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*NONLINEAR PHENOMENA IN A PLASMA LOCATED IN AN ALTERNATING ELECTROMAGNETIC FIELD**

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3. NONLINEAR EFFECTS IN THE PROPAGATION OF RADIO WAVES IN A PLASMA (IONOSPHERE)

THE nonlinearity of electrodynamic processes in a plasma manifests itself particularly clearly in the propagation of sufficiently strong radio waves. Thus, the propagation of one wave in a plasma gives rise to a nonlinear "self interaction" effect wherein the frequency spectrum of the wave (appearance of harmonics of the principal frequency), its absorption, and its phase all change. When several waves are propagated, the superposition principle is violated: the incident and reflected, the ordinary and extraordinary, or in general any two waves, cease being independent — they interact in a nonlinear manner because they themselves change the property of the medium (plasma) in which they propagate.†

The ordinary theory of propagation of radio waves in a plasma (in the ionosphere or in the corona of the sun)[15] neglects the influence of the wave on the plasma. This is true, as a first approximation, if the field of the wave is weak, i.e., if its amplitude satisfies the condition

$$E_0 \ll E_p = \sqrt{3kT\frac{m}{e^2}\delta_{\text{eff}}\,(\omega^2 + \nu_{\text{eff}}^{(0)2})}\,. \tag{0.1}$$

The influence of the field of a weak wave on a plasma can be taken into account in the next approximation and should naturally result only in small corrections. In spite of this, nonlinear effects can be observed even for weak waves, and are of practical significance. For example, cross modulation of radio waves in the ionosphere is readily observed even if condition (0.1) is satisfied (see Sec. 3.4).

Naturally, the effect of strong radio waves ($E_0 \gtrsim E_p$), and more so of very strong ones ($E_0 \gg E_p$), on the plasma can no longer be neglected. Therefore ordinary theory of radio-wave propagation does not hold for such waves.

The principal results of the nonlinear theory of propagation of radio waves in a plasma will be developed below essentially as applied to the earth's ionosphere. Therefore, supplementing the statements made in the introduction, we list in Table III the values of the "plasma field" E_p for the ionosphere. The table lists also the estimated maximum change in electron temperature in the ionosphere under the influence of the electric field of the waves radiated by stations of different power.* It is seen from the table that strong radio waves in the medium and long wave bands can change substantially the energy of the electrons in the lower portion of the E layer. To the contrary, the effect of short waves on the ionosphere, as well as that of low-power medium and long waves, is quite insignificant — these waves are weak.

The propagation of electromagnetic waves in a medium is described by Maxwell's equations

$$\left.\begin{aligned}
&\operatorname{curl}\mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} = \frac{4\pi}{c}\mathbf{j}_t + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t},\\
&\operatorname{div}\mathbf{D} = 4\pi\varrho,\\
&\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial B}{\partial t},\\
&\operatorname{div}\mathbf{H} = 0,\\
&\qquad \mathbf{B} = \hat{\mu}\mathbf{H},\ \ \mathbf{D} = \hat{\varepsilon}\mathbf{E},\ \ \mathbf{j} = \hat{\sigma}\mathbf{E},
\end{aligned}\right\} \tag{3.1}$$

*For part 1 see Usp. Fiz Nauk 70, 202 (1960), Soviet Physics—Uspekhi 3, 147,(1960).

†In an inhomogeneous medium, the term "interaction of waves" is also used in a different sense.[69] Here we take interaction to mean only the effects connected with nonlinearity.

*Estimates of the electron temperature, listed in table III, are based on formula (1.24), with the amplitude of the wave on the boundary of the ionosphere assumed to be $E_0 = 300\sqrt{w}/r$ millivolt/m (see reference 15, Sec. 74) (w is the station power in kilowatts, r is the distance from the broadcast station to the ionosphere in kilometers, and the field E_0 is in millivolts per meter).

It should be noted that the earth's magnetic field, generally speaking, weakens the influence of the radio waves on the ionosphere (disregarding the gyro-resonance region). This influence is therefore strongest when the effect of the magnetic field is insignificant, as must occur when E is parallel to H_0, i.e., for an ordinary wave in transverse propagation; the values of T_e listed in Table III, are calculated precisely for this case.

We note also that the effect of radio waves on the ionosphere (the possibility of producing artificial glow of the ionosphere by means of radio waves) is discussed in references 70 and 71.

TABLE III

Ionosphere	ω	$E_p \cdot \frac{\text{mv}}{\text{m}}$	$T_e/T-1$							
			10 kw	100 kw	200 kw	500 kw	1000 kw	5000 kw	10^5 kw	10^6 kw
D layer (daytime)	$\omega \leqslant 3\cdot 10^6$	320	0,002	0,02	0,04	0.1	0.2	0,7	4	15
$\nu_0=10^7$	10^7	470	0,001	0,01	0.02	0,05	0.1	0.4	—	—
$T=300°$	10^8	3200	0,00002	0,0002	0,0004	0,001	0.002	0.01	—	—
$\delta=2\cdot 10^{-3}$										
$h=60$ km										
Lower part of the E layer (night)	$\omega \leqslant 2\cdot 10^5$	19	0.3	1.4	2,1	3,6	5.7	12	54	170
	10^6	32	0,1	0.8	1.4	2.7	4,3	11	—	—
$\nu_0=7\cdot 10^5$	$2\cdot 10^6$	56	0,04	0.3	0.6	1.5	2,7	9	—	—
$T=200°$	$5\cdot 10^6$	130	0,006	0,06	0.1	0.3	0.6	3	—	—
$\delta=2\cdot 10^{-3}$	10^7	270	0.002	0,02	0.03	0.08	0.2	0,8	—	—
$h=90$ km	10^8	2700	0.00002	0.0002	0.0003	0.0008	0.002	0,008	—	—
F layer	$\omega \leqslant 3\cdot 10^2$	0.02	—	—	—	—	—	—	—	—
$\nu_0=10^3$	10^3	1900	0.000004	0.00004	0.00008	0,0002	0,0004	0.002	—	—
$T=2000°$										
$\delta=10^{-4}$										
$h=300$ km										

where the operators $\hat{\mu}$, $\hat{\epsilon}$, and $\hat{\sigma}$ depend on the properties of the medium.

In a plasma the permeability $\hat{\mu}$ can usually be set equal to unity (see reference 15, sec 57). The conductivity $\hat{\sigma}$ and the permittivity $\hat{\epsilon}$ are determined by the velocity distribution of the plasma electrons (the ion current can usually be neglected; see Sec. 1), namely

$$\mathbf{j}_t = \hat{\sigma}\mathbf{E} + \frac{\partial}{\partial t}\left[\frac{\hat{\varepsilon}-1}{4\pi}\mathbf{E}\right] = e\int \mathbf{v} f(\mathbf{v}, \mathbf{r}, t)\, d\mathbf{v}, \tag{3.2}$$

where the distribution function f is determined by the Boltzmann equation (2.2). The system (3.1), (3.2), and (2.2) describes the propagation of radio waves in plasma.

It is important to note that although the field of the wave is indeed inhomogeneous in space, yet when determining the distribution function, and consequently the operators $\hat{\epsilon}$ and $\hat{\sigma}$, this inhomogeneity can sometimes be neglected, i.e., the term $\mathbf{v}\,\text{grad}_\mathbf{r} f$ in Eq. (2.2) can be neglected. This is tantamount to assuming that the operators $\hat{\epsilon}$ and $\hat{\sigma}$ are local, i.e., that the total current density at a given point $\mathbf{r}$ is determined by the field $\mathbf{E}$ at the same point. In a weak field this condition is not satisfied if the field amplitude $\mathbf{E}$ changes substantially over the electron mean free path, and also in a few other cases (see references 72 and 73).* In a strong field it is necessary to satisfy a more stringent requirement: the amplitude of the field $\mathbf{E}$ must change little over the electron-energy relaxation length $l/\sqrt{\delta_{\text{eff}}}$ (the latter varies much less than the mean free path l, because in a single impact the electron gives up, on the average, only a small part of its energy δ_{eff}, see Sec. 1).

It is also important that usually the electric field of a wave in a plasma can be considered rapidly varying, i.e., as satisfying the condition

$$\frac{\delta_{\text{eff}}\,\nu_{\text{eff}}}{\omega} \ll 1. \tag{1.16}$$

In fact, this condition holds in the ionosphere for all waves of length λ less than 100 or even 1000 km. It is satisfied for $\lambda < 10^{10}$ cm in the solar corona and for $\lambda < 10$ to 100 m in electronic instruments and experimental apparatus.

Under conditions (1.16), as shown in Secs. 1 and 2, the electron temperature in a field of any strength is constant, to first approximation, and the current density $\mathbf{j}_t$ varies with the frequency of the field $\mathbf{E}$. Therefore, if the conditions of locality are also satisfied, the wave propagation problem is solved in two steps. First, as in the case of the weak field, we find the current $\mathbf{j}_t$ as a function of $\mathbf{E}$. This current is used in the next step to solve the field equations.

*In a weak field, if the operators $\hat{\varepsilon}$ and $\hat{\sigma}$ are not local, the use of the local quantities $\varepsilon(\omega)$ and $\sigma(\omega)$ is obviously no longer permissible. At the same time, it is possible and sometimes convenient to use the functions $\varepsilon(\omega, \mathbf{k})$ and $\sigma(\omega, \mathbf{k})$, which depend not only on ω but also on the wave vector $\mathbf{k}$ and which do not have, consequently, a local character. Non-locality in a weak field reduces thus to a dependence of ε and σ on $\mathbf{k}$, and is therefore called spatial dispersion.

3.1. Propagation of Radio Waves in a Plasma with Allowance for Nonlinearity (Interaction of Radio Waves)

Let us now consider propagation (in a plasma) of radio waves with a field $\mathbf{E}_0(0)\cos\omega t$ on the boundary of the medium (in the plane $z = 0$). If condition (1.16) is satisfied, the electron distribution function

f_0 cannot vary as rapidly as the electric field, and in the first approximation the value of f_0 is established at a constant level, independent of the time (the time-variable corrections are small and have amplitudes of order $\delta\nu/\omega$, see Secs. 1.2, 2.3d, and 2.4). Accordingly, the principal parts of the conductivity and of the dielectric permittivity of the plasma also remain independent of the time. Thus, a wave of frequency ω, arriving at the boundary of an isotropic plasma, will propagate in a medium whose ϵ and σ are constant in time. The wave frequency ω therefore remains constant, and Eqs. (3.1), which describe the propagation of the wave reduce, as can be readily seen,[20] to the wave equation

$$\Delta \mathbf{E} - \text{grad div}\, \mathbf{E} + \frac{\omega^2}{c^2} \varepsilon' (\mathbf{r}, \omega, E_0) \mathbf{E} = 0. \qquad (3.3)$$

The expressions for $\epsilon' = \epsilon - i4\pi\sigma/\omega$ were derived in Secs. 1 and 2: ϵ' depends both on the frequency and on the amplitude E_0 of the alternating electric field of the wave. As a result, the wave equation (3.3) is nonlinear, and differs essentially from the usually considered linear wave equation.

In view of the complexity of Eq. (3.3), let us proceed to solve it under several simplifying assumptions. That is to say, we consider an isotropic plasma. We assume that it consists of flat layers and varies only in the z direction. The normal to the wave front will also be directed along the z axis (normal incidence). Then Eq. (3.3) for the components E_x and E_y assume the form $d^2E/dz^2 + \omega^2\epsilon' E/c^2 = 0$, and if the properties of the medium change sufficiently slowly, the geometrical-optics approximation $E = C \times \exp\{i\omega t \pm i(\omega/c)\int\sqrt{\epsilon'}\, dz\}$ remains valid (for more details see reference 15, Sec. 65).* Assuming, as usual,

$$\varepsilon' = \varepsilon - i\frac{4\pi\sigma}{\omega} = (n - i\varkappa)^2 \qquad (3.4)$$

and considering only a wave traveling along the z axis, we can thus write out a formal solution for the wave equation (3.3) in the form

$$E = Ce^{i\left(\omega t - \frac{\omega}{c}\int_0^z n\, dz\right) - \frac{\omega}{c}\int_0^z \varkappa\, dz}, \qquad (3.5)$$

where C can be considered constant in the zeroth approximation. It is essential to note that since ϵ', and consequently also n and κ, depends on the amplitude E_0, the relation (3.5) is really an integral equation in the nonlinear theory. For the amplitude E_0 we therefore obtain

$$E_0 = C \exp\left\{-\frac{\omega}{c}\int_0^z \varkappa(z, E_0)\, dz\right\},$$

or

$$\frac{dE_0}{dz} + \frac{\omega}{c}\varkappa(z, E_0) E_0 = 0. \qquad (3.6)$$

*In addition to the ordinary conditions of the applicability of the geometrical-optics approximation (see reference 15, Sec. 65), it is necessary in the case of a strong field that the field amplitude E_0 change little over a wavelength, i.e., that the condition $\kappa/n \ll 1$ be satisfied for $E_0 \gtrsim E_p$.

To obtain an explicit expression for the wave amplitude E_0 we shall employ henceforth for ϵ and σ the simple formulas (1.8) of the "elementary theory." In addition, we shall assume that the plasma temperature T and the electron-collision frequency in the unperturbed plasma ν_0 are homogeneous, i.e., independent of z (for a high-frequency wave, $\omega^2 \gg \nu_0^2$, the latter assumption is not essential).* In a plasma usually $|\epsilon| \gg 4\pi\sigma/\omega$ (this condition is violated near the "point of reflection," where $\epsilon = 0$; at the same time, as $\epsilon \to 0$, the geometrical-optics approximation employed here is also inapplicable). If the foregoing condition is satisfied, $\kappa = 2\pi\sigma/\omega n \cong 2\pi\sigma/\omega\sqrt{\epsilon}$, and in a weak field [see (1.8)] we have

$$\varkappa_0(z) = \frac{2\pi e^2 N \nu_0}{m\omega(\omega^2 + \nu_0^2)\sqrt{1 - \frac{4\pi e^2 N}{m(\omega^2 + \nu_0^2)}}}, \qquad (3.7a)$$

where $\nu_0 \equiv \nu_{\text{eff}}^{(0)}(T)$ is the effective number of collisions in the equilibrium plasma (for $E_0 \leq E_p$ and $T_e = T$).

Within the framework of the elementary theory (see Sec. 1), the expressions for ϵ and σ remain the same in an arbitrary field as in a weak field, except for replacement of ν_0 by $\nu_{\text{eff}}(T_e) \equiv \nu(T_e)$. The absorption coefficient κ can therefore be written in the form

$$\varkappa(z, E_0) = \varkappa_0(z)\frac{\frac{\nu(T_e)}{\nu_0}\left(\frac{\omega^2}{\nu_0^2} + 1\right)}{\frac{\omega^2}{\nu_0^2} + \frac{\nu^2(T_e)}{\nu_0^2}}, \qquad (3.7)$$

where for simplicity the dependence of the index of refraction $n = \sqrt{\epsilon}$ on T_e is disregarded.† The dependence of the electron temperature on the amplitude E_0 of the wave field is given by (1.22).

Let us consider first the case when the principal role is played by collisions with molecules. The ratio $\nu(T_e)/\nu_0$ is in this case equal to $\sqrt{T_e(E_0)/T}$, and it is convenient to use in Eq. (3.6), [with absorbtion coefficient (3.7)] a new variable $\tau = \sqrt{T_e(E_0)/T}$ instead of E_0. Relation (1.22) then becomes

$$\left(\frac{E_0}{E_p}\right)^2 = (\tau^2 - 1)\frac{\omega^2 + \nu_0^2\tau^2}{\omega^2 + \nu_0^2}. \qquad (1.22a)$$

Expressing E_0 and dE_0/dz in terms of τ and $d\tau/dz$, with the aid of (1.22a), we rewrite (3.6) in the form

$$\frac{d\tau}{dz}\left(\frac{1}{\tau^2 - 1} + \frac{2\nu_0^2}{\omega^2 + \nu_0^2}\right) + \frac{\omega}{c}\varkappa_0(z) = 0. \qquad (3.6a)$$

Its solution is

$$\frac{\tau - 1}{\tau + 1}\exp\left\{\frac{4\nu_0^2}{\omega^2 + \nu_0^2}\tau\right\} = C\exp\{-2K(z)\}, \qquad (3.8)$$

*Equation (3.6) can, naturally, be readily solved numerically for an arbitrary dependence of κ on z and E_0. A graphic method for this solution is indicated, for example, in reference 74, and the corresponding numerical calculations for the ionosphere were made in reference 20. The question of the influence of the field of a wave on its absorption in the ionosphere is also considered in reference 76, but not fully enough (see reference 77).

†Such an assumption corresponds to reality far away from the point of reflection of the wave (when $n \cong \sqrt{\varepsilon} \cong 1$), and also always for $\omega^2 \gg \nu^2$.

where $K(z) = (\omega/c)\int_0^z \kappa_0(z)\,dz$ is the total absorption of the weak wave from the start of the layer to the point z. On the plasma boundary (at $z = 0$) the wave amplitude $E_0(0)$ is given. Consequently,

$$C = \frac{\tau_0 - 1}{\tau_0 + 1}\exp\left\{\frac{4\nu_0^2}{\omega^2+\nu_0^2}\tau_0\right\}, \tag{3.8a}$$

where $\tau_0 = \sqrt{T_e[E_0(0)]/T}$. It is seen from (3.8) that τ_0 is the maximum value of τ; with increasing z, or more accurately with increasing $K(z) = (\omega/c)\int_0^z \kappa_0\,dz$, τ diminishes monotonically. Deep in the plasma, where $K(z) \gg 1$, τ tends to unity, i.e., the wave becomes weak, as it should.

The solution (3.8) determines τ at each point z inside the plasma. Knowing $\tau(z)$ from (1.22a), we can readily determine also the unknown amplitude $E_0(z)$. It is best represented in the form

$$E_0(z) = E_0(0)\exp\{-K(z)\}\,P, \tag{3.9}$$

where $K(z) = (\omega/c)\int_0^z \kappa_0\,dz$ is the weak-wave absorption, and P is a factor that shows the result of the interaction between the wave and the plasma. In a weak field, the factor P is naturally close to unity. In the general case, P depends on the amplitude of the wave field at the plasma boundary, on the frequency, and on the depth of penetration of the wave into the plasma, i.e., $P = P[E_0(0)/E_p,\ \omega/\nu_0,\ K(z)]$.

The expression for P is particularly simple deep in plasma, i.e., where $K(z) \gg 1$. In fact, it is clear from (3.8) that $\tau(z)$ is close to unity in this case:

$$\tau(z) = 1 + 2\exp\left\{-\frac{4\nu_0^2}{(\omega^2+\nu_0^2)}\right\} C\exp\{-2K(z)\}.$$

Consequently

$$E_0(z) \simeq E_p\sqrt{\tau^2-1} = 2E_p\exp\left\{\frac{-2\nu_0^2}{(\omega^2+\nu_0^2)}\right\}\sqrt{C}\exp\{-K(z)\}$$

$$= 2E_p\sqrt{\frac{\tau_0-1}{\tau_0+1}}\exp\left\{\frac{2\nu_0^2}{\omega^2+\nu_0^2}(\tau_0-1) - K(z)\right\} \tag{3.10}$$

and

$$P = 2\frac{E_p}{E_0(0)}\sqrt{\frac{\tau_0-1}{\tau_0+1}}\exp\left\{\frac{2\nu_0^2}{\omega^2+\nu_0^2}(\tau_0-1)\right\}. \tag{3.11}$$

It is seen from (3.11) that at high frequencies ($\omega \gg 2\nu_0^2\tau_0$) the factor P is less than unity, which is quite natural, since at high frequencies the absorption coefficient of the wave increases with increasing T_e, i.e., with increasing amplitude $E_0(0)$. In a very strong field [when $E_0(0) \gg E_p$, i.e., $\tau_0 \gg 1$] P diminishes with increasing $E_0(0)$ *in proportion to* $E_p/E_0(0)$. In this case the amplitude of the wave penetrating deep into the plasma, as is clear from (3.10), does not increase with increasing $E_0(0)$, but tends to a constant value, $E_0(z) = 2E_p\exp\{-K(z)\}$, which is independent of $E_0(0)$. This is also seen from Fig. 9 (curve 1).

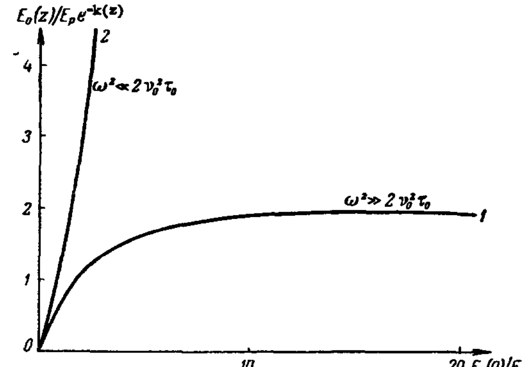

FIG. 9

If the converse condition $\omega^2 \ll 2\nu_0^2\tau$ is satisfied, the factor P, to the contrary, increases with $E_0(0)$. This is also quite understandable, for at low frequencies the absorption coefficient of the wave in the plasma diminishes with increasing T_e. In a very strong field in this case the self-interaction factor

$$P = \frac{2E_p}{E_0(0)}\exp\left\{\sqrt{4\left(\frac{\nu_0^2}{\omega^2+\nu_0^2}\right)^{\frac{3}{2}}\frac{E_0(0)}{E_p}}\right\}$$

increases exponentially with increasing $E_0(0)$ (see curve 2 on Fig. 9). In this case the absorption is greatly attenuated (owing to self-interaction).

Expressions (3.10) and (3.11) are correct only inside the plasma, where $K(z) \gg 1$, and where the wave has already become weakened by attenuation. A simple expression is obtained for the factor P for any $K(z)$ only at high frequencies, namely

$$P = \frac{2E_p}{E_0(0)}\sqrt{\frac{\tau_0-1}{\tau_0+1}}\left(1 - \frac{\tau_0-1}{\tau_0+1}\exp\{-2K(z)\}\right)^{-1}. \tag{3.12}$$

It is seen from (3.12) that at high frequencies the factor P diminishes sharply with increasing $K(z)$. It is important that the stronger the field, the faster the rate of decrease of P, so that when $\omega^2 \gg 2\nu_0^2\tau_0$ the thickness of the layer in which the wave still remains strong is independent of the amplitude of the wave field on the plasma boundary. This thickness is determined by the condition $K(z) \sim 1$. A method of calculating P for an arbitrary frequency is indicated in reference 74.

The problem is solved quite analogously when the principal role is played by collisions with ions. It is enough to examine here only high frequency waves, for only such waves can propagate in a strongly ionized plasma [since $\omega \gtrsim \omega_0 = \sqrt{4\pi e^2N/m} \gg \nu_{0i} \sim e^4N/(kT)^{3/2}m^{1/2}$ by virtue of the condition $e^2N^{1/3}/kT \ll 1$].* The absorption coefficient of such waves diminishes sharply with increasing electron temperature. Accordingly, the self-interaction coefficient P increases rapidly with increasing amplitude $E_0(0)$ $\left[\text{as } \exp\left\{\frac{1}{3}\left(\frac{E_0(0)}{E_p}\right)^3\right\}\right]$. Because of this, the strong wave, the field of which is greater than "critical,"

*Only when $\omega \gtrsim \omega_0$ is the permittivity $\varepsilon = 1 - \omega_0^2/(\omega^2 + \nu_{0i}) > 0$, since $\omega_0^2 \gg \nu_{0i}^2$.

$$E_0(0) > E_{K_s} \cong 1.5\sqrt[3]{K_s}\,E_p,$$

passes through a layer of plasma of thickness K_s with practically no attenuation [here $K_s = (\omega/c)\int_0^s \kappa_0 ds$ is the total absorption of a weak wave in a layer s]. The magnitude of the critical field is actually close to E_p and increases very slowly with increasing thickness of the layer s. The dependence of the coefficient of transmission of the wave through a layer of plasma on the wave amplitude at its boundary is shown in Fig. 10 for different values of K_s.

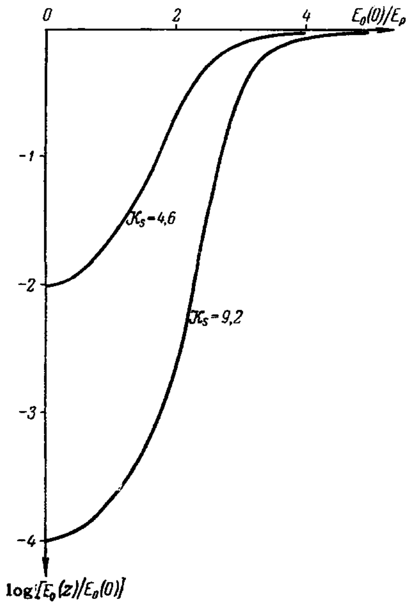

FIG. 10

The corrections to the phase shift of the wave, resulting from the self-interaction, have also been derived in references 74 and 77, in the geometrical-optics approximation. Thus, in the case of collisions with neutral particles

$$\Delta\varphi = \tan^{-1}\frac{\nu_0\tau_0}{\omega} - \tan^{-1}\frac{\nu_0\tau}{\omega} + \left(\frac{\bar{\varkappa}_0}{n_0}\right)\left\{\frac{\tau_0(\omega^2+\nu_0^2)}{\omega^2+\nu_0^2\tau_0^2} - \frac{\tau(\omega^2+\nu_0^2)}{\omega^2+\nu_0^2\tau^2}\right\}, \tag{3.13}$$

where τ is defined as before by (3.8), and $(\bar{\kappa}_0/n_0)$ is the average value of the ratio of the absorption and refraction indices of a weak wave along the path in the perturbed region. It is clear from (3.13) that the phase-shift changes due to self-interaction are insignificant at both high and low frequencies. $\Delta\varphi$ reaches its maximum value (on the order of $\pi/2$) in a sufficiently strong field, when $\nu_0^2 \ll \omega^2 \ll \nu_0^2\tau_0^2$.

We have already seen that the amplitude of a signal passing deep in a plasma has essentially a nonlinear dependence on the amplitude of the signal arriving at the boundary of the plasma. It is clear that if the wave arriving at the boundary of the plasma is amplitude-modulated at a low frequency Ω, then its modulation may change substantially inside the plasma. Here, in the quasi-stationary case, when the modulation frequency Ω is very low (much less than $\delta\nu_0$), the problem of propagation of an amplitude-modulated wave in a plasma is essentially identical with the problem already considered, that of propagation of an unmodulated wave. It is merely necessary to bear in mind that the amplitude of the plasma boundary varies slowly with time: $E_0(0, t) = E_0(0)(1 + \mu_0 \cos \Omega t)$, where μ_0 is the depth of modulation at the plasma boundary. For the depth of the modulation μ of the wave in the plasma we obtain in this case the following simple expression:[75]

$$\mu = \mu_0 \frac{\tau}{\tau_0}\frac{\omega^2+\nu_0^2\tau_0^2}{\omega^2+\nu_0^2\tau^2}. \tag{3.14}$$

This formula has been obtained by expansion in powers of μ_0 and μ and is therefore valid only at a low depth of modulation. Here τ has the same meaning as in the case considered above ($\tau = \sqrt{T_e(E_0)/T}$); it is defined, as previously, by the relation (3.8).

It is seen from (3.14) that at high frequencies the depth of modulation of a wave in a plasma is less than that of the wave arriving at the boundary of the plasma, meaning that the wave is "demodulated" in the plasma. Furthermore, for a very strong wave, the "demodulation" may be quite considerable, almost complete, for $\mu = \mu_0/\tau_0 \cong \mu_0 E_p/E_0(0)$ when $E_0(0) \gg E_p$. At low frequencies ($\omega^2 \ll \nu_0^2$) the depth of modulation of the wave in the plasma, to the contrary, increases by self-interaction. The dependence of the depth of modulation of the wave in the plasma on $E_0(0)/E_p$ is shown in Fig. 11.

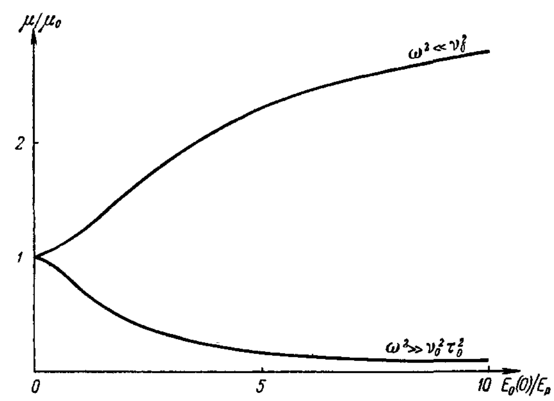

FIG. 11

In the case of an arbitrary depth of modulation μ_0, the amplitude of the wave in the plasma is determined from (3.9):

$$E_0(z, t) = E_0(0, t)\exp\{-K(z)\}\,P\left(\frac{E_0(0, t)}{E_p}, \frac{\omega}{\nu_0}, K(z)\right), \tag{3.9a}$$

where $E_0(0, t) = E_0(0)(1 + \mu_0 \cos \Omega t)$ is the amplitude of the wave on the plasma boundary, and P is the self-interaction factor [see (3.11) and (3.12)]. It is seen from (3.9a) that in general, self-interaction changes not only the depth of modulation of the wave in the plasma, but also distorts the form of the modulation curve (i.e., harmonics with frequencies 2Ω, 3Ω,

. . . appear). In strong fields, the distortion of the modulation is insignificant at small μ_0 and very large at $\mu_0 \sim 1$. Characteristic curves that show the form of modulation of strong radio waves deep in plasma at $\mu_0 = 1$ are shown in Fig. 12.

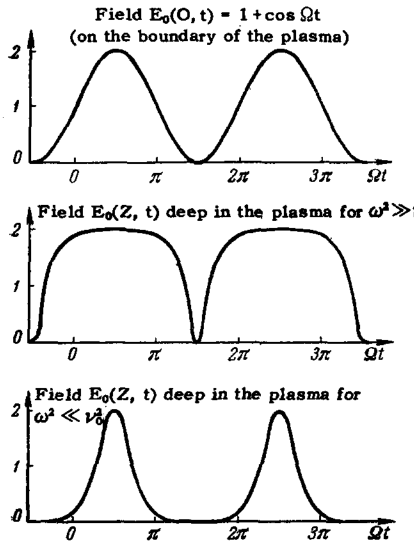

FIG. 12

If the modulation frequency Ω is not small compared with $\delta\nu_0$, then the calculation of the variation of the wave modulation in the plasma is somewhat more complicated, since we can no longer assume the electron temperature to be quasi-stationary, i.e., to be given by (1.22). It is therefore necessary to solve simultaneously Eqs. (1.11) for the electron temperature and Eq. (3.6) for the wave amplitude. The solution is obtained by expansion in powers of μ_0, which is valid naturally only for small μ_0 (see references 75 and 20). The expressions for the depth and phase of the modulation become quite complicated in this case. At high frequencies $\omega^2 \gg \nu_0^2$, and the equations become simpler

$$\left.\begin{aligned} \mu &= \mu_0 \sqrt{\frac{(\delta\nu_0\tau)^2+\Omega^2}{(\delta\nu_0\tau_0)^2+\Omega^2}},\\ \varphi &= \tan^{-1}\frac{\Omega}{\delta\nu_0\tau_0} - \tan^{-1}\frac{\Omega}{\delta\nu_0\tau}. \end{aligned}\right\} \quad (3.15)$$

It is seen therefore that as the modulation frequency increases, the "demodulation" becomes weaker; at high modulation frequencies the changes in the wave modulation are generally insignificant (when $\Omega \gg \delta\nu_0$ we have $\mu \rightarrow \mu_0$ and $\varphi \rightarrow 0$). Furthermore, the changes in the phase tend to zero even at low modulation frequencies. The maximum value $\varphi_{\max} = \tan^{-1}\sqrt{\tau_0/\tau} - \tan^{-1}\sqrt{\tau/\tau_0}$ is reached when $\Omega_{\max} = \delta\nu_0\sqrt{\tau\tau_0}$. The dependence of μ/μ_0 and φ on Ω is shown in Fig. 13.

In the case of a weak wave it is easy to obtain expressions for the depth of modulation of the fundamental and of the harmonics at any value of μ_0. They have the following forms:[77-79]

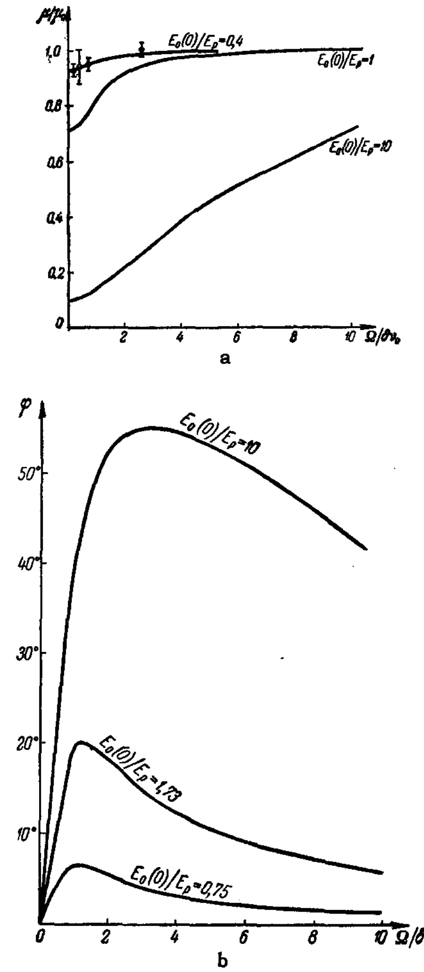

FIG. 13

$$\mu_\Omega = \mu_0\left(1 - q\left[\frac{2(\delta\nu_0)^2\left(1-\frac{\mu_0^2}{2}\right)}{\Omega^2+(\delta\nu_0)^2} - \frac{\mu_0^2(\delta\nu_0)^2}{4\{(\delta\nu_0)^2+4\Omega^2\}}\right]\right),$$

$$\mu_{2\Omega} = \frac{3\mu_0^2}{2} q \sqrt{\frac{\left|(\delta\nu_0)^2+\frac{25}{9}\Omega\right|(\delta\nu_0)^2}{[4\Omega^2+(\delta\nu_0)^2][\Omega^2+(\delta\nu_0)^2]}},$$

where

$$q = \frac{1}{4}\left(\frac{E_0(0)}{E_p}\right)^2 \frac{\omega^2-\nu_0^2}{\omega^2+\nu_0^2}(1-\exp\{-2K(z)\}).$$

From this we see in particular that in a weak field the changes in the depth of modulation are small, as is the distortion of modulation ($q \ll 1$), as should be.*

*Self interaction of a weak radio wave in a plasma can, naturally, be evaluated by successive approximations.[77-79] In the first approximation, the effect of the wave on the plasma is neglected. One then determines the perturbations that the field of the wave, calculated in the first approximation, produces in the electron temperature, and consequently in the conductivity and the dielectric permittivity of the plasma. Taking these perturbations into account in the wave equation, we obtain the resultant changes in the amplitude, phase, and depth of modulation of the wave. In the limiting case of a weak field, naturally, the results of such a calculation agree with those obtained by nonlinear theory.

As a result of the self-interaction, both the amplitude and the phase of the wave change in the plasma; therefore, when an amplitude-modulated wave propagates in a plasma, phase modulation should also occur: $\Delta\varphi = \Delta\varphi_0 + \beta_\Omega \cos \Omega t + \ldots$, where β_Ω is the index of phase modulation. The expressions for the index of phase modulation are in general quite complicated.[75,77,80] The index of phase modulation is found to be small both for high frequencies ($\omega^2 \gg \nu_0^2$) and for low ones ($\omega^2 \ll \nu_0^2$); its maximum value

$$\beta_{max} \cong 0.7\,\mu_0 \frac{\tau_0^2 - 1}{\tau_0 \sqrt{2\tau_0^2 - 1}} \tag{3.16}$$

is reached when $\omega_{max} = \nu_0\sqrt{2\tau_0^2 - 1}$ and $\Omega \ll \delta\nu_0$.

For convenience, let us recapitulate the principal effects that nonlinearity produces in the propagation of radio waves in a plasma.

1) The frequency spectrum of the wave is distorted. The distortions are insignificant if the field of the wave alternates rapidly, i.e., if $\omega \gg \delta\nu_0$. In this case the amplitudes of the harmonics are small compared with the amplitude of the fundamental wave (their ratio does not exceed $\delta\nu_0/\omega$).[20] No exact expressions have been obtained for the amplitudes of the harmonics.*

2) The absorption of the wave in the plasma may change very strongly (compared with the absorption of a weak wave).

In this case, for very strong radio waves, with amplitudes much greater than the "plasma field" [$E_0(0) \gg E_p$], the absorption differs from that of a weak field even qualitatively. That is to say, such waves either pass through the layer of the plasma, experiencing hardly any absorption (regardless of how the weak wave would be absorbed, see Fig. 10), or, to the contrary, are almost completely absorbed in the plasma (in the latter case the wave, after passing through the layer of plasma, always becomes weak: its amplitude is determined only by the plasma field and is in general independent of the amplitude of the incident wave; see Fig. 9).

These questions have been investigated in sufficient detail, and the absorption can be determined for waves of any amplitude.[74]

3) The phase of the wave changes insignificantly, at most by an order of $\pi/2$. The changes in phase are strongest for waves whose frequency is close to the effective number of electron collisions.

4) The amplitude modulation of the wave may change very greatly.

*The amplitude of the first harmonic (of a wave of frequency 3ω) was calculated in reference 77 in the weak-field approximation. In that paper, however, only the symmetrical part of the distribution function, f_0, was taken into account, whereas for an exact solution of the problem it is necessary to take into account also the asymmetrical part, f_2. This is connected with the fact that neglect of the function f_2 is accurate only to terms of order δ. Therefore the calculation of terms of order $\delta\nu/\omega$ with neglect of the function f_2 is legitimate only if $\delta\nu/\omega \gg \delta$ (see Sec. 2.1). At the same time, the results obtained in reference 77 are of the correct order of magnitude.

In the case of collisions with molecules, a very strong high-frequency wave is almost completely demodulated after passing through the plasma; the depth of modulation of a low-frequency wave, to the contrary, increases. The modulation distortion (the appearance of harmonics) depends substantially on the depth of modulation μ_0, being insignificant at small μ_0 and very large when μ_0 is close to unity. In addition to the amplitude modulation due to the self-interaction, phase modulation is also produced. All the foregoing changes in the modulation of the wave in the plasma are essential only for low modulation frequencies; at high modulation frequencies ($\Omega \gg \delta\nu_0$) these changes are insignificant.

Changes in modulation can also be determined for a wave of any amplitude.[75,77,78]

5) The wave reflection may change substantially in the plasma. For example, for strong radio waves, geometrical optics is not valid in the region where $\kappa/n \gtrsim 1$. In this region, apparently, there is considerable reflection of strong waves. The reflection of waves from layers of an inhomogeneous plasma, as far as we know, has not been considered in the nonlinear theory.

6) Owing to the influence of the waves on the plasma, the principle of superposition of waves is violated, as is well known, for weak radio waves. Because of this, in particular, self-interaction effects can become reinforced through interaction between the incident and reflected waves. The same applies also to the ordinary and extraordinary waves in a magnetoactive plasma.

The effect of interaction between the ordinary and extraordinary waves is discussed in references 20, 81, and 82. The interaction between waves of different frequencies is considered below, in Secs. 3.3 and 3.5.

3.2. Role of Self Interaction in the Propagation of a Radio Wave in the Ionosphere

Let us consider now the role of nonlinear self-interaction effects in the propagation of radio waves in the ionosphere. This role is essentially different for short, medium, and long radio waves.

a) Short Waves ($\lambda \le 200$ m, $\omega > 10^7$). The plasma field E_p for such waves as shown earlier, (Table III), is quite large: $E_p \sim$ (300 to 4000) mv/m. Therefore, even if the broadcast station has a power of 1000 kw, the short radio waves become weak, $[E_0(0)/E_p]^2 \le 10^{-1}$ to 10^{-3}, and the self-interaction effects are insignificant.

b) Medium Waves (200 m < λ < 2000 m; $10^6 < \omega < 10^7$). For such waves, the plasma field in the lower part of the E layer is relatively weak: $E_p \sim 25\,(\omega/10^6)$ mv/m. Therefore at large broadcast-station power, the amplitude of the medium-wave field in the ionosphere is comparable with the plasma field, and may even exceed it considerably. Numerical estimates of the size of the nonlinear effects for waves of different frequency and power are listed in Table IV (P is the self interaction

TABLE IV

f, kcs	W, kw	R=0			R=400 kw		
		$E_0(0)/E_p$	P	μ/μ_0	$E_0(0)/E_p$	P	μ/μ_0
160 ($\omega = 10^6$)	100	1.14	0,94	0,94	0.5	0.98	0.97
	500	2.54	0.92	0,98	1.12	0,94	0.94
	1000	3.56	0.93	1.07	1.57	0.93	0.93
	5000	8.05	1.12	1.21	3.54	0.93	1.06
320 ($\omega = 2\cdot 10^6$)	100	0.65	0.93	0.87	0.29	0.98	0,97
	500	1.45	0.78	0.68	0.65	0.93	0.87
	1000	2.02	0.69	0.62	0.91	0.89	0.80
	5000	4.52	0.49	0.55	2.04	0.69	0.61
480 ($\omega = 3\cdot 10^6$)	100	0.43	0.96	0.92	0.20	0.99	0.98
	500	0.95	0.86	0.75	0.44	0.96	0.92
	1000	1,35	0.76	0,64	0.62	0.93	0.85
	5000	3.04	0,49	0,43	1.4	0,75	0.62

factor [Eq. (3.9)] which shows how the wave amplitude is changed by self-interaction in the ionosphere; the ratio μ/μ_0 shows the change in the depth of modulation of the wave).

It is seen from the table that for strong radio waves of medium wave lengths the role of self interaction can be quite considerable.* For example, when the power of a station broadcasting at 500 kcs changes from $w^{(1)} = 1000$ kw to $w^{(2)} = 5000$ kw, the amplitude of the wave reflected from the ionosphere increases, as is clear from Table IV, by a factor

$$\frac{E_0^{(2)}(0)\,P\,(E_0^{(2)}(0))}{E_0^{(1)}(0)\,P\,(E_0^{(1)}(0))} = \sqrt{\frac{w^{(2)}}{w^{(1)}}}\,\frac{0.49}{0.76} \cong 1.44$$

(if one could neglect the effects of self interaction, then to obtain the same increase in the amplitude of the reflected wave, the broadcast-station power would have to be not 5 times greater, but only double). The modulation of the wave can also be substantially distorted by self interaction in the ionosphere.

The foregoing self interaction effects, and also the other nonlinear features of wave propagation noted above, can be important in the analysis of problems of high-power radio broadcasting at medium wavelengths. In fact, because of self interaction of high frequency waves ($\omega^2 \gg \nu_0^2$) in the plasma, the amplitude of the signal reflected from the ionosphere increases very slowly with increasing power of the radiating station, and at the same time the depth of modulation decreases. Furthermore, when the wave becomes very strong [$E_0(0) \gg E_p$], the amplitude of the reflected wave in general stops increasing with increasing radiating power, and the depth of modulation of the wave tends to zero. It is therefore clear that it is not advantageous to increase the power of such stations above a certain limit. The latter is determined by the permissible modulation distortion: for example, if not more than 30% of modulation distortion is permissible, then, as can be seen from Table IV, the limiting power of stations broadcasting at 300 to 500 kcs is on the order of 2000 to 5000 kw.

On the other hand, an increase in the power of a low-frequency wave ($\omega^2 \ll \nu_0^2$) leads to a sharp increase in the amplitude of the signal reflected from the ionosphere. As can be seen from Table IV, the frequency that separates these opposite cases is on the order of 100 to 300 kcs (i.e., $\lambda = c/f \approx 1$ to 3 km).

To be able to estimate the significance of self interaction effects to radio broadcasting, Table IV lists the values of P and μ/μ_0 not only for $R = 0$, but also for $R = 400$ km (R is the land distance between the transmission and reception points; for radio broadcasting at $R < 100$ km, the wave reflected from the ionosphere is of no importance, but when $R \gtrsim 200$ or 300 km, it becomes decisive). It is seen from Table IV that although the effects of self interaction, generally speaking, become weaker with increasing R (although sometimes they may become reinforced), in the case of very strong stations they can still distort the signal substantially.

In spite of numerous detailed theoretical indications,[83,84] attempts at an experimental detection of self interaction effects in the ionosphere have for a long time led to no conclusive results (we do not speak here of the self-demodulation resonance near the gyro frequency, a question which will be discussed in detail later). A report has appeared recently[79,85] of a series of measurements, performed in England, in which the self-demodulation effect was observed. The power of the 200 kcs transmitter ranged from 75 to 400 kw in these experiments. The measurement results are in good agreement with theory; this is seen, for example, from Fig. 13a, which shows, in addition to the theoretical curves, the results of a measurement of the depth of modulation as a function of the modulation frequency. Strong self interaction effects in waves broadcast by a high-power transmitter were also observed recently.[133]

It must be noted that an experimental investigation of self interaction in the ionosphere also yields the

*It is interesting to note that the "strongest" (for equal radiation power) radio waves are those whose frequency is close to the effective electron-collision frequency in the lower part of the E layer ($\omega \sim \nu_0 \sim 10^6$). However, self interaction affects such waves less, because when $\omega \gtrless \nu_0$ the self interaction factor P and the ratio μ/ν_0 do not depend monotonically on the power of the wave (this reflects the non-monotonic dependence of σ and κ on ν_{eff}).

In the D layer the plasma field E_p is large ($E_p \geq 300$ millivolt/m) and the effects of self interaction are accordingly insignificant.

electron temperature in the ionosphere, in the field of a strong wave, and permits determination of the essential parameter δ_{eff}. In fact, for example, the dependence of the modulation phase on the modulation frequency, as shown above, reaches a maximum value, $\varphi_{max} = \tan^{-1}\sqrt{\tau_0} - \tan^{-1} 1/\sqrt{\tau_0}$, when $\Omega_{max} = \delta\nu_0 \times \sqrt{\tau_0}$. By measuring the dependence of φ on Ω and by determining φ_{max} and Ω_{max} experimentally, we can find τ_0 and consequently also $T_e = \tau_0^2 T$ and $\delta_{eff} = e^2E^2(0)/3kTm\omega^2(\tau_0^2 - 1)$ [see (1.25)]. Estimates show that $\varphi_{max} \sim 15$ to 20° when $\omega \sim (2$ to $3) \times 10^6$ and w = 500 kw.

c) Long Waves ($\lambda > 2000$ m; $\omega < 10^6$). The plasma field E_p for long waves is small (in the lower part of the E layer, $E_p \sim 25$ to 30 mv/m). The effects of self interaction can therefore play a substantial role in the case of strong long radio waves.*

It must be borne in mind, however, that the propagation of such waves in the ionosphere, as is well known, cannot be described in the approximation of geometric optics, while the nonlinear theory has been developed thus far only for this case. Therefore, to be able to estimate the role of self interaction in the case of long waves, a separate analysis is necessary, but has not been made as yet. Nor have these effects been experimentally investigated.

d) Resonant Self Demodulation Near the Gyrofrequency. A considerable reduction in the depth of modulation has been observed[88-94] in the reception of radio waves reflected from the ionosphere, at frequencies close to gyromagnetic, $\omega_H = |e| H_0/mC$ [in the ionosphere $\omega_H \cong (6$ to $8) \times 10^6$].

The following fundamental features of this effect are noted:

1) The demodulation increases resonantly near the gyro frequency but is insignificant away from the gyrofrequency (see Fig. 14a).

2) The maximum demodulation is on the order of 50% or more (see Figs. 14a and 14b).

3) The demodulation increases with increasing modulation frequency, reaching a maximum at $\Omega \cong (5$ to $6) \times 10^3$ (see Fig. 14b).

4) No dependence whatever on the transmitter power was observed for the demodulation. Near the gyro frequency, a very strong demodulation is observed even for w = 3 kw.[88]

The question of the mechanism of this strong nonlinear effect has as yet not been resolved. In many papers[82,88,91,95] a suggestion is advanced that self modu-

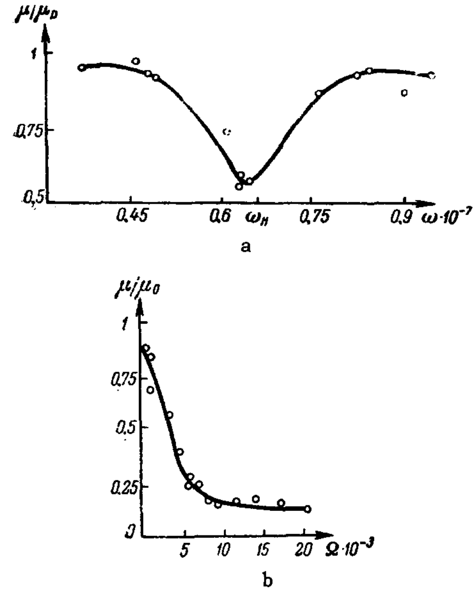

FIG. 14

lation near the gyro frequency is a result of the action exerted by the extraordinary wave on the ordinary one in the ionosphere. Such an assumption, however, is subject to serious objections. Firstly, a quantitative calculation[80,81,75,101] (see also Sec. 3.3b) shows that the demodulation induced in the ordinary wave by the extraordinary one in the ionosphere is extremely slight. Even at an interfering-station power of 100 kw, the maximum demodulation should not exceed 1%. Secondly, the demodulation due to the action of the extraordinary wave on the ordinary one attenuates with increasing modulation frequency: it has a maximum at low modulation frequency, and should fall off sharply at $\Omega > 2000$. Finally, there should be a clearly pronounced dependence of the demodulation on the power of the transmitting station, and, to the contrary, there should be no sharp resonant increase in the effect near the gyro frequency. Thus, the theoretically derived properties of the demodulation, due to the effect of the extraordinary wave on the ordinary one in the ionosphere, are in complete disagreement with the features of the observed effect.

It is therefore worth while paying attention to the fading mechanism of demodulation, the possibility of which was indicated in references 96, 97, 75, and 91a. In fact, it is well known that if strong interference fading takes place for the unmodulated wave, the depth of modulation of the received signal is decreased in the transmission of modulated waves (see, for example, references 98 and 99). The basic outlines of this mechanism of demodulation consist of the following:

1) The demodulation may be very strong, even complete. With this, the magnitude of demodulation depends not on the power of the radiated signal, but on the relative amplitudes of the fields of two (or several) interfering rays, arriving at the point of reception along different paths S_1, S_2, ... (i.e., having different phases at the reception point).

*Attention must be called to one peculiarity of self interaction of long radio waves: whereas for medium and short radio waves the amplitudes of the harmonics due to self interaction in a plasma are small (since $\delta\nu_0/\omega \ll 1$), they may be considerably greater in a case of long waves. In the D layer, for example, $\delta\nu$ and ω are in general of the same order of magnitude for very long radio waves.

This circumstance can influence, in particular, the frequency spectrum of atmospherics, which has recently been investigated (see references 86 and 87); we note that the power of an atmospheric is very large, on the order of 10^5 to 10^6 kw, so that it always represents a strong radio wave (see Table III).

2) The demodulation increases with increasing Ω, reaching a maximum (in the case of interference of two rays) when the modulation phase difference of the interfering rays equals π, i.e., when $\Omega_{max} \cong \pi c/(S_1 - S_2)$ (when $S_1 - S_2 = 150$ km, $\Omega_{max} = 6000$).

These properties of the fading mechanism of demodulation are in good agreement with the aforementioned peculiarities of the observed effect. The resonant amplification of the interference fading near the gyro frequency can be due to the fact that the extraordinary wave arrives at the reception point in addition to the ordinary wave.* The extraordinary wave may be reflected from a lower level of the ionosphere if sufficiently strong electron-density gradients are present in the inhomogeneities at the edge of the layer (it is necessary that the electron concentration change substantially over distances on the order of 100 m). It is possible that such reflections were observed, for example, in the work described in reference 100.

We note that reference 101 indicates also the possibility of demodulation due to the difference in conditions of propagation of the fundamental wave (frequency ω) and sideband waves (frequencies $\omega - \Omega$ and $\omega + \Omega$). This assumption, however, has not been corroborated in reference 101 by a detailed analysis.

3.3. Nonlinear Interaction of Modulated Radio Waves (Cross Modulation)

Disturbances produced in a plasma by a strong radio wave should also influence other waves propagating in the disturbed region. It is possible to separate in this case three types of disturbances produced by a strong wave in a plasma. Firstly, if a strong wave is amplitude modulated at a low frequency Ω, then the disturbances produced by the wave in the plasma will also be modulated, as will be consequently other waves passing through the disturbed region. This phenomenon is usually called *cross modulation* or the *Luxemburg effect*. It is observed in the propagation of waves in the ionosphere and is of practical importance to radio broadcasting at medium waves. Cross modulation was investigated in many theoretical and experimental works,[101–120,56,70,80] the results of which will be reported below.

Unmodulated waves also disturb the plasma, and this leads to two effects. Firstly, the rapidly varying disturbing wave produces in the plasma constant (time invariant) changes in the electron distribution functions, and consequently, in the conductivity and in the dielectric permittivity of the plasma. Because of the latter, the conditions for the propagation of other waves to the disturbed region also change, causing the amplitude and phase of these waves to change. Secondly, in addition to constant disturbances in ϵ and σ, there are produced weak disturbances which vary rapidly with time at a frequency that is a multiple of the perturbing wave. The presence of such disturbances in the propagation of the other waves in the plasma should result in waves with combination frequencies. Interactions with unmodulated waves will be considered in Sec. 3.5.

a) Cross Modulation in an Isotropic Plasma. In order to calculate the effect of cross modulation, it is necessary to determine first the magnitude of the low frequency perturbations produced in the plasma by a strong wave (E_1 wave), and then find how these perturbations influence the other wave (the E_2 wave), which propagates in the disturbed region. Naturally, if the E_1 wave is strong, its self interaction must be taken into account in the calculation of the perturbations that it produces in the plasma. For the sake of simplicity, however, we shall first disregard self interaction, i.e., we shall consider the case of a relatively weak perturbing wave.

The amplitude of the electric field of an amplitude-modulated weak wave E_1 in a plasma has at a point z, in the approximation of geometric optics, a value (see reference 15, Sec. 65)

$$E_1 = \sqrt[4]{\frac{\varepsilon_1(0)}{\varepsilon_1(z)}}\, E_1(0)\,(1 + \mu_0 \cos \Omega t) \exp\{-K_1(z)\}. \quad (3.17)$$

Here $\epsilon_1(0)$ and $\epsilon_1(z)$ are the dielectric permittivities for the E_1 wave at the beginning of the layer (at $z = 0$) and at the point z; $K_1(z) = (\omega_1/c)\int_0^z \kappa_1 dz$ is the absorption of the E_1 wave in the layer. Substituting the field (3.17) in (1.11), we obtain the electron temperature disturbances produced by the E_1 wave in the plasma. The time-variable component of these disturbances, having frequencies Ω and 2Ω, is:*

$$\frac{\Delta_\Omega T_e}{T} = \frac{2\mu_0 e^2 E_1^2(0)}{3kTm\delta\,(\omega_1^2 + \nu_0^2)} \sqrt{\frac{\varepsilon_1(0)}{\varepsilon_1(z)}} \exp\{-2K_1(z)\} \times \left\{\frac{\delta\nu_0}{\sqrt{(\delta\nu_0)^2 + \Omega^2}} \cos(\Omega t - \varphi_\Omega) + \frac{\mu_0 \delta\nu_0 \cos(2\Omega t - \varphi_{2\Omega})}{4\sqrt{(\delta\nu_0)^2 + 4\Omega^2}}\right\}, \quad (3.18)$$

where $\varphi_\Omega = \tan^{-1}(\Omega/\delta\nu_0)$ and $\varphi_{2\Omega} = \tan^{-1}(2\Omega/\delta\nu_0)$. Corresponding periodic disturbances appear also in the electron collision frequency, $\nu = \nu_0 + \Delta_\Omega\nu$. Considering here only collisions with neutral particles, we have $\Delta_\Omega\nu = (\nu_0/2)(\Delta_\Omega T_e/T)$, since $\nu = \nu_0\sqrt{T_e/T}$ (it is precisely this case that occurs in the lower atmosphere, where cross modulation is always produced).†

The amplitude of any other weak wave, which we consider unmodulated, is determined at the point z, naturally, again by the expression (3.17)

*Cutolo[88] emphasizes that the fading was not very strong during the time of the measurements; to the contrary, Aitchinson and Goodwin[91,91a] indicate the presence of strong interference fading.

*The condition $\omega \gg \Omega$ is used in the solution and the terms that have frequencies Ω and 2Ω are disregarded in the equation for T_e. One can therefore use the equation for T_e in the form (1.13), replacing E_0^2 by the square of the amplitude (3.17) and disregarding the terms with cosine Ωt and cosine $2\Omega t$.

†It is clear from (1.5) that in collisions with ions $\Delta_\Omega\nu = (-3\nu_0/2)(\Delta_\Omega T_e/T)$, i.e., the signs of $\Delta_\Omega\nu$ and $\Delta_\Omega T_e$ are different. Therefore the coefficients introduced below, μ_Ω and $\mu_{2\Omega}$, are of different sign in the case of collisions with ions than in the case of collisions with molecules.

$$E_2(z)=\sqrt[4]{\frac{\varepsilon_2(0)}{\varepsilon_2(z)}}\,E_2(0)\exp\left\{-\frac{\omega_2}{c}\int_0 \varkappa_2\,ds\right\}, \quad (3.17')$$

where ϵ_2 is the permittivity and κ_2 the absorption coefficient of the E_2 wave.

The *absorption coefficient depends on the number* of collisions of the electron. Therefore, in the propagation of the E_2 wave in the disturbed region of the plasma, part of the absorption coefficient varies periodically with time. Separating this time-variable part, κ_2 $\left[\kappa_2 = \kappa_2(\nu_0) + \frac{\partial\kappa_2(\nu_0)}{\partial\nu_0}\Delta_\Omega\nu\right]$, we find that the amplitude of the wave passing through the disturbed plasma layer S can be represented in the form:*

$$E_2=E_2(0)\exp\left\{-\frac{\omega_2}{c}\int_S \varkappa_2(\nu_0)\,ds\right\}$$

$$\times\left\{1-\frac{\omega_2}{2c}\int_S \nu_0\frac{\partial\varkappa_2(\nu_0)}{\partial\nu_0}\frac{\Delta_\Omega T_e}{T}ds\right\}, \quad (3.19)$$

where $\Delta_\Omega T_e$ is the variable part of the electron-temperature disturbances in the plasma, given by Eq. (3.18). It is clear therefore that the E_2 wave, after passing through the disturbed layer, is found to be amplitude modulated at frequencies Ω and 2Ω, i.e., it has the form $E_2 = \text{const}\{1-\mu_\Omega\cos(\Omega t-\varphi_\Omega) - \mu_{2\Omega}\cos(2\Omega t - \varphi_{2\Omega})\}$; in this case the depth and the phase of the cross modulation are determined by the expressions

$$\mu_\Omega=\frac{\omega_2}{c}\int_S \frac{\mu_0 e^2E_1^2(0)}{3kTm\delta(\omega_1^2+\nu_0^2)}\nu_0\frac{\partial\varkappa_2(\nu_0)}{\partial\nu_0}\frac{\delta\nu_0}{\sqrt{(\delta\nu_0)^2+\Omega^2}}$$

$$\times\sqrt{\frac{\varepsilon_1(0)}{\varepsilon_1(s)}}\exp\{-2K_1(s)\}\,ds, \quad (3.20a)$$

$$\mu_{2\Omega}=\frac{\omega_2}{c}\int_S \frac{\mu_0^2e^2E_1^2(0)}{12kTm\delta(\omega_1^2+\nu_0^2)}\nu_0\frac{\partial\varkappa_2(\nu_0)}{\partial\nu_0}\frac{\delta\nu_0}{\sqrt{(\delta\nu_0)^2+4\Omega^2}}$$

$$\times\sqrt{\frac{\varepsilon_1(0)}{\varepsilon_1(s)}}\exp\{-2K_1(s)\}\,ds, \quad (3.20b)$$

$$\varphi_\Omega=\tan^{-1}\frac{\Omega}{\delta\nu_0}, \quad \varphi_{2\Omega}=\tan^{-1}\frac{2\Omega}{\delta\nu_0}. \quad (3.21)$$

To obtain the final expression for the depth of the cross modulation, it is necessary to integrate in (3.20) over ds. Let us assume here first that both waves, E_1 and E_2, are normally incident on the plasma (i.e., that the normals to the waves are parallel to the z axis, along which the properties of the plasma vary). Then the plasma layer from the start of the layer ($z = 0$) to the point of reflection of the E_1 wave ($z = z_{01}$) is disturbed. We shall consider this point, for the sake of simplicity, to be considerably below the point of reflection of the E_2 wave, so that in the disturbed region $\epsilon_2(s) \cong \epsilon_2(0) \cong \epsilon_1(0) = 1$. Then, considering that

$$\varkappa_2(s)=\varkappa_2(z)=\frac{2\pi e^2N(z)\nu_0}{m\omega_2(\omega_2^2+\nu_0^2)\sqrt{\varepsilon_2}}$$

[see (3.7a)], we find that

$$\frac{\omega_2}{c}\nu_0\frac{\partial\varkappa_2(\nu_0)}{\partial\nu_0}\frac{1}{\omega_1^2+\nu_0^2}\sqrt{\frac{\varepsilon_1(0)}{\varepsilon_1(z)}}$$

$$=\frac{2\pi e^2N(z)\nu_0(\omega_2^2-\nu_0^2)}{mc(\omega_2^2+\nu_0^2)^2(\omega_1^2+\nu_0^2)}\frac{1}{\sqrt{\varepsilon_1(z)}}=\frac{\omega_2^2-\nu_0^2}{(\omega_2^2+\nu_0^2)^2}\frac{\omega_1}{c}\varkappa_1(z), \quad (3.22)$$

where $\kappa_1(z)$ is the absorption coefficient of the E_1 wave. Let us assume now also that the plasma temperature T and the collision frequency ν_0 do not change in the disturbed region (for high-frequency waves the assumed constancy of ν_0 is not an essential condition). Then, substituting (3.22) in (3.20) and considering that $\omega_1\kappa_1(z)/c = dK_1/dz$ and that the E_1 wave can be represented, up to its points of reflection, in the form of a sum of an incident and reflected wave (see reference 15, Sec. 71) with a sufficient degree of accuracy, we can readily perform the integration with respect to dz in (3.20):

$$\mu_\Omega=\int_S \frac{\mu_0e^2E_1^2(0)}{3kTm\delta}\frac{\delta\nu_0}{\sqrt{(\delta\nu_0)^2+\Omega^2}}\frac{\omega_2}{c}\nu_0\frac{\partial\varkappa_2(\nu_0)}{\partial\nu_0}\frac{1}{\omega_1^2+\nu_0^2}$$

$$\times\sqrt{\frac{\varepsilon_1(0)}{\varepsilon_1(s)}}\exp\{-2K_1(s)\}ds$$

$$=\frac{\mu_0e^2E_1^2(0)}{3kTm\delta}\frac{\delta\nu_0}{\sqrt{(\delta\nu_0)^2+\Omega^2}}\frac{\omega_2^2-\nu_0^2}{(\omega_2^2+\nu_0^2)^2}\left[\int_0^{z_{01}}\frac{\omega_1}{c}\varkappa_1(z)\exp\{-2K_1(z)\}\,dz\right.$$

$$\left.+\exp\{-2K_1(z_{01})\}\int_0^{z_{01}}\frac{\omega_1}{c}\varkappa_1(z)\exp\{-2K_1(z)\}\,dz\right]$$

$$=\frac{\mu_0e^2E_1^2(0)}{6kTm\delta}\frac{\omega_2^2-\nu_0^2}{(\omega_2^2+\nu_0^2)^2}\frac{\delta\nu_0}{\sqrt{(\delta\nu_0)^2+\Omega^2}}(1-\exp\{-2K_1^0\}). \quad (3.23a)$$

Here $K_1^0 = 2K_1(z_{01}) = 2\int_0^{z_{01}}\frac{\omega_1}{c}\kappa_1(z)\,dz$ is the total absorption of the E_1 wave in the plasma. We have disregarded in (3.23a) that usually not only the incident ray of the E_2 wave, but also the reflected ray are propagated through the disturbed region. This naturally doubles the depth of cross modulation. We note also that Eq. (3.23a) can also be readily generalized to include the case of oblique incidence of the E_1 and E_2 waves on the plasma (using the well-known theorem on the connection between the absorption in the case of oblique incidence and the absorption in the case of normal incidence; see reference 15, Sec. 74). Thus, if the wave E_2 is incident at an angle ψ_2, it is enough to multiply μ_Ω in (3.23a) by $\cos\psi_2$ and to replace ω_2 by $\omega_2\cos\psi_2$. When the E_1 wave is incident at an angle ψ_1 and $\omega_1^2 \gg \nu_0^2$, it is enough to multiply μ_Ω

---

*In expression (3.19) it has been assumed that

$$\exp\left\{-\frac{\omega_2}{2c}\int_S \nu_0\frac{\partial\varkappa_2}{\partial\nu_0}\frac{\Delta_\Omega T_e}{T}ds\right\}=1-\frac{\omega_2}{2c}\int_S \nu_0\frac{\partial\varkappa_2}{\partial\nu_0}\frac{\Delta_\Omega T_e}{T}ds.$$

Therefore Eq. (3.19) is valid only if the depth of the cross modulation is considerably less than unity. A case when this condition is not satisfied is considered, for example, in reference 20.

by $\cos\psi_1$. We thus obtain finally*

$$\mu_\Omega = \frac{\mu_0 e^2 E_1^2(0)}{3kTm\delta}\frac{\omega_2^2\cos^2\psi_2-\nu_0^2}{(\omega_2^2\cos^2\psi_2+\nu_0^2)^2}\cos\psi_1\cos\psi_2 \times \frac{\delta\nu_0}{\sqrt{(\delta\nu_0)^2+\Omega^2}}(1-\exp\{-2K_1^0\}). \quad (3.23)$$

Expression (3.23) shows that the depth of cross modulation is proportional to the depth of modulation and to the power of the interfering wave. The depth of cross modulation for $\omega_2^2\cos^2\psi_2 \gg \nu_0^2$ also increases with increasing angle of incidence of the E_2 wave on the plasma layer (which is true, naturally, only so long as the point of reflection of the E_2 wave lies above the point of reflection of the E_1 wave; otherwise a change in the angle of incidence of the E_2 wave results also in a change in the dimensions of the interaction region). It is also seen from (3.23) that the depth of cross modulation has a maximum at low frequencies of modulation, $\Omega \ll \delta\nu_0$; when $\Omega \gg \delta\nu_0$, it diminishes as $\delta\nu_0/\Omega$. The phase of the cross modulation, to the contrary, is small at small $\Omega/\delta\nu_0$ and increases with $\Omega/\delta\nu_0$ (it increases up to $\pi/2$). The dependence of the depth and phase of the cross modulation on $\Omega/\delta\nu_0$ is plotted in Fig. 15 (the points on Fig. 15 are the experimental results, see Sec. 3.4b).

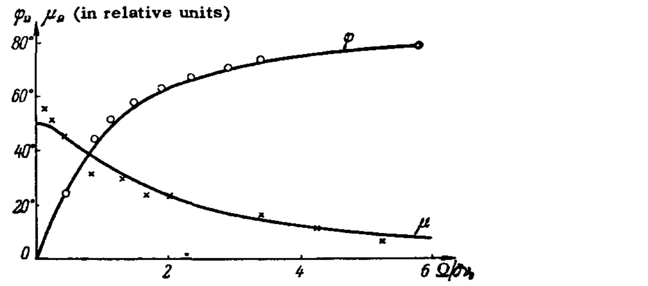

FIG. 15

The dependence of the depth of cross modulation on the frequency of the interfering wave is determined by the factor $F(K_1^0) = 1 - \exp\{-2K_1^0\}$. If the interfering wave is considerably absorbed in the interaction region ($K_1^0 \gtrsim 1$), then the factor F is always close to unity and the depth of cross modulation is independent of the frequency ω_1. In this case, so to speak, complete cross modulation is produced. If, to the contrary, the E_1 wave is only slightly absorbed in the plasma, then the depth of cross modulation is proportional to its total absorption K_1^0. The dependence of the depth of cross modulation on the frequency ω_2 has a simple form

$$\mu_\Omega \sim \frac{\left(\omega_2^2-\frac{\nu_0^2}{\cos^2\psi_2}\right)}{\left(\omega_2^2+\frac{\nu_0^2}{\cos^2\psi_2}\right)^2}.$$

It was assumed earlier that the point of reflection of the E_2 wave is considerably above the point of reflection of the interfering E_1 wave. Equation (3.23) is valid, however, even when this condition is not satisfied. The form of the factor $F(K_1^0)$ merely changes somewhat. For example, if the point of reflection of the E_2 wave lies considerably below the point of reflection of the interfering E_1 wave, then the factor $F = F_1(K_1^0)$ has the form shown in Fig. 16.* It is seen from the figure that, in this case, the factor F reaches a maximum $F_{max} \cong 1.4$ when $K_1^0 \cong 1$. We note that the height of this maximum depends little on the character of variation of the electron concentration $N(z)$ in the interaction region.[120] The dotted curve on the same diagram represents the factor $F(K_1^0) = 1 - \exp\{-2K_1^0\}$, which, as we have already seen, is valid in the inverse limiting case. It is important that the difference between the foregoing limiting functions F and F_1 is small.

The depth of cross modulation for the second har-

*We note that earlier, prior to references 110, 105, and 120, Eqs. (3.20) were thought to be final; they were usually written in the form (references 102 and 15, Sec. 64)

$$\mu_\Omega = \frac{e^2\overline{E_1^2(s)}}{3kTm\delta}K_2(s)\left\{\frac{\cos^2\beta}{\omega_1^2+\nu_0^2}+\frac{\sin^2\beta}{2[(\omega_1-\omega_H)^2+\nu_0^2]} + \frac{\sin^2\beta}{2[(\omega_1+\omega_H)^2+\nu_0^2]}\right\}\frac{\delta\nu_0}{\sqrt{(\delta\nu_0)^2+\Omega^2}}, \quad (3.20')$$

where $\overline{E_1^2(s)}$ is a certain average value of the amplitude of the E_1 wave in the interaction region and $K_2(s) = \int_S \frac{\omega_2}{c}\varkappa_2(s)\,ds$ is the total absorption of the E_2 wave in the interaction region. In formula (3.20′), account is also taken of the presence of a constant magnetic field H_0; here $\omega_H = |e|H_0/mc$ and β is the angle between $\mathbf{E}$ and $\mathbf{H}_0$.

Expression (3.20′) does not yield the absolute magnitude of the depth of cross modulation, since the quantity $\overline{E_1^2(s)}$ remains somewhat indeterminate in this case. A more important fact, however, is that Eq. (3.20′) does not describe correctly the variation of the depth of cross modulation with the frequency ω_1 of the perturbing wave. For example, it follows from (3.20′), for a specified $\overline{E_1^2}$ and K_2, that as the frequency ω_1 of the perturbing wave approaches the gyro frequency ω_H, the depth of cross modulation should always increase resonantly (under ionospheric conditions μ_Ω should increase by a factor of 50 to 100!). This conclusion, however, is not true: it must be taken into account that actually the dimensions of the disturbed layer also change substantially with changing frequency ω_1 [i.e., the values of $\overline{E_1^2(s)}$ and $K_2(s)$ in (3.20′) also change]. As a result, there is in general no resonant increase in depth of cross modulation when $\omega_1 \cong \omega_H$; this increase is possible only under certain special conditions (see Secs. 3.3b and 3.4e). This latter circumstance remained unclear for a long time and was the subject of many discussions.[70,103-106,110-114a,120] We note that the formulas similar to (3.20′) also fail to describe correctly the variation of the depth of cross modulation with the power of the interfering station in the case of a strong interfering E_1 wave. Expression (3.23), in which an exact integration has been carried out over the region of interaction, is naturally free of the foregoing shortcomings of expression (3.20′).

*Naturally, K_1^0 is in this case the total attenuation of the interfering wave up to the point of reflection of the wave E_2, or $K_1^0 = (\omega_1/c)\int_0^{z_{02}}\varkappa_1\,dz$ (we have neglected here, for the sake of simplicity, the influence of the reflecting ray of the E_1 wave). We note that the factor F is largest when the points of reflection of the E_1 and E_2 waves coincide. Furthermore, if $K_1^0 \gg 1$, then F is again close to unity; at small K_1^0, however, F increases more rapidly than $F_1(K_1^0)$, and its maximum is somewhat higher.

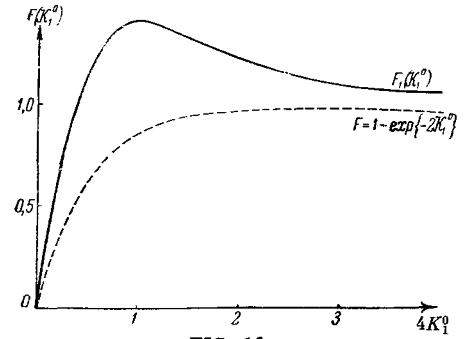

FIG. 16

monic is always small compared with μ_Ω:

$$\mu_{2\Omega}=\mu_\Omega\frac{\mu_0}{4}\sqrt{\frac{(\delta\nu_0)^2+\Omega^2}{(\delta\nu_0)^2+4\Omega^2}}\,.$$

The calculation performed is correct only in the case of a weak disturbing wave [when $E_1^2(0) \ll E_{p1}^2 = 3kT\delta m(\omega_1^2+\nu_0^2)/e^2$]. If the interfering wave is strong, its field changes substantially the absorption and the cross modulation, as well as the absorption of the E_2 wave in the interaction region. All this should affect the depth of cross modulation. The corresponding cross-modulation problem for the case when the interfering wave is strong was considered in reference 120. For the case when the interfering wave attenuates sufficiently strongly in the interaction region, so that "total" cross modulation is produced, the following simple expression is obtained for the depth of cross modulation at normal incidence of the E_1 and E_2 waves on the layer:

$$\mu_\Omega=\mu_0\frac{e^2E_1^2(0)}{3kTm\delta}\,\frac{2}{\tau_0(\tau_0+1)}\,\frac{\omega_2^2-\nu_0^2\tau_0}{(\omega_2^2+\nu_0^2)(\omega_2^2+\nu_0^2\tau_0^2)}\,, \qquad (3.24)$$

where the modulation frequency is considered to be low ($\Omega \ll \delta\nu_0$) and all the quantities have the same meaning as in (3.23), while $\tau_0^2 = T_eE_1(0)/T$ is the change in the electron temperature on the plasma boundary, due to the interfering wave; this change is determined, as usual, by Eq. (1.24):

$$\tau_0^2=1+\frac{\omega_1^2+\nu_0^2}{2\nu_0^2}\left(\sqrt{1+\frac{4\nu_0^2}{\omega_1^2+\nu_0^2}\left(\frac{E_1(0)}{E_{p1}}\right)^2}-1\right), \qquad (1.24)$$

where E_{p1} is the "plasma field" for the E_1 wave. Comparing (3.24) with (3.23) for a weak perturbing wave [for $\Omega = 0$ and $F(K_1^0) = 1$], we see that the greatest change takes place in the dependence of the depth of cross modulation on the amplitude of the interfering wave $E_1(0)$ at the plasma boundary, i.e., on the power W of the interfering station [since $E_1(0) \sim \sqrt{W}$]. For example, when ω_1 and ω_2 are high, the depth of cross modulation (3.24) does not increase linearly with increasing power (as in the case of a weak field), but tends to a constant limit: $\mu_\Omega = 2\mu_0\omega_1^2/\omega_2^2$ (see curve 1, Fig. 17). Even stronger is the change in the character of the dependence of μ_Ω on W when the frequency of the E_2 wave cannot be considered high ($\omega_2 \sim \nu_0$). The depth of cross modulation may even diminish here with increasing W (see curve 2, Fig. 17).

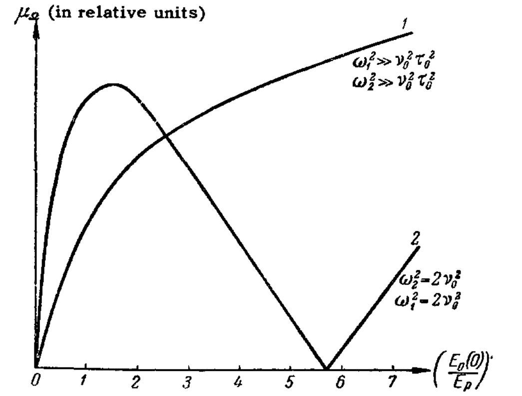

FIG. 17

We have calculated the disturbances produced in a plasma by the interfering wave by using elementary theory. A kinetic-theory calculation of the same effects, as given in references 56 and 80, leads naturally to almost the same results.*

b) Calculation of the Effect of a Constant Magnetic Field. Resonance Effects near the Gyro Frequency. The equations derived here for an isotropic plasma are correct, naturally, in a magnetoactive plasma, provided the frequencies of the waves E_1 and E_2 are much greater than the gyro frequency ($\omega_1^2 \gg \omega_H^2$; $\omega_2^2 \gg \omega_H^2$).

The effect of the magnetic field must, generally speaking, be taken into consideration in the opposite case ($\omega_1^2 \ll \omega_H^2$, $\omega_2^2 \ll \omega_H^2$), which is the most interesting from the point of view of cross modulation in the ionosphere. In fact, for example, the perturbations of the electron temperature in the plasma $\Delta_\Omega T_e$, produced by a strong E_1 wave in the presence of a magnetic field $\mathbf{H}_0$, are given, as previously, by Eq. (3.18), in which we must, however, replace $E_1^2(0)/(\omega_1^2+\nu_0^2)$ by the expression

$$\frac{E_1^2(0)\cos^2\beta}{\omega_1^2+\nu_0^2}+\frac{E_1^2(0)\sin^2\beta}{2[(\omega_1-\omega_H)^2+\nu_0^2]}+\frac{E_1^2(0)\sin^2\beta}{2[(\omega_1+\omega_H)^2+\nu_0^2]}\,, \qquad (3.25)$$

where β is the angle between $\mathbf{E}_1$ and $\mathbf{H}_0$ [compare with (1.26)]. It is clear therefore that when $\omega_1^2 \ll \omega_H^2$ the different components of the field $\mathbf{E}_1$ are not of equal weight, and the principal disturbances in the plasma are produced by the field component of $\mathbf{E}_1$ parallel to $\mathbf{H}_0$ (i.e., the principal role is played by the ordinary wave under conditions of quasi-transverse propagation, see reference 15, Sec. 75). The average value of $\cos^2\beta$ is $1/3$; consequently, when averaged over the entire orientation of $\mathbf{E}_1$ relative to $\mathbf{H}_0$, the magnetic field reduces the disturbances produced in the plasma by the low-frequency E_1 wave by $1/3$.

Further, the depth of the cross modulation of the E_2 wave in the magnetoactive plasma depends essentially on the character of polarization of the wave. Thus, in quasi-transverse propagation, the ordinary

*In the case considered here, of collisions with neutral particles, the closeness of the results of the elementary and kinetic calculations follows from the considerations given in Sec. 2.5b.

wave is not affected at all by the magnetic field. Expressions (3.20) and (3.23) are therefore still valid for the depth of cross modulation of the ordinary E_2 wave, with replacement of $E_1^2(0)$ by $E_1^2(0)\cos^2\beta$. On the other hand, the absorption of the extraordinary wave in quasi-transverse propagation when $\omega_H^2 \sin^2\alpha \gg \omega_0^2 = 4\pi e^2 N/m$, is known (see reference 15, Sec. 75) to be given by the same equations as the absorption of a wave in an isotropic plasma, except that the wave frequency ω_2 must be replaced by $\omega_H \sin\alpha$ where α is the angle between the normal to the front of the wave and the direction of the magnetic field. A suitable replacement of ω_2 by $\omega_H \sin\alpha$ should be carried out also in expressions (3.20) and (3.23) for the depth of cross modulation. Thus, the magnetic field reduces the depth of the cross modulation for the extraordinary wave E_2 substantially (since $\omega_H^2 \sin^2\alpha \gg \omega_2^2$ in quasi-transverse propagation) but affects the depth of cross modulation of the ordinary wave only in connection with the appearance of the factor $\cos^2\beta$.

Let us discuss now the question of resonant effects near the gyro frequency,* i.e., when $\omega_1 \sim \omega_H$.[70,110,114a,120] We assume first that the interfering wave propagates longitudinally. It is then enough to discuss only the extraordinary interfering wave, since the disturbances produced by the ordinary wave have no resonant properties. In this case the disturbances produced in the plasma by the extraordinary wave are given, as before, by Eq. (3.18) in which, naturally, it is necessary to replace $E_1^2(0)/(\omega_1^2 + \nu_0^2)$ in Eq. (3.25) by $\dfrac{E_{1X}^2(0)}{[\omega_1 - \omega_H)^2 + \nu_0^2]}$, where $E_{1X}(0)$ is the amplitude of the extraordinary wave on the plasma boundary. Analogously, when going from the E_1 wave to the E_{1X} wave, the absorption coefficient κ_1 changes, too. As a result, the transformation (3.22), and hence Eqs. (3.23a) and (3.23), retain in this case their previous form. It is merely necessary to replace in them $E_1^2(0)$ by $E_{1X}^2(0)$, and also to replace $1 - \exp\{-2K_1^0\}$ by the factor $F = F_1(K_{1X}^0)$, shown in Fig. 16 (since the point of reflection of the wave E_2 is usually lower than the point of reflection of the extraordinary wave E_{1X} propagating along the magnetic field); K_{1X}^0 is the total absorption of the E_{1X} wave up to the point of reflection of the E_2 wave.

It follows from (3.23) that the depth of cross modulation at the gyro frequency (i.e., when $\omega_1 = \omega_H$) does not exceed the depth of cross modulation on any other frequency ω_1, provided total cross modulation is produced in the latter case (i.e., $K_1 \gtrsim 1$). The reason for this lies in the fact that although the extraordinary wave E_{1X} does indeed produce very strong disturbances in the plasma when $\omega_1 \cong \omega_H$, it is itself attenuated within a very thin layer. To the contrary, if the frequency ω_1 differs substantially from the gyro frequency, then although the disturbances it produces are considerably weaker than when $\omega_1 \cong \omega_H$, the disturbed

*As far as we know, resonant effects at $\omega_2 \cong \omega_H$ have not been discussed in the literature.

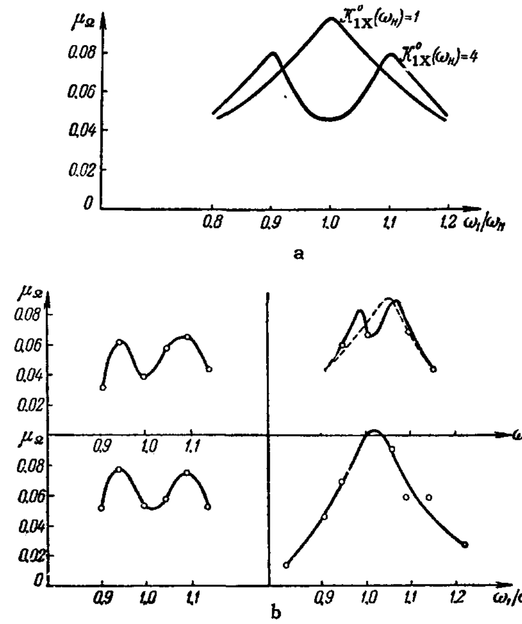

FIG. 18

layer is correspondingly much thicker. Therefore the summary ("total") depth of cross modulation is the same in either case (provided the E_2 wave passes through the entire disturbed layer, i.e., $K_1^0 \gtrsim 1$).

At the same time, if the E_2 wave propagates in a sufficiently thin layer of plasma (i.e., if the frequency ω_2 is low or the angle of incidence ψ_2 on the layer is large), then total cross modulation will be produced only for that E_{1X} wave whose frequency is close to the gyro frequency (for only such a wave attenuates sufficiently within a thin layer of plasma). In addition, as is clear from (3.23), under these conditions the "total" depth of cross modulation can assume relatively large values, since $\mu_\Omega \sim 1/\omega_2^2 \cos\psi_2$. Therefore, at low frequencies ω_2 or large angles ψ_2, the depth of cross modulation has a clearly pronounced resonant peak as the frequency ω_1 is varied near the gyro frequency.* This is seen from Fig. 18a, which shows the dependence of μ_Ω on ω_1, calculated for the ionosphere from Eq. (3.23), for $\omega_1 \sim \omega_H = 9.6 \times 10^6$, $\omega_2 = 3.7 \times 10^6$, $\cos\psi_2 = 0.23$, and $w = 36$ kw. It is seen from the diagram that the resonant frequency has a single hump or double hump, depending on the value of $K_{1X}^0(\omega_H)$, the total absorption of the E_{1X} wave (when $\omega_1 = \omega_H$) up to the point of reflection of the E_2 wave; the double hump is clearly pronounced when $K_{1X}^0(\omega_H) \gg 1$.

We have considered the resonance of the cross modulation only when the interfering wave propagates longitudinally. In the case of nonlongitudinal propaga-

*When the E_2 wave has a high frequency, the depth of cross modulation is relatively small. It follows therefore, in particular, that the changes in the modulation of the ordinary wave in the plasma, induced by the extraordinary wave, are also small when $\omega_1 \cong \omega_H$ (since the frequency of the ordinary wave is high, of the order of ω_H). For example, in the ionosphere the depth of modulation of the ordinary wave is changed by the extraordinary wave, as is clear from (3.23), by not more than 1% if the power of the transmitting station does not exceed 100 kw (see Sec. 3.2d).

tion, owing to the influence of the polarization of the plasma, the resonant effects occur not at the gyro frequency ω_H, but at a frequency

$$\omega_{res} = \sqrt{\frac{\omega_H^2+\omega_0^2}{2} \pm \sqrt{\frac{(\omega_H^2+\omega_0^2)}{4} - \omega_H^2\omega_0^2\cos^2\alpha}}.$$

Here $\omega_0 = \sqrt{4\pi e^2 N/m}$ is the plasma (Langmuir) frequency, and α is again the angle between the normal to the front of the wave and the direction of the magnetic field (see reference 14, Sec. 75).

The frequency ω_{res} varies from $\omega_{res} = \omega_H$ (when $\alpha = 0$, i.e., in longitudinal propagation) to $\omega_{res} = \sqrt{\omega_H^2 + \omega_0^2}$ (when $\alpha = \pi/2$, i.e., in transverse propagation).

It must be kept in mind, however, that at the resonant frequency the corresponding wave usually is attenuated at the very beginning of the plasma layer. Therefore, if the layer does not have a sharp boundary, the disturbing wave is completely attenuated in the region of small ω_0 (such conditions are realized, in particular, in the ionosphere). In this case, the difference in cross modulation for nonlongitudinal propagation from that for longitudinal propagation is slight near resonance (for example, in the ionosphere the resonant frequency is shifted by at most 1 or 2%[121,120]).

3.4. Results of Experimental Investigations of Cross Modulation in the Ionosphere

a) Absolute Magnitude of the Depth of Cross Modulation. Cross modulation is observed in the ionosphere at night in the medium-wave band ($\lambda \sim 200$ to 2000 m), and the interaction between waves takes place in the lower part of the E layer, at an altitude of 80 to 100 km. The greatest value of the depth of cross modulation, $\mu_{\Omega\,max}$, is reached (from average data) under the following conditions: $300 \le \Omega \le 500$; $\omega_1 \sim 10^6$; $\omega_2 \sim 5 \times 10^6$; $\cos\psi_2 \sim 0.25$; $\cos\psi_1 \sim 0.7$; $r_1 = 140$ km (r_1 is the distance from the interfering station to the interaction region).[104] When the interfering-station power is $w = 100$ kw and $\mu_0 = 1$, a value $\mu_{\Omega\,max} \cong 0.05$ is obtained. Calculation based on (3.23) (with allowance for the magnetic field) yields, under the same conditions, $\mu_{\Omega\,max} \cong 0.045$, which is in good agreement with experiment.

b) Dependence of μ_Ω and φ_Ω on μ_0 and Ω. The dependence of μ_Ω on μ_0, based on the experimental data, is always linear,[107,108] in agreement with the theoretically deduced Eqs. (3.20) and (3.23).

The dependence of the phase of the cross modulation on the modulation frequency was investigated in references 107 and 108. The experimental data are in good agreement with the theory (see Fig. 15).* The value of $\delta\nu_0$ measured in these experiments is in the range $10^3 \lesssim \delta\nu_0 \lesssim 2 \times 10^3$; consequently, the electron-collision frequency in the interaction region (i.e., at an altitude of 80 to 85 km) is $\nu_0 \sim 5 \times 10^5$ to 10^6.

The dependence of the depth of cross modulation on the modulation frequency, according to references 107 and 108, is in good qualitative agreement with the deductions of the theory (see Fig. 15), but the deviations from the theoretical curve (3.23) are greater here. It is noted firstly that μ_Ω falls off more rapidly at large Ω (reference 105), and secondly that the depth of cross modulation diminishes somewhat as $\Omega \to 0$.[108,111] The first of the foregoing peculiarities has apparently a simple explanation[105] (see Sec. 3.5b), while the cause of the second (low-frequency) anomaly is unclear.

c) Dependence of μ_Ω on the Power of the Interfering Station. The dependence of the cross modulation on the radiating power of the interfering station was investigated in references 105, 107, 108, and 110; it was found there that μ_Ω is proportional to w up to the maximum radiation power employed (w = 560 kw). Calculations for the conditions under which the experiments were performed show that in these experiments the E_1 wave was not strong [the ratio $(E_1(0)/E_{p1})^2$ is always small, less than 0.5], so that these experimental results agree with the theory.[120]*

d) Dependence of μ_Ω on the Frequencies ω_1 and ω_2. The dependence of the depth of cross modulation on the frequency of the interfering wave and on the frequency of the E_2 wave is greatly influenced by the earth's magnetic field. An important role is played here by the character of the polarization of the wave, and it is therefore necessary to separate in the experiment the ordinary and extraordinary components. In particular, the depth of cross modulation of the ordinary component of the E_2 wave in quasi-transverse propagation should be proportional to $1/\omega_2^2 \cos\psi_2$ (provided the point of reflection of the E_2 wave lies above the point of reflection of the interfering wave; when $\omega_1 \sim 10^6$, such a dependence can apparently be readily observed experimentally).

So far, no special measurements of the dependence of μ_Ω on ω_1 and ω_2 have been made (with the exception of the gyro resonance region). According to averaged data, the depth of cross modulation μ_Ω has a maximum at $\omega_1 \sim 10^6$ and $\omega_2 \sim 5 \times 10^6$; it diminishes by a factor of 2 or 3 as ω_2 is decreased to 10^6 or as ω_1 is increased to 3×10^6. When $\omega_1 < 10^6$ and $\omega_2 < 10^6$, no cross modulation is observed. At high

*We note that although the calculated kinetic-theory dependence of φ_Ω on $\Omega/\delta\nu_0$ leads only to a small discrepancy in the results of the elementary theory (the values of φ_Ω differ by at most 20%), it may still be possible to verify this discrepancy experimentally.[120]

*It is indicated in reference 120 that as the power of the interfering station is varied up to 500 kw, one can already observe an essential nonlinearity in the dependence of μ_Ω on ω. For this purpose, however, it is necessary that: a) the region of interaction be as close as possible to the interfering station (above the interfering station); b) the E_2 wave be an ordinary one and if possible of low frequency, $\omega_2 \sim 10^6$ [for it is precisely in this case that the nonlinearity should be particularly clearly pronounced (see Fig. 17)]; c) the measurements be performed at a low modulation frequency $\Omega \ll \delta\nu_0$; d) the measurements be performed at sufficiently large number of values of w, and several measurements must be made at small w, approximately 50 to 100 kw (to display the behavior of the linear plot of μ_Ω vs. w).

values of ω_1 and ω_2, μ_Ω diminishes rapidly with increasing frequency (with the exception of the gyroresonance region); the maximum frequency of the E_2 wave, at which cross modulation had been observed, was $\omega_2 \sim 2.5 \times 10^7$; the maximum value of ω_1 was approximately 10^7 (reference 104).

e) Resonance of Cross Modulation at $\omega_1 \cong \omega_H$. The dependence of the depth of cross modulation on the frequency of the interfering wave at $\omega_1 \cong \omega_H$ was investigated in references 110, 111, 113, and 114. It was observed in references 113 and 114 that the depth of cross modulation increases resonantly near the gyro frequency, whereas no considerable increase in depth of modulation was observed in references 110 and 111. The absolute value of the maximum depth of cross modulation as a function of the frequency ω_2, according to the data of all these investigations,[110,111,113,114,105] is shown in Fig. 19, while the course of the resonant curve $\mu_\Omega(\omega_1)$ is shown in Fig. 19b (from the data of reference 114). The solid line in Fig. 19 and the curves of Fig. 18a have been calculated from (3.23) for the conditions under which the experiments were performed. All the experimental results are in good agreement with the theory.

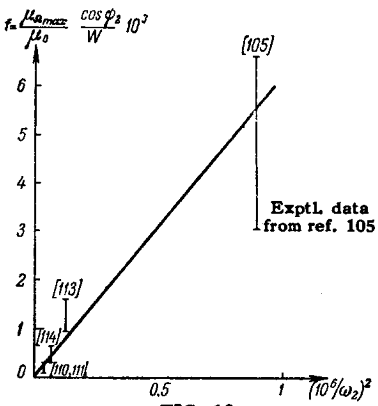

FIG. 19

3.5. Nonlinear Interaction of Unmodulated Radiowaves

a) Change in the Conditions of Propagation of the E_2 Wave. A strong unmodulated E_1 wave produces in the plasma electrons time-invariant temperature changes, which influence, in particular, the absorption of other waves propagating in the disturbed region. The amplitude of the E_2 wave passing through the disturbed region of the plasma is naturally represented in the form

$$E_2(z) = E_2(0) \exp\{-K_2(z)\} P_{12},$$

where $K_2(z)$ is the absorption of the E_2 wave in the case of the unperturbed plasma, and P_{12} is a factor that accounts for the effect of the interfering E_1 wave on the amplitude of the E_2 wave. The factor P_{12} is calculated in the same manner as the self-interaction factor (3.9); the resultant expression for P_{12} is[20]

$$P_{12} = \exp\left\{\sqrt{\frac{\varepsilon_1(0)}{\overline{\varepsilon_2(z)}}}\left[\frac{\omega_1^2+\nu_0^2}{\omega_2^2+\nu_0^2}\ln\frac{\tau_1+1}{\tau_0+1}\right.\right.$$
$$\left.\left.+\left(1-\frac{\omega_1^2+\nu_0^2}{2(\omega_2^2+\nu_0^2)}\right)\ln\frac{\omega_2^2+\nu_0^2\tau_1^2}{\omega_2^2+\nu_0^2\tau_0^2}+\frac{2\nu_0^2(\tau_0-1)}{\omega_2^2+\nu_0^2}\right]\right\}. \qquad (3.26)$$

Here, as usual, $\tau_0 = \sqrt{T_e(E_1(0))/T}$ is the value of τ at the start of the interaction region, i.e., on the plasma boundary, while τ_1 is the value at the end of the interaction region; $\tau(E_1)$ is again given by Eqs. (3.8) and (1.24). Further, $\epsilon_1(0)$ is the value of the dielectric constant for the E_1 wave on the plasma boundary, while $\overline{\epsilon_2(z)}$ is the average value of the dielectric constant for the E_2 wave in the interaction region.

No detailed analysis was made of the magnitude of the factor P_{12} and of its dependence on the parameters. We note merely that in the case of high frequencies ($\omega_2^2 \gg \nu_0^2$, $\omega_1^2 \gg \nu_0^2$) the factor P_{12} is very simply related to the previously-investigated self-interaction factor P of the interfering wave: $P_{12} = (P)^{\sqrt{\epsilon_1}\,\omega_1^2/\sqrt{\overline{\epsilon_2}}\,\omega_2^2}$. It is clear therefore that the absorption of the E_2 wave can change in some cases substantially under the influence of the interfering wave (see Sec. 3.1). We note that if either E_1 or E_2 is an unmodulated pulse, then Eq. (3.26) can be used only when the pulse duration is longer than $1/\delta\nu_0$. In the opposite case it becomes necessary to take into account the fact that there is not enough time to establish a stationary electron temperature. Certain calculations of the interaction of pulses were carried out in references 123 and 125.

Interaction of unmodulated pulses in a plasma was also observed experimentally under laboratory conditions,[122–124] as well as in the ionosphere.[125,125a] It is important to emphasize that reduction of the results of these observations yields the following essential parameters: the electron collision frequency, ν_0, and the fraction of the transmitted energy δ. In addition, if the duration of the pulse is sufficiently small, we can even localize the region of interaction between pulses in the ionosphere. Nonlinear interaction between short pulses may thus prove to be a very effective method of investigating the ionosphere.[125,126*]

b) "Sideband" Waves (Waves with Combination Frequencies). A strong unmodulated wave E_1 produces in a plasma not only constant disturbances, but also alternating ones, at frequencies that are multiples of ω_1. These give rise to combination-frequency waves whenever radio waves interact in a plasma.[57]

The frequency of the time-variable electron-temperature disturbances, as is clear from (1.13), is twice the frequency of the interfering wave ($2\omega_1$). A similar time variation holds naturally for the electron-collision frequency and for the conductivity of the plasma in the case of any other wave $E_2 = E_{20}\exp(i\omega_2 t)$, propagating in the disturbed region, i.e.,

*We note that the formulas used in reference 125 to reduce the experimental data were not always sufficiently correct.

$$\sigma=\sigma_0+\sigma_1^+e^{2i\omega_1 t}+\sigma_1^- e^{-2i\omega_1 t}, \quad (3.27)$$

where σ_1^+ and σ_1^- are the amplitudes of the alternating perturbations of the conductivity. Calculation of the amplitudes σ_1^+ and σ_1^-, carried out in references 57, 127, and 20 with the aid of kinetic theory, leads in the general case to rather complicated expressions.* These expressions become simple in the case of high frequencies ω_1 and ω_2 ($\omega_1^2 \gg \nu_0^2$, $\omega_2^2 \gg \nu_0^2$); in this case, if $(\omega_2 - 2\omega_1)^2 \gg \nu_0^2$, then in the case of collisions with molecules

$$\sigma_1^+=\sigma_1^-=-\frac{3e^4NE_1^2(z)\,\nu_0}{80m^2\omega_1^2kT\tau}\left(\frac{1}{(\omega_2+2\omega_1)^2}+\frac{1}{(\omega_2-2\omega_1)^2}\right), \quad (3.28)$$

where $E_1(z) = E_1(0)\exp\{-ik_1z\}$ is the field of the E_1 wave in the plasma $[k_1 = \omega\sqrt{\epsilon'}/c = \omega(n-i\kappa)/c]$. To simplify the formulas, the plasma is henceforth considered homogeneous. When $(\omega_2-2\omega_1)^2 \ll \omega_2^2$, a resonance increase takes place in the amplitudes σ_1^+ and σ_1^-; their maximum value (when $\omega_2 = 2\omega_1$) is

$$\sigma_1^+=\sigma_1^-=\frac{2}{15\pi}\,\frac{e^4NE_1^2(z)}{m^2\omega_1^2\nu_0kT\tau^3}\,. \quad (3.29)$$

As usual, τ is given here by Eq. (3.8). In the case of a weak interfering wave, naturally, $\tau \cong 1$. It is important that in a strong field the amplitudes σ_1^+ and σ_1^- are small compared with σ_0, provided condition (1.16) is satisfied, i.e., provided $\delta\nu_0/\omega_1 \ll 1$ (compare with Sec. 3.1).†

Let us determine now the changes in the propagation conditions of the E_2 wave in the presence of high-frequency disturbances in the conductivity of the plasma. In the case when these disturbances are quasi-stationary, i.e., constant or varying sufficiently slowly with time (as occurs, for example, in the case of cross modulation), the usual approximation of geometric optics holds for the amplitude of an E_2 wave propagating in the disturbed region [see (3.19)]:

$$E_2\sim E_2(0)\exp\left\{-\frac{\omega_2}{c}\int_0^z\varkappa_2\,dz\right\}\cong E_2(0)\exp\left\{-\frac{\omega_2}{c}\int_0^z\frac{2\pi\sigma_0}{\omega_2\sqrt{\varepsilon_2}}dz\right\}$$
$$\times\left(1-\frac{\omega_2}{c}\int_0^z\frac{2\pi\sigma_1^+}{\omega_2\sqrt{\varepsilon_2}}e^{2i\omega_1 t}dz-\frac{\omega_2}{c}\int\frac{2\pi\sigma_1^-}{\omega_2\sqrt{\varepsilon_2}}e^{-2i\omega_1 t}dz\right), \quad (3.30)$$

where $\kappa_2 = 2\pi\sigma/\omega_2\sqrt{\epsilon_2}$ is the coefficient of absorption of the E_2 wave. It follows from (3.30) that the amplitude of the "sideband" wave of frequency $\omega_2 + 2\omega_1$ is, in the quasi-stationary approximation,

$$E_{21}^+=E_2(0)\exp\left\{-\frac{\omega_2}{c}\int_0^z\frac{2\pi\sigma_0}{\omega_2\sqrt{\varepsilon_2}}dz\right\}\left|\frac{\omega_2}{c}\int_0^z\frac{2\pi\sigma_1^+}{\omega_2\sqrt{\varepsilon_2}}dz\right|, \quad (3.31)$$

where σ_1^+ is the amplitude of the alternating disturbances of the plasma conductivity, as determined above.

It is important, however, that when unmodulated radio waves interact, the disturbances in the conductivity vary rapidly with time, generally speaking, and the quasi-stationary approximation can frequently not be employed. It is therefore necessary to investigate the wave equation

$$\frac{1}{c^2}\frac{\partial^2E_2}{\partial t^2}+\frac{\partial^2E_2}{\partial z^2}+\frac{4\pi}{c^2}\frac{\partial j_2}{\partial t}=0, \quad (3.32)$$

which describes the propagation of the E_2 wave in the general case. In (3.32) it is assumed for simplicity that the properties of the plasma change only in the z direction and that the E_2 wave propagates in the same direction. We assume, furthermore, that the E_2 wave is of high frequency ($\omega_2^2 \gg \nu^2$), so that the alternating disturbances of the ϵ can be neglected. In this case

$$\frac{\partial j_2}{\partial t}=\frac{\partial}{\partial t}\left(\sigma E_2+\frac{\partial P_2}{\partial t}\right)=\frac{\partial}{\partial t}(\sigma E_2)+\frac{\varepsilon-1}{4\pi}\frac{\partial^2E_2}{\partial t^2}=\sigma_0\frac{\partial E_2}{\partial t}$$
$$+\sigma_1^+\frac{\partial}{\partial t}(E_2e^{2i\omega_1 t})+\sigma_1^-\frac{\partial}{\partial t}(E_2e^{-2i\omega_1 t})+\frac{\varepsilon-1}{4\pi}\frac{\partial^2E_2}{\partial t^2}\,. \quad (3.33)$$

Substituting (3.33) in the wave equation (3.32), we can solve this equation by successive approximation (using the fact that $|\sigma_1^\pm| \ll \sigma_0$).* This solution has the form

$$E_2=E_2(0)\exp\{-ik_0z\}\,e^{i\omega_2 t}$$
$$+E_{21}^+(z)^{i(\omega_2+2\omega_1)t}+E_{21}^-(z)\,e^{i(\omega_2-2\omega_1)t}, \quad (3.34)$$

where

$$E_{21}^+(z)=-\frac{2\pi(\omega_2+2\omega_1)}{k_1^+c^2}E_2(0)\left\{e^{ik_1^+z}\int_z^\infty\sigma_1^+(z)\,e^{-i(k_1^++k_0)z}dz\right.$$
$$\left.+e^{-k_1^+z}\left(\int_0^z\sigma_1^+(z)\,e^{i(k_1^+-k_0)z}dz-\int_0^\infty\sigma_1^+(z)\,e^{-i(k_1^++k_0)z}dz\right)\right\},$$

$$k_0=\frac{\omega_2}{c}(n_2-i\varkappa_2)=\frac{\omega_2}{c}\sqrt{\varepsilon(\omega_2)-i\frac{4\pi\sigma(\omega_2)}{\omega_2}}\,;$$
$$k_1^+=\frac{\omega_2+2\omega_1}{c}\sqrt{\varepsilon(\omega_2+2\omega_1)-i\frac{4\pi\sigma(\omega_2+2\omega_1)}{\omega_2+2\omega_1}}\,. \quad (3.35)$$

An analogous expression is obtained for the second "sideband" wave $E_{21}^-(z)$ (for more details see reference 15, Sec. 64, and references 57 and 20).†

Comparing now the exact solution of the wave equation (3.35) with the quasi-stationary approximation (3.31), let us determine the conditions of applicability of the quasi-stationary approximation:

*In calculations of high-frequency plasma disturbances, and in particular of σ_1^+ and σ_1^-, it is necessary to take into account not only the alternating disturbances of the symmetrical portion of the distribution function f_0, but also the asymmetrical function f_1 (see Sec. 2). In other words, the system of (2.7) and (2.20) is inadequate in this case, and the more complete system (2.5) must be used. This was not taken into account in references 57 and 127 and an error, by a factor 1.8, in the calculation of the amplitudes σ_1^+ and σ_1^- resulted for an isotropic plasma (see reference 20). As far as we know, no analysis of a magnetoactive plasma has been made with allowance for the function f_1.

†In a weak field the ratio σ_1/σ_0 is naturally even smaller; thus, we have $\frac{\sigma_1}{\sigma_0}\sim\delta\left(\frac{E_1}{E_{p1}}\right)^2\sim\frac{e^2E_1^2}{kTm\omega_1^2}$ when $\omega_1^2\gg\nu_0^2$ and $\frac{\sigma_1}{\sigma_0}\sim\frac{\delta\nu_0}{\omega_1}\left(\frac{E_1}{E_{p1}}\right)^2\sim\frac{e^2E_1^2}{kTm\omega_1\nu_0}$ when $\omega_1^2\ll\nu_0^2$.

*In this problem the boundary conditions have the form $E_2 = E_2(0)\exp\{i\omega_2 t\}$ when $z = 0$ and $E_2 \to E_2(t)\exp\{-ikz\}$ as $z\to\infty$.

†We note that in reference 15 the problem was formulated somewhat differently from the very outset: the field of the wave E_1 was considered homogeneous in space (field in a capacitor).

$$2\omega_1 \ll \omega_2, \tag{3.36a}$$

$$\Delta z\,(k_1^+ - k_0) \cong \frac{4\omega_1 \Delta z}{c} \ll 1, \tag{3.36b}$$

where c is the velocity of light and Δz are the dimensions of the region of interaction of the waves E_1 and E_2 [in the last condition, (3.36b), we put for simplicity $\epsilon \cong 1$]. In the ionosphere, the condition (3.36b) is not satisfied for the radio band [in fact, if it is assumed that $\Delta z \sim 10$ km, condition (3.36b) is satisfied only when the frequency of the interfering wave is less than 1 kcs]. Consequently, in analyzing the interaction between unmodulated radio waves in the ionosphere it is always necessary to use the exact solution (3.34) and (3.35), and not the quasi-stationary approximation (3.31). For cross modulation, the frequency $2\omega_1$ must be replaced in the conditions (3.36) by the modulation frequency Ω. In this case condition (3.36b) is not satisfied only for high modulation frequencies, $\Omega \gtrsim 10^4$, which apparently causes the rapid fall-off in the depth of cross modulation at high frequencies noted in Sec. 3.4b.

Estimates of the ratio of the amplitude of the "sideband" wave (3.35) to the amplitude of the principal wave E_2 for radio waves in the ionosphere show that when the interfering station has a power of 100 kw and when $\omega_1 \approx \omega_2 \sim 10^6$ to 10^7,

$$\eta = \left|\frac{E_{21}(z)}{E_2(z)}\right| \sim 10^{-2} \frac{e^4 E_1^2 N \nu_0}{m^2 \omega_1^2 kT (\omega_2 - 2\omega_1)^2} \sim 5\cdot 10^{-6} \text{ to } 10^{-8}. \tag{3.37}$$

Under resonance conditions, when $\omega_2 \cong 2\omega_1$, the value of η increases by one-and-one-half or two orders of magnitude. For example, when $\omega_2 = 2\omega_1 = 4\times 10^6$ we have $\eta \cong 3\times 10^{-5}$, whereas when $\omega_1 = \omega_2 = 2\times 10^6$, we have $\eta \cong 10^{-6}$. No "sideband" waves were experimentally observed.

c) Nonlinear Effects Connected with Changes in the Electron Concentration. The nonlinear phenomena previously considered were all due, in final analysis, to a change in the electron velocity under the influence of the field. Corresponding changes were produced, for the same reason, in such quantities as T_e, ν_{eff}, σ, and ϵ. The nonlinear interaction between waves in a plasma (and, in general, the nonlinear dependence of the current $\mathbf{j}_t$ on the field $\mathbf{E}$) may, however, have also an entirely different nature — it may be due to a change in the electron concentration (see reference 128). Actually, the tensor $\epsilon'_{ik} = \epsilon_{ik} - i4\pi\sigma_{ik}/\omega$ depends on N; for example, for an isotropic plasma $\epsilon'_{ik} = \epsilon'\delta_{ik}$, and $\epsilon' = \epsilon - i4\pi\sigma/\omega = 1 - 4\pi e^2 N/m\omega(\omega - i\nu)$. Therefore under conditions when the electric field causes a certain change ΔN in the electron concentration, the properties of the plasma also begin to depend on the field, i.e., the medium becomes nonlinear.

Assuming that the ions are immobile and merely compensate for the equilibrium electron charge eN, the value of $e\Delta N$ is obviously merely equal to the density of the average microscopic charge $\bar{\rho}$ of the plasma. In turn*

$$\bar{\varrho} = e\Delta N = \frac{1}{4\pi}\,\mathrm{div}\,\mathbf{E}. \tag{3.38}$$

Thus, the nonlinear effect considered here always exists, disregarding the quantitative aspect of the matter for the time being, whenever $\mathrm{div}\,\mathbf{E} \neq 0$.

In a homogeneous field, naturally, $\mathrm{div}\,\mathbf{E} = 0$. The same takes place in the propagation of transverse (electromagnetic) waves in a homogeneous and isotropic plasma. But even for plasma (longitudinal) waves in a homogeneous and isotropic plasma, $\mathrm{div}\,\mathbf{E} \neq 0$. The result is, for example, the scattering of electromagnetic waves by plasma waves and in general a nonlinear interaction of waves of both types (see references 129-132). In an isotropic but inhomogeneous plasma, $\mathrm{div}\,(\epsilon'\mathbf{E}) = 0$ [this follows, for example, from the field equation $\mathrm{curl}\,\mathbf{H} = (i\omega/c)(\mathbf{D} - i4\pi\mathbf{j}/\omega) = (i\omega/c)\,\epsilon'\mathbf{E}$, where "sideband currents" are assumed absent]. Therefore, considering also (3.38),

$$\Delta N = -\frac{\mathbf{E}\,\mathrm{grad}\,\varepsilon'}{4\pi e\varepsilon'}. \tag{3.39}$$

In a magnetoactive plasma $\mathrm{div}\,\mathbf{E}$ is generally speaking not equal to zero even for a homogeneous medium. The point is that in an unisotropic medium $\mathrm{div}\,(\mathbf{D} - i4\pi\mathbf{j}/\omega) = \partial(\epsilon'_{ik}E_k)/\partial x_i = 0$ and consequently, $\mathrm{div}\,\mathbf{E} \equiv \partial E_k/\partial x_k \neq 0$ even when $\epsilon'_{ik} = \mathrm{const}$. In particular, for a plane electromagnetic wave in a homogeneous magnetoactive medium, whose general form is, by virtue of its electric polarization, $\mathbf{E} = \mathbf{E}_{0a}\cos\varphi + \mathbf{E}_{0b}\sin\varphi$, we have

$$\Delta N = \frac{\mathrm{div}\,\mathbf{E}}{4\pi e} = -\frac{\omega n}{4\pi ec}\{E_{0a}\cos\theta_a \sin\varphi + E_{0b}\cos\theta_b\cos\varphi\}, \tag{3.40}$$

where $\varphi = \omega t - \mathbf{k}\cdot\mathbf{r}$, $n = ck/\omega$ is the index of refraction for the considered ordinary or extraordinary wave (we neglect absorption, for the sake of simplicity), and θ_a or θ_b are the angles between E_{0a} or E_{0b} and $\mathbf{k}$, respectively. If the external magnetic field $\mathbf{H}_0$ is not too weak, or if no exclusive propagation directions (for example, along the field $\mathbf{H}_0$) are considered, then $\cos\theta_a$ and $\cos\theta_b$ on the order of unity (the angles θ_a and θ_b can be readily calculated from known formulas; see, for example, reference 15, Secs. 62 and 75). In such cases the effect of (3.40) is greater than the effect of (3.39), if the length $\lambda\!\!\!^{-} = cn/\omega = \lambda/2\pi$ is less than the characteristic length L on which the properties of the plasma change significantly (in other words, $L \sim |\epsilon'/\mathrm{grad}\,\epsilon'|$).

Thus, a field E_1 of frequency ω_1 changes ϵ and σ in the plasma by amounts on the order of $\Delta\sigma/\sigma$ $\Delta\epsilon/\epsilon \sim \Delta N/N \sim E_1/4\pi eNL$ or $\Delta\sigma_{ik}/\sigma_{ik} \sim \Delta\epsilon_{ik}/\epsilon_{ik}$ $\sim \omega n E_1/4\pi ecN$ (it is assumed for simplicity that $\epsilon \sim \epsilon_{ik} \sim 1$). The frequency of these changes coincides, naturally, with the field frequency ω_1. If another wave E_2,

*The charge $\bar{\rho}$ is not equal to the density of the "free charge" $\rho = \mathrm{div}\,\mathbf{D}/4\pi$, usually introduced in electrodynamics. The charge ρ in a plasma can differ from zero whenever, say, a charged body is introduced. In addition, it is clear from the continuity equation $\partial\rho/\partial t + \mathrm{div}\,\mathbf{j} = 0$ that $\rho \neq 0$ when $\mathrm{div}\,\mathbf{j} \neq 0$.

We arrive at (3.38) directly by averaging the equation $\mathrm{div}\,\mathbf{E}_{micro} = 4\pi\rho_{micro}$ for the microfield, since in our case $\mathbf{E}$ is always precisely the average macroscopic field, $\mathbf{E} = \overline{\mathbf{E}}_{micro}$ so that $\bar{\rho} = \bar{\rho}_{micro}$ is obviously equal to $\overline{e\Delta N}$.

of frequency ω_2, propagates in a medium disturbed by E_1, "sideband" waves are produced with frequencies $\omega_2 \pm \omega_1$. In the case discussed in Sec. 3.5b, the frequency of the "sideband" waves was $\omega_2 \pm 2\omega_1$. This is obviously connected with the fact that the effect described by (3.39) and (3.40) is linear in the field E_1, whereas the changes in the random electron velocities and the resultant changes in ν_{eff}, ϵ, and σ depend only on E_1^2.

The ratio of the amplitude of the "sideband" waves, produced by a change in the electron concentration, to the amplitude of the sideband waves of (3.35) is of order ξ where

$$\xi \sim \frac{kT_e m\omega_1^2 n}{ce^3 N E_1(z)} \sim \frac{n\omega_1^2}{\omega_0^2} \sqrt{\frac{kT_e}{\delta mc^2}} \left(\frac{E_{p1}}{E_1(z)}\right); \qquad (3.41)$$

here n is the index of refraction for the E_1 wave, $\omega_0 = \sqrt{4\pi e^2 N/m}$ is the plasma frequency, and E_{p1} is "plasma" field (1.23) for the E_1 wave. If $\sqrt{kT_e/\delta mc^2} \sim 10^{-2}$ ($T_e \sim 500°$, $\delta \sim 10^{-3}$), $\omega_1^2/\omega_0^2 \sim 1$ to 10, and $E_{p1}/E_1(z) \sim 1$, then the $\xi \sim 10^{-1}$ to 10^{-2}, i.e., this effect is one or two orders of magnitude smaller in the ionosphere than the effect discussed in Sec. 3.5b. In other words, the amplitude of the "sideband" harmonic of frequency $\omega_2 \pm \omega_1$ will be from 10 to 100 times weaker than the amplitude of the harmonic of frequency $\omega_2 \pm 2\omega_1$. At the same time, when $\xi > 1$, the harmonic $\omega_2 \pm \omega_1$ is stronger; the condition $\xi \sim 1$ corresponds to a weak field, $E_1(z)/E_{p1} \sim 10^{-1}$ to 10^{-2} (when $\omega_1^2/\omega_0^2 \sim 1$ to 10 and $\sqrt{kT_e/\delta mc^2} \sim 10^{-2}$). If the E_1 wave is amplitude modulated, then the additional cross modulation of the wave E_2, produced by the effect (3.40), has a depth on the order of the ratio of the amplitude of the "sideband" wave of frequency $\omega_2 \pm \omega_1$ to the amplitude of the E_2 wave. It is easy to see that even when $E_1(z)/E_{p1} \sim 1$ this depth of cross modulation is very small, on the order of 10^{-6} to 10^{-8} [$\mu' \sim \eta \cdot \xi$, where η and ξ are defined by (3.37) and (3.41) respectively]. This result (the smallness of μ') is quite understandable, for the change of concentration with the field, unlike the change in the electron temperature, does not have a dc component or one slowly varying with time.

CONCLUDING REMARKS

In future research on nonlinear phenomena in a plasma located in an electromagnetic field, it is appropriate to note the following.

The problem of the behavior of a stationary plasma in a homogeneous field of arbitrary frequency and intensity can be considered, in general, as solved. Future work should therefore be concentrated on nonstationary processes. Even under stationary conditions, however, it is necessary to investigate in more detail the behavior of the electrons in a plasma, with allowance for inelastic processes. It is also desirable to obtain a more exact form for the electron distribution function in the region of high velocities, $v \gg \sqrt{kT_e/m}$, and to calculate the high-frequency corrections proportional to δ (these terms have been calculated only in the absence of a constant magnetic field). These latter problems are essentially connected with the same factor — the need of going beyond the confines of the system of equations (2.7) and (2.20), i.e., of taking into account the function f_2 etc in the general expansion (2.5).

The theory of propagation of waves in a plasma with allowance for nonlinearity is, generally speaking, in a less satisfactory state. The analysis has been carried out here only in the approximation of geometric optics, whereas for many interesting cases (for example, for the propagation of long waves in the ionosphere), geometric optics is inadequate or even utterly inapplicable. Further study of nonlinear effects on the propagation of radio waves in a plasma (ionosphere) is hindered, however, by imperfections in the theory, and by lack of sufficiently reliable and complete experimental data. It is necessary above all to investigate the singularities in the propagation of strong waves (station power $w \gtrsim 300$ kw at frequencies $\omega \lesssim 5 \times 10^6$) for the purpose of measuring the demodulation of waves and other self-interaction effects (see Secs. 3.1 and 3.2). The cause of resonant demodulation of waves near the gyro frequency still is unclear (see Sec. 3.2d). As regards the nonlinear interaction of radio waves in the ionosphere, particular interest attaches to pulse methods, which permit a detailed investigation of the effects of interaction and thus obtain valuable information on the structure of the ionosphere (see Sec. 3.5a); the same methods can also be used to investigate the self interaction of waves. It is also important to observe "sideband" waves with combination frequencies $\omega_2 \pm 2\omega_1$ and $\omega_2 \pm \omega_1$ (see Secs. 3.5b and c).

In addition to the work on the ionosphere, the investigation and use of nonlinear phenomena in a plasma produced under laboratory conditions assumes at the present time an ever increasing significance. Experiments in this field have been carried out only with short pulses. It is clear, however, from the argument in Sec. 3 that much broader possibilities exist here. Finally, it is obvious that nonlinear phenomena, quite analogous to those described above, could be investigated also in semiconductors (see reference 7).

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