \frac{H_1(s) \exp(ys + \lambda_1(s)t)}{s \sqrt{2\pi\lambda_1''(s)} t},$$

$$\{P\}^p = \frac{H_1(s) \exp(ys + \lambda_1(s)t)}{E \sqrt{2\pi\lambda_1''(s)} t},$$

$$\{\Gamma(E_0, t, E)\}^p = \frac{H_3(s) \exp(ys + \lambda_1(s)t)}{E \sqrt{2\pi\lambda_1''(s)} t},$$

$$\{N_\Gamma(E_0, t, E)\}^p = \frac{H_3(s) \exp(ys + \lambda_1(s)t)}{s \sqrt{2\pi\lambda_1''(s)} t}.$$

Here

$$H_3(s) = \frac{C(s)}{\lambda_1(s) - \lambda_2(s)}, \quad \lambda_1'(s) = -\frac{y}{t}.$$

It is possible to write the following relationships between functions:

$$\{P(E_0, t, E)\}^p = \frac{\{N_p\}^p s}{E};$$

$$\{\Gamma(E_0, t, E)\}^p = \frac{\{N_p\}^p C(s) \cdot s}{E(\lambda_1(s) + \sigma_0)};$$

$$\{N_\Gamma(E_0, t, E)\}^p = \frac{\{N_p\}^p C(s)}{\lambda_1(s) + \sigma_0}.$$

Here throughout $y = \ln(E_0/E)$

b) Primary photon of energy E_0 :

1. With allowance for ionization losses. If we use³ the approximate expression for the function $B(s) = 2\sigma_0/(s+1)^3$, which differs from the accurate expression by only a few percent, then we can obtain the following expression for the distribution functions:

$$\{N_p(E_0, t, E)\}^\Gamma = \frac{H_1(s) D_2(s) G(s, \epsilon) \exp(ys + \lambda_1(s)t)}{s \sqrt{2\pi\lambda_1^*(s)t}}$$

$$\{P(E_0, t, E)\}^\Gamma =$$

$$\{N_\Gamma(E_0, t, E)\}^P = \frac{H_1^*(s) D(s) G_2(s, \epsilon) \exp(ys)}{s \sqrt{2\pi\lambda_1^*(s)t}}$$

$$\{\Gamma(E_0, t, E)\}^\Gamma =$$

$$\delta(E_0 - E) e^{-\sigma_0 t} + \frac{H_1(s) D_3(s) G_2(s, \epsilon) \exp(ys + \lambda_1(s)t)}{Es \sqrt{2\pi\lambda_1^*(s)t}}$$

Here

$$\{N_\Gamma(E_0, t, E)\}^\Gamma = e^{-\sigma_0 t} + \frac{H_1(s) D_3(s) G_2(s, \epsilon) \exp(ys + \lambda_1(s)t)}{s \sqrt{2\pi\lambda_1^*(s)t}}$$

$$D_2(s) = \frac{B(s) [f(\lambda_1(s))]^s}{(\lambda_1(s) + \sigma_0) \Gamma(s+1)}; D_3(s) = \frac{D_2(s)}{\lambda_1(s) + \sigma_0}; \lambda_1^*(s) = -\frac{y}{t}$$

The following relationships between functions can be written:

$$\{N_p(E_0, t, E)\}^\Gamma = \frac{\{N_p\}^P B(s)}{\lambda_1(s) + \sigma_0}; \{P(E_0, t, E)\}^\Gamma = \frac{\{P\}^P B(s)}{\lambda_1(s) + \sigma_0};$$

$$\{\Gamma(E_0, t, E)\}^\Gamma = \delta(E_0 - E) e^{-\sigma_0 t} + \frac{\{N_p\}^P B(s)}{E (\lambda_1(s) + \sigma_0)^2};$$

$$\{N_\Gamma(E_0, t, E)\}^\Gamma = e^{-\sigma_0 t} + \frac{\{N_p\}^P B(s) G_2(s, \epsilon)}{G(s, \epsilon) (\lambda_1(s) + \sigma_0)^2}.$$

2. Without accounting for ionization losses

$$\{N_p(E_0, t, E)\}^\Gamma = \frac{H_4(s) \exp(ys + \lambda_1(s)t)}{s \sqrt{2\pi\lambda_1^*(s)t}}$$

$$\{P(E_0, t, E)\}^\Gamma = \frac{H_4(s) \exp(ys + \lambda_1(s)t)}{E \sqrt{2\pi\lambda_1^*(s)t}}$$

$$\{\Gamma(E_0, t, E)\}^\Gamma = \delta(E_0 - E) e^{-\sigma_0 t} - \frac{H_2(s) \exp(ys + \lambda_1(s)t)}{E \sqrt{2\pi\lambda_1^*(s)t}}$$

$$\{N_\Gamma(E_0, t, E)\}^\Gamma = e^{-\sigma_0 t} - \frac{H_2(s) \exp(ys + \lambda_1(s)t)}{s \sqrt{2\pi\lambda_1^*(s)t}}$$

Here

$$H_4(s) = \frac{B(s)}{\lambda_1(s) - \lambda_2}, \lambda_1^*(s) = \frac{y}{t}$$

The following relationships between functions can be written:

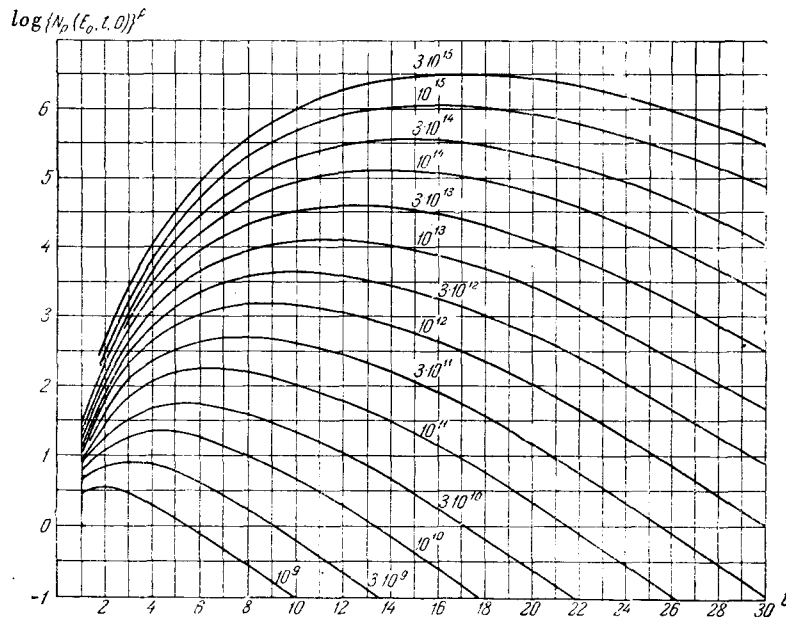


FIG. 1. The dependence on depth of the number of electrons with an energy greater than zero $\{N_p(E_0, t, 0)\}^P$ in a shower caused by a primary electron of energy E_0 . The numbers near each curve indicate the energy of the primary electron in ev. Ionization losses calculated to order $1/t^2$ are included.

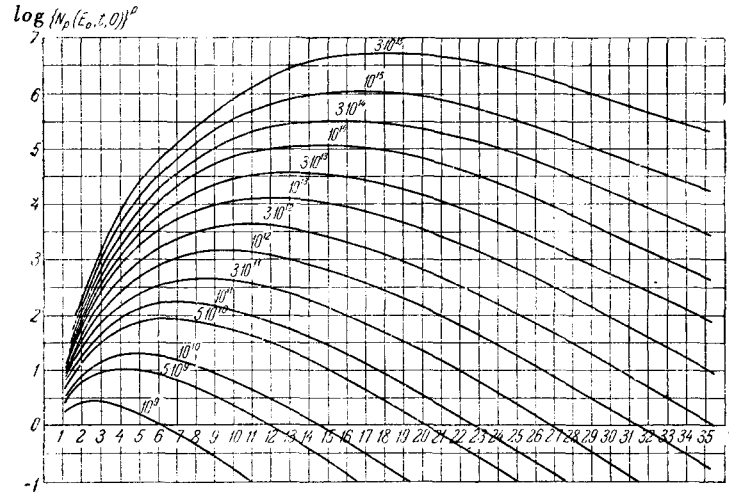


FIG. 2. Same as in FIG. 1, but in a shower from a primary photon.

$$\{P(E_0, t, E)\}^r = \frac{\{N_p\}^r s}{E},$$

$$\{\Gamma(E_0, t, E)\}^r = \delta(E_0 - E) e^{-\sigma_0 t} - \frac{\{N_p\}^r s (\lambda_2(s) + \sigma_0)}{EB(s)},$$

$$\{N_r(E_0, t, E)\}^r = e^{-\sigma_0 t} - \frac{\{N_p\}^r (\sigma_0 + \lambda_2(s))}{B(s)}.$$

Figures 1 and 2 show the cascade curves $\{N_p(E_0, t, 0)\}^P$ and $\{N_0(E_0, t, 0)\}^r$ for different values of the variables $E_0, t,$ and E including ionization losses. These data together with those in Table II which also gives the quantities $G_1(s, \epsilon)$ and $G_2(s, \epsilon)$ can be useful for various calculations. Thus, the method of double functional transformations and the further approximation (2.7) make it possible to obtain a complete description of the one-dimensional development of a cascade shower in light substances.

5. The Scott, Snyder, Bhabha and Chakrabarty Method

We recall that expression $P(\lambda, s)$ admits of an inverse transformation in s and that the series (2.6) can be integrated term by term. But the n -th term of the series behaves like λ^n , and the inverse Laplace transformation cannot be applied to it.

Thus we must sum the series (2.6) or write down its general term in a form in which the integer n can be replaced by the complex variable σ , and then present the series in the form of a contour integral (see, for instance, reference 2). The denominator of the n -th term of (2.6) contains n factors $(s+1) \dots (s+n+1)$; we write them as a ratio of two gamma functions. The

numerator of the n -th term of (2.6) contains $(n-1)$ factors $\psi(\lambda, s+1) \dots \psi(\lambda, s+n)$. W. T. Scott³³ also introduces in the numerator the ratio of two new functions, defined in terms of infinite products, thus getting rid the variable number of factors in the n -th term of the series. We introduce the new functions

$$\psi(\lambda, s+1) \psi(\lambda, s+2) \dots \psi(\lambda, s+n) = \frac{L(\lambda, s+n+1)}{L(\lambda, s+1)},$$

where the function $L(\lambda, s)$ must satisfy recurrence relation

$$L(\lambda, s) \psi(\lambda, s) = L(\lambda, s+1) \text{ or } L(\lambda, s) = \psi(\lambda, s-1) L(\lambda, s-1). \quad (2.13)$$

It can be shown³⁴ that equation (2.13) has a solution

$$L(s, \lambda + Z) = L(s, \lambda) [\psi(s, \lambda)]^Z \prod_{n=0}^{\infty} \left\{ \frac{\psi(s, \lambda+n)}{\psi(s, \lambda+n+Z)} \left(\frac{\psi(s, \lambda+n+1)}{\psi(s, \lambda+n)} \right)^Z \right\} \quad (2.14)$$

provided the infinite product in (2.14) converges and $\lim_{N \rightarrow \infty} [\psi(s, \lambda+N)/\psi(s, N)] = 1$.

Using the asymptotic expression for $A(s)$, it is easy to show that these two conditions are fulfilled. The function $L(s, \lambda)$ satisfies the boundary condition $L(s, 0) = 1$. The equality (2.14) is valid for any Z ; for $Z = 1$ we have $L(s, \lambda+1) = \psi(s, \lambda) L(s, \lambda)$ which coincides with (2.13) λ is changed to $\lambda+1$. $L(s, \lambda)$ is an analytic function. From the

expressions (2.4) for the function $\psi(s, \lambda)$ and the quantities $A(s)$, $B(s)$ and $C(s)$ it is obvious that the function $\psi(\lambda, s)$ has finite non-vanishing values for all finite λ , excluding zero at the points $\lambda = -s - n, n = 1, 2, \dots$. Consequently $L(s, \lambda)$ is a nonvanishing analytic function for all finite λ ex-

cluding the isolated poles at the points $\lambda = -s - n$. The infinite product in (2.14) converges slowly; it is inconvenient for numerical calculations. Therefore in doing numerical calculations the following convenient approximation of this function was used:

$$L(s, \lambda + z) = L(s, \lambda) \lim_{N \rightarrow \infty} [\psi(s, \lambda + N + 1)]^{\alpha(Z)} [\psi(s, \lambda + N)]^{\beta(Z)} \times [\psi(s, \lambda + N + 1)]^{\gamma(Z)} [\psi(s, \lambda + N + 2)]^{\delta(Z)} \prod_{n=0}^{\infty} \psi(s, \lambda + n) / \psi(s, \lambda + n + Z), \tag{2.15}$$

where

$$\alpha(Z) = -\frac{Z(Z+1)(Z-1)(Z-2)}{24}; \quad \gamma(Z) = -\frac{Z(Z+1)(Z+2)(3Z-7)}{24};$$

$$\beta(Z) = \frac{Z(Z-1)(Z-2)(3Z-7)}{24}; \quad \delta(Z) = \frac{Z(Z-1)(Z+1)(Z+2)}{24}.$$

The polynomials $\alpha, \beta, \gamma, \delta$ are chosen such that (2.15) coincides with (2.14) for all N with $Z = 0, +1, \pm 2$. The condition $\alpha(Z) + \beta(Z) + \gamma(Z) + \delta(Z) = Z$ guarantees that (2.15) converges at the same values of λ , as in expression (2.14). At the same time for a given number of terms N in the product, the expression (2.15) is more accurate for small Z than (2.14). A useful asymptotic expression for the function $L(\lambda, s)$ can be obtained by writing for large s $\ln \psi(\lambda, s) = \ln L(s+1, \lambda) - \ln L(s, \lambda) \cong \frac{\partial}{\partial s} \ln L(\lambda, s)$ and using the relation $\psi(\lambda, s) = c_1 \ln s$. After integration we obtain:

$$L(\lambda, s) \cong (c_1 \ln s)^s, \quad \text{when } |s| \rightarrow \infty.$$

The series (2.6) for the function $P(\lambda, s)$ can now be rewritten:

$$P(\lambda, s) = \sum_{n=0}^{\infty} \frac{(-1)^n \varphi(s+n+1, \lambda) \Gamma(s+1) L(s+1, n, \lambda)}{\beta^n \Gamma(s+n+2)}.$$

It is easy to verify that the latter expression can be written in the form of a contour integral:

$$P(\lambda, s) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{d\sigma \cdot \pi \psi(\sigma+s+1, \lambda) L(s+1, \sigma, \lambda)}{\sin \pi \sigma \cdot \Gamma(s+\sigma+2) \beta^n}, \tag{2.16}$$

where the contour of integration is a straight line parallel to the imaginary axis with $-1 < \text{Re } \gamma < 0$. The function $L(s+1, \sigma, \lambda)$ behaves like λ^σ ; for large values of $|\lambda|$; consequently the integral (2.16) over all σ must behave like λ^γ , where $-1 < \gamma < 0$. Therefore we can now apply the inverse Laplace and Mellin transformations to the expression $P(\lambda, s)$

$$P(E, t) = -\frac{1}{(2\pi i)^3} \int_{c_\lambda} e^{\lambda t} d\lambda \int_{c_s} E^{-s-1} ds \int_{c_\sigma} d\sigma \frac{\pi}{\sin \pi \sigma} \frac{\Gamma(s+1)}{\Gamma(s+\sigma+2) \beta^n} L(s+1, \sigma, \lambda) \varphi(s+n+1, \lambda). \tag{2.17}$$

$c_{\lambda, s, \sigma}$ are straight lines parallel to the imaginary axis in the planes λ, s , and σ to the right of all poles of the integrand. Thus we find a solution to (1.1) in the form of triple complex integrals. The calculation of these integrals is no simple matter. Different authors have calculated them in different ways; often an unsuccessful method of computing integrals has greatly reduced

the accuracy of the numerical results or led to serious errors.

We can derive the Snyder results²⁸ from (2.17), which has been obtained by the method of double functional transformations. We examine the shower caused by a primary electron with the energy E_0 . After changing the order of integration we will compute first the integral over λ .

After taking the residues of the integrand (2.14) at the points $\lambda = \lambda_1(s + n + \sigma + 1)$ and $\lambda = \lambda_2(s + n + \sigma + 1)$ making all the necessary simplifications and

changing variables $\sigma + n + 1 = -y$, $s + \sigma + n + 1 = s$ (on the right we have Snyder's "s") we obtain equations (26) and (27) of reference 28:

$$P(E, t) = -\frac{1}{(2\pi)^2 \beta} \sum_{n=0}^{\infty} \int_D ds \int_C dy \frac{\Gamma(-y) \Gamma(s+y+n+1)}{\Gamma(s+n+1) [\lambda_1(s+n) - \lambda_2(s+n)]} \times$$

$$\times \left(\frac{E_0}{\beta}\right)^s \left(\frac{\beta}{E}\right)^{s+y+n+1} \cdot \{A_n(s+n) [\lambda_1(s+n) + \sigma_0] K_{\lambda_1}(s+n, y) e^{\lambda_1(s+n)t} -$$

$$- B_n(s+n) [\lambda_2(s+n) + \sigma_0] K_{\lambda_2}(s+n, y) \cdot e^{\lambda_2(s+n)t}\} \tag{2.17}$$

and analogously for the function $\Gamma(E, t)$. The following notation was introduced here:

$$L(s+1, \sigma+n+1, \lambda_1(s+n+\sigma+1)) = L(s+y-1, -y, \lambda_1(s)) = \frac{K_{\lambda_1}(s, y)}{\Gamma(y+1)},$$

$$L(s+1, \sigma+n+1, \lambda_2(s+n+\sigma+1)) = \frac{K_{\lambda_2}(s, y)}{\Gamma(y+1)}.$$

The quantities A_n and $B_n(s+y)$ are determined from the conditions

$$A_N(s+N) =$$

$$= \sum_{k=1}^N (-1)^{k+1} \Gamma(s+N+1) \{[\lambda_1(s+N-k) - \lambda_2(s+N)] K_{\lambda_1}(s+N-k, k) \times$$

$$\times A_{N-k}(s+N-k) - [\lambda_2(s+N-k) - \lambda_2(s+N)] K_{\lambda_2}(s+N-k, k) B_{N-k} \times$$

$$\times (s+N-k)\} / \Gamma(k+1) \Gamma(s+N-k+1) [\lambda_1(s+N-k) - \lambda_2(s+N-k)],$$

$$B_N(s+N) =$$

$$= \sum_{k=1}^N (-1)^{k+1} \Gamma(s+N+1) \{[\lambda_1(s+N-k) - \lambda_2(s+N)] K_{\lambda_1}(s+N-k, k) \times$$

$$\times A_{N-k}(s+N-k) - [\lambda_2(s+N-k) - \lambda_1(s+N)] K_{\lambda_2}(s+N-k, k) \times$$

$$\times B_{N-k}(s+N-k)\} / \Gamma(k+1) \Gamma(s+N-k+1) [\lambda_1(s+N-k) - \lambda_2(s+N-k)]. \tag{2.18}$$

If a primary photon of energy E_0 falls on the boundary of a substance, then:

$$A_j(s) = B_0(s) = 1,$$

if a primary particle is an electron, then

$$A_0^*(s) = \frac{-(\lambda_2(s) + \sigma_0)}{C(s)}; \quad B_0^*(s) = \frac{-(\lambda_1(s) + \sigma_0)}{C(s)}.$$

In the case of a primary-photon spectrum of the form $1/E$ with maximum energy E_0 ,

$$A_0^*(s) = \frac{-(\lambda_2(s) + \sigma_0)}{sC(s)}; \quad B_0^*(s) = \frac{-(\lambda_1(s) + \sigma_0)}{sC(s)}.$$

The expressions (2.18) make it possible to express successively the value of $A_N(s+N)$ and $B_N(s+N)$ in terms of the rational functions $s - \lambda_1(s+N)$, $\lambda_2(s+N)$ and the quantities $A_0(s)$ and $B_0(s)$.

To present the solution in the most convenient form for numerical work we investigate the behavior of the functions $A_N(s)$ and $B_N(s)$. The functions $\lambda_1(s)$ and $\lambda_2(s)$ have branch points in the half-plane $Re s \ll 0$. In going around the branch points, the functions $\lambda_1(s)$ and $\lambda_2(s)$ alternate. The same characteristics are possessed by the functions $K_{\lambda_1}(s, y)$ and $K_{\lambda_2}(s, y)$.

Investigation of (2.18) shows that $A_n(s)$ and $B_n(s)$ are analytic functions in the region $Re s > 0$ and that they have branch points only where the

functions $\lambda_1(s)$ and $\lambda_2(s)$ do. In going around the branch points, the functions $A_n(s)$ and $B_n(s)$ alternate. The analyticity of the functions $A_n(s)$ and $B_n(s)$ with $Re s > 0$ makes it possible for us to

move the contour of integration for n units to the left in the n -th term of (2.17). As a result we obtain:

$$P(E, t) = \sum_{n=0}^{\infty} \left(\frac{\beta}{E_0}\right)^n P_n(E, t),$$

$$P_n(E, t) = -\frac{1}{4\pi^2\beta} \int_D ds \int_C dy \frac{\Gamma(-y)\Gamma(s+y+1)}{\Gamma(s+1)[\lambda_1(s)-\lambda_2(s)]} \left(\frac{E_0}{\beta}\right)^s \left(\frac{\beta}{E}\right)^{s+y+1} \times$$

$$\times \{A_n(s)[\lambda_1(s)+\sigma_0] K_{\lambda_1}(s, y) e^{\lambda_1(s)t} - B_n(s)[\lambda_2(s)+\sigma_0] K_{\lambda_2}(s, y) e^{\lambda_2(s)t}\}. \quad (2.19)$$

An analogous expression can be written for function $\Gamma(E, t)$. The first term of the expansion of the solution in powers of β/E_0 was used in the earlier works of Snyder²⁴ and Serber.³⁵ The integrand expressions of (2.19) do not have branch points as functions of s and y .

We recall that the functions $K_{\lambda_1}(s, y)$, $K_{\lambda_2}(s, y)$ and $C(s+y)$ have poles at $y = -s - n$. When $E \ll \beta$ this makes it possible to compute the integral over s in (2.19) by residues. We compute the function $N(E, t)$, the number of particles with an energy higher than E at given depth t ;

$$N(E, t) = \sum_{n=0}^{\infty} \left(\frac{\beta}{E_0}\right)^n N_n(E, t),$$

where

$$N_n(E, t) = \frac{1}{2\pi i} \int_C ds \left(\frac{E_0}{\beta}\right)^s \times$$

$$\times \left[\frac{A_n(s)[\lambda_1(s)+\sigma_0] K_{\lambda_1}(s, -s) e^{\lambda_1(s)t} - B_n(s)[\lambda_2(s)+\sigma_0] K_{\lambda_2}(s, -s) e^{\lambda_2(s)t}}{s[\lambda_1(s)-\lambda_2(s)]} - \right.$$

$$- \frac{1}{\lambda_1(s)-\lambda_2(s)} \left\{ A_n(s)[\lambda_1(s)+\sigma_0] K_{\lambda_1}(s, -s) \left[\frac{s+y+1}{G_{\lambda_1}(s, y)} \right]_{y=-s-1} e^{\lambda_1(s)t} \left(\frac{E}{\beta}\right) \times \right.$$

$$\times \left(\ln \frac{\beta}{E} + f(y) \right) - B_n(s)[\lambda_2(s)+\sigma_0] K_{\lambda_2}(s, -s) \left[\frac{s+y+1}{G_{\lambda_2}(s, y)} \right]_{y=-s-1} \times$$

$$\left. \left. \times e^{\lambda_2(s)t} \left(\frac{E}{\beta}\right) \left(\ln \frac{\beta}{E} + g(y) \right) + O\left(\frac{E}{\beta}\right)^2 \left\{ \ln^3 \frac{\beta}{E} + \dots \right\} \right\} \right]. \quad (2.20)$$

The functions $f(y)$ and $g(y)$ included in (2.20) are given in reference 28. The first term in (2.20) defines $N(0, t)$, the total number of particles with an energy greater than zero at given depth t . For $t > 1/2$, the term with $\lambda_1(s)$ is much

larger than the term with $\lambda_2(s)$ (see also references 25 and 36). The integral that depends on $\lambda_1(s)$ can be calculated approximately by the saddle point method

$$N_0(E, t) = \frac{\exp(\lambda_1(s)t + ys)}{[\beta(s) + b(s)t]^{1/2}} A_0(s) \left\{ H(s) - M(s) \frac{E}{\beta} \left[\ln \frac{\beta}{E} + f(s) \right] + \right.$$

$$\left. + O\left(\frac{E}{\beta}\right)^2 \left[\ln^3 \frac{\beta}{E} + \dots \right] \right\}. \quad (2.21)$$

Here

$$t = (ys - \alpha(s))/(-s\lambda_1'(s)); \quad b(s) = s^2\lambda_1''(s);$$

$$H(s) = \frac{[\lambda_1(s)+\sigma_0] K_{\lambda_1}(s, -s)}{(2\pi)^{1/2} [\lambda_1(s)-\lambda_2(s)]}; \quad M(s) = \frac{2\sigma_0 H(s)}{\lambda_1(s)+\sigma_0}.$$

The functions $f(s)$ and $\alpha, \beta(s)$ contained here were determined earlier. The quantities $\lambda_1(s), a(s), b(s), H(s), M(s), A(s)$, and so forth are given in table 1 of reference 28 for different values of s .

Analogously, the following relation is obtained for $\Gamma(E, t)$

$$\Gamma(E, t) = \frac{N(0, t)}{E(\lambda_1(s) + \sigma_0)} + O\left(\frac{E}{\beta}\right) \left[\ln^3 \frac{\beta}{E} + \dots \right]. \quad (2.21')$$

Differentiating (2.21) we obtain

$$P(E, t) = \frac{2\sigma_0 N(0, E) \left\{ \ln \frac{\beta}{E} + f(s) - 1 \right\}}{\beta [\lambda_1(s) + \sigma_0]},$$

that is, as E tends to zero, the differential spectrum of the electrons diverges as $\ln E$ and the differential spectrum of the photons diverges as $1/E$. We note that the saddle point method can be used only in calculating $N_0(t), N_0^+(t), N_0^*(t)$, and $N_1^*(t)$. The saddle point method cannot be used to compute the remaining quantities $N_n(t), N_n^+(t)$, and so forth, since the functions $A_n(s)$ vanish for $s = 1, 2, 3, \dots, n$. It is possible to compute the integral (2.20) over s from the residues of the poles in the left half-plane. As a result we obtain a power series in t , which can be used in numerical computations only at very small depths (see also reference 25). The author calculated $N(E, t) = N_0(E, t)$ for a primary electron or photon and for a spectrum of photons at energies $\gamma = 2, 3, 4, 5, 6, 7, 8$. Using tables 1 and 2 of reference 28 we can compute the ratio N_1/N_0 for the case of the primary electron as a function of depth at certain values of γ . Table III shows the results of calculations for $\gamma = 2$ and 3. From the table it can be seen, for instance, that for $\gamma = 2$ at a depth of 4 t -units, where $N/N_{\max} \sim 1/3$, the accuracy of Snyder's curve is of the order of 10 to 20% and the error increases with increasing depth. The area under the curve for $\gamma > 2$ does not differ by more than 3 to 5% from E_0/β ,

Table III.

s	1.05	1.55	2.15	2.65	
t	1.91	4.49	9.4	17.3	$\gamma=2$
t	3.14	6.75	14.1		$\gamma=3$
N_1/N_0	0.3	0.7	1.3	1.9	

The author considers this as a criterion of applicability of the saddle point method for the

evaluation of the integrals. However, at small γ the first, second, and greater moments of the Snyder curves differ considerable from the accurate ones, i. e., the curves are deformed. They give somewhat more particles up to the maximum and somewhat fewer particles at the greater depths.

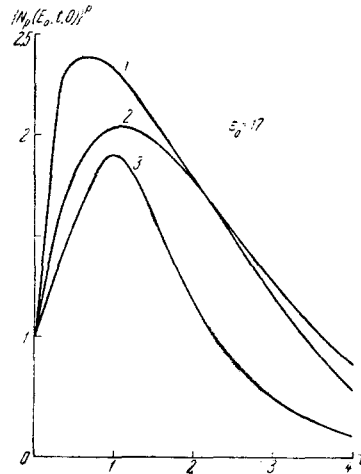


FIG. 3. Dependence on depth of the total number of electrons $\{N_p(E_0, t, 0)\}^{\gamma}$ in a shower caused by a primary electron of energy $\epsilon_0 = 17$. 1--from reference 28, 2--by moments, reference 37, 3--from reference 29.

Figure 3 shows the Snyder curve 1 for $\gamma = 2$ and curve 2 computed by moments.³⁷ From a comparison of the curves it is obvious that the first moments in t of Snyder's curves are less than the accurate values. Thus, $\bar{t} = \int_0^4 N t dt / \int_0^4 N dt$ (*), computed from curve 1, are 9% less than the accurate value, $t^2 = \int_0^4 N t^2 dt / \int_0^4 N dt$ (*) is 13% less than the accurate value. From Fig. 3 it is obvious that the difference between the moments $\bar{t}^n = \int_0^\infty N t^n dt / \int_0^\infty N dt$ according to Snyder and the accurate ones will be more than for the (*) moments. Thus, the curves in reference 28 are accurate enough, starting with $\gamma \gg 4$ or 5.

Using the data in tables 1 and 2 of reference 28 which give the values of the functions H, M, f, \dots , we can show that for $E/\beta = 0.01$ the term $O(E/\beta)^2 \ln^3(\beta/E)$ in (2.21) is about 3% of the first two, for $E/\beta = 0.05$ it is already about 20%; that is to say, (2.21) cannot be used to calculate the energy particle spectrum at $E/\beta > 0.05$. Snyder's method makes it possible to do a detailed calculation of the integral energy spectrum for E/β from 0 to 0.05. But a detailed calculation of the spectrum in this energy region, under the initial suppositions, does not make physical sense. Thus Snyder's method allows us to find the total

number of particles as a function of the depth at primary-particle energies $\gamma \gg 4$ or 5, but actually it does not make it possible to compute the energy spectrum. Since we are ignoring the term $e^{\lambda_2(s)t}$ in the calculation of the integrals in (2.20), the expressions we obtain are sufficiently accurate only starting with a depth $t > 1$. It is shown in reference 38 that the saddle point method can be applied with reasonable accuracy (10%) to the calculation of $N(E, t)$ down to $t = 0$, if we present the expression for $N(E, t)$ in the form

$$N(E, t) = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} e^{\varphi(s, E)} ds,$$

To obtain the results of Bhabha and Chakrabarty²⁹ we put $s + \sigma + 2 = s'$, $ds = ds'$ (s' on the right is identical with s in reference 29) in (2.17) and, fixing the contour of integration over λ , we move the contour of integration c_s to the right, and the contour c_σ to the left. First we calculate the integral over σ from the residues at the poles $\sigma = -m - 1$, $m = 0, 1, 2, \dots$. The series obtained diverges, but if we take only the final number of poles, ignoring the remainder, and apply to each term of the series the inverse Laplace transformation, then the resulting series will converge. We write the result of integration over σ in the form:

$$P(E, t) = \sum_{n=0}^{\infty} \frac{E^{-n}}{2\pi i E_0 \beta^{-n+1}} \int_{c_s} ds \left(\frac{E_0}{E}\right)^s \frac{\Gamma(s+n)}{\Gamma(s)} (-1)^n \frac{1}{2\pi i} \times \int_{c_\lambda} e^{\lambda t} d\lambda L(s+n, n-1, \lambda). \quad (2.22)$$

By using the recurrence relation (2.13) and the boundary condition for function $L(\lambda, s)$ repeatedly, we obtain

$$L(s+n, -n-1, \lambda) = \prod_{k=0}^n \frac{1}{\psi(s+k-1, \lambda)}. \quad (2.23)$$

The function

$$\psi_n(s, t) = \frac{1}{2\pi i} \int_{c_\lambda} e^{\lambda t} L(s+n, -n-1, \lambda) d\lambda \quad (2.24)$$

can be found from the expression

$$\psi_0(s, t) = \frac{1}{2\pi i} \int_{c_\lambda} e^{\lambda t} \frac{\lambda + \sigma_0}{[\lambda - \lambda_1(s-1)][\lambda - \lambda_2(s-1)]} d\lambda. \quad (2.25)$$

Integrating (2.25), we obtain

$$\psi_0(s, t) = \frac{(\sigma_0 + \lambda_1(s)) e^{\lambda_1(s)t}}{\lambda_1(s) - \lambda_2(s)} + \frac{(\sigma_0 + \lambda_2(s)) e^{\lambda_2(s)t}}{\lambda_1(s) - \lambda_2(s)}, \quad (2.25')$$

an expression which coincides with formula (8) of reference 29. It is not difficult to verify that by using (2.23) and (2.25) we can rewrite (2.24) in the form

$$\left. \begin{aligned} \psi_n(s, t) &= \int_0^t \psi_0(s+n, t-t') \psi_{n-1}(s, t') dt' \\ \text{or} \\ \psi_n(s, t) &= \int_0^t \psi_0(s, t') \psi_{n-1}(s+1, t-t') dt', \end{aligned} \right\} \quad (2.24')$$

which coincides with (11) and (13) of reference 29. It is not difficult to show that by using (2.25) and (2.24), we can rewrite (2.22) in the form

$$P(E, t) = \frac{1}{2\pi E_0 i} \int_{c_s} \left(\frac{E_0}{E}\right)^s \sum_{n=0}^{\infty} \left(-\frac{\beta}{E}\right)^n \frac{\Gamma(s+n)}{\Gamma(s)} \psi_n(s, t) ds, \quad (2.22')$$

which coincides with (14) of reference 29. In reference 36 it was shown that

$$\psi_n(s, t) < \frac{(1+\varepsilon)^{n+1} t^n}{n!}.$$

Therefore the series (2.22') converges uniformly and absolutely when $E > \beta t$; thus we can interchange in (2.22') the order of summation and integration. (2.22') is an accurate solution of the basic equations of the theory, under the conditions $E > \beta t$.

Let us introduce the new functions

$$G(s, t) = \frac{\psi_1(s, t)}{\psi_0(s, t)} \quad \text{и} \quad f_n(s, t) = \quad (2.26)$$

$$\psi_0(s, t) \frac{G^n(s, t)}{n!} - \psi_1(s, t) \frac{G^{n-1}(s, t)}{(n-1)!} + \dots$$

We rewrite the integrand of (2.22') by replacing E^{s+n} by $\{(E + \beta G(s, t)) - \beta G(s, t)\}^{s+n}$ and expand the

latter expression in powers of $\beta G/(E + \beta G)$. As a result we obtain:

$$E_0^s \sum_{n=0}^{\infty} (-\beta)^n \frac{\Gamma(s+n)}{\Gamma(s)} \frac{\Psi_n(s, t)}{\{(E + \beta G) - \beta G\}^{n+s}} =$$

$$= E_0^s \sum_{n=0}^{\infty} (-\beta)^n \frac{\Gamma(s+n)}{\Gamma(s)} \Psi_n(s, t) \frac{1}{(E + \beta G)^{n+s}} \sum_{m=0}^{\infty} \left(\frac{\beta G}{E + \beta G}\right)^m \frac{\Gamma(n+s+m)}{\Gamma(m+1)\Gamma(n+s)}.$$

The double series converges absolutely, according to (2.26), and therefore its terms can be changed in position. As a result we obtain the following expression for the function $P(E, t)$:

$$P(E, t) = \frac{1}{2\pi i E_0} \int_{C_s} \left(\frac{E}{E + \beta G}\right)^s \sum_{n=0}^{\infty} \left(\frac{\beta}{E + \beta G}\right)^n \frac{\Gamma(s+n)}{\Gamma(s)} f_n(s, t) ds. \quad (2.27)$$

From the condition of analyticity and continuity of (2.27), it follows that it is an accurate solution of (1.1) for all $E \gg 0$. For $N(E, t)$ we obtain

$$N(E, t) = \sum_{n=0}^{\infty} N_n(E, t), \text{ where } N_n(E, t) = \frac{1}{2\pi i} \int_{C_s} \frac{1}{s+n-1} \left(\frac{E_0}{\beta}\right)^{s-1} \left(\frac{\beta}{E + \beta G}\right)^{s+n-1} \frac{\Gamma(s+n)}{\Gamma(s)} f_n(s, t) ds. \quad (2.28)$$

We introduce the function (see also reference 39)

$$\Phi_n(s, \lambda) = \int_0^{\infty} e^{-\lambda t} f_n(s, t) dt, \text{ then } f_n(s, t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t} \Phi_n(s, \lambda) d\lambda, \quad (2.29)$$

where σ is a large real number such that all poles Φ_n lie to the left of the path of integration. The integral (2.28) can now be computed by the double saddle point method. As a result we obtain

$$N_n(E, t) = \frac{\exp(\Phi_n(s_n, \lambda_n))}{2\pi \left\{ \frac{\partial^2 \Phi_n}{\partial \lambda_n^2} \frac{\partial^2 \Phi_n}{\partial s_n^2} - \left(\frac{\partial^2 \Phi_n}{\partial \lambda_n \partial s_n} \right)^2 \right\}^{1/2}}. \quad (2.30)$$

The saddle point is determined from the equations

$$t + \frac{\partial \ln \Phi_n(s, \lambda)}{\partial \lambda} = 0, \quad (2.31)$$

$$y_0 + \left[\ln \frac{\beta}{E + \beta G} - (s+n-1) \frac{\beta}{E + \beta G} \frac{\partial G}{\partial s} \right] - \frac{1}{s+n-1} + \frac{\partial}{\partial s} \{ \ln \Gamma(s+n) - \ln \Gamma(s) \} + \frac{\partial}{\partial s} \ln \Phi_n(s, \lambda) = 0.$$

The quantities essential for numerical calculations are given in reference 29. When $E \gg \beta$, the second term in the square brackets of (2.31) is so small compared with the first that it can be ignored. Therefore the variation of $G(s, t)$ with s practically does not change the saddle point. Consequently, for the high-energy part of the spectrum the inclusion of ionization losses leads only to a lowering of the energy of the particles by $\beta G(s_n, t)$. For low energies, $E/\beta < 1$, it is impossible to ignore the second term in the square brackets. However the saddle point changes very slowly with E , so that it can be defined with good accuracy at $E = 0$. At low E we then obtain the following expression for the function $N_n(E, t)$:

$$N_n(E, t) = \left\{ \frac{\beta G(s_n, t)}{E + \beta G(s_n, t)} \right\}^{s_n+n-1} N_n(0, t). \quad (2.30')$$

We note, however, that (2.30') should be used only when $E_0/\beta \gg 1$; when $E_0/\beta \sim 1$ the more accurate expression (2.30) should be used. From (2.30) it is easy to obtain an expression for the differential electron energy spectrum

$$P(E, t) = \sum_{n=0}^{\infty} P_n(E, t),$$

$$P_n(E, t) = - \frac{\partial N_n(E, t)}{\partial t} \cong \frac{s_n+n-1}{\beta G(s_n, t)} \left\{ \frac{\beta G(s_n, t)}{E + \beta G(s_n, t)} \right\}^{s_n+n} N_n(0, t).$$

Using the double saddle point method we can calculate in practice only the first two terms of the series. Thus for $P(E, t)$ we obtain the expression:

$$P(E, t) \cong \text{const} \left\{ (s_0 - 1) \left(\frac{\beta G}{E + \beta G}\right)^{s_0} N_0(0, t) + (s_0 + 1) \left(\frac{\beta G}{E + \beta G}\right) N_2(0, t) \right\} \quad (2.32)$$

From (2. 32) it can be seen that the contribution of the second term becomes larger than that of the first term, beginning at an energy less than

$$\frac{(E + \beta G)^2}{\beta^2 C^2} \cong \frac{(s_0 + 1) N_2}{(s_0 - 1) N_0}.$$

Thus, at $\gamma = 10$ and $t = 20$, N_2 makes a large contribution to the electron spectrum at an energy less than 0.1β . The differential electron spectrum, according to (2. 32), remains finite for $E = 0$, while in accordance with results of references 2, 3, 28, and others it diverges as $1/n$. With $E = 0$ the first two terms of the series give a value of $N(t, 0)$ 30 to 40% too low the maximum of the curve. We note, that according to reference 29 $\int_0^\infty N dt = E_0/\beta$ while $\int_0^\infty (N_1(0, t) + N_2(0, t)) dt = 0.7E_0/\beta$. This difference is caused by the fact that expression (2. 32) underestimates the electrons of low energy, $E < \beta$. It is known that the relative number of soft electrons increases with depth; therefore the inaccuracy of the curves in reference 29 increases rapidly as the shower develops. Figure 3 shows the following: curve 3

for $\gamma = 2$ according to reference 29 and curve 1 according to reference 28. The difference between the curves at the maximum is 25% and at 4 t -units it is already 60%. Obviously the series (2. 28) and (2. 32) converge more rapidly the greater the value of E . The Bhabha and Chakrabarty solution is therefore convenient for computing the electron spectrum for energies of the order of the critical energy and higher.

In this way the authors of the references cited above have developed new improved mathematical methods in recent years. However, they did not succeed in obtaining new physical results. New physical results were obtained in reference 31, the principal results of which will now be discussed.

6. The Chakrabarty and Gupta Method

We start from the expression for the function $P(E, t)$ given by (2. 22') or (2. 27). Further, as in reference 33, the Laplace transformation with respect to t in Eq. (2. 25) yields

$$P(E, t) = -\frac{1}{4\pi^2\beta} \int ds \int d\lambda \left(\frac{E_0}{\beta}\right)^{s-1} \sum_{n=0}^\infty (-1)^n \left(\frac{\beta}{E}\right)^{s+n} \frac{\Gamma(s+n)}{\Gamma(s)} e^{\lambda t} \Phi_n(s, \lambda), \tag{2. 33}$$

where

$$\Phi_n(s, \lambda) = \Phi_{n-1}(s, \lambda) \Phi_0(s+n, \lambda); \quad \Phi_0(s, \lambda) = \frac{\sigma_0 + \lambda}{(\lambda - \lambda_1(s))(\lambda - \lambda_2(s))}.$$

Following reference 31, it is easy to write down the solution of the difference equation in the form

$$\Phi_p(s, \lambda) = \lim_{N \rightarrow \infty} \{\Phi_0(s+N+1, \lambda)\}^{p+1} \prod_{i=0}^N \frac{\Phi_0(s+i, \lambda)}{\Phi_0(s+p+i+1, \lambda)},$$

where p is an arbitrary number. The series in (2. 33) converges if $E > \beta t$. Our problem is to find a solution suitable for all E . As in reference 34, we represent the series (2. 33) in the form of a contour integral* which coincides with series

(2. 33) for $E > \beta t$ and is its analytical continuation in the region $E > \beta t$. We have

$$\sum_{n=0}^\infty (-1)^n \left(\frac{\beta}{E}\right)^{s+n} \Gamma(s+n) \Phi_n(s, \lambda) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(s+p) \Gamma(p+1) \Gamma(-p) \left(\frac{\beta}{E}\right)^{s+p} \Phi_p(s, \lambda) dp,$$

where $-1 < \gamma < 0$. By closing the contour of integration on the right with a semicircle of infinite radius, and taking the residues of the integrand at points $p = n$, we readily verify the correctness of the equation. Substituting the previous equation

*We note that S. Z. Belen'kiĭ was the first to use a similar device in calculating the energy spectrum of particles, with allowance for ionization losses.² In reference 30, this method of solution, applicable for any E , was used inconsistently.

into (2.33) and integrating over λ we obtain, after certain simplifications, Expression (2.34) differs considerably from (2.27). We recall that $Re \lambda_1(s)$ increases with an

$$P(E, t) = \sum_{m=0}^{\infty} P_m(E, t), \text{ where } P_m(E, t) =$$

$$= -\frac{1}{4\pi^2\beta} \int_{\sigma-i\infty}^{\sigma+i\infty} ds \left(\frac{E_0}{\beta}\right)^{s-1} \int_{\nu-i\infty}^{\nu+i\infty} dp \frac{\Gamma(s+p)\Gamma(-p)}{\Gamma(s)} \left(\frac{\beta}{E}\right)^{s+p} \times$$

$$\times \left[\frac{\sigma_0 + \lambda_1(s+m)}{\lambda_1(s+m) - \lambda_2(s+m)} G_m(s, p) \exp \lambda_1(s+m)t - \right.$$

and
$$\left. - \frac{\sigma_0 + \lambda_2(s+m)}{\lambda_1(s+m) - \lambda_2(s+m)} F_m(s, p) \exp \lambda_2(s+m)t \right],$$

$$G_m(s, p) = \Gamma(p+1) \lim_{N \rightarrow \infty} \left\{ \Phi_0(s+N+1, \lambda_1(s+m)) \right\}^{p+1} \times$$

$$\times \prod_{\substack{i=0 \\ i \neq m}} \Phi_0(s+i, \lambda_1(s+m)) \prod_{i=0}^N \cdot 1 / \Phi_0(s+p+i+1, \lambda_1(s+m)),$$

$$F_m(s, p) = \Gamma(p+1) \lim_{N \rightarrow \infty} \left\{ \Phi_0(s+N+1, \lambda_2(s+m)) \right\}^{p+1} \times$$

$$\times \prod_{\substack{i=0 \\ i \neq m}} \Phi(s+i, \lambda_2(s+m)) \prod_{i=0}^N \cdot 1 / \Phi_0(s+p+i+1, \lambda_2(s+m)).$$

increasing $Re s$. Consequently the contribution of the term with $\exp \lambda_1(s+1)t$ is much smaller than the contribution of the term containing $\exp \lambda_1(s)t$ and so forth. The term $P_m(E, t)$ is proportional to $\exp \lambda_1(s+m)t$, while in expression (2.27) it is proportional to $\exp \lambda_1(s)t$. Thus, series (2.34) converges much more quickly than (2.27). With $\gamma_0 > 1$ in (2.34), the contribution of the first term is much larger than the contribution of all the others for the energies $E \gg 0$. It is easy to see that (2.34) with a change of notation coincides with Snyder's expression (2.19). From (2.34) it is easy to obtain the expression for $N(E, t)$. Moreover, the integrals in the expressions for $P(E, t)$ and $N(E, t)$ were calculated by the double saddle point method. In the limiting case $(s+p) \rightarrow 0$, which corresponds to $E \rightarrow 0$, the expressions obtained coincide with the corresponding formulas of Snyder. When $-p \rightarrow 0$, which corresponds to $E \rightarrow \infty$, the expressions obtained coincide with the solution of the basic equations of the theory without allowance for ionization losses. The values of $P(E, t)$ and $N(E, t)$ are easy to obtain for intermediate values of E from tables 1 and 2 of reference 31. The attempts made by the authors of reference 31

to justify the use of the Bhabha and Chakrabarty²⁹ solution in heavy elements (Pb and others) for $E = 0$, on the grounds that it is close to the value of $N(E_1, t)$ obtained in reference 31 for $\gamma = \ln E_1 / \beta = -2$ (in lead this value is $E_1 \approx 2m_e c^2$), seem to us unsatisfactory for the following reasons. From the basic equations of the theory it is easy to obtain the "law of conservation of energy" of cascade theory, $\int_0^\infty N(O, t) dt = E_0 / \beta$. This relation is satisfied by the solution $N(O, t) = \sum_{m=0}^\infty N_m(O, t)$ cf. (2.28) obtained by Bhabha and Chakrabarty, and is not satisfied by its first two terms, which underestimate the overwhelming number of electrons at low energies. We cannot therefore consider that their solution for $E = 0$ describes the cascade process in heavy elements with $E_1 > 2m_e c^2$ (because of the inaccuracy of the basic physical premises in the energy region $E = (0 \text{ to } 2) m_e c^2$), because their solution does not allow for the energy variation of the total coefficient of photon absorption and Rutherford scattering of the charged particles, which play a definite pole in the development of showers in heavy substances. Thus the method of Chakrabarty and Gupta allows us to compute the cascade curves and the energy spectra of particles at various depths, including ionization losses, in light substances with $\gamma > 1$. That is to say, it allows us to obtain an accurate complete solution of the one-dimensional problem of cascade theory. It is interesting to compare these results with the results of S. Z. Belen'kii and other authors. As already mentioned, the solution obtained by Bhabha and Chakrabarty, which does not allow for ionization losses, agrees with the results of other authors. The dependence of the total number of particles on depth, obtained

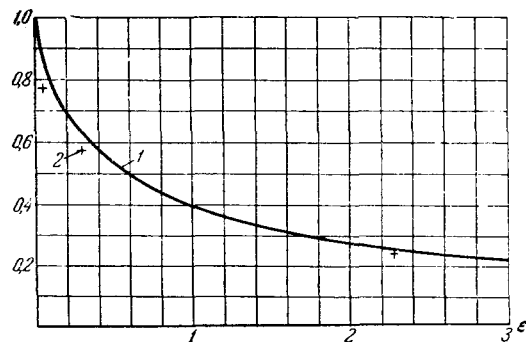


FIG. 4. Tamm-Belen'kii energy equilibrium spectrum²⁶ (solid curve). Crosses--electron spectrum in the maximum of the cascade after Chakrabarty and Gupta.³¹

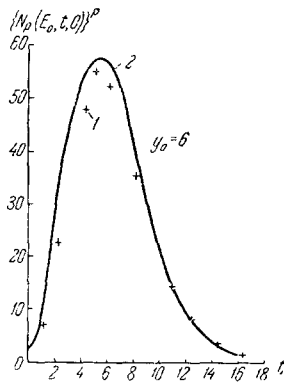


FIG. 5. The dependence on depth of the total number of electrons $\{N_p(E_0, t, 0)\}^P$ in a shower from a primary electron of energy $\gamma_0 = 6$. The solid curve was drawn according to data in reference 2. The crosses indicate values taken from reference 31.

in reference 31, coincides with the results of Snyder and of Belen'kii. Figure 4 shows (1) the equilibrium energy spectrum and (2) the spectrum at the maximum of the shower, drawn according to data of reference 31. Curves 1 and 2 are sufficiently close to each other, except for the energy region $E \sim 0.1\beta$ where the difference is 20%. In this energy region, however, the basic physical premises of the theory are not accurate. Figure 5 shows the dependence of the number of particles of energy greater than E on the depth in a shower caused by a primary electron of energy $\gamma_0 = 6$, according to data of reference 31 (curve 1) and the data of reference 2 (curve 2). From the figure it can be seen that curves 1 and 2 are in good agreement. This comparison shows that the approximated complete solution of the one-dimensional problem obtained by S. Z. Belen'kii (see reference 2, for instance) agrees very well with the exact one.

7. Solution of Generalized Equations of the Theory

We shall examine the generalized basic equations of the cascade theory, which account for the generation of electrons and photons continuously along the entire path of development by some kind of penetrating radiation. Let $S_p(E, t)dEdt$ be the number of electrons and $S_\pi(E, t)dEdt$ the number of photons of energy $(E, E + dE)$ produced by the external radiation in a layer dt . Then the basic equations of the cascade theory can be written

$$\left. \begin{aligned} \frac{\partial P(E, t)}{\partial t} &= L_1[P, \Gamma] + \beta \frac{\partial P}{\partial E} + S_p(E, t), \\ \frac{\partial \Gamma(E, t)}{\partial t} &= L_2[P, \Gamma] + S_\Gamma(E, t). \end{aligned} \right\} (1.1')$$

Here L_2 and L_1 are linear integral operators that take into account bremsstrahlung and pair production; in low-energy regions the operators approximately account for the Compton effect. It is interesting to examine the equations in (1.1') by choosing a source function in the form:

a) $S_p(E, t) = 0, S_\Gamma(E, t) = \delta(E_0 - E) e^{-\mu t}$

and

b) $S_\Gamma(E, t) = \begin{cases} E^{-\nu-1} e^{-\mu t} & \text{for } E > E_h, \\ 0 & \text{for } E < E_h. \end{cases}$

After performing the necessary transformations, analogous to those of Sec. 4, we obtain the first term of the expansion of $P(\lambda, s)$ in powers of β/E_0 , in the form

$$P(\lambda, s) = \frac{2\sigma_0 E_0 s \Gamma(s+1)}{\beta(\lambda + \mu)(\lambda + \sigma_0) \Gamma(s+2)} e_0^{-s+s+1} E_0^s \int_0^1 \tau^{s-1} (1-\tau)^{-s+s+1} d\tau.$$

Going from the variables λ and s to the variables t and E , we obtain the following expressions for the function $N_p(t, E)$:

$$N_p(t, E) = \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{H_1(s) D_2(s)}{s} G(s, \epsilon) \exp(\lambda_1(s)t + ys - \ln(\lambda_1(s) + \mu)) ds,$$

where $D_2 = 2\sigma_0 [f(\lambda_1(s))]^s / (\lambda_1(s) + \sigma_0) \Gamma(s+2)$, and the function $G(s, 2)$ is defined by (2.11). If $\epsilon \gg 1$, then $G(s, \epsilon) = \Gamma(s+1)/\epsilon^s$ and

$$N_p(t, E) = \frac{H_1(s) D_2'(s) \exp\{ys + \lambda_1(s)t - \ln(\lambda_1(s) + \mu)\}}{s \sqrt{2\pi \lambda_1'(s) t - \frac{[\lambda_1''(s)(\lambda_1(s) + \mu) - \lambda_1'^2(s)]}{(\lambda_1(s) + \mu)^2}}}, \quad (2.35)$$

where $D_2'(s) = B(s)/(\lambda_1(s) + \sigma_0)$ and $\gamma = \ln E_0/E$; expression (2.35) coincides with the solution of (1.1') without allowance for ionization losses; if $\gamma \gg 1$ (here $\gamma = \ln E_0/\beta$) and E is of the order of the critical energy β and less, we obtain the following expression for $N_p(t, E)$:

$$N_p(t, E) = \frac{H_1(s) D_2'(s) \exp\{ys + \lambda_1(s)t - \ln(\lambda_1(s) + \mu)\}}{s \left\{ 2\pi \lambda_1'(s) t - \frac{[\lambda_1'(s)(\lambda_1(s) + \mu) - \lambda_1'^2(s)]}{(\lambda_1(s) + \mu)^2} \right\}^{1/2}} G(s, \epsilon), \tag{2.36}$$

where $\epsilon = Ef(\lambda, (s))/\beta$. The parameter s in (2.35) and (2.36) is determined from the equation

$$\lambda_1'(s) = - \frac{y}{t - \frac{1}{\lambda_1(s) + \mu}}. \tag{2.37}$$

From (2.37) it can be seen that as t approaches infinity, s does not increase without limit (as in the equations without a source), but approaches $\lambda_1(s) = -\mu$. Thus, beginning at a certain depth, equilibrium begins to be established between the electron-photon component and its generating radiation, which is absorbed exponentially with an absorption coefficient $1/\mu$. Equilibrium between the generating and shower components will be attained at large depths near the maximum range of the electron-photon shower. Equilibrium was examined in references 3 and 40. The results obtained can be generalized to a generating function of the power-law type. In analogy with (2.36), we obtain for $N_p(E_0, t, E)$ an expression in the form of a sum of two terms $N_{p1}(E_0, t, E)$ and $N_{p2}(E_0, t, E)$. For $N_{p1}(E_0, t, E)$ we obtain for $E < E_k$ the following expression:

$$N_{p1}(E_0, t, E) = \frac{H_1(s) D_3(s) \exp\{ys + \lambda_1(s)t - \ln(\lambda_1(s) + \mu)\}}{s (2\pi)^{1/2} \left\{ \lambda_1'(s) t - \frac{[\lambda_1'(s)(\lambda_1(s) + \mu) - \lambda_1'^2(s)]}{(\lambda_1(s) + \mu)^2} \right\}^{1/2}} G(s, \epsilon), \tag{2.38a}$$

where

$$D_3(s) = \frac{2\sigma_0 E_0^\gamma \{f(\lambda_1(s))\}^s}{(\lambda_1(s) + \sigma_0) \Gamma(s+2)(s-\gamma)}; \quad y = \ln E_0/\beta; \quad \text{Re } s < \gamma.$$

The parameter s is determined from (2.37). With $E > E_k$, allowing for the pole $s = \gamma$, we obtain

$$N_{p1}(E_0, t, E) = - \frac{H_1(\gamma) B(\gamma) \Gamma(\gamma+1)}{\gamma (\lambda_1(\gamma) + \sigma_0) (\lambda_1(\gamma) + \mu)} e^{\lambda_1(\gamma)t} \frac{1}{E^\gamma} \tag{2.38 a}$$

For the second term we obtain

$$N_{p2}(E_0, t, E) = - \frac{H_1(s) D(s) \exp\{\lambda_1(s)t - \ln(\lambda_1(s) + \mu)\} \{A_1(\epsilon) + A_2(\epsilon)\}}{s (2\pi)^{1/2} \left\{ \lambda_1'(s) t - \frac{[\lambda_1'(s)(\lambda_1(s) + \mu) - \lambda_1'^2(s)]}{(\lambda_1(s) + \mu)^2} \right\}^{1/2}} \tag{2.38b}$$

Here

$$A_1(\epsilon) = \frac{2\sigma_0 \beta^{-\gamma}}{\Gamma(\gamma+2)} \epsilon \int_0^1 \frac{(1-\tau)^\gamma}{\tau^2} \sum_{n=0}^\infty \left[\frac{\epsilon(1-\tau)}{\tau} \right]^n \frac{(-1)^n}{n!} \Gamma(-n-\gamma) d\tau,$$

$$A_2(\epsilon) = \frac{2\sigma_0 \beta^{-\gamma} \epsilon^{-\gamma+1}}{\Gamma(\gamma+2)} \int_0^1 \tau^{\gamma-2} \sum_{n=0}^\infty \left[\frac{\epsilon(1-\tau)}{\tau} \right]^n \frac{(-1)^n}{n!} \Gamma(-n+\gamma) d\tau,$$

$$D(s) = \frac{[f(\lambda_1(s))]^\gamma \Gamma(\gamma-s)}{\lambda_1(s) + \sigma_0};$$

s is determined from the equation

$$t = \frac{1}{\lambda_1(s) + \mu}.$$

Thus formulas (2.38) determine $N_p(E_0, t, E)$ in the case of a power-law generating function. We note that the term $N_{p2}(t, E)$ is considerably less than $N_{p1}(t, E)$.

So far, all the methods of solving the basic equations of the theory have been based on the use of functional transformations. We recall that the success of the functional transformation method is due to the use of asymptotic expressions for the bremsstrahlung and pair-production cross sections which are homogeneous functions of the energies of the primary and secondary particles. Were we to use the more exact cross sections, which are not homogeneous, the formalism of functional transformations becomes cumbersome and unmanageable. Even for light elements the use of asymptotic expressions for cross sections instead of the exact ones involves some error in determining the function $N(E_0, t, E)$.⁴¹ In reference 41 the basic equations of the theory were solved by the perturbation method. The Snyder solution,²⁸ obtained in the Born approximation, was the first solution. The correction to it was

calculated by using the more accurate approximation of the Bethe-Heitler cross section

$$R(E, E') = \frac{R_0(E, E')}{1 + k \left| \frac{E'}{E(E-E')} \right|},$$

which differed from the exact cross section by not more than 2%. Here $K = \frac{mc^2}{Z^{1/3}} \cdot \frac{255}{15.6 - \frac{4}{3} \ln Z}$, and R_0

is the cross section in case of complete screening. The results obtained by accurate numerical calculations are correct only for light elements at high primary-particle energies, $\ln E_0/\beta > 1$. In comparison with the Snyder curves, the formally more exact curves have a maximum somewhat lower and at a greater depth. The number of particles in the first few t -units of the absorber is some 10 to 15% less and the number of particles at greater depths increases correspondingly. The numerical results are given in such a way that it is possible to compute the correction to the solution in the case of complete screening in any light substance. Reference 41 does not allow for the Compton effect. We recall that in light substances the total coefficient of photons absorption due to pair production and the Compton effect remains constant, with good accuracy, up to 10^6 ev. Therefore the results of the theory with asymptotic cross sections are more accurate in light substances. We can expect that they differ from the exact results by not more than 10%.

8. Other Methods of Solution

Bhabha and Heitler⁴² developed another method of solving the basic equations of the theory. It consists of the following. Let an electron of energy E_0 fall on the boundary of a layer at $t = 0$. First one calculates the probability $f_0(E_0, E, t)$ that this electron will reach a depth t with an energy greater than E . Then one determines the number of photons of energy greater than E emitted by the electron at different points in its path, and one calculates the number of electrons of the first generation $f_1(E_0, E, t)$ formed by photons and attaining depth t with an energy more than E . Analogously one calculates the number of electrons of the following generations reaching depth t with an energy greater than E . The total number of electrons is determined in the form of the sum of electrons of the various generations

$$N(E_0, E, t) = f_0(E_0, E, t) + f_1(E_0, E, t) + f_2(E_0, E, t) + \dots$$

The series converges rather quickly if E is of the order of the energy of the primary particle E_0 , and if t is not larger than several cascade units. In all the calculations by the method of consecutive collisions, simplified cross sections are used for

bremsstrahlung and pair production, and complete screening is assumed. Ionization losses and the Compton effect are ignored. The Bethe-Heitler formula¹³ is used for the straggling in the energy loss during radiation. The formula does not account for electron loss to ionization. References 42 and 43 calculated the quantities

$\{N_p(E_0, E, t)\}^P$. Reference 44 included calculations of the quantities $\{N_p(E_0, E, t)\}^n$, $\Gamma_p(E_0, E, t)$, and $\Gamma(E_0, E, t)$. The distribution functions of N, P and for a spectrum of primary photons of the type $1/E$ for $E < E_0$ and 0 for $E > E_0$ were determined in reference 45 with an electronic computer by the method of consecutive collisions. The distribution functions were computed in references 42 to 44 for the depth intervals $l = t/\ln 2$ from 0.2 to 5 and for $\ln E_0/E$ from 2 to 10 and in reference 45 for intervals of t from 0.125 to 20.0 and for E_0/E from 0.9 to 0.00675. The integral spectra of electrons and photons in a shower from a primary electron or photon in the energy range $\log E_0/E$ from 1 to 10 and depths t from 0.1 to 5.0 were determined in reference 46 with an electronic computer, without ionization loss and with exact cross sections in the completely screened form. As usual, the solution was obtained in the form of an integral in the complex plane which was evaluated with the electronic computer. The values obtained for the same quantities in earlier references (42 to 44) were found to be in error by 10%. This inaccuracy can be attributed to two causes: the use of simplified cross sections and the approximateness of the numerical methods used. * Reference 47 contains equations for the distribution function of the number of particles produced in a layer of substance between t and $t + dt$ with energies between E and $E + dE$ at the place of production. The evaluations are solved for the average number of particles $\epsilon\{P(E_0, E, t)\}$ produced in layer dt with energy in the interval dE at the place of production, not including ionization losses. A solution of the equations with ionization losses, integrated over t , was obtained in the form of an absolutely converging series. This last solution, it is true, was barely investigated and was not solved numerically. We note that the quantity $\epsilon\{P(E_0, E, t)\}$ is more convenient for comparison with experiments with photoplates and diffusion chambers. A solution for these equations, including ionization losses, is given in reference

*We used the saddle point method to calculate the same functions at approximately the same energy intervals and depths. We found the saddle point method to cause an error of not more than 10%.

48. In addition, reference 48 shows that without including ionization losses the number of particles $\epsilon\{P(E, t)\}$, produced in the layer from t_1 to t_2 with energy E (or more than E) at the place of production, is equal to the area under the corresponding cascade curve from t_1 to t_2 . When ionization losses are included, the first quantity is always bigger than the second. In the limiting case with $E \rightarrow 0$, $t_1 = 0$ and $t_2 = 0$, we have $\epsilon\{N_p(E_0, 0, t)\} \rightarrow \infty$ while the second quantity remains bounded. An analytical expression for the average energy of one charged particle arriving at a given depth of the shower was obtained. Detailed tables of the quantities $\epsilon\{N_p(E, t)\}$ including ionization losses are shown.

9. The Cascade Theory for Heavy Substances

The cascade theory encounters some special difficulties in heavy substances, where the shower production is particularly intense. The first of the difficulties is that in heavy substances the total absorption coefficient for photons is strongly dependent on the energy. Up to now we have considered this coefficient as a constant, which follows from the asymptotic expression for the probability of pair production. The second difficulty is that the scattering of cascade particles in heavy substances is very great. For instance, the critical energy in lead is $\beta = 6.4$ Mev. It is known that in the region of the maximum of the cascade more than two-thirds of the particles have an energy less than critical; their mean-squared angle multiple-scattering angle is $\overline{\theta^2} = (E_k/E)^2 \gg 1$, where $E_k = 21$ Mev. Therefore in heavy substances it is necessary to examine the equations with allowance for scattering, which become in the multiple-scattering approximation (the approximation of L. D. Landau⁴⁹)

$$\left. \begin{aligned} \cos \vartheta \frac{\partial P}{\partial t} &= L^1 [P(t, E, \vartheta) \Gamma(t, E, \vartheta)] + \frac{E_s^2 \Delta_{\vartheta} P(t, E, \vartheta) / 4E^2}{\beta \frac{\partial P}{\partial E}}, \\ \cos \vartheta \frac{\partial \Gamma}{\partial t} &= L_2 [P(t, E, \vartheta), \Gamma(t, E, \vartheta)]. \end{aligned} \right\} (2.39)$$

Here $\Delta_{\vartheta} = \frac{\partial}{\partial \vartheta} (\sin \vartheta \frac{\partial}{\partial \vartheta}) / \sin \vartheta$ is the Laplacian operator, $E_k = E_s (L_{res} / 2L_{rad})^{1/2}$, $E_s = mc^2 \sqrt{4\pi \cdot 137} = 21$ Mev. If an electron or photon of energy E_0 falls vertically on a layer of matter at $t = 0$, then the boundary conditions are written in the form:

or
$$P(0, E, \vartheta) = \delta(E_0 - E) \delta(\vartheta), \Gamma(0, E, \vartheta) = 0$$

$$P(0, E, \vartheta) = 0, \Gamma(0, E, \vartheta) = \delta(E - E_0) \delta(\vartheta).$$

In a number of articles attempts were made to overcome the foregoing difficulties in the theory for heavy substances and to solve (1.1) for variable $\sigma(E)$ or to solve (2.39) with scattering for variable $\sigma(E)$. The cascade curves in lead were calculated in reference 50. The energy dependence of $\sigma(E)$ was roughly approximated by straight lines in different energy regions. With E_1 so large than one can consider $\sigma = \sigma_0(E_1)$ was taken as equal to 3×10 ev), the function $N(E_1, t)$ was taken from reference 3 and introduced as a source into the equations for the function $N(E, t)$ corresponding to $E < E_1$. The equations obtained for $N(E, t)$ were solved by very complicated semi-numerical methods. The authors obtained a qualitatively correct conclusion on the high penetration of a shower in heavy substances because of the high penetration of the photons, whose energy was near to the critical energy in the given substance. However the calculations of these authors are inherently contradictory, since the curves do not satisfy the law of "conservation of energy" of cascade theory, $\int_0^{\infty} N(0, t) dt = E_0 / \beta$. The areas under their curves, for primary electrons of energies 10^9 and 10^{10} ev, are equal to about $0.7E_0 / \beta$. In addition, the electron spectra in the maximum of the cascade in air and in lead, according to reference 50, differ considerably from one another and from the "equilibrium" spectrum which is almost the same in the maximum of the cascade, for any energy dependence of the photon absorption coefficient.²⁶ The cascade curves obtained in reference 50 underestimate the considerable number of low energy electrons. They are more correct at lesser depths. An increase in depth entails a marked increase in the relative number of soft electrons and consequently an inaccuracy in the curves. Figure 6 shows curves for primary 10^9 and 10^{10}

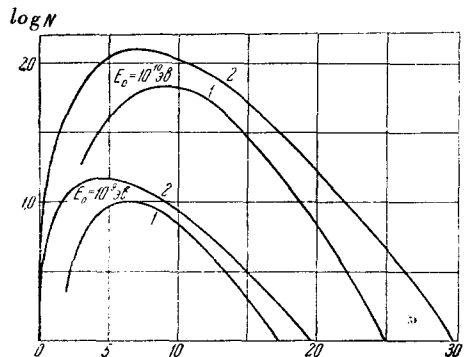


FIG. 6. Cascade curves in lead from primary electrons of energies $E_0 = 10^9$ and 10^{10} ev drawn according to moments⁵¹ (2) and according to data from reference 50 (1).

ev electrons, obtained from reference 50 and obtained on the basis of moments from reference 51. The cascade curves differ from one another by more than 40%. We note that actually the difference is even greater, since different values for the critical energy for lead were used in references 50 and 51. Considering all this, these cascade curves are not quantitatively satisfactory.

In reference 52 a method analogous to that used in reference 53 was used to derive an approximate formula for $N(E, t)$. It is affirmed there that this formula can be applied with any Z , provided $E_0/\beta \gg 1$. The Bethe-Heitler approxi-

mation cross-section proposed in reference 41 was used in the derivation. Ionization losses were calculated approximately:

$$N(E_0, E, t) = \frac{e^{K_0}}{\sqrt{t + K_1}} \exp(-t + 2\sqrt{(t + K_1)(y + K_2)}), \tag{2.40}$$

where $K_n = [K_{n0} + \beta K_{n1}(Z/2.3)]/(E + \beta)$;
 $y = \ln\{E_0/(E + \beta/2.3)\}$; $\beta = 700/(Z + 1.2)$
 is the critical energy. The quantities K_{n0} and K_{n1} are given in Table IV. Figures 7 and 8 show curves of $N(E_0, E, t)$ in air and in lead, drawn according to (2.40) with $E_0/\beta = 100$.

TABLE IV.

K_{00}	K_{01}	K_{10}	K_{11}	K_{20}	K_{21}
-1.22	$-0.9 \ln\left(\frac{Z}{10} + 1\right)$	0.02	$0.13 \ln\left(\frac{Z}{10} + 1\right)$	-0.9	$0.77 \ln\left(\frac{Z}{10} + 1\right)$

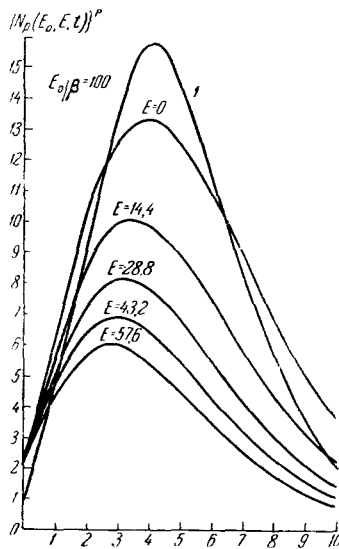


FIG. 7. Dependence on depth of the number of electrons with an energy greater than E in a shower caused by an electron of energy $E_0/\beta = 100$, from reference 52. Curve (1) is $\{N_p(E_0, t, 0)\}^P$ is based on reference 28.

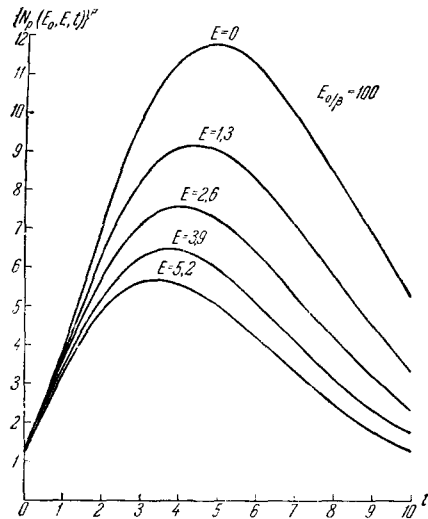


FIG. 8. Same as FIG. 7, but in lead.

At $t=0$ $N(E, t)$ in air changes with increasing E from 2.29 to 2.35; in lead $N(E, t)|_{t=0}$ changes with increasing E from 1.15 to 1.33. $dN(0, t)/dt|_{t=0}$ is 4.8 in lead and 2.5 for air, while the exact value of $dN/dt|_{t=0}$ is 0. Curve 1 for air is drawn in the same figure according to Snyder's formulas. It differs from curves of reference 52

at the maximum by 25 to 35% and more. In addition, cascade curves in lead for primary electrons at high energies of the order of 10^{10} to 10^{11} ev, drawn according to (2.40), differ considerably at greater depths from the curves drawn according to the method of moments.

Figure 9 shows (1) the electron equilibrium

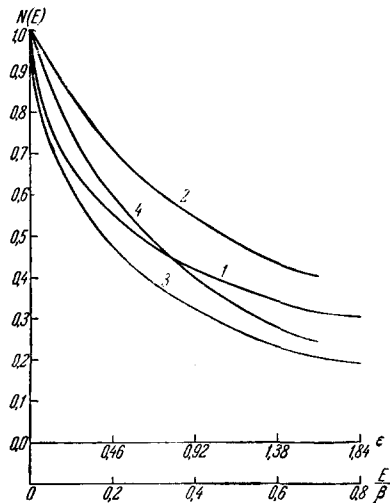


FIG. 9. 1--equilibrium spectrum of electrons,²⁶ curve 2 and the crosses near it are the spectra in the maximum of a cascade in air and lead respectively⁵² 3 and 4--spectra of electrons at depth 8 t -units, according to references 2 and 52 respectively.

spectrum and (2) spectra at the maximum of the cascade in air and lead, according to reference 52. The spectra at the maximum for air and lead coincide with one another but differ from the equilibrium spectrum by 30% and more. Reference 3 shows that the equilibrium spectrum obtained by I. E. Tamm and S. Z. Belen'kii for $E \gg 10^7$ ev differs from the exact spectrum by not more than 4%, while the spectrum obtained in reference 52 for $E = 10^7$ ev differs from the equilibrium spectrum by 28%. In addition, the authors of reference 52 do not allow for scattering, which is great in heavy substances. Figure 9 also shows (curves 3 and 4) the energy spectra of electrons according to references 2 and 52 respectively, at a depth of 8 t -units (cascade parameter $s = 1.4$) in air; these spectra differ considerably from each other, for example, by 22% at $E = 10^7$ ev.

Reference 52 develops the iteration method of solving the basic equations of the theory. Complicated numerical methods were used to calculate the first correction to the solutions $N(E_0, 0, t)$ obtained by using asymptotic cross sections. In calculating the corrections, Bethe-Heitler approximation of the cross sections, proposed in reference 41 was used. The author calculated only two curves for $\ln E_0/\beta = 3.69$ and 5, for a primary electron in lead. Scattering was not included. The method is applicable if $\ln E_0/\beta > 1$, while it is evidently impossible to obtain the energy spectra of particles because the method is so complicated. Cascade curves in lead in the

energy region of 10^9 to 10^{11} ev were obtained in reference 54, as in reference 52, by numerical methods. However, scattering was incorrectly accounted for, and this resulted in noticeable errors, especially at greater depths.

By solving the basic equations of the theory with the help of the foregoing methods, most of which are based on the method of functional transformations, we have thus succeeded in obtaining a rather complete description of the average one-dimensional picture of a shower in a light substance. This problem was most completely solved in reference 2 and 31, under the conditions $\ln E_0/\beta > 1$ and $t > 1$. However these methods do not make it possible to obtain a solution in the energy region of $E_0/\beta \sim 1$ in light substances, or a satisfactory solution with a more accurate accounting for scattering in heavy substances at any E_0 , especially in the energy region $E_0/\beta \sim 1$. Another approach can be used in solving the problem, namely, the function can be defined according to its moments. The next chapter is devoted to the application of the method of moments to cascade theory.

III. SOLUTION OF BASIC EQUATIONS OF THE THEORY. THE METHOD OF MOMENTS.

10. Calculating the Moments of Distribution Functions of Cascade Particles.

We recall again that the calculation of the cascade curve and the energy spectra of particles meets with considerable difficulties when asymptotic expressions are used illegitimately for the pair production and bremsstrahlung cross sections, when it is necessary to include scattering of cascade particles, and also when the energy of the primary particle is commensurate with the critical energy. All these cases are of considerable practical interest. The calculation of moments is a very promising method of solving the problem.

We define the moments of a distribution function by the following relation:

$$\bar{t}^n(E_0, E) = \frac{\int_0^{\infty} f(E_0, t, E) t^n dt}{\int_0^{\infty} f(E_0, t, E) dt}. \quad (3.1)$$

S. Z. Belen'kii obtained recurrence formulas with which to compute consecutively all the moments of the cascade curve ⁵⁵

$$\begin{aligned} \{\bar{t}_p^n(E_0, 0)\}^p &= \frac{n}{E_0} \int_0^{E_0} [P_p(E_0, E) \{t_p^{n-1}(E, 0)\}^p \\ &+ \Gamma_p(E_0, E) \{t_p^{n-1}(E, 0)\}^r E dE \end{aligned} \quad (3.2)$$

and the analogous formula for $\{t_r^m(E_0, 0)\}^r$. Here $\{t_p^n(E_0, 0)\}^{p,r}$ are the n -th moments of the distribution functions of electrons with an energy more than zero in a shower started by a primary electron or photon. $P_p(E_0, E)$ and $\Gamma_p(E_0, E)$ are the equilibrium spectra of electrons and photons in a shower started by a primary electron of energy E_0 . The functions P_p and Γ_p enter into the recurrence formula as zero moments. In reference 5 this formula was generalized to include the case of an arbitrary spectrum of primary particles. The scattering of cascade particles is great in heavy substances, where shower production is especially intense. The effect of scattering on the moments and on the form of the curve in heavy substances was investigated in reference 56. A recurrence formula was derived there to calculate consecutively all the moments of the function, with allowance for the dependence of $\sigma(E)$ on the energy, the scattering, and the ionization losses in the case of a δ -like spectrum of primary particles:

$$\left. \begin{aligned} \{\bar{t}_p^n(E_0, 0)\}^p &= \frac{n}{E_0} \int_0^{E_0} \{P_{p,0,n}(E_0, E) \{t_p^{n-1}(E, 0)\}^p + \\ &+ \Gamma_{p,0,n}(E_0, E) \{t_p^{n-1}(E, 0)\}^r\} E dE, \\ \{\bar{t}_r^n(E_0, 0)\}^r &= \frac{n}{E_0} \int_0^{E_0} \{P_{r,0,n}(E_0, E) \{t_r^{n-1}(E, 0)\}^p + \\ &+ \Gamma_{r,0,n}(E_0, E) \{t_r^{n-1}(E, 0)\}^r\} E dE, \end{aligned} \right\} (3.3)$$

where $P_{p,0,n}(E_0, E) = \int_0^E \int_{(\omega)} P_p(t, E_0, E, \theta) \cos^n \theta d\omega dt$. Analogous formulas were obtained for the moments $\{t_r^n(E_0, 0)\}^{p,r}$. We note that (3.3) determines the moments of the distribution functions by depth, with allowance for scattering for the total number of particles included in the solid angle $\phi = 4\pi$; therefore the conclusions of the theory must be compared with experiments in which the measuring device is located in lead (see reference 58, for example) to exclude boundary effects not allowed for in (3.3). In addition, the derivation of (3.3) ignored the reverse current of particles through the boundary layer of the substance at $t = 0$. It is shown in reference 58 that the number of electrons returning through the boundary layer of the substance at $t = 0$ is less than 5% of the incident beam and therefore the reverse current can be ignored. The formulas given for the moments can be generalized. It is possible to obtain recurrence formulas for the moments of the distribution functions of the number of particles with an energy more than the given E^0 , and not only the total number of particles with an energy more than zero : ⁵⁹

$$\left. \begin{aligned} \{\bar{t}_p^n(E_0, E^0)\}^p &= \frac{n \int_{E^0}^{E_0} \{\bar{t}_p^{n-1}(E', E^0)\}^p \int_{E^0}^{E'} P_p(E_0, E) dE \cdot P_p(E_0, E') + \\ &+ \{\bar{t}_p^{n-1}(E', E^0)\}^r \int_{E^0}^{E'} P_r(E_0, E) dE \cdot \Gamma_p(E_0, E') dE'}{\int_{E^0}^{E_0} P_p(E_0, E) dE}, \\ \{\bar{t}_r^n(E_0, E^0)\}^r &= \frac{n \int_{E^0}^{E_0} \{\bar{t}_p^{n-1}(E', E^0)\}^p \int_{E^0}^{E'} P_p(E_0, E) dE \cdot P_r(E_0, E') + \\ &+ \{\bar{t}_p^{n-1}(E', E^0)\}^r \int_{E^0}^{E'} P_r(E_0, E) dE \cdot \Gamma_r(E_0, E') dE'}{\int_{E^0}^{E_0} P_r(E_0, E) dE} \end{aligned} \right\} (3.4)$$

Analogous formulas were deduced for the moments $\{t_r^n(E_0, E^0)\}^{p,r}$. As in reference 56, we generalize formulas (3.4) to include the case of an arbitrary spectrum of primary particles. Thus, for boundary conditions of the type $P(E, 0) = 0$, $\Gamma(E, 0) = \Phi_\gamma(E_0, E)$ (where E_0 is the upper boundary of the spectrum), which do not differ essentially from an arbitrary spectrum, we obtain

$$\left. \begin{aligned} \{\bar{t}_p^n(E_0, E^0)\}^p &= \frac{\int_{E^0}^{E_0} \Phi_\gamma(E_0, E) \{\bar{t}_p^n(E, E^0)\} \int_{E^0}^E P_r(E, E') dE' dE}{\int_{E^0}^{E_0} P(E_0, E) dE}, \\ \{\bar{t}_r^n(E_0, E^0)\}^r &= \frac{\int_{E^0}^{E_0} \Phi_\gamma(E_0, E) \{\bar{t}_r^n(E, E^0)\}^r \int_{E^0}^E \Gamma_r(E, E') dE' dE}{\int_{E^0}^{E_0} \Gamma(E_0, E) dE} \end{aligned} \right\} (3.5)$$

where

$$P(E_0, E) = \int_{E'}^{E_0} \Phi_{\gamma}(E_0, E') P_{\Gamma}(E', E) dE', \Gamma(E_0, E) = \int_{E'}^{E_0} \Phi_{\gamma}(E_0, E') \Gamma_{\Gamma}(E', E) dE'.$$

In the same way we can generalize the recurrence formulas with scattering:⁵⁹

$$\begin{aligned} & n \int_{E_0}^{E_0} \left[\{t_p^{n-1}(E', E^0)\}^p \int_{E_0}^{E'} P_{p,0,0}(E_0, E) dE \cdot P_{p,0,n}(E_0, E') \right. \\ & \left. + \{t_p^n(E_0, E^0)\}^p \right. \\ & \left. + \{t_p^{n-1}(E', E^0)\}^p \int_{E_0}^{E'} P_{\Gamma,0,0}(E_0, E) dE \cdot \Gamma_{p,0,n}(E_0, E') \right] dE' \\ & = \frac{\int_{E_0}^{E_0} P_{p,0,0}(E_0, E) dE}{\int_{E_0}^{E_0} P_{p,0,0}(E_0, E) dE} \text{ etc.} \end{aligned}$$

Consequently, starting from the known function for the equilibrium spectrum, which is the zero moment, our recurrence formulas allow us to calculate all the following moments of distribution functions of the shower particles. The moments \bar{t} and \bar{t}^2 were first approximately computed for lead in reference 60. An attempt was made there to relate the magnitudes of the first moments with the position and number of particles in the maximum of the shower. Following reference 60, we write the cascade curve in the form

$$N(E_0, t, 0) = \exp \left\{ \varphi \left(t, \frac{E_0}{\beta} \right) \right\}.$$

Suppose the number of particles in the shower has a sharp maximum at a certain point t_{max} . Expanding the function ϕ near the maximum and limiting ourselves to only the first two terms, we compute the integrals $\int_0^{\infty} N(E_0, t, 0) t^n dt$ ($n=0, 1, 2$) which relate the first moments with the position and number of particles at the maximum. However, this approach to calculating the integrals is equivalent to the saddle point method usually used in the theory of showers. Therefore the formulas obtained are correct only if $\ln E_0/\beta \gg 1$. After some simple calculations we obtain

$$t_{max} = \bar{t}; N_{max} = \frac{E_0}{\beta \sqrt{2\pi(\bar{t}^2 - \bar{t}^2)}}.$$

The results of calculating the dependence of t_{max} and N_{max} on E_0 for lead can be found in references 2 and 3. However, the relations are approximate and become correct only at very high^{51, 74} energies, $E_0 \sim 10^{11}$ ev. We note that, in the presence

of recurrence formulas, the task of calculating the distribution functions of electrons and photons in the shower or of their more important characteristics involves, in principle, finding an equilibrium spectrum integrated over depth, which enters in all recursion formulas as a zero moment. One must also consider that a small error in the expression for the zero moments will later on increase in the calculation of the higher moments. It is very important, therefore, to estimate the accuracy of the equilibrium spectrum when applying the method of moments to the solution of basic equations of the cascade theory.

11. The Electron and Photon Equilibrium Spectrum

Let us integrate the equations of (1.1), which do not include scattering, with respect to t from 0 to ∞ , with allowance for boundary conditions (one primary electron of energy E_0). Eliminating the function $\Gamma_p(E_0, E)$ from the second equation, we obtain

$$-\delta(E_0 - E) = L[P_p(E_0, E)] + \beta \frac{\partial P_p(E_0, E)}{\partial E}, \quad (3.6)$$

where $L[P_p(E_0, E)]$ is the integral operator, acting on the variable E and connected with bremsstrahlung and pair production. A solution of (3.6) for the equilibrium spectrum was obtained by I. E. Tamm and S. Z. Belen'kii,²⁶ who replaced the operator L by the following approximate operator:

$$L_1[P(E_0, E)] = \frac{qN(E_0, E)}{E} + \frac{q \partial N(E_0, E)}{\partial E}. \quad (3.6')$$

where $N(E_0, E) = \int_E^{\infty} P(E_0, E) dE$, $q = 2.29$. It was shown that when $E/\beta \gg 1$ and the last term in (3.6) can be ignored the solution of (3.6) is practically the same as the solution of (3.6) with $L_1(P)$ instead of $L(P)$. However, the replacement of L_1 by L only influences bremsstrahlung and pair production. So long as the equation accurately accounts for the ionization losses, one can hope that the solution of (3.6) with L_1 and the inclusion of ionization losses will also lead to good results. It is not difficult to find a solution for the approximate equation (3.6):

$$N(E_0, E) = \frac{E_0}{\beta} \left\{ 1 - \epsilon e^{\epsilon} (\text{Ei}(-\epsilon_0) - \text{Ei}(-\epsilon)) - \frac{\epsilon}{\epsilon_0} e^{\epsilon - \epsilon_0} \right\}, \quad (3.7)$$

here $\epsilon = qE/\beta$, β is the critical energy and $\text{Ei}(-\epsilon) = -\int_{\epsilon}^{\infty} x^{-1} e^{-x} dx$ is the exponential integral. As follows from reference 61, the approximation

(3. 6') is equivalent to assuming the following quantity to be independent of the energy;

$$q(E) = q = [N_p(E_0, E)]^{-1} \int_E^{\infty} \frac{\partial N_p(E_0, E')}{\partial E'} f\left(\frac{E}{E'}\right) dE',$$

where the function $f(x)$ depends weakly on the argument x :

$$f(x) = -1.36 + 0.327x + 1.21x^2 + 0.68x^3 - 1.36 \ln \frac{1-x}{x}$$

$$-1.17x^2 \ln x - 1.76x \ln x.$$

Numerical calculations show that a change of E from $E/\beta \gg 1$ to $E/\beta \approx 0.1$, causes $q(E)$ to change by only 2.5%. In reference 3 Eq. (3. 6) was solved by the method of adjunct equations. The approximate solution obtained coincided with (3. 7). This is evidently due to the fact that the two different solution methods are based on one premise, namely that $N(E) \sim 1/E$ in the very high-energy regions, $E/\beta \gg 1$.

The mathematical approximations made in the solution of (3. 6) were estimated by the method of consecutive approximations developed in reference 3. It was shown there that with $\epsilon_0 = 1$ the correction to the Tamm-Belen'kiĭ solution does not exceed 4.5%, while with an increase in ϵ_0 the correction is even less. Consequently, within the framework of the initial assumptions, the approximate equilibrium spectrum of Tamm and Belen'kiĭ differs from the accurate solution of (3. 6) by less than 4.5%.

The error in the Tamm and Belen'kiĭ spectrum can be evaluated only by comparison with the results of calculations that are free of the following simplifying suppositions: (1) the use of asymptotic expressions for the bremsstrahlung radiation and pair production cross sections, which are strictly accurate only for very high energies; (2) the Compton effect is approximated; (3) high-energy recoil electrons resulting from collision between shower electrons and electrons of the medium are ignored. Integrating (3. 3) numerically, Rossi and Clapman evaluated $N(E_0, E)$, for a fixed value of E , from the primary energy E_0 . The calculations were done for air ($t = 43 \text{ g/cm}^2$, $\beta = 98 \text{ Mev}$) at $E = 10^7 \text{ ev}$. In the calculation they accounted for all the radiative processes occurring in the substance. Exact formulas were taken for the cross sections of elementary processes. Not included was pair production and radiation in the field of atomic electrons, and the influence of the density effect on ionization losses. The Tamm-Belen'kiĭ spectrum differs by not more than 4% from the spectrum calculated by Rossi and Clapman. Thus

we can consider that beginning with $E \gg 10^7 \text{ ev}$ in light substances the approximate equilibrium spectrum differs from the exact one by not more than 4%. We note that mathematically the exact solution of (3. 6) differs from the spectrum calculated by Rossi and Clapman by 8.5%. Consequently the approximate equilibrium spectrum is closer to the real cascade than the formally more accurate solution of the equations.

Richards and Nordheim⁶² worked out another numerical method for determining the electron equilibrium spectrum. Their method can be applied under the conditions $E_0/\beta \gg 1$ and $E \ll E_0$ and gives values of $N(E_0, E)$ for a given value of E_0 as a function of the energy of the secondary particles. They used accurate expressions for the probabilities of all the processes taking place in the substance with radiation. From the good agreement of the equilibrium spectrum with the numerical calculations one can draw the conclusion that the Compton effect is sufficiently well accounted for by the approximate equations (3. 3), and the processes of collision with production of high-energy electrons have little effect on the development of the shower.

An expression for the photon equilibrium spectrum was obtained in reference 2. The author started from the fact that expression (3. 7) represents the electron equilibrium spectrum sufficiently accurately. An equation was then written for the photon distribution function $\Gamma(E_0, E)$ with allowance for the Compton effect, bremsstrahlung and pair production. The cross section for Compton scattering was determined from the following approximate expression

$$W_k(E', E) dE = \frac{g dE}{EE'},$$

where g is a certain constant, equal in the case of air to 1.32 Mev. The equation for $\Gamma(E_0, E)$ is easily solved. As a result, the following expression was obtained for the photon equilibrium spectrum:

$$\Gamma(E_0, E) = \frac{N(E_0, E)}{E\sigma(E)} + \Gamma_2(E_0, E), \quad (3. 8)$$

where $\Gamma_2(E_0, E)$ is a complicated correction term. Analysis of the expression showed that up to energies of 0.07β in air the correction introduced in the differential spectrum by the second term does not exceed 18%. Inclusion of the second term changes the value of the integrated spectrum only several percent. In reference 63 the equilibrium spectrum of photons in air was calculated with a more accu-

rate expression for the Compton-effect cross section

$$W_k(E', E) dE = \frac{g dE}{E \cdot E'} \left[1 + \left(\frac{E}{E'} \right)^2 \right].$$

These calculations show that the basic conclusions obtained in reference 2 remain valid. The photon equilibrium spectra for carbon, aluminum, iron, copper, and lead were calculated in reference 64. The correction to expression (3.8), obtained by including an accurate account of the Compton effect can be considered small for energies $\sim 0.1\beta$ up to $Z = 30$. However, for lead the correction to the differential spectrum at $E = 0.5\beta$ reaches 70%, but here it is already impossible to speak of the suitability of the method of calculation used in reference 64, where the correction is considered small. Even in lead at $E > 2$ Mev, the expression for the integrated spectrum obtained from (3.8') differs insignificantly from the exact one. Thus, the validity of replacing the coefficient of photon absorption due to pair production by the total absorption coefficient is also confirmed by the fact that the expression

$$\Gamma(E_0, E) = \frac{N(E_0, E)}{E\sigma(E)} \quad (3.8')$$

describes the equilibrium spectrum of photons in light substances sufficiently accurately.

For heavy substances it is necessary to take into account the energy dependence of the total photon absorption coefficient and Rutherford scattering of shower electrons. Equations with a variable photon absorption coefficient were solved only for the case of equilibrium. After integrating (2.39) over t from 0 to ∞ , with allowance for boundary conditions (one primary electron of energy E_0), and eliminating $\Gamma_p(E_p, E, \vartheta)$ from the second equation we obtain:

$$\begin{aligned} & -\cos \vartheta \cdot \delta(E_0 - E) \delta(\vartheta) \\ & = L [P_p(E_0, E, \vartheta)] + \frac{E_0^2}{4E^2} \Delta_0 P_p(E_0, E, \vartheta) + \beta \frac{\partial P}{\partial E}, \end{aligned} \quad (3.9)$$

where $L [P_p(E_0, E, \vartheta)]$ is the integral operator associated with pair production and bremsstrahlung. A solution of (3.9) was approximated in reference 65 by dropping the last term (as in the solution of (3.6), the operator L was changed to L_1):

$$\begin{aligned} N_p(E_0, E, \vartheta) &= \sum_{n=0}^{\infty} f_n(\varepsilon_0, \varepsilon) P_n(\cos \vartheta), \\ f_n(\varepsilon_0, \varepsilon) &= \begin{cases} \frac{2n+1}{4\pi q} \frac{\varepsilon_0^n}{V \varepsilon_0^2 + a_n^2} \frac{1}{V \varepsilon^2 + a_n^2} & \text{for } \varepsilon < \varepsilon_0, \\ 0 & \text{for } \varepsilon > \varepsilon_0, \end{cases} \end{aligned}$$

where $a_n = \frac{1}{2} \varepsilon_k \sqrt{n(n+1)}/q$, $E_k = 21$ Mev, and $P_n(\cos \vartheta)$ are the Legendre polynomials. In reference 57 an approximate solution of (3.9) was found, which includes scattering and ionization losses:

$$\begin{aligned} N_p(E_0, E, \vartheta) &= \sum_{n=0}^{\infty} f_n(\varepsilon) P_n(\cos \vartheta), \\ f_n &= \frac{2n+1}{4\pi q} \varepsilon_0 \frac{1}{V \varepsilon^2 + a_n^2} \left[1 - \frac{1}{1+a_n} \varphi(y) \right], \end{aligned} \quad (3.10')$$

where

$$\varphi(y) = e^{-y} (y+1) \sqrt{y^2+1}; \quad y = \frac{\varepsilon}{a_n}.$$

Using the adjunct-equation method⁵ an accurate solution was obtained for (3.9) in the form of series of Legendre polynomials:

$$P_n(E_0, E, \vartheta) = \sum_{n=0}^{\infty} \varphi_n(E_0, E) P_n(\cos \vartheta).$$

Substituting in (3.9), we obtain the following equation for the function $\varphi_n(E_0, E)$:

$$\left\{ L + \beta \frac{\partial}{\partial E} \right\} \varphi_n(E_0, E) = -\frac{2n+1}{4\pi} \delta(E_0 - E),$$

where L is a certain operator connected with bremsstrahlung and pair production and acting on the variable E . We define an operator L^* such that

$$\int_{E_1}^{\infty} u_n(E, E_1) L \varphi_n(E_0, E) dE = \int_{E_1}^{\infty} \varphi_n(E_0, E) L^* u_n(E, E_1) dE,$$

where $u_n(E, E_1)$ is an arbitrary function satisfying the condition

$$u_n(E, E_1) = 0 \text{ for } E < E_1.$$

From the equation for the function $\varphi_n(E_0, E)$ it is possible to obtain the following relation:

$$\begin{aligned} \int_0^{\infty} \varphi_n(E_0, E) \left[L^* - \beta \frac{\partial}{\partial E} \right] u_n(E, E_1) dE \\ = -\frac{2n+1}{4\pi} u_n(E_0, E_1). \end{aligned} \quad (*)$$

If function $u_n(E, E_1)$ is given, then the foregoing relation can be considered as an integral equation for $\varphi_n(E_0, E)$. The arbitrariness of the choice of u_n can be used to make equation (*) easier to solve than the original equation for φ_n . Suppose then $u(E, E)$ satisfies the equation

$$\left[L^* - \beta \frac{\partial}{\partial E} \right] u_n(E, E_1) = -\delta(E - E_1) \frac{2n+1}{4\pi}.$$

The last equation is said to adjunct to the equation for the function φ_n . The operators in this equation act on the variable E_0 (in the notation of the original equation for φ_n) and not on E . From equation (*) it follows that $u_n(E_0, E_1) = \varphi_n(E_0, E_1)$. The solution of the adjunct equation and consequent-

ly the equation for $\phi_n(E_0, E)$ can be found in the form

$$= \frac{2n+1}{4\pi} \frac{\epsilon_0}{q} \epsilon e^{V\sqrt{\epsilon^2+a_n^2}} \left(\frac{\epsilon}{a_n+V\sqrt{\epsilon^2+a_n^2}} \right)^{a_n} \int_{\epsilon}^{\epsilon_0} \frac{e^{-V\sqrt{x^2+a_n^2}}}{x^2} \left(\frac{a_n+V\sqrt{x^2+a_n^2}}{x} \right)^{a_n} dx. \tag{3.10'}$$

With $a_n = 0$, (3.10') coincides with the expression for the equilibrium spectrum without scattering. We note that the solution (3.10') is still an approximation, since the initial equations were approximations. However, an expression for the equilibrium spectrum was obtained in reference 26 from these equations, except without the scattering, which did not differ from the exact one by more than 4%. Multiple scattering in (3.9) is allowed for accurately so that (3.10) evidently also differs from the exact expression by not more than 4%. This error, due to the approximate nature of the initial equations of Tamm and Belen'kiĭ, has been evaluated several times.^{2, 3, 6} On the other hand, the agreement between the values of the function $N_p(E, \vartheta)$ given by (3.10) and (3.10') is quite satisfactory, as they coincide within 6%. For $\epsilon \ll a_n$, (3.10') becomes

$$f_0^p(E_0, E) = \frac{2n+1}{4\pi} \frac{\epsilon_0}{q} \frac{1}{a_n+1} \quad (n = 1, 2, 3, \dots),$$

whereas for $f_0^p(E_0, E)$ we obtain the following asymptotic expansion:

$$f_0^p(E_0, E) = \frac{1}{4\pi} \frac{E_0}{\beta} (1 + \epsilon \ln \epsilon + \epsilon c + \dots), \quad \epsilon \ll 1.$$

From the expressions for f_0^p and f_n^p at low energies it follows that any device that detects particles moving forward within a solid angle $\phi < 4\pi$ will detect less particles in an energy region $E < 2$ to 3 Mev than by the equilibrium spectrum. In reference 66 an experimental spectrum of charged particles in the maximum of the cascade was obtained and compared with the equilibrium spectrum. About up to 3 Mev the spectra coincided. The difference between the theoretical and experimental spectra can be explained by the fact that the authors have underestimated the considerable quantity of slow electrons going backwards. The reverse current in the maximum of the cascade in lead was about 0.41. Plates up to 1.5 t-units of lead were placed in the cloud chamber. About half of the low-energy particles making angles more than 30° from the axis of the shower could be detected by the authors. In reference 67 the electron energy spectrum was investigated in the region of the maximum of the cascade produced in lead by a spectrum of type $1/E$ of primary photons. The authors note good agreement with the equilibrium spectrum in the energy region from 3 to 180 Mev.

They explain the divergence in the 0 to 3 Mev region by the large back scattering.

Thus, this discussion permits us to conclude that the equilibrium spectrum of the particles is correct, with good accuracy up to about 4%, for light and heavy substances beginning with a particle energy of about 2 or 3 Mev. This makes it possible to calculate with the necessary accuracy the first several moments of the distribution function of cascade particles.

12. The Standard Method of Plotting Cascade Curves by Moments.

We recall that for high-energy showers ($E_0/\beta \gg 1$) knowledge of only the first two moments of the cascade curves makes it possible to estimate the position and number of particles in the maximum of the shower. For a low-energy shower, when E_0/β is of the order of unity, similar relations cannot be obtained since the method used in the derivation of the relations t_{\max} and $N_{\max} \sim f(E_0/\beta, t, t^2)$ is equivalent to the saddle point method usually used in the cascade theory. One can pose the problem of calculation of the cascade curve from its known moments. Knowing all the moments \bar{t}^n it is possible, in principle, to find also the distribution function itself (the Stieltjes moment problem⁶⁸). If the function $N(t)$ decreases at very high t faster than $e^{-k\sqrt{x}}$, where k is a positive number, then the following relation is correct:⁶⁸

$$YN(Y^2) = \frac{1}{\pi} \lim_{\lambda \rightarrow \infty} \int_0^\lambda \left(1 - \frac{s}{\lambda}\right) \varphi(s) \cos(sY) ds,$$

where

$$\varphi(s) = \sum_{n=0}^{\infty} \frac{(-1)^n E_0 \bar{t}^n s^{2n}}{\beta (2n)!}.$$

These relations make it possible to calculate the function $N(t)$ by its moments. For example, let us examine several extreme cases.⁵⁵ Let the energy of the primary particle be $\epsilon_0 \gg 1$, with ϵ_0 so great that $\ln \epsilon_0 \gg 1$. In this case it is easy to obtain the following expressions for the first moments:

$$\{\bar{t}_r(\epsilon_0, 0)\}^p = \frac{1 + \frac{1}{\sigma_0}}{q} \ln \epsilon_0; \quad \{\bar{t}_p(\epsilon_0, 0)\}^p = \frac{1 + \frac{1}{\sigma_0}}{q} \ln \epsilon_0$$

and for the n-th moment:

$$\{\bar{t}^n(\epsilon_0, 0)\}^p = \left(1 + \frac{1}{\sigma_0}\right)^n \frac{\ln^n \epsilon_0}{q^n}.$$

Thus, in this approximation, $\bar{t}^n = \bar{t}^n$. The function $\phi(s)$ is then equal to

$$\varphi(s) = \frac{E_0}{\beta} \cos(s\bar{t}^{\frac{1}{2}}).$$

This value of $\phi(s)$ corresponds to the following distribution function $N(t)$:

$$N(t) = \frac{E_0}{\beta} \delta(t - \bar{t}).$$

Thus in the extreme case of very high energies the distribution function approaches a δ -function. We note that in reference 60 the distribution function was approximated by the Gaussian function

$$N(t) = \frac{E_0}{\beta} \frac{0.3}{\sqrt{\ln \frac{E_0}{\beta}}} \exp \left\{ -\frac{1}{3.2} \frac{(t - \bar{t})^2}{\ln \frac{E_0}{\beta}} \right\}.$$

It is not difficult to see that as $\ln E_0/\beta$ approaches infinity the foregoing expression becomes the δ -function. Let the energy of the primary particle be $\epsilon_0 \ll 1$. Suppose that at such low energies photon absorption is negligible. Then it is easy to obtain the following expression for the n -th moment. $\bar{t}^n = (E_0/\beta)^n (n+1) \dots$. In this case the function $\phi(s)$ is equal to

$$\begin{aligned} \varphi(s) &= \frac{E_0}{\beta} \sum_{n=0}^{\infty} \frac{(-1)^n \left[\left(\frac{E_0}{\beta} \right)^{1/2} s \right]^{2n}}{(n+1)n!} \\ &= \frac{2}{s^2} \left\{ \left(\frac{E_0}{\beta} \right)^{1/2} s \cdot \sin \left[\left(\frac{E_0}{\beta} \right)^{1/2} \cdot s \right] \right. \\ &\quad \left. + \cos \left[\left(\frac{E_0}{\beta} \right)^{1/2} \cdot s \right] - 1 \right\}, \end{aligned}$$

and the distribution function $N(t)$ is equal to

$$N(t) = \begin{cases} 1, & \text{if } t < \frac{E_0}{\beta}, \\ 0, & \text{if } t > \frac{E_0}{\beta}. \end{cases}$$

This distribution corresponds to the primary charged particle losing energy only by ionization in passing through a substance. Suppose that $\epsilon_0 \lesssim 1$, but that photon absorption plays an important role. This case corresponds to the passage of electrons of energy of the order of the critical energy through heavy substances. It can be shown that in this case, for $n \gg 1$,

$$\bar{t}^n = \frac{n!}{q} \int_0^{\epsilon_0} \frac{\chi_1(\epsilon_0, \epsilon)}{\sigma_{\text{ion}}(\epsilon)} d\epsilon, \text{ where } \chi_1(\epsilon_0, \epsilon) = \epsilon e^{\epsilon} \int_{\epsilon}^{\epsilon_0} \frac{e^{-x}}{x^2} dx.$$

This signifies that for $t > \bar{t}$ the distribution function looks like

$$N(t) = \frac{E_0}{\beta} \int_0^{\epsilon_0} e^{-\sigma_{\text{min}} t} \chi_1(\epsilon_0, \epsilon) \sigma(\epsilon) d\epsilon.$$

However, in all practically important cases it is possible to calculate only the first several moments. In practice, therefore, it is more expedient to approximate the function $N(t)$ by some other function whose first several moments coincide with the accurate moments of the function $N(t)$. This means that one must select such approximation formulas which make it possible to describe sufficiently accurately the desired distribution functions by means of two or three first moments. In reference 2 the cascade curves for primary particles of very high energies are approximated by an expression of the type

$$\exp \left\{ + at^{\frac{1}{2}} - \gamma t \right\}.$$

The coefficients a and γ were chosen to make this approximation give the correct value for the area under the cascade curve (conservation of energy) and under the first moment. The cascade curves for primary particles of lower energies are approximated by an expression of the type

$$\exp \left\{ at^{\frac{1}{2}} - \gamma t \right\} + \chi \exp(-\gamma t).$$

the coefficients a , γ , and χ were selected to obtain the correct values for the areas of the first two moments of the cascade curves. In reference 69 there were obtained cascade curves in lead, including the dependence of $\sigma(E)$ on energy and with an approximate allowance for scattering. The curves for primary electrons were approximated by an expression of the type $(1 + \beta) \exp \{ at^{\frac{1}{2}} - \gamma t \} - \beta \exp(-2\gamma t)$, analogous to the one proposed in reference 2. However, this approximation unsatisfactorily describes the cascade process at primary-particle energies $E_0 < 10^8$ ev and at depths $t < t_{\text{max}}$ of the cascade. At great depths this expression behaves like e^{-kt} , where the coefficient k about doubles with a change in E_0 from 10^8 to 10^{11} ev, while the exact curves behave at greater depths like $e^{-\sigma_{\text{min}} t}$, where σ_{min} is the minimum value of the total photon absorption coefficient. In order to determine the coefficients entering into the approximation formula it is necessary to solve a system of three transcendental equations. This complicates the calculation considerably. We also note that to plot a curve from three moments it is necessary to select another type of approximation relation, which cannot be determined without a detailed investigation. To determine the coefficients in this case it is necessary to solve a system of four transcendental equations, which in itself is a complex computational problem.

It is possible to propose the standard method of drawing cascade curves, using a set of polynomials orthogonal in the interval $(0, \infty)$.⁷⁰ It is known that in light and heavy substances the range for photon

absorption remains of the order of one cascade unit up to energies on the order of 10^6 ev. The range of electrons at energy E less than the critical energy is equal in order of magnitude to E/β .

Therefore it is natural to consider that the cascade curve for large values of t will behave like $e^{-\sigma_{\min}t}$, where σ_{\min} is the minimum value of the total photon absorption coefficient. In addition, we know the values of cascade curves with $t = 0$, given by the boundary conditions; we know from the basic equations (1.1) the values of the derivatives of the cascade curves with respect to the depth at $t = 0$; and we know the numerical values of the first several moments of the cascade curves. Using this information on the behavior of cascade curves we construct a weighting function $w(x)$ [with which the set of polynomials is orthogonal in the interval $(0, \infty)$] representing approximately the basic characteristics of the sought cascade curves. With the chosen weighting function we construct by the usual methods of the theory of orthogonal polynomials the necessary set of orthogonal polynomials.

We approximate the function $\phi(x)$ with the aid of the sum of polynomials:

$$\varphi(x) = A_0 + A_1 L_1^i(x) + A_2 L_2^i(x) + A_3 L_3^i(x) + \dots$$

The polynomials $L_n(x)$ are orthogonal in the interval $(0, \infty)$ with weight $w(x)$. The approximation will be better the smaller the mean-squared error

$$M = \int_0^\infty w(x) (A_0 + A_1 L_1^i(x) + A_2 L_2^i(x) + \dots - \varphi(x))^2 dx.$$

The coefficients A_n , determined from the condition of minimum mean-squared error $\frac{\partial M}{\partial A_n} = 0$, are

$$A_n = \frac{1}{N_n} \int_0^\infty w(x) \varphi(x) L_n^i(x) dx,$$

here

$$N_n = \int_0^\infty w(x) L_n^i(x) dx.$$

In the expansion of $\phi(x)$ we replace the variable x by γt (where γ is any positive number) and multiply both sides of the equation by $w(\gamma t)$. Introducing the notation $w(\gamma t) \phi(\gamma t, E_0, E) = N(E_0, E, t)$ we write

$$N(E_0, E, t) = \sum_{n=0}^\infty A_n(E_0, E) L_n^i(\gamma t). \quad (3.11)$$

The coefficients A_n are equal in our case* to

$$A_n(E_0, E) = \frac{1}{N_n} \int_0^N N(E_0, E, t) L_n^i(\gamma t) dt^*. \quad (3.12)$$

*The coefficients A_n can be determined from the condition of orthogonality of the polynomials $L_n^i(x)$. Multiplying (3.11) by $L_k^i(x)$ and integrating over x from 0 to ∞ we find for $A_n(E_0, E)$ expressions that coincide with (3.12).

Let $w(x) \gg 0$ and let the moments of the function $w(x)$ of any order exist:

$$a_n = \int_0^\infty x^n w(x) dx, a_0 = 1.$$

Then, following the usual methods of the theory of orthogonal polynomials,⁷¹ we determine the polynomials $L_n^i(x)$ as follows:

$$L_n^i(x) = \frac{1}{D_{n-1}} \begin{vmatrix} a_0 & a_1 & \dots & a_n \\ a_1 & a_2 & \dots & a_{n+1} \\ \dots & \dots & \dots & \dots \\ a_{n-1} & a_n & \dots & a_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix}, \quad \text{here } D_n = \begin{vmatrix} a_0 & a_1 & \dots & a_n \\ a_1 & a_2 & \dots & a_{n+1} \\ \dots & \dots & \dots & \dots \\ a_n & a_{n+1} & \dots & a_{2n} \end{vmatrix}. \quad (3.13)$$

We write the condition of orthogonality of the polynomials in the form

$$\int_0^\infty L_n^i(x) L_k^i(x) w(x) dx = \begin{cases} 0, & n \neq k, \\ \frac{D_n}{D_{n-1}}, & n = k. \end{cases} \quad (3.14)$$

Expression (3.11), with the function $w(\gamma t)$ suitably defined, gives a convenient approximation of cascade curves, and the coefficients $A_n(E_0, E)$ are simply connected with the moments of the sought distribution functions.

13. Cascade Curves and Energy Spectra in Light Substances

To compute the cascade curves and energy spectra of particles in light substances from (3.11) and (3.12), it is necessary to calculate the first several moments of the function $N(E_0, E, t)$. In reference 37 explicit analytical expressions were obtained for the first two moments of the cascade curves.

Table V

z_0	$\left\{ \frac{1}{p} \right\}^p$	$\left\{ \frac{1}{p} \right\}^1$	$\left\{ \frac{12}{p} \right\}^p$	$\left\{ \frac{12}{p} \right\}^1$	$\left\{ \frac{73}{p} \right\}^p$	$\left\{ \frac{73}{p} \right\}^1$
0.2	0.096	1.357	0.148	3.694		
0.4	0.188	1.421	0.301	3.899		
0.6	0.273	1.480	0.459	4.119		
0.8	0.353	1.536	0.619	4.375		
1.0	0.429	1.590	0.780	4.609	2.240	19.79
1.2	0.500	1.641	0.943	4.850		
1.4	0.568	1.691	1.107	5.083		
1.6	0.633	1.740	1.271	5.313		
1.8	0.694	1.785	1.434	5.540		
2.0	0.752	1.829	1.599	5.762	6.364	26.51
3.0	1.006	2.027	2.392	6.803	9.817	32.78
4.0	1.213	2.195	3.132	7.746	13.37	38.79
5.0	1.386	2.340	3.845	8.625	16.84	44.52
6.0	1.536	2.468	4.503	9.429	20.29	50.02
7.0	1.666	2.582	5.192	10.26	29.53	61.29
8.0	1.782	2.685	5.691	10.87	32.73	65.86
9.0	1.886	2.778	6.233	11.53	42.83	77.59
10.0	1.979	2.862	7.140	12.53	48.16	86.07

Table V lists the values of the first three moments of the electron cascade for values of the primary energy ϵ_0 from 0.2 to 10. The third moment was calculated with the help of recurrence formulas by numerical integration. For $E_0/\beta \gg 1$, the formulas for t and t^2 can be rewritten accurate up to terms of order $1/\epsilon_0$ as follows:

$$\begin{aligned} \{\bar{t}_p(E_0, 0)\}^p &= \ln \frac{E_0}{\beta} + 0,41; \{\bar{t}_p^2(E_0, 0)\}^p \\ &- \{\bar{t}_p^2(E_0, 0)\}^p = 1.76 \ln \frac{E_0}{\beta} - 0,21, \\ \{\bar{t}_p(E_0, 0)\}^r &= \ln \frac{E_0}{\beta} + 1,2; \{\bar{t}_p^2(E_0, 0)\}^r \\ &- \{\bar{t}_p^2(E_0, 0)\}^r = 1.76 \ln \frac{E_0}{\beta} + 2,32. \end{aligned}$$

Table VI gives the values of these moments for ϵ_0 from 15 to 2290 and also the values of the moments calculated by formulas from reference 3.

It is clear from the table that the first and second moments differ by no more than 1% over a wide range of energies. Thus the expressions for moments obtained in reference 37 in the high-energy region differ insignificantly from the moments obtained starting from the solution to the basic equations of the theory by the Snyder method. We recall that the approximations we made in calculating the moments relate to the processes that play a major role precisely at high energies.

The values of the moments obtained were used to plot the cascade curves. We approximate cas-

Table VI

ϵ_0	According to reference 3	According to reference 2	According to reference 3	According to reference 2	According to reference 3	According to reference 2	According to reference 3	According to reference 2
	$\{\bar{t}_p\}^p$	$\{\bar{t}_p\}^p$	$\{\bar{t}_p\}^r$	$\{\bar{t}_p\}^r$	$\{\bar{t}_p^2\}^p$	$\{\bar{t}_p^2\}^p$	$\{\bar{t}_p^2\}^r$	$\{\bar{t}_p^2\}^r$
15	2.290	2.298	3.080	3.098	8.343	8.108	13.40	13.52
20	2.577	2.598	3.367	3.389	10.24	9.992	15.74	15.87
25	2.800	2.814	3.590	3.614	11.84	11.56	17.66	17.81
30	2.983	2.999	3.773	3.799	13.22	12.94	19.32	19.47
40	3.270	3.289	4.060	4.089	15.52	15.22	22.06	22.22
63.6	3.734	3.757	4.524	4.557	19.58	19.27	26.84	27.02
229	5.015	5.051	5.805	5.851	33.04	32.73	42.64	42.55
2290	7.318	7.377	8.408	8.177	65.5	65.34	78.78	78.78

cade curves for primary photons by a sum of Laguerre polynomials $L_n^1(x)$:

$$\{N_p(E_0, t, 0)\}^r = \gamma t e^{-\gamma t} \sum_{n=0}^h A_n(E_0, 0) L_n^1(\gamma t). \quad (3.15)$$

The coefficients A_n are simply connected with the moments of the sought distribution function

$$\begin{aligned} A_0(E_0, 0) &= \frac{\gamma E_0}{\beta}; \quad A_1(E_0, 0) = \frac{\gamma E_0 (2 - \gamma \{\bar{t}_p(E_0, 0)\}^r)}{2\beta}; \\ A_2(E_0, 0) &= \frac{\gamma E_0 (3 - 3\gamma \{\bar{t}_p(E_0, 0)\}^r + \gamma^2 \{\bar{t}_p^2(E_0, 0)\}^r)}{3\beta} \quad \text{etc.} \end{aligned} \quad (3.16)$$

We assume the coefficient γ to be equal to the minimum value of the total photon absorption coefficient. The cascade curves for primary photons were approximated also by sums of Laguerre polynomials $L_n^0(x)$

$$\{N_p(E_0, t, 0)\}^r = e^{-\gamma t} \left\{ \sum_{n=0}^h A_n L_n^0(\gamma t) + c_1 L_{h+1}^0(\gamma t) \right\}. \quad (3.17)$$

We define the constant c by requiring that the value of $N_p(E_0, t, 0)$ in the approximate curves (3.17) be equal to its exact value $t = 0$ in references 2 and 37 it is shown that the approximation (3.17) of the cascade curves is completely equivalent to approximation (3.16). Since we wish to obtain cascade curves for primary particles of energy of the order of the critical energy, where the number of particles in the maximum of the cascade is of the order of unity and the maximum lies at the depth of the order of one cascade unit, it is very important that the approximation curve accurately satisfy boundary conditions, that is, that the approximation curve be as accurate as possible for the first t -units of the absorber. Cascade curves for a primary electron were approximated with the help of sum (3.17). It was shown that in the energy region $E_0/\phi \gg 1$ the approximate cascade curves do not differ from the exact curves by more than 5%. The

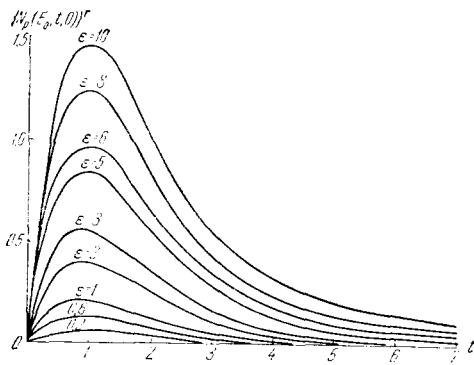


FIG. 10. Dependence on depth of the total number of electrons $\{N_p(E_0, t, 0)\}^r$ in a shower caused by a primary photon of energy ϵ_0 from 0.2 to 10.

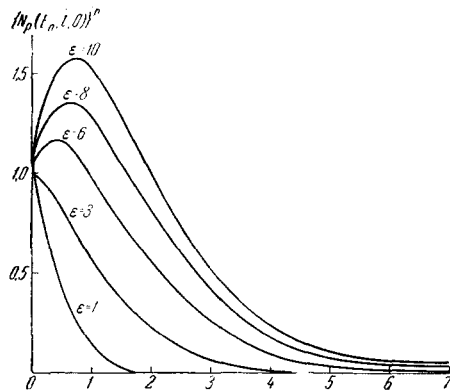


FIG. 11. Same as FIG. 10, but in a shower from a primary electron.

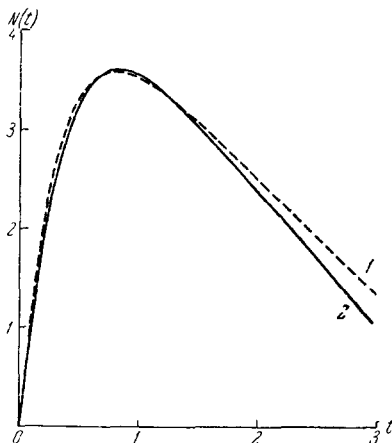


FIG. 12. 1--experimental cascade curve in carbon from the bremsstrahlung spectrum of primary photons with a maximum energy 330 Mev;⁵⁸ 2--obtained by averaging over the spectrum of primary photons the cascade curves calculated by moments.⁵⁷ Curves (1) and (2) were normalized to $t = 0.1$.

cascade curves calculated according to (3.17), using the first two moments, differ by not more than 10--15% from the curves obtained by using the first three moments, while they differ considerably from the curves plotted from only the first moment. This indicates that series (3.17) converges rapidly. In addition, the accuracy of the curves obtained at very low values of the primary-particle energy was especially investigated. Considering all this it is possible to confirm that only the first three terms of the series (the first two moments are used) approximate the accurate solution with an error of the order of 10%. Figure 10 shows the distribution functions $\{N_p(E_0, t, 0)\}^r$ of electrons with energy greater than zero in showers caused by a primary photons of energies ϵ_0 from 0.2 to 10. Figure 11 shows the distribution functions $\{N_p(E_0, t, 0)\}^r$ of electrons with energy greater than zero in a shower caused by primary electrons with energies ϵ_0 from 1 to 10. We note that cascade curves for a primary electrons or photons of energies $\epsilon_0 \gg 1$ do not differ greatly from one another, while at $\epsilon_0 \sim 1$ the difference between the curves is great. This is not difficult to understand: high-energy electrons and photons expend a greater part of their energy on the production of secondary particles capable of continuing the shower. Low-energy electrons expend most of their energy irreversibly on ionization.

An experimental cascade curve in carbon was obtained in reference 58 from a type $1/E$ spectrum of primary photons with $E < E_{max}$. By averaging curves (3.17) over the $1/E$ spectrum it is possible to compute the corresponding theoretical curve. The results of the calculations are shown in Fig. 12. The curves coincide within 5% and a certain difference in the tail can be explained by the fact that in the calculation the photon absorption coefficient was considered to be equal to the asymptotic coefficient, while in the energy region under examination it is 15% less, which leads to a higher penetrating ability for the cascade.

By the foregoing method it is possible to obtain not only the dependence of the total number of particles on the depth, but also the depth dependence of the number of particles with energy more than a certain E , that is, to obtain energy spectra of particles at any depth. In reference 72 recurrence formulas were obtained for the moments $\{t_r^n(E_0, E_1)\}^p$ and $\{t_r^n\}^1$ of the distribution function of photons of energy greater than E in a shower caused by a primary electron or photon of energy E_0 . The results of calculating t and t^2 for

Table VII.

		$\{\bar{i}_r(E_0, E)\}^P$				$\{\bar{i}_r(E_0, E)\}^F$			
ϵ_0	ϵ	0.11	0.2	0.6	1	0.11	0.2	0.6	1
0.6		1.386	1.361	1.294		1.451	1.358	1.294	—
1		1.480	1.445	1.410	1.294	1.664	1.514	1.312	1.294
5		2.207	2.092	1.942	1.847	2.816	2.624	2.216	1.946
11		2.806	2.705	2.467	2.318	3.496	3.347	2.958	2.694
20		3.323	3.212	2.939	2.774	4.047	3.905	3.536	3.293
30		3.692	3.576	3.287	3.109	4.432	4.293	3.933	3.698
50		4.171	4.050	3.753	3.557	4.927	4.791	4.444	4.210

		$\{\bar{i}_r^2(E_0, E)\}^P$				$\{\bar{i}_r^2(E_0, E)\}^F$			
ϵ_0	ϵ	0.11	0.2	0.6	1	0.11	0.2	0.6	1
0.6		3.95	3.82	3.35		4.22	3.70	3.35	—
1		4.22	4.05	3.67	3.35	5.36	4.54	3.44	3.35
5		7.63	7.03	6.03	5.41	12.18	10.99	8.44	6.79
11		11.66	10.92	9.24	8.23	17.40	16.24	13.37	11.53
20		15.74	14.96	12.70	11.38	22.40	21.11	18.01	16.01
30		19.05	17.92	15.51	14.02	26.29	24.91	21.58	19.46
50		23.63	22.48	19.66	17.89	31.74	30.28	26.35	24.35

values of ϵ_0 from 0.6 to 50 and ϵ_1 from 0.1 to 1.0 are given in Table VII:

The functions $\{N_r(E_1, t)\}^P$ and $\{N_r(E_1, t)\}^F$ were approximated by sums of the polynomials $L_n^1(x)$ and $L_n^0(x)$. The results of calculations for air are shown in Figs. 13 to 19. The values of the computed curves coincide within 10% with the

values calculated by the accurate formulas of the theory, with $\epsilon_0 \gg 1$. The energy spectrum in the maximum of the approximated curves coincide within 10% with the equilibrium spectrum. In reference 73 the same method was used as in reference 72 to calculate the energy spectra of electrons in showers caused by a primary electron or photon.

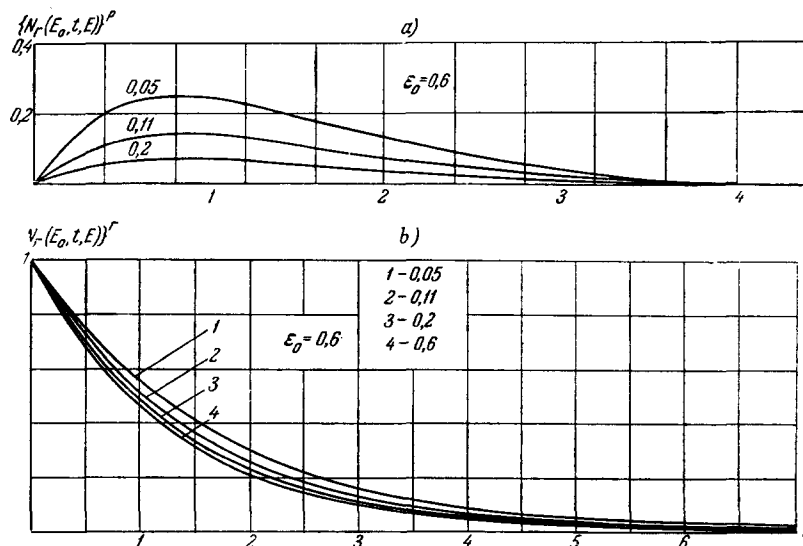


FIG. 13. Dependence on depth of the number of photons with energy greater than ϵ (from 0.05 to 1) $\{N_r(E_0, t, E)\}^{P,F}$ in a shower caused by a primary electron (a) or photon (b) with energy ϵ_0 from 9.6 to 50.⁷²

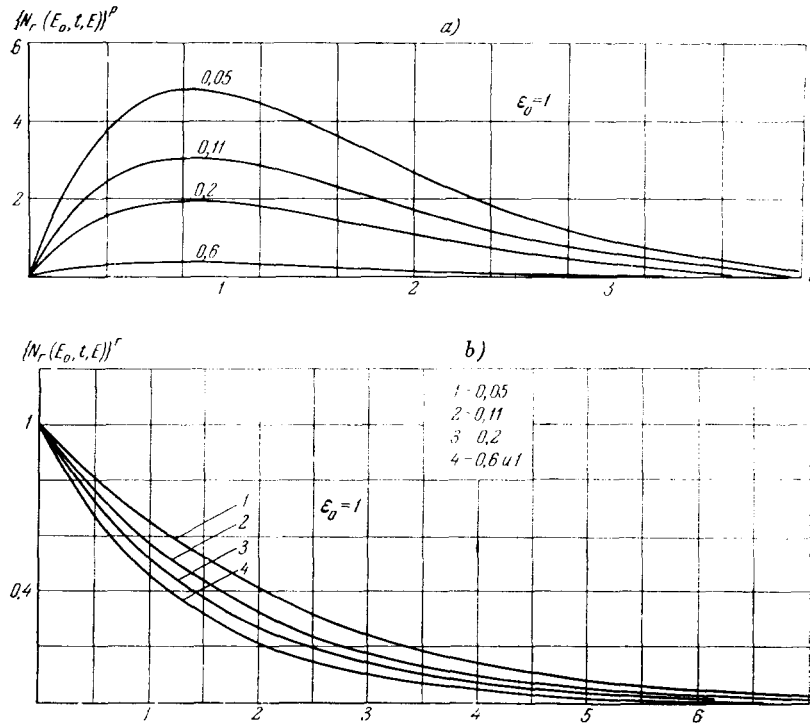


FIG. 14. Same as FIG. 13.

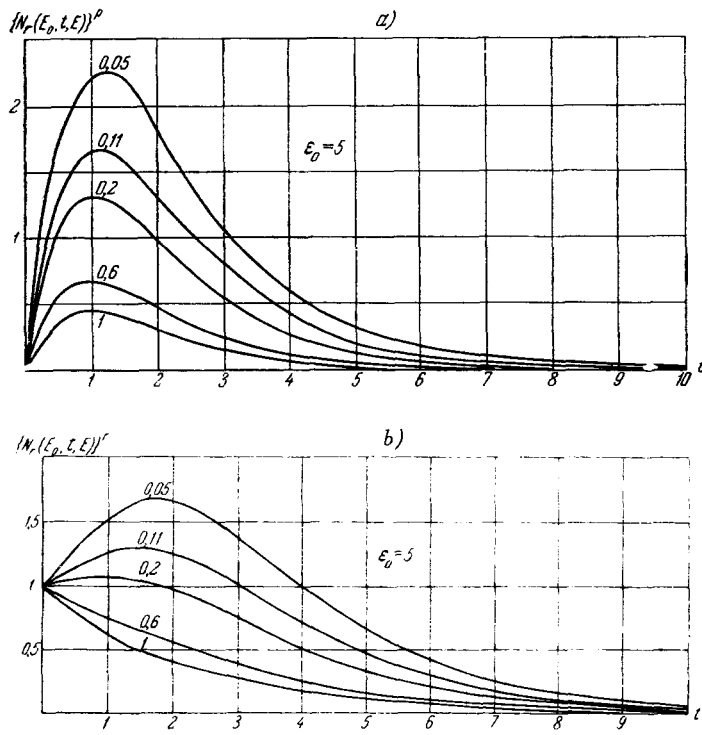


FIG. 15. Same as FIG. 13.

FIG. 16. Same as FIG. 13.

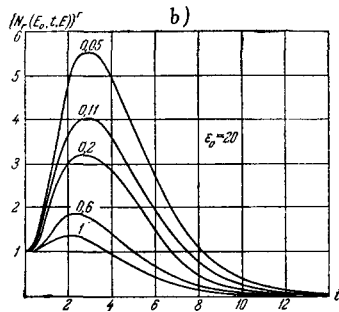
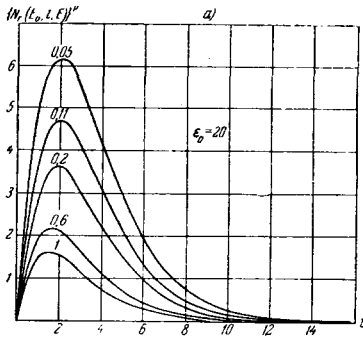
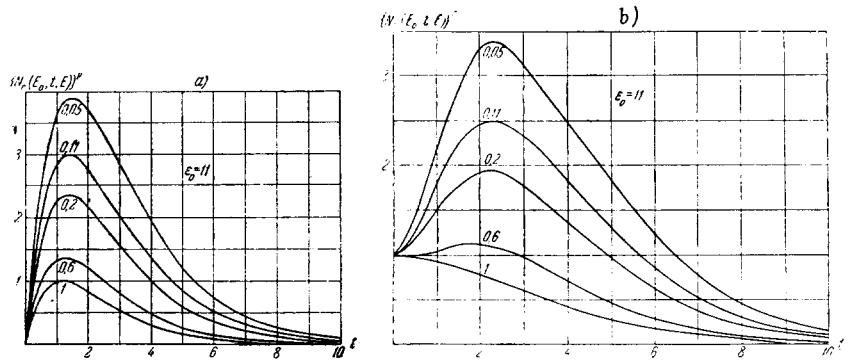


FIG. 17. Same as FIG. 13.

FIG. 18. Same as FIG. 13.

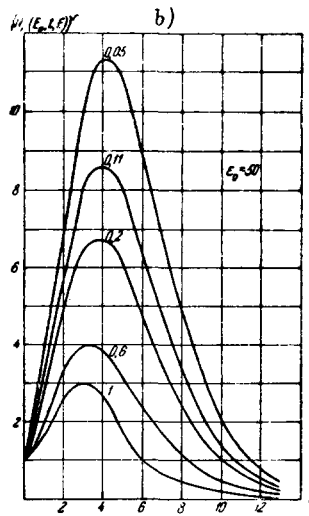
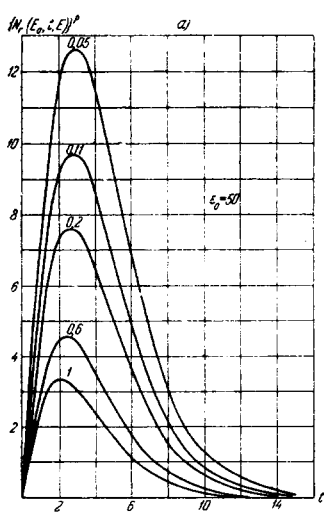
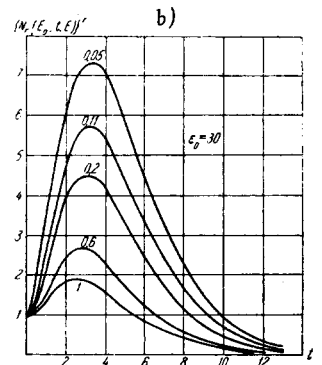
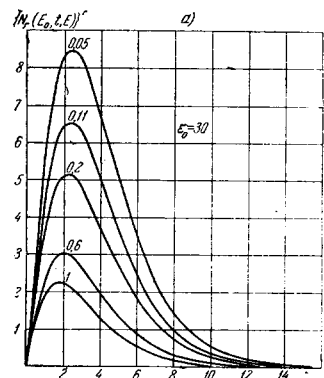


FIG. 19. Same as FIG. 13.

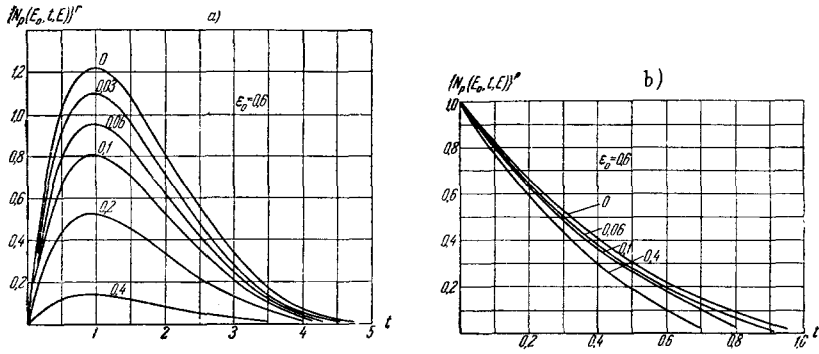


FIG. 20. Dependence on depth of the number of electrons with energy greater than ϵ (from 0.03 to 2) $\{N_p(\epsilon_0, t, E)\}^F$ in a shower caused by a primary photon (a) or electron (b) with energy ϵ_0 from 9.6 to 15.⁷³

FIG. 21. Same as FIG. 20.

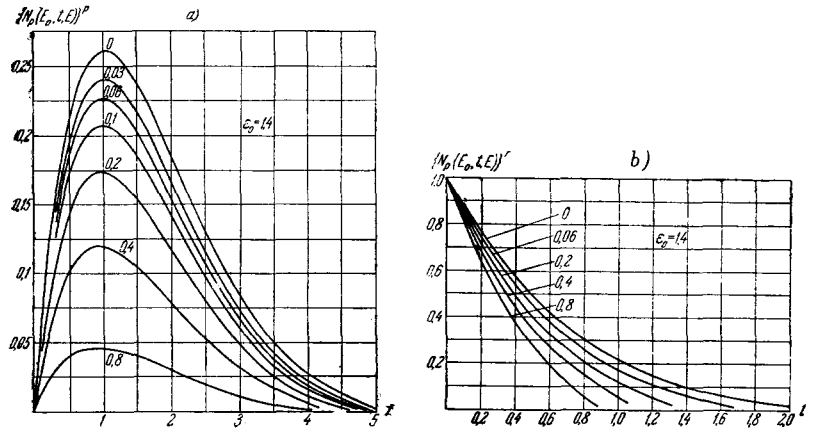


FIG. 22. Same as FIG. 20.

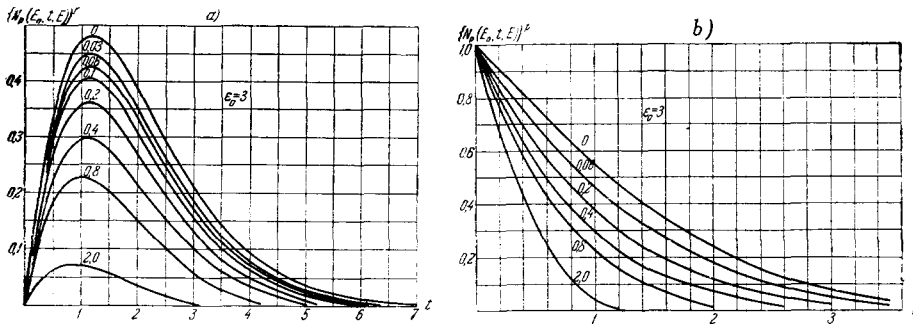
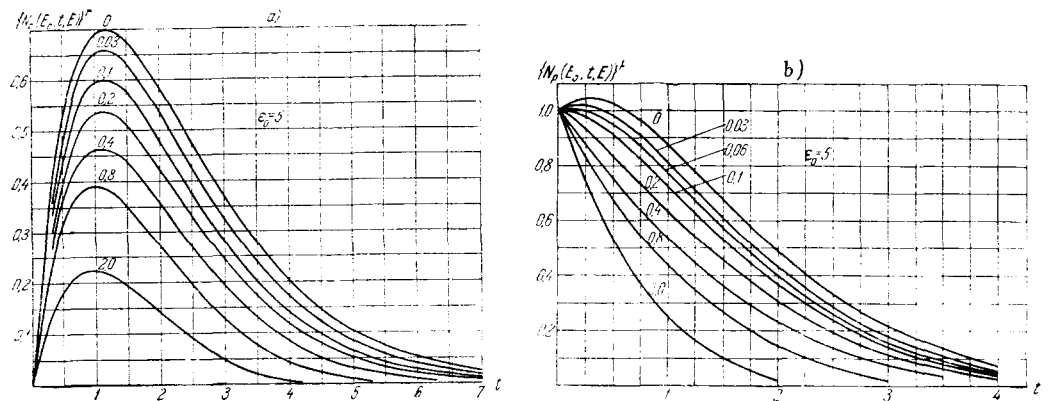


FIG. 23. Same as FIG. 20.



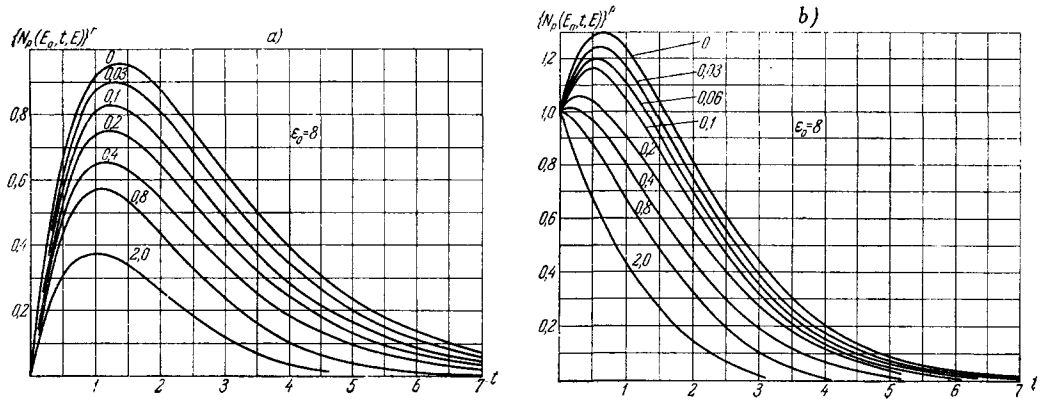


FIG. 24. Same as FIG. 20.

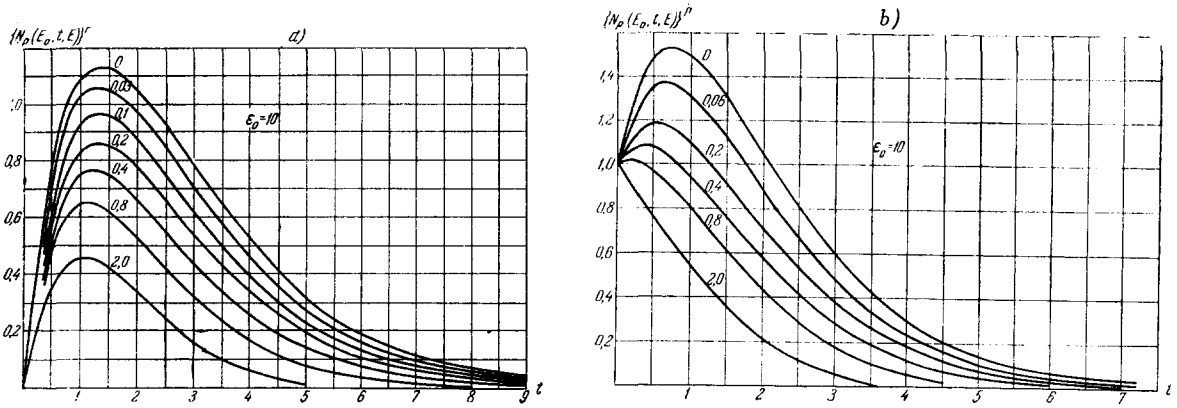


FIG. 25. Same as FIG. 20.

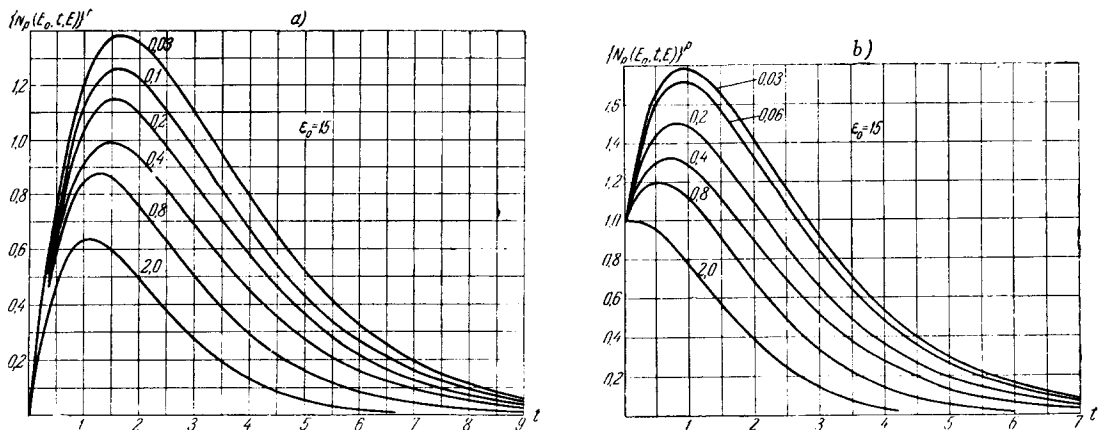


FIG. 26. Same as FIG. 20.

Table VIII

		$\{\bar{t}_p(E_0, E)\}^P$							$\{\bar{t}_p(E_0, E)\}^P$						
$\epsilon_0 \backslash \epsilon$		0.03	0.06	0.1	0.2	0.4	0.8	2.0	0.03	0.06	0.1	0.2	0.4	0.8	2.0
0.6		0.189	0.160	0.147	0.098	0.043			1.393	1.387	1.385	1.335	1.321		
1.4		0.493	0.435	0.408	0.311	0.237	0.126		1.62	1.57	1.56	1.50	1.44	1.38	
3.0		0.934	0.886	0.838	0.746	0.620	0.316	0.061	1.96	1.92	1.88	1.80	1.70	1.42	1.07
5.0		1.299	1.259	1.208	1.130	0.978	0.660	0.230	2.25	2.22	2.17	2.10	1.97	1.68	1.26
8.0		1.706	1.654	1.60	1.54	1.37	1.03	0.63	2.61	2.56	2.51	2.46	2.29	1.98	1.63
10		1.90	1.85	1.80	1.73	1.56	1.23	0.82	2.78	2.74	2.69	2.62	2.46	2.15	1.79
15		2.27	2.22	2.17	2.08	1.94	1.63	1.19	3.13	3.08	3.03	2.93	2.80	2.48	2.07
		$\{\bar{t}_p^2(E_0, E)\}^P$							$\{\bar{t}_p^2(E_0, E)\}^P$						
$\epsilon_0 \backslash \epsilon$		0.03	0.06	0.1	0.2	0.4	0.8	2.0	0.03	0.06	0.1	0.2	0.4	0.8	2.0
0.6		0.335	0.197	0.145	0.065	0.0084			3.83	3.69	3.55	3.49	3.42		
1.4		0.873	0.753	0.615	0.415	0.202	0.082		4.74	4.53	4.38	4.17	3.84	3.67	
3.0		2.09	1.90	1.73	1.41	0.997	0.479	0.030	6.40	5.04	5.85	5.52	4.94	3.94	2.68
5.0		3.55	3.35	3.11	2.72	2.10	1.28	0.31	8.13	7.90	7.65	7.15	6.33	5.05	3.42
8.0		5.16	5.00	4.81	4.33	3.63	2.52	1.12	10.4	10.0	9.77	9.19	8.22	6.61	4.76
10		6.65	6.05	5.15	5.28	4.52	2.74	1.72	11.9	11.4	11.0	10.3	9.15	7.56	5.52
15		8.47	8.27	7.90	7.30	6.45	4.98	3.05	14.2	14.0	13.5	12.7	11.6	9.7	7.1

The results of the calculation of the first two moments are given in Table VIII. Cascade curves $\{N_p(E_1, t)\}^{P, \nu}$ obtained by moments are shown in Figs. 20 to 26. The accuracy of these curves is of the order of 10%.

Thus, using the method of moments based on the equilibrium spectrum of Tamm and Belen'kii, it is possible to calculate cascade curves and energy spectra of particles in light substances in showers caused by primary electrons or photons in the range of energy ϵ_0 from 0.2 to ~ 100 .

14. Cascade Curves and Energy Spectra in Heavy Substances

We restate briefly the basic difficulties of the cascade theory of showers in heavy substances: (1) the total photon absorption coefficient depends strongly on energy; (2) scattering of shower particles in heavy substances is very great. Therefore in the theory it is not possible to obtain expressions for $N(E_0, E, t)$, since the method of functional transformations is not applicable, because its usefulness is connected with the application of asymptotic expressions for the cross sections of the basic processes.

In references 26, 55 and 57 a method was developed for solving the basic equations integrated over the depth, with variable σ and including scattering. This made it possible to obtain expressions for the equilibrium spectra of electrons and photons. Using the equilibrium spectrum, it is possible in principle to calculate all the moments of distribution functions of shower particles. We can

estimate the accuracy of the cascade theory for heavy substances by comparing the calculated moments with experiment. In reference 56 a recurrence formula was derived to calculate consecutively all the moments of distribution functions from an arbitrary spectrum of primary particles. The first two moments of the distribution function of electrons was also calculated in lead for a spectrum of primary photons of the type $1/E$. The moments obtained were compared with experiment. Since this work did not account for scattering, it calls for a revision.

In reference 37 there was derived a recurrence formula for consecutive calculation of all the moments of distribution functions, with allowance for scattering. The first two moments of the distribution functions of electrons in lead were calculated for a type $1/E$ spectrum of primary photons. The results of the calculations are shown in Table IX.

Table IX.

$\bar{t}_{theoret.}$	$\bar{t}_{exper.}$	$\bar{t}_{theoret.}^2$	$\bar{t}_{exper.}^2$
5.89 \pm 0.12	5.9 \pm 0.2	53.4 \pm 2	55 \pm 4

The average experimental and theoretical values of the first moment differ by 1%, and of the second moment by 4%. However, the difference is the limits of error of experiment and calculation. Without taking scattering into account, the magnitude of the first moment for $\sigma = \sigma_0$ is less than the

experimental value by about 30%, and for $\sigma = \sigma(E)$ it is larger by about 10%. Consequently the first two moments of the distribution function, calculated with the most accurate accounting of scattering and ionization losses possible, is in very good agreement with experiment.

We note that in light substances the method of moments is essential for calculating the cascade curves and energy spectra at primary particle energy $\epsilon_0 \sim 1$. The method of functional transformations works well for $\epsilon_0 \gg 1$. The situation is different in the theory for heavy substances, where it is necessary to calculate cascade curves and energy spectra in the widest range of primary-particle energies possible. Therefore in choosing a weighting function it is necessary to use more information on the behavior of the cascade curve. Earlier we selected a weighting function, with allowance for the behavior of curves at large t and for the boundary conditions at $t = 0$. Now we shall also use the magnitude of the first moment to construct the weighting function. Reference 51 gives calculations of the first two moments of the distribution function of electrons in lead in a shower caused by a primary electron or photon of energy E_0 . The results are given in Table X. Cascade curves for a primary photon were approximated with the help of a sum of polynomials orthogonal in the interval $(0, \infty)$ with the weight ⁷⁴

$$w(\gamma t) = \frac{\gamma^{i+1} t^i e^{-\gamma t}}{\Gamma(i+1)}.$$

Here γ is the minimum value of the total photon absorption coefficient, and i is determined from the condition that the first moment of the weighting function be equal to the first moment of the sought distribution function

$$i = \gamma \bar{t}_r(E_0, 0) - 1.$$

We note that for $\epsilon_0 \sim 1$ we obtain $i < 1$. In this case $\frac{dN}{dt} \Big|_{t=0} = \infty$, therefore it is more convenient to consider i equal to 1 and to define γ from the corresponding condition. Thus, with $\epsilon_0 \sim 1$, the method of approximation used here automatically turns into the method used earlier to calculate cascade curves in light substances. Cascade curves for primary electrons were approximated with the help of polynomials orthogonal in the interval $(0, \infty)$ with the weight

$$w(\gamma t) = \frac{\gamma^{i+1} (a+t)^i e^{-\gamma t}}{\Gamma(i+1) + a\gamma^i}.$$

Here the quantities i and a are determined from the condition that the first moment of the weighting function be equal to the value $\bar{T}(E_0, 0)$ of the sought for distribution function, and from the boundary condition $\{N_p(\epsilon_0, 0, 0)\}^P = 1$. For $\epsilon_0 \sim 1$ we obtain

$0 < i < 1$. To avoid an infinite derivative at $t = 0$, we consider that for the first several curves i is equal to the nearest integer and from this we determine the quantity a . We note that, with slight complication, we can calculate in this way cascade curves that are more accurate for small values of t . For this it is sufficient to introduce into the weighting function, in a suitable manner, one more parameter whose value can be determined from the boundary condition for the derivation of the cascade curve with respect to t . The electron cascade curves in lead calculated this way for a primary photon and

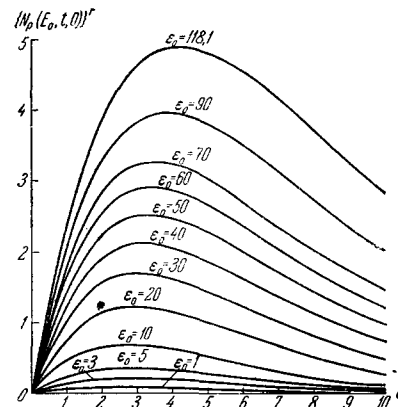


FIG. 27. Cascade curves $\{N_p(E_0, t, 0)\}^P$ in lead from primary photons of energy ϵ_0 from 1 to 118.1. ⁵¹

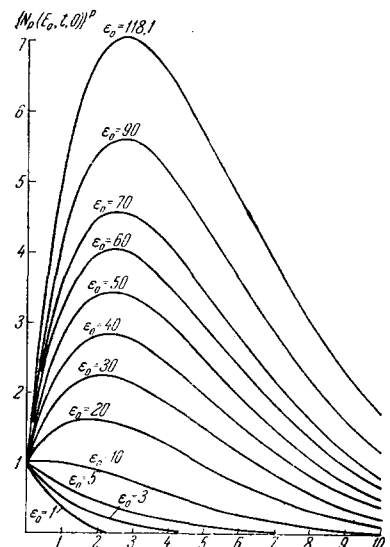


FIG. 28. Same as FIG. 27 but for a primary electron.

electron are shown in Figs. 27 and 28 and also in Table XI, which gives $\{N_p(E_0, t, 0)\}^P$ for $E_0 = 10^9$ and 10^{10} ev. The coefficient was assumed to be equal to 0.24. The moments for $E_0 = 10^9$ ev were

Table XI.

ev	t	0.5	1.0	2.0	3.0	3.5	4.0	5.0	5.5
10 ⁹	$\{N_p(E_0, t, 0)\}^r$	5.77	9.35	13.76	15.6	15.9	15.8	15.1	14.5
10 ¹⁰		16.3	33.3	65.4	91.8	102.2	110.6	—	125.2
ev	t	6.0	8.0	10.0	12.0	14.0	16.0	20.0	
10 ⁹	$\{N_p(E_0, t, 0)\}^r$	13.8	10.64	7.66	5.26	3.48	2.23	0.84	
10 ¹⁰		125.8	119.3	101.4	79.9	59.5	42.3	18.9	

taken from reference 57 and those for $E_0 = 10^{10}$ ev from reference 69. We note that in all the cascade curves obtained the absorption coefficient in the tail was near to the experimental value 0.24 of reference 9. The accuracy of the curves obtained as in the case of light substances is $\sim 10\%$.

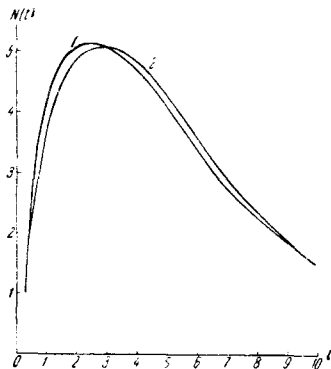


FIG. 29. 1--experimental cascade curve in lead from the bremsstrahlung spectrum of primary photons with maximum energy 330 Mev. 2--obtained by averaging over the spectrum of photons of cascade curves calculated by moments.³⁷ The curves normalized to $t = 0.5$.

Figure 29 shows the experimental cascade curve in lead (1), obtained in reference 58 from a $1/E$ spectrum of primary photons. Curve 2 is the result of averaging the cascade curves shown in Figs. 27 and 28 over the same photon spectrum. The calculated and experimental curves are within 5% of each other.

Thus the curves shown describe satisfactorily the cascade process in lead.

Cascade curves in lead were obtained in reference 75 by the Monte Carlo method, for selected individual cases, with a special machine. However, electrons from photons of energy less than 10 Mev and Rutherford scattering of charged particles were approximately taken into account in reference 75. The authors obtained the number of electrons at a given depth t in the form of the sum of two terms, $n = n_1 + n_2$, where n_1 is the number of electrons from photons of energy greater than 10 Mev, obtained by the Monte-Carlo method and n_2 is the number of electrons from photons of an energy less than 10 Mev. For determining n_2 the author used the following formula:

$$n_2(t) = \int_0^t \frac{w(t')}{\beta} e^{-(t-t')\sigma_m} dt',$$

where $w(t')$ is the average energy lost by photons of energy less than 10 Mev on radiation in one event, σ_m is the average photon absorption coefficient in the energy region 1 to 10 Mev, and β is the critical energy in lead. Such a choice of n_2 is based on the assumption, known to be inaccurate, that electrons obtained from photons of energy less than 10 Mev lose much less of their energy by ionization along a path that is much shorter than the photon mean free path; (see Fig. 28). We note that in the majority of cases of Compton scattering the secondary electrons carry off a significant part (more than half) of the energy of the incident photon. The calculation for scattering is done the following way: All the

points of the cascade curves obtained without accounting for scattering are moved to the left by a distance

$$\Delta t = r(1 - e^{-t}). \tag{3.18}$$

This way of accounting for scattering involves a practically constant shifting of the entire curve by $\Delta t = r$ for $t > 2$, which is known to be inaccurate. In addition, such a calculation of scattering does not change the height of the maximum of the curve*

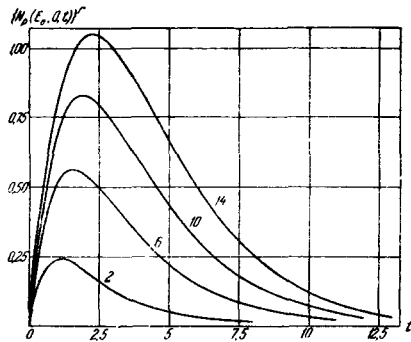


FIG. 30. Cascade curves for copper for a primary photon. The numbers near the curves indicate the energy of the primary photon in units β/q .

whereas according to reference 57 even with $E_0 = 10$ ev the calculation of scattering involves an increase in the height of the maximum by 2.5%. Figure 30 shows cascade curves for a primary electron of energy $E_0 = 10^8$ ev calculated by the method of moments; curve 1 includes scattering, curve 2 does not. From the figure it can be seen that a calculation of scattering involves a more complex deformation of the entire curve and in addition involves an increase in the height of the maximum by about 10%. Therefore the cascade curves obtained in reference 75 are satisfactory if $E_0/\beta \gg 1$. In this energy region they coincide, within 10 to 15%, with the curves shown in Figs. 27 and 28.

The electron cascade curves for copper, for a primary electron and photon in the energy interval $10^7 \lesssim E \lesssim 10^{12}$ ev, were calculated by the method of moments in reference 74. The calculations included the energy dependence of the total photon absorption coefficient and the Rutherford scattering of the charged particles. The coefficient γ was taken equal to 0.38. The calculated first two moments are given in Table XII. Cascade curves are shown in Figs. 31 to 33. The accuracy of the curves obtained in $\sim 10\%$.

Table XII.

ϵ_0	$\{ \bar{t}_p(\epsilon_0, 0) \}^p$	$\{ \bar{t}_p^2(\epsilon_0, 0) \}^p$	$\{ \bar{t}_p(\epsilon_0, 0) \}^r$	$\{ \bar{t}_p^2(\epsilon_0, 0) \}^r$	ϵ_0	$\{ \bar{t}_p(\epsilon_0, 0) \}^p$	$\{ \bar{t}_p^2(\epsilon_0, 0) \}^p$	$\{ \bar{t}_p(\epsilon_0, 0) \}^r$	$\{ \bar{t}_p^2(\epsilon_0, 0) \}^r$
2	0.96	2.99	3.13	17.8	$3.5 \cdot 10^2$	6.62	56.4	7.48	69.7
5	2.02	8.22	3.61	21.2	$5.6 \cdot 10^2$	7.10	64.0	7.92	77.1
10	2.64	12.1	3.94	23.8	$7.7 \cdot 10^2$	7.42	69.2	8.22	82.7
14	3.06	15.1	4.24	26.2	$9.8 \cdot 10^2$	7.66	73.3	8.47	87.2
35	4.13	24.9	5.16	36.3	$1.4 \cdot 10^3$	8.01	79.5	8.83	94.1
56	4.66	30.6	5.65	42.3	$7.4 \cdot 10^3$	9.68	112	10.5	129
77	5.01	34.6	5.97	46.7	$1.4 \cdot 10^4$	10.3	126	11.1	144
98	5.27	37.9	6.22	50.1	$7.4 \cdot 10^4$	12.0	166	12.8	187
$1.4 \cdot 10^2$	5.66	42.9	6.58	55.3	$1.4 \cdot 10^5$	12.6	183	13.4	205

Thus, using the method of moments, it is possible to calculate the function $N(E_0, t, 0)$, over a wide range of energies and depths in heavy substances. The first two moments of the distribution function of electrons $\{N_p(E_0, t, E^0)\}^r$ for an energy more than E^0 in lead were calculated in reference 59. The results of the calculation of moments are given in Table XIII. The function $\{N_p\}^r$ was

*We note that after applying transformation (3.18) to the cascade curve the author then renormalizes its area to E_0/β which can change the height of the maximum somewhat.

approximated by a sum of polynomials $L_n^1(x)$. The results of the calculations of $\{N_p\}^r$ for various values E_0 and E^0 are shown in Figs. 34 to 36. Figure 37 (1) shows the equilibrium spectrum and the energy spectrum in the region of the maximum of the cascade. From the figure it can be seen that the spectra in the region of the maximum of the cascade with $\epsilon_0 = 50, 90,$ and 118.1 agree with one another within 2% and agree within 10% with the equilibrium spectrum whose accuracy has been evaluated a number of times. The average energy of one charged particle arriving at a given depth,

determined from the calculated spectra, agrees with the exact values within 20%. In addition, it was shown that beyond the maximum the cascade dies out more slowly in heavy substances than in light ones. All this allows us to state that the energy spectra in heavy substances, as calculated in reference 59, are accurate to about 10%.

This review of works on the one-dimensional cascade theory of showers shows that the method of functional transformations, together with the method of moments, makes it possible to obtain practically a complete description of the average one-dimensional behavior of electron-photon cascades in light and heavy substances.

TABLE XIII

$\epsilon_0=50$			$\epsilon_0=90$			$\epsilon_0=118.1$		
z_0	$\{t_p(z_0, z_0)\}^{\Gamma}$	$\{\bar{t}_p^2(z_0, z_0)\}^{\Gamma}$	z_0	$\{t_p\}^{\Gamma}$	$\{\bar{t}_p^2\}^{\Gamma}$	z_0	$\{t_p\}^{\Gamma}$	$\{\bar{t}_p^2\}^{\Gamma}$
0	6.30	58.5	0	7.08	70.3	0	7.43	74.6
0.183	6.12	54.9	0.183	6.89	66.6	0.183	7.25	70.9
0.738	5.59	45.4	0.932	6.22	53.6	0.838	6.57	59.5
1.292	5.18	40.2	1.680	5.74	45.4	1.493	6.12	50.8
2.402	4.66	31.3	2.428	5.40	38.9	2.476	5.67	43.9
3.511	4.30	26.7	4.674	4.79	31.6	4.114	5.19	36.3
5.175	3.90	22.0	6.919	4.33	27.5	6.734	4.54	29.1
6.839	3.61	18.6						

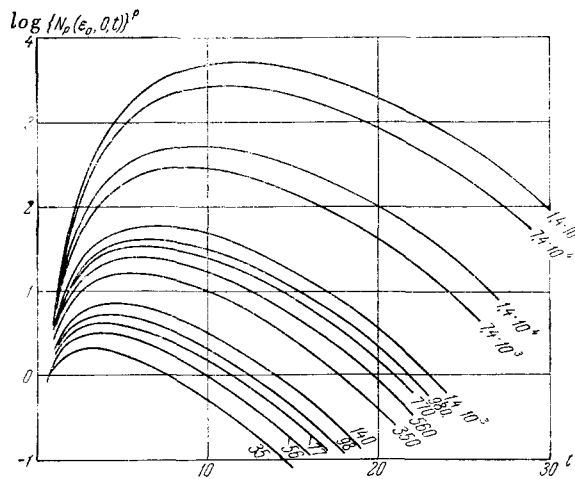


FIG. 31. Same as FIG. 30.

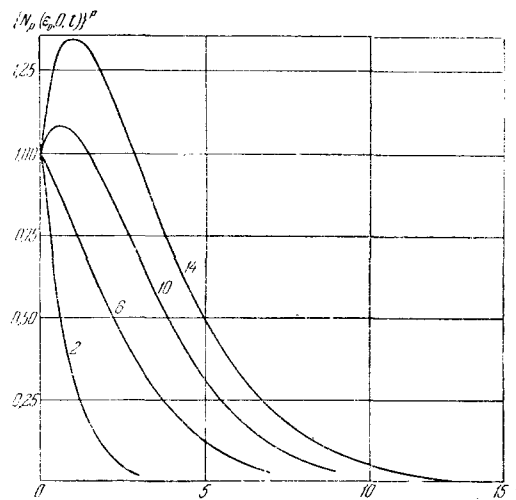


FIG. 32. Same as FIG. 30 but for a primary electron.

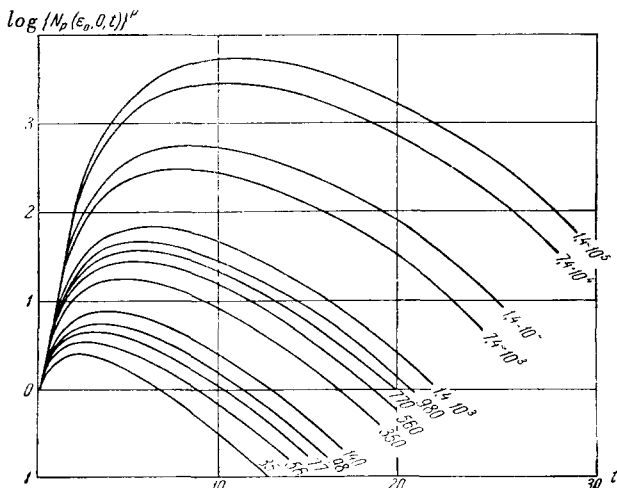


FIG. 33. Same as in FIG. 32.

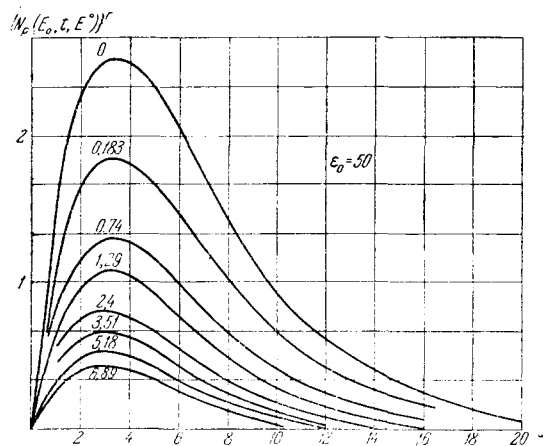


FIG. 34. Dependence on depth of the number of electrons of energy greater than ϵ (from 0 to ~ 7) in lead in a shower caused by a primary photon of energy $\epsilon_0 = 50, 90, \text{ and } 118.1$.

The applicability of the method of functional transformations is limited by the requirement that the functions for the probabilities of all processes be homogeneous with respect to the energies of the primary and secondary particle. To obtain a solu-

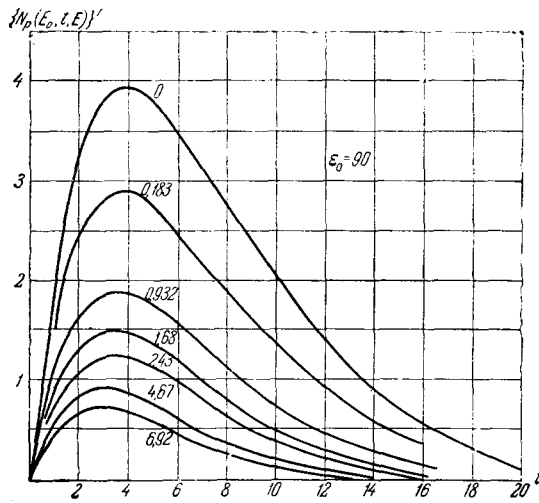


FIG. 35. Same as FIG. 34.

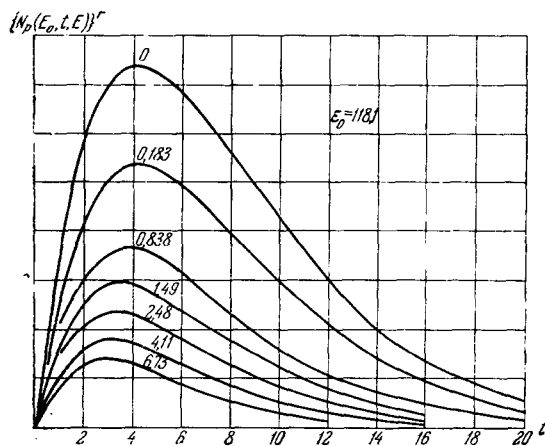


FIG. 36. Same as FIG. 34.

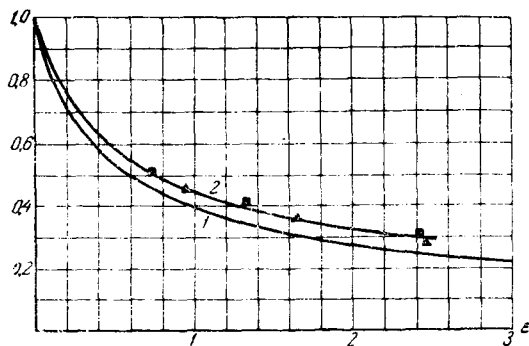


FIG. 37. 1--equilibrium spectrum, curve 2 and the triangles and squares near it are the spectra in the maximum of a cascade of a primary electron of energy $\epsilon_0 = 50, 90, \text{ and } 118.1$.

tion to the equations by the method of functional transformations it is necessary in practice to fulfill one more condition, $\ln E_0/\beta > 1$. In addition it is very difficult in practice to obtain a solution that includes scattering by this method.

The method of moments yields a solution for the light substances in the energy region $\epsilon_0 \sim 1$ and also a solution for heavy substances with allowance for the energy dependence of the total photon absorption coefficient and for Rutherford scattering of the charged particles in the cascade. The method of moments makes it possible to obtain with good accuracy a solution for a rather wide range of depths and energies of the primary particles. However, it is an approximate method. The tables and graphs in this review can be useful in a variety of numerical calculations in the cascade theory.

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