New Instrument and Measurement Methods

COMPARISON OF FREQUENCIES BY THE SINUSOIDAL AND ELLIPTIC SWEEP METHODS

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THE cathode-ray tube can be used to compare frequencies, over a wide range, with great flexibility and sensitivity (up to 1.5×10^{-6} cps at an observation time of 3 minutes). Unlike methods of frequency comparison, oscilloscopic methods permit the use of higher frequency ratios, so that fewer harmonics and subharmonics need be generated.

The cathode-ray oscilloscope can indicate the equality of two frequencies, or the existence of a definite ratio between them, only within a narrow frequency range (at a 1 : 1 frequency ratio this range does not exceed approximately ± 2 cps) in which the human eye is capable of discerning with certainty changes that occur in the outline of the pattern on the screen.

The most convenient way of comparing frequencies is to use stationary patterns. This is possible if the frequency of one generator can be varied continuously. By comparing two fixed frequencies, the difference frequency is determined from the measured repetition period of the moving figure, i.e., the interval of time during which a pattern goes through a complete sequence of transformations.

Oscillographic methods of frequency comparison are usually used in the range from approximately 10 cps to 30 Mcs.

1. THE SINUSOIDAL SWEEP METHOD*

This method features a principal circuit of exceeding simplicity: potentials of the measured and known frequency are applied to the deflecting plates of an cathode-ray tube.[1-7] In ordinary frequency measurements, as long as the short-period phase instabilities of the oscilloscope amplifiers do not play a substantial role, and the compared frequencies can still be handled by the amplifiers, both frequencies can be applied to the input terminals of the instrument. This permits frequency comparison at low voltages and makes for convenient regulation of the size and shape of the figure.

The phase difference between the voltages of

*The author avoids the use of the ambiguous term "Lissajous figure."

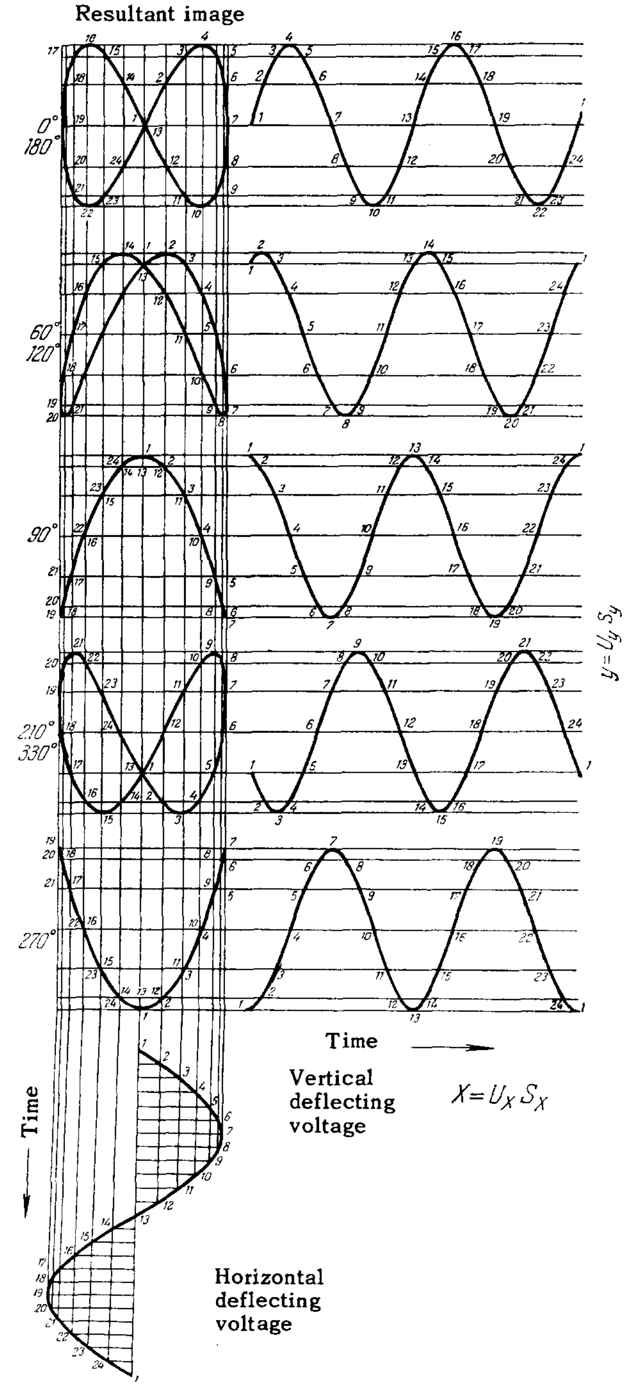


FIG. 1. Sinusoidal-sweep pattern at a 2:1 frequency ratio, for different phase angles between the deflecting voltages U_x and U_y.

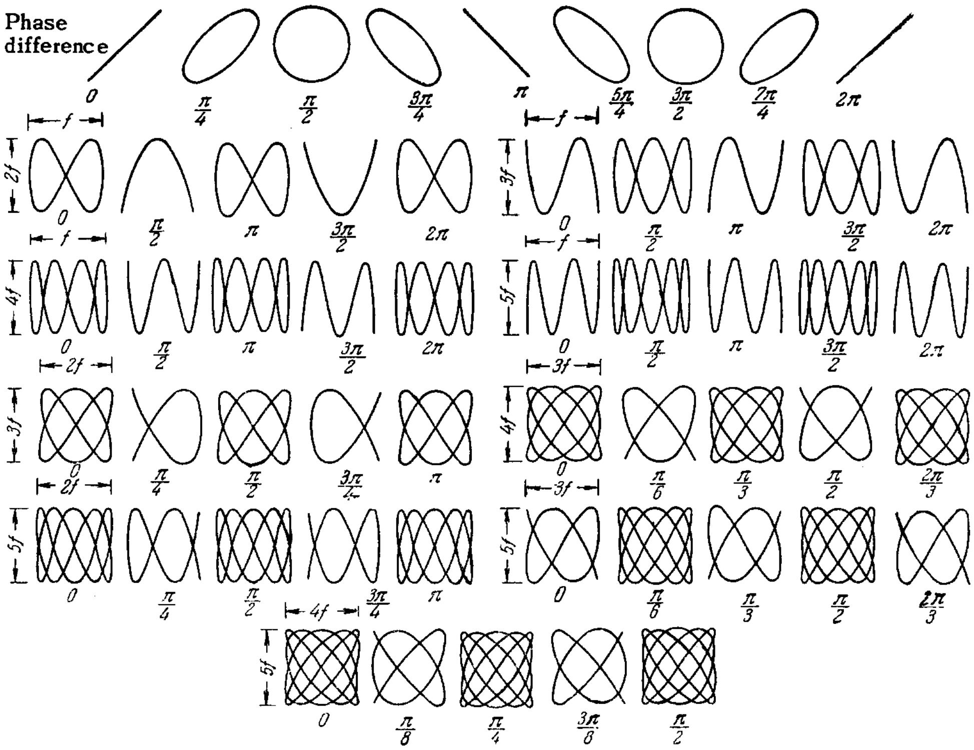


FIG. 2. Various sinusoidal-sweep figures for equal horizontal and vertical deflections of the beam.

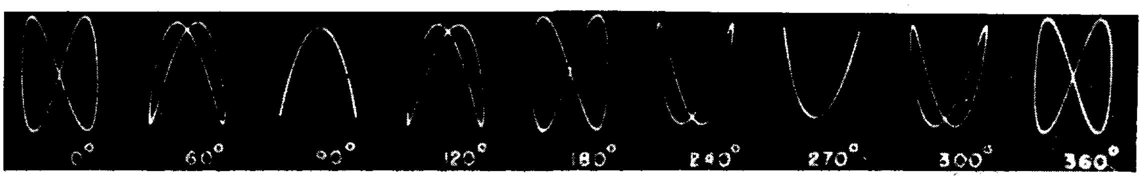


FIG. 3. Sinusoidal-sweep figures at a 2:1 frequency ratio.

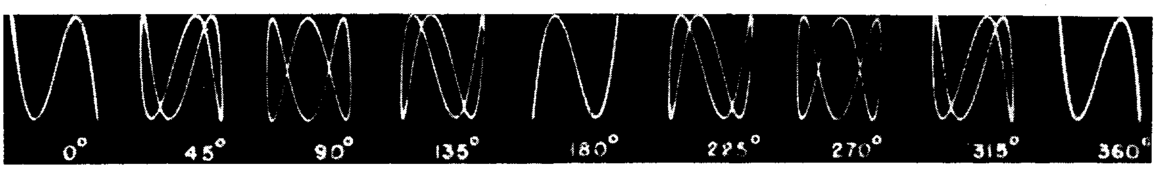


FIG. 4. Sinusoidal-sweep figures at a 3:1 frequency ratio.

the compared frequencies is measured in fractions of a cycle of the highest frequency. A 180° phase difference means that the time of start of the positive half wave of the higher-frequency voltage is separated from the time of start of the positive half wave of the lower-frequency voltage by an interval equal to half the period of the higher frequency.

Let us consider the formation of sinusoidal-sweep figures at a frequency ratio 2 : 1 in the case when the horizontal and vertical beam deflections are of equal amplitude. The vertical deflection of the beam is $U_y S_y$, where U_y is the voltage on the vertical plates and S_y is the sensitivity of the cathode-ray tube in the vertical direction. The horizontal deflection of the beam is $U_x S_x$, where U_x is the voltage on the horizontal plates and S_x is the sensitivity of the tube in the horizontal direction. The lower half of Fig. 1 shows the time variation of the dependence of the deflection of the bright spot on its initial position at point 1, under the influence of a horizontal deflecting voltage. The right portion of Fig. 1 shows an analogous diagram for the vertical deflection. The resultant pattern shows the successive positions occupied by the spot on the screen tube during one cycle of the lower frequency.

If the frequencies are exactly related as the ratio of two integers, then the trace of the beam is superimposed on itself, and a stationary pattern appears on the screen.

The shape of the oscillogram depends not only on the ratio of compared frequencies, which is of interest to the experimenter, but also on the amplitude ratio, on the phase shift between the two voltages, and on the presence of higher harmonics. Various methods for interpreting the patterns make it possible to establish the frequency ratio uniquely.

Figures 2 – 4 show sinusoidal-sweep figures for equal vertical and horizontal deflection amplitudes and for a sinusoidal waveform. When the frequencies are equal, simple figures, without intersecting lines, appear on the cathode-ray screen – straight lines, ellipses with various axis inclinations, or circles. As the frequency ratio increases, the figure becomes more complicated. Fractional frequency ratios result in loops or peaks, the number of which increases with the numerator and denominator of the irreducible fraction that expresses this ratio (Figs. 2 and 5). When the horizontal and vertical voltages are interchanged, the pattern is rotated by 90°.

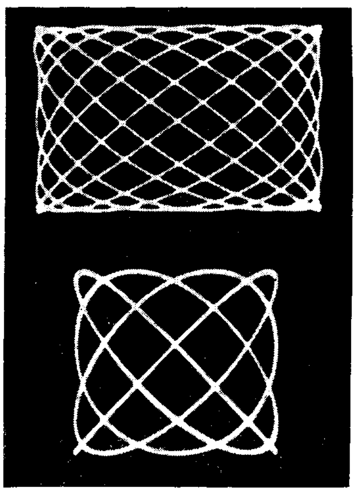


FIG. 5. Sinusoidal-sweep figures at a frequency ratio 8:7 (different phase positions).

If the frequency ratio is nearly the quotient of two integers, a cycling pattern is produced, which passes in sequence through all the possible phase positions. An impression is gained that the pattern is drawn on a transparent cylinder, the axis of which is parallel to the plane of the screen.

If the frequency ratio differs from an integer or from a rational fraction in such a manner, that the difference between the measured frequency and the nearest frequency that gives a stationary pattern exceeds approximately 20 cycles, a faint uniformly-glowing rectangle is seen on the oscilloscope screen. This is due to the fact that the spot continuously changes its trajectory. As the frequency ratio approaches that of two integers, the trajectories come closer and the flickering lines gradually merge into one moving figure, which stops when this ratio is reached exactly. The repetition period of the pattern produced in the case of a rational-fraction frequency ratio is shorter than that of a whole-number frequency ratio.[8,9] During the time when the pattern goes through its complete set of transformations, the phase of the upper frequency changes by an amount which is determined by dividing 2π by the smaller of the numbers contained in the irreducible fraction that expresses the frequency ratio (see Fig. 2).

The best way to interpret sinusoidal-sweep figures is the "tangency method."[10] Tangents are drawn to two sides of the pattern, as shown dotted in Fig. 6a, and the number of peaks or loops tangent to the lines is counted. The ratio of the number of tangencies equals the ratio of the compared frequencies. When the loops of the figure are superimposed on each other, each loop or peak is counted as two, and the free end of the curve is counted as one (Fig. 6b). With a little skill it is possible to count the number of loops instead of the number of tangencies, or the number of peaks and free ends of the curve on the side of the pattern.

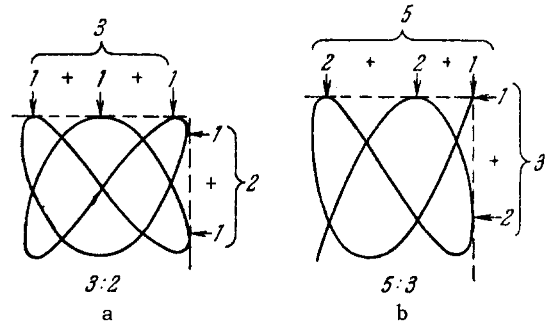


FIG. 6. Interpretation of sinusoidal-sweep figures: a – figure with complete loops (ratio 3:2), b – figure with incomplete loops (frequency ratio 5:3).

A judicious choice of the dimensions of the figure makes its interpretation easier. Thus, the oscillogram of Fig. 7b is easier to read than that of Fig. 7a. It is advisable to spread out adjacent loops or peaks on one side of the figure and to bring together excessively spaced loops or peaks on the other side. The pattern is stretched or compressed by adjusting the sensitivity of the oscilloscope amplifier.

If the measured frequency is known approximately, it can be determined uniquely for a whole-number ratio of frequencies. In other cases a bad

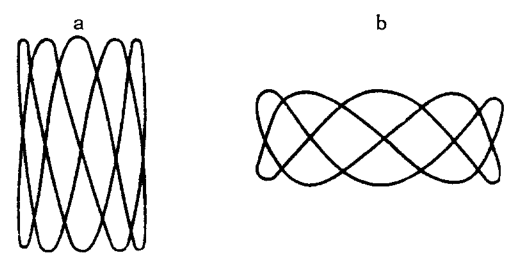


FIG. 7. Different forms of a sinusoidal sweep figure with a frequency ratio 5:2: a – difficult to read, b – easy to read.

error can be committed. For a unique determination of the unknown frequency it is necessary to take into account the directions in which both the deflecting voltages act on the electron beam.[7] The horizontal tangent to the figure touches the peaks and loops produced by the vertical deflection of the beam. The vertical tangent touches the peaks and loops produced by the horizontal deflecting voltage. Thus, the following rule can be established for the unknown frequency

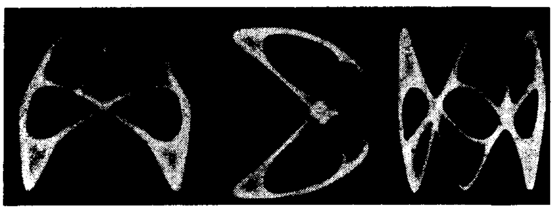
$$\frac{f_{\text{hor}}}{f_{\text{vert}}} = \frac{K_{\text{vert}}}{K_{\text{hor}}}, \qquad (1)$$

where f_{hor} is the frequency applied to the horizontal deflecting plates, f_{vert} the frequency applied to the vertical deflecting plates, K_{hor} the number of points of tangency between the figure and the horizontal tangent, and K_{vert} the number of points of tangency between the figure and the vertical tangent.

More complicated patterns are of less practical value than the simpler ones, since the low stability of the frequency makes it difficult to count the number of peaks, and the large number of lines makes analysis of the figure difficult. At the stability attainable in ordinary audio-frequency generators, it is possible to interpret patterns at frequency ratios up to 15 : 1, 11 : 5, or 8 : 7. Let us note that an oscilloscope with a tube 13 cm in diameter and a vertical amplifier with good overload ability can, in conjunction with a pattern that is stretched out beyond the limits of the screen, determine a frequency ratio up to 50 : 1 (the peaks are counted and the figure is shifted simultaneously in a vertical direction).

At radio frequencies, only simple patterns can be interpreted, unless the absolute stability of the compared frequencies is high.

If one of the frequencies is amplitude-modulated, the oscillogram assumes a peculiar shape, but can still be interpreted by the tangency method (Fig. 8). The presence of higher harmonics in one or both voltages distorts the pattern and makes its analysis difficult (Figs. 9 and 10).



a b c

FIG. 8. Sinusoidal-sweep figures with one of the frequencies amplitude-modulated: a and b – frequency ratio 2:1, c – frequency ratio 3:1.

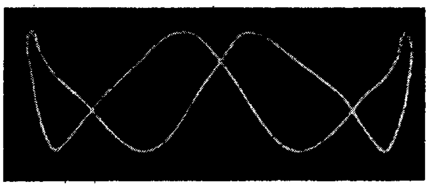


FIG. 9. Distorted sinusoidal-sweep figure, frequency ratio 4:1.

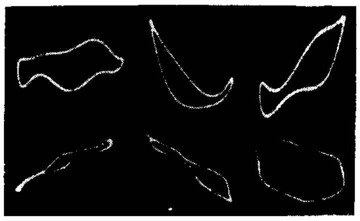


FIG. 10. Various phase positions of a distorted sinusoidal-sweep figure, frequency ratio 1:1.

Rotating Sinusoidal-Sweep Figures

Rotating figures are used to compare fixed frequencies, when the unknown frequency is almost equal to a standard frequency or to a certain harmonic of the latter. What is determined here is the difference frequency, obtained by measuring the repetition rate of the figure with a stopwatch or chronoscope (differential measurement method).

Work with rotating figures is difficult at very small frequency differences, because the instants of the start and finish of the repetition period of the figure are indefinite. If the difference frequency exceeds approximately 0.3 – 2 cps (the smaller figure corresponds to fractional frequency ratios that are characterized by a short repetition period of the figure), it becomes impossible to observe the pattern since it breaks up into a series of flickering lines.

The interpretation of rotating figures, unless they are very simple, is difficult even at a frequency difference on the order of 0.15 – 0.2 cycles. The frequency ratio can be determined by stopping the pattern with the aid of an auxiliary generator. The latter is tuned (to a known frequency) until a stationary ellipse appears on the oscilloscope screen. The known frequency is then replaced by the measured one, and the frequency of the auxiliary generator is adjusted carefully until the motion of the pattern is smoothly slowed down and eventually stopped.

At rational-fraction frequency ratios the patterns are frequently very close to each other. To avoid possible errors, a rational-fraction frequency ratio should be measured four or five times if this method is used.

If the frequency ratio is known to be an integer, it is possible to measure the unknown frequency approximately (see next section) and, by dividing the greater of the compared frequencies by the smaller one, to set the quotient equal to the next nearest integer.

The difference frequency f_d is determined in terms of the repetition period T of the figure from the following expression

$$f_{d(cps)} = \frac{1}{mT_{(sec)}}. \tag{2}$$

The coefficient m equals the smaller of the two numbers comprising the frequency-ratio. If the frequency ratio is an integer, m = 1.

If the difference frequency exceeds approximately 0.05 cycles, it is necessary, to insure the required accuracy in the measurement of the repetition period of the figure, to determine the time interval τ during which the figure goes through n complete repetition periods. In this case the difference frequency is

$$f_{d(cps)} = \frac{n_{(per)}}{m\tau_{(sec)}}. \tag{3}$$

It is easy to make mistakes in the number of repetition periods of a rapidly rotating sinusoidal-sweep figure. It is more convenient to count the number of positions at which the lines of the figure coincide with each other. Patterns of this type, with incomplete loops, can be called "open figures." In spite of the fact that open figures differ from each other unless the phase shifts are identical, the possibility of error is decreased, if they are not made different and if the instants when the bright lines are superimposed on each other are fixed. The count begins and the stopwatch is started at the instant when the first open figure is produced on the screen.

In this case the difference frequency f_d becomes

$$f_d = \frac{p-1}{K\tau}, \tag{4}$$

where p is the number of open figures counted by the observer, τ the observation time (seconds). and K the number of open figures per cycle of the higher frequency (at a given frequency ratio).

The values of K for the frequency ratios most frequently encountered in practice is given in Table I.

TABLE I

	Whole-number frequency ratio				Fractional frequency ratio				
$f_1 : f_2$	1 : 2	1 : 3	1 : 4	1 : 5	2 : 3	3 : 4	2 : 5	3 : 5	4 : 5
K	2	2	2	2	4	6	4	6	8

It is seen from Table I that K has twice the value of the smaller of the two integers in the irreducible fraction of the frequency ratio.

According to (3) and (4), the number of repetition periods of the figure is

$$n = \frac{p-1}{2}. \tag{5}$$

When working with rotating sinusoidal-sweep figures, the difference frequency f_d is

$$f_d = f_g - Nf_s, \tag{6}$$

f_g is the greater of the compared frequencies, f_s the smaller of the compared frequencies, N the frequency ratio (ratio of two integers). For the case when the measured frequency f_x is greater than the standard frequency f_0, we obtain from Eq. (6)

$$f_x = Nf_0 \pm f_d. \tag{7}$$

If $f_x < f_0$ we have

$$f_x = \frac{f_0 \mp f_d}{N}. \tag{8}$$

Thus, to measure the unknown frequency f_x against a fixed standard frequency f_0, it is necessary to establish the frequency ratio N, to measure[12,13] the difference frequency f_d, and to determine the sign of f_d.

The sign of the difference frequency can be determined by various methods. If the measured frequency can be varied, the pattern is stopped by slowly varying the frequency. If it is necessary to increase the measured frequency in order to stop the figure, the measured frequency is less than the standard frequency or its harmonic (Nf_0 or f_0/N). In this case a minus sign should precede f_d. Analogously, if the measured frequency must be decreased, the sign of f_d is plus.

If both compared frequencies are fixed, an auxiliary generator is used, which is alternately tuned to f_x and to Nf_0 or f_0/N with the aid of the oscilloscope. The relation between the two scale settings of the auxiliary generator establishes which of the frequencies, f_x or Nf_0 (or else f_0/N), is greater, and this determines the sign of f_d.

The sign of a very small difference frequency is established by modulating the brightness of the image and observing the direction of motion of the bright spots along a circle or an ellipse. First an auxiliary generator is used to establish the direction of rotation of the figure corresponding to a positive frequency increment. If oscilloscope amplifiers with unknown phase characteristics are used then, to avoid errors, the frequency of the auxiliary generator must be set equal to one of the compared frequencies.

If the difference frequency is very small, the time required for the measurement becomes excessive. In this case, in accordance with Eq. (6), it is advisable to use a higher standard frequency. Thus, when measuring the frequency of a reference 1000 cps oscillator, deviating from the standard 1000-cycle frequency by 10^{-8}, the figure goes through a complete repetition cycle in approximately 30 hours. If the standard frequency used is 100 kilocycles, the repetition period of the figure is reduced to 20 minutes.

If, to the contrary, the difference frequency is so great that examination of the figure becomes difficult or impossible, a lower reference frequency must be used when $f_x < f_0$, and the repetition period of the figure must be correspondingly increased. When $f_x > f_0$, f_x can be divided by a suitable factor (see, for example, references 14 – 16), or differential-generator method can be used.[17]

The relative error in the measurement of the difference frequency is

$$\frac{\Delta f_d}{f_d} = \pm \sqrt{\left(\frac{\Delta n}{n}\right)^2 + \left(\frac{\Delta \tau}{\tau}\right)^2}, \tag{9}$$

where $\Delta f_d / f_d$ is the relative error in the measurement of the difference frequency, $\Delta n/n$ is the relative error in the determination of the number of repetition cycles of the figure, $\Delta\tau/\tau$ is the relative error in the measurement of the time interval τ, during which the figure completes n repetition periods, of duration T ($\tau = nT$).

Denoting by Δf_0 the absolute error in the reference frequency, we obtain the following expressions for the relative error in the measured frequency

$$\frac{\Delta f_x}{f_x} = \pm \sqrt{\left(\frac{\Delta f_d}{f_x}\right)^2 + \left(N\frac{\Delta f_0}{f_x}\right)^2} \qquad (f_x > f_0);$$

$$\frac{\Delta f_x}{f_x} = \pm \frac{1}{N} \sqrt{\left(\frac{\Delta f_d}{f_x}\right)^2 + \left(\frac{\Delta f_0}{f_x}\right)^2} \qquad (f_x < f_0).$$

Inserting the value of Δf_d from (9), and bearing in mind that $f_d = n/m\tau$, we obtain when $f_x > f_0$

$$\frac{\Delta f_x}{f_x} = \pm \frac{1}{f_x} \sqrt{\left(\frac{\Delta n}{m\tau}\right)^2 + \left(\frac{n\Delta\tau}{m\tau^2}\right)^2 + (N\Delta f_0)^2}\ . \tag{10}$$

If $f_x < f_0$, we have

$$\frac{\Delta f_x}{f_x} = \pm \frac{1}{N f_x} \sqrt{\left(\frac{\Delta n}{m\tau}\right)^2 + \left(\frac{n\Delta\tau}{m\tau^2}\right)^2 + \Delta f_0^2}. \tag{11}$$

The observer can be assumed to fix the instant when the lines of the image coincide accurate to half the width of the line (both at the beginning and at the end of the repetition period of the figure). In comparing nearly equal frequencies, the figure at the beginning and the end of the repetition period is a straight line in one of two possible positions. In this case we obtain

$$\Delta n \cong \pm \sqrt{2}\,\frac{\frac{d}{2}}{a} \cong 0.7\frac{d}{a}, \tag{12}$$

where d is the width of the line of the ellipse, and a the amplitude of the oscillations of the beam on the oscilloscope screen. For d = 1 mm and a = 130 mm (5 inch oscilloscope tube), $\Delta n \cong \pm 0.005$ of the repetition period. The error in the number of repetition periods of a sinusoidal-sweep figure can be reduced considerably by stretching the figure far beyond the limits of the screen and by filtering carefully the higher harmonics of both voltages, so as to obtain an exact congruence of the bright lines.

The error $\Delta\tau$ in the measurement of the time interval τ, during which the figure goes through n repetition cycles, consists of several errors inherent in the stopwatch and in the observer himself. Let us start with the former errors.

In the calibration of the stopwatch, the error proportional to the time, δ_{vm}, can be calculated from the following formulas, based on the standards established for stopwatches[18] and on the assumption that the variation in the movement can amount to ⅓ the value of the correction

$\delta_{vm} = 7 \times 10^{-5}\,\tau$ (for stopwatches of first class);
$\delta_{vm} = 15 \times 10^{-5}\,\tau$ (for stopwatches of second class);
$\delta_{vm} = 25 \times 10^{-5}\,\tau$ (for stopwatches of third class).

It is necessary to take into account next the error due to the time of jump of the second hand of the stopwatch, δ_j (at the beginning and end of the measured time interval), and the error δ_{ecc} due to the eccentricity of the dial. For stopwatches of the first class these errors, amount to approximately ±0.1 seconds. A vertical dial position should be avoided, since it may introduce an additional error due to imperfect balancing of the wheel.

The reaction time of the observer differs from individual to individual as well as from measurement to measurement made by the same observer. The difference between the maximum and minimum reaction times comprises the so-called "psychotechnical error," δ_{ps}. This error, for an average observer under ordinary conditions, amounts to approximately ±0.1 seconds.[19,20]

The error $\Delta\tau$ in a stopwatch measurement of the time interval τ can be calculated approximately, by assuming δ_{vm} and δ_{ecc} to be random errors (actually δ_{ecc} is a systematic error, and

TABLE II

Class of stopwatch \ Time interval (minutes)	Error in seconds					
	up to 5	10	15	20	25	30
First class	±0.2	±0.2	±0.21	±0.22	±0.23	±0.24
Second class (second hand advances every 0.1 sec)	±0.2	±0.22	±0.24	±0.26	±0.29	±0.34
Second class (second hand advances every 0.2 sec)	±0.32	±0.33	±0.34	±0.37	±0.39	±0.42
Third class	±0.37	±0.39	±0.42	±0.47	±0.52	±0.57

δ_{vm} is due both to random and regular processes), in accordance with the formula

$$\Delta\tau \cong \pm\sqrt{\delta_{ecc}^2 + 2\delta_j^2 + \delta_{vm}^2 + \delta_{ps}^2}\ . \qquad (13)$$

The measurement errors of stopwatches of all three classes are listed in Table II.

The error of the 1000-cycle standard long-wave broadcast signal can be assumed to be $\Delta f_0 \approx \pm 3 \times 10^{-5}$ cps.

For a whole-number frequency ratio, the error in the repetition period of the figure is somewhat greater when $N > 1$ than when $N = 1$, whereas in the case of a fractional frequency ratio the error for $N > 1$ is less.

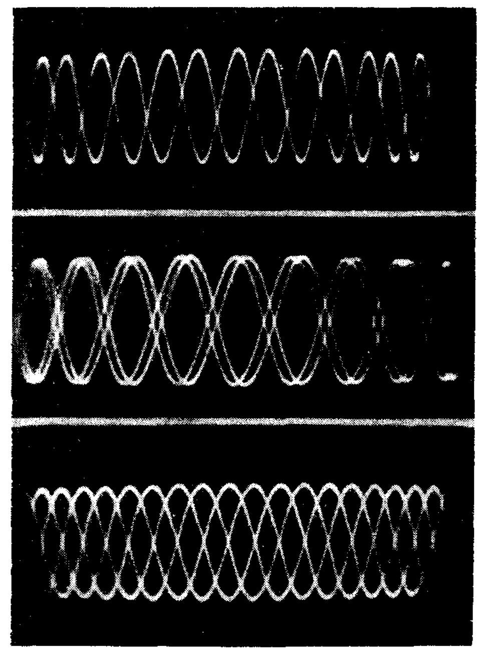
If the unknown frequency is compared against a standard frequency transmitted over a communication cable, the error due to phase fluctuations in the line should be taken into account.[22,21]

Use of Sinusoidal-Sweep Figures with a High Ratio

In cases when the pattern is not interpreted, but it is merely noted that the frequency ratio is an integer, a high frequency ratio can be used. Audio frequencies can thus be measured with a precision radio-frequency generator, or the scale of a generator can be calibrated over a wide frequency range against a few known frequencies.[24,23]

The "multiple figure" produced in the case of a whole-number frequency ratio, when both voltages are sinusoidal, has the form of two sine curves moving relative to each other and coalescing from time to time into one curve (if one frequency is an exact multiple of the other, the figure is stationary). To recognize a multiple figure in the case of a large frequency ratio, the oscillogram is stretched out, and only a small number of loops is left on the screen, by increasing greatly the gain of the amplifier to which the lower frequency is applied.

A whole-number frequency ratio that gives a pattern of simplest form is readily distinguishable from any fractional ratio, even with denominator 2



a

b

c

FIG. 11. Stretched sinusoidal-sweep figures at large frequency ratios: a and b – various phase shifts in multiple figures; c – fractional figure (denominator of the frequency ratio equals 2).

(Fig. 11, a, b, c). The distinguishing feature of the multiple figure is that there is no more than one intersection between two neighboring peaks.

When operating with stretched-out figures, it is preferable to use tubes producing bright green lines. If the compared frequencies have good stability and the oscilloscope vertical amplifier has considerable overload ability, it is possible[25] to recognize multiple figures with a frequency ratio up to 1600 : 1.

To determine large whole-number frequency ratios, the unknown frequency is measured approximately. The higher frequency is then divided by the lower one and the quotient is set equal to the nearest integer. The measured frequency is then calculated by multiplying or dividing the known frequency by the frequency ratio.

In this procedure it is essential to estimate correctly the measurement accuracy required in the preliminary measurement of the unknown frequency. Let f_X denote the measured low frequency, F the known high frequency and by N the frequency ratio, ΔN the absolute error in the determination of the frequency ratio, and $\Delta f_X / f_X$ the relative error in the preliminary measurement of f_X. If the error in the high frequency F can be neglected, then

$$\frac{F}{f_x \pm \Delta f_x} = N \mp \Delta N.$$

Hence

$$\Delta N = \frac{F \, \Delta f_x}{(f_x \pm \Delta f_x) \, f_x}.$$

Neglecting the small quantity Δf_X in the denominator, we have

$$\frac{\Delta f_x}{f_x} \cong \frac{\Delta N}{N}.$$

To prevent large errors it is necessary to insure $\Delta N < 0.5$. Taking $\Delta N \leq 0.4$, we obtain

$$\left[\frac{\Delta f_x}{f_x}\right]_{\lim} \leqslant \frac{40}{N}\,\%. \tag{14}$$

In choosing an instrument for the preliminary measurement of f_X, it must be borne in mind that the reduced measurement of the instrument may be considerably less than the limiting measurement error of interest to us in this case.

Measurement of Audio Frequencies with the Aid of a Heterodyne Frequency Meter and Oscilloscope

In view of the relative complexity and high cost of audio-frequency measuring apparatus with an accuracy on the order of $\pm(0.01 \text{ to } 0.05)\%$, considerable practical interest attaches to the use of a widely employed radio-frequency instrument — the heterodyne frequency meter — for this purpose. Audio frequencies can be measured with one heterodyne frequency meter, several frequency dividers, and a tunable intermediate generator.[26] An even better setup is one without an intermediate generator.[27] Livshitz[26] describes also a simple frequency measurement method, suitable above approximately 4.5 kcs, in which the unknown frequency is applied to the vertical amplifier of the oscilloscope, and the output voltage of a heterodyne frequency meter is applied to the horizontal amplifier. Near the lowest frequency of the instrument (usually 125 kcs) a pattern is established with a ratio N

$$F_1 = N f_x.$$

In view of the inadequate stability of the heterodyne frequency meter and in view of the complexity of the pattern, it is impossible to count the number of peaks. The frequency of the instrument is then gradually increased until the appearance of the next higher multiple figure, for which

$$F_2 = (N+1) f_x.$$

This yields $N = F_1/(F_2 - F_1)$ (which is set equal to the nearest integer), and the unknown frequency is

$$f_x = \frac{F_1}{N}. \tag{15}$$

In measurements of the lowest frequencies it is important that both settings, F_1 and F_2, be within the range of a single crystal control point.

To reduce the lower measurement limit it is necessary to reduce the error in the preliminary measurement of f_X by increasing the frequency difference, $F_2 - F_1$. The simplest procedure is to tune the equipment successively to obtain multiple figures. However, at very low frequencies a large number of settings is required: this method is therefore applicable either if lower accuracy is permissible or in conjunction with a supplementary calculation for the purpose of finding a more remote multiple figure.[28]

The heterodyne frequency meter is first adjusted for multiple figures $m+1$ times (Fig. 12).

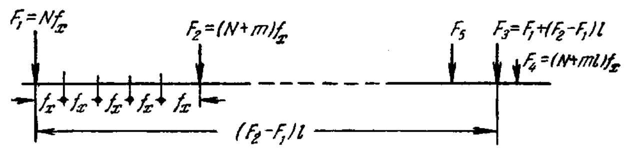


FIG. 12. Arrangement of settings of heterodyne frequency meter.

The frequency difference $F_2 - F_1$ (which approximately equals mf_X) is multiplied by a suitable integer l and the frequency $F_3 = F_1 + (F_2 - F_1)l$ is calculated. In view of the error in the determination of the F_3 and the errors in the heterodyne frequency meter at a given point on its scale, no multiple figure is usually observed. The multiple figure closest to F_3 will correspond to the sought exact tuning $F_4 = (N + ml) f_X$. The approximate value of f_X, which is later corrected after the frequency ratio is determined, is

$$f_x \cong \frac{F_4 - F_1}{ml}. \tag{16}$$

The number of intervals between multiple figures, m, the coefficient l, and the procedure for determining F_4 must all be so chosen as to exclude the possibility of an error.[28]

Frequencies on the order of several times ten cps and higher can be measured with a heterodyne

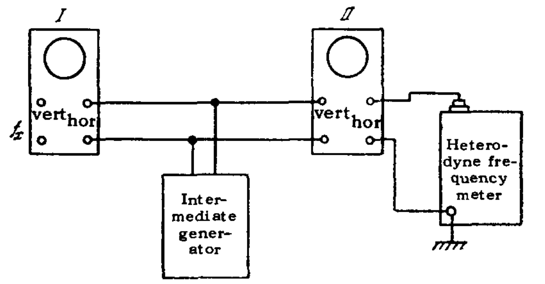


FIG. 13. Block diagram of the apparatus for the measurement of audio frequencies with a heterodyne frequency meter, an auxiliary generator, and two oscilloscopes.

frequency meter, an auxiliary audio generator, and two oscilloscopes (Fig. 13). The frequency of the auxiliary generator f_{int} is set to be an integral multiple of the measured frequency,

$$N_1 = \frac{f_{int}}{f_x}.$$

Oscilloscope I serves to monitor and determine the ratio N_1. Oscilloscope II is used to tune the heterodyne frequency meter to an integral multiple of the frequency of the intermediate generator

$$N_2 = \frac{F_1}{f_{int}}.$$

The measured frequency is calculated from

$$f_x = \frac{F_1}{N_1 N_2}. \tag{17}$$

To determine the frequency ratios of the heterodyne frequency meter and of the intermediate generator N_2, the scale of the former is tuned for $(m+1)$ – multiple patterns near the lower limit of the low-frequency range.

Then

$$N_2 = \frac{F_1 m}{F_{m+1} - F_1} \tag{18}$$

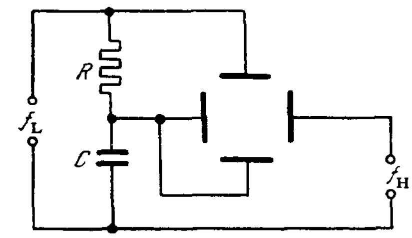
(which is set equal to the nearest integer). Here F_1 is the scale reading of the heterodyne frequency meter when set for the first multiple figure; m is the number of intervals between the multiple figures; F_{m+1} is the scale reading of the heterodyne frequency meter when set to the last multiple figure. The coefficients N_1 and m are taken from a table.[29]

II. THE ELLIPTICAL SWEEP METHOD*

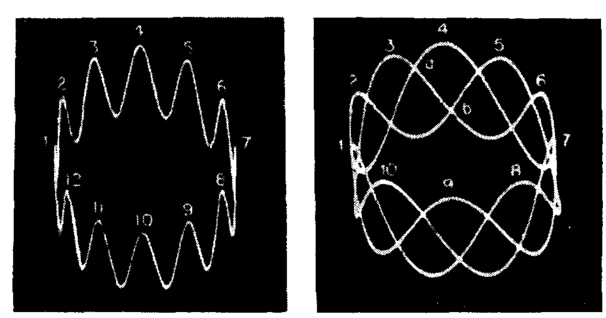
To simplify the interpretation of patterns in the case of a rational-fraction frequency ratio, the "front" and "rear" portions of the figure are separated, and the latter is scanned not along a straight line but along an ellipse or a circle.

Elliptic sweep is produced by applying to the deflecting plates two sinusoidal voltages approximately 90° out of phase.[3,30,31] The phase shift is produced by an RC network, to which the lower frequency f_L is connected. The higher-frequency voltage f_H is connected in series with one of the phase-shift network voltages (Fig. 14).

FIG. 14. Simplified diagram of the elliptic sweep method.



If the frequency of the sine wave superimposed on the ellipse is an integral multiple of the lower frequency, a stationary one-line pattern is produced (Fig. 15a). The frequency ratio equals the number of figure peaks, which are counted as shown on the diagram. If the frequency ratio is some irreducible rational fraction, the number of lines in the figure equals the denominator of the fraction. The numerator of the fraction is determined by counting the number of peaks in the pattern, as shown in Fig. 15b. The height of the curve produced by the higher frequency should be adjusted independently of the dimensions of the ellipse.



a b

FIG. 15. Elliptic-sweep patterns: a – 12:1 frequency ratio, b – 11:3 frequency ratio.

If the frequency ratio differs slightly from an integer, the single-line figure rotates. The same occurs with a multiple figure, when the frequency ratio is not equal exactly to the ratio of two integers. There is no gradual transition from a one-line figure to a multiple-line figure and vice versa. The one-line figure remains intact as it rotates as long as the frequency ratio is closer to an integer than to the nearest rational fraction. The change in the form of the pattern is abrupt.

Figure 16 shows the sequence of multiple-line elliptical-sweep patterns as the higher frequency

*Elliptic-sweep patterns are frequently called "gear-wheel patterns" in the American literature. –Tr.

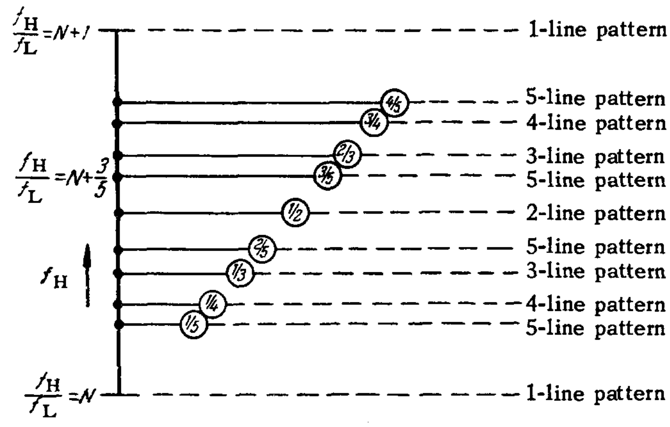


FIG. 16. Sequence of elliptic-sweep patterns with up to 5 lines.

increases above a certain whole-number frequency ratio. For the sake of clarity, the diagram does not show patterns with more than five lines.

Multiple-line elliptic-sweep figures (Fig. 17) give a considerable number of intermediate points between two neighboring whole-number frequency ratios. If patterns with up to ten lines are used, 31 intermediate points are obtained. The use of these patterns is possible with both low (4 : 1 or 5 : 1) and high (100 : 1 or 200 : 1) frequency ratios. The number of lines with which it is possible to operate reliably depends on the absolute stability of the compared frequencies.

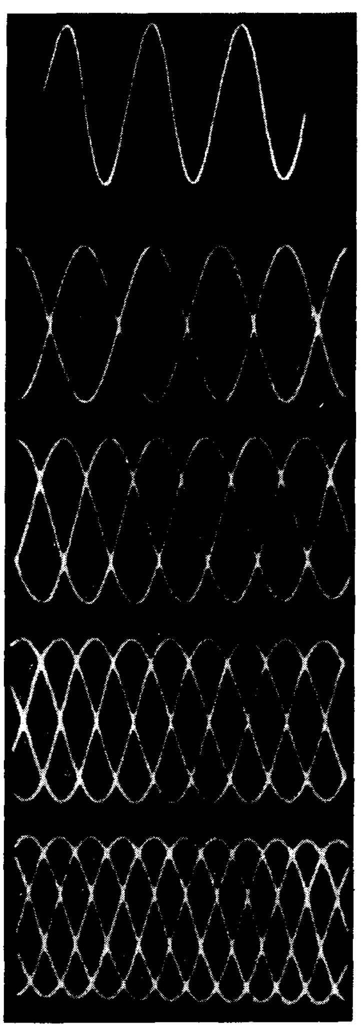


FIG. 17. Various stretched elliptic-sweep patterns: a – single-line pattern, b – double-line pattern, c – three-line pattern, d – four-line pattern, e – five-line pattern.

Multiple-line elliptic-sweep patterns are integrated in the following manner: A circle is produced on the oscilloscope screen in the absence of the high frequency. The gain of one amplifier is increased so as to leave on the screen two almost parallel straight lines. The gain of the second amplifier is then increased until the lines are at the edges of the screen. By changing beam-position adjustment, one of the lines is shifted to the center of the oscilloscope screen. The high frequency is then applied and the loops of the pattern are adjusted in size by varying this voltage. The ease with which a multiple-line figure is interpreted depends on its dimensions and on a suitable choice of the beam intensity and focus. It is preferable to use tubes with green phosphor.

In the case of a complex pattern of low stability, it is easier to count the number of lines along which the loops intersect (Fig. 18) rather than count the number of lines in the pattern. The number of lines in the pattern equals the number of loop-intersection lines plus one.

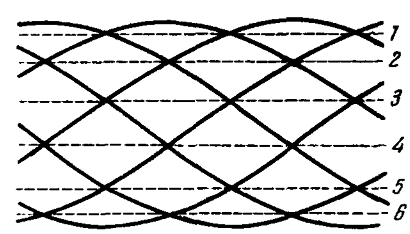


FIG. 18. Determination of the number of loop intersection lines of an elliptic-sweep pattern.

In multiple-line elliptic-sweep figures, the numerator of the frequency ratio is usually of no interest, since it is desirable to obtain a series of intermediate values between two neighboring whole-number frequency ratios. The ambiguity that results from the fact that the oscilloscope screen contains only a small portion of the entire pattern (for example, a five-line pattern may correspond to a frequency difference equal to $\frac{1}{5}$, $\frac{2}{5}$, or $\frac{3}{5}$ of the lower frequency) is not a serious hindrance in the calibration of an oscillator or of a frequency meter. Actually, the observer determines a successive set of values, of which those obtained with the aid of the single-line and double-line patterns are unique and serve as references for the remaining ones. Values obtained with the aid of three-line patterns can also serve as reference points, since the $\frac{1}{3}$ and $\frac{2}{3}$ values are readily distinguishable from each other.

The use of figures with more than five lines

TABLE III

Number of lines in pattern	Deviation from the next lower whole-number multiple frequency			Number of lines in pattern	Deviation from the next lower whole-number multiple frequency		
	As a fraction of the lower frequency	Cycles, at 50 cps line frequency	Cycles, at f_0 = 1000 cps		As a fraction of the lower frequency	Cycles, at 50 cps line frequency	Cycles, at f_0 = 1000 cps
1	0	0	0	2	$^1/_2$	25.0	500.0000
10	$^1/_{10}$	5.0	100.0000	9	$^5/_9$	27.8	555.5556
9	$^1/_9$	5.6	111.1111	7	$^4/_7$	28.6	571.4286
8	$^1/_8$	6.3	125.0000	5	$^3/_5$	30.0	600.0000
7	$^1/_7$	7.1	142.8571	8	$^5/_8$	31.3	625.0000
6	$^1/_6$	8.3	166.6667	3	$^2/_3$	33.3	666.6667
5	$^1/_5$	10.0	200.0000	10	$^7/_{10}$	35.0	700.0000
9	$^2/_9$	11.1	222.2222	7	$^5/_7$	35.7	714.2857
4	$^1/_4$	12.5	250.0000	4	$^3/_4$	37.5	750.0000
7	$^2/_7$	14.3	285.7143	9	$^7/_9$	38.9	777.7778
10	$^3/_{10}$	15.0	300.0000	5	$^4/_5$	40.0	800.0000
3	$^1/_3$	16.7	333.3333	6	$^5/_6$	41.7	833.3333
8	$^3/_8$	18.8	375.0000	7	$^6/_7$	42.9	857.1429
5	$^2/_5$	20.0	400.0000	8	$^7/_8$	43.8	875.0000
7	$^3/_7$	21.4	428.5714	9	$^8/_9$	44.4	888.8889
9	$^4/_9$	22.2	444.4444	10	$^9/_{10}$	45.0	900.000

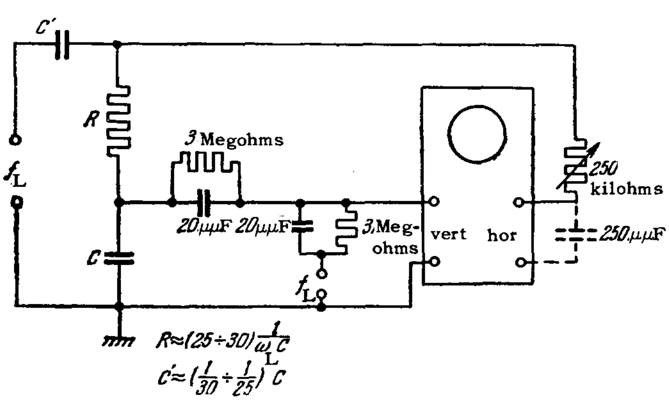


FIG. 19. Practical version of the circuit of the elliptic-sweep method.

is advantageous in the calibration of a high-grade standard oscillator or frequency meter. Two-line, three-line, and four-line patterns are used principally for ordinary laboratory practice.

In practical applications of these patterns it is convenient to use Table III, which contains data on elliptic-sweep patterns with up to 10 lines. The numbers in the table are arranged in the same sequence as they occur on the oscilloscope screen as the higher of the compared frequencies increases from a certain whole-number ratio (the known frequency is taken to be the lower frequency).

The use of the oscilloscope amplifiers permits the use of low input voltages and convenient adjustment of the dimensions and shape of the pattern. In view of the unequal phase shifts of the amplifiers, a narrow ellipse may be produced at certain frequencies instead of a circle (which insures the best legibility of the patterns). An additional phase-shifting network, usually in the grid circuit of the horizontal amplifier, insures a circular sweep (Fig. 19). If the low frequency exceeds 250 or 300 cycles, the required phase shift is produced by varying the input resistance or capacitance of the horizontal amplifier. At lower frequencies an additional capacitor must be used (shown dotted in the figure).

The reduction in the input voltage of the vertical oscilloscope amplifier, produced by the large resistor R, is offset by the greater gain of this amplifier, compared with the horizontal amplifier. To reduce the coupling between the two generators, the higher-frequency voltage is applied to a small capacitor shunted by a large resistor. A small capacitor in series with the phase-shifting network attenuates the "background" voltage produced by the rectifier of the low-frequency oscillator at the terminals of the vertical amplifier.

The use of the simple circuit shown in Fig. 14 is undesirable, in view of the considerable parasitic couplings that may occur between the two oscillators.

In conclusion we note that the elliptic-sweep method should not be used for relatively simple frequency ratios, which produce patterns that can be interpreted by the sinusoidal-sweep method (9:2, 11;2, 10:3, etc.). This reduces the chance of error due to partial locking of the lower-frequency oscillator.

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