

*SOME PROBLEMS IN THE PROPAGATION OF ULTRASONIC WAVES OF FINITE
AMPLITUDE IN LIQUIDS*

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Usp. Fiz. Nauk **68**, 687-715 (August, 1959)

1. INTRODUCTION

RECENTLY, it has become customary to divide acoustic waves into waves of small (actually, infinitesimally small) amplitude and waves of finite amplitude. Such a division is a result of the fact that the set of hydrodynamic equations in the case of waves of infinitesimally small amplitude can be reduced to the linear wave equation, which is not possible in the case of waves of finite amplitude. Waves of finite amplitude occupy the intermediate region between waves of infinitesimally small amplitude and strong shock waves.

Using the one-dimensional Euler equation

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x}$$

as an example, it can be shown that for the wave $v = v_{10} \sin(\omega t - kx)$ the nonlinear term $v dv/dx$ has a maximum value of the order of kv_{10}^2 while the other terms are of order ωv_{10} . The nonlinear term will be of the same order as the linear one for a displacement $v_{10}/\omega = \xi_{10} \sim \lambda$ or for $v_{10} \sim c_0$, or, finally, for sound pressures $p_{10} \sim \rho_0 c_0^2$ which are of the order of the pressure that determines the elasticity of the medium (the atmospheric pressure P_0 in the case of gases or the internal pressure P_* in the case of liquids).

Until comparatively recently, it has been assumed that nonlinear phenomena are not significant in the propagation of sound in liquids, and the fundamental reason for this was considered to be the smallness of the sound pressures (the maximum sound pressure in liquids now obtained experimentally does not exceed several tens of atmospheres) in comparison with the internal pressure in the liquid, which is of the order of several thousand atmospheres.

For a nondissipative medium, the exact Riemann solutions show that different points of the disturbance profile should be propagated in space with different velocities, which depend on the displacement velocity of the particle. The latter velocity is added to the local sound velocity and leads to a distortion of the shape of the disturbance as it propagates. In the finite case this should lead to the formation of a discontinuity. It should be emphasized that in a nondissipative medium the

number of wavelengths out to the discontinuity, is determined, for an initially harmonic wave, by the Mach number,* and for Mach numbers that are sufficiently small, but not zero, the distance to the formation of the discontinuity is finite. Thus, in a nondissipative medium, a harmonic wave even of small amplitude must be considered as a wave of finite amplitude at distances that are large in comparison with the wavelength. In this sense, all disturbances in an ideal medium are disturbances of finite amplitude, because the nonlinear effects accumulate as the wave propagates. The distortion of the waveform, which can be represented as the generation and growth of harmonics during the course of propagation, is slowed down in lossy media by the different nature of the dissipative processes. Therefore, an initially harmonic wave in a dissipative medium, depending on the energy density, can either lead to the formation of an acoustic discontinuity† (at high energy densities) or be so absorbed during its propagation that the nonlinear effects produce no noticeable effect. Thus, in dissipative media, in contrast with the nondissipative ones, conditions can be created that correspond to waves of infinitesimally small amplitude.

Proceeding to real media, it should be pointed out that the absorption per wavelength $\alpha\lambda$ is much greater in gases than in liquids (at a frequency of 1 Mcs in air, $\alpha\lambda \cong 6.8 \times 10^{-3}$; for water, $\alpha\lambda \cong 3.7 \times 10^{-5}$). Liquids, especially those of low viscosity, are closer to nondissipative media in this regard than are gases; therefore, the nonlinear effects in liquids can accumulate as the disturbance progresses.

We also note that in liquids it is technically much simpler to obtain high ultrasonic intensities. Intensities of the order of several watts per sq. cm at frequencies $\sim 0.3 - 5$ Mcs are not difficult

*The Mach number can be defined in the acoustic case as $v_0/c_0 = M_0$, where v_0 is the amplitude of the displacement velocity and c_0 is the velocity of propagation of the wave in the undisturbed medium.

†By acoustic discontinuity we mean the production of a wave close to sawtooth in shape, with a front thickness much less than $\lambda/2$.

to produce in liquids today. A continuous intensity of $\sim 300 \text{ w/cm}^2$ was recently obtained at a frequency of 1.5 Mcs in an unfocused beam for a short time (several seconds). Pulsed operation permits still higher ultrasonic intensities. Focusing systems, when steps are taken to prevent or reduce the effect of cavitation, can produce intensities of the order of several kilowatts/cm² in the focus.

It has been established rather recently that the mechanism of waveform distortion in liquids leads to a significant increase in the dissipation of acoustic energy (this phenomenon has been little studied in gases). For comparatively low intensities of ultrasound (of the order of a few w/cm²), in such low-viscosity liquids as alcohols, the absorption coefficient increases by two orders of magnitude over the absorption coefficient determined for conditions corresponding to waves of infinitesimally small amplitude.

At the present time, a study of various nonlinear effects in liquids has yielded many data which require generalization. In this review, we speak only of the distortion and absorption of waves of finite amplitude, without considering such interesting phenomena as acoustic streaming, cavitation, etc. In this case, chief attention will be paid to distortion in dissipative media and to the increased absorption brought about by this distortion.

2. THEORY OF DISTORTION AND ABSORPTION OF WAVES OF FINITE AMPLITUDE

a) Nondissipative medium. For a general analysis of the equations of hydrodynamics of an ideal liquid, it is appropriate to use the methods of similarity theory. The methods of similarity theory have been applied in acoustical problems to estimate the different terms of the hydrodynamic equations in references 2, 3, and also for the general analysis of the equations.⁴ By means of a linear transformation of variables and parameters, in other words, by changing the scales of the measurements for these quantities, it is possible to find conditions for which the hydrodynamic equations are invariant under such transformations. For example, the transformation of the primed quantities into the unprimed ones

$$\left. \begin{aligned} \frac{\mathbf{v}'}{U} = \mathbf{v}; \quad \frac{x'\Omega}{U} = x; \quad \frac{y'\Omega}{U} = y; \quad \frac{z'\Omega}{U} = z; \\ \Omega t' = t; \quad \frac{p'}{P} = p; \quad \frac{\rho'}{R} = \rho; \quad \frac{c'}{C} = c \end{aligned} \right\} \quad (2.1)$$

(here U , Ω , P , R , C are certain constants, \mathbf{v} is the velocity, x , y , and z are coordinates, t is the time, p the pressure, ρ the density, and c

the velocity of sound), transforms the equation of continuity, Euler's equation, and the adiabatic equation for the primed quantities into the equations

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \rho \mathbf{v} = 0, \quad \rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \nabla) \mathbf{v} \right] = - \frac{C^2}{U^2} c^2 \nabla p, \\ c^2 = \frac{P}{RC^2} \left(\frac{dp}{d\rho} \right). \end{aligned} \quad (2.2)$$

Thus, the equations are invariant relative to the transformation (2.1) if $C^2/U^2 = 1$ and $P/RC^2 = 1$, or, what amounts to the same thing, if the local values of the Mach number $\mathbf{M} = \mathbf{v}/c$ and of the number $E = p/\rho c^2$ are identical. When these numbers are equal and the boundary conditions of the problem are invariant relative to the transformation (2.1), waves propagating in different media will be identical for different values of the wave parameters. It should be noted that arbitrary combinations of these dimensionless numbers will be equal for similar waves, for example, $E/|\mathbf{M}|^2 = p/\rho |\mathbf{v}|^2$ ("the cavitation number"*), and also $\mathbf{M}/E = \rho c \mathbf{v}/p$, etc. Modeling of waves of finite amplitude can be achieved for any two numbers. In a loss-free medium, as was pointed out above, distortion of the waveform takes place during propagation up to the formation of a discontinuity. The presence of two numbers demonstrates that there are two sources of nonlinear distortion, one due to nonlinearities of the continuity and Euler equations, and the other due to the nonlinearity in the equation connecting the density with the pressure.† Expanding the equation connecting density and pressure, $p = \varphi(\rho)$ in a series about the equilibrium density, we obtain

$$p = P_0 + \rho_0 c_0^2 \left(\frac{\rho - \rho_0}{\rho_0} \right) + \frac{\rho_0^2}{2} \frac{dc^2}{d\rho} \Big|_{\rho=\rho_0} \left(\frac{\rho - \rho_0}{\rho_0} \right)^2, \quad (2.3)$$

where $P_0 \equiv \varphi(\rho_0)$ is the pressure and $c_0^2 = \frac{d\varphi}{d\rho} \Big|_{\rho=\rho_0}$ is the velocity of sound propagation in the undisturbed medium.

In the case of shock waves of weak intensity, the change in the entropy can be represented by a quantity of third order of smallness in comparison with the pressure jump, if dissipative processes in the shock layer are not taken into consideration. Therefore, with accuracy up to terms of second order of smallness, we can use the adiabatic equation for gases:

*See, e.g., G. D. Birkhoff, *Hydrodynamics*, Russian translation, IL, 1954, p. 92, [Dover, New York, 1955].

†It should be noted that such a division is to some extent arbitrary, inasmuch as the set of hydrodynamic equations is solved simultaneously. However, it is appropriate for a comparison of liquids and gases.

TABLE I

Liquid	ρ_0 (20°C), g/cm ³	$c_0 \cdot 10^{-3}$ (20°C), cm/sec	$K_0 \cdot 10^{-3}$ (20°C), kg/cm ²	$P_* \cdot 10^{-3}$ (20°C), kg/cm ²	B/A
Mercury	13.595	1.45	256.0 ¹⁾	24.5 ¹⁾	10.5 ²⁾
Glycerine	1.261	1.92	41.5 ¹⁾	4.75 ¹⁾	8.8 ²⁾
Water	0.998	1.49	21.8 ¹⁾	3.25 ¹⁾	6.7 ²⁾ ; 4.8 ⁴⁾ ; 5.06 ⁵⁾
Ethyl alcohol	0.789	1.17	9.1 ¹⁾	1.00 ¹⁾	9.1 ²⁾ ; 8.0 ⁴⁾ ; 10 ⁵⁾
Methyl alcohol	0.791	1.12	7.3 ¹⁾	1.0 ³⁾	7.3 ²⁾ 3); 7.6 ⁴⁾
n-Propyl alcohol	0.806	1.22	12.3 ⁸⁾	1.54 ⁸⁾	8.0 ⁴⁾
n-Butyl alcohol	0.81	1.26	13.2 ⁸⁾	1.57 ⁸⁾	8.4 ⁴⁾
Carbon tetra- chloride	1.59	0.94	9.6 ¹⁾	1.04 ⁸⁾	9.2 ⁴⁾
Carbon bisulfide	1.26	1.16	17.4 ⁸⁾	3.22 ⁸⁾	5.4 ⁴⁾ ; 11.0 ⁸⁾ ; 12±5 ⁷⁾
Benzene	0.879	1.32	10.5 ¹⁾	1.54 ⁸⁾	6.8 ⁴⁾

1) Data of static measurement.⁶
2) B/A computed from $K_0/P_* = B/A$.
3) This value of P_* was determined from the plot of $K = \Phi(p' + P_*)$.⁶
4) Data of a thermodynamic calculation with use of the experimental dependence of the velocity of low-amplitude ultrasound on the temperature and pressure.⁷
5) Data determined from the interaction of two ultrasonic waves.⁸
6) Data used for the comparison of theory with experiment on the absorption of ultrasonic waves of finite amplitude.⁵
7) The same for comparison with harmonics.⁹
8) Calculation from the dynamic modulus $K_0 = \rho_0 c_0^2$; $P_* = K_0 \frac{A}{B}$.

$$p = P_0 \left(\frac{\rho}{\rho_0} \right)^\gamma \approx P_0 + \gamma P_0 \left(\frac{\rho - \rho_0}{\rho_0} \right) + \frac{\gamma(\gamma-1)}{2} P_0 \left(\frac{\rho - \rho_0}{\rho_0} \right)^2 + \dots \quad (2.4)$$

Here P_0 is the atmospheric pressure, ρ_0 is the density of the undisturbed medium, and $\gamma = c_p/c_v$ is the ratio of specific heats at constant pressure and volume.

The equation of state for liquids is more complicated; however, for the compressions in an acoustic wave of finite amplitude, it is satisfactorily represented in the form⁵

$$p = P_* + A \left(\frac{\rho - \rho_0}{\rho_0} \right) + \frac{B}{2} \left(\frac{\rho - \rho_0}{\rho_0} \right)^2. \quad (2.5)$$

Here P_* is the internal pressure in the liquid. The coefficients A and B , which are temperature dependent, can be determined in terms of the bulk modulus of the liquid $K = \rho_0 \frac{dp}{d\rho}$. Using (2.5), we have

$$K = (A^2 + 2p'B)^{1/2} \cong A + \frac{B}{A} p',$$

where $p' = p - P_*$ is the external pressure, and use is made of the condition $2p' \frac{B}{A^2} \ll 1$, which is satisfied for all liquids up to $\sim 10^3$ atmos. On the other hand,

$$K = \kappa_1 (P_* + p') \text{ and } \frac{B}{A} = \kappa_1 = \frac{K_0}{P_*}, \text{ where } K_0 = K|_{p'=0}.$$

Another method to determine the nonlinear characteristics of a liquid was used in reference 7: the relative velocity of a certain point in the profile of the wave was assumed to be determined by the displacement velocity and by the change produced in the local sound velocity by the departures of the temperature and pressure from the undisturbed values. The values of B/A for different liquids are given in Table I. Some of these data were determined under static conditions. However, inasmuch as in liquids $c_p/c_v \cong 1.0$

and $K_0^\infty/K_0^{(0)} = c_p/c_v$, where $K_0^{(0)} \equiv K^{(0)}|_{p'=0}$ and $K_0^{(\infty)} = K^{(\infty)}|_{p'=0}$ are respectively the isothermal and adiabatic elastic moduli, we can use with sufficiently good approximation the static value of B/A in the analysis of waves of finite amplitude, bearing in mind that $\kappa_1 = B/A$ does not depend on the temperature.⁶

From a comparison of the equation for gases (2.4) and Eq. (2.5) for liquids, we obtain the result that γ corresponds formally to $1+B/A$.^{5,10} In this sense, a liquid that obeys Eq. (2.5) is "gas-like:" for equal compressions, the propagation of sound in the liquid will be the same as the propagation in a gas with a ratio of specific heats which is equal to $1+B/A$ at an undisturbed pressure P_* equal to the internal pressure in the liquid. Keeping this circumstance in mind we shall no longer differentiate between γ and $1+B/A$.

Turning our attention to similarity methods, it is easy to see that if we mean by p , ρ , and c in E the total pressure (applied plus internal), and the total values of the density and velocity, then $E \equiv \gamma \frac{B}{A} + 1$. Thus, all the dimensionless quantities in a wave of finite amplitude must depend on the parameter that characterized the nonlinearity of the equation of state and on the Mach number. In particular, the number of waves up to the formation of the discontinuity is $N = x_{cr}/\lambda$:

$$N = \Phi(\gamma, M).$$

An exact solution of the problem of a traveling acoustic plane wave of finite amplitude was first obtained by Poisson. Then Riemann¹¹ found the general solution of a one dimensional system of hydrodynamic equations of a nonviscous liquid, under the assumption that the equation of state of the medium could be represented in the form $p = \varphi(\rho)$, with φ an arbitrary function. Not pausing on the details of the solution, because this solution is given in a number of textbooks on wave propagation and mechanics of continuous media, we note that Riemann introduced the function

$$\sigma(\rho) = \int \frac{c \, d\rho}{\rho}, \text{ where } c = \left(\frac{dp}{d\rho}\right)^{1/2}, \text{ a function}$$

which reduces a set of Euler equations to a set with a solution in the form $v(\rho) = \sigma(\rho)$. This solution for the displacement velocity can be written in the form

$$v = \psi_1[x - t(c + v)] + \psi_2[x + t(c - v)], \quad (2.6)$$

where ψ_1 and ψ_2 are arbitrary functions. The solution (2.6) represents two disturbances, one propagated in the direction $x > 0$ with velocity

$c_1 = c + v$, and the second in the direction $x < 0$ with velocity $c_2 = c - v$. We emphasize that the relation between the displacement velocity and the density (and consequently the pressure) in a Riemann wave can be easily established if the equation of state $p = \varphi(\rho)$ is known:

$$v(\rho) = \int_{\rho_0}^{\rho} \frac{c \, d\rho}{\rho}. \quad (2.7)$$

For an adiabatic propagation of the disturbance, we have from (2.4), and also from (2.7) for $\gamma \neq 1$,

$$\left. \begin{aligned} c^2 &= \left(\frac{dp}{d\rho}\right)_s = c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}; & c_0^2 &= \frac{\gamma P_0}{\rho_0}; \\ v(\rho) &= \frac{2c_0}{\gamma-1} \left[\left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}} - 1 \right]; \\ v(\rho) &= \frac{2c_0}{\gamma-1} \left[\left(\frac{p}{P_0}\right)^{\frac{\gamma-1}{2\gamma}} - 1 \right]. \end{aligned} \right\} \quad (2.7a)$$

The exact relation

$$c = c_0 + \frac{\gamma-1}{2} v. \quad (2.7b)$$

follows from Eq. (2.7a) for c^2 and $v(\rho)$. For small compressions $(\rho - \rho_0)/\rho_0 = \rho'/\rho_0$, and also for ratios of the sound pressure $p' = p - P_0$ to the pressure in the medium p'/P_0 , the usual relations of the linear approximation for a plane wave $v = c_0 \frac{p'}{\rho_0}$ and $v = P'/\rho_0 c_0$ follow from (2.7a).

We note that for a linear relation between the pressure and density ($\gamma = 1$) we have

$$\left. \begin{aligned} c^2 &= c_0^2 = P_0/\rho_0; & v(\rho) &= c_0 \ln \frac{\rho}{\rho_0}; \\ v(\rho) &= c_0 \ln \frac{p}{P_0}. \end{aligned} \right\} \quad (2.7c)$$

The wave is distorted, but this distortion is less than in the case when a nonlinear equation of state is considered. It should be pointed out that the equations of (2.7a), strictly speaking, cannot be used for isothermal propagation of sound, because in this case it would be necessary to take into consideration losses by heat conduction (see Sec. 1b). In the particular case of a harmonic sound source which radiates the wave in the direction $x > 0$, (2.6) can be written in the form

$$v = v_{10} \sin\left(\omega t - \frac{\omega x}{c+v}\right).$$

Here v_{10} is the amplitude of the velocity at the source. If we assume that the medium obeys Eq. (2.4), a case possible only if the motion is isentropic, we obtain by using (2.7b)

$$v = v_{10} \sin\left(t - \frac{x}{c_0 + \frac{\gamma+1}{2} v}\right). \quad (2.8)$$

It is seen from (2.8) that the different points of the wave profile propagate with different velocities

$c_1 = c_0 + \frac{\gamma+1}{2} v$, and if $\gamma > -1$ we have $v > 0$ where $c_1 > c_0$, and $v < 0$ where $c_1 < c_0$. This causes the waveform to distort as it propagates. The distortion can be interpreted as the generation, growth, and interaction of harmonics in the process of wave propagation. Actually, for small Mach numbers, $M_{10} = v_{10}/c_0 \ll 1$, (2.8) can be represented in the form

$$v = v_{10} \sin \omega \left(t - \frac{x}{c_0} \right) + \frac{(\gamma+1)v_{10}^2 \omega x}{4c_0^3} \sin 2\omega \left(t - \frac{x}{c_0} \right) + \dots \\ = v_{10} \left[\sin \omega \left(t - \frac{x}{c_0} \right) + \frac{\gamma+1}{4} M_{10} k x \sin 2\omega \left(t - \frac{x}{c_0} \right) + \dots \right], \quad (2.9)$$

where $k = \omega/c_0$ is the wave number. It is significant that the amplitude of the fundamental tone does not decrease, in spite of the growth of the harmonics. In an ideal medium no forces oppose the distortion of the wave shape; independent of the size of the initial disturbance, the Riemann solution leads, as propagation proceeds, to the formation of a strong discontinuity (in which the positions of the maximum and minimum velocity coincide in space, or $\frac{\partial x}{\partial v}$ vanishes), and thereupon to an "entanglement" of the wave, i.e., to three values of the pressure (or velocity) at a single point in space. This latter circumstance does not have any physical meaning. Moreover, upon formation of a discontinuity with zero thickness, the equations of hydrodynamics describing the sound field cannot be applied. Finally, and what is very important, the motion ceases to be isentropic upon formation of the discontinuity, because in the narrow thickness of the front the processes of heat conduction begin to play an important role and one can no longer assume that the pressure can be represented as a function of the density alone.

It has already been mentioned that the condition for the formation of the discontinuity is $\frac{\partial v}{\partial x} = \infty$. Making use of this condition and of (2.8), we can determine the distance to the point of formation of the discontinuity for sound waves that are harmonic at the source

$$x_{cr} = \frac{\lambda c_0}{\pi(\gamma+1)v_{10}}$$

or, in dimensionless form

$$2\pi N = \frac{2}{(\gamma+1)M_{10}} \quad (2.10)$$

It is easy to see that (2.9) and (2.10), which have been obtained for the one-dimensional case, are

invariant relative to the transformation (2.1) for equal values of $(\gamma+1)M_{10}/2$. In terms of x_{cr} , (2.9) can be rewritten as

$$v = v_{10} \left[\sin(\omega t - kx) + \frac{x}{2x_{cr}} \sin 2(\omega t - kx) + \dots \right] \quad (2.11)$$

We note that these relations do not depend on the properties of the medium in which the wave is propagated. It follows from (2.11) that the amplitude of the second harmonic at the critical distance $x = x_{cr}$ is $v_2 = v_{10}/2$, and similarly, $p_2 = p_{10}/2$ etc.

For small Mach numbers, the distance to the formation of the discontinuity can also be determined in the case of a spherical diverging wave. For a pulsating sphere of radius r_0 , we can, at $r \gg r_0$ (when the different parts of the spherical wave can be considered plane), write the expression for the wave in the form¹⁴

$$vr = A \cos(kr - \omega t) - \frac{\gamma+1}{4} \frac{k}{c_0} \ln \left(\frac{r}{r_0} \right) A^2 \sin 2(kr - \omega t). \quad (2.12)$$

Upon formation of a discontinuity at a great distance, when it is possible to consider the various parts of the wave as plane, $v_2/v_1 = 1/2$; therefore, the distance r_{cr} at which the discontinuity of the spherical wave cumulates is obtained from (2.12):

$$kr_0 \ln \frac{r_{cr}}{r_0} = \frac{2}{(\gamma+1)M_{10}}. \quad (2.13)$$

Here $M_{10} = v_{10}/c_0$, where v_{10} is the amplitude of the displacement velocity at the surface of the sphere. Keeping in mind (2.10), we can write (2.13) in the form

$$r_{cr} = r_0 e^{\frac{x_{cr}}{r_0}}.$$

Here x_{cr} is the critical distance of a plane wave with the same Mach number in that medium in which the spherical wave is propagated. As the Mach number decreases, the critical distance for the spherical wave increases rapidly. For $M_{10} \sim 10^{-3}$ to 10^{-4} , it greatly exceeds the critical distance of the plane wave. On this basis, we can conclude that in the case of a diverging spherical wave nonlinear distortions are much less significant than in the case of plane waves.

Summing up, it can be said that an exact solution of the hydrodynamic equations in a nondissipative medium leads, in contrast to the linear equations, to waveform distortion. The waveform distortion is sooner or later accompanied by the formation of a discontinuity, independent of the amplitude of the acoustical quantities. For $kx_{cr} \gg 1$ or $kr_{cr} \gg 1$, we can say that the shape

changes little within a distance equal to a single wavelength, which indicates the smallness of interaction of harmonics at these distances. Therefore, at any rate, at distances of the order of a wavelength, although no discontinuity is formed, the wave can be considered nonmonochromatic with weak interaction of the harmonic components. For the case of a plane wave, the condition $kx_{cr} \gg 1$ is equivalent to the condition

$$\frac{\gamma+1}{2} M_{10} \ll 1. \quad (2.14)$$

For gases $\frac{\gamma+1}{2} \cong 1.1$ to 1.3; for liquids, $\frac{\gamma+1}{2}$

$\cong 4$ to 6 (see Table I, and also the formal connection between γ and B/A on p. 583). Therefore, the condition (2.14) is simply equivalent to specifying smallness of the Mach number: $M_{10} = v_{10}/c_0 \ll 1$. For the maximum intensities obtained at the present time (~ 300 w/cm² in water¹), $v_{10} \sim 2 \times 10^3$ cm/sec and $M_{10} \sim 1.5 \times 10^{-2}$. Therefore, for all intensities experimentally obtainable today, the interactions of the harmonic components of the wave can be assumed weak and the wave can be considered in the quasilinear approximation.

b) Dissipative medium. The propagation of a wave of finite amplitude in a dissipative medium is much more complicated than in a medium without dissipation. The presence of dissipative processes makes the propagation of the acoustic wave, strictly speaking, no longer a reversible process: the equations of a viscous liquid are not invariant relative to the transformation $t' = -t$ and $\mathbf{v}' = -\mathbf{v}$. If the medium, in addition to the viscosity, has a nonvanishing heat conductivity, then the process of propagation, strictly speaking, also ceases to be isentropic even in the absence of a discontinuity. As is well known, the hydrodynamic equations of a viscous gas can be obtained from the Maxwell-Boltzmann kinetic equation by assuming that the distribution functions of the particles in a phase space do not differ appreciably from equilibrium (see, for example, reference 12). The condition that the deviation of the process from equilibrium is small means, in the case of a gas, that the change in temperature is small in comparison with the absolute temperature at a distance equal to the mean free path, that the change in the velocity is small in comparison with the velocity of sound,¹² and also that the period of the wave is small in comparison with the relaxation time or that the mean free path is small in comparison with the wavelength.¹³ Whether or not one can consider the departures from the equilibrium state in a wave of finite amplitude to be small is still a moot point,

in our opinion. Without mentioning such rapid nonequilibrium processes as the collapse of cavitation bubbles upon formation of a shock wave, the absorption of waves of finite amplitude in the case of appreciable nonlinear distortions must also be regarded from the viewpoint of the thermodynamics of nonequilibrium processes with large departures from equilibrium. In the case of shock waves, there is no doubt of the necessity of accounting for the large departures of the process from equilibrium. At the present time, several liquids are known to have relaxation frequencies of 10^5 cps (for example, toluene). When a wave of finite amplitude and a frequency $\sim 10^6$ cps propagates in these liquids, one cannot consider the departure from equilibrium to be small in processes occurring in such waves. We shall return to this question when we compare the experimental material with the theoretical results; however, it should be pointed out here that the available experimental material and the status of the theory of propagation of waves of finite amplitude in dissipative media are such that it is impossible to decide with certainty on the completeness of the hydrodynamic equations of a viscous, heat-conducting liquid.

We apply the transformation of variables

$$\left. \begin{aligned} \mathbf{v}' &= \mathbf{v}; & \frac{\Omega x'_i}{C} &= x_i; & \frac{\rho'}{R} &= \rho; & \frac{p'}{P} &= p; & \frac{c'}{C} &= c; \\ \Omega t' &= t; & \frac{\eta'}{N} &= \eta; & \frac{\xi'}{N} &= \xi; & \frac{T'}{\theta} &= T; & \frac{S'}{\Sigma} &= S; & \frac{\kappa'}{K} &= \kappa, \end{aligned} \right\} \quad (2.15)$$

(where η and ξ are the shear and bulk viscosities, respectively, T is the temperature, S the entropy, and κ the coefficient of thermal conductivity) to the set of hydrodynamic equations of a viscous liquid.

Then we have, in the unprimed variables,

$$\frac{\partial \rho}{\partial t} + \frac{U}{C} \nabla \rho \mathbf{v} = 0, \quad (2.16)$$

$$\begin{aligned} & \rho \left[\frac{\partial \mathbf{v}}{\partial t} + \frac{U}{C} (\mathbf{v} \nabla) \mathbf{v} \right] \\ &= -\frac{C}{U} c^2 \Delta \rho + \frac{N\Omega}{RC} \left[\eta \nabla \mathbf{v} + \left(\xi + \frac{1}{3} \eta \right) \nabla \nabla \mathbf{v} \right] \end{aligned} \quad (2.17)$$

and the equation of heat transfer is

$$\begin{aligned} \frac{\partial S}{\partial t} + \frac{U}{C} \mathbf{v} \nabla S &= \frac{N\Omega}{\theta \Sigma R} \frac{U^2}{C^2} \left[\frac{\eta}{2} \left(\frac{\partial v_i}{\partial x_h} + \frac{\partial v_h}{\partial x_i} \right)^2 + \xi (\nabla \mathbf{v})^2 \right] \\ &+ \frac{K\Omega}{\Sigma RC^2} \kappa \nabla T. \end{aligned} \quad (2.18)$$

The equation of state and the dependence of the entropy on the characteristic thermodynamic coordinates are also subject to the transformation (2.15). Assuming that the equation of state is known, and also that the pressure dependence of

the entropy is known, we have the transformation

$$c^2 = \frac{P}{RC^2} \left(\frac{dP}{d\rho} \right). \quad (2.19)$$

As is seen from (2.16) – (2.19), the equations are invariant relative to the transformation (2.15) provided the following five numbers are constant: the local Mach number $M = v/c$ and the numbers $b\omega/\rho c^2$, $p/\rho c^2$, $b\omega|v|^2/TS\rho c^2$ and $\kappa\omega/S\rho c^2$.

Here $b = \frac{4}{3}\eta + \xi$. As is well known, the increase

of entropy in a shock wave of weak intensity is a quantity of third order of smallness (see, for example, reference 14). Therefore, with accuracy up to terms of second order of smallness, we can assume $S = \text{const.}$, i.e., we assume the propagation of waves of finite amplitude to be adiabatic. As was shown in references 3 and 15 for the case of a plane wave, the linearization of the equation of heat transfer leads to linear dissipative forces. This makes it possible to reduce the totality of similarity numbers to three: M ,

$b_1\omega/\rho c^2$ and $p/\rho c^2$, where $b_1 = \left(\frac{4}{3}\eta + \xi\right) + \kappa\left(\frac{1}{c_v} - \frac{1}{c_p}\right)$. These numbers have a simple physical meaning: as before, in the case of a nonviscous liquid, M and $p/\rho c^2$ are numbers determining the distortion (M accounts for the nonlinearity of (2.16) and (2.17) and $p/\rho c^2$ for the nonlinear equation of state). The number $b_1\omega/\rho c^2$ determines the relative effect of the losses due to viscosity and heat conduction.

For nonmetallic liquids, the contribution of the heat conduction to the absorption is much less than that of the viscous absorption; we therefore use the number $b\omega/\rho c^2$ in what follows. For similar waves the ratio of the dimensionless numbers will also

be identical. For example, $Re = \frac{p}{\rho c^2} \times \frac{\rho c^2}{b\omega} = \frac{p}{b\omega}$

determines the ratio of the distortion due to the nonlinear equation of state to the number that defines the dissipation of the energy as a result of

viscous friction; the number $Re' = M \frac{\rho c^2}{b\omega} = \frac{\rho c v}{b\omega}$ defines the ratio of the distortion due to the nonlinearities of the equations of continuity (2.16) and of motion (2.17) to the number defining the dissipation of energy due to viscous friction.* In the

*We note here that for small viscosities and large nonlinear distortions, as will be shown in what follows, we can use the Riemann relation (2.7a) for the determination of the number Re' :

$$Re' = \frac{\rho c v}{b\omega} = \frac{2\gamma p}{(\gamma-1)b\omega} \left[1 - \left(\frac{p}{P_0} \right)^{\frac{1-\gamma}{2\gamma}} \right] \cong \frac{p}{b\omega} = Re$$

with accuracy up to $(p/P_0)^2$.

acoustic case, Re and Re' can be called the acoustic Reynolds' numbers.

We note that several nonlinear effects (for example, the distortion and absorption at large Re , the turbulence of the acoustic streaming, not to speak of cavitation) observed at the present time, cannot be explained by considering only second-order terms in the hydrodynamic equations. Therefore, the linearization of (2.18) is far from evident, and even incorrect, strictly speaking. Unfortunately, a solution of the complete problem of the propagation of waves of finite amplitude with account of the nonlinearity of the equation of heat transfer entails difficulties that are currently insurmountable.

In what follows, we shall consider a set of hydrodynamic equations for a case in which linearization of the equation of heat transfer is possible. In this case, as was shown in reference 3, there are two limiting modes, depending on the acoustic Reynolds number $Re = p/b\omega$. For $Re \ll 1$, by using the method of successive approximations (which is frequently applicable in the consideration of the various problems of nonlinear acoustics¹⁶) we obtain for the sound pressure of the first and second harmonic, with accuracy up to quantities of second order of smallness,^{17,18}

$$p_1 = p_{10} e^{-\alpha_1^0 x} \sin(\omega t - kx), \quad (2.20)$$

$$p_2 = \frac{(\gamma+1)p_{10}^2}{4b\omega} (e^{-2\alpha_1^0 x} - e^{-4\alpha_1^0 x}) \sin 2(\omega t - kx). \quad (2.21)$$

Here p_{10} is the amplitude of the sound pressure at the source, and α_1^0 is the absorption coefficient of a wave of small amplitude having the frequency of the fundamental wave. The amplitude of the second harmonic reaches a maximum value at a distance

$$x_m^{(2)} = \frac{\ln 2}{2\alpha_1^0}. \quad (2.22)$$

From (2.20) and (2.21) we find that the ratio

$$\frac{p_2}{p_1} = \frac{(\gamma+1)Re_{10}}{2} (e^{-\alpha_1^0 x} - e^{-3\alpha_1^0 x}) \cos(\omega t - kx), \quad (2.23)$$

(where $Re_{10} = p_{10}/b\omega$) reaches a maximum at the distance

$$x_m = \frac{\ln 3}{2\alpha_1^0}. \quad (2.24)$$

This distance can be called the range of relative stabilization of the wave; p_2/p_1 begins to fall off at distances $x > x_m$. In the formation of a sawtooth wave, $p_2/p_1 = 1/2$, and it follows from (2.24), (2.23) and the smallness of Re that the stable form of the wave in this case is far from sawtooth. For $Re \sim 1$, values of v_2 , ρ_2 , and p_2 were obtained in reference 18 with amplitudes increasing with time

and distance. For $Re \gg 1$, the dissipative coefficients (viscosity and thermal conductivity) drop out from an analogous increasing second-approximation solution. In the latter case ($Re \gg 1$), the second approximation coincides with the second approximation of the Riemann solution. This gives grounds for assuming that (for $Re \gg 1$) the viscosity of the medium cannot prevent the formation of the discontinuity. The stable shape of the wave in this case will be close to sawtooth.

Thus, as is usually done in hydrodynamics for large Reynolds numbers, we can neglect the viscosity in the initial equations. We note, however, that complete neglect of the viscosity for $Re \gg 1$ led in hydrodynamics to a number of paradoxes which served as the point of departure for the introduction of the boundary layer by Prandtl. Similarly, viscosity and thermal conductivity can be neglected here only where the gradients of the velocity and temperature are small; in an analysis of the narrow front of the sawtooth wave, it is naturally impossible to neglect the viscosity and thermal conductivity.

In the case of $Re \sim 1$, the propagation of a plane wave emitted by a harmonically vibrating piston in a dissipative medium was studied in reference 19. Starting from the equations in Lagrangian coordinates, and applying the method of Hopf-Cowl (for details, see reference 15), we obtain an expression for the velocity

$$v(x, t) = \frac{2b_1}{\gamma(\gamma+1)\rho_0 c_0} \frac{\partial}{\partial t} \ln \left[I_0 \left(-\frac{\gamma+1}{2} Re \right) + 2 \sum_{n=1}^{\infty} I_n \left(-\frac{\gamma+1}{2} Re \right) e^{-n^2 \alpha_1^2 x} \cos n\omega \left(t - \frac{x}{c} \right) \right] \quad (2.25)$$

where $I_n \left(-\frac{\gamma+1}{2} Re \right)$ is the Bessel function of imaginary argument. As was shown in reference 15, beginning with the distance x which satisfies the condition $\exp(-2\alpha_1^2 x) \ll 1$, this solution coincides with the solution of Fay.²⁰ The solution of the equation in Lagrangian coordinates was found by him for the "comparatively stable wave shape," i.e., for waves whose form does not change as a result of the simultaneous action of nonlinear distortion and absorption. The expression obtained for the sound pressure in the region of stabilization of the sawtooth wave has the following form

$$p = \frac{2b_1 \omega}{\gamma+1} \sum_{n=1}^{\infty} \frac{\sin n(\omega t - kx)}{\sinh(\alpha_0 + \alpha_1^2 x)^*} \quad (2.26)$$

*We note that in reference 20 the value given for p is four times larger. This was corrected in reference 15.

$$\text{Here } \alpha_0 = \sinh^{-1} \left[\frac{\pi}{(\gamma+1) Re} \right].$$

At the present time there is no exact solution of the problem of the propagation of a plane harmonic wave in a dissipative medium for large nonlinear distortions. This is connected on the one hand with mathematical difficulties, and on the other with difficulties of fundamental character, mentioned at the beginning of this section.

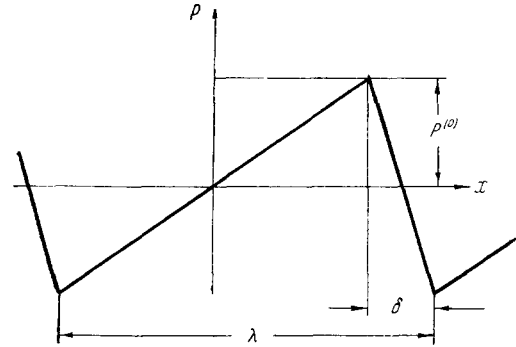


FIG. 1. Sawtooth wave.

Thus, for $Re \gg 1$, the initially sinusoidal wave is gradually transformed to sawtooth. Subsequently, the wave is so propagated that its form remains close to sawtooth: the sawtooth wave is relatively stable in a certain region of propagation. At large distances from the source of the sound, the wave (because of damping) transforms into a sinusoidal wave of small amplitude. The spectral composition of the sawtooth wave with front thickness δ (see Fig. 1) can be written in the form

$$p_n = \frac{2P^{(0)}}{n\pi \left(1 - \frac{\delta}{\lambda}\right)} \frac{\sin n\pi \frac{\delta}{\lambda}}{n\pi \frac{\delta}{\lambda}} \quad (n = 1, 2, 3, \dots), \quad (2.27)$$

where $P^{(0)}$ is the amplitude of the sawtooth wave.

The thickness of the front of the sawtooth can be determined from the relation for the thickness of a shock wave in the case of a small pressure discontinuity.¹⁴ It is easy to see that for a periodic wave the front thickness is

$$\delta = \frac{2\lambda}{\pi(\gamma+1) Re}, \quad (2.28)$$

where $Re = P^{(0)}/b\omega$. We note that this relation is obtained under the assumption that the dissipative forces are linear. In liquids, for Re on the order of several times ten (acoustic pressure ~ 10 atmos at frequencies ~ 1 Mcs), the thickness of the front in (2.28) ought to be $\sim 10^{-3}\lambda$. As follows from (2.27), the thickness of the front can be determined if the spectral composition of the wave is known. Current measurements of the spectral composition (see Sec. 3) give δ approximately

two orders larger than given by (2.28); however, this may be due to errors in measurement.

c) Absorption of waves of finite amplitude.

It has already been noted in Sec. b) that there does not exist today a more or less complete theory of the propagation of waves of finite amplitude in a dissipative medium, even in the one-dimensional case. Therefore, all the existing methods of the determination of absorption are approximate. A common feature of all these methods is that the dissipative forces are assumed linear. In the case of the cumulation of a discontinuity, this assumption makes the absorption coefficient of the discontinuity (as in the case of weak shock waves) independent of the linear coefficients of viscosity and of the thermal conductivity. In addition, even these approximate theories of absorption demonstrate the possibility of an appreciable increase in the dissipation of energy in a wave of finite amplitude in comparison with waves of infinitesimally small amplitude. This is connected with the increase in the dissipation of energy in the higher harmonics of the wave of finite amplitude. We note here that the absorption of waves of finite amplitude differs from the absorption of waves of infinitesimally small amplitude not only in that it is essentially larger, but also in that it depends on the spectral composition of the wave. For example, for initially harmonic waves, the harmonics grow and the absorption coefficient changes with increasing distortion. This leads to the result that, in contrast to the absorption of waves of small amplitude, the absorption of waves of finite amplitude is not exponential. Before turning to the problem of absorption, we consider several general questions. It has been observed above that for small Mach numbers and $\gamma \sim 1$ the waveform distortion at distances on the order of a wavelength is small. Therefore, in an ideal medium, the finite-amplitude wave can be considered at these distances, as a nonmonochromatic wave with weakly interacting harmonics. The introduction of linear dissipative forces in the equation cannot lead to a change in the interaction of harmonic components of the wave. This permits us to proceed to the problem of the absorption of waves of finite amplitude from a quasi-linear point of view.

In the case in which $Re \ll 1$, the absorption coefficient for a plane wave, initially sinusoidal, can be obtained²¹ (see also reference 22) from Eqs. (2.20) and (2.21) by thermodynamic calculations:

$$\frac{\alpha}{\alpha_1^0} = 1 + \frac{3(\gamma+1)^2 p_{10}^2}{16bb_1\omega^2} (e^{-\alpha_1^0 x} - e^{-3\alpha_1^0 x})^2. \quad (2.29)$$

Here $b_1 = b + \kappa \left(\frac{1}{c_v} - \frac{1}{c_p} \right)$. If we can neglect the heat conduction ($b_1 = b$), the additional absorption from (2.29) is proportional to Re_{10}^2 . It also follows from (2.29) that even small distortion of the wave form leads to nonexponential absorption; at the sound source, $\alpha = \alpha_1^0$, after which α increases in the propagation direction and reaches at the point* $x_m = \frac{\ln 3}{2\alpha_1^0}$ a maximum value

$$\frac{\alpha}{\alpha_1^0} = 1 + 0.028(\gamma+1)^2 Re_{10}^2. \quad (2.30)$$

$\alpha \rightarrow \alpha_1^0$ as $x \rightarrow \infty$.

It should be noted that the accuracy of the experimental determination of the absorption coefficient of a wave of small amplitude is currently about 10%. The distortion of the wave can introduce errors in the measurement of the absorption coefficient of a wave of small amplitude. Using the relations derived, we can determine the conditions under which "nonlinear" errors are possible which exceed the limits of the determination of the absorption coefficient of a wave of small amplitude by 10%:

$$Re_{10} \gtrsim \frac{1.9}{\gamma+1}. \quad (2.31)$$

An increase in the accuracy of the measurement of α_1^0 requires smaller sound pressures. This rather stringent limitation on the sound pressure in the determination of small amplitude absorption can be relaxed if the measurements are carried out far from the point $x_m = \ln 3/2\alpha_1^0$.†

In the other limiting case, when $Re > 1$, formation of a discontinuity is possible in the medium, as follows from what was pointed out above. In this case the relatively stable waveform will be a sawtooth wave. Thus the absorption determined under these conditions in the region of stabilization of the wave is related to the absorption of acoustic discontinuities.

Writing down the condition for the decrease of amplitude of the sawtooth wave, under which the wave remains a sawtooth during the process of

*This distance is larger than the distance of stabilization of the second harmonic and is equal to the distance of stabilization of the wave where p_2/p_1 is maximum [see (2.2)].

†Keeping in mind that the intensity of ultrasound is $I \sim \nu^2 U_{\text{eff}}^2$ (where ν is the frequency and U_{eff} is the effective voltage on the piezoradiator), the voltage obtained from (2.31) for a quartz radiating on one side is $U_{\text{eff}} \lesssim 7 \times 10^3 [b/(\gamma+1)]$, where it is seen that beyond the relaxation region (where $b = \frac{1}{2}\eta + \xi$ does not depend on the frequency), the voltage U at different frequencies ought to be identical in the measurement of absorption. For example, $U_{\text{eff}} \lesssim 30$ v for water, and $\lesssim 13$ v for methyl alcohol. In viscous liquids (keeping in mind also the large bulk viscosity) this limit lies at much higher voltages.

propagation, we can obtain the relation¹⁴

$$\frac{\alpha}{\alpha_1^0} = \frac{\gamma+1}{\pi} \frac{p_x}{b\omega} = \frac{\gamma+1}{\pi} \text{Re}_x. \quad (2.32)$$

We note that p_x is the acoustic pressure at the point of measurement of the coefficient of absorption. The coefficient of absorption of such a stable acoustic discontinuity does not depend on linear dissipative coefficients. This does not mean at all that the dispersion of energy is not determined by the viscosity and thermal conductivity, for it is just these dissipative processes that bring about the possibility of the existence of a wave that is stable in form. We also note that the coefficient α is proportional to the acoustic pressure, which is typical of weak shock waves.

A quasilinear method was used in reference 5 for the determination of the absorption. The absorption coefficient of a plane wave of finite amplitude was determined in the following way. The distance traveled by the wave in the formation of the discontinuity in a real medium [see (2.10)] was divided into ten sections $\Delta x = 0.1 x_{CR}$. Writing down the change of amplitude of the pressure of the n -th harmonic in the n -th section in the form

$$\begin{aligned} p_1^{h+1} &= p_1^h \exp(-\delta_1^h - \alpha_1^0 \Delta x), \\ p_n^{h+1} &= p_n^h \exp(+\delta_n^h - n^2 \alpha_1^0 \Delta x) \quad (n = 2, 3, \dots), \end{aligned}$$

where δ_n^k are factors that take into account the interaction of the harmonics, we can obtain the condition under which $p_n^s/p_1^s = A_n$ ($n = 2, 3, \dots$; $1 \leq s \leq 10$), where the A_n do not depend on the number of the section. The wave under these conditions becomes relatively stable, i.e., it is propagated so that its relative spectral composition does not change with propagation, although the "intensity of the individual spectral lines" does change. Under these conditions, it was shown to be possible to determine the absorption coefficient in the region of the stabilized wave:

$$\frac{\alpha}{\alpha_1^0} = \frac{\delta_1^s + \alpha_1^0 \Delta x}{\alpha_1^0 \Delta x} = 1 + 10 \delta_1^s (\gamma + 1) \text{Re}_{10}. \quad (2.33)$$

The factor δ_1^k was determined in reference 5 from the distortion of the waveform in the ideal medium. It should be noted that the similarity of the wave of finite amplitude in the ideal medium was employed in reference 5; this also made it possible to introduce essentially a new coordinate, a fraction of the total distance to the place of formation of the discontinuity, for waves of different frequencies propagated in different media. The limits of applicability of (2.33) are rather difficult to establish. For $\text{Re}_{10} < 1$, stabilization of the wave takes place for $s < 10$, where δ_1^s depends on the

number of the section in which the wave becomes stabilized. The larger the intensity, the higher the values of s at which stabilization occurs. Thus δ_1^s depends on the amplitude of the acoustic pressure, and (2.33) is not the right equation for the dependence of $\frac{\alpha}{\alpha_1^0}$ on the pressure. For $\text{Re}_{10} > 1$, as was pointed out above, the stable form of the wave is close to sawtooth and $s = 10$. For $s = 10$, according to the data given in reference 5, $\delta_1^{10} \cong 0.03$ and (2.33) gives a value of α/α_1^0 close to (2.32), provided one can neglect the change of amplitude of the acoustic pressure up to the point of formation of the discontinuity. It should be noted that the lack of an analytical expression for δ_1^s makes it difficult to use (2.33). It is more suitable to use (2.29) for $\text{Re} < 1$ and (2.32) for $\text{Re} > 1$. We also note that the absorption is determined in reference 4 by a method that is analogous in principle to that of reference 5. If E_n is the mean density of energy in the n -th harmonic, then the change of this energy is determined by the linear dissipation of energy by and the interaction with the other harmonic components of the wave:

$$\frac{dE_n}{dx} = -\alpha_n^0 E_n + \sum_{s=1}^{\infty} \epsilon_{ns} = -\alpha_n E_n \quad (n = 1, 2, 3, \dots). \quad (2.34)$$

Here ϵ_{ns} is the wave energy transferred from the s -th to the n -th harmonic per unit distance traveled by the wave (with appropriate sign); α_n is the partial coefficient of absorption. If*

$\alpha = -\frac{1}{E} \frac{dE}{dx}$ and $E = \sum_{n=1}^{\infty} E_n$, it is easy to see that

$$\alpha = \sum_{n=1}^{\infty} \alpha_n^0 \sigma_n = \sum_{n=1}^{\infty} \alpha_n \sigma_n, \quad (2.35)$$

since $\sum_{n=1}^{\infty} \sum_{s=1}^{\infty} \epsilon_{ns} = 0$. Here, $\sigma_n = E_n/E$. It is seen from (2.35) that the differential absorption coefficient is determined by the spectral composition of the wave. Using (2.35) it is possible to show that α has a maximum in the region of stabilization, where $E_n/E_1 = B_n$ (B_n does not depend on x), and that in this region the absorption coefficient of the wave of finite amplitude is equal to an arbitrary partial coefficient $\alpha = \alpha_n$ ($n = 1, 2, 3, \dots$).

*The equality of the energy density of the wave to the sum of the energy densities of the individual harmonics is not self-evident. However, as experimental data show, in ordinary non-relaxing liquids, in the case of a plane wave, the phase relations between the harmonic components remain unchanged in the process of wave propagation.

In other words one can show that in the case of plane waves of finite amplitude, for $Re \ll 1$, the absorption coefficient of the wave of finite amplitude is proportional to Re^2 ; for $Re > 1$, the absorption coefficient in the region of stabilization of the wave is proportional to Re . In this latter region, as it is not difficult to see, the absorption coefficient α does not depend on the dissipative coefficients of the medium (the shear and bulk viscosities, and also the thermal conductivity). We also note that in contrast to the absorption of waves of small amplitude, the absorption coefficient of waves of finite amplitude depends on the distance, i.e., the absorption is nonexponential. Therefore, outside of the region of stabilization it is meaningful to speak only of a differential absorption coefficient.

At the present time there is great interest in the absorption of waves of finite amplitude in relaxing media. The problem of linear relaxation, for a wave close in shape to sawtooth, was considered in reference 23, where it was shown that for a change in the Reynolds' number from ~ 1 to ~ 17 , the absorption at the relaxation maximum increased approximately 1.5 times, while the relaxation maximum shifted in the direction of lower frequencies from $\omega\tau = 1$ to $\omega\tau \cong 0.4$. For a wave close to sawtooth in shape, the use of (2.26) yielded

$$\frac{\alpha_r''}{\alpha_r'} = (1 + \omega^2\tau^2) \frac{\sum_{n=1}^{\infty} \frac{n^2}{(1 + n^2\omega^2\tau^2) \sinh^2 n(\alpha_0 + \alpha_1^0 x)}}{\sum_{n=1}^{\infty} \frac{1}{\sinh^2 n(\alpha_0 + \alpha_1^0 x)}} \quad (2.36)$$

Here α_r'' is the relaxing part of the absorption of a wave close to sawtooth, and α_r' is the relaxing part of the absorption of a wave of small amplitude. As is seen, the relaxing part of the absorption depends on the Reynolds number and on $\omega\tau$.

3. DISTORTION OF WAVES OF FINITE AMPLITUDE IN LIQUIDS

In this section we shall consider the experimental methods for the determination of nonlinear distortions of the wave shape, and also a qualitative comparison of theory with experiment. In one of the first works,²⁴ which notes the effect of nonlinearity on the propagation of ultrasonic waves in liquids, it is established that the interaction of two ultrasonic waves of different frequency, propagating in the same direction, leads to sum and difference frequencies. Later, a similar experiment was performed on the interaction of waves

at right angles,²⁵ as a result of which amplitude and phase modulation of one wave by the other were noted. This method was applied for the determination of the nonlinear characteristics of liquids in reference 8. Distortion of the shape of monochromatic traveling waves of finite amplitude was shown in references 10 and 26. The measurement setup is shown in Fig. 2. An ultra-

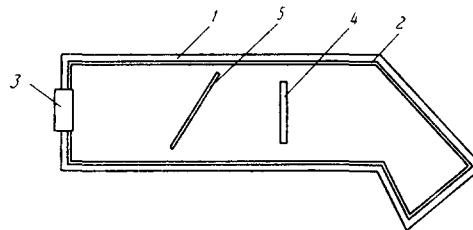


FIG. 2. Arrangement for the determination of the spectral composition of a wave of finite amplitude.

sonic source 3 (quartz transducer) and a quartz receiver 4 are placed in the tank 1 covered (to reduce wall reflections) by an absorber 2 made up of layers of soft rubber. An acoustic filter is placed between the source and a receiver whose resonant frequency coincided with the frequency of one of the harmonics of the wave. This filter had the form of a thin metallic plate 5, turned at such an angle to the direction of propagation of the ultrasound as to markedly isolate the desired harmonic from the signal past the filter. The use of the filter also eliminated the possibility of the formation of standing waves by reflection from receiver 4. The amplitudes of the harmonic components of the wave in this case could be determined with the receiver 4 stationary; only the filter 5 was displaced along the axis of the ultrasonic beam. The dependences of the amplitudes of the second and third harmonics on the distance to the source of the sound (for a frequency of 1.5 Mcs) is shown in Figs. 3a and 3b. In these drawings, curves 1, 2, and 3 were determined for source intensities I_{10} of $\cong 0.5$, 2, and 7.8 w/cm^2 , respectively, in tap water. In Fig. 3, the right ordinate scale applies to transformer oil, the second harmonic in which (curve 4) was determined for $I_{10} \cong 7.8 w/cm^2$. The ordinates represent voltages at the output of a tuned amplifier attached to the receiver (with a gain of 50 for the second and 100 for the third harmonic). In addition to the method just described, the results of which are indicated in Fig. 3 by crosses, a method was used in which the wave reflected from the receiver was attenuated by covering the receiver with sheets of rubber; further amplification with the use of a tuned amplifier permitted a

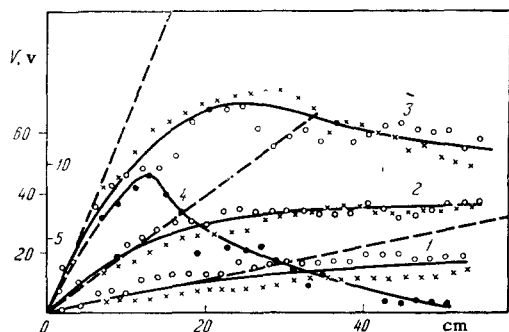


FIG. 3a. Second harmonic vs. the distance to the sound source in water (1, 2, 3) and in transformer oil (4).

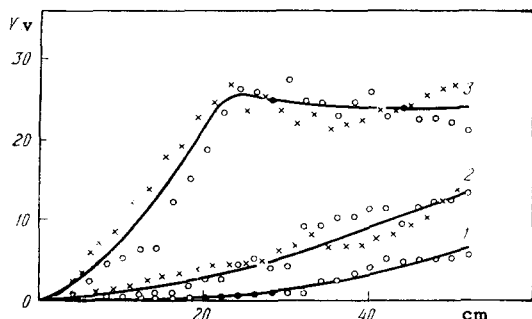


FIG. 3b. Third harmonic vs. the distance to the sound source in water.

sufficiently accurate separation of the harmonic. The results of the measurement by this method are shown in the drawing by circles.

We note that the tuned method described above permits the measurement of the harmonic content for small intensities. For example, at 1 Mcs, the second harmonic could be observed at voltages on the radiating quartz of the order of several volts.

In contrast to the method of tuned receivers, which might be called the spectral method, a broadband piezo receiver, in conjunction with a broadband amplifier, permits direct observation of the wave. This was done in references 27, 28. The waveform at 1 Mcs, for an initial intensity $I_{10} \cong 40 \text{ w/cm}^2$ obtained by means of a broadband quartz transducer (with a resonant frequency of 11.5 Mcs), is shown in Fig. 4.²⁷ The first oscillogram pertains to a distance of 2 cm from the source; the waveform differs somewhat from sinusoidal, but it is quite different from sawtooth. The following oscillograms pertain to 10, 20, and 48 cm from the source. These oscillograms show how the wave gradually transforms to one that is nearly sawtooth in shape. The maximum curvature of the leading front on the wave was obtained at 20 cm; further on this wave propagates without appreciable change in shape. Use of either an electric or mechanical (as was done in reference 27) analyzer permits one to obtain the spectral composition of the wave. A very important factor

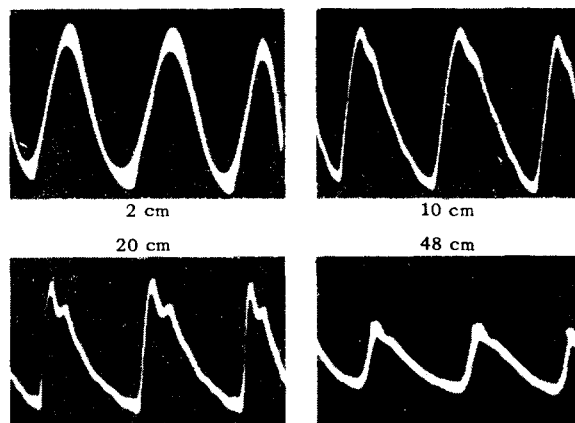


FIG. 4. Oscillograms showing the gradual change of waveform along the direction of propagation.²⁷

is that these photographs have been obtained within time intervals not exceeding 1 second from the moment of switching on the ultrasonic field.

A variety of optical methods have been developed for the determination of the waveform distortion of waves of finite amplitude in liquids.²⁹⁻³³ It should be noted here that optical methods, as in other acoustic measurements, possess the advantage that they allow measurement without introducing any distortion into the acoustic field. However, the optical methods give in most cases only indirect information on the change in the wave shape. In the observation of the diffraction of light by an ultrasonic wave of finite amplitude, an asymmetry about the diffraction maximum has been observed³⁰ in the distribution of the intensity of the light. The envelope of the distribution has two maxima, located on two sides of the zeroth-order spectrum. Upon increase in the sound intensity, one of the maxima decreases and moves away from the zeroth order spectrum; at the same time, the other maximum increases and approaches the zeroth order spectrum. Photographs are shown in Fig. 5 of the mercury line ($\lambda = 4358 \text{ \AA}$), obtained at 583 kcs in distilled water at 5 cm from the sound source (the intensity at the position of the light beam is shown on each photograph).^{*} The theory of the method was developed in reference 31 under the assumption that the ultrasonic diffraction grating only phase-modulates the light. It follows from this theory, in particular, that for an ideal sawtooth wave (a wave with zero front thickness) only a single maximum will be observed around the k -th order ($k = \frac{2\Delta n L}{\lambda}$, where Δn is the

*These photographs, as well as those of Fig. 8, were kindly furnished to the authors by I.G. Mikhaïlov and V. A. Shutilov.

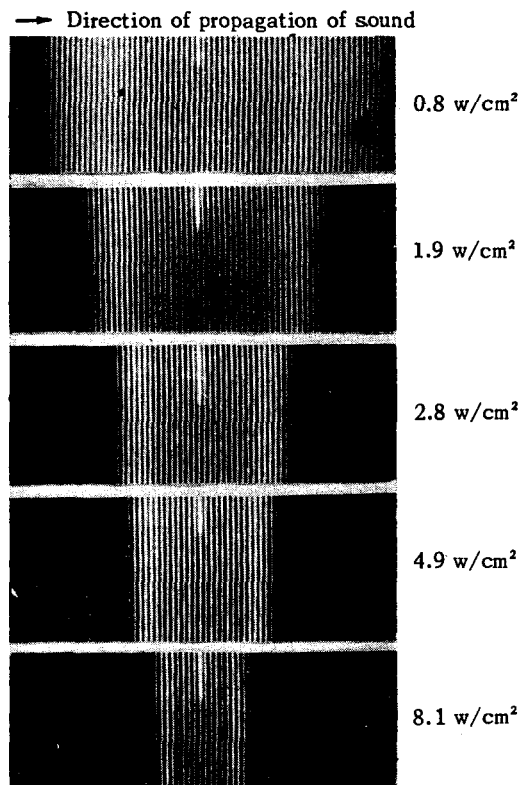


FIG. 5. Diffraction of light on a wave of finite amplitude. Asymmetry was obtained in the distribution of the light intensity over the diffraction maxima.

change in the index of refraction and L is the depth of the sound field).

For a sawtooth wave with a front thickness δ , two maxima are observed around the orders

$$k_1 = \frac{2\Delta n L}{\lambda} \frac{\Lambda}{d} \quad \text{and} \quad k_2 = \frac{2\Delta n L}{\lambda} \frac{\Lambda}{\delta},$$

where Λ is the acoustic wavelength and $d = \Lambda - \delta$. The theory advanced is valid up to $d/\delta < 3$; for higher values of d/δ it is necessary to take the amplitude modulation of the light into account. We note that, for small thickness of the wavefront, the need also arises of investigating the higher-order diffraction spectra, which can lead to certain experimental difficulties.

Figure 6 shows the optical setup used in reference 32 to determine waveform distortion. The image of illuminated slit S_1 is focused by a long focus lens L on slit S_3 . Slit S_2 limits the width of the light beam to $\sim 0.1 - 0.4 \Lambda$. Therefore, no diffraction on the ultrasonic wave produced by the quartz Q was observed in the plane of the slit S_3 . However, for traveling ultrasonic waves, the image of the slit was displaced, owing to the change in the index of refraction, which led to the smearing of the image of slit S_1 in comparison with the image in the absence of the ultrasound. For very small width of the slit S_3 , by displacing S_3 in

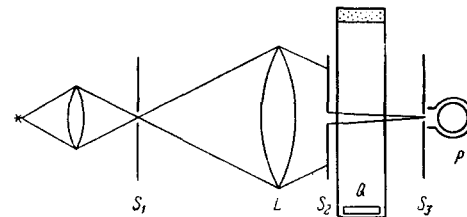
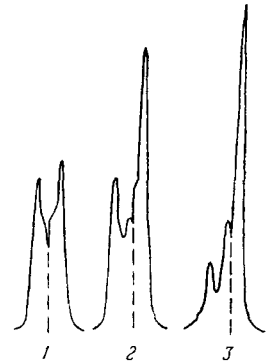


FIG. 6. Optical setup for the observation of wave distortion.³²

its plane, it is possible to determine the distribution of light intensity in the smeared image from the constant component of the photocurrent, produced by the photomultiplier P . A method similar to this was used in reference 29 to determine the waveform of standing waves. In reference 32 this method, with an ultrasonic optical shutter which permitted stroboscopic illumination, was also used to determine the waveform of a traveling wave of finite amplitude.

The distribution of the light intensity in the smeared slit image, obtained in reference 32 for the propagation of ultrasound (1 Mcs) in water at different distances from the source, is shown in Fig. 7 (1 - 10 cm; 2 - 50 cm; and 3 - 90 cm, for 500 volts applied to the quartz). The dotted lines indicate the mean position of the slit. The qualitative similarity in the behavior of the maxima in the smeared image of the slit in Fig. 7 and the maxima which surround the distribution of light intensity in the diffraction of light on a wave of finite amplitude, should be noted.

FIG. 7. Distribution of light intensity for a broader slit image.³²



We also note the possibility of combining the acoustic spectral method with the optical one. Making use of a filter plate for separating the harmonics of the wave, as pointed out above, one can use as the indicator of the harmonic the diffraction of light by the ultrasound^{33*} rather than a piezoelectric receiver tuned to the harmonic. We recall that in the diffraction of light on the

*This method was also reported by I. G. Mikhaïlov and V. A. Shutilov at the All-Union Conference on acoustics in 1958.

second harmonic, the distance between the diffraction maxima increases at a double rate, and so forth. The diffraction of light is shown in Fig. 8 for a distance of 25 cm from the source, for $I_{10} = 7.8$ watts/cm² and $\nu = 583$ kcs; the first of the photographs pertains to diffraction on the distorted wave, the second and third to diffraction on the second and third harmonics of this wave, separated by filter plates made of heavy flint glass.

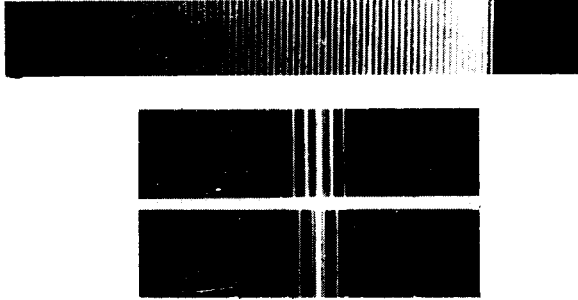


FIG. 8. Diffraction of light on a distorted wave and on harmonics separated with the help of a filter plate.

Thus a number of different experimental methods confirm the nonlinear distortion of the waveform in liquids. At the present time the distortion has been observed in the frequency range 0.5 – 4.5 Mcs. This is connected with the fact that at these comparatively low ultrasonic frequencies it is sufficiently simple to obtain waves with high acoustic Reynolds numbers; these frequencies are sufficiently high to allow for the production of plane waves. In this range of frequencies, the distortion has been observed at such low intensities that such measurements can also be carried out in principle at much higher frequencies.

The amount of experimental data on the distortion of the waves of finite amplitude is still small today, and it is not possible to regard it as sufficiently accurate to permit valid comparison with theoretical results. The maximum accuracy with which the harmonic content can be measured today (second and third harmonics) is no greater than 10 – 15%. Moreover, comparison with theoretical data is essentially complicated by the fact that it is often extremely difficult to create the idealized conditions of the theory experimentally. This is especially true of such phenomena as cavitation, heating of the medium, etc.

For all that, we shall, where it is possible, attempt to compare, even if only qualitatively, the experimental results with the theoretical. At distances small in comparison with the distance of formation of a wave of stable shape, the

experimental growth of the second and third harmonic agrees qualitatively with that which is given by the Riemann solution; in Fig. 3a, the dashed lines show the growth of the second harmonic in accord with (2.9) for $B/A = 5.0$. As is seen, at small distances the second harmonic is proportional to x and p_{10}^2 and the third to x^2 and p_{10}^3 .²⁶ The agreement is still better if dissipative losses are taken into consideration.⁹ We note that the method of direct determination of the distortion, in view of its sufficient sensitivity, can (the same as was done in reference 8 by using the interaction of two waves) be used to determine the dynamic nonlinear constants of the liquid.

For small Reynolds numbers, the stabilization range of the second harmonic, determined from (2.22), agrees very well with the experimental value. In particular, this has been verified for acetic acid.³⁴

The existence of a region of stabilization of the wave has been confirmed experimentally. Data of harmonic analysis²⁷ of oscillograms of the type shown in Fig. 4 are plotted in Fig. 9.

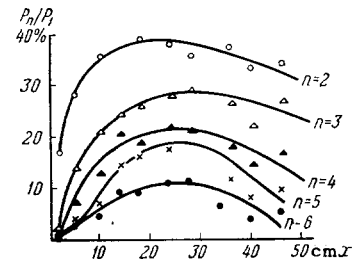


FIG. 9. Relative spectral composition of finite amplitude waves at different distances from the source of sound.²⁷

The relative content of the harmonic p_n/p_1 is plotted along the ordinate, and the distance from the source along the abscissa; the numbers of the curves correspond to the numbers of the harmonics. The spectral composition of the wave at distances $\sim 15 - 40$ cm from the source changes very slightly ($\sim 10\%$), which permits us to speak of a region where the waveform is stable.

All the experimental facts given correspond qualitatively to the theoretical results. However, certain phenomena cannot be explained from the viewpoint of the theory given above. In the first place, there is the difference in the distortion of the positive and negative phases of the pressure. As established in reference 27, the waveform of an intense ultrasonic wave changes with time. The oscillograms in Fig. 4 were obtained, as pointed out, within 1 sec after applying the voltage to the source of sound. Within 5 – 7 sec, the amplitude decreases, and on the side of negative pressures there is a shearing off of the sharp tooth of the wave. Thereafter, this phenomenon becomes still more clearly outlined. Such an asymmetry in the

distortion of the positive and negative phases of the pressure is evidently explained by cavitation or precavitation phenomena. It can be interpreted as the difference in the absorption of the different phases of the wave, which cannot be explained by the equations of hydrodynamics of a viscous liquid.

Moreover, the front thickness of the sawtooth wave, determined from these data, is appreciably larger than it should be according to (2.28).³⁵ Here it suffices to note that the accuracy of the determination of the thickness of the front, according to the data in the literature, is extremely small because of the small accuracy of determination of the spectral composition. A front thickness $\sim 10^{-3}\lambda$, obtained in pressure jumps of the order of tens of atmospheres for a sawtooth wave in liquids, can scarcely be determined by the method of harmonic analysis.

The reason for the lack of correspondence with hydrodynamic theory must evidently be sought in the nonlinear character of the dissipative processes that take place in a wave of finite amplitude.

4. ABSORPTION OF WAVES OF FINITE AMPLITUDE IN LIQUIDS

It has long been known that, when working with comparatively intense ultrasonic waves, absorption coefficients can be obtained which differ appreciably from those measured at low intensities. In the development of the relaxation theory of absorption attempts were even made at attributing the excess absorption (over Stokes') to nonlinear distortions.³⁶ The fundamental reason for the appreciable increase in the absorption of waves of finite amplitude in comparison with waves of small amplitude is the nonlinear distortion of the wave, which can be represented as the conversion of the wave into a non-monochromatic one. The amount of experimental data today is still insufficient to be able to say with assurance that the dissipative processes in the wave are nonlinear, but, from our point of view, some experimental results can be regarded as pointing to the nonlinear character of the dissipative processes.

In speaking of the absorption of waves of finite amplitude, we should keep it in mind that, in contrast to the absorption of waves of small amplitude, the absorption of waves of finite amplitude is nonexponential. This is connected with the fact that [as follows from (2.35)] the absorption coefficient depends on the spectral composition of the wave. Inasmuch as the spectral composition of the wave changes during propagation, the absorption

coefficient also changes. As was pointed out above, there is a region in which the change in the spectral composition changes but slightly (the stabilization region), where the absorption coefficient changes slowly. Near the source which radiates monochromatic waves, the absorption coefficient is close to the coefficient for waves of small amplitude; in the stabilization region, it is maximal; at large distances, when the wave returns to sinusoidal, the absorption coefficient tends to the coefficient for a wave of small amplitude.

Before proceeding to the specific results of the determination of the absorption of waves of finite amplitude, we note several experimental peculiarities of the measurement of finite amplitude waves. These peculiarities are, in the first place, associated with the fact that the finite amplitude wave is nonmonochromatic. Strictly speaking, it is necessary to use a broadband receiver to determine the absorption coefficient of a finite-amplitude wave. If the measurements are carried out in the region of stabilization of the wave, then we can determine any partial absorption coefficient, which is equal in this case to the absorption coefficient of a wave of finite amplitude. At high ultrasonic frequencies (~ 4 Mcs), the stabilization of the wave takes place rather rapidly and, keeping in mind the rather large experimental error, it is possible to assume that the measurements are carried out in the stabilization region.

We also note that acoustic streaming, the possible appearance of cavitation, and also the heating of the medium under the action of the ultrasound have an effect on the results of measurement. Very significant difficulties at low (< 1 Mcs) frequencies can be avoided by measurements carried out ahead of the stabilization region. In this case, the absorption coefficient will be less than what follows from (2.30) and (2.32). As a result of the effect of all the factors mentioned, the error of measurement of the absorption coefficient of finite-amplitude waves ($\sim 10 - 20\%$) is greater than the error of small-amplitude measurements ($\sim 10\%$). Mention should also be made of one more difficulty. In order to obtain high acoustic Reynolds numbers, it is advantageous to work at comparatively low ultrasonic frequencies. Along with this, investigation of the entire range (excluding that of small Re) is made difficult by the large errors in the measurement of small absorption coefficients. This is explained by the fact that the region of small Re has been investigated up to now at frequencies above ~ 3 Mcs, while the region of high Re has been studied at frequencies below ~ 2 or 3 Mcs. The equivalence of the different frequen-

cies, keeping in mind some difference in the experimental and theoretical data (see below), cannot be assumed as self-evident.

A systematic investigation of the dependence of the absorption on the intensity was initiated by Fox.³⁷ At a frequency of 9.916 Mcs, for intensities $0.01 - 5 \text{ w/cm}^2$, he made measurements in boiled distilled water, in water saturated with air, and in water saturated with CO_2 . The intensity, determined with an absorbing radiometer protected against acoustic streaming, is shown in Fig. 10 as a function of the distance. Curve 1 cor-

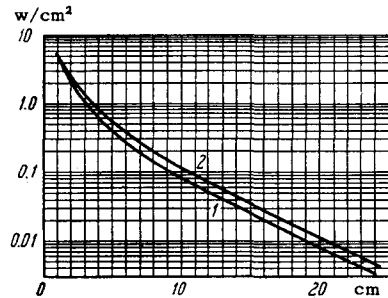


FIG. 10. Absorption in water at 10 Mcs according to reference 37; 1 - water saturated with air; 2 - water, saturated with CO_2 .

responds to water saturated with air (the curve for degassed water, obtained at distances up to 8 cm, coincides with this curve); curve 2 corresponds to water saturated with CO_2 . In the semi-logarithmic plot, the slope of these curves determines the absorption coefficient. It is seen from this picture that up to $\sim 0.04 \text{ w/cm}^2$ the absorption coefficient is equal to 44×10^{-3} and does not depend on the intensity.* At 5 w/cm^2 , the absorption coefficient increased to 190×10^{-3} . Insertion of an electric filter to eliminate harmonics in the signal applied to the radiating crystal gave the same results, within the limits of measurement errors. Thereafter, the measurement of the absorption of waves of finite amplitude was made in a number of researches^{4,5,10,28,38-42,46} in different liquids and by various methods. The majority of the researches indicate that the increase in the absorption coefficient depends on the viscosity of the liquid: for otherwise equal conditions, the relative absorption coefficient in viscous liquids† is less than in low-viscosity liquids. This agrees qualitatively with the values of both for $\text{Re} \ll 1$ (2.30) and for $\text{Re} > 1$ (2.32). The dependence of the absorption coefficient on the viscosity of the liquid was verified in reference 42, where measurements

*We note that these intensities correspond to a Reynolds number $\sim 1/6$; at a value of $\gamma \sim 7.7$ for water, this result is in satisfactory agreement with (2.31).

†By relative absorption coefficient we mean here and in what follows the ratio of the coefficient of absorption of a finite-amplitude wave to the coefficient of absorption of a wave of small amplitude.

were made of the absorption coefficient in transformer oil in the region of its maximum value. Inasmuch as in this liquid the density and velocity change insignificantly with changing temperature, while the shear viscosity changes appreciably, one can estimate the dependence of the absorption coefficient on the viscosity from the temperature dependence of the absorption at an identical frequency and intensity. These measurements were carried out for $\text{Re} \sim 2 - 10$. It follows from (2.32) that the absorption coefficient of the finite-amplitude wave α should not depend on the viscosity in the region of stabilization. The results of the measurements are shown in Fig. 11, from which it is seen that the absorption at higher temperatures increases somewhat. This can be attributed both to the inapplicability of (2.32) for Re close to 1, and to a temperature dependence of the bulk viscosity which is different from the dependence of the shear viscosity.

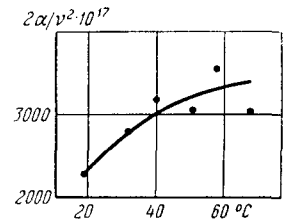
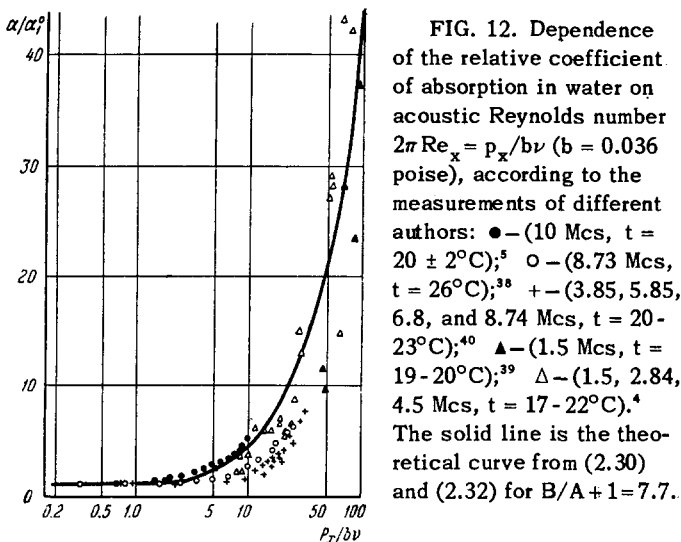


FIG. 11. Dependence of α/ν^2 on temperature in transformer oil at 1.5 Mcs.⁴²

As concerns the frequency dependence of the absorption of a wave of finite amplitude, a comparison of the absorption in water^{5,37,10} shows that at lower frequencies (for otherwise equal conditions) the increase in the relative absorption coefficient is appreciably greater than at the higher frequencies. This also agrees qualitatively with (2.30) and (2.32). Later, this was demonstrated in measurements of the absorption at different frequencies.⁴⁰ At relatively small intensities, $\alpha \sim p^2$; as the intensity is increased in the region of maximum absorption, $\alpha \sim p$.

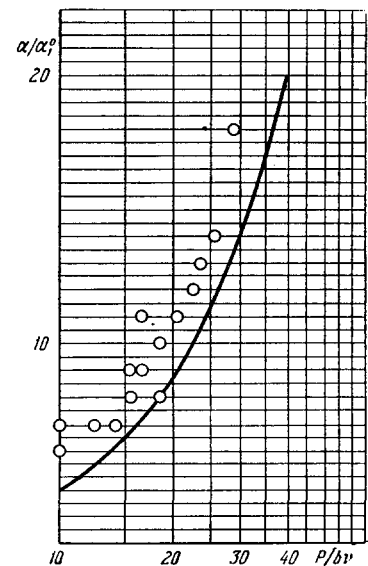
Thus, it has been shown by different authors, working with different methods, that the experimental values of the relative absorption coefficients of finite-amplitude waves are in qualitative agreement with (2.30) and (2.32). Turning to a quantitative comparison with theory, it should be noted that there are still too few experimental data and, on the other hand, that the existing material was obtained at a time when a number of absorption problems were not yet clear. Therefore, the literature lacks all the data necessary for comparison with theory. Data are given in Fig. 12, from various sources on the absorption in water and its dependence on the Reynolds number $2\pi \text{Re}_x = p_x/b\nu$, where p_x is the amplitude of



the sound pressure at the point of determination of the absorption coefficient. On this graph the solid line is the theoretical curve computed from (2.30) for $Re_x \leq 1$ and from (2.32) for $Re_x > 1$; the values $b = 0.036$ poise and $B/A = 6.7$ have been used. It should be remarked that in most references from which the data are taken, it is not noted whether the data refer to the sound pressure at the point of measurement of the coefficient α or to the sound pressure at the source. It is possible that the deviation of the experimental data for $2\pi Re_x \sim 6-30$ from the theoretical curve is due to this fact. Another reason for the rather large scatter of the experimental results may be that not all the data refer to the region of stabilization of the wave. As already observed above (see page 589) the absorption coefficient in the region of stabilization reaches a maximum value. Thus the solid curve in Fig. 12 refers to the maximum coefficient of absorption. We note that only deviations on the low side can be explained in this way. Bearing all this in mind, we can say that the experimental results of absorption of water agree quite satisfactorily with the theoretical values if we use the largest B/A (see Table I).

Figure 13 shows the dependence of the relative coefficient of absorption on the acoustic Reynolds number in methyl alcohol.⁴ The solid line corresponds to the theoretical curve of (2.32) with $b = 0.0174$ poise and $B/A = 7.3$. In view of the rather large error of measurement ($\sim 20\%$), the agreement with theory can be regarded as satisfactory. These measurements were carried out at 4.5 Mcs for relatively small Reynolds numbers; the measurements at 1.5 Mcs for relatively large Re show that there is a substantial difference between theory and experiment.

FIG. 13. Dependence of the relative coefficient of absorption α/α_1^0 on the acoustic Reynolds number $2\pi Re_x$ in methyl alcohol at 4.5 Mcs.



In Table II data are given on absorption in transformer oil. For calculations here, use has been made of a shear viscosity $\eta_{20^\circ} = 0.024$ poise and a Stokes' absorption $(2\alpha_1^0/\nu^2)_{20^\circ} = 430 \times 10^{-17}$.

TABLE II. Absorption in transformer oil⁴

$\frac{p_x}{\eta\nu}$	$\frac{2\alpha}{2\alpha_1^0}$	$\frac{p_x}{\eta\nu}$	$\frac{2\alpha}{2\alpha_1^0}$	$\frac{p_x}{\eta\nu}$	$\frac{2\alpha}{2\alpha_1^0}$
1.1	1.3	2.1	3.0	6.2	5.5
1.1	1.2	2.6	3.1	6.8	9.7
1.4	1.3	2.9	3.7	7.0	6.2
1.5	2.1	3.9	5.9	7.6	9.2
1.7	2.5	4.1	4.3	8.2	9.4
1.7	2.4	4.8	6.8	9.4	9.8
1.9	2.6	4.8	5.6		

It is easy to see that such a choice of the viscosity does not affect the slope of the curve $\alpha/\alpha_1^0 = \varphi(Re_x)$. It follows from the data of Table II that for transformer oil, $B/A \cong 15 \pm 3$, which is also the highest value for liquids (see Table I).

Thus, we can assume that in some liquids, the absorption, while qualitatively in agreement with Eq. (2.32), which is derived from the equations of hydrodynamics, does not conform quantitatively to the theoretical absorption. We note that the measurement of the absorption under conditions in which the production of cavitation is rendered difficult (by raising the hydrostatic pressure to 15 kg/cm^2)¹⁰ shows that the development of cavitation processes cannot play a significant role in the increase of absorption.

With regard to the absorption of finite-amplitude waves in liquids possessing a relaxation frequency in the frequency range under study, such measurements were carried out in aqueous solutions of

acetic acid of varying concentration (0.1–4 M) at 6.8 Mcs and in solutions of magnesium sulfate at 3.85 and 6.8 Mcs.⁴⁰ In solutions of acetic acid, an increase in the concentration of acid led to a decrease in the relative absorption coefficient, which also agreed with references 34 and 38. In view of the relaxation frequency of acetic acid,* it should be said that the measurements of references 38 and 40 refer to $\omega\tau > 1$; the maximum value of the Reynolds numbers in these measurements is $Re \sim 3.5$. It is possible that the smallness of the acoustic Reynolds number was the reason for the fact that the bulk viscosity behaved the same as the shear viscosity. No difference from absorption in other liquids was discovered.

Unfortunately, data on the absorption of finite-amplitude waves in various liquids (other than water) are extremely scanty at the present time; measurements of absorption in ethyl alcohol, toluene,³⁹ and transformer oil were carried out by the same method (the systematic error associated with the method can thus be considerable). It is quite difficult to explain the reason for the increase in the absorption coefficient of waves of finite amplitude over the theoretical value given by Eq. (2.32). All that is evident is that this increase takes place for intense sound. On the other hand, this is not energy dissipation due to the usual nonlinear distortions connected with nonlinear hydrodynamic equations. It is known that intense ultrasonic waves are accompanied by cavitation, which is not described by the equations of hydrodynamics. However, the absorption was determined under conditions in which cavitation did not play an essential role. It is highly probable that there occur in the medium various irreversible processes, the effect of which increases with increasing amplitude, and which are different from cavitation.

In this connection, the research of reference 45 should be pointed out, where a study has been carried out on solutions of polymers. In this work it has been established that when an intense sound pulse acts on polymer solutions, the absorption above certain threshold intensities begins to depend on the time of action of the ultrasound on the solution. After the passage of a sufficiently long time interval (several times ten minutes), "saturation" takes place; the absorption does not change with time. It is very important that this absorption is not determined by the nonlinear distortions, because transmission of sound pulses of alternately

*Various values have been obtained for the relaxation frequency of acetic acid, from $\sim 5 \times 10^5$ to 3×10^6 cps (see, for example, references 43 and 44).

large and small amplitude has shown that a similar change takes place in the logarithm of the voltage on the receiver. After the ultrasound was turned on for a certain time ($\sim 5 - 10$ min under the conditions of the experiment), the solution "relaxed," i.e., it completely regained its initial acoustical properties. We note that for a solution of polyisobutylene in benzene at 5–35 Mcs and 1.8 kilovolts on the radiating quartz, the absorption in the disturbed and undisturbed solution differed in a ratio of about five to one. The authors attribute this phenomenon as a break in the Van der Waals links of the polymer chain.

Without mentioning the specific mechanism of this phenomenon, it should be noted that this work indicates in some measure that the mechanism of nonlinear distortion is not the only one responsible for the absorption of waves of finite amplitude. In a medium under the influence of intense sound, changes take place which can have a strong effect on the dissipation of energy.

Summarizing, it must be said that the maximum coefficient of absorption in ordinary, not especially viscous liquids at intensities of several w/cm^2 can exceed the small-amplitude coefficient by two orders of magnitude.^{10,46} Thus, absorption due to nonlinear distortion is materially larger than the super-Stokes part of the absorption. It should be remarked that the theoretical results on the absorption of waves of finite amplitude correctly describe the observed phenomena qualitatively and give values of the relative coefficient of absorption that are correct in order of magnitude. However, in liquids other than water the absorption coefficient in the stabilization region is evidently somewhat larger than it should be from (2.32).* The reason for this, as also for some of the aforementioned deviations in the distortion, must evidently be sought in the nonlinear character of the dissipation processes. In addition to the study of the absorption of waves of finite amplitude in liquids, certain researches were devoted to the velocity of propagation of finite-amplitude waves. In reference 38, the velocity was measured in degassed water, glycerin, and weak solutions of acetic acid and sodium acetate with an ultrasonic interferometer with two crystals, at sound pressures of 0.1–5 atmos and at 4.7 Mcs. Within the limits of accuracy of measurement ($\pm 0.1\%$), no change of velocity with increasing intensity was established. Measurement of the velocity of finite-

*It should be noted that, for many liquids, the nonlinear parameter B/A is not known at the present time. This complicates the comparison of experimental absorption data with theory.

amplitude waves in a number of liquids (ethyl alcohol, benzene, and others), by the method of light diffraction by ultrasound, was reported in reference 30. The errors of these measurements did not exceed 0.05%. No difference in the velocity of the finite-amplitude wave from the velocity of a wave of small amplitude was discerned within the limits of measurement errors. In this connection, we point out what is noted in reference 16, that the velocity of propagation of the "zeros" (if we speak, for example, of acoustic pressure, then the "zeros" correspond to points of the wave where the pressure is equal to the hydrostatic pressure in the medium) must be equal to c_0 , the velocity of sound. It should be noted that the propagation velocity of a weak shock must also be equal to the sound velocity (see, for example, reference 14; the velocity of a shock, as is well known, is determined by the secant of the Hugoniot adiabat and differs from the sound velocity). Keeping in view the possibility of increase in the accuracy of determination of the velocity, the performance of such experiments with a sawtooth wave at higher Reynolds numbers is of interest.

CONCLUSION

Until very recently it was assumed that nonlinear distortions of acoustic waves in liquids, associated with the nonlinear equation of state, could not take place because of the small value of the experimentally attainable sound pressures in comparison with the internal pressure in the liquid. Investigations of the distortion by different methods have shown that these nonlinearities possess a much larger effect in liquids than in gases (for equal compressions). Keeping in mind the relatively small (in comparison with gases) absorption of sound in most liquids, the nonlinear distortions lead to the cumulation of a weak shock at any wavelength. Theoretical researches show that the formation of a shock is possible in the case in which the acoustic Reynolds number $Re > 1$. We note here that acoustic methods, with application of waves of finite amplitude, uncover several experiment possibilities; in particular, they permit us to determine the nonlinear constants of a barotropic liquid. It is very significant that different characters of weak shocks in liquids can be investigated by acoustic methods.

Investigation of the absorption of waves of finite amplitude in liquids shows that, in contrast to absorption of waves of infinitesimal amplitude, this absorption bears a nonexponential character

and depends on the Reynolds number. The absorption coefficient in the region of stabilization of the wave, for relatively small intensities (order of several w/cm^2), can exceed the absorption coefficient of a wave of small amplitude by two orders of magnitude in low viscosity liquids. The mechanism of the increase of absorption is fundamentally determined by the nonlinear distortion of the waveform.

It should be emphasized that a number of the experimental results set forth do not fit into the framework of existing theory and can serve as the basis for further development of theory and for setting up of experiments.

We note in closing that the processes of waveform distortion of an acoustic wave in gases are sufficiently well understood. However, the absorption of such waves in gases has been investigated much less than in liquids.

In solids, the appearance of singularities in the propagation of waves of finite amplitude is also possible; however, this question has been investigated almost not at all.

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Translated by R. T. Beyer

Translator's Comments—This paper provides an excellent review of the research work, carried on mainly in America and the Soviet Union, on the propagation of finite amplitude ultrasonic waves in liquids. This field was first explored theoretically by Riemann and others one hundred years ago, but the modern revival of interest has stemmed largely from the theoretical papers of Fay (1931) and Eckart (1948). The use of perturbation theory in the latter paper has been picked up and exploited by the Russian school, led by Andreev and his students at the Acoustics Institute, Moscow. In particular, they obtained expressions for the second harmonic content in the wave and for the effective absorption coefficient, and studied the conditions for the appearance of a discontinuity.

Although some American experimental work appeared in the middle 1950's, beginning with Fox and Wallace (1955), the experimental work of Krasil'nikov and co-workers at Moscow University, and of Mikhailov and his group at Leningrad University have led the field.

One useful contribution of the paper is the compilation of

the nonlinearity index B/A for all available cases. This reviewer¹ has been able to extend this table to show the temperature dependence of the ratio for a number of liquids.

The most recent work in finite propagation studies (not reported in this article) has centered about theoretical² and experimental³ investigations of finite amplitude propagation in relaxing media, and also on attempts to examine the harmonic content in more detail.⁴

One omission should be noted—that of the paper by Fubini-Ghiron⁵ in which the harmonics were calculated explicitly for a finite amplitude wave in a dissipation-free medium. This paper appears to be equally unknown to American researchers.

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