Letter to the Editor

SPECTRAL DISTRIBUTION OF RADIANT ENERGY

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The energy distribution in a continuous spectrum describes mostly the dependence on the wavelength of the spectral density, which is the derivative of the radiant flux P, i.e., the intensity, with respect to the wavelength λ:

$$p(\lambda)=\frac{dP}{d\lambda}.$$

This function (frequently called the spectral intensity) can vary in form. In the special case $p(\lambda)=$ const, the spectrum is usually called a constant-energy spectrum, because equal intervals $\Delta\lambda$ of the spectrum scale contain equal fluxes (Fig. 1).

It is also possible to write down the energy distribution in the spectrum as a function of any other quantity which bears a single-valued relationship with the wavelength,[1] taking into account the derivative of the flux with respect to this quantity. If we take the wave number[2] $n = 1/\lambda$ as such a quantity, then the function characterizing the energy distribution in the spectrum will be

$$p(n)=\frac{dP}{dn}.$$

This function is connected with $p(\lambda)$ by the relation

$$p(n)=-\lambda^2 p(\lambda)=-\frac{p(\lambda)}{n^2}.$$

For $p(n)$ = const, (Fig. 2), the spectra can be called constant energy in the same sense as in Fig. 1. However, in a constant-energy spectrum characterized by constant $p(\lambda)$, the energy distribution, which is a function of $p(n)$, has the form shown in Fig. 2,[3] while for $p(n)$ = const, the spectral density $p(\lambda)$ is inversely proportional to the wavelength (Fig. 1).

Let us consider the radiant flux in the interval λ, $\lambda + \Delta\lambda$ of the constant-energy spectrum of Fig. 1:

$$\Delta P=p(\lambda)\,\Delta\lambda;$$

it is represented by the shaded area. In Fig. 2, this same flux, also shown by the shaded area, is given by

$$\Delta P=p(n_{av})\,\Delta n.$$

for sufficiently small $\Delta\lambda$, where

$$\Delta n=\frac{1}{\lambda}-\frac{1}{\lambda+\Delta\lambda}.$$

and n_{av} is the average wave number in this interval. We now change the wavelength λ while keeping the interval $\Delta\lambda$ fixed. We then find the same flux ΔP in the constant energy spectrum of Fig. 1. The previous flux is also obtained for Fig. 2, if we change the interval Δn and the average wave number in correspondence to the new value of the wavelength. If the interval Δn remains unchanged, then the flux found in Fig. 2 will change, increasing with increasing wavelength.

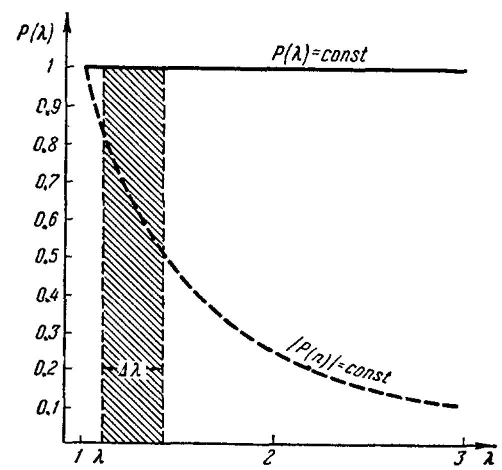

Figure 1

If the energy distribution over the spectrum is such that $p(\lambda)$ has a maximum at some wavelength, then the function $p(n)$ has a maximum at another wavelength. Considering the flux in the interval $\Delta\lambda$, which is unchanged over the entire spectrum, we find the maximum value of this flux in the interval Δn, which is also constant over the spectrum, we find its maximum value at the point of the maximum $p(n)$. Both are, of course, valid: in the first case, we find the maximum flux in the interval $\Delta\lambda$ which is constant over the spectrum, and in the second, in the constant interval Δn. These maxima are at different points in the spectrum, a constant

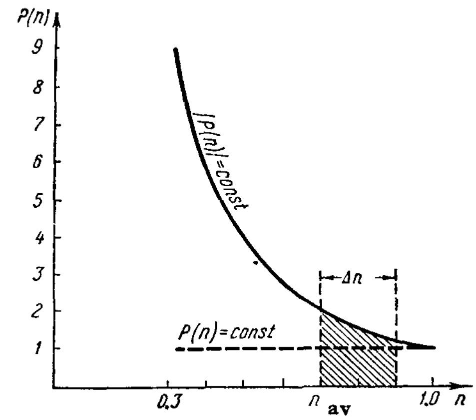

Figure 2

interval $\Delta\lambda$ corresponds to variable intervals Δn over the spectrum, and vice versa.[4]

In the general case, the spectral scale can be represented by some function $f(\lambda)$, while the energy distribution over the spectrum – the spectral intensity – is given by

$$p(f)=\frac{dP}{df}\,. \tag{1}$$

Depending on the form of the function $f(\lambda)$, the spectral density will have different values at a given point in the spectrum (as was pointed out by Fabry[5]), and its maximum will be located at one wavelength, or another, as pointed out by Planck in considering scales of wavelength and frequency.[6] Nevertheless, spectral densities composed in a different fashion are of equal usefulness for the calculation of the flux, since

$$P=\int p(f)\,df. \tag{2}$$

which permits us to find the flux as the area[7] bounded by the curve p(f) and the axis $f(\lambda)$ in the limits under consideration. But the maximum flux, like the flux in the interval d f, takes on different values and is found in the different parts of the spectrum, depending on the form of the spectral scale, i. e., the function $f(\lambda)$. This has led to a search for a spectral scale in which the maximum of the spectral density would be found at the position of the assumed "real" maximum flux.

However, as Gershun[1] has pointed out, such a problem cannot be validly set up, because the flux at some point or other of the spectrum is conceivable only as a flux in a certain range of some particular spectral scale. It is possible that the lack of precision of terminology contributed to an incorrect statement of the problem, as the consequence of which the maximum of the spectral density was called the maximum of the energy curve.[8]

To each point of the spectral scale one can assign a certain radiant flux, by considering the interval f, $f+\Delta f$ of the spectrum and computing the flux $p(f)\Delta f$ in this interval or the ratio of the same flux to the total flux of the entire spectrum

$$\eta=\frac{p(f)\,\Delta f}{P}\,. \tag{3}$$

which has the meaning of the efficiency in the given interval.[9] In the latter case the spectral coordinate f will be assigned the flux ηP. For Δf = const, the efficiency will have a maximum for the same value of f as the density $p(f)$. If $f(\lambda)=\lambda$, then the interval $\Delta\lambda$ should be small in comparison with the wavelength λ; in the opposite case, the flux in it, i.e., the numerator of (3), must be computed in the part of the spectrum under consideration in which the efficiency η is computed, for $\Delta\lambda$ = const. Then the efficiency will have a maximum at the point of the maximum of the spectral density $p(\lambda)$. In particular, for an absolutely black body, the wavelength for which the efficiency is maximum will be determined in this case by the displacement law in the form obtained by Wien.

For an absolutely black body one can study the maximum of the efficiency coefficient η in another manner,[10] by considering η as a function of the temperature T and of the wavelength. By introducing the Stefan-Boltzmann constant σ and the Planck constants c_1 and c_2, we obtain

$$\eta=\frac{c_1}{\sigma}\cdot\frac{\lambda^{-5}T^{-4}\Delta\lambda}{\exp\frac{c_2}{\lambda T}-1}\,.$$

Setting λ = const, we find the temperature T_m for which the efficiency η is a maximum. From

$$\frac{d\eta}{dT}=0$$

it is not difficult to obtain[9]

$$T_m=\frac{C}{\lambda}\,. \tag{4}$$

where

$$C=\frac{c_2}{x}\,.$$

while x is determined by the equation[10]

$$\frac{x}{1-\exp(-x)}=4.$$

Then the maximum efficiency for the wavelength λ is

$$\eta_{\max}=k\,\frac{\Delta\lambda}{\lambda}\,. \tag{5}$$

where

$$k=\frac{c_1}{\sigma}\,\frac{C^{-4}}{\exp x-1}\,.$$

Equation (5) shows that the maximum efficiency η which is possible for each wavelength λ will be the same over the entire spectrum if the relative value $\Delta\lambda/\lambda$ of the interval is constant over the spectrum, the fraction of the total flux of the radiator occurring in which interval is determined by this coefficient. In this case, each wavelength corresponds to a value of T_m of its own, as given by (4). For the wavelength λ, this will be the temperature T_m^1, for which $\eta < \eta_{max}$ for all wavelengths $\lambda \lessgtr \lambda$. Consequently, in the relative interval determined by the wavelength λ', the flux will be maximum for this case, i.e., for a temperature T_m^1. But the constancy of the relative interval $\Delta\lambda/\lambda$ takes place in the spectral scale $f(\lambda) = \ln\lambda$. The flux in an interval of such a scale should be a maximum for that wavelength for which the spectral density is maximum:

$$p(\ln\lambda) = \frac{dP}{d(\ln\lambda)} = \lambda p(\lambda). \tag{6}$$

As a consequence, the spectral density $p(\ln\lambda)$ of the radiation of an absolutely black body at a temperature T_m is a maximum for a wavelength connected with T_m by Eq. (4).[10] The maxima of the efficiency η and of the spectral density $p(\ln\lambda)$ are obtained at the same point in the spectrum only when, as would be expected, we consider the efficiency of the radiator in a relative wavelength interval that is constant over the spectrum, i.e., in a constant interval of the logarithmic spectral scale. This can be obtained directly from the expression for the efficiency. Setting

$$\delta\lambda = \frac{\Delta\lambda}{\lambda} = \text{const}$$

and keeping (6) in mind, we find that

$$\eta = \frac{p(\ln\lambda)}{P}\delta\lambda = \frac{c_1}{\sigma}\cdot\frac{\lambda^{-4}T^{-4}\delta\lambda}{\exp\frac{c_2}{\lambda T} - 1} \tag{7}$$

Differentiation of this expression with respect to λ for T = const leads to (4) and (5) with the previous[4] values of C and k. We note that Eq. (7) differs by only a constant factor from the expression introduced by Worthing[11] for the number of photons in a unit spectral interval emitted per unit time per unit area of a perfectly black body. This number, consequently, is a maximum under those conditions for which the η as given by Eq. (7) is a maximum.

Any spectral scale[1] can be used, the preferable one depending on the problem solved. The energy distribution in the spectrum can be described by the spectral density of radiant flux in an interval of this scale. In a given spectrum, each spectral density expressed as a function of the chosen spectral scale, has, as Gershun[1] pointed out, its own characteristic maximum, if it has one at all. In particular, this refers to the description of the energy distribution in the spectrum of a perfectly black body. Wien's law in its classical expression, which corresponds to the description of the energy distribution by the function $p(\lambda)$, is just as valid as in the other expressions which it obtains in the application of other functions, in particular, $p(\ln\nu)$ or $p(\ln\lambda)$. Although in the latter two cases the maximum of the spectral density occurs at the same wavelength,[3,12] there is no basis for assuming the logarithmic scale to be privileged for the determination of the "maximum energy" in the spectrum, as has become popular in recent publications.[4,10,13] The logarithmic scale is suitable, as Rayleigh has shown, for the graphic representation of spectra, for it makes all octaves of equal length.[14] The frequency scale ν which preserves its values in the transition from one medium to another can be preferable in problems where this is the case.[15] If there are no special conditions, then one should use for the description of the energy distribution in the spectrum the usual scale of wavelength, which is most suitable in that corresect data are customarily given in it.

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