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INTERACTION OF DEUTERONS WITH NUCLEI

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1. INTRODUCTION

NUCLEAR reactions produced by deuterons play an important part in nuclear physics. The cross sections for such reactions are considerably greater than those for the corresponding reactions by other charged particles. For this reason deuterons are widely used for obtaining radioactive isotopes.

The peculiar features of nuclear reactions produced by deuterons result from the properties of the deuteron: its loose structure which is related to its low binding energy, and the asymmetric distribution of electric charge in the deuteron.

Because of the low binding energy of the deuteron, the neutron and proton in the deuteron spend a considerable fraction of the time outside the range of the nuclear forces. Thus, in the collision of a deuteron with a nucleus, formation of a compound nucleus (in which case the deuteron as a whole is absorbed by the nucleus) need not occur. The more probable processes are those in which only one of the particles constituting the deuteron is absorbed by the nucleus; the other particle then appears directly as a reaction product. Such a process, in which one of the particles in the deuteron is absorbed by a nucleus while the other is set free, is called a breakup or stripping reaction.

The mechanism of the stripping reaction may be different for different energies of the incident deuteron. If the energy of the incident deuteron is less than the height of the Coulomb barrier, the repulsive Coulomb force acting on the proton will allow only the neutron to enter the region of action of the nuclear forces. In this case the final nucleus is formed as the result of neutron capture, while the proton emerges with an excess kinetic energy resulting from the release of energy in the breakup of the deuteron and from the Coulomb repulsion.

The asymmetric distribution of the electric charge in the deuteron makes it possible also for an electrical breakup of the deuteron to occur, in which both the neutron and proton are set free. This type of disintegration can occur at any energy of the incident deuterons above the disintegration threshold.

In the region of energies above the Coulomb barrier, the stripping reaction occurs principally from direct interaction of one of the particles in the deuteron with the nucleus. Since the dimensions of the deuteron are large, in such a case the other particle does not in general enter the region of action of the nuclear forces. Thus the capture of one of the particles in the deuteron is accompanied by the liberation of the other particle. The angular distribution of the particles which are set free is determined by the state of the final nucleus which is formed as a result of the reaction. Thus, if the energy is not very high the stripping reaction can be used for studying the properties of nuclei. At present the spins and parities of many states of light nuclei are determined by using the stripping reaction.

The picture of the stripping process is especially simple at high energies, when the quasiclassical approximation is applicable. In this case the momentum carried off by the particle which is set free is equal to its momentum at the time of the collision, and consists of the momentum of the center of mass motion of the deuteron and the momentum of the relative motion of the particles in the deuteron.

The stripping reaction with high energy deuterons is used for obtaining fast and practically monoenergetic neutrons.

In addition to the stripping reaction at high energies we may mention another mechanism of interaction of deuterons with nuclei, which results in an additional yield of neutrons and protons. This mechanism is the diffraction breakup of the deuteron, occurring far away from the nucleus.

A great deal of both theoretical and experimental work is being done at present on the problems of interaction of deuterons with nuclei. But these problems have not been clarified sufficiently in the Russian literature. For this reason it seems desirable to give a survey of the theoretical papers concerning processes of interactions of deuterons with nuclei. (Experimental papers are not considered in the survey; the references to experimental papers are for the most part haphazard.)

Special attention is given in this survey to the processes of direct interaction of deuterons with nuclei, which have been the subject of intensive study recently. We shall limit ourselves to the range of deuteron energies in which the formation of mesons does not play an essential role. For convenience we shall divide the energy range into two parts: the region of low and medium energies ($E_d < 20$ Mev), and the high energy region ($20 \text{ Mev} < E_d < 300$ Mev).

In the region of low and medium energies, we consider the following processes: elastic scattering of deuterons, the effect on the elastic scattering of the spatial extent of the deuteron and the absorption of the deuteron, the stripping reaction resulting from direct interaction, the interference between direct processes and processes involving compound nucleus formation, and finally the inelastic scattering of deuterons by nuclei, which is accompanied by excitation of the nucleus and disintegration of the deuteron.

We shall treat separately those processes in which the Coulomb interaction plays a decisive role: the Coulomb breakup of the deuteron and (d, p) reactions on heavy nuclei. These processes are important in the low-energy region, especially for heavy nuclei.

In the high energy region, we shall give special attention to the diffractive interaction of deuterons with nuclei. In particular, we shall treat the dissociation of the deuteron in the electromagnetic field of the nucleus and the formation of deuterons in the collision of fast nucleons with nuclei.

I. INTERACTION OF DEUTERONS WITH NUCLEI IN THE LOW AND MEDIUM ENERGY REGIONS

2. Elastic Scattering of Deuterons

1. The role of the Coulomb interaction. For low and medium energies of the incident deuterons, the elastic scattering is determined mainly by the Coulomb interaction. Scattering of deuterons through compound nucleus formation is extremely improbable. The reason for this is the high excitation energy of the compound nucleus formed after absorption of the deuteron. The decay of such a nucleus with emission of a deuteron is extremely unlikely because of the competition of other possible decay processes.

The Coulomb potential barrier surrounding the nucleus results from the combined action of the nuclear forces which are effective for small distances between nucleons and the repulsive Coulomb forces outside the nucleus. The height B of the Coulomb barrier for deuterons can be defined as follows:

$$B = \frac{Ze^2}{R},$$

where e is the charge on the deuteron, Ze is the charge on the nucleus, and R is the radius of the region of nuclear interaction, which should be set equal to the sum of the nuclear radius R_A and the deuteron radius R_d:

$$R = R_A + R_d.$$

Using the fact that $R_A = r_0 A^{1/3}$ (where A is the mass number and $r_0 = 1.2 \times 10^{-13}$ cm) and $R_d = 2.1 \times 10^{-13}$ cm, we find for B the expression

$$B = 1.2ZA^{-1/3}(1 + 1.75A^{-1/3})^{-1} \text{ Mev}.$$

The important quantity for the passage of the deuteron through the Coulomb barrier is the relative kinetic energy, which is equal to $\frac{M_A}{M_A + M_d} E_d$, where E_d is the kinetic energy of the incident deuteron relative to an infinitely heavy nucleus, and M_d and M_A are the masses of the deuteron and the nucleus. The barrier is unimportant if $E_d \gg B'$, where B' is the effective height of the barrier, equal to

$$B' = \frac{M_A + M_d}{M_A} B$$
$$= 1.2Z(A+2)A^{-\frac{4}{3}}(1 + 1.75A^{-\frac{1}{3}})^{-1} \text{ Mev}. \quad (2.1)$$

Below we give values of the effective barrier height B' in Mev for various nuclei

Nucleus	He_4^2	Be_9^4	Ne_{20}^{10}	Ca_{40}^{20}	Zn_{60}^{30}	Sn_{112}^{50}	Yb_{174}^{70}	U_{238}^{92}
B'	1.0	1.5	2.8	4.8	6.4	9.2	11.5	14.0

The Coulomb interaction of the deuteron with nucleus is conveniently characterized by the parameter

$$n = \frac{Ze^2}{\hbar v},$$

where v is the velocity of the incident deuteron. For low values of this parameter ($n \ll 1$), the Coulomb interaction can be treated by perturbation theory. In the opposite limiting case of $n \gg 1$ the quasi-classical approximation is applicable. For heavy nuclei the parameter n is already greater than unity for medium values of the deuteron energy.

The scattering of charged particles in the Coulomb field is given by the Rutherford formula. For elastic scattering of deuterons, deviations from the Rutherford formula may occur for two reasons. Firstly, there may be deviations resulting from the spacial extension of the deuteron, and secondly, if the energy of the incident deuteron is above the Coulomb barrier, there may be deviations resulting from penetration of the deuteron through the barrier leading to absorption of the deuteron.

2. Structure of the deuteron and elastic scattering. Since the deuteron consists of a neutron and proton, it is a complex nucleus and has a spatial structure. The spatial dimensions of the deuteron are characterized by the average separation of the neutron and proton which compose it. This separation is usually called the radius of the deuteron because of the low binding energy (ϵ = 2.23 Mev), the radius of the deuteron is greater than the range of the nuclear forces acting between the neutron and proton. A second peculiarity of the deuteron structure is the extreme asymmetry of its electrical charge distribution — the center of mass and center of charge of the deuteron do not coincide. Because of this, even when the energy of the incident deuteron is well below the Coulomb barrier we may expect to see deviations from the Rutherford formula. The nature of these deviations was explained in a paper of French and Goldberger.[63]

The motion of the deuteron in the Coulomb field of the nucleus, which for simplicity we may assume to be a point source, can be described by the Schrödinger equation

$$\left\{-\frac{\hbar^2}{4M}\Delta_d - \frac{\hbar^2}{M}\Delta_r + V(r) + \frac{Ze^2}{\left|\mathbf{r}_d - \frac{1}{2}\mathbf{r}\right|} - E\right\}\Psi(\mathbf{r}, \mathbf{r}_d) = 0. \quad (2.2)$$

Δ_d and Δ_r are the Laplacians with respect to the coordinates of the center of mass of the deuteron, $\mathbf{r}_d$, and the relative coordinates $\mathbf{r}$; V(r) is the potential of the nuclear interaction between the neutron and proton; $Ze^2/|\mathbf{r}_d - \frac{1}{2}\mathbf{r}|$ is the energy of the Coulomb interaction of the deuteron and the nucleus, and depends on the radius vector of the proton, $\mathbf{r}_d - \frac{1}{2}\mathbf{r}$; E is the total energy of the deuteron, and is equal to $E = \frac{\hbar^2 k^2}{4M} - \epsilon$, where $\mathbf{k}$ is the wave vector of the incident deuteron.

For finding the solution, it is convenient to rewrite (2.2) in the form

$$\left\{-\frac{\hbar^2}{4M}\Delta_d - \frac{\hbar^2}{M}\Delta_r + V(r) + \frac{Ze^2}{r_d} - E\right\}\Psi$$
$$= Ze^2\left\{\frac{1}{r_d} - \frac{1}{\left|\mathbf{r}_d - \frac{1}{2}\mathbf{r}\right|}\right\}\Psi. \quad (2.3)$$

We now expand the function Ψ on the left side of (2.3) in eigenfunctions of the relative motion of the neutron-proton system,

$$\Psi(\mathbf{r}, \mathbf{r}_d) = \varphi_0(r)\psi(\mathbf{r}_d) + \text{orthogonal term},$$

where $\varphi_0(r)$ is the wave function of the ground state of the deuteron. We multiply (2.3) by $\varphi_0(r)$ and integrate over $\mathbf{r}$. Treating the right side of (2.3) as a perturbation, we replace Ψ by $\Psi_0 = \varphi_0(r)\psi_{\mathbf{k}}(\mathbf{r}_d)$, where $\psi_{\mathbf{k}}(\mathbf{r}_d)$ is the wave function of the deuteron in a state of definite momentum **k** in the Coulomb field. We thus obtain

$$\left\{\Delta_d - \frac{2kn}{r_d} + k^2\right\}\psi(\mathbf{r}_d)$$
$$= 2kn \int \left\{\frac{1}{\left|\mathbf{r}_d - \frac{1}{2}\mathbf{r}\right|} - \frac{1}{r_d}\right\}\varphi_0^2(r)\,\psi_{\mathbf{k}}(\mathbf{r}_d)\,d\mathbf{r}, \qquad (2.4)$$

where $n = Ze^2/\hbar v$, and v is the velocity of the incident deuteron.

The function $\psi_{\mathbf{k}}(\mathbf{r}_d)$, normalized to unit flux of particles in the incident beam, is

$$\psi_{\mathbf{k}}(\mathbf{r}) = e^{-\frac{\pi}{2}n}\Gamma(1+in)\,e^{i\mathbf{kr}}F(-in,\ 1,\ i(kr-\mathbf{kr})), \quad (2.5)$$

where $\Gamma(x)$ is the gamma function and $F(\alpha, \gamma, z)$ is the confluent hypergeometric function. The function $\psi_{\mathbf{k}}(\mathbf{r}_d)$ is a solution of (2.4) when the right hand side is set equal to zero, and describes the scattering of deuterons by the Coulomb field of the nucleus when we neglect the spatial extension of the deuteron. At infinity $\psi_{\mathbf{k}}(\mathbf{r}_d)$ is a sum of a plane wave and an outgoing spherical wave.

The function $\psi(\mathbf{r}_d)$ which is defined by the inhomogeneous equation (2.4) describes the elastic scattering of deuterons taking account of the spatial extension of the deuteron.

For large values of the parameter n, the inhomogeneous term in (2.4) will be extremely small. Because of the spherical symmetry of the ground state of the deuteron, the only non-zero contribution to the integral in (2.4) comes from $r > 2r_d$. However, since the effective values of r are of the order of the deuteron radius R_d, while a deep penetration of the deuteron toward the Coulomb center is impossible for large Z, the corrections to the Rutherford formula can be neglected when $n \gg 1$.

Using the asymptotic Green's function for the Coulomb field

$$G(\mathbf{r}, \mathbf{r}') \to -\frac{1}{4\pi r}\,e^{i(kr - n\ln 2kr)}\,\psi^*_{\mathbf{k}'}(\mathbf{r}'),$$

where $\mathbf{k}' = (\mathbf{r}/r)k$, and

$$\psi_{\mathbf{k}'}(\mathbf{r}) = e^{-\frac{\pi}{2}n}\Gamma(1-in)\,e^{i\mathbf{k}'\mathbf{r}}F(in,\ 1,\ -i(kr-\mathbf{k}'\mathbf{r})), \quad (2.6)$$

we can find the asymptotic form of the solution of Eq. (2.4). The coefficient of the outgoing wave in this expression will determine the amplitude of the elastic scattering of the deuterons. This elastic scattering amplitude has the form

$$f(\vartheta) = f_R(\vartheta)$$
$$-\frac{nk}{2\pi}\int \psi^*_{\mathbf{k}'}(\mathbf{r}_d)\left\{\frac{1}{\left|\mathbf{r}_d - \frac{1}{2}\mathbf{r}\right|} - \frac{1}{r_d}\right\}\psi_{\mathbf{k}}(\mathbf{r}_d)\,\varphi_0^2(r)\,d\mathbf{r}\,d\mathbf{r}_d. \quad (2.7)$$

where ϑ is the angle of scattering (the angle between the vectors $\mathbf{k}'$ and $\mathbf{k}$), and $f_R(\vartheta)$ is the Coulomb scattering amplitude

$$f_R(\vartheta) = -\frac{Ze^2}{4Mv}\,\frac{e^{-in\ln\sin^2\frac{\vartheta}{2}}}{\sin^2\frac{\vartheta}{2}}\,\frac{\Gamma(1+in)}{\Gamma(1-in)}.$$

Using a Hulthèn function as the wave function of the ground state of the deuteron,

$$\varphi_0(r) = N\,\frac{e^{-\alpha r} - e^{-\beta r}}{r}, \quad \beta = 7\alpha,$$
$$N^2 = \frac{\alpha}{2\pi(1-\alpha r_t)}, \quad \alpha = \frac{\sqrt{M\varepsilon}}{\hbar} \qquad (2.8)$$

(where $r_t = 1.6 \times 10^{-13}$ cm is the effective range of the nuclear forces in the triplet state), we can write the integral appearing in (2.7) in the form

$$-8\pi N^2\left[\int\limits_{4\alpha}^{\infty} + \int\limits_{4\beta}^{\infty} - 2\int\limits_{2\alpha+2\beta}^{\infty}\right]\frac{d\gamma}{\gamma^2}\int d\mathbf{r}\,\frac{e^{-\gamma r}}{r}\,\psi^*_{\mathbf{k}'}(\mathbf{r})\,\psi_{\mathbf{k}}(\mathbf{r}).$$

The integration over $\mathbf{r}$ can be carried out using formula (10.1) of the Appendix.

We thus find for the differential cross section for elastic scattering of deuterons the formula

$$d\sigma = \Big|1 - 32\pi N^2 k^2 e^{\frac{2\pi n}{2\pi n}} - 1\sin^2\frac{\vartheta}{2}\exp\left(in\ln\sin^2\frac{\vartheta}{2}\right)$$
$$\times\left[\int\limits_{4\alpha}^{\infty} + \int\limits_{4\beta}^{\infty} - 2\int\limits_{2\alpha+2\beta}^{\infty}\right]\frac{d\gamma}{\gamma^2}(\gamma^2 - 2i\gamma k)^{2n}\left(\gamma^2 + 4k^2\sin^2\frac{\vartheta}{2}\right)^{-2n-1}$$
$$\times F\left(-in,\ -in,\ 1;\ -\frac{4k^2}{\gamma^2}\sin^2\frac{\vartheta}{2}\right)\Big|^2 d\sigma_R. \qquad (2.9)$$

The quantity $d\sigma_R$ is the cross section for scattering of point particles by the Coulomb field, and is given by the Rutherford formula

$$d\sigma_R = \left(\frac{Ze^2}{4Mv^2}\right)^2\frac{do}{\sin^4\frac{\vartheta}{2}}.$$

If $n \ll 1$, we find from (2.9) for the ratio $d\sigma/d\sigma_R$ the value

$$\frac{d\sigma}{d\sigma_R} = \left\{\frac{1}{1-\alpha r_t}\,\frac{4\alpha}{q}\left(\tan^{-1}\frac{q}{4\alpha}\right.\right.$$
$$\left.\left. + \tan^{-1}\frac{q}{4\beta} - 2\tan^{-1}\frac{q}{2\alpha+2\beta}\right)\right\}^2,$$
$$q = 2k\sin\frac{\vartheta}{2}. \qquad (2.10)$$

This ratio is equal to unity for small scattering angles and decreases with increasing angle. For example, for $E_d = 4$ Mev and $\vartheta = 180°$, we find a ratio of 0.3. Formula (2.10) corresponds to using the Born approximation.

In the case of large n, formula (2.9) gives unity for the ratio $d\sigma/d\sigma_R$ independently of the angle of scattering.

For arbitrary values of n, the integration in (2.9) can only be done numerically. For example, for the scattering of 14-Mev deuterons by aluminum ($n = 0.8$), numerical integration gives

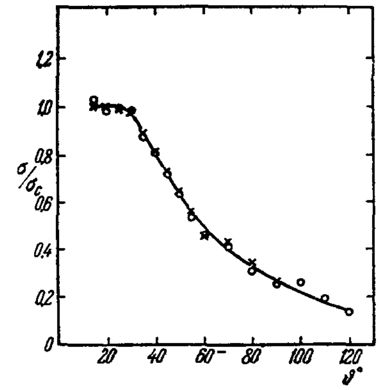


FIG. 1. Dependence of the ratio of elastic scattering of deuterons to Rutherford scattering on scattering angle, for E_d = 15.2 Mev (circles – Pb^{208}, crosses – Bi^{209}).

$d\sigma/d\sigma_R = 0.67$ for $\vartheta = 140°$ (while the Born approximation gives 0.11).

Thus for deuteron energies below the Coulomb barrier, the ratio $d\sigma/d\sigma_R$ for small n decreases monotonically with increasing angle. With increasing n this falloff becomes less marked, and for $n \gg 1$ we get Rutherford scattering.

3. Barrier penetration and scattering. Experiments on elastic scattering of medium-energy deuterons by heavy nuclei show that at small angles the scattering is purely Rutherford scattering, but starting at an angle which depends on the deuteron energy the cross section drops markedly from the value given by the Rutherford formula. For example, in the elastic scattering of 15.2-Mev deuterons by Pb^{208}, the ratio of the elastic scattering cross section to the Rutherford cross section is equal to unity out to scattering angles of $\vartheta = 30°$, while for greater angles the ratio falls off exponentially.[73] As shown by Porter,[105] this exponential falloff of cross section with angle can be explained as the effect of absorption of the deuterons in the incident beam.

In the energy range we are considering, the deuteron wavelength is much less than the nuclear radius (for $E_d = 15$ Mev, the ratio $R/\lambdabar$ is $\cong 10$), so we can use the quasi-classical approximation. To simplify matters we shall assume that the trajectory of the deuteron in the Coulomb field of the nucleus is not distorted by the nuclear forces. Then the decrease of scattering cross section with increasing angle can be explained by considering the absorption of the deuterons along their trajectory in the Coulomb field.

We can write the elastic scattering cross section in the form

$$d\sigma = T(\vartheta)\, d\sigma_R, \tag{2.11}$$

where $T(\vartheta)$ is the coefficient for transmission of the deuteron through the nucleus at a fixed scattering angle, and is equal to

$$T = \exp\left(-\int \frac{dx}{l(x)}\right) \tag{2.12}$$

(x is the coordinate of the deuteron along the trajectory and $l(x)$ is the mean free path of the deuteron in nuclear matter).

Introducing the distance of closest approach b, which is related to the scattering angle ϑ by the formula

$$b(\vartheta) = \frac{Ze^2}{E_d}\left(1 + \frac{1}{\sin\frac{\vartheta}{2}}\right), \tag{2.13}$$

we find

$$\int \frac{dx}{l(x)} = 2\int_{b(\vartheta)}^{\infty} dr\,\frac{dx}{dr}\,\frac{1}{l(r)},$$

where r is the radius in the plane of the deuteron orbit. Integration by parts gives

$$\int \frac{dx}{l(x)} = 2\int_{b(\vartheta)}^{\infty} dr\,x(r)\,\frac{d}{dr}\left(-\frac{1}{l(r)}\right),$$

since $x = 0$ for $r = b$, and $l^{-1} \to \rho$ for $r \to \infty$. The mean free path is inversely proportional to the density of nuclear matter.

Assuming that the variation of the density of nuclear matter with radius is given by

$$\rho(r) = \frac{1}{2}\left(1 - \tanh\frac{r-R}{d}\right)\rho_0, \tag{2.14}$$

where R is the nuclear radius and d the width of the diffuse boundary of the nucleus, we find

$$\frac{1}{l(r)} = \frac{1}{2l_0}\left(1 - \tanh\frac{r-R}{d}\right)$$

(where l_0 is the mean free path at the center of the nucleus).

Assuming for simplicity a straight-line trajectory of the deuteron inside the nucleus, we finally get

$$T(\vartheta) = \exp\left\{-\frac{1}{l_0 d}\int_{b(\vartheta)}^{\infty} dr \times (r^2 - b^2(\vartheta))^{\frac{1}{2}}\,\mathrm{sech}^2\frac{r-R}{d}\right\}. \tag{2.15}$$

If the boundary of the nucleus is assumed to be sharp, so that $\frac{d}{dr}\left(-\frac{1}{l(r)}\right)\xrightarrow[d\to 0]{}\frac{1}{l}\,\delta(r-R)$, we find

$$T(\vartheta) = \exp\left\{-\frac{2R}{l_0}\left(1 - \frac{b^2(\vartheta)}{R^2}\right)^{\frac{1}{2}}\right\}.$$

By suitable choice of the parameters R, d, and l_0, we can fit the angular dependence of the cross section ratio given by formula (2.15) to the experimentally observed dependence (Fig. 2). The best fit is obtained for the following choice of pa-

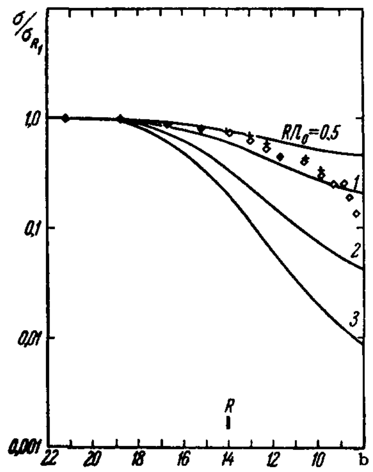


FIG. 2. Dependence of σ/σ_R on distance of closest approach $b(\vartheta)$ for various values of R/l_0. (E_d = 12.2 Mev, $R = 14 \times 10^{-13}$ cm, $d = 3.5 \times 10^{-13}$ cm).

rameter values: $R = 14 \times 10^{-13}$ cm, $d = 3.5 \times 10^{-13}$ cm and $R/l_0 \simeq 1$) (for the case of elastic scattering of deuterons by Pb^{208} and Bi^{209} at $E_d = 15.2$ Mev).[105]

It should also be kept in mind that the decrease of the elastic scattering cross section with angle may also be related to the possibility of electrical dissociation of the deuteron.

3. (d,p) and (d,n) Stripping Reactions.

1. Introduction. Of primary interest in the low and medium regions are the (d,p) and (d,n) reactions, which are widely used at present in nuclear spectroscopy for studying nuclear properties. These reactions can proceed in two different ways.

First, under the action of the deuterons, the formation of a compound nucleus can occur; then the compound nucleus decays with the emission of a proton or a neutron. Schematically such a two-stage process can be represented as follows:

$$A + d \to C \to B + p.$$

In this case, for sufficiently low energies of the incident deuterons, one can observe resonance phenomena (especially for light nuclei), which are due to the quasi-discrete structure of the spectrum of the compound nucleus. In the center of mass system, the angular distribution is then symmetric about a line perpendicular to the direction of incidence of the deuteron.[4]

Secondly, direct transitions (breakup or stripping reactions) are possible in which the nucleus absorbs only one of the particles constituting the deuteron,

$$A + d \to B + p.$$

Such direct processes are possible because of the low binding energy of the deuteron. The angular distribution of the reaction products from direct transitions is characterized by a very definite shape, from which one can deduce the spin and parity of the final state of the residual nucleus if the spin and parity of the initial nuclear state are known.

The possibility of using deuteron reactions for obtaining data on the spectroscopy of nuclei was first pointed out by Butler.[43] The theory of the stripping reaction with medium energy deuterons was also given by Butler,[44] who determined the angular distribution of the products of the stripping reaction by using the condition of continuity of the wave function at the nuclear surface. The results of the theory were in good agreement with the experimental data.

Butler's derivation of the angular distribution of the products of a stripping reaction was extremely complicated, so that there have been a whole series of papers[35,92,87,70,61,116,121] in which the angular distribution has been found by other methods. Bhatia, Huang, Huby and Newns[35] determined the stripping angular distribution using the Born approximation. Although there is little justification for applying such an approximation in this energy range, the results were extremely close to those of Butler. Later Daitch and French[57] showed that the Born approximation gives the same results as Butler's theory. (cf. also references 30 and 118).

A more consistent theory of stripping reactions on the basis of perturbation theory, which takes into account the scattering of the deuteron and proton waves, was developed in a paper of Tobocman.[116]

In this paragraph we shall treat the stripping reaction by using a method which is due to Landau and Lifshitz,[17] and was applied by them to the dissociation of the deuteron in the Coulomb field of a heavy nucleus.

For definiteness we shall consider the (d,p) stripping reaction, although the results will be applicable also to (d,n) reactions since for the case of light nuclei the Coulomb interaction can be neglected.

2. Energy relations. Energy relations play an important part in stripping reactions at low and medium energies of the incident deuterons. On the assumption that the initial nucleus was in its ground state ($E_A = 0$), the energy balance for the A(d,p)B reaction can be written in the center of mass system as

$$E_d - \varepsilon = E_p - S_n + E_B,$$

where E_d and E_p are the kinetic energies of the incident deuteron and the emitted proton, ϵ is the bonding energy of the deuteron, S_n is the energy with which the captured neutron is bound in the nucleus B if the latter is in its ground state, and E_B is the excitation energy of nucleus B in its final state. (To take account of the finiteness of the nuclear masses, E_d and E_p should be taken as the total kinetic energy of the system before and after the collision.)

The change in the total kinetic energy of the system (the Q value) for the stripping reaction is

$$Q = E_p - E_d = S_n - \varepsilon - E_B.$$

The most interesting stripping reactions are those in which the nucleus is formed in its ground state or a state of low excitation energy. If the nucleus B is formed in its ground state ($E_B = 0$), the Q of the reaction will be ~ 6 Mev.

Assuming that the state of the nucleons which constitute nucleus A is not changed when nucleus B is formed, we can ascribe the energy $E_n = E_B - S_n = E_d - E_p - \epsilon$ to the absorbed neutron.

This energy can be either negative or positive. If $E_n < 0$ the neutron will be in a bound state in the nucleus. If $E_n > 0$ the state will be virtual, i.e., the nucleus B will be unstable with respect to decay with emission of a neutron.

The energy relations we have given will also be applicable to the (d, n) stripping reaction, if n and p are interchanged in all the formulas.

3. Angular distribution in a stripping reaction. Let us determine the angular distribution of the particles formed in the stripping reaction A(d, p) B. We shall assume that the mass of nucleus A is infinitely large compared to the mass of the deuteron. Then the Schrödinger equation describing the motion of the deuteron (system of neutron + proton) in the field due to the presence of a nucleus A can be written as

$$\left\{H_A - \frac{\hbar^2}{2M}\Delta_n - \frac{\hbar^2}{2M}\Delta_p + V_n + V_p + V_{np} - E\right\} \times \Psi(\zeta, \mathbf{r}_n, \mathbf{r}_p) = 0, \qquad (3.1)$$

where H_A is the Hamiltonian for the internal motion of the initial nucleus A, and ζ is the coordinate describing its motion; Δ_n and Δ_p are the Laplace operators with respect to the neutron coordinates $\mathbf{r}_n$ and proton coordinates $\mathbf{r}_p$; V_n and V_p are the potentials of the interaction of the neutron and proton with the nucleus A, V_{np} is the potential for the nuclear interaction of the neutron and proton, and E is the total energy of the system.

To solve Eq. (3.1), we expand the wave function Ψ in terms of the wave functions of the residual nucleus B. These wave functions which we denote by $\varphi_b(\zeta, \mathbf{r}_n)$, with quantum number b, satisfy the equation

$$\left\{H_A - \frac{\hbar^2}{2M}\Delta_n + V_n - E_b\right\}\varphi_b(\zeta, \mathbf{r}_n) = 0. \qquad (3.2)$$

We shall assume that the function φ_b is subjected to the normalization

$$\int \varphi_b(\zeta, \mathbf{r}_n)\varphi_{b'}^*(\zeta, \mathbf{r}_n)\,d\zeta\,d\mathbf{r}_n = \delta_{bb'}. \qquad (3.3)$$

The solution of (3.1) can be represented as

$$\Psi(\zeta, \mathbf{r}_n, \mathbf{r}_p) = \sum_b \psi_b(\mathbf{r}_p)\varphi_b(\zeta, \mathbf{r}_n) + \text{orthogonal terms}, \qquad (3.4)$$

where the expansion coefficients ψ_b, which depend on the proton coordinates, can be regarded as the wave functions of the proton, freed as a result of the interaction, corresponding to definite states φ_b of the residual nucleus B.

After substituting (3.4) in (3.1) and using the orthogonality of the functions φ_b, we get the following equation for determining the function ψ_b:

$$\left\{\Delta_p + k_p^2 - \frac{2M}{\hbar^2}V_p\right\}\psi_b(\mathbf{r}_p) = \frac{2M}{\hbar^2}\int \varphi_b^*(\zeta, \mathbf{r}_n)V_{np}\Psi(\zeta, \mathbf{r}_n, \mathbf{r}_p)\,d\zeta\,d\mathbf{r}_n, \qquad (3.5)$$

where $k_p^2 = \frac{2M}{\hbar^2}(E - E_b)$. By using the Green's function, this differential equation can be converted to an integral equation.

To find the cross section for the (d, p) reaction it is necessary to know only the asymptotic form of $\psi_b(\mathbf{r}_p)$. It is easily found by using the asymptotic expression for the Green's function[5] of Eq. (3.5):

$$G(\mathbf{r}_p, \mathbf{r}_p') \to -\frac{1}{4\pi}\frac{e^{ik_p r_p}}{r_p}\psi_{\mathbf{k}_p}^*(\mathbf{r}_p'),$$

$$\mathbf{k}_p = \frac{\mathbf{r}_p}{r_p}k_p, \quad r_p \to \infty. \qquad (3.6)$$

Here $\psi_{\mathbf{k}_p}(\mathbf{r}_p)$ is the wave function of the emerging proton in a state with a definite wave vector $\mathbf{k}_p$, taking into account the scattering of the proton in the field of the residual nucleus B. At infinity $\psi_{\mathbf{k}_p}$ is the sum of a plane wave and an outgoing spherical wave.

Using (3.6) we get the following asymptotic expression for the function ψ_b, valid for large r:

$$\psi_b(\mathbf{r}_p) \to f\frac{e^{ik_p r_p}}{r_p}, \quad r_p \to \infty, \qquad (3.7)$$

$$f = -\frac{M}{2\pi\hbar^2} \times \int \psi_{\mathbf{k}_p}^*(\mathbf{r}_p)\varphi_b^*(\zeta, \mathbf{r}_n)V_{np}\Psi(\zeta, \mathbf{r}_n, \mathbf{r}_p)\,d\zeta\,d\mathbf{r}_n\,d\mathbf{r}_p. \qquad (3.8)$$

The coefficient f of the outgoing wave in (3.7) is the amplitude for the (d, p) reaction. The differential cross section is related to the amplitude by the formula

$$d\sigma = \frac{v_p}{v_d} |f|^2 dO, \tag{3.9}$$

where v_d is the velocity of the incident deuteron and v_p is the velocity of the outgoing proton.

Formula (3.8) gives the exact value of the reaction amplitude, but to compute it we must know the exact wave function for the whole system, $\Psi(\zeta, \mathbf{r}_n, \mathbf{r}_p)$. The reaction amplitude can be computed approximately by replacing the exact function Ψ in (3.8) by the "incident" wave

$$\Psi\zeta_0(\zeta, \mathbf{r}_n, \mathbf{r}_p) = \varphi_a(\zeta)\varphi_0(r)\psi_{\mathbf{k}_d}(\mathbf{r}_d),$$

where $\varphi_a(\zeta)$ is the wave function of the initial nucleus A, $\varphi_0(r) = \sqrt{\alpha/2\pi}\ e^{-\alpha r}/r$ is the wave function of the ground state of the deuteron ($\alpha = \sqrt{M\epsilon/\hbar^2}$, ϵ is the binding energy of the deuteron), and $\psi_{\mathbf{k}_d}(\mathbf{r}_d)$ is the wave function for the motion of the deuteron center of mass in the field of the nucleus A. At infinity, $\psi_{\mathbf{k}_d}$ is the sum of the incident plane wave with wave vector $\mathbf{k}_d$ and the scattered outgoing spherical wave. (This replacement is actually equivalent to using the first approximation of perturbation theory.)

Thus we find the following expression for the reaction amplitude:

$$f = -\frac{M}{2\pi\hbar^2}\int \psi^*_{\mathbf{k}_p}(\mathbf{r}_p) F^*(\mathbf{r}_n) V_{np}\varphi_0(r)\psi_{\mathbf{k}_d}(\mathbf{r}_d)\, d\mathbf{r}_n d\mathbf{r}_p, \tag{3.10}$$

where

$$F^*(\mathbf{r}_n) = \int \varphi_0^*(\zeta, \mathbf{r}_n)\varphi_a(\zeta)\, d\zeta. \tag{3.11}$$

We see that $F(\mathbf{r}_n)$ can be considered to be the function of the neutron in the final state.

Because of the short range character of the nuclear forces, in evaluating the integral appearing in (3.10) we can use the realtion (cf. the Appendix)

$$V_{np}\varphi_0(r) = -\frac{4\pi\hbar^2}{M}\sqrt{\frac{\alpha}{2\pi}}\,\delta(\mathbf{r}_n - \mathbf{r}_p). \tag{3.12}$$

(This equation holds for zero range of the nuclear forces between neutron and proton in the deuteron.) We thus finally get for the reaction amplitude

$$f = 2\sqrt{\frac{\alpha}{2\pi}}\int \psi^*_{\mathbf{k}_p}(\mathbf{r}) F^*(\mathbf{r})\psi_{\mathbf{k}_d}(\mathbf{r})\, d\mathbf{r}. \tag{3.13}$$

The main contribution to this integral comes from the region outside the nucleus ($r > R$ where R is the nuclear radius), since in the energy region we are considering ($E_d < 20$ Mev) the mean free path of deuterons and protons in nuclear matter is very short, so that the wave functions $\psi_{\mathbf{k}_d}$ and $\psi_{\mathbf{k}_p}$ which describe the free states of the deuteron and proton go to zero in the interior of the nucleus.

The inclusion of the possibility of penetration of deuterons and protons into the interior of the nucleus corresponds to treating the (d, p) process as occurring via compound nucleus formation.

It is convenient to expand the wave function of the neutron in the residual nucleus in a series of spherical harmonics:

$$F(\mathbf{r}) = \sum_{l,m} \mathfrak{R}_l(r) Y_{lm}(\vartheta, \varphi). \tag{3.14}$$

The individual terms in this expansion correspond to different states of the neutron with definite values of the orbital angular momentum. We may note that according to the shell model there should be only one term in the sum over l, i.e., the neutron in the nucleus should be in a state with a definite value of l.

In the external region $r > R$, an exact wave function can be found for the neutron. If the neutron energy E_n is negative, the radial wave function for a neutron in a state of orbital angular momentum l will have the following form in the region outside the nucleus:

$$\mathfrak{R}_l(r) = C_l \mathfrak{k}_l(k_n r), \quad r > R,$$

where $\mathfrak{k}_l(x) = \sqrt{\pi/2x}\ K_{l+\frac{1}{2}}(x)$ is the spherical MacDonald function, $k_n = \frac{\sqrt{2M|E_n|}}{\hbar^2}$ and C_l is a normalization constant. The constant C_l is conveniently expressed in terms of the reduced width γ_l of the state, which is given in terms of the value of the radial wave function of the neutron at the surface of the nucleus by the formula

$$\gamma_l = \frac{\hbar^2 R}{2M}|\mathfrak{R}_l(R)|^2.$$

In the case of a virtual neutron state ($E_n > 0$), the reduced width γ_l is proportional to the intrinsic neutron width Γ_l which characterizes the probability of decay of the residual nucleus B with emission of a neutron carrying off orbital angular momentum l. Thus, expressing C_l in terms of γ_l, we have

$$\mathfrak{R}_l(r) = \sqrt{\frac{2M}{\hbar^2 R}\gamma_l}\,\frac{\mathfrak{k}_l(k_n r)}{\mathfrak{k}_l(k_n R)}, \quad r > R. \tag{3.15}$$

Since in (3.13) the region of integration in the interior of the nucleus is unimportant, in calculating the amplitude f we can use the expansion (3.14) and replace the radial fuctions $\mathfrak{R}_l(r)$ by the expressions (3.15). We then find

$$f = \sqrt{\frac{4M\alpha}{\pi\hbar^2 R}}\sum_{l,m}\sqrt{\gamma_l}\, I_l^m, \tag{3.16}$$

where

$$I_l^m = \int \psi_{\mathbf{k}_p}^*(\mathbf{r}) \frac{\mathfrak{f}_l(k_n r)}{\mathfrak{f}_l(k_n R)} Y_{lm}^*(\vartheta, \varphi) \psi_{\mathbf{k}_d}(\mathbf{r})\, d\mathbf{r}. \quad (3.17)$$

Substituting this expression for the amplitude into (3.9), we find for the differential cross section

$$d\sigma = \frac{k_p}{k_d} \frac{8M\alpha}{\pi\hbar^2 R} \Big| \sum_{l,\, m} \sqrt{\gamma_l}\, I_l^m \Big|^2 dO. \quad (3.18)$$

4. Inclusion of spin. If we take into account the spins of the nuclei and the spins of the deuteron, neutron and proton, we get an additional factor in (3.16) which depends on the spins and their projections.

In fact when the spin is taken into account, we should take for the wave function of the initial state Ψ which appears in the general expression (3.8) for the reaction amplitude the function

$$\Psi_0 = \varphi(\zeta)\, \varphi_0(r)\, \psi_{\mathbf{k}_d}(\mathbf{r}_d)\, \chi_{i\mu_i} \chi_{1\mu_1},$$

where $\chi_{i\mu_i}$ and $\chi_{1\mu_1}$ are the spin functions of the initial nucleus (i and μ_i are the spin and its projection for the initial nucleus) and the deuteron (μ_1 is the projection of the deuteron spin). The spin wave functions will be assumed to be orthonormal:

$$\sum \chi_{i\mu_i} \chi_{i'\mu_i'} = \delta_{ii'} \delta_{\mu_i \mu_i'}.$$

For the wave function of the final state we should take

$$\psi_{\mathbf{k}_p}(\mathbf{r}_p)\, \chi_{\frac{1}{2}\mu}\, \varphi_{j\mu_j}(\zeta, \mathbf{r}_n),$$

where $\chi_{\frac{1}{2}\mu}$ is the spin function of the liberated proton and $\varphi_{j\mu_j}$ is the total wave function of the residual nucleus in the state with spin j and spin projection μ_j. Obviously the spin j of the residual nucleus is the sum of the spin i of the initial nucleus, the orbital angular momentum l of the captured neutron and the spin of the neutron.

The wave function of the residual nucleus can be expanded in spin functions of the initial nucleus, spin functions of the neutron and eigenfunctions of the orbital angular momentum of the neutron:

$$\varphi_{j\mu_j}(\zeta, \mathbf{r}_n) = \sum_{\substack{l,\, s,\, m,\, \mu_s \\ \mu_i,\, \mu_n}} \varphi_{jls}(\zeta, \mathbf{r}_n) \left(\mu_i \mu_n \Big| i\, \tfrac{1}{2}\, s\mu_s\right) (\mu_s m \mid sl j\mu_j)\, \chi_{i\mu_i} \chi_{\frac{1}{2}\mu_n} Y_{lm}, \quad (3.19)$$

where $(\mu_i\mu_n \mid i\tfrac{1}{2}s\mu_s)$ and $(\mu_s m \mid slj\mu_j)$ are Clebsch-Gordan coefficients.

We note that the integral

$$\int \varphi_{jls}(\zeta, \mathbf{r}_n)\, \varphi_i^*(\zeta)\, d\zeta = \mathfrak{R}_{jl}(r_n)$$

can be regarded as the wave function of the captured neutron and can be represented in the region outside the nucleus in the form

$$\mathfrak{R}_{jl}(r_n) = \sqrt{\frac{2M}{\hbar^2 R} \gamma_{jls}}\, \frac{\mathfrak{f}_l(k_n r_n)}{\mathfrak{f}_l(k_n R)}, \quad r_n > R,$$

where γ_{jls} is the reduced width of the state in which the absorbed neutron has orbital angular momentum l and the nucleus has total spin j.

We now use (3.19) and the expansion of the deuteron spin function in terms of spin functions for the neutron and proton

$$\chi_{1\mu_1} = \sum_{\mu_n \mu_p} \left(\tfrac{1}{2}\, \tfrac{1}{2}\, \mu_n \mu_p \Big| 1\mu_1\right) \chi_{\mu_n} \chi_{\mu_p}.$$

After carrying out the integration and the summation over the spin variables we get for the reaction amplitude:

$$f = \sqrt{\frac{4M\alpha}{\pi\hbar^2 R}} \sum_{l,\, s,\, m,\, \mu_s,\, \mu_n} \sqrt{\gamma_{jls}} \left(\mu_i\mu_n \Big| i\, \tfrac{1}{2}\, s\mu_s\right) (\mu_s m \mid slj\mu_j) \times \left(\tfrac{1}{2}\, \tfrac{1}{2}\, \mu_n\mu_p \Big| 1\mu_1\right) I_l^m, \quad (3.20)$$

where I_l^m is defined as before in (3.17).

The cross section will be given by the square modulus of (3.20). The cross section must be averaged over spin projections in the initial state and summed over spin projections in the final state:

$$\frac{1}{3(2i+1)} \sum |f|^2 = \frac{4M\alpha}{\pi\hbar^2 R} \frac{1}{3(2i+1)} \sum_{\substack{\mu_i,\, \mu_1 \\ \mu_j,\, \mu_p}} \Big| \sum_{l,\, s,\, m,\, \mu_s,\, \mu_n} \sqrt{\gamma_{jls}} \left(\mu_i\mu_n \Big| i\, \tfrac{1}{2}\, s\mu_s\right) \times (sl\mu_s m \mid j\mu_j) \left(\tfrac{1}{2}\, \tfrac{1}{2}\, \mu_n\mu_p \Big| 1\mu_1\right) I_l^m \Big|^2.$$

The summation can be done by using the following relations:

$$\left.\begin{aligned} &\sum_{\mu_d,\, \mu_p} \left(\tfrac{1}{2}\, \tfrac{1}{2}\, \mu_n\mu_p \Big| 1\mu_d\right) \left(\tfrac{1}{2}\, \tfrac{1}{2}\, \mu_n'\mu_p \Big| 1\mu_d\right) = \tfrac{3}{2}\, \delta_{\mu_n\mu_n'}, \\ &\sum_{\mu_i,\, \mu_n} \left(i\, \tfrac{1}{2}\, \mu_i\mu_n \Big| s\mu_s\right) \left(i\, \tfrac{1}{2}\, \mu_i\mu_n \Big| s'\mu_s'\right) = \delta_{ss'} \delta_{\mu_s\mu_s'}, \\ &\sum_{\mu_s,\, \mu_j} (sl\mu_s m \mid j\mu_j)(sl'\mu_s m' \mid j\mu_j) = \frac{2j+1}{2l+1} \delta_{ll'} \delta_{mm'}, \end{aligned}\right\} \quad (3.21)$$

which result from the orthogonality properties of the Clebsch-Gordan coefficients.

Finally, after averaging and summing over spin states, we get the following expression for the (d,p) cross section:

$$d\sigma = \frac{2j+1}{2i+1} \frac{k_p}{k_d} \frac{4M\alpha}{\pi\hbar^2 R} \sum_l \frac{\gamma_{jl}}{2l+1} \sum_m |I_l^m|^2\, dO. \quad (3.22)$$

Here i and j are the spins of the initial state of nucleus A and the final state of nucleus B and $\gamma_{jl} = \sum_s \gamma_{jls}$.

The summation in (3.22) extends only over those values of l which satisfy the selection rule

$$\left||i-j|-\frac{1}{2}\right| \leqslant l \leqslant i+j+\frac{1}{2}.$$

Also, if the parity of the initial state of nucleus A and the final state of nucleus B are the same then only even values of l are possible, while if the initial and final states have opposite parity only odd values of l are possible.

The amplitude I_l^m which appears in (3.22) is given by

$$I_l^m = \int \psi_{k_p}^*(\mathbf{r}) \frac{\mathfrak{k}_l(k_n r)}{\mathfrak{k}_l(k_n R)} Y_{lm}(\vartheta, \varphi) \psi_{k_d}(\mathbf{r})\, d\mathbf{r}, \quad (3.23)$$

where $\psi_{\mathbf{k}_d}$ is the wave function for the center of mass motion of the deuteron in the field of nucleus A, and $\psi_{\mathbf{k}_p}$ is the wave function for the motion of the proton in the field of nucleus B. The value of k_n is related to the energy E_n of the absorbed neutron by the equation $k_n = \sqrt{-2ME_n/\hbar^2}$.

Formula (3.22) determines the angular distribution of the protons from the (d, p) stripping reaction.

5. The plane-wave approximation. The calculation of the angular distribution of the protons produced in a stripping reaction reduces to the computation of the integral (3.17). This integral can be calculated in explicit form if we neglect the scattering of the deuteron and proton waves in the field of the nucleus, i.e., if we replace the wave functions $\psi_{\mathbf{k}_d}$ and $\psi_{\mathbf{k}_p}$ in the integral (3.17) by plane waves $e^{i\mathbf{k}_d\mathbf{r}}$ and $e^{i\mathbf{k}_p\mathbf{r}}$, and integrate only over the region outside the nucleus, $r \geq R$. Thus I_l^m can be represented approximately as

$$I_l^m = \int_{r>R} e^{i\mathbf{kr}} Y_{lm}^*(\vartheta, \varphi) \frac{\mathfrak{k}_l(k_n r)}{\mathfrak{k}_l(k_n R)}\, d\mathbf{r}, \quad \mathbf{k} = \mathbf{k}_d - \mathbf{k}_p. \quad (3.24)$$

Obviously the plane wave approximation can be used only if the energy E_d of the incident deuteron and the energy E_p of the emerging proton are considerably above the Coulomb barrier Ze^2/R.

Using the expansion of the plane wave in spherical harmonics,

$$e^{i\mathbf{kr}} = 4\pi \sum_{l,m} i^l j_l(kr) Y_{lm}^*(\vartheta_k, \varphi_k) Y_{lm}(\vartheta, \varphi),$$

and the orthogonality property of the spherical harmonics, we find

$$I_l^m = 4\pi i^l Y_{lm}^*(\vartheta_k, \varphi_k) \int_R^\infty j_l(kr) \frac{\mathfrak{k}_l(k_n r)}{\mathfrak{k}_l(k_n R)} r^2 dr = 4\pi i^l Y_{lm}^*(\vartheta_k, \varphi_k)$$

$$\times \frac{R^2}{k^2+K_n^2} \left\{ \frac{dj_l(kR)}{dR} - j_l(kR) \frac{d}{dR} \ln \mathfrak{k}_l(k_n R) \right\}. \quad (3.24')$$

Substituting this expression for I_l^m in (3.22) and carrying out the summation over m by using the relation

$$\sum_m |Y_{lm}|^2 = \frac{2l+1}{4\pi},$$

we obtain finally for the cross section of the stripping reaction in the plane wave approximation the following expression:

$$d\sigma = \frac{2j+1}{2i+1} \frac{k_p}{k_d} \frac{4M\alpha R^3}{\hbar^2} \frac{1}{\left\{\alpha^2 + \left(\frac{1}{2}\mathbf{k}_d - \mathbf{k}_p\right)^2\right\}^2} \sum_l \gamma_{jl} \left| \frac{dj_l(kR)}{dR} \right.$$

$$\left. - j_l(kR) \frac{d}{dR} \ln \mathfrak{k}_l(k_n R) \right|^2 dO. \quad (3.25)$$

We have made use of the relation $k^2 + k_n^2 = 2\{\alpha^2 + (\frac{1}{2}\mathbf{k}_d - \mathbf{k}_p)\}^2$, which follows from conservation of energy.

The proton angular distribution given by formula (3.25) depends on the energy E_d of the incident deuteron, the energy E_p of the emerging proton, and on the orbital angular momentum of the captured neutron.

For fixed initial state of nucleus A and final state of nucleus B, the permissible values of l are given by the selection rules:

(a) j is the vector sum of i, l, and $\frac{1}{2}$, i.e.,

$$\left||j-i|-\frac{1}{2}\right| \leqslant l \leqslant j+i+\frac{1}{2}.$$

(b) If the initial and final states have the same parity, l is even. If the parities are opposite, the admissible l values are odd.

Formula (3.25) contains two factors which depend on the angle ϑ of emergence of the proton (the angle between the vectors $\mathbf{k}_p$ and $\mathbf{k}_d$).

(1) The deuteron factor $\{\alpha^2 + (\frac{1}{2}\mathbf{k}_d - \mathbf{k}_p)^2\}^{-2}$.

The proton, having an initial momentum $\frac{1}{2}\mathbf{k}_d$, is emitted with momentum $\mathbf{k}_p$. The difference $\mathbf{k}_p - \frac{1}{2}\mathbf{k}_d$ is the momentum of the relative motion of the proton in the deuteron at the moment when the neutron is stripped off. The factor $\{\alpha^2 + (\frac{1}{2}\mathbf{k}_d - \mathbf{k}_p)^2\}^{-2}$ is proportional to the probability for a given value of the relative momentum in the deuteron. As a function of the angle ϑ between $\mathbf{k}_p$ and $\mathbf{k}_d$, this factor has a maximum in the forward direction. The greater the angle at which the proton emerges, the greater must be the momentum of the relative motion in the deuteron and the less the probability. The deuteron factor is the same for transitions with different values of l. The dependence of the deuteron factor on angle is shown in Fig. 3.

(2) The neutron factor $\left| \frac{dj_l(kR)}{dR} - j_l(kR) \times \frac{d}{dr} \ln \mathfrak{k}_l(k_n R) \right|^2$.

The neutron leaves the deuteron with momentum $\mathbf{k} = \mathbf{k}_d - \mathbf{k}_p$. The neutron transfers this momentum to the nucleus. The factor

$$\left| \frac{dj_l(kR)}{dR} - j_l(kR)\frac{d}{dR}\ln \mathfrak{f}_l(k_n R)\right|^2$$

is proportional to the probability that a neutron with momentum **k** will be found on the surface of the nucleus and have orbital angular momentum l. This factor, which contains spherical Bessel functions, is an oscillating function of the angle ϑ, whose oscillations decrease with increasing ϑ. If $l = 0$, the neutron factor has its principal maximum in the forward direction $\vartheta = 0$. For all other values of l, there is a minimum at $\vartheta = 0$. The position of the first maximum for $l \neq 0$ can be found from the quasi-classical condition for capture of the neutron: $kR = l$, where $k = [(k_d - k_p)^2 + 4k_d k_p \sin^2(\vartheta/2)]^{1/2}$. The greater the angular momentum k which the neutron must have to penetrate to the distance R.

As l increases, the first maximum of the neutron factor shifts toward larger angles ϑ and decreases in magnitude. The angular dependence of the neutron factor for different l values is shown in Fig. 3.

Figure 3 also shows the characteristic dependence of the differential cross section on angle ϑ for various values of l.

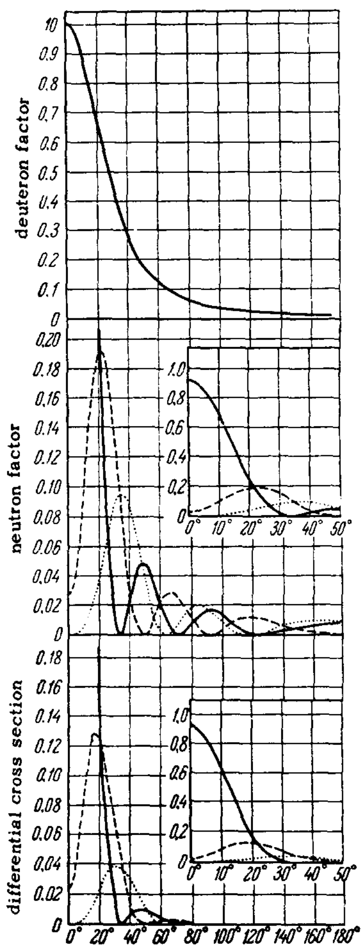


FIG. 3. Angular distribution for a stripping reaction. The solid curve is for $l = 0$, the dashed curve for $l = 1$, and the dotted curve for $l = 2$. $E_d = 6.9$ Mev, $E_p = 10.8$ Mev, $R = 7 \times 10^{-13}$ cm.

If the selection rules permit several different l values, the differential cross section will be the sum (without interference) of contributions from different l values. The weight factors for the various terms will be determined by the reduced widths γ_{jl}.

6. Transition to the Serber model. If the energy of the incident deuteron is sufficiently high ($E_d \gg \epsilon$), the neutron will be captured in a virtual state ($k_n = i\kappa_n$ corresponds to an energy of the final nucleus which lies in the continuous spectrum). If we denote the density of final states of the nucleus by ρ_{jl}, the cross section for stripping leaving the final nucleus with energy in the interval dE_j can be written in the form

$$d\sigma = \frac{2j+1}{2i+1}\frac{8M\alpha R^3}{\hbar^2}\frac{k_p}{k_d}\frac{1}{\left\{\alpha^2 + \left(\frac{1}{2}k_d - k_p\right)^2\right\}^2}\sum_l \gamma_{jl}\left|\frac{dj_l(kR)}{dR} - j_l(kR)\frac{d}{dR}\ln h_l^{(1)}(\varkappa_n R)\right|^2 \rho_{jl}\,dE_j\,do. \tag{3.26}$$

If there are a large number of terms with different l values in (3.26), the main contribution will come from angles for which the difference $k^2 - \kappa_n^2$ is small, so that

$$\left\{\frac{dj_l(kR)}{dR}h_l^{(1)}(\varkappa_n R) - j_l(kR)\frac{dh_l^{(1)}\varkappa_n R}{dR}\right\} \approx -\frac{i}{kR^2}.$$

Using the principle of detailed balancing, which states that the reduced width γ_{jl} is related to the neutron sticking probability ζ_l and the density ρ_{l_j} of final states of the nucleus by the formula[2]

$$\gamma_{jl} = \frac{(2l+1)(2i+1)\zeta_l}{2\pi\varkappa_n R(2j+1)\rho_{jl}},$$

and noting that for $\kappa_n R \gg 1$ we have approximately $|h_l^{(1)}(\kappa_n R)|^2 \simeq 1/\kappa_n^2 R^2$, we get the following expression:

$$d\sigma = \frac{2M\alpha}{\pi\hbar^2 k_d}\frac{1}{\left\{\alpha^2 + \left(\frac{1}{2}k_d - k_p\right)^2\right\}^2}\sum_l (2l+1)\zeta_l\,dE_j\,do.$$

We see that the energy of the deuteron is shared approximately equally between the neutron and proton.

In the case of fast neutrons we may assume that absorption occurs only at impact parameters smaller than the nuclear radius. Since we are interested in a stripping process we must consider only those neutrons which are bound to protons which do not interact with the nucleus. If the radial projection of the distance between neutron and proton is ρ, it is obvious that such neutrons will have impact parameters $l\lambda\!\!\!^{-}$ ($\lambda\!\!\!^{-} = 2/k_d$) which are contained in the interval between $R - \rho$ and R. Carrying out the summation over impact parameters in this interval, averaging over different values of ρ and integrating with respect to energy and angle of the emerging protons, we get the Serber[111] formula for the total cross section for the stripping reaction

$$\sigma_p = \frac{\pi}{2} R R_d, \quad R_d \ll R. \tag{3.27}$$

The distribution of the emerging protons in energy and angle then corresponds to the "transparent" model of Serber,[111]

$$d\sigma(\vartheta) = \sigma_p \sqrt{\frac{\varepsilon}{E_d}} \frac{\vartheta\, d\vartheta}{\left(\frac{\varepsilon}{E_d} + \vartheta^2\right)^{\frac{3}{2}}},$$

$$d\sigma(E_p) = \frac{\sigma}{\pi} \frac{\sqrt{\varepsilon E_d}\, dE_p}{\left(E_p - \frac{1}{2} E_d\right)^2 + \varepsilon E_d}. \tag{3.28}$$

7. The effect of finite nuclear mass. For the case of stripping reactions on light nuclei the fact that the nuclear mass is finite can lead to sizeable corrections. We shall show how the results of the preceding paragraphs should be changed to take account of the finite mass of the nucleus.

(1) In Eq. (3.10) for the reaction amplitude, the proton mass M should be replaced by the reduced mass $MM_B/(M+M_B)$. (We assume the neutron and proton masses to be equal and denote them by M, the mass of the initial nucleus A is M_A, the mass of the final nucleus B is M_B.)

(2) The reduced mass of the neutron appears in the reduced width γ_{jl}, so that the neutron mass M in (3.15) should be replaced by the reduced mass of the neutron $MM_A/(M+M_A)$.

(3) In the expression for the deuteron velocity $v_d = \hbar k_d/M_d$, which appears in the cross section (3.9), the deuteron mass $M_d = 2M$ should be replaced by the reduced mass $2MM_A/(2M+M_A)$.

Inclusion of these corrections gives the additional factor

$$\left(1+\frac{M}{M_B}\right)^{-1}\left(1+\frac{M}{M_A}\right)^{-1}\left(1+\frac{2M}{M_A}\right)^{-1} = \left(1+\frac{2M}{M_A}\right)^{-2}.$$

in the cross section formula (3.22).

(4) The expression (3.10) contains the vectors $\mathbf{r}_n$ and $\mathbf{r}_p$, which determine the coordinates of the neutron and proton relative to the center of mass of the initial nucleus. Let us introduce the vector $\mathbf{r}'_p = \mathbf{r}_p - (M/M_B)\mathbf{r}_n$, which gives the coordinates of the proton relative to the center of mass of the residual nucleus B. It is obvious that when we take account of the finite mass of the nucleus the wave function $\psi_{\mathbf{k}_p}$ in (3.10), which describes the motion of the proton in the field of the residual nucleus B, should depend on $\mathbf{r}'_p$. Thus, in the plane wave approximation we get for the reaction amplitude

$$f = 2\sqrt{\frac{\alpha}{2\pi}} \int e^{i\mathbf{k}\mathbf{r}} F(\mathbf{r})\, d\mathbf{r}, \quad \mathbf{k} = \mathbf{k}_d - \frac{M_A}{M_B}\mathbf{k}_p.$$

Noting also that

$$k^2 + k_n^2 = 2\frac{M_A}{M_B}\left\{\alpha^2 + \left(\frac{1}{2}\mathbf{k}_d - \mathbf{k}_p\right)^2\right\},$$

we finally get, in the plane wave approximation taking account of the finite nuclear mass, the following expression for the differential cross section for the stripping reaction:

$$d\sigma = \frac{2j+1}{2i+1}\frac{k_p}{k_d}\frac{\left(1+\frac{M}{M_A}\right)^4}{\left(1+\frac{M}{M_B}\right)^2}\frac{4M\alpha R^3}{\hbar^2}\frac{1}{\left\{\alpha^2+\left(\frac{1}{2}\mathbf{k}_d-\mathbf{k}_p\right)^2\right\}^2}$$

$$\times \sum_l \gamma_{jl}\left|\frac{dj_l(kR)}{dR} - j_l(kR)\frac{d}{dR}\ln \mathfrak{h}_l(k_n R)\right|^2 do,$$

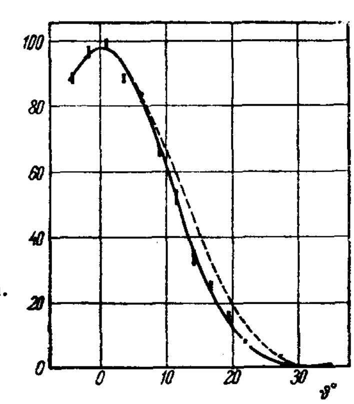
$$\mathbf{k} = \mathbf{k}_d - \frac{M_A}{M_B}\mathbf{k}_p. \tag{3.29}$$

This formula gives the angular distribution of the protons in the center of mass system.

8. Comparison with experiment. The angular distribution of the products of a stripping reaction given in formula (3.25) was first found by Butler. Despite the large number of assumptions made in deriving formula (3.25) (the assumption of zero range of the nuclear forces between neutron and proton, the replacement of the exact wave function Ψ by the approximate wave function Ψ_0 in the expression (3.8) for the amplitude, the neglect of scattering of the deuteron and proton waves in the field of the nucleus, and the neglect of the possibility of penetration of the deuteron and proton into the nucleus), the angular distribution given by this formula is in good agreement with experimental data for a large number of reactions (especially for light nuclei).

Figures 4, 5, 6, and 7 show the angular distributions observed in various reactions. The observed angular distributions and the theoretical distributions given by formula (3.25) are in good agreement in the region of small angles. At large angles there is a discrepancy which results from the possibility of processes involving compound nucleus formation.

FIG. 4. Angular distribution of protons from $Al^{27}(d, p)Al^{28}$, $E_d = 8$ Mev, $Q = 5.49$ Mev, $l = 0$. The solid curve is for $R = 6.15 \times 10^{-13}$ cm, the dashed curve for $R = 5.4 \times 10^{-13}$ cm.



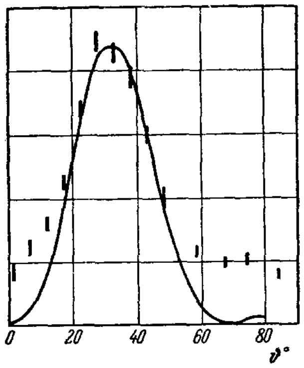


FIG. 5. Angular distribution of protons from Si^{28} (d, p) Si^{29}, $E_d = 8.18$ Mev, $Q_1 = 4.97$ Mev, $E_1 = 1.28$ Mev, $l = 2$, $R = 4.4 \times 10^{-13}$ cm.

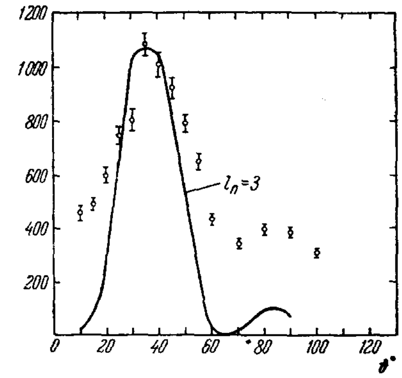


FIG. 6. Angular distribution of protons from Ca^{42} (d, p) Ca^{43}, $E_d = 7$ Mev, Ca^{43} in its ground state, $l = 3$, $R = 7.5 \times 10^{-13}$ cm.

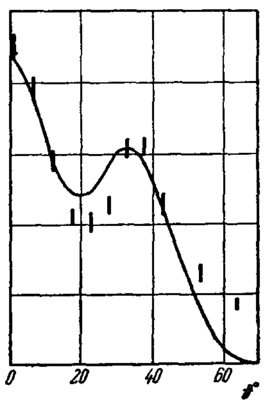


FIG. 7. Angular distribution of protons from Mg^{25} (d, p) Mg^{26}, $E_d = 8.21$ Mev, $Q_1 = 7.05$ Mev, $E_1 = 1.83$ Mev, $l = 0,2$ mixture, $R = 5.3 \times 10^{-13}$ cm.

In comparing the experimental data with (3.25), one should choose the best value of the parameter R; this value may differ somewhat from the usually accepted value of the nuclear radius R_0. A good fit with experiment is obtained by choosing R somewhat larger than the nuclear radius R_0 as given by the empirical formula

$$R_0 = \left(1.7 + 1.22A^{\frac{1}{3}}\right)\cdot 10^{-13}\ \text{cm}.$$

where A is the mass number of the nucleus.

Figure 4 shows how the angular distribution changes if the value of R is changed by 10%.

However, in many cases one observes considerable deviation of experiment from the predictions of the Butler theory. These deviations show the importance of taking into account both the nuclear and Coulomb scattering of the particles participating in the stripping reaction.[32]

9. Inclusion of scattering of deuteron and proton waves. If we take into account Coulomb and nuclear scattering, the wave functions of the deuteron and proton in the region outside the nucleus, can be chosen in the form

$$\psi_{k_d}(\mathbf{r}) = 4\pi \sum_{l,\,m} i^l e^{i\eta_l(n_d)} \{F_l(n_d k_d r) - \alpha_l^d H_l(n_d, k_d r)\}\, r^{-1} Y^*_{lm}(\vartheta_{k_d}, \varphi_{k_d})\, Y_{lm}(\vartheta, \varphi), \quad (3.30)$$

$$\psi_{k_p}(\mathbf{r}) = 4\pi \sum_{l,\,m} i^l e^{-i\eta_l(n_p)} \{F_l(n_p, k_p r) - \alpha_l^{p*} H_l^*(n_p, k_p r)\}\, r^{-1} Y^*_{lm}(\vartheta_{k_p}, \varphi_{k_p})\, Y_{lm}(\vartheta, \varphi), \quad (3.31)$$

where $F_l(n, kr)$ and $G_l(n, kr)$ are the regular and irregular radial Coulomb functions, which are solutions of the equation

$$\frac{d^2u_l}{dr^2} + \left[k^2 - \frac{l(l+1)}{r^2} - \frac{2MZe^2}{\hbar^2 r}\right] u_l = 0,$$

$H_l = F_l - iG_l$; $\eta_l = \arg\Gamma(1 + l + in)$ is the Coulomb scattering phase; $n = Ze^2/\hbar v$ where v is the velocity of the particle; the amplitudes α_l^d and α_l^p describe the purely nuclear scattering of the partial deuteron and proton waves.

The amplitudes α_l^d and α_l^p can be expressed in terms of the logarithmic derivative of the radial wave function f_l at the nuclear surface[4]

$$\alpha_l = \frac{1}{2}\left\{1 + \frac{f_l - \Delta + is}{f_l - \Delta - is}\,\frac{H_l^*(R)}{H_l(R)}\right\},$$

where

$$\Delta_l = R\left[\frac{G_l G_l' + F_l F_l'}{G_l^2 + F_l^2}\right]_{r=R}, \quad s_l = R\left[\frac{G_l F_l' - F_l G_l'}{G_l^2 + F_l^2}\right]_{r=R}.$$

The expressions for the amplitudes α_l simplify in various limiting cases.

(a) For an absolutely impenetrable nucleus

$$\alpha_l = \frac{F_l(R)}{H_l(R)}.$$

(b) In the neighborhood of a resonance energy

$$\alpha_l^s = -\frac{i}{2}\,\frac{\Gamma_l^s}{E - E^s + \frac{i}{2}\Gamma_l}\,\frac{H_l^*(R)}{H_l(R)}.$$

(c) For a black nucleus

$$\alpha_l = \begin{cases} \frac{1}{2} & l \leqslant kR \\ 0 & l > kR. \end{cases}$$

However when we use the functions of (3.30) and (3.31), the integral I_l^m cannot be calculated in explicit form. Numerical computations carried out by Tobocman and Kalos[117] have shown that inclusion of Coulomb and nuclear scattering of the deuteron and proton can result in marked deviations from the results of the Butler theory.

In Figs. 8 and 9 we give graphs showing the influence of Coulomb and nuclear scattering on

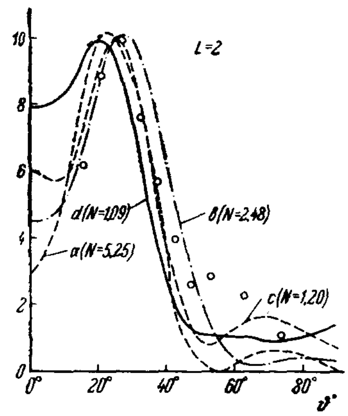


FIG. 8. Angular distribution of protons from $F^{19}(d, p)F^{20}$, for $E_d = 14.3$ Mev, $Q_0 = 4.37$ Mev, $l = 2$, $R = 5.05 \times 10^{-13}$ cm; a) plane wave approximation; b) Coulomb scattering included; c) Coulomb scattering and absorption of protons with $l_p \leq 4$ included; d) Coulomb scattering and hard sphere scattering of protons included. N is the normalization factor.

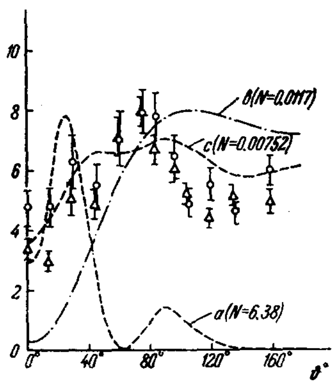


FIG. 9. Angular distribution of protons from $Ti^{48}(d, p)Ti^{49}$, $E_d = 2.6$ Mev, $Q_0 = 4.46$ Mev, $l = 1$, $R = 6.49 \times 10^{-13}$ cm; a) plane wave approximation; b) Coulomb scattering included; c) Coulomb scattering and absorption of protons with $l \leq$ included.

the angular distribution of protons from the (d, p) reaction.

Coulomb scattering of the deuteron and proton waves results in a shift of the maxima of the angular distribution toward larger angles and a broadening and reduction in height of the maxima. There is also a reduction of the total cross section. In the case of low-energy incident deuterons, the Coulomb effects can change the angular distribution completely. If the deuteron energy is considerably above the top of the Coulomb barrier, the Coulomb effects may produce a noticeable change in the angular distribution but they will not spoil the unique assignment of the l value for the captured neutron.

The effect of nuclear scattering of the deuteron and proton waves on the angular distribution is opposite to that of the Coulomb scattering. As a result of nuclear scattering the maxima in the angular distribution are shifted toward smaller angles and the width of the maxima decreases. The size of the total cross section decreases, just as it does because of the Coulomb effects.

10. The investigation of nuclear structure by means of the stripping reaction. The (d,p) and (d,n) stripping reactions on light nuclei with medium energy deuterons are a powerful tool for studying nuclear properties. Of principal interest are the stripping reactions which lead to formation of the residual nucleus in its ground state or a low-lying excited state.

The passage of a monoenergetic beam of deuterons through a layer of material A results in the formation among the reaction products of monoenergetic groups of protons or neutrons. Each such group corresponds to a definite level of the residual nucleus B. By measuring the Q of the reaction for the different proton groups, we can determine the level energy E_B of a state of the residual nucleus B from the relation

$$E_B = S_n - \varepsilon - Q.$$

But the real importance of stripping reactions for nuclear spectroscopy is related to the characteristic angular distribution from such reactions. The study of the angular distribution of the protons (or neutrons) of a given group enables one to draw conclusions concerning the spin and parity of the corresponding state of the residual nucleus.

If the spin and parity of the initial state of nucleus A are known, we can find the spin and parity of the final state of nucleus B by comparing the experimentally observed angular distribution of the protons with the distribution given by (3.25). This comparison enables us to find the possible values of the orbital angular momentum l of the neutron absorbed by the nucleus. Very often one can obtain satisfactory agreement with experiment for only one definite value of l. A first indication of the possible values of l can be found by studying the experimental distribution at small angles. A forward maximum shows that $l = 0$ is present; a forward minimum shows that $l = 0$ is absent.

If l has been found, the selection rule determines the parity of the final state uniquely, while the spin j is one of the possible values obtainable from vector addition of i, l, and $\frac{1}{2}$. It is convenient to choose a target nucleus A which has zero spin or a low value of the spin, to make the number of choices for j a minimum. If $i = 0$, only two values of j are possible (while for $l = 0$, j is uniquely determined).

The stripping reaction can also be used to find reduced widths of levels of the residual nucleus by examining the intensities of the groups of protons emitted in the reaction. In fact, after having determined the possible l values from the shape of the angular distribution, one can by a suitable choice of the radius R make a fit at small angles of the curve determined from (3.25) to the experimental curve. Then from the measured absolute value of the cross section at the first maximum, one can use formula (3.25) to calculate the reduced width γ_{jl} of the corresponding level of the residual nucleus B.[114,65]

If the captured neutron in the final state of the residual nucleus can have several possible values of l, we can in this same fashion determine the reduced widths γ_{jl} corresponding to the different l values.

According to the shell model, a nucleon in the nucleus must be in a state of definite orbital angular momentum. The possible l values can be enumerated from the shell model. Thus the values of the reduced widths γ_{jl} determined from a stripping reaction can be used to test the validity of the shell model.[34,45]

The values of reduced widths obtained from stripping data using formula (3.25) are several times smaller than those obtained by other methods (for example, from experiments on (p, p) scattering, etc). This is due to the approximations made in (3.25). It has been shown[87,117] that taking account of scattering of the deuteron and proton waves leads to a reduction of the factor $\sum_m |I_l^m|^2$ which appears in the more exact formula (3.22) for the cross section. Thus the inclusion of scattering of the deuteron and proton waves enables one to obtain a more correct value for the reduced width. Despite the fact that the plane wave approximation which is used in (3.25) gives too small values for the reduced widths, the ratios of the reduced widths for different levels are given correctly by this approximation.[65]

11. Polarization in stripping reactions. From general symmetry considerations it is evident that the particles liberated in a stripping reaction may be polarized in a direction perpendicular to the plane containing the wave vectors of the incident deuteron and the emerging particle. A determination of the polarization produced in a stripping reaction can give additional information concerning the spin of the residual nucleus.

In the plane wave approximation, the reaction products are unpolarized. In fact in this case the neutrons [if we consider the reaction A(d, p)B] are absorbed by nucleus A independently of the polarization of the incident deuteron, so that the emerging protons are unpolarized. However, when we take into account the interaction of the emerging proton with the nucleus, we get a polarization.

The possibility of polarization from a stripping reaction was first pointed out by Newns,[99] who determined the polarization on the assumption that the nucleus is opaque for protons.

The possibility of absorption of the proton by the nucleus causes the average value of the projection (along the vector $\mathbf{k}_p \times \mathbf{k}_d$) of the orbital angular momentum of the neutron, which was originally bound to the proton in the deuteron and later absorbed by the nucleus, to be positive. This results in a polarization of the protons. The total angular momentum of the absorbed neutron can assume the values $l+\frac{1}{2}$ and $l-\frac{1}{2}$, i.e., the orbital angular momentum and spin can be either parallel or antiparallel. Since the spins of the proton and neutron are parallel in the deuteron, and since in the capture a positive value of the projection of the orbital angular momentum is more probable, the protons will be partially polarized along the direction of the vector $\mathbf{k}_p \times \mathbf{k}_d$ if $j_n = l+\frac{1}{2}$, and will be polarized in the opposite direction for $j_n = l-\frac{1}{2}$. The magnitude of the polarization will be given by the expression

$$P = \pm \frac{2}{3(2j_n+1)} \left(\frac{\sum\limits_m m\,|I_l^m|^2}{\sum\limits_m |I_l^m|^2} \right), \quad j_n = l \pm \frac{1}{2}. \qquad (3.32)$$

Thus the sign of the polarization gives an indication of the value of j_n. Since the spin j of the residual nucleus is the vector sum of i and j_n, knowledge of j_n simplifies the problem of finding j. For example, if $i = 0$, $j = j_n = l \pm \frac{1}{2}$, so that j is uniquely determined by the sign of the polarization.

Horowitz and Messiah[88] have determined the polarization of the protons from the stripping reaction, using a hard sphere model of the nucleus. They found the same sign for the polarization as did Newns.

Cheston[49] determined the polarization of the protons from stripping reactions which results from spin-orbit interaction of the proton and the residual nucleus. The parameters for the potential of this interaction were chosen on the basis of data on scattering of low energy protons by nuclei. It turned out that the polarization due to the spin-orbit coupling is opposite to the polarization which results for a black nucleus or a hard-sphere nucleus.

Polarization of the protons was observed experimentally by Hillman[84] in the $C^{12}(d, p)C^{13}$ reaction. The experimentally determined sign of the polarization agrees with that given by Cheston, but the absolute value of the polarization is three times as large as the calculated value. It was later shown by Tobocman, Newns, and Refai[123] that the correct sign of the polarization of the protons from a stripping reaction can be obtained if the scattering of the deuteron by the nucleus is taken into account. The experimental results[124,125] agree with reference 123.

12. Angular correlations in (d, pγ) and (d, nγ) reactions. Additional information on the spin of the final state of the nucleus in a stripping reaction A(d,p)B can be gotten by studying the angular correlation between the protons and the γ-quanta emitted by nucleus B if it is formed in an excited state. The theory of the angular correlations for (d, pγ) and (d, nγ) reactions was given by Biedenharn, Boyer, and Charpie[36] (Cf. also references 68, 107, and 89).

The determination of the angular correlation in a (d, pγ) reaction reduces to finding the angular distribution of the γ radiation for a fixed direction of emergence of the proton. The matrix element for such a stripping process with subsequent emission of a γ quantum having angular momentum L and projection M will be proportional to the product of the stripping amplitude (3.20) and the matrix element of the multipole moment $(Q_{LM})_{j\mu_j;j_f\mu_f}$, for the transition of the residual nucleus from the state j, μ_j to the state j_f, μ_f by emission of a γ quantum. Using (3.20) and (3.24′) and omitting factors depending on the angular momentum projections (which do not affect the angular distribution), we get

$$M=\sum_{\substack{lsm\mu_s\\ \mu_n\mu_j}} \sqrt{\gamma_{js}}\, i^l \left(\tfrac{1}{2}\tfrac{1}{2}\mu_p\mu_n \,|\, 1\mu_d\right)\left(i\,\tfrac{1}{2}\mu_i\mu_n \,|\, s\mu_s\right)$$

$$\times (sl\mu_s m \,|\, j\mu_j)\, q_l(\mathbf{k})\, Y^*_{lm}(\vartheta_\mathbf{k}\varphi_\mathbf{k})\,(Q_{LM})_{j\mu_j;\,j_f\mu_f}, \quad (3.33)$$

where we have introduced the abbreviation

$$q_l(\mathbf{k})=\left\{\frac{dj_l(kR)}{dR}-j_l(kR)\frac{d}{dR}\ln\mathfrak{P}_l(k_nR)\right\}.$$

The summation in (3.33) extends over all possible values of the projection μ_j in the "intermediate" state.

Since the operator Q_{LM} of the multipole moment is an L-vector, i.e., a quantity transforming according to the (2L+1)-dimensional irreducible representation of the rotation group, and since the wave functions $\varphi_{j\mu_j}$ and $\varphi_{j_f\mu_f}$ are also L-vectors (with L = j and L = j_f), the matrix elements of Q_{LM} coincide (except for factors independent of the angular momentum projections) with the coefficients of the expansion

$$\varphi_{j\mu_j}\varphi_{j_f\mu_f}=\sum_L (jj_f\mu_j\mu_f \,|\, LM)\,\varphi_{LM}.$$

The squared modulus of the matrix element (3.33) gives the probability of emission of a γ quantum with given angular momentum L and projection M. The angular distribution of such radiation is uniquely determined by the well-known functions F_{LM}, which are given, for example, in reference 1. We thus obtain for the angular distribution of the γ quanta, averaged over the polarizations of the angular momenta in the initial and final states, the following expression:

$$W(\mathbf{k},\mathbf{k}_\gamma)\sim\sum_{\substack{\mu_d\mu_i\mu_p\\ \mu_f M}}\Bigg|\sum_{\substack{lsm\mu_s\\ \mu_n\mu_j}}\sqrt{\gamma_{jls}}\, i^l$$

$$\times\left(\tfrac{1}{2}\tfrac{1}{2}\mu_p\mu_n \,|\, 1\mu_d\right)\left(i\,\tfrac{1}{2}\mu_i\mu_n \,|\, s\mu_s\right)(sl\mu_s m \,|\, j\mu_j)$$

$$\times (jj_f\mu_j\mu_f \,|\, LM)\, Y^*_{lm}(\vartheta_\mathbf{k},\varphi_\mathbf{k})\, q_l(\mathbf{k})\Bigg|^2 F_{LM}(\mathbf{k}_\gamma). \quad (3.34)$$

This expression is usually called the correlation function.

The summation over μ_d, μ_p, μ_i, and μ_n in (3.34) can be carried out by using the orthogonal properties (3.21) of the Clebsch-Gordan coefficients. If we now use the formula for expansion of a product of spherical harmonics in a series of spherical harmonics

$$Y^*_{lm}Y_{l'm'}=(-1)^m\sum_\nu\sum_{\mu=-\nu}^{\nu}\left[\frac{(2l+1)(2l'+1)}{4\pi(2\nu+1)}\right]^{\frac{1}{2}}$$

$$\times (ll'00 \,|\, \nu 0)(ll'-mm' \,|\, \nu\mu)\, Y_{\nu\mu}, \quad (3.35)$$

and also use the formula for summing over angular momentum projections:[38]

$$\sum_{mm'\mu_s}(-1)^m(sl\mu_s m \,|\, j\mu_j)(sl'\mu_s m' \,|\, j\mu_j)(ll'-mm' \,|\, \nu\mu)$$

$$=(-1)^{-l+l'-s+\mu_j}(2j+1)(jj\mu_j-\mu_j \,|\, \nu 0)\, W(l_j l'_j;\, s\nu), \quad (3.36)$$

we finally get the correlation function in the form

$$W(\mathbf{k},\mathbf{k}_\gamma)=\sum_{ll's\nu}\sqrt{\gamma_{ls}\gamma_{l's}}\, i^{l-l'}$$

$$\times(-1)^s(2l+1)^{\frac{1}{2}}(2l'+1)^{\frac{1}{2}}(ll'00 \,|\, \nu 0)(LL1-1 \,|\, \nu 0)$$

$$\times W(ljl'j;\, s\nu)\, W(jLjL;\, j_f\nu)\, q_l(\mathbf{k})\, q_{l'}(\mathbf{k})\, P_\nu(\cos\theta), \quad (3.37)$$

where θ is the angle between the direction of emission of the γ ray and the direction of the momentum $\mathbf{k}$ transferred to the nucleus formed in the stripping process; W(abcd; ef) denotes the Racah coefficients;[106] the factor $i^{l-l'}$ in (3.37) is real since the conservation of parity forces $l-l'$ to take on only even values.

Because of the law of combination of angular momenta, the highest degree, ν_{max}, of the Legendre polynomials appearing in (3.37) is an even integer less than or equal to 2j, $2l_{max}$, and 2L.

If the angular distribution of the γ rays is not isotropic, l is different from zero. From the shape of the angular distribution one can, in general, determine the relative magnitudes of the reduced widths γ_{jls}.

If the spin of the initial nucleus is $i = 0$, the angular correlation depends only on j, j_f, l, and L, and is independent of γ_{jls}. In this case one can, from the observed correlation, make a unique choice of one of the two possible j values determined from the angular distribution of the protons.

Figure 10 shows the angular distribution of γ rays observed[53] in the $Be^9(d,p)Be^{10}$ reaction.

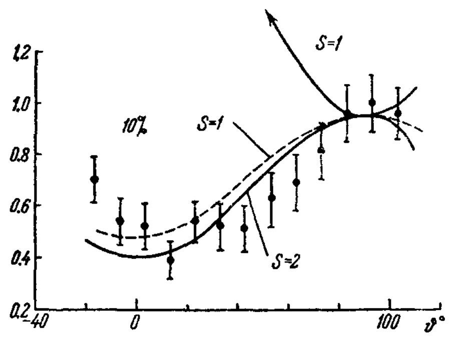


FIG. 10. Angular distribution of γ rays from the transition of B^{10} from its first excited state (2^+) at 3.37 Mev to the ground (0^+) state, for $E_d = 3.5$ Mev, $\theta_p = 20°$. S is the channel spin.

13. Formation of deuterons in the collision of nucleons with nuclei. The inverse process to the stripping reaction is the so-called pickup (capture) reaction, in which a proton incident on the nucleus pulls out a neutron to form a deuteron. The pickup process, like the stripping reaction, occurs as a result of direct interaction, in which the transition from the initial to the final state proceeds without formation of a compound nucleus. By using the principle of detailed balancing for inverse processes, we can relate the cross section for the pickup reaction to the stripping cross section. Thus the cross section for the B(p,d)A reaction will be given by the formula

$$d\sigma_{pd} = \frac{3(2i+1)}{2(2j+1)} \frac{k_d^2}{k_p^2} d\sigma_{dp}, \qquad (3.38)$$

where $d\sigma_{dp}$ is given by (3.22).

By using formula (3.38) we can determine the spin and parity of nuclei by studying the angular distribution of the deuterons formed in the reaction. (p,d) and (n,d) reactions at medium energies have been observed experimentally for several nuclei. It should be pointed out that there are experimental difficulties in the use of the pickup reaction because of the large negative Q value for such reactions.

14. Other direct processes involving deuterons. Stripping reactions can occur not only in collisions of deuterons with nuclei, but also for collisions of other light nuclei with nuclei. For example, in the collision of tritons or He^3 with nuclei, as a result of the stripping process deuterons can be formed whose angular distribution is the same as for (d,p) and (d,n) reactions. The theory of the (t,d) and (He^3,d) reaction was treated by Newns[98] and Butler and Salpeter.[47] The differential cross section for the (t,d) and (He^3,d) reactions is given by formulas of the same type as (3.22), except that the deuteron factor $\{\alpha^2 + (\frac{1}{2}k_d - k_p)^2\}^{-2}$ is replaced by a factor which gives the probability for finding the relative momentum $\frac{2}{3}k_t - k_d$ in the ground state of the triton or He^3. Qualitatively this factor gives the same angular dependence as the deuteron factor.

Just as in the stripping reaction A(d,p)B, the reduced neutron width γ_{jl} appears as a parameter in the cross section for the A(t,d)B reaction. Simultaneous investigation of the transition $A \to B$ using both deuterons and tritons enables one to eliminate this undetermined parameter from the theory.

Because of the practical difficulties in obtaining beams of tritons or He^3 nuclei, the inverse reactions (d,t) and (d,He^3) using deuterons are more important. These reactions also proceed without compound nucleus formation. A neutron or proton in the bombarded nucleus is captured by the deuteron in flight, without the deuteron penetrating into the nucleus. The angular distribution of the products of such reactions has the same character as that for pickup reactions induced by protons.

4. (d,p) and (d,n) Reactions with Compound Nucleus Formation

1. Determination of the reaction amplitude. The angular distribution for the (d,p) reaction calculated on the basis of the stripping mechanism is usually in good agreement with the experimentally observed angular distribution at small angles. But in the region of large angles deviations may occur due to the possibility of processes occurring via the formation of a compound nucleus. The formation of a compound plays a specially important role for deuteron energies in the neighborhood of resonances. However, in many cases even at small angles the observed experimental angular distribution deviates from that predicted on the basis of either mechanism. This is an indication of the importance of interference between the two processes, which may be important in the case of low energies and very light nuclei, where the quasidiscrete spectrum

of the compound nucleus manifests itself.[32,96]

Interference between the stripping process and the process with formation of a compound nucleus was treated by Thomas,[115] and independently in references 55 and 22.

To find the angular distribution of protons from the (d, p) reaction when both direct transitions and transitions with compound nucleus formation are taken into account, it is convenient to use the method of Bethe, as presented, for example, in reference 3.

Let us consider the reaction A(d, p) B. The total wave function of the system will satisfy the Schrödinger equation

$$\{H-E\}\Psi=0, \qquad (4.1)$$

where E is the total energy of the system. The total Hamiltonian can be written in the form

$$H=-\frac{\hbar^2}{2M}\Delta_p+H_B+V_{pB}, \qquad (4.2)$$

where H_B is the Hamiltonian for the internal motion of the residual nucleus B, and V_{pB} is the potential for the interaction between the proton and the residual nucleus B, which includes the interaction of the proton with the absorbed neutron.

To find the solution of (4.1) we represent the wave function Ψ in the form

$$\Psi=\varphi_a\varphi_d\psi_d+\varphi_b\psi_p+c\varphi_c+\text{ orthogonal terms} \qquad (4.3)$$

φ_a, φ_d, and φ_b are internal wave functions for the initial nucleus, the deuteron and the residual nucleus, all normalized to unity; φ_c is the wave function of the compound nucleus and differs from zero only within a finite region which is determined by the nuclear radius R_c, and c is a coefficient to be determined later. For simplicity we treat the case where there is one level E_c of the compound nucleus. The function ψ_d describes the relative motion of the deuteron and nucleus A ($\psi_d \neq 0$ for r_n, $r_p > R_A$) and the spin s in the initial channel. (The spin s of the entrance channel is given by the vector sum of the deuteron spin and the spin of nucleus A.) The function ψ_p describes the relative motion of the proton and the residual nucleus B ($\psi_p \neq 0$ for $r_p > R_B$), as well as the spin s′ in the exit channel (where s′ is the vector sum of the spin of the emergent proton and the spin of the residual nucleus). If (4.1) is valid, the following equations must be satisfied:

$$\int \varphi_a^*\varphi_d^*(H-E)\Psi\, d\mathbf{r}\, d\tau_A=0,$$

$$\int \varphi_b^*(H-E)\Psi\, d\tau_B=0, \qquad (4.4)$$

$$\int \varphi_c^*(H-E)\Psi\, d\tau_c=0. \qquad (4.5)$$

The equations (4.4) are differential equations for the wave functions ψ_d and ψ_p. Equation (4.5) enables us to determine the coefficient c of the wave function of the compound nucleus in (4.3).

Let us determine the wave function ψ_p of the emergent proton. Substituting (4.1) and (4.3) in (4.4), we have

$$\{\Delta_p+k_p^2-U_p\}\psi_p=\frac{2M}{\hbar^2}\int \varphi_b^*V_{pB}(\varphi_a\varphi_d\psi_d+c\varphi_c)\,d\tau_B, \qquad (4.6)$$

$$U_p=\frac{2M}{\hbar^2}\int \varphi_b^*V_{pB}\varphi_b\,d\tau_B. \qquad (4.7)$$

Using the asymptotic Green's function (3.6), we can find for the solution of (4.5) the following asymptotic expression, valid for large r_p:

$$\psi_p \longrightarrow \frac{e^{ik_pr_p}}{r_p}\chi_{s'\mu_s'}f_{s\mu_s;\,s'\mu_s'}. \qquad (4.8)$$

In this formula, the reaction amplitude f is the sum $f = f^B + f^c$, where the first term f^B is the amplitude for the direct transition (stripping reaction)

$$f^B=-\frac{M}{2\pi\hbar^2}\int \psi_{\mathbf{k}_p}^*\chi_{s'\mu_s'}^*\varphi_b^*V_{pB}\psi_{\mathbf{k}_d}\chi_{s\mu_s}\varphi_d\varphi_a\,d\tau, \qquad (4.9)$$

while the second term f^c is the amplitude for the (d, p) reaction with compound nucleus formation

$$f^c=-c\frac{M}{2\pi\hbar^2}\int \psi_{\mathbf{k}_p}^*\chi_{s'\mu_s'}^*\varphi_b^*V_{pB}\varphi_c\,d\tau. \qquad (4.10)$$

($\chi_{s\mu_s}$ and $\chi_{s'\mu_s'}$ are the spin wave functions for the entrance and exit channels, respectively.)

For the calculation of f^B, we note that $V_{pB} = V_{pn} + V_{pA}$. But the contribution of V_{pA} to f^B can be neglected, since ψ_d is different from zero only for $r_p > R_A$, whereas because of the short range of the nuclear forces V_{pA} is effective only for $r_p < R_A$. Thus we get

$$f^B=-\frac{M}{2\pi\hbar^2}\int \psi_{\mathbf{k}_p}^*\chi_{s'\mu_s'}^*\varphi_b^*V_{pn}\varphi_a\varphi_d\psi_{\mathbf{k}_d}\chi_{s\mu_s}\,d\tau.$$

Expanding the spin wave function $\chi_{s'\mu_s'}$ of the final state of the system in eigenfunctions Y_{lm} of the orbital angular momentum of the neutron absorbed by the nucleus and using the condition (3.12) for zero range of the nuclear forces, we get finally

$$f^B=\sqrt{\frac{M\alpha}{\pi\hbar^2R}}\sum_{l,m} i^l\,\overline{\gamma_l}\,(lsm\mu_s\,|\,s'\mu_s')\,I_l^m,$$

$$|s-s'|\leqslant l\leqslant s+s', \qquad (4.11)$$

where I_l^m is given by (3.17).

The coefficient c multiplying the compound nucleus wave function in (4.3) can be determined by using (4.5) in the same fashion as in reference 3:

$$c = \frac{\int \varphi_c^* V_{\mathrm{dA}} \varphi_a \varphi_{\mathrm{d}} \psi_{\mathbf{k}_{\mathrm{d}}} \chi_{s\mu_s}\, d\tau}{E - E_c + \frac{i}{2}\Gamma_c}. \tag{4.12}$$

V_{dA} is the potential acting between the deuteron and nucleus A; Γ_c is the total width of the resonance level of the compound nucleus at energy E_c, in a state with definite angular momentum I_c. Γ_c is equal to the sum of the deuteron and proton widths, Γ^d and Γ^p, which are given by the formulas $\Gamma^d = k_d \sum_{l_d} \left| U^{I_c}_{l_d s} \right|^2$ and $\Gamma^p = k_p \sum_{l_p} \left| U^{I_c}_{l_p s'} \right|^2$, where

$$U^{I_c}_{l_d s} = \frac{2\sqrt{2M}}{\hbar} \int i^{-l_d} e^{-i\eta l_d} f_{l_d}(r_d)\, Y^*_{l_d s I_c \mu_s} \varphi_d^* \varphi_a^* V_{\mathrm{dA}} \varphi_c\, d\tau,$$

$$U^{I_c}_{l_p s'} = \frac{2\sqrt{M}}{\hbar} \int i^{-l_p} e^{i\eta l_p} f_{l_p}(r_p)\, Y^*_{p s' I_c \mu_s} \varphi_l^* V_{\mathrm{pB}} \varphi_c\, d\tau. \tag{4.13}$$

Thus the amplitude for the (d, p) reaction with formation of a compound nucleus is given by the expression

$$f^c = -\frac{M}{2\pi\hbar^2} \frac{\int \psi^*_{\mathbf{k}_p} \chi^*_{s'\mu'_s} \varphi_b^* V_{\mathrm{pB}} \varphi_c\, d\tau \int \varphi_c^* V_{\mathrm{dA}} \varphi_a \varphi_d \chi_{s\mu_s} \psi_{\mathbf{k}_d}\, d\tau}{E - E_c + \frac{i}{2}\Gamma_c}. \tag{4.14}$$

Using the expressions for $\psi_{\mathbf{k}_d}$ and $\psi_{\mathbf{k}_p}$, and the law of vector addition of angular momenta, we can write the amplitude (4.14) as

$$f^c = -\sum_{l_d,\, l_p,\, m_p} 2^{-\frac{1}{2}} \pi^{\frac{1}{2}} (2l_d + 1)^{\frac{1}{2}} (l_d s 0 \mu_s \mid I_c \mu_s)(l_p s' m_p \mu'_s \mid I_c \mu_s)$$

$$\times \frac{U^{I_c *}_{l_d s} U^{I_c}_{l_p s'}}{E - E_c + \frac{i}{2}\Gamma_c} Y_{l_p m_p}(\vartheta, \varphi). \tag{4.15}$$

where the z axis is taken along the vector $\mathbf{k}_d$.

2. <u>The reaction cross section</u>. The differential cross section for the (d, p) reaction with unpolarized particles is given by the square modulus of the reaction amplitude, averaged over the spin projections for the entrance channel and summed over spin projections in the exit channel:

$$d\sigma_{s;\,s'} = \frac{1}{2s+1} \frac{v_p}{v_d} \sum_{\mu_s,\, \mu'_s} |f_{s\mu_s;\, s'\mu'_s}|^2\, do. \tag{4.16}$$

Noting that the amplitude for the (d, p) reaction is the sum of the amplitude for direct transitions and the amplitude (4.9) for transitions with compound nucleus formation, we can write the cross section as a sum of three terms:

$$d\sigma_{ss'} = d\sigma^B_{ss'} + d\sigma^c_{ss'} + d\sigma^{interf}_{ss'}. \tag{4.17}$$

Finally the differential cross section for the (d,p) reaction when we disregard the channel spins s and s′ is obtained by averaging (4.16) over all possible values of s and summing over all s′:

$$d\sigma = \sum_{s,\, s'} \frac{2s+1}{3(2i+1)} d\sigma_{ss'}. \tag{4.18}$$

(The fraction in (4.18) gives the statistical weight of the spin s in the entrance channel.)

The term $d\sigma^B$ in the total cross section (4.17) gives the contribution of direct transitions. When this term is averaged and summed over the various values of s and s′, we find, as expected, that it coincides with (3.22).

The term $d\sigma^c$ in (4.17) is the contribution of transitions with compound nucleus formation. In order to simplify the expression for $d\sigma^c$, we use the expansion (3.35) of a product of spherical harmonics in a series of spherical harmonics. Then the sum of absolute square moduli f^c can be written as

$$\sum_{\mu_s \mu'_s} |f^c|^2 = \sum_{l_d l'_d l_p l_p} \frac{\pi}{2} (2l_d + 1)^{\frac{1}{2}} (2l'_d + 1)^{\frac{1}{2}} \frac{U^*_{l_d s} U_{l'_d s} U_{l_p s'} U^*_{l'_p s'}}{(E - E_c)^2 + \frac{1}{4}\Gamma_c^2}$$

$$\times \sum_L \left[\frac{(2l_p + 1)(2l'_p + 1)}{4\pi(2L + 1)} \right]^{\frac{1}{2}} (l_p l'_p 00 \mid L0) \times \sum_{\mu_s \mu'_s m_p m'_p M} (-1)^{m_p}$$

$$\times (l_d s 0 \mu_s \mid I_c \mu_s)(l'_d s 0 \mu_s \mid I_c \mu_s)(l_p s' m_p \mu'_s \mid I_c \mu_s)(l'_p s' m'_p \mu'_s \mid I_c \mu_s)$$

$$\times (l_p l'_p - m_p m'_p \mid LM)\, Y_{LM}.$$

The summation over angular momentum projections can be done using (3.36). We then find for the differential cross section, which determines the angular distribution of the protons from the (d, p) reaction with compound nucleus formation, the formula of Blatt and Biedenharn:[40]

$$d\sigma^c_{ss'} = \frac{\lambda\!\!\!^{-2}_d}{2s+1} \sum_{L=0}^{\infty} R_L(s, s') P_L(\cos\vartheta)\, do, \tag{4.19}$$

$$R_L(s, s') = \frac{1}{4\left[(E - E_c)^2 + \frac{1}{4}\Gamma_c^2\right]}$$

$$\times \sum_{l_d l'_d l_p l'_p} (2I_c + 1)^2 (2l_d + 1)^{\frac{1}{2}} (2l'_d + 1)^{\frac{1}{2}} (2l_p + 1)^{\frac{1}{2}} (2l'_p + 1)^{\frac{1}{2}}$$

$$\times (l_d l'_d 00 \mid L0)(l_p l'_p 00 \mid L0)\, W(l_d I_c l'_d I_c;\, sL)$$

$$\times W(l_p I_c l'_p I_c;\, s'L)\, k_d k_p \operatorname{Re}\{U^*_{l_{ds}} U_{l'_{ds}} U_{l_p s'} U^*_{l'_p s'}\}. \tag{4.20}$$

The summation in (4.20) over l_d and l'_d runs from $|I_c - s|$ to $I_c + s$, while the sum over l_p and l'_p goes from $|I_c - s'|$ to $I_c + s'$.

Integrating (4.19) over angles, we easily get the Breit-Wigner formula

$$\sigma^c = \pi \lambda\!\!\!^{-2}_d \frac{2I_c + 1}{2s + 1} \frac{\Gamma_d \Gamma_p}{(E - E_c)^2 + \frac{1}{4}\Gamma_c^2}. \tag{4.21}$$

for the total cross section. Substituting (4.12) and

(4.15) in (4.17), we find for the interference term in the cross section,

$$d\sigma^{interf} = -\frac{1}{2s+1}\frac{v_d}{v_d}\sqrt{\frac{2M\alpha}{\hbar^2 R}}$$

$$\times \sum_{l_d l_p l_n}\sum_{\mu_s \mu_s' m_p m_n} (2l_d+1)^{\frac{1}{2}}(l_n s m_n \mu_s | s'\mu_s')(l_d s 0 \mu_s | I_c \mu_s)$$

$$\times (l_p s' m_p \mu_s' | I_c \mu_s)\cdot\sqrt{\gamma_{jl_n}}$$

$$\times \mathrm{Re}\left\{ I_{l_n}^{m_n} \frac{U_{l_d s} U_{l_p s'}^*}{E-E_c-\frac{i}{2}\Gamma_c} Y_{l_p m_p}^*(\vartheta,\varphi)\right\} do. \quad (4.22)$$

In specific cases formula (4.22), and similarly (4.20), simplifies considerably. As an example, let us consider the case when the orbital angular momentum of the captured neutron is equal to zero, $l_n = 0$. Then the spins of the entrance and exit channels are the same, since $(l_n s m_n \mu_s | s'\mu_s') \rightarrow (0s0\mu_s | s'\mu_s') = \delta_{ss'}\delta_{\mu_s\mu_s'}$. Carrying out the summation over μ_s $(m_p = 0)$ in (4.22) by means of the relation

$$\sum_{\mu_s} (l_d s 0 \mu_s | I_c \mu_s)(l_p s 0 \mu_s | I_c \mu_s) = \frac{2I_c+1}{2l_d+1}\delta_{l_d l_p},$$

we get

$$d\sigma^{interf} = -\frac{2I_c+1}{2s+1}\frac{v_p}{v_d}\sqrt{\frac{M\alpha}{2\pi\hbar^2 R}}\sqrt{\gamma_{jo}}$$

$$\sum_{l=|I_c-s|}^{I_c+s} \mathrm{Re}\left\{ I_0^0 \frac{U_l^d U_l^{p*}}{E-E_c-\frac{1}{2}\Gamma_c}\right\} P_l(\cos\vartheta)\, do. \quad (4.23)$$

In the plane wave approximation the interference term for the case of $l_n = 0$ has the form

$$d\sigma^{interf} = -\frac{2I_c+1}{2s+1}\frac{v_p}{v_d}\sqrt{\frac{M\alpha}{2h^2 R}}\frac{R^2\sqrt{\gamma_{jo}}}{\alpha^2+\left(\frac{1}{2}\mathbf{k}_d-\mathbf{k}_p\right)^2}$$

$$\times \left\{\frac{dj_0(kR)}{dR} - j_0(kR)\frac{d}{dR}\ln \mathfrak{f}_0(K_n R)\right\}$$

$$\times \sum_{l=|I_c-s|}^{I_c+s} \mathrm{Re}\left\{\frac{U_l^d U_l^{p*}}{E-E_c-\frac{i}{2}\Gamma_c}\right\} P_l(\cos\vartheta)\, do. \quad (4.24)$$

Because of the interference between the direct process (stripping reaction) and the process with compound nucleus formation, the angular distribution can differ drastically, even at small angles, from the distribution given by stripping theory, if the energy of the compound nucleus is in the region of the quasi-discrete spectrum.

If the energy spread of the incident deuterons is large compared to the distance between neighboring levels of the compound nucleus, the interference term (4.22) which results from superposition of the two amplitudes vanishes as a result of the averaging over energy. Thus the average cross section for the (d, p) reaction will be the sum of the individual cross sections for the stripping process and the (d, p) reaction via compound nucleus formation. This same situation will occur if the energy of the compound nucleus is in the region of the quasi-continuous spectrum.

5. Inelastic Scattering of Deuterons

1. Inelastic scattering processes. When deuterons collide with nuclei, we can also have inelastic scattering processes: scattering of the deuteron accompanied by excitation of the nucleus, A (d, d′) A*; scattering accompanied by breakup of the deuteron, A (d, np) A; and finally scattering in which the deuteron breaks up and the nucleus is excited, A (d, np) A*. Like the stripping reaction, these processes may occur without formation of a compound nucleus. The angular distribution in such inelastic scattering processes, just as in the stripping reactions, is characterized by a complex structure from which one can draw conclusions concerning the spin and parity of the final state of the nucleus.

The mechanism of the inelastic scattering processes is similar to that for the stripping reaction. The inelastic scattering can be described particularly simply if we assume that in the collision of the deuteron with the nucleus only of the constituents of the deuteron (say, the neutron) interacts with the nucleus, while the other (the proton) is outside the range of the nuclear forces. Then the interaction occurs essentially only with the surface of the nucleus. The transfer of energy from the interacting particle (the neutron) to the nucleus can occur either with or without disruption of the binding of the neutron and proton in the deuteron. In the latter case we can have inelastic scattering of the deuteron accompanied by excitation of the nucleus,[91] while in the former case the scattering is accompanied by breakup of the deuteron and we may also have simultaneous excitation of the nucleus.[23]

2. Excitation of the nucleus in deuteron scattering. In treating inelastic scattering of deuterons, we start from (3.1) but now neglect the interaction of the proton with the nucleus. Then

$$\left\{H_A - \frac{\hbar^2}{4M}\Delta_d - \frac{\hbar^2}{M}\Delta + V_n + V_{np} - E\right\}\Psi(\zeta, \mathbf{r}, \mathbf{r}_d) = 0, \quad (5.1)$$

where $E = E_d - \epsilon$. We are assuming that the initial nucleus is in its ground state $E_{a_0} = 0$. We shall try to find a solution of (5.1) of the form

$$\Psi(\zeta, \mathbf{r}, \mathbf{r}_d) = \sum_a \varphi_a(\zeta)\varphi_0(r)\psi_a(\mathbf{r}_d)$$
$$+ \text{orthogonal terms} \quad (5.2)$$

where $\varphi_a(\zeta)$ and $\varphi_0(r)$ are solutions of the equation

$$(H_A - E_a)\varphi_a(\zeta) = 0,$$
$$\left(-\frac{\hbar^2}{M}\Delta + V_{np} + \varepsilon\right)\varphi_0(r) = 0. \quad (5.3)$$

We then get from (5.1) the following equation for the wave function describing the center of mass motion of the deuteron after scattering:

$$\{\Delta_d + k'^2\}\psi_a(\mathbf{r}_d)$$
$$= \frac{4M}{\hbar^2}\int \varphi_a^*(\zeta)\varphi_0(r)V_n\Psi(\zeta, \mathbf{r}, \mathbf{r}_d)\,d\zeta\,d\mathbf{r}, \quad (5.4)$$

where $k'^2 = (4M/\hbar^2)(E_d - E_a)$; E_a is the excitation energy of the nucleus in the final state. Replacing the exact wave function Ψ on the right side of (5.4) by the incident wave $\Psi_0 = e^{i\mathbf{k}\mathbf{r}_d} \times \varphi_0(r)\varphi_{a_0}(\zeta)$ (where $\mathbf{k}$ is the wave vector of the incident deuteron), we find for the asymptotic behavior of the solution of (5.4),

$$\psi_a(\mathbf{r}_d) \to f\frac{e^{ik'r_d}}{r_d},$$
$$f = -\frac{M}{\pi\hbar^2}\int e^{-i\mathbf{q}\mathbf{r}_d}\varphi_0^2(r)\varphi_a^*(\zeta)V_n\varphi_{a_0}(\zeta)\,d\zeta\,d\mathbf{r}\,d\mathbf{r}_d. \quad (5.5)$$

The differential cross section for scattering of the deuteron accompanied by excitation of the nuclear level E_a, is

$$d\sigma = \frac{k'}{k}|f|^2\,dO. \quad (5.6)$$

To simplify the calculation of the scattering amplitude, we shall neglect the neutron spin. Assuming that the neutron-nucleus interaction occurs only at the nuclear surface, we can express the integral over the internal coordinates of the nucleus, which appears in (5.6), as

$$\int \varphi_a^*(\zeta)V(\mathbf{r}_n, \zeta)\varphi_{a_0}(\zeta)\,d\zeta$$
$$= \frac{\delta(r_n - R)}{R^2}\sum_{l,m}\langle\mu_j|V|\mu_i lm\rangle Y_{lm}^*(\vartheta_n, \varphi_n), \quad (5.7)$$

where R is the nuclear radius, l is the angular momentum transferred from the neutron to the nucleus, and

$$\langle\mu_j|V|\mu_i lm\rangle = \int \varphi_a^*(\zeta)V(r_n, \zeta)\varphi_{a'}(\zeta)Y_{lm}(\vartheta_n, \varphi_n)\,d\zeta\,dO_n.$$

(We assume that the functions φ_{a_0} and φ_0 refer to nuclear states with spins and spin projections i, μ_i and j, μ_j.)

Transforming from the variables $\mathbf{r}$ and $\mathbf{r}_d$ to $\mathbf{r}$ and $\mathbf{r}_n$ in (5.5), and using the spherical harmonic expansion of the plane wave, we find

$$f = -\frac{4M}{\hbar^2}\int e^{\frac{i}{2}\mathbf{q}\mathbf{r}}\varphi_0^2(r)\,d\mathbf{r}$$
$$\times\sum_{l,m}(-i)^l j_l(qR)\langle\mu_j|V|\mu_i lm\rangle Y_{lm}^*(\vartheta_q, \varphi_q).$$

The cross section (5.6) must be summed over spin projections μ_j in the final state and averaged over spin projections μ_i in the initial state. We then get

$$d\sigma = \frac{k'}{k}\left|\int e^{\frac{i}{2}\mathbf{q}\mathbf{r}}\varphi_0^2(r)\,d\mathbf{r}\right|^2\sum_l|B_l|^2 j_l^2(qR)\,dO, \quad (5.8)$$

where

$$|B_l|^2 = \frac{4M^2}{(2i+1)\pi\hbar^4}\sum_{\mu_i\mu_j m}|\langle\mu_j|V|\mu_i lm\rangle|^2.$$

If we include the neutron and proton spins, the expression for $|B_l|^2$ is replaced by

$$|B_l|^2 = \frac{4M^2}{3\pi(2i+1)\hbar^4}\sum_{\mu_i\mu\mu_j\mu' m}\left|\int \varphi_{j\mu_j}^*\chi_{1\mu'}^*V\chi_{1\mu}\varphi_{i\mu_i}Y_{lm}\,d\zeta\,dO\right|^2$$

($\chi_{1\mu}$ and $\chi_{1\mu'}$ are the deuteron spin functions before and after scattering.) However, since the theory is not able to calculate the absolute value of the cross section, we should regard $|B_l|^2$ as a free parameter of the theory.

Noting that $\int\varphi_0^2(r)e^{\frac{1}{2}\mathbf{q}\mathbf{r}}\,d\mathbf{r} = \frac{4\alpha}{q}\tan^{-1}\frac{q}{4\alpha}$, we finally get for the cross section for deuteron scattering accompanied by transition of the nucleus from a state with spin i to one with spin j,[91]

$$d\sigma_{ij} = \frac{k'}{k}\left(\frac{4\alpha}{q}\tan^{-1}\frac{q}{4\alpha}\right)^2\sum_l|B_l|^2 j_l^2(qR)\,dO,$$
$$\mathbf{q} = \mathbf{k}' - \mathbf{k}. \quad (5.9)$$

The angular momentum transferred from the deuteron to the nucleus in the scattering is given by l. The summation in (5.9) extends over all integral values of l given by the selection rule

$$\mathbf{j} = \mathbf{i} + \mathbf{l} + \mathbf{1}$$

(if i = 0 and l = 0, then j = 0 or 1). The values of l are odd or even according as the parity of the nucleus does or does not change in the transition.

Formula (5.9) gives the angular distribution of the deuterons at small angles. (Obviously, at large angles we must include the inelastic scattering which occurs via compound nucleus formation.) We can determine l from the experimental angular distribution by using (5.9). (If the selection rule allows several values of l, the lowest l will be the most important one.) Having found l, and knowing the spin and parity of the initial state, we can determine the spin and parity of the final state.

Just as in the case of the stripping reaction, the best agreement with experiment is obtained by choosing a value of R which is somewhat larger than the nuclear radius R_0.

Figures 11 and 12 show the angular distributions of deuterons inelastically scattered[78] from Li^7 and Mg^{24}. The energy of the incident deuterons was $E_d = 15.1$ Mev. The comparison shows satisfactory agreement of theory and experiment.

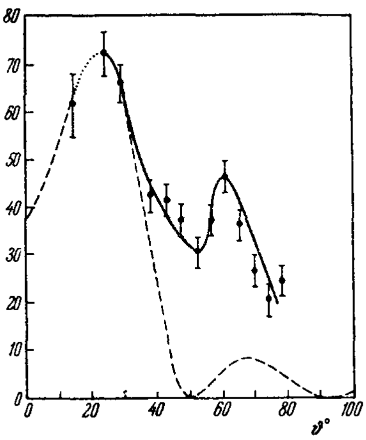


FIG. 11. Angular distribution of inelastically scattered deuterons from $Li^7(d, d')Li^{7*}$; $E_d = 15.1$ Mev, $Q = -4.61$ Mev, $l = 1$, $R = 4.8 \times 10^{-13}$ cm.

3. Breakup of the deuteron in the scattering process. The scattering of the deuteron by a nucleus may be accompanied by breakup of the deuteron, and we may also have simultaneous transition of the nucleus to an excited state. We can use Eq. (5.1) to describe the breakup process, but now we must look for a solution of the form

$$\Psi(\zeta, \mathbf{r}, \mathbf{r}_d) = \sum_{\mathbf{f}a} \varphi_a(\zeta)\varphi_{\mathbf{f}}(\mathbf{r})\psi_{\mathbf{f}a}(\mathbf{r}_d) + \text{orthogonal terms}, \quad (5.10)$$

where $\varphi_{\mathbf{f}}(\mathbf{r})$ is the wave function for the relative motion of the neutron and proton in the unbound state; $\mathbf{f}$ is the wave vector of the relative motion of the system.

The function $\varphi_{\mathbf{f}}(\mathbf{r})$ is a solution of the equation

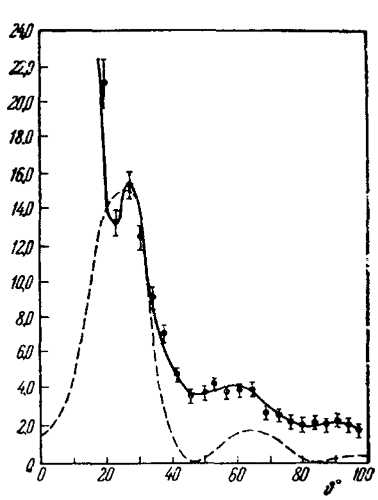


FIG. 12. Angular distribution of inelastically scattered deuterons from $Mg^{24}(d, d')Mg^{24*}$; $E_d = 15.1$ Mev, $Q = -1.37$ Mev, $l = 2$, $R = 6.2 \times 10^{-13}$ cm.

$$\left(-\frac{\hbar^2}{M}\Delta + V_{np} - \varepsilon_{\mathbf{f}}\right)\varphi_{\mathbf{f}}(\mathbf{r}) = 0, \quad (5.11)$$

where $\epsilon_{\mathbf{f}} = \hbar^2 f^2/M$ is the energy of relative motion of the neutron-proton system. If we assume that the neutron and proton interact only in an S state (which is justified for low energies of relative motion), the wave function can be written as the sum of a plane wave and an outgoing spherical wave

$$\varphi_{\mathbf{f}}^{(s)}(\mathbf{r}) = e^{i\mathbf{f}\mathbf{r}} + \frac{a^{(s)}}{r}e^{-ifr}, \quad s = 0, 1, \quad (5.12)$$

where $a^{(s)} = -1/(\alpha_s - if)$ is the neutron-proton scattering length in an S state and depends on the spin state of the neutron-proton system. If the neutron-proton system is in the triplet $s = 1$ state, $\alpha_1 = \alpha = \sqrt{M\epsilon/\hbar^2}$, where $\epsilon = 2.23$ Mev is the binding energy of the deuteron. If the system is in the singlet $s = 0$ state, $\alpha_0 = \alpha' = \sqrt{M\epsilon_0/\hbar^2}$, where $\epsilon_0 = 69$ kev is the energy of the virtual state of the deuteron.

The outgoing spherical wave in (5.12) corresponds to the production of particles.

It is easy to verify that the wave functions $\varphi_{\mathbf{f}}^{(1)}(\mathbf{r})$ are orthogonal to the wave function $\varphi_0(\mathbf{r}) = \sqrt{\alpha/2\pi}\,(e^{-\alpha r}/r)$, which describes the bound state of the neutron-proton system,

$$\int \varphi_0(r)\varphi_{\mathbf{f}}^{(1)}(\mathbf{r})\,d\mathbf{r} = 0.$$

The functions $\varphi_{\mathbf{f}}^{(1)}(\mathbf{r})$ together with the function $\varphi_0(\mathbf{r})$ form a complete set of orthonormal functions, satisfying the relation

$$\varphi_0(r)\varphi_0(r') + \int \varphi_{\mathbf{f}}^*(\mathbf{r})\varphi_{\mathbf{f}}(\mathbf{r}')\frac{d\mathbf{f}}{(2\pi)^3} = \delta(\mathbf{r} - \mathbf{r}'). \quad (5.13)$$

We note that the wave functions $\varphi_{\mathbf{f}}^{(0)}(\mathbf{r})$ which correspond to singlet states of the neutron-proton system are not orthogonal to the wave function $\varphi_0(\mathbf{r})$. The orthogonality of the total wave functions in this case comes from the orthogonality of the spin wave functions for the singlet and triplet states.

Using the expansion (5.10) and choosing the incident wave to be $\Psi_0 = e^{i\mathbf{k}\mathbf{r}_d}\varphi_0(\mathbf{r})\varphi_{a_0}(\zeta)$, we get the following expression for the amplitude for breakup of the deuteron:

$$f = -\frac{M}{\pi\hbar^2}\int e^{-i\mathbf{q}\mathbf{r}_d}\varphi_{\mathbf{f}}^*(\mathbf{r})\varphi_a^*(\zeta)V\varphi_0(r)\varphi_{a_0}(\zeta)\,d\zeta\,d\mathbf{r}\,d\mathbf{r}_d, \quad (5.14)$$

where $\mathbf{q} = \mathbf{k}' - \mathbf{k}$, and the modulus of $\mathbf{k}'$ is given by

$$k'^2 = \frac{4M}{\hbar^2}(E_d - \epsilon - E_a - \varepsilon_{\mathbf{f}}).$$

Taking into account the nuclear spin and the spins of neutron and proton, and performing operations similar to those in the preceding case, we finally get the following formula for the differential cross section for breakup of the deuteron in scattering by a nucleus:

$$d\sigma^{(s)} = \frac{\alpha}{\pi^2}\frac{k'}{k}\left|\frac{1}{\alpha^2+\left(\mathbf{f}-\frac{1}{2}\mathbf{q}\right)^2}\right.$$

$$\left.+\frac{1}{q\,(i\alpha^{(s)}-f)}\ln\frac{f+\frac{1}{2}q+i\alpha}{f-\frac{1}{2}q+i\alpha}\right|^2\sum_l|B_l^{(s)}|^2 j_l^2(qR)\,d\mathbf{f}\,dO. \quad (5.15)$$

In deriving (5.15) we have used the relation

$$\int\varphi_{\mathbf{f}}^{(s)*}(\mathbf{r})\,\varphi_0(r)\,e^{\frac{i}{2}\mathbf{q}\mathbf{r}}d\mathbf{r}$$

$$=\sqrt{8\pi\alpha}\left\{\frac{1}{\alpha^2+\left(\mathbf{f}-\frac{1}{2}\mathbf{q}\right)^2}+\frac{1}{q\,(i\alpha^{(s)}-f)}\ln\frac{f+\frac{1}{2}q+i\alpha}{f-\frac{1}{2}q+i\alpha}\right\}.$$

The summation in (5.15) runs over all values of l given by the selection rule $\mathbf{j}=\mathbf{i}+\mathbf{l}+\mathbf{1}$. Only even values of l are taken if the parity of the nucleus is unchanged, and conversely. If the state of the nucleus does not change in the deuteron breakup ($j=i$, $E_a=0$), the sum in (5.15) consists of a single term corresponding to $l=0$.

The wave vector of the center of mass, $\mathbf{k}'$, and the wave vector of the relative motion $\mathbf{f}$ can be expressed in terms of the wave vectors of the liberated neutron and proton, $\mathbf{k}_n$ and $\mathbf{k}_p$, by means of the equations

$$\mathbf{k}'=\mathbf{k}_n+\mathbf{k}_p,\quad \mathbf{f}=\frac{1}{2}(\mathbf{k}_n-\mathbf{k}_p).$$

Formula (5.15) gives the differential distribution in angle and energy of the neutrons and protons emitted in the dissociation of the deuteron.

Expression (5.15) is extremely complicated. However, if we limit ourselves to the region of small angles between the wave vectors of the center of mass of the neutron-proton system before and after the breakup, $qr_{eff}\ll 1$ ($r_{eff}\sim\alpha^{-1}$), the results become much simpler. Making the additional assumption that $f_{eff}\ll k$, we get the following formula for the momentum distribution of the protons from the breakup:

$$d\sigma^{(1)}(k_p)=\frac{8\alpha}{3\pi}\frac{\left(\frac{1}{4}k^2-k_p^2-\alpha^2\right)^{3/2}(2\mathbf{k}_p-\mathbf{k})^2}{\left(\frac{1}{4}k^2-k_p^2\right)^4}$$

$$\times|B^{(1)}|^2 j_0^2(|2\mathbf{k}_p-\mathbf{k}|R)\,d\mathbf{k}_p,\quad s=1, \quad (5.16)$$

$$d\sigma^{(0)}(k_p)=\frac{8\alpha}{\pi}\frac{(\alpha-\alpha_0)^2\left(\frac{1}{4}k^2-k_p^2-\alpha^2\right)^{\frac{1}{2}}}{\left(\frac{1}{4}k^2-k_p^2\right)^2\left(\frac{1}{4}k^2-k_p^2-\alpha^2-\alpha_0^2\right)}$$

$$\times|B^0|^2 j_0^2(|2\mathbf{k}_p-\mathbf{k}|R)\,d\mathbf{k}_p,\,s=0. \quad (5.17)$$

Formulas (5.16) and (5.17) are given for the case where the state of the nucleus does not change during the breakup.

Similar formulas will apply for the neutrons formed from the breakup.

6. Interaction of Deuterons with Heavy Nuclei

1. Deuteron reactions in a Coulomb field. In the preceding paragraphs, in treating the collision of deuterons with nuclei we neglected the Coulomb interaction of the deuteron with the charge on the nucleus. This is valid when the energy of the deuteron is considerably above the top of the Coulomb barrier. But if the energy of the incident deuteron is comparable to or less than the height of the barrier, the Coulomb interaction plays an essential role.

For medium energy deuterons ($E_d > 5$ Mev) colliding with light nuclei, Coulomb effects can be neglected. But for collisions of deuterons with heavy nuclei, the Coulomb interaction is extremely important. The Coulomb interaction is especially important for low deuteron energies, when the classical distance of closest approach $b = Ze^2/E_d$ is much greater than the nuclear radius R.

Because the center of mass and center of charge of the deuteron do not coincide, the Coulomb interaction can lead to various processes of dissociation of the deuteron. The following processes are possible: liberation of both neutron and proton, capture of the neutron and emission of the proton, capture of the proton with emission of the neutron, and capture of both particles. All these processes are possible even when the energy of the incident deuteron is below the Coulomb barrier. In fact, because of the relatively low binding energy of the deuteron, a process of "dissociation" of the deuteron can occur outside the nucleus, which later leads to the reactions enumerated. The probability of the process occurring with this "preliminary" breakup of the deuteron turns out to be considerably greater than the probability for the same process occurring via formation of a compound nucleus.

The mechanism of "preliminary" breakup was pointed out by Oppenheimer and Phillips[102] in connection with (d,p) reactions. (The (d,p) reaction at low energies is sometimes called the Oppenheimer-Phillips process.) The theory of all the processes enumerated was given in the quasi-classical approximation (when the deuteron energy is much less than the height of the Coulomb barrier) by Lifshitz[18] (cf. also references 33 and 119). However, the quasi-classical approximation, in which one considers only "head-on" collisions of the deuteron with the nucleus (collisions with zero orbital angular momentum of the deuteron relative to the nucleus) could give only the variation of the effective cross section with deuteron energy.

Later Landau and Lifshitz[17] developed a method which enabled one to calculate the effective cross sections for the various processes. In their paper, Landau and Lifshitz treated the (d, np) reaction on heavy nuclei. The theory of the (d, p) reaction on heavy nuclei was given by Ter-Martirosyan[25] and Biedenharn, Boyer, and Goldstein.[37]

2. The (d, p) reaction on heavy nuclei. Let us treat the (d, p) reaction on heavy nuclei, assuming that the energy of the incident deuteron is less than the height of the Coulomb barrier, $E_d < Ze^2/R$. In this case the angular distribution of the protons emitted in the reaction is determined mainly by the Coulomb field of the nucleus. Then, unlike the case of the (d, p) reaction on light nuclei, the angular distribution depends very little on the orbital angular momentum l of the captured neutron and has typically a maximum in the backward direction. The treatment is very much simplified in the limiting case of $n_d = Ze^2/\hbar v_d \gg 1$ and $n_p = Ze^2/\hbar v_p \gg 1$, when the quasi-classical approximation is applicable.

In finding the differential cross section we can use the general theory of the (d, p) reaction given in Sec. 3, according to which

$$d\sigma = \frac{2j+1}{2i+1}\frac{4M\alpha}{\pi\hbar^2 R}\frac{k_p}{k_d}\sum_{l,m}\frac{\gamma_{jl}}{2l+1}|I_l^m|^2\,dO. \tag{6.1}$$

But now in calculating the coefficients I_l^m we must use Coulomb wave functions for the deuteron and proton.

For the deuteron wave function we take the Coulomb function

$$\varphi_{\mathbf{k}_d}(\mathbf{r}) = e^{-\frac{\pi}{2}n_d}\Gamma(1+in_d)\,e^{i\mathbf{k}_d\mathbf{r}} \times F(-in_d,\,1,\,i(k_d r - \mathbf{k}_d\mathbf{r})), \tag{6.2}$$

which is made up at infinity of a plane wave with wave vector $\mathbf{k}_d$ and an outgoing spherical wave.

For the wave function of the proton we should take a Coulomb function which at infinity contains a plane wave with wave vector $\mathbf{k}_p$ and an incoming spherical wave,

$$\psi_{\mathbf{k}_p}(\mathbf{r}) = e^{-\frac{\pi}{2}n_p}\Gamma(1-in_p)\,e^{i\mathbf{k}_p\mathbf{r}} \times F(in_p,\,1,\,-i(k_p r - \mathbf{k}_p\mathbf{r})). \tag{6.3}$$

The integral (3.17) can be calculated approximately in the limiting case of $n_d \gg 1$ and $n_p \gg 1$. Then the integration in (3.17) can be extended over the whole space $\mathbf{r}$, since the contribution of the region $r < R$ is very small. In fact in the quasi-classical approximation $n_d \gg 1$ and $n_p \gg 1$, the main contribution comes from distances greater than both the distances of closest approach $b_d = Ze^2/E_d$ and $b_p = Ze^2/E_p$, which for low energies of the incident deuteron gives a distance much greater than the nuclear radius R.

Using the series expansion of $\mathfrak{k}_l(x)$,

$$\mathfrak{k}_l(x) = \frac{\pi}{2}\frac{e^{-x}}{x}\sum_{k=0}^{l}\frac{(l+k)!}{k!\,(l-k)!\,(2x)^k}, \tag{6.4}$$

we note that in the integral (3.17) the factor

$$e^{-k_n r}\psi^*_{\mathbf{k}_p}(\mathbf{r})\,\varphi_{\mathbf{k}_d}(\mathbf{r}) = \exp\{-k_n r + \ln\psi^*_{\mathbf{k}_p}(\mathbf{r})\,\varphi_{\mathbf{k}_d}(\mathbf{r})\}$$

is a rapidly varying function of r. The value of an integral over such a rapidly varying function comes principally from the region near the saddle point $\mathbf{r}_1(r_1, \vartheta_1, \varphi_1)$ at which the function $F(\mathbf{r}) = -k_n r + \ln\psi^*_{\mathbf{k}_p}(\mathbf{r})\,\varphi_{\mathbf{k}_d}(\mathbf{r})$ has an extremum. Therefore the slowly varying spherical harmonic in (3.17) can be taken outside the integral and evaluated at $\vartheta = \vartheta_1$ and $\varphi = \varphi_1$. In the integral which remains, the function $\mathfrak{k}_l(k_n r)$ can be replaced approximately by the first term of the expansion (6.4), if $k_n r_1 > l(l+1)/2$. This condition is almost always satisfied, if l is not very large and $|E_n|$ is not very small.[18] We thus get

$$I_l^m = \frac{\pi}{2k_n}\frac{Y^*_{lm}(\varphi_1, \vartheta_1)}{\mathfrak{k}_l(k_n R)}\int\frac{e^{-k_n r}}{r}\psi^*_{\mathbf{k}_p}(\mathbf{r})\,\varphi_{\mathbf{k}_d}(\mathbf{r})\,d\mathbf{r}.$$

Substituting this expression for I_l^m in (6.1), we find

$$d\sigma = \frac{2j+1}{2i+1}\frac{k_p}{k_d}\frac{M\alpha}{4\hbar^2 k_n^2 R}\left|\int\frac{e^{-k_n r}}{r}\psi^*_{\mathbf{k}_p}(\mathbf{r})\,\varphi_{\mathbf{k}_d}(\mathbf{r})\,d\mathbf{r}\right|^2 \times\sum_l\frac{\gamma_{jl}}{|\mathfrak{k}_l(k_n R)|^2}\,dO. \tag{6.5}$$

It follows from (6.5) that only the absolute value of the cross section depends on l; in this approximation the angular distribution is independent of l.

The integral over the Coulomb functions in the cross section (6.5) can be evaluated exactly:

$$\int\frac{e^{-k_n r}}{r}\psi^*_{\mathbf{k}_p}(\mathbf{r})\,\varphi_{\mathbf{k}_d}(\mathbf{r})\,d\mathbf{r}$$

$$= 8\pi^2\left[\frac{n_p n_d}{(e^{2\pi n_p}-1)(e^{2\pi n_d}-1)}\right]^{\frac{1}{2}}\left[\frac{(k_n - ik_p)^2 + k_d^2}{(\mathbf{k}_d-\mathbf{k}_p)^2 + k_n^2}\right]^{in_p}$$

$$\times\left[\frac{(k_n - ik_d)^2 + k_p^2}{(\mathbf{k}_d-\mathbf{k}_p)^2 + k_n^2}\right]^{in_d}\frac{1}{(\mathbf{k}_d-\mathbf{k}_p)^2 + k_n^2}\,\frac{F(-in_p,\,-in_d,\,1,\,-\zeta)}{1+\zeta}.$$

Here $\zeta = \zeta_0\sin^2\vartheta/2$, $\zeta_0 = \dfrac{4k_p k_d}{(\mathbf{k}_d - \mathbf{k}_p)^2 + k_n^2}$, and ϑ is the angle between the vectors $\mathbf{k}_d$ and $\mathbf{k}_p$. The square modulus of the integral is equal to

$$\left|\int\ldots\right|^2=\frac{64\pi^4 n_p n_d}{(e^{2\pi n_p}-1)(e^{2\pi n_d}-1)}$$

$$\times\frac{\exp\{2n_d(\pi-\varphi_d)+2n_p\varphi_p\}}{[(k_d-k_p)^2+k_n^2]^2}\left|\frac{F(in_p,in_d,1,-\zeta)}{1+\zeta}\right|^2.$$

where the angles φ_p and φ_d are given by the formulas

$$\varphi_p=\tan^{-1}\frac{2k_n k_p}{k_d^2-k_p^2+k_n^2},\quad \varphi_d=\tan^{-1}\frac{2k_n k_d}{k_d^2-k_p^2-k_n^2}\quad \text{for } E_n<0,$$
$$\varphi_p=0,\qquad\qquad \varphi_d=0\qquad\qquad \text{for } E_n>0.$$

In the limiting case of $n_p \gg 1$, $n_d \gg 1$, we can use the following asymptotic formula for the hypergeometric functions:

$$|F(in_p,in_d,1,-\zeta)|^2\simeq\frac{1+\zeta}{2\pi n_d\zeta}\frac{\exp\{2n_d\psi_d+2n_p(\pi-\psi_p)\}}{\sqrt{(4\rho/\zeta)-(1-\rho)^2}},$$

$$\psi_p=\cos^{-1}\frac{(1-\rho)\zeta-2\rho}{2\rho\sqrt{1+\zeta}},$$

$$\psi_d=\cos^{-1}\frac{(1-\rho)\zeta+2}{2\sqrt{1+\zeta}},\qquad \rho=\frac{n_p}{n_d}. \tag{6.6}$$

Using this asymptotic formula we find for the cross section[25]

$$d\sigma=\frac{2j+1}{2i+1}\frac{k_p}{k_d}\frac{M_\alpha}{\hbar^2 k_n R}\sum_l\frac{\gamma_{jl}}{|I_l(k_n R)|^2}\frac{32\pi^3 n_p}{[(k_d-k_p)^2+k_n^2]^2}$$

$$\times\exp\{2n_p\varphi_p-2n_d\varphi_d\}N(\zeta)\,dO,$$

$$N(\zeta)=\frac{\exp\{-2n_p\psi_p+2n_d\psi_d\}}{\zeta(1+\zeta)\sqrt{(4\rho/\zeta)-(1-\rho)^2}}. \tag{6.7}$$

The expression in the exponential in $N(\zeta)$ increases with increasing ζ, i.e., with angle ϑ; therefore the cross section $d\sigma$ increases exponentially with increasing angle ϑ. The function $N(\zeta)$ is a maximum at $\vartheta=\pi$, and for small values of $\pi-\vartheta$ the dependence of $N(\zeta)$ on $\pi-\vartheta$ is close to a Gaussian. This is easily shown by expanding the argument of the exponential in $N(\zeta)$ in powers of $\zeta_0-\zeta\simeq\zeta_0(\pi-\vartheta)^2/4$. In this way one finds

$$N(\zeta)\sim\exp\left\{-\frac{(\pi-\vartheta)^2}{\delta^2}\right\}, \tag{6.8}$$

where $\delta^2=\dfrac{(k_d-k_p)^2+k_n^2}{n_d\alpha k_d}$. The width δ of the Gaussian distribution in the backward direction is the smaller the greater the value of Z, the smaller the energy E_d of the incident deuteron and the greater the energy E_n of the captured neutron.

In reference 37 a specific set of parameters was considered and made it possible to see how the character of the proton angular distribution changes when the parameter n_d changes. Since the angular distribution of the protons depends

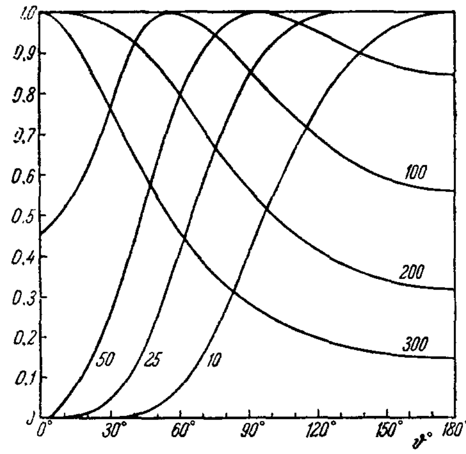


FIG. 13. Angular distribution of protons as a function of incident deuteron energy E_d (Z = 92).

only slightly on the energy of the level into which the neutron is captured, the energy of this level was taken arbitrarily to be 2.23 Mev. The angular distributions of the protons, normalized to unity at their maxima, are shown in Fig. 13 for various energies of the incident deuterons, for Z = 92. The parameter n_d varies from $n_d = 7.1$ at $E_d = 10$ Mev to $n_d = 1.3$ at $E_d = 300$ Mev. With increasing energy, there is a qualitative change in the shape of the angular distribution. Even though for energies E_d of the order of tens of Mev the angular distribution has a maximum in the forward direction, already for an energy of 200 Mev the distribution has a maximum in the forward direction.

The total cross section for the (d, p) reaction on heavy nuclei is found by integrating (6.7) over solid angle dO. Noting that $dO = (4\pi/\zeta_0)\,d\zeta$, we get

$$\int N(\zeta)\,dO=\frac{4\pi}{\zeta_0}\int_0^{\zeta_0}\frac{\exp\{-2n_p\psi_p+2n_d\psi_d\}}{\zeta(1+\zeta)\sqrt{(4\rho/\zeta)-(1-\rho)^2}}\,d\zeta.$$

Because of the rapid falloff of $N(\zeta)$ with increasing $\zeta_0-\zeta$, the important values of ζ in the integral are those near to ζ_0. Expanding the expression under the exponential in powers of $x=\zeta_0-\zeta$ and extending the integral to infinity, we find

$$\int N(\zeta)\,dO=\frac{\pi}{4n_d}\left[\frac{(k_d-k_p)^2+k_n^2}{\alpha k_d}\right]^2$$
$$\times\exp\{-2n_p\psi_p(0)+2n_d\psi_d(0)\},$$

where $\psi_p(0)$ and $\psi_d(0)$ are the values corresponding to $\zeta=\zeta_0$. (The factor multiplying the exponential has been taken out of the integral and evaluated at $\zeta=\zeta_0$.) We finally get the following formula for the total cross section for the (d, p) reaction:[25]

$$\sigma = \frac{2j+1}{2i+1}\frac{4\pi^4 M}{\hbar^2 k_{\mathrm{d}}^2 k_{\mathrm{n}}^2 R_\alpha}\sum_l \frac{\gamma_l}{|\mathfrak{f}_l(k_{\mathrm{n}}R)|^2}\exp\{-\Phi(E_{\mathrm{d}}, E_{\mathrm{n}})\},$$

$$\Phi(E_{\mathrm{d}}, E_{\mathrm{n}}) = -2n_{\mathrm{p}}(\varphi_{\mathrm{p}} - \psi_{\mathrm{p}}(0)) + 2n_{\mathrm{d}}(\varphi_{\mathrm{d}} - \psi_{\mathrm{d}}(0))$$
$$= 2\mathrm{Re}\left\{-n_{\mathrm{p}}\tan^{-1}\frac{\sqrt{-E_{\mathrm{n}}} - \sqrt{2\varepsilon}}{\sqrt{E_{\mathrm{p}}}} + n_{\mathrm{d}}\tan^{-1}\frac{\sqrt{-2E_{\mathrm{n}}} - \sqrt{\varepsilon}}{\sqrt{E_{\mathrm{d}}}}\right\}. \qquad (6.9)$$

(We note that the energies E_d, E_n, and E_p are related by the formula $E_d - \epsilon = E_p + E_n$.) The exponential factor in (6.9), which gives the dependence of the cross section on incident deuteron energy, can also be gotten from the quasi-classical approximation.[18,93]

Formula (6.9) gives the cross section for the (d,p) reaction with capture of the neutron into a definite energy level. The neutron energy E_n corresponding to maximum cross section for the (d,p) reaction with fixed energy E_d of the incident deuteron can be found by determining the minimum of the function $\Phi(E_d, E_n)$ appearing in the exponential. This most probable energy for the captured neutron E_n is a function of the energy E_d of the incident deuteron, and decreases with increasing E_d so long as $E_d < 1.7\,\epsilon$. (Over this region, E_n lies in the range $1.5 - 0.5\,\epsilon$.) If $E_d > 1.7\,\epsilon$, the most probable energy of the captured neutron is zero.

3. Breakup of the deuteron in the Coulomb field of the nucleus. A deuteron passing at some distance from a nucleus can, under the influence of the Coulomb field of the nucleus, dissociate into a neutron and a proton. If the energy of the incident deuteron is less than the height of the Coulomb barrier, the probability for such an electric breakup is much greater than the probability for the (d, np) process occurring via compound nucleus formation.

On the assumption of zero range of the nuclear forces between neutron and proton, the amplitude for breakup of the deuteron in the Coulomb field of the nucleus is equal, according to (3.13), to

$$f = 2\sqrt{\frac{\alpha}{2\pi}}\int e^{-i\mathbf{k}_{\mathrm{n}}\mathbf{r}}\psi^*_{\mathbf{k}_{\mathrm{p}}}(\mathbf{r})\,\psi_{\mathbf{k}_{\mathrm{d}}}(\mathbf{r})\,d\mathbf{r}. \qquad (6.10)$$

In this formula the neutron wave function is taken to be a plane wave, while the Coulomb functions (6.2) and (6.3) are used for the deuteron and proton. Using (6.10), we get the following expression for the cross section:

$$d\sigma = \frac{2M\alpha k_{\mathrm{n}}k_{\mathrm{p}}}{\pi^2\hbar^2 k_{\mathrm{d}}}\frac{n_{\mathrm{d}}n_{\mathrm{p}}}{(e^{2\pi n_{\mathrm{d}}}-1)(e^{2\pi n_{\mathrm{p}}}-1)}|I|^2 dE_{\mathrm{n}}\,dO_{\mathrm{n}}\,dO_{\mathrm{p}}, \qquad (6.11)$$

where I denotes the integral

$$I = \int e^{i\mathbf{q}\mathbf{r}}F(-in_{\mathrm{d}}, 1, i(k_{\mathrm{d}}r - \mathbf{k}_{\mathrm{d}}\mathbf{r}))\,F(-in_{\mathrm{p}}, 1, i(k_{\mathrm{p}}r - \mathbf{k}_{\mathrm{p}}\mathbf{r}))\,d\mathbf{r},$$
$$\mathbf{q} = \mathbf{k}_{\mathrm{d}} - \mathbf{k}_{\mathrm{p}} - \mathbf{k}_{\mathrm{n}}. \qquad (6.12)$$

This integral can be calculated exactly:

$$\left.\begin{aligned} I &= \frac{d}{d\lambda}(BF(-in_{\mathrm{d}}, -in_{\mathrm{p}}, 1, \zeta))|_{\lambda=0},\\ B &= -4\pi i(q^2 - 2\mathbf{q}\mathbf{k}_{\mathrm{d}} - 2\lambda k_{\mathrm{d}})^{in_{\mathrm{d}}}\\ &\quad\times(q^2 + 2\mathbf{q}\mathbf{k}_{\mathrm{p}} - 2\lambda k_{\mathrm{p}})^{in_{\mathrm{p}}}q^{-2(in_{\mathrm{d}}+in_{\mathrm{p}}+1)},\\ \zeta &= 2\frac{q^2(k_{\mathrm{d}}k_{\mathrm{p}} + \mathbf{k}_{\mathrm{d}}\mathbf{k}_{\mathrm{p}}) - 2(\mathbf{q}\mathbf{k}_{\mathrm{d}} + \lambda k_{\mathrm{d}})(\mathbf{q}\mathbf{k}_{\mathrm{p}} - \lambda k_{\mathrm{p}})}{(q^2 - 2\mathbf{q}\mathbf{k}_{\mathrm{d}} - 2\lambda k_{\mathrm{d}})(q^2 + 2\mathbf{q}\mathbf{k}_{\mathrm{p}} - 2\lambda k_{\mathrm{p}})}. \end{aligned}\right\} \qquad (6.13)$$

If the energies of the deuteron and proton are sufficiently low, the expression for the cross section can be simplified considerably. If $n_d \gg 1$ and $n_p \gg 1$, we can use the asymptotic expansion (6.6) for the hypergeometric function $F(-in_d, -in_p, 1, \zeta)$. Then the cross section as a function of the energy of the liberated neutron will have its maximum value at $E_n = 0$ and will decrease exponentially with increasing E_n. As a function of the direction of the outgoing proton, the cross section is a maximum for motion of the proton opposite to the direction of incidence of the deuteron, and falls off exponentially when we move away from this direction. Thus in calculating the integral I, we may set

$$k_{\mathrm{n}} = 0, \qquad \mathbf{q} = \mathbf{k}_{\mathrm{d}} - \mathbf{k}_{\mathrm{p}}, \qquad \mathbf{k}_{\mathrm{d}}\mathbf{k}_{\mathrm{p}} = -k_{\mathrm{d}}k_{\mathrm{p}}.$$

in all the non-exponential expressions [according to (6.6), the exponential factor is contained in the hypergeometric function]. Under these conditions the derivative $d\zeta/d\lambda|_{\lambda=0}$ is zero, so that the term in I which contains the derivative of the hypergeometric function drops out. Thus we find for the absolute square of the amplitude,

$$|I|^2 = \frac{\beta^2 M_\varepsilon}{\hbar^2}\frac{64\pi^2}{(k_{\mathrm{d}}+k_{\mathrm{p}})^6(k_{\mathrm{d}}-k_{\mathrm{p}})^2}|F(-in_{\mathrm{d}}, -in_{\mathrm{p}}, 1, \zeta)|^2,$$

where we have introduced the notation $\beta = \frac{Ze^2}{\hbar}\times\left(\frac{M}{\epsilon}\right)^{1/2}$, and k_p refers to the energy $E_p = E_d - \epsilon$. The argument ζ of the hypergeometric function, which appears in the exponential in the asymptotic formula, must be expanded in powers of the neutron energy E_n and the angle ϑ_p between the vectors $\mathbf{k}_p$ and $-\mathbf{k}_d$:

$$\zeta = -\frac{4k_{\mathrm{d}}k_{\mathrm{p}}}{(k_{\mathrm{d}}-k_{\mathrm{p}})^2}\left\{1 - \frac{\vartheta_{\mathrm{p}}^2}{4} + \frac{3k_{\mathrm{p}}^2 - k_{\mathrm{d}}^2}{2k_{\mathrm{p}}^2(k_{\mathrm{d}}-k_{\mathrm{p}})^2}k_{\mathrm{n}}^2 - \frac{k_{\mathrm{n}}^2\sin^2\theta_{\mathrm{n}}}{(k_{\mathrm{d}}-k_{\mathrm{p}})^2} + \frac{k_{\mathrm{n}}\vartheta_{\mathrm{p}}\sin\theta_{\mathrm{n}}\cos\varphi}{k_{\mathrm{d}}-k_{\mathrm{p}}}\right\}$$

(θ_n is the angle between the vectors $\mathbf{k}_n$ and $-\mathbf{k}_d$;

φ is the difference of the azimuths of $\mathbf{k}_n$ and $\mathbf{k}_p$ taken relative to the polar axis $-\mathbf{k}_d$).

Using the asymptotic formula (6.6) for the absolute square of the hypergeometric function, we get the following expression for the cross section:

$$d\sigma = \beta^3 \frac{\hbar^2}{M\varepsilon} \frac{\varepsilon^2 (\varepsilon E_n)^{\frac{1}{2}} \exp(-\beta\Phi)}{\pi \sqrt{2} E_d (E_d+\varepsilon)^2 [(2E_d)^{\frac{1}{2}} + (E_d-\varepsilon)^{\frac{1}{2}}]^2} \, dE_n \, d\theta_n \, d\theta_p, \tag{6.14}$$

where

$$\Phi = \Phi_0 + E_n\Phi_1 + E_n \sin^2\theta_n \Phi_2 + \theta_p^2 \Phi_3 + \theta_p E_n^{\frac{1}{2}} \sin\theta_n \cos\varphi \Phi_4,$$

and

$$\Phi_0 = \left(\frac{8\varepsilon}{E_d-\varepsilon}\right)^{\frac{1}{2}} \cos^{-1}\left(\frac{E_d-\varepsilon}{E_d+\varepsilon}\right)^{\frac{1}{2}} - 4\left(\frac{\varepsilon}{E_d}\right)^{\frac{1}{2}} \cos^{-1}\left(\frac{E_d}{E_d+\varepsilon}\right)^{\frac{1}{2}},$$

$$\Phi_1 = \frac{(2\varepsilon)^{\frac{1}{2}}}{(E_d-\varepsilon)^{\frac{3}{2}}} \cos^{-1}\left(\frac{E_d-\varepsilon}{E_d+\varepsilon}\right)^{\frac{1}{2}} - \frac{2\varepsilon(E_d-3\varepsilon)}{(E_d+\varepsilon)^2(E_d-\varepsilon)},$$

$$\Phi_2 = \frac{4\varepsilon}{(E_d+\varepsilon)^2},$$

$$\Phi_3 = \frac{\varepsilon}{[(2E_d)^{1/2}+(E_d-\varepsilon)^{1/2}]^2},$$

$$\Phi_4 = \frac{4\varepsilon}{(E_d+\varepsilon)[(2E_d)^{1/2}+(E_d-\varepsilon)^{1/2}]}.$$

The total cross section $\sigma(E_d)$, as a function of deuteron energy, is obtained by integrating (6.14) over the energy of the neutron (where, because of the rapid convergence of the integral, we can integrate from 0 to ∞), and over all directions of the neutron and proton (where again the integration over θ_p can be taken between the limits 0 and ∞). As a result of the computation, we get

$$\sigma = (4\pi)^{\frac{3}{2}} \beta^{\frac{1}{2}} \frac{\hbar^2}{M\varepsilon} \varepsilon^{\frac{3}{2}} E_d^{-1} (E_d+\varepsilon)^{-2} \Phi_1^{-\frac{3}{2}} e^{-\beta\Phi_0}. \tag{6.15}$$

For example, for Bi ($Z = 83$), the cross section for breakup is $\sigma = 4.5 \times 10^{-26}\ \mathrm{cm}^2$ at $E_d = 8.2$ Mev, and $\sigma = 0.3 \times 10^{-26}\ \mathrm{cm}^2$ at $E_d = 6.3$ Mev.

After integration of (6.14) over proton directions only (i.e., over dO_p), the angle θ_n drops out of the resultant expression, i.e., the distribution of the neutrons over direction (uncorrelated with the direction of the protons) is isotropic. We get the following expression for the energy distribution of the neutrons:

$$d\sigma(E_n) = 2\sqrt{\pi}\,\sigma(\beta\Phi_1)^{\frac{3}{2}} E_n^{\frac{1}{2}} e^{-\beta E_n \Phi_1} dE_n. \tag{6.16}$$

The angular distribution of the protons can be found by integrating (6.14) with respect to $dE_n dO_n$, giving

$$d\sigma(\theta_p) = \sigma \frac{\beta\Phi_1\Phi_3}{\pi(\Phi_1+\Phi_2)} \exp\left\{-\beta \frac{\Phi_1\Phi_3}{\Phi_1+\Phi_2} \theta_p^2\right\} dO_p. \tag{6.17}$$

Thus the distribution of the protons with respect to angle θ_p is a Gaussian with a maximum in the direction opposite to the direction of motion of the deuteron.

II. INTERACTIONS OF DEUTERONS WITH NUCLEI IN THE HIGH-ENERGY REGION

7. Diffractive Interaction of Deuterons with Nuclei

1. Nuclear diffraction. In treating the interaction of deuterons with nuclei for deuteron energies of the order of tens of Mev and above, we can use the optical model, according to which the nucleus is treated phenomenologically as a medium characterized by definite optical properties (refractive index and absorption coefficient). If the mean free path of nucleons in nuclear matter is small compared to nuclear dimensions, the nucleus can be treated as a black absorbing body. The treatment becomes especially simple for the case of an absolutely black nucleus.

We know that the absorption of particles scattered by the nucleus leads to an additional perturbation of the incident wave and thus to additional elastic scattering which is not associated with compound nucleus formation. For point particles (such as neutrons) whose wave length is small compared to nuclear dimensions, this scattering is analogous to the diffraction of light by an absolutely black sphere.

The diffraction scattering of complex particles like deuterons must show some special features. In addition to absorption and diffraction elastic scattering, which occur for point particles, the following processes can occur with deuterons: stripping of a neutron or a proton, and diffraction breakup of the deuteron.

In the case of the stripping reaction, when a fast deuteron passes the nucleus a proton or neutron may bump into the nucleus while the other particle goes by outside the nucleus. The result is that a proton or neutron is instantaneously captured by the nucleus while the second particle constituting the deuteron is liberated and continues on its path outside the nucleus. The theory of the stripping reaction at high energy was given by Serber.[111]

Because of the low binding energy of the deuteron, the diffractive interaction of deuterons with nuclei can result in diffraction breakup of the deuteron occurring far from the nucleus. This dissociation which results in liberation of a neutron and proton occurs when the change in deuteron momentum resulting from the diffraction scattering is sufficiently large. The possibility of diffraction breakup of the deuteron was established independently by Akhiezer and Sitenko,[9,10,28] Feinberg,[26,7] and Glauber.[71]

The diffraction scattering of particles by absorbing nuclei can be investigated by an optical method using Huygens' principle, which can be generalized to take account of the Coulomb interaction and the complex structure of the particles undergoing scattering.

First let us consider the simplest problem of diffraction scattering of point particles (for example neutrons) by absorbing nuclei. For simplicity we shall limit ourselves to the case of an absolutely black spherical nucleus whose radius we denote by R. We shall assume that the wavelength λ of the incident particle is small compared to nuclear dimensions, $\lambda \ll R$. This condition will be satisfied for deuterons with energy greater than 10 Mev.

The free motion of the particles in the plane perpendicular to the wave vector of the incident particle (the z axis) is described by the wave function $\psi_\kappa = e^{i\kappa\rho}$, where κ and ρ are the projections of the wave vector and radius vector of the particle on a plane perpendicular to the z axis. The functions ψ_κ are normalized by the condition $\int \psi_\kappa^* \psi_{\kappa'} d\rho = \delta_{\kappa\kappa'}$.

The wave function for the incident particle is $\psi_0 = 1$. The presence of the absorbing nucleus results in the absorption of the part of this function at $\rho < R$. The diffraction pattern can be obtained by expanding the part of the wave function which corresponds to scattered particles, $\Psi = \{\Omega(\rho) - 1\}\psi_0$, where

$$\Omega(\rho) = \begin{cases} 0, & \rho \leqslant R, \\ 1, & \rho > R, \end{cases}$$

in terms of the functions ψ_κ:

$$\Psi = \sum_\kappa a_\kappa \psi_\kappa. \qquad (7.1)$$

The differential cross section for diffraction scattering in which the wave vector κ of the scattered particle lies in the range $d\kappa$ is related to a_κ by the formula

$$d\sigma = |a_\kappa|^2 \frac{d\kappa}{(2\pi)^2}.$$

If k is the magnitude of the wave vector of the particle, $\kappa = k \sin \vartheta$ and $d\kappa = k^2 dO$, where dO is the element of solid angle. The scattering amplitude $f(\vartheta)$ is related to the expansion coefficient a_κ by

$$f(\vartheta) = -i\frac{k}{2\pi} a_\kappa. \qquad (7.2)$$

From (7.1) it follows that

$$a_\kappa = \int \psi_\kappa^* \{\Omega(\rho) - 1\} \psi_0 \, d\rho.$$

Carrying out the integration and using (7.2), we get the well known formulas:

$$f(\vartheta) = iR\frac{J_1(kR\vartheta)}{\vartheta},$$

$$d\sigma_e = \frac{R^2 J_1^2(kR\vartheta)}{\vartheta^2} dO, \qquad \sigma_e = \pi R^2 \qquad (7.3)$$

(Since the diffraction treatment is valid for small angles, $\sin \vartheta$ can be replaced by ϑ.) The cross section for absorption of the neutrons is also

$$\sigma_a = \pi R^2.$$

The total cross section for interaction of fast neutrons with nuclei can be found from the forward elastic scattering amplitude,

$$\sigma_t = 4\pi\lambda\!\!\!^{-} \operatorname{Im} f(0); \qquad (7.4)$$

For neutrons, $f(0) = ikR^2/2$, and $\sigma_t = 2\pi R^2$, as it should be.

In the case of scattering of fast neutrons by nonspherical nuclei, in addition to elastic scattering we may also have scattering of the neutrons accompanied by excitation of the nucleus. In this case the function Ω should be taken equal to zero in the region of the shadow cast by the nucleus on the plane perpendicular to the wave vector of the incident neutron, and equal to unity outside this region. Obviously the area of shadow will depend on the relative orientation of the symmetry axis of the nucleus and the wave vector of the incident neutron. Then the diffraction wave function must be built up from products of the functions ψ_κ with eigenfunctions of rotational states of the nonspherical nucleus. The excitation of rotational states of nonspherical nuclei in diffraction scattering was treated by Drozdov.[15]

At high energies, when the mean free path of the particles in nuclear matter becomes comparable with the dimensions of the nucleus, the nucleus should be treated as a semitransparent body which is characterized by a complex absorption coefficient

$$b = b_1 - i2(\nu - 1)k,$$

where b_1 is the absorption coefficient and ν the refractive index of nuclear matter. Then the factor Ω should be assumed to be

$$\Omega(\rho) = \begin{cases} e^{-b\sqrt{R^2-\rho^2}}, & \rho \leqslant R, \\ 1 \quad , & \rho > R. \end{cases}$$

For neutrons the nucleus begins to be semitransparent at energies above 100 Mev.

2. Diffraction scattering and diffraction breakup of deuterons. Our treatment of the diffraction of point particles can be generalized to the case of diffraction scattering of weakly bound composite particles like the deuteron by absolutely black nuclei. To do this we again use an expansion of the diffracted wave function, but now we have two factors Ω_n and Ω_p for the neutron and the proton. (In this treatment we are of course neglecting the Coulomb interaction of the deuteron and the nucleus.)

In studying the diffraction of deuterons we must consider both the motion of the center of mass and the relative motion of the neutron and proton in the deuteron. The motion of the center of mass of the deuteron in the plane perpendicular to the wave vector of the incident deuteron (the z axis) is described by the wave function $\psi_\kappa = e^{i\kappa\rho_d}$, where κ and ρ_d are the projections of the wave vector and the radius vector to the center of mass of the deuteron on the plane perpendicular to the z axis. The relative motion of the particles in the deuteron $\varphi_0(\mathbf{r})$, and the relative motion of the neutron and proton emitted as a result of the breakup is described by $\varphi_f(\mathbf{r})$. The functions φ_f together with the function φ_0 form a complete system of orthonormal functions.

Since the deuteron is a weakly bound system, in which the neutron and proton spend a large fraction of the time outside the range of the nuclear forces, the diffraction of deuterons by an absolutely black nucleus is given by expanding the function $\Psi = (\Omega_n\Omega_p - 1)\psi_0\varphi_0$ in the complete set of functions $\psi_\kappa\varphi_0$ and $\psi_\kappa\varphi_f$:

$$\Psi = \sum_{\kappa} a_\kappa \psi_\kappa \varphi_0 + \sum_{\kappa, f} a_{\kappa f} \psi_\kappa \varphi_f. \tag{7.5}$$

The expansion coefficients a_κ and $a_{\kappa f}$ can be regarded as probability amplitudes for diffraction scattering and diffraction breakup, respectively. From (7.5) it follows that

$$a_\kappa = -\int\int \varphi_0(r)\psi_\kappa^*(\rho_d)\{\omega_n + \omega_p - \omega_n\omega_p\} \times \psi_0(\rho_d)\varphi_0(r)\,d\rho_d\,d\mathbf{r}, \tag{7.6}$$

$$a_{\kappa f} = -\int\int \varphi_f^*(\mathbf{r})\varphi_\kappa^*(\rho_d)\{\omega_n + \omega_p - \omega_n\omega_p\} \times \psi_0(\rho_d)\varphi_0(r)\,d\rho_d\,d\mathbf{r}, \tag{7.7}$$

where, for convenience, we have introduced the notation $\omega(\rho) = 1 - \Omega(\rho)$.

Using the expansion

$$\omega(\rho) = \frac{1}{2\pi}\int \frac{RJ_1(gR)}{g} e^{ig\rho}\,dg, \tag{7.8}$$

we can write the elastic scattering amplitude $f(\vartheta)$, which is related to a_κ by (7.2), as

$$f(\vartheta) = ik\Big\{2\frac{4\alpha}{\kappa}\tan^{-1}\frac{\kappa}{4\alpha}\frac{RJ_1(\kappa R)}{\kappa} - \frac{1}{2\pi}\int\frac{4\alpha}{|2\mathbf{g}-\boldsymbol{\kappa}|} \times \tan^{-1}\frac{|2\mathbf{g}-\boldsymbol{\kappa}|}{4\alpha}\frac{RJ_1(gR)}{g}\frac{RJ_1(|\boldsymbol{\kappa}-\mathbf{g}|R)}{|\boldsymbol{\kappa}-\mathbf{g}|}dg\Big\}. \tag{7.9}$$

The differential cross section for elastic scattering of the deuterons is then:

$$d\sigma_e = R^2\Big|2\frac{2p}{\kappa'}\tan^{-1}\frac{\kappa'}{2p}\frac{J_1(\kappa')}{\kappa'} - \frac{1}{2\pi}\int\frac{2p}{|2\mathbf{g}'-\boldsymbol{\kappa}'|} \times \tan^{-1}\frac{|2\mathbf{g}'-\boldsymbol{\kappa}'|}{2p}\frac{J_1(g')}{g'}\frac{J_1(\boldsymbol{\kappa}'-\mathbf{g}')}{|\boldsymbol{\kappa}'-\mathbf{g}'|}dg'\Big|^2 d\boldsymbol{\kappa}',$$

where we have introduced the dimensionless quantities $\boldsymbol{\kappa}' = \boldsymbol{\kappa}R$, $\mathbf{g}' = \mathbf{g}R$ and $p = R/R_d$. The formula becomes much simpler in the limiting case of large p:

$$d\sigma_e = 2\pi R^2\Big\{\Big(\frac{2p}{\kappa'}\tan^{-1}\frac{\kappa'}{2p}\Big)^2\frac{J_1^2(\kappa')}{\kappa'} + \frac{1}{2p}J_1(\kappa')J_0(\kappa')\Big\}d\kappa', \quad \kappa' \ll p,\ p \gg 1.$$

To get the total elastic scattering cross section, we use the condition of completeness of the functions ψ_κ. From formula (7.6), we get

$$\sigma_e = \int I^2(\rho_d)\,d\rho_d, \quad I(\rho_d) = \int\{\omega_n + \omega_p - \omega_n\omega_p\}\varphi_0^2(r)\,d\mathbf{r}.$$

If $p \gg 1$, the contribution of the region $\rho_d < R$ to the cross section is equal to πR^2 to terms of order $1/p^2$. The product $\omega_n\omega_p$ is equal to zero when $\rho_d > R$, so

$$I(\rho_d) = 2\int_0^\infty \frac{2p}{g}\tan^{-1}\frac{g}{2p}J_1(g)J_0\Big(\frac{g\rho_d}{R}\Big)dg$$

$$= 4p\int_0^1\frac{dy}{y}I_1\Big(\frac{2p}{y}\Big)K_0\Big(\frac{\rho_d}{R}\frac{2p}{y}\Big), \rho_d > R.$$

Using the asymptotic expressions for $I_1(x)$ and $K_0(x)$ when $x \gg 1$, we get

$$I(\rho_d) = \sqrt{\frac{R}{\rho_d}}\int_1^\infty\frac{d\zeta}{\zeta^2}e^{-4ab\zeta}, \quad b = \rho_d - R > 0, \quad p \gg 1,$$

and consequently the contribution of the region $\rho_d > R$ to σ_e is

$$2\pi R\int_0^\infty db\Big|\int_1^\infty e^{-4ab\zeta}\frac{d\zeta}{\zeta^2}\Big|^2 = \frac{2\pi}{3}(1 - \ln 2)RR_d.$$

Thus the total cross section for diffraction elastic scattering of deuterons is

$$\sigma_e = \pi R^2 + \frac{2\pi}{3}(1-\ln 2) R R_d, \quad R_d \ll R. \quad (7.10)$$

In addition to purely elastic scattering which is analogous to the diffraction scattering of point particles, for composite particles like the deuteron we can also have diffraction breakup. Using (7.7), we write the amplitude $a_{\kappa f}$ for diffraction breakup in the form

$$a_{\kappa f} = -\frac{(2\pi)^{\frac{3}{2}} R}{\alpha^{\frac{5}{2}}} \left\{ \frac{J_1(pz)}{z} [\Phi(\mathbf{u}, \mathbf{z}) + \Phi(\mathbf{u}, -\mathbf{z})] - \frac{1}{2\pi} \int d\mathbf{g} \frac{J_1(g)}{g} \frac{J_1\left(p\left|\mathbf{z}-\frac{\mathbf{g}}{p}\right|\right)}{\left|\mathbf{z}-\frac{\mathbf{g}}{p}\right|} \Phi\left(\mathbf{u}, \frac{2\mathbf{g}}{p} - \mathbf{z}\right) \right\},$$

$$\Phi(\mathbf{u}, \mathbf{z}) = \frac{1}{1+(\mathbf{z}-\mathbf{u})^2} + \frac{1}{2z(1-u)} \ln \frac{u+z+i}{u-z+i},$$

where $\mathbf{z} = \boldsymbol{\kappa}/2\alpha$ and $\mathbf{u} = \mathbf{f}/\alpha$. The cross section for diffraction breakup is related to the amplitude $a_{\kappa f}$ by

$$d\sigma_d = |a_{\kappa f}|^2 \frac{d\boldsymbol{\kappa}\, d\mathbf{f}}{(2\pi)^5}. \quad (7.11)$$

The total cross section for diffraction breakup is given by the formula:

$$\sigma_d = \frac{R^2}{\pi^2} \int\int d\mathbf{z}\, d\mathbf{u} \left| \frac{J_1(pz)}{z} [\Phi(\mathbf{u}, \mathbf{z}) + \Phi(\mathbf{u}, -\mathbf{z})] - \frac{1}{2\pi} \int d\mathbf{g} \frac{J_1(g)}{g} \frac{J_1\left(p\left|\mathbf{z}-\frac{\mathbf{g}}{p}\right|\right)}{\left|\mathbf{z}-\frac{\mathbf{g}}{p}\right|} \Phi\left(\mathbf{u}, \frac{2\mathbf{g}}{p} + \mathbf{z}\right) \right|^2. \quad (7.12)$$

If $p \gg 1$,

$$\sigma_d = \frac{2}{3} R R_d \int_0^\infty u I(u)\, du,$$

where $I(u)$ gives the distribution of the relative energy of the products of the breakup, and has the form

$$I(u) = \frac{3}{(1+u^2)^2} \left[u + \frac{2u}{1+u^2} - \sin^{-1} \frac{u}{\sqrt{1+u^2}} \right] - 16(1-\ln 2) \frac{u}{(1+u^2)^3} \quad (7.13)$$

The total cross section for the diffraction breakup of deuterons is

$$\sigma_d = \frac{\pi}{3}\left(2\ln 2 - \frac{1}{2}\right) R R_d. \quad R_d \ll R. \quad (7.14)$$

Like the stripping reactions, the diffraction breakup of the deuteron results in emission of a neutron and a proton, i.e., it increases the yield of neutrons from the interaction of fast deuterons with nuclei.

3. The stripping reaction at high energies. We can also use the diffraction method in treating the stripping reaction at high energies of the incident deuterons (energies exceeding several tens of Mev).

For the case of an absolutely black nucleus one can then develop a theory of the stripping reaction which takes into account the finite radius of the nucleus.[12]

Let us determine the cross section for a process in which one of the particles originally constituting the deuteron is liberated, while the other particle is captured by the nucleus. To be specific, let us consider the reaction in which the neutron is liberated while the proton is absorbed by the nucleus. This process can be described by the wave function

$$\Psi = \Omega_n \psi_0(\rho_d) \varphi_0(r).$$

Expanding Ψ in an integral over the functions $e^{-i\mathbf{k}\mathbf{r}_n}$ (where $\mathbf{r}_n$ is the radius vector of the neutron), we get the probability amplitude for the neutron to have wave vector $\mathbf{k}$ and the proton to be at the point $\mathbf{r}_p$. This probability amplitude is obviously equal to

$$a_k(\mathbf{r}_p) = \int e^{-i\mathbf{k}\mathbf{r}_n} \Omega_n \varphi_0(r)\, d\mathbf{r}_n. \quad (7.15)$$

Integrating $|a_k(\mathbf{r}_p)|^2$ over $d\boldsymbol{\rho}_p$ between the limits $\rho_p = 0$ and $\rho_p = R$, we get the differential cross section $d\sigma_n$ for stripping in which the wave vector of the emitted neutron is in the interval $d\mathbf{k}$:

$$d\sigma_n = \frac{d\mathbf{k}}{(2\pi)^3} \int_{\rho_p < R} d\boldsymbol{\rho}_p |a_k(\boldsymbol{\rho}_p)|^2 = \frac{d\mathbf{k}}{(2\pi)^3} \int d\boldsymbol{\rho}_p \{1-\Omega_p\} |a_k(\boldsymbol{\rho}_p)|^2. \quad (7.16)$$

In the limiting case of $p \gg 1$ (when the nuclear boundary is a plane), the amplitude $a_k(\mathbf{r}_p)$ can be found in explicit form. Except for an irrelevant phase factor, this amplitude is

$$a_k(\mathbf{r}_p) = \frac{\sqrt{2\pi\alpha}}{P(k_x - iP)} e^{P x_p},$$

where $P = (\alpha^2 + k_y^2 + k_z^2)^{1/2}$ and the x axis is along the normal to the nuclear boundary. In this limiting case the differential cross section for the stripping reaction is given by the expression:

$$d\sigma_n = \frac{d\mathbf{k}}{(2\pi)^3} \frac{\pi R\alpha}{(\alpha^2+k^2)} \int_0^{2\pi} \frac{d\varphi}{(\alpha^2 + k_z^2 + \varkappa^2 \sin^2\varphi)^{\frac{3}{2}}}, \quad R_d \ll R. \quad (7.16')$$

Using the completeness of the system of functions $e^{i\mathbf{k}\mathbf{r}_n}$, we can write the total cross section for stripping as

$$\sigma_n = \int\int d\boldsymbol{\rho}_p\, d\mathbf{r}_n \{1-\Omega_p\} \Omega_n \varphi_0^2(r).$$

Substituting the expansions (7.8) for Ω_n and Ω_p, we get finally

$$\sigma_n = \pi R^2 \left\{ 1 - 2 \int_0^\infty \frac{p}{\zeta} \tan^{-1} \frac{\zeta}{p} \frac{J_1^2(\zeta)}{\zeta} d\zeta \right\}, \quad p = \frac{R}{R_d}. \quad (7.17)$$

This expression simplifies in the limiting case of large p. Using the fact that for $p \gg 1$,

$$\int_0^\infty \frac{p}{\zeta} \tan^{-1} \frac{\zeta}{p} \frac{J_1^2(\zeta)}{\zeta} d\zeta$$

$$= \frac{1}{2} - \int_0^1 d\zeta K_1\left(\frac{p}{\zeta}\right) I_1\left(\frac{p}{\zeta}\right) \approx \frac{1}{2} - \frac{1}{4p},$$

we get Serber's approximate formula for the total stripping cross section:

$$\sigma_n^s = \frac{\pi}{2} R R_d, \quad R_d \ll R. \quad (7.18)$$

The dependence of σ_n on p is shown in Fig. 14. For lead, $p = 4.2$ and formula (7.17) gives $\sigma_n = 3.2 \times 10^{-25}$ cm^2, while Serber's formula gives $\sigma_n^S = 2.7 \times 10^{-25}$ cm^2. For $p = 1$, $\sigma_n = 5.8 \times 10^{-26}$ cm^2 and $\sigma_n^S = 6.9 \times 10^{-26}$ cm^2.

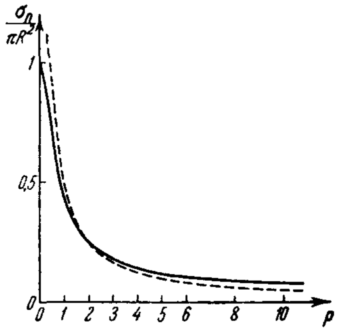


FIG. 14. Dependence of σ_n on $p = R/R_d$. (The dashed curve is a plot of $\sigma_n^s = \frac{\pi}{2} RR_d$.)

The cross section for stripping off of a proton will also be given by (7.17) and (7.18).

To find the energy distribution of the emitted neutrons, we must integrate (7.16) over the perpendicular components of the vector **k**:

$$d\sigma_n(k_z) = \frac{dk_z}{2\pi} \int \frac{d\boldsymbol{\varkappa}}{(2\pi)^2} \int d\boldsymbol{\rho}_p \{1 - \Omega_p\}$$

$$\times \left| \int d\boldsymbol{\rho}_n e^{-i\boldsymbol{\varkappa}\boldsymbol{\rho}_n} \Omega_n \int dz e^{ik_z z} \varphi_0(r) \right|^2.$$

Using the completeness of the functions $e^{i\kappa\rho_n}$, and the expansions (7.8) for Ω_n and Ω_p, we get finally

$$d\sigma_n = \sigma_n(\boldsymbol{k})\, d\boldsymbol{k}, \qquad \sigma_n(k) = \frac{4p^2R^2}{\pi}$$

$$\times \int_0^1 K_0^2(p\zeta\sqrt{1+k^2}) (\sin^{-1}\zeta + \zeta\sqrt{1-\zeta^2})\, \zeta d\zeta, \quad (7.19)$$

where the dimensionless quantity k is related to the energy E_n of the liberated neutron by the formula

$$K = \frac{E_n - \frac{1}{2}E_d}{\sqrt{\epsilon E_d}}$$

(E_d is the energy of the incident deuteron.) Formula (7.19) determines the energy distribution of the emerging neutrons for arbitrary values of the parameter $p = R/R_d$.

In the limiting case of $p \gg 1$, formula (7.19) goes over into Serber's formula

$$\sigma_n(k) = \frac{\pi}{4} R R_d \frac{1}{(1+K^2)^{3/2}}, \quad R_d \ll R. \quad (7.20)$$

We see that the center of the distribution is at a neutron energy equal to half the energy of the incident deuteron. The width of the distribution is $\Delta = \sqrt{2^{2/3}-1}\sqrt{\epsilon E_d}$, which amounts to 31 Mev for a deuteron energy of 190 Mev.

To get the angular distribution of the neutrons, we must integrate (7.16) over dk_z. Restricting ourselves to the limiting case of $p \gg 1$, we get:

$$d\sigma = \frac{RR_d}{\pi(1+\zeta^2)^{\frac{3}{2}}} \left\{ 1 - \frac{1}{2\zeta^3}[(1+\zeta^2)\tan^{-1}\zeta - \zeta] \right\} dO_\zeta, \quad (7.21)$$

where $\zeta = \vartheta/\vartheta_0$, $\vartheta_0 = (\epsilon/E_d)^{1/2}$ and $dO_\zeta = 2\pi\zeta\, d\zeta$. We see that most of the neutrons move inside a cone whose axis coincides with the direction of the initial deuteron beam and whose opening angle is equal in order of magnitude to $\vartheta_0 = (\epsilon/E_d)^{1/2}$, which amounts to about 6° for 190 Mev deuterons.

The experimentally observed angle and energy distributions of neutrons liberated from stripping reactions at high energies are in agreement with the theory. The neutron yield for incident deuteron energies ~ 200 Mev is $1\frac{1}{2}$ to 2 times the value given by (7.18).[110] Formula (7.17) only partially explains this discrepancy. The remaining difference may be due to Coulomb or diffraction breakup of the deuterons, which have not been studied experimentally.

4. Total cross section for diffractive interaction of deuterons with nuclei. The total cross section σ_t for interaction of fast neutrons with nuclei can be determined by using (7.4) if we know the elastic scattering amplitude at zero angle. The amplitude for scattering of deuterons at zero angle from an absolutely black nucleus is

$$f(0) = i\frac{k}{2\pi} \int\int \varphi_0^2(r) \{\omega_n + \omega_p - \omega_n\omega_p\}\, d\rho_d d\mathbf{r}.$$

Consequently the total cross section is

$$\sigma_t = 2 \int\int \varphi_0^2(r) \{\omega_n + \omega_p - \omega_n\omega_p\}\, d\rho_d\, d\mathbf{r}. \quad (7.22)$$

Using formula (7.8), we get

$$\sigma_t = 4\pi R^2 \left\{ 1 - \int_0^\infty \frac{p}{\zeta} \tan^{-1} \frac{\zeta}{p} \frac{J_1^2(\zeta)}{\zeta} d\zeta \right\}, \quad p = \frac{R}{R_d}. \quad (7.23)$$

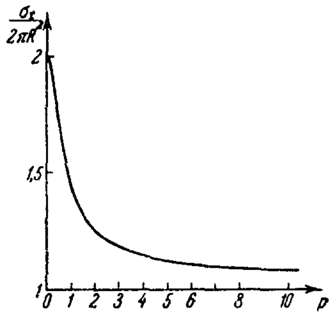


FIG. 15. Dependence of total cross section σ_t on the parameter $p = R/R_d$.

(The cross section σ_t naturally does not include the Coulomb scattering.) The dependence of σ_t on p is shown in Fig. 15.

In the limiting case of $p \gg 1$ we have

$$\sigma_e = 2\pi R^2 + \pi R R_d, \quad R_d \ll R. \tag{7.24}$$

It can be shown that the following relations hold for arbitrary values of $p = R/R_d$:

$$\sigma_e + \sigma_d = \frac{1}{2}\sigma_t, \quad \sigma_n + \sigma_p + \sigma_a = \frac{1}{2}\sigma_t, \tag{7.25}$$

where σ_a is the cross section for absorption of deuterons by the nucleus. In fact substituting (7.7) in (7.11) and integrating over $\boldsymbol{\kappa}$ and $\mathbf{f}$, we get

$$\sigma_e + \sigma_d = \iint \varphi_0^2(r)\{\omega_n + \omega_p - \omega_n\omega_p\}\, d\rho_d d\mathbf{r}.$$

Comparing this expression with (7.22) we arrive at (7.25).

Using (7.17) for the stripping cross sections σ_n and σ_p, and the expression (7.23) for the total cross section, we easily find for the cross section for absorption of the deuteron by the nucleus the formula

$$\sigma_a = 2\pi R^2 \int_0^\infty \frac{p}{\zeta}\tan^{-1}\frac{\zeta}{p}\,\frac{J_1^2(\zeta)}{\zeta}\,d\zeta, \quad p = \frac{R}{R_d}. \tag{7.26}$$

In the limiting case of $p \gg 1$, this expression gives

$$\sigma_a = \pi R^2 - \frac{\pi}{2} R R_d, \quad R_d \ll R. \tag{7.27}$$

The cross section for absorption of one particle by the nucleus is πR^2, but since the cross section for the process in which one of the particles of the deuteron enters the nucleus while the other passes by outside the nucleus is $(\pi/2) R R_d$, the cross section for absorption of both particles is $\pi R^2 - (\pi/2) R R_d$.

5. Interaction of fast nucleons with deuterons. The characteristic feature of nucleon-nucleon scattering at high energies (greater than 400 Mev) is inelastic scattering, i.e., scattering accompanied by the production of π mesons.

In the energy range 800 — 1400 Mev, it was found[52,48] that the elastic and inelastic cross sections are practically equal to one another and constant. Therefore the interaction between nucleons in this energy range can be described using the diffraction model, according to which the total interaction cross section will be $2\pi R^2$, where R is the radius of the region of interaction. Taking for the cross section the value $\sigma_0 \sim 45$ millibarns, we get for this radius the value $R \simeq 0.85 \times 10^{-13}$ cm.

In this same energy range (800 — 1400 Mev), the total cross section for interaction of a nucleon with the deuteron is noticeably less than the sum of the interaction cross sections with a free neutron and proton.[52] This effect was explained using a diffraction mechanism for the interaction of nucleons at very high energies.[72] Obviously the scattering or absorption of the incident particle by one of the nucleons in the deuteron will be reduced if this nucleon enters the shadow cast by the other nucleon in the deuteron (eclipsing effect).

Let us consider the scattering of a fast nucleon by a bound system of nucleons (a deuteron). If the velocities of the nucleons in the deuteron are small compared to the velocity of the incident nucleon, their motion can be neglected during the time of passage of the nucleon through the deuteron. The scattering of the nucleon by the fixed neutron and proton with coordinates $\mathbf{r}_n$ and $\mathbf{r}_p$ can be characterized by functions Ω_n and Ω_p with centers at the positions of the neutron and proton:

$$\Psi = \Omega_n\Omega_p\psi_0.$$

Expanding Ψ in terms of the functions $\psi_{\boldsymbol{\kappa}} = e^{i\boldsymbol{\kappa}\boldsymbol{\rho}}$ and averaging the amplitude over all possible relative separations of the neutron and proton in the deuteron, we get an expression for the elastic scattering amplitude which coincides with (7.9), but now R represents the radius of the region of interaction of two nucleons. Then using (7.4) we can get the following expression for the total cross section for interaction of a nucleon with the deuteron:

$$\sigma_t = 2\sigma_0\left\{1 - \int_0^\infty \frac{p}{\zeta}\tan^{-1}\frac{\zeta}{p}\,\frac{J_1^2(\zeta)}{\zeta}\,d\zeta\right\},$$

$$\sigma_0 = 2\pi R^2, \quad p = \frac{R}{R_d}. \tag{7.28}$$

If $p \ll 1$, it is easy to show that

$$\sigma_t = 2\sigma_0\left\{1 - \frac{2}{3}p\right\}, \quad R \ll R_d. \tag{7.29}$$

The main contribution to the total cross section comes from processes in which the deuteron is dissociated. One can show that for $p \ll 1$, the cross section for elastic scattering of nucleons by deuterons is

$$\sigma_l = \frac{\pi^3}{2} R^2 p^2 \ln \frac{1}{p}, \quad p \ll 1. \tag{7.30}$$

Actually, p is only a little less than unity ($p \cong 0.85/2.1 = 0.4$). Using the graph in Fig. 15, we find for the total cross section the value $\sigma_{t0} \simeq 1.8\,\sigma_0 \simeq 81$ millibarns, which in satisfactory agreement with the experimental data.[52]

8. Breakup of Fast Deuterons in the Coulomb Field of the Nucleus.

1. Electric and magnetic breakup of the deuteron. The interaction of a fast deuteron with the Coulomb field of a nucleus can also result in the breakup of the deuteron into a neutron and a proton. Even though the Coulomb breakup of the deuteron at high energies is less important for most nuclei than the breakup resulting from direct nuclear collisions and the diffraction breakup, for the case of very heavy nuclei the cross section for Coulomb breakup is of the same order of magnitude as the cross section for nuclear breakup.

The Coulomb breakup of high energy deuterons was treated by Dancoff,[58] who found the cross section for the process and also obtained the angle and energy distribution of the products of the disintegration (cf. also reference 97). Relativistic corrections to the Coulomb breakup, and the magnetic breakup of the deuteron, which is accompanied by transition of the n-p system from the triplet to the singlet state, were treated in reference 20.

Let us consider the interaction of a fast deuteron with the Coulomb field of the nucleus. If the condition $n = Ze^2/\hbar v \ll 1$ is satisfied, we can use perturbation theory, treating the interaction of the deuteron with the nuclear Coulomb field as a small perturbation.

In finding the cross section for disintegration of the deuteron, it is convenient to use a coordinate system in which the deuteron is at rest before the collision while the nucleus moves with velocity $\mathbf{v}$. The electromagnetic potentials of the moving nucleus are then given by

$$\varphi = \frac{Ze}{r(t)}, \quad \mathbf{A} = \frac{\mathbf{v}}{c}\varphi, \quad r(t) = \{(1-\beta^2)\rho^2 + (z-vt)^2\}^{\frac{1}{2}}.$$

The time dependent perturbation is

$$V(t) = e\varphi(\mathbf{r}_p) - \frac{e}{2Mc}\{\mathbf{p}_p\mathbf{A}(\mathbf{r}_p) + \mathbf{A}(\mathbf{r}_p)\mathbf{p}_p\}$$
$$- \mathbf{m}_p\mathbf{H}(\mathbf{r}_p) - \mathbf{m}_n\mathbf{H}(\mathbf{r}_n),$$

$$\mathbf{p}_p = \frac{\hbar}{i}\frac{\partial}{\partial \mathbf{r}_p}, \quad \mathbf{H} = -\frac{1}{c}(1-\beta^2)[\mathbf{v}\times\nabla\varphi], \quad \mathbf{m}_p = \frac{e\hbar}{2Mc}\mu_p\boldsymbol{\sigma}_p,$$
$$\mathbf{m}_n = \frac{e\hbar}{2Mc}\mu_n\boldsymbol{\sigma}_n \tag{8.1}$$

(μ_p and μ_n are the magnetic moments of the proton and neutron, expressed in nuclear magnetons).

The initial and final wave functions of the system are

$$\Psi_i = \psi_i e^{-\frac{i}{\hbar}E_i t}, \quad \psi_i = \varphi_0(r)\chi_{1\mu_d}, \qquad E_i = -\varepsilon,$$

$$\Psi_f = \psi_f e^{-\frac{i}{\hbar}E_f t}, \quad \psi_f = e^{i\mathbf{k}\mathbf{r}_d}\varphi_{\mathbf{f}}^{(s)}(\mathbf{r})\chi_{s\mu_s}, \quad E_f = \frac{\hbar^2k^2}{4M} + \varepsilon_f, \tag{8.2}$$

where $\mathbf{k}$ is the wave vector of the motion of the center of mass and $\mathbf{f}$ is the wave vector of the relative motion of the n-p system after the breakup.

With the normalization we have used for the wave functions, the differential cross section for a breakup in which $\mathbf{k}$ and $\mathbf{f}$ are in the intervals $d\mathbf{k}$ and $d\mathbf{f}$ is given by

$$d\sigma = |a|^2\frac{d\mathbf{k}\,d\mathbf{f}}{(2\pi)^6}, \tag{8.3}$$

where a is the probability amplitude for the transition,

$$a = -\frac{i}{\hbar}(\psi_f, V\psi_i), \quad V = \int_{-\infty}^{\infty} V(t)e^{i\omega t}dt, \quad \omega = \frac{E_f - E_i}{\hbar}.$$

Noting that $\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{r(t)}dt = \frac{2}{V}K_0\left(\frac{\omega}{V}\sqrt{1-\beta^2}\rho\right)e^{i\frac{\omega}{V}z}$, we get

$$V = \frac{2Ze^2}{v}\left\{e^{i\frac{\omega}{v}z_p}\left(1 + i\frac{\hbar v}{Mc^2}\frac{\partial}{\partial z_p}\right)K_0\left(\frac{\omega}{v}\sqrt{1-\beta^2}\rho_p\right)\right.$$
$$-\frac{\hbar\omega}{2Mc^2}\sqrt{1-\beta^2}\left(\mu_p[\mathbf{n}_p\times\boldsymbol{\sigma}_p]_z K_1\left(\frac{\omega}{v}\sqrt{1-\beta^2}\rho_p\right)e^{i\frac{\omega}{v}z_p}\right.$$
$$\left.\left.+\mu_n[\mathbf{n}_n\times\boldsymbol{\sigma}_n]_z K_1\left(\frac{\omega}{v}\sqrt{1-\beta^2}\rho_n\right)e^{i\frac{\omega}{v}z_n}\right)\right\}.$$

If we also carry out the integration in a over the coordinates of the center of mass of the deuteron, we find

$$a = -i\frac{2ze^2}{\hbar v}\frac{(2\pi)^2\delta\left(k_z - \frac{\omega}{v}\right)}{\varkappa^2 + (1-\beta^2)k_z^2}\left(\varphi_f\chi_{s\mu_s}, \left\{e^{-\frac{i}{2}\mathbf{kr}}\left(1 - i\frac{\hbar v}{Mc^2}\frac{\partial}{\partial z}\right)\right.\right.$$
$$+ i\frac{\hbar}{2Mc^2}\left([\mathbf{v}\times\mathbf{k}]\,\mathbf{S}\left(\mu_p e^{-\frac{i}{2}\mathbf{kr}} + \mu_n e^{\frac{i}{2}\mathbf{kr}}\right)\right.$$
$$\left.\left.\left.-\frac{1}{2}[\mathbf{v}\times\mathbf{k}](\sigma_n - \sigma_p)\left(\mu_p e^{-\frac{i}{2}\mathbf{kr}} - \mu_n e^{\frac{i}{2}\mathbf{kr}}\right)\right)\right\}\varphi_0\chi_{1\mu_d}\right),$$

where $\mathbf{S} = \frac{1}{2}(\boldsymbol{\sigma}_n + \boldsymbol{\sigma}_p)$ is the spin operator for the n-p system.

With regard to the calculation of the integral over the relative coordinate $\mathbf{r}$, we not that we should cut off the range of k, since very large values of k correspond to small values of the impact parameter, which should however not be less than the nuclear radius R. In fact, for impact parameters which are less than the nuclear radius R, a collision with the nucleus will occur, and then the Coulomb interaction is unimportant. Therefore, in treating the Coulomb breakup of the deuteron, we should assume that k is bounded and the maximal value of k should be of the order of R^{-1}.

The effective value of the separation between neutron and proton in the deuteron is of order R_d; thus the effective value of the product **kr** is of order R_d/R and is thus much less than unity.

Expanding the exponentials $e^{\pm\frac{1}{2}\mathbf{kr}}$ in series and stopping at the first non-vanishing term, we get the following expression for the probability amplitude:

$$a = a_E\delta_{1s} + a_M\delta_{0s},$$

where a_E and a_M are the probability amplitudes for electric and magnetic transitions, and are equal respectively to

$$a_E = i\frac{2Z}{\hbar v}\frac{e^2(2\pi)^2\sqrt{8\pi\alpha}}{\varkappa^2+(1-\beta^2)k_z^2}\,\delta\left(k_z-\frac{\omega}{v}\right)\frac{kf}{(\alpha^2+f^2)^2}(\cos\vartheta\cos\vartheta'$$

$$+\sin\vartheta\sin\vartheta'\cos\varphi-\beta^2\cos\vartheta\cos\vartheta')\,\delta_{\mu\mu_d}, \qquad (8.4)$$

$$a_M = i\frac{Ze^2}{2Mc^2}(\mu_n-\mu_p)\frac{(2\pi)^2\sqrt{8\pi\alpha}}{\varkappa^2+(1-\beta^2)k_z^2}$$

$$\times\delta\left(k_z-\frac{\omega}{v}\right)\frac{\alpha-\alpha'}{(f^2+\alpha^2)(f-i\alpha')}\left|\mathbf{k}\frac{\mathbf{v}}{v}\right|\delta_{0\mu_p}, \qquad (8.5)$$

where ϑ is the angle between **k** and **v**, ϑ' is the angle between **f** and **v**, and φ is the difference of the azimuthal angles of **k** and **f**.

In the electric breakup of the deuteron, the spin of the n-p system does not change; in the magnetic breakup the neutron-proton system makes a transition from the triplet to the singlet state.

Taking the absolute square of (8.4) and noting that $\delta^2 \to \frac{1}{2\pi}\delta\left(k_z-\frac{\omega}{v}\right)$, we integrate over **dk** and get the following expression for the differential cross section for electric disintegration of the deuteron:

$$d\sigma_E = \left(\frac{Ze^2}{\hbar V}\right)^2\frac{2\alpha f^2}{\pi(\alpha^2+f^2)^4}\left\{\sin^2\vartheta'\ \ln\frac{\Gamma^2-\beta^2}{1-\beta^2}\right.$$

$$\left.+[2(1-\beta^2)\cos^2\vartheta-\sin^2\vartheta']\frac{\Gamma^2-1}{\Gamma^2-\beta^2}\right\}df, \qquad (8.6)$$

where $\Gamma = \hbar v/(\epsilon+\epsilon_f)R$. (In view of our previous remarks, we have limited the ϑ integration to the region from 0 to ϑ_{max}, where $\cos\vartheta_{max} = \Gamma^{-1}$.)

Integrating (8.6) over angle, we find the energy distribution of the disintegration products:

$$d\sigma_E(\varepsilon_f) = \frac{8}{3}\left(\frac{Ze^2}{Mc^2}\right)^2\frac{Mc^2\sqrt{\varepsilon}\,\varepsilon_f^{\frac{3}{2}}}{\beta^2(\varepsilon+\varepsilon_f)^4}\ln\frac{\Gamma^2}{1-\beta^2}\,d\varepsilon_f. \qquad (8.7)$$

Since the upper limit k_{max} is determined only to order of magnitude, formula (8.7) makes sense only if the argument of the logarithm is large ($\Gamma \gg 1$). This condition is satisfied for high energies of the deuteron. The factor $(1-\beta^2)^{-1}$ under the logarithm takes account of the relativistic dilation of the cross section for electric disintegration with increasing energy of the deuteron.

Using (8.5) it is easy to find the energy distribution of the products for magnetic disintegration:

$$d\sigma_M(\varepsilon_f) = \frac{2}{3}\left(\frac{Ze^2}{Mc^2}\right)^2$$

$$\times(\mu_n-\mu_p)^2\frac{\sqrt{\varepsilon\varepsilon_f}(\sqrt{\varepsilon}+\sqrt{\varepsilon'})^2}{(\varepsilon+\varepsilon_f)^2(\varepsilon'+\varepsilon_f)}\ln\frac{\Gamma^2}{1-\beta^2}\,d\varepsilon_f. \qquad (8.8)$$

(In the case of magnetic disintegration, the angular distribution of the products is isotropic.)

The integration of (8.7) and (8.8) over energy ϵ_f can be done numerically. Figure 16 shows the behavior of the total cross sections σ_E and σ_M in the energy interval $E_d = 0.2 - 10$ Bev, for $R = 1.1 \times 10^{-13}$ cm. In the extreme relativistic region, the magnetic disintegration cross section σ_M is an order of magnitude less than the electric cross section σ_E.

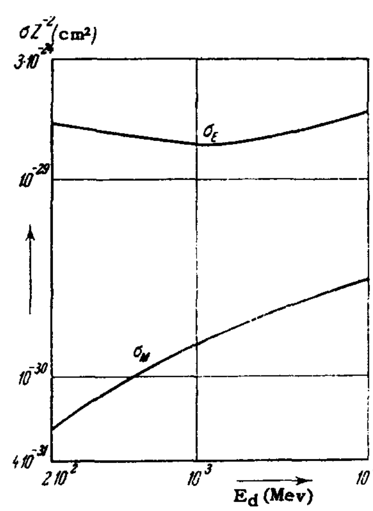


FIG. 16

2. Polarization of neutrons from electromagnetic breakup of deuterons. Despite the relatively small value of the cross section for magnetic breakup, the latter can be detected easily since the interference between the electric and magnetic processes leads to a polarization of the disintegration products. The polarization of the neutrons formed in the disintegration of the deuteron in the electromagnetic field of a nucleus was treated by Sawicki.[108]

For fixed values of the wave vectors **k** and **f**, the polarization of the neutrons will obviously be proportional to the following expression:

$$\frac{1}{3}\sum_{\mu}(a_E\chi_{1\mu}+a_M\chi_{00})^*\boldsymbol{\sigma}_n(a_E\chi_{1\mu}+a_M\chi_{00}).$$

When we take account of the normalization, it is easy to see that the polarization of the neutrons will be

$$P(\mathbf{f},\mathbf{k})=\frac{\frac{2}{3}\operatorname{Re}(a^E a^{M*})}{|a_E|^2+\frac{1}{3}|a_M|^2}.$$

Using (8.4) and (8.5) and transforming to the laboratory system (this corresponds to replacing $\mathbf{k}$ by $\mathbf{k}-\mathbf{k}_0$), we can obtain the following expression for the polarization of the neutrons:

$$P=\frac{\frac{1}{6}\Lambda^2(\mu_p-\mu_n)(\alpha-\alpha')\frac{f(\mathbf{fk}-\mathbf{fk}_0)\,|[\mathbf{k},\mathbf{k}_0]|}{(\alpha^2+f^2)(\alpha'^2+f^2)}}{\frac{(\mathbf{fk}-\mathbf{fk}_0)^2}{(\alpha^2+f^2)^2}+\frac{1}{48}\Lambda^4(\mu_p-\mu_n)^2(\alpha-\alpha')^2\frac{|[\mathbf{k},\mathbf{k}_0]|^2}{\alpha'^2+f^2}}, \quad (8.9)$$

where $\mathbf{k}_0$ and $\mathbf{k}$ are the wave vectors of the center of mass of the n-p system in the laboratory frame before and after the breakup, and $\Lambda = \hbar/Mc$ is the Compton wave length of the nucleon. In the case we are considering, where $n \ll 1$, the neutron polarization is independent of the nuclear charge Z and is large for small values of the cosine of the angle between the vectors $\mathbf{f}$ and $\mathbf{k}-\mathbf{k}_0$. We note that the polarization is an extremely sensitive function of the angles of emergence of the neutron and proton. If we assume $E_n = E_p = E_d/2$, then for $E_d = 100$ Mev, when the angle of emergence of the neutron is $\vartheta = 10°$, and the angle between the directions of the neutron and proton is $\theta = 18°$, the polarization $P = -0.21$.

9. Formation of Deuterons in the Collision of Fast Nucleons with Nuclei.

1. Methods for producing deuterons. In the collision of fast nucleons with nuclei, the production of fast deuterons can occur. Production of deuterons was first observed experimentally in the bombardment of nuclei with 90 Mev neutrons.[77] The beam of deuterons which was observed was emitted in the forward direction with a halfwidth $\sim 25-30°$, while the maximum of the energy distribution of the deuterons was at $60-65$ Mev. The total cross section for carbon was 2.6×10^{-26} cm^2, and increased for heavier nuclei. Later, production of deuterons was also observed from proton bombardment of nuclei (cf. for example, reference 83). The sharp peaking in direction of the deuterons and their high energy show that the observed deuterons are not products of evaporation from a compound nucleus.

There are two possible ways in which deuterons can be produced in the collision of fast nucleons with nuclei, without formation of a compound nucleus.

First there is direct capture (pick-up), in which the deuterons are formed as a result of the direct capture by the incident nucleon of one of the nucleons in the nucleus. Deuterons which are formed by such a direct capture are characterized by being peaked sharply in the forward direction and can have energies of the same order as the incident nucleon. Chew and Goldberger,[50] using the Born approximation, gave the theory of the direct capture, which was later developed by Heidmann.[82]

Secondly, indirect capture is possible. The incident nucleon colliding with some nucleon in the nucleus loses only a part of its energy. The nucleon in the nucleus which takes up this energy may form a deuteron by capturing some other nucleon inside the nucleus along its path. The mechanism of indirect capture was proposed by Bransden.[41] For energies of the incident nucleons above 300 Mev, indirect capture is more important than direct capture.

2. Direct capture. Let us treat the formation of deuterons in the collision of fast neutrons with nuclei as a result of direct capture. Let $\mathbf{r}_0$ be the radius vector of the incident neutron, $\mathbf{r}_1$ the radius vector of the proton which is captured, and $\mathbf{r}_2$ etc the radii-vectors of the other nucleons in the nucleus. If we treat the interaction of the incident neutron and the proton which it captures as a small perturbation, we can write the transition amplitude as

$$f=-\frac{M}{2\pi h^2}\int e^{-i\mathbf{k}_d(\mathbf{r}_0+\mathbf{r}_1)/2}\varphi_0(\mathbf{r}_0-\mathbf{r}_1)$$
$$\times\varphi_f^*(2,\ldots A)V_{01}e^{i\mathbf{k}\mathbf{r}_0}\varphi_i(1,\ldots A)\,d\tau, \qquad (9.1)$$

where $\mathbf{k}$ is the wave vector of the incident neutron, $\mathbf{k}_d$ is the wave vector of the deuteron which is formed, φ_0 is the deuteron wave function, φ_i and φ_f are the wave functions of the nucleus in the initial and final states.

If we limit ourselves to heavy nuclei, we can use the Fermi model, in which the nucleus is treated as an assembly of non-interacting particles contained in a spherical well of nuclear dimensions. Then the initial wave function can be taken as

$$\varphi_i(1,\ldots A)=V^{-\frac{1}{2}}e^{i\mathbf{p}\mathbf{r}_1}\varphi_i(2,\ldots A),$$

where $\mathbf{p}$ is the wave vector of the proton and V is the volume of the nucleus. The integration in (9.1) with respect to $\mathbf{r}_2\ldots\mathbf{r}_A$ gives unity if these nucleons remain in their initial state; the integration over $\mathbf{r}_1$ gives a result different from zero if $\mathbf{k}-\mathbf{k}_d=\mathbf{p}$. Since the Fermi distribution p is restricted to lie below L, the cross section will be different from zero only if $|\mathbf{k}-\mathbf{k}_d| \ll L$. Actually the fast nucleon transfers a momentum greater than L. To take this possibility into account, we make corrections to the Fermi model. The correction to the Fermi model reduces to taking account of the interaction of the proton which is to be captured with another nucleon in the nucleus. Then the wave function of the initial state can be taken as

$$\varphi_i(1,\ldots A) = V^{-\frac{1}{2}} e^{i\mathbf{P}(\mathbf{r}_1+\mathbf{r}_2)} \varphi_{\mathbf{p}}(\mathbf{r}_1-\mathbf{r}_2)\, \varphi_i(3,\ldots A). \quad (9.2)$$

For the final state wave function we can take

$$\varphi_f(2,\ldots A) = V^{-\frac{1}{2}} e^{i\mathbf{p}'\mathbf{r}_2} \varphi_f(3,\ldots A), \quad (9.3)$$

Transforming from $\mathbf{r}_0$, $\mathbf{r}_1$ and $\mathbf{r}_2$ to new variables $\mathbf{r} = \mathbf{r}_0 - \mathbf{r}_1$, $\mathbf{r}' = \mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{r}_2$, the amplitude for the reaction takes the form

$$f = -\frac{M}{2\pi\hbar^2} F\left(\mathbf{k} - \frac{1}{2}\mathbf{k}_d\right) G_{\mathbf{pP}}(\mathbf{k}-\mathbf{k}_d)\, V^{-1} \times \int e^{i(\mathbf{k}-\mathbf{k}_d+2\mathbf{P}-\mathbf{p}')\mathbf{r}_2}\, d\mathbf{r}_2, \quad (9.4)$$

where the last integral should be taken over the region of the nucleus, and

$$F\left(\mathbf{k} - \frac{1}{2}\mathbf{k}_d\right) = \int e^{i\left(\mathbf{k}-\frac{1}{2}\mathbf{k}_d\right)\mathbf{r}} \varphi_0(r)\, V(r)\, d\mathbf{r},$$

$$G_{\mathbf{pP}}(\mathbf{k}-\mathbf{k}_d) = \int e^{i(\mathbf{k}-\mathbf{k}_d+\mathbf{P})\mathbf{r}'} \varphi_{\mathbf{p}}(\mathbf{r}')\, d\mathbf{r}'.$$

If the neutron-proton interaction is described by the Yukawa potential

$$V(r) = V_0 \frac{e^{-\mu r}}{\mu r}, \quad V_0 = 67.8 \text{ Mev}, \quad \mu = 0.847\cdot 10^{13} \text{ cm}^{-1},$$

and (2.8) is used for the deuteron wave function,

$$F(\mathbf{l}) = \frac{4\pi N V_0}{\mu l}\left\{\tan^{-1}\frac{l}{\alpha+\mu} - \tan^{-1}\frac{l}{\beta+\mu}\right\},$$

$$\mathbf{l} = \mathbf{k} - \frac{1}{2}\mathbf{k}_d. \quad (9.5)$$

Choosing the wave function for the relative motion of the neutron and proton interacting with one another in the nucleus in the form

$$\varphi_{\mathbf{p}}(\mathbf{r}') = B\frac{\alpha}{(\alpha^2+p^2)^{\frac{1}{2}}}\left[\frac{\sin(pr'+\delta)}{\sin\delta} - e^{-\mu r'}\right] e^{-\gamma r'},$$

$$B = V^{-\frac{1}{2}}\alpha^{-1}, \qquad \delta = -\cot^{-1}\frac{\alpha}{p}, \quad \gamma = \rho^{\frac{1}{3}}R,$$

we also get the function

$$G(\mathbf{q}) = \frac{4\pi V^{-\frac{1}{2}}}{(\alpha^2+p^2)^{\frac{1}{2}}}\left\{-\frac{1}{q^2+(\mu+\gamma)^2} + \frac{\gamma^2+q^2-\alpha\gamma-p^2}{(\gamma^2+q^2+p^2)^2-4q^2p^2}\right\},$$

$$\mathbf{q} = \mathbf{k} - \mathbf{k}_d + \mathbf{P}. \quad (9.6)$$

The differential cross section for formation of the deuteron is given by the square modulus of the amplitude (9.4), integrated over all possible values of $\mathbf{p}'$. We note that this integration gives

$$\int \frac{V}{(2\pi)^3} d\mathbf{p}' \left| V^{-1}\int e^{i(\mathbf{k}-\mathbf{k}_d+2\mathbf{P}-\mathbf{p}')\mathbf{r}_2}\, d\mathbf{r}_2\right| = \int d\mathbf{p}'\delta(\mathbf{k}-\mathbf{k}_d+2\mathbf{P}-\mathbf{p}') = 1.$$

Thus if the initial state of the nucleons in the nucleus is characterized by the vectors $\mathbf{p}$ and $\mathbf{P}$, the cross section for formation of a deuteron with wave vector $\mathbf{k}_d$ is equal to

$$\sigma_{\mathbf{pP}} = \frac{k_d}{2k}\left(\frac{M}{2\pi\hbar^2}\right)^2 F^2\left(\mathbf{k}-\frac{1}{2}\mathbf{k}_d\right)\frac{3}{4} w G^2_{\mathbf{pP}}(\mathbf{k}-\mathbf{k}_d). \quad (9.7)$$

The factor $\frac{3}{4}$ is the statistical weight factor for the triplet state of the neutron and proton forming the deuteron. The factor w gives the number of different proton-proton or proton-neutron pairs in the nucleus.

The wave vectors $\mathbf{p}$ and $\mathbf{P}$ can be expressed in terms of the wave vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ of nucleons 1 and 2 in the initial state, when the separation of the nucleons is large compared to μ^{-1}, by the formula

$$\mathbf{p} = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2), \quad \mathbf{P} = \frac{1}{2}(\mathbf{p}_1+\mathbf{p}_2). \quad (9.8)$$

The energy of the outgoing deuteron is determined from energy conservation:

$$\frac{\hbar^2}{2M}(k^2+p_1^2+p_2^2) - U_0 = \frac{\hbar^2 k_d^2}{4M} + \frac{\hbar^2}{2M}(\mathbf{k}-\mathbf{k}_d+\mathbf{p}_1+\mathbf{p}_2)^2 - \varepsilon, \quad (9.9)$$

where U_0 is the depth of the potential well for the nucleon in the nucleus, which is taken to be 29 Mev.

To obtain the cross section for deuteron formation regardless of the initial state of nucleons 1 and 2, (9.7) must be multiplied by the probability $\mathfrak{P}(\mathbf{p}, \mathbf{P})$ for definite values $\mathbf{p}$ and $\mathbf{P}$ and then integrated over all possible values of $\mathbf{p}$ and $\mathbf{P}$ subject to the condition (9.9) of energy conservation. This probability is

$$\mathfrak{P}(\mathbf{p}, \mathbf{P})\, d\mathbf{p}\, d\mathbf{P} = \frac{9}{2\pi^2 L^6} d\mathbf{p}\, d\mathbf{P}, \quad (9.10)$$

if p_1 and $p_2 < L$, and is equal to zero otherwise. The coefficient in (9.10) is obtained from the normalization condition $\int \mathfrak{P}\, d\mathbf{p}\, d\mathbf{P} = 1$.

In the expression (9.7) for the cross section, only the factor $G^2_{\mathbf{pP}}$ depends on $\mathbf{p}$ and $\mathbf{P}$.

We may mention that the integral $\int G^2\mathfrak{P}\, d\mathbf{p}\, d\mathbf{P}$ for fixed value of $\mathbf{k}_d$ can be replaced approximately by

$$\int G^2\mathfrak{P}\, d\mathbf{p}\, d\mathbf{P}\,|_{k_d} \simeq \int G^2\mathfrak{P}\, d\mathbf{p}\, d\mathbf{P} \int \mathfrak{P}\, d\mathbf{p}\, d\mathbf{P}\,|_{k_d} = \int G^2\mathfrak{P}\, d\mathbf{p}\, d\mathbf{P}\, \frac{dE}{40}.$$

where the energy E is measured in Mev. We introduce the notation

$$\int wG_{\mathrm{p}}^{2}\mathrm{p}\mathfrak{P}d\mathrm{p}\,d\mathbf{P}=N(Q),\quad \mathbf{Q}=\mathbf{k}-\mathbf{k}_{\mathrm{d}}. \tag{9.11}$$

This function determines the momentum distribution of the nucleons in the nucleus. If Q is very small, the correction to the Fermi model can be neglected, and $N(Q) \simeq V$. In the general case, the function $N(Q)$ of (9.11) cannot be obtained in explicit form. Numerical integration gives the following values for $n(Q) = NQ/A$:

$n = 17 \times 10^{-39}$ cm³ for $Q \to 0$; $n = 3.6 \times 10^{-39}$ cm³ for $Q = 1.3 \times 10^{-13}$ cm^{-1}; $n = 7.6\ 10^{65}/Q^{8}\ 10^{-39}$ cm for Q large.

These values refer to a temperature of the Fermi distribution equal to $\theta \sim 9$ Mev

$(L = 1.0 \times 10^{13}$ cm^{-1}, $V = 17 \cdot A \times 10^{-39}$ cm³,

$\alpha^{-1} = 5.39 \times 10^{-13}$ cm).

Thus the differential cross section for formation of a deuteron by direct capture, per Mev of energy, is

$$\sigma = A\frac{k_{\mathrm{d}}}{2k}\left(\frac{M}{2\pi\hbar^{2}}\right)^{2}\frac{3}{4}F^{2}\left(\mathbf{k}-\frac{1}{2}\mathbf{k}_{\mathrm{d}}\right)n(\mathbf{k}-\mathbf{k}_{\mathrm{d}})\frac{1}{40}. \tag{9.12}$$

In Figs. 17 and 18 we give the differential cross sections per nucleon for different energies of the emitted deuterons, and the energy spectra of the deuterons at various angles. (The energy of the incident nucleon is 90 Mev.) The yield of the faster deuterons drops off more rapidly with angle than that for slower deuterons. The most probable energy of the deuterons decreases with increasing angle of emergence. These regularities of the deuteron spectra are in agreement with the experimental data.

By numerical integration[82] the following values were found for the total cross section for deuteron formation:

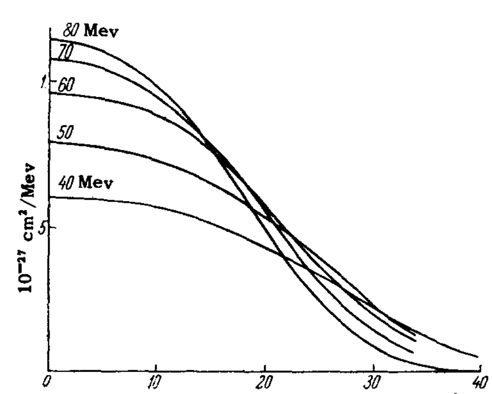


FIG. 17

$\sigma = 3.6 \cdot A^{2} \cdot 10^{-28}$ cm², $E_n = 100$ Mev; $\sigma = 5.5 \cdot A^{2} \cdot 10^{-28}$ cm², $E_n = 200$ Mev; $\sigma = 4.5 \cdot A^{2} \cdot 10^{-28}$ cm², $E_n = 300$ Mev.

For high energies of the incident nucleons ($E_n >$ 0.5 Bev), the following asymptotic formula was found for the cross section:

$$\sigma \longrightarrow 7.7 \cdot A \cdot \left(\frac{100}{E_n\ \mathrm{Mev}}\right)^{6} 10^{-25}\ \mathrm{cm}^{2}, \tag{9.13}$$

i.e., at high energies the total cross section is inversely proportional to the sixth power of the energy of the incident nucleons.

The cross section per nucleon for formation of a deuteron is of order r_0^2, where r_0 is the radius of the volume ascribed to a single nucleon in the nucleus. For this reason, of the total volume $\frac{4\pi}{3} r_0^3 A$ of the nucleus only the volume $\pi r_0 (r_0 A^{1/3})^2$ will be effective for formation of deuterons. This means that the effective number A in (9.12) should be taken equal to $\frac{3}{4}A^{2/3}$. Thus, for carbon at 90 Mev, we get a cross section of 8×10^{26} cm², which is approximately three times as large as the measured cross section.

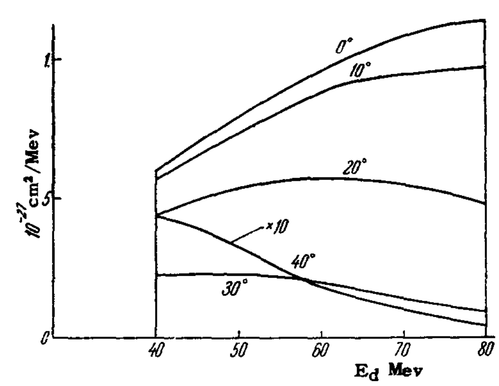


FIG. 18

3. Indirect capture. Again treating the interaction of the incident neutron with the nucleon in the nucleus as a small perturbation, we write the amplitude for an indirect transition which leads to formation of a deuteron in the form

$$f = -\frac{M}{2\pi\hbar^{2}}\int e^{-i\mathbf{k}'\mathbf{r}_0}\,e^{-i\mathbf{k}_{\mathrm{d}}(\mathbf{r}_1+\mathbf{r}_2)/2}$$

$$\times\varphi_0(\mathbf{r}_1-\mathbf{r}_2)\,\varphi_f(3,\ldots A)\,V_{01}e^{i\mathbf{k}\mathbf{r}_0}\varphi_i(1,\ldots A)\,d\tau, \tag{9.14}$$

where $\mathbf{k}$ and $\mathbf{k}'$ are the wave vectors of the incident neutron before and after the collision, and $\mathbf{k}_d$ is the wave vector of the deuteron which is formed.

Using the Fermi model with the correction for the interaction between nucleons 1 and 2, and choosing new variables $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $\mathbf{r}_d = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$ and $\mathbf{r}' = \mathbf{r}_0 - \mathbf{r}_1$, we write the amplitude in the form

$$f = -\frac{M}{2\pi\hbar^2} F(\mathbf{k}-\mathbf{k}') G_p(\mathbf{k}-\mathbf{k}') V^{-\frac{1}{2}} \times \int e^{i(\mathbf{k}-\mathbf{k}'-\mathbf{k}_d+2\mathbf{P})\mathbf{r}_d} d\mathbf{r}_d, \qquad (9.15)$$

where

$$F(\mathbf{k}-\mathbf{k}') = \int e^{i(\mathbf{k}-\mathbf{k}')\mathbf{r}'} V(r')\, d\mathbf{r}',$$

$$G_p(\mathbf{k}-\mathbf{k}') = \int e^{i(\mathbf{k}-\mathbf{k}')\mathbf{r}/2} \varphi_0(r)\varphi_p(\mathbf{r})\, d\mathbf{r}.$$

We note that $F(\mathbf{k}-\mathbf{k}')$ also determines the cross section for scattering of the free neutron by nucleon 1, which in the center of mass system has the form

$$\sigma(\mathbf{k}-\mathbf{k}') = \left(\frac{M}{4\pi\hbar^2}\right)^2 F^2(\mathbf{k}-\mathbf{k}').$$

The factor G_p can be obtained in explicit form by using (2.8) and (9.6):

$$G_p(\mathbf{k}-\mathbf{k}') = \frac{8\pi N}{|\mathbf{k}-\mathbf{k}'|(\alpha^2+p^2)^{1/2}} \left\{ \frac{1}{4}\cot\delta \ln \frac{(\alpha^2+q_1^2)(\beta^2+q_2^2)}{(\alpha^2+q_2^2)(\beta^2+q_1^2)} + \frac{1}{2}\tan^{-1}\frac{(\beta-\alpha)(\alpha\beta+q_1q_2)(q_1+q_2)}{(\alpha^2-q_1q_2)(\beta^2-q_1q_2)+(q_1+q_2)^2\alpha\beta} + \frac{1}{2}\tan^{-1}\frac{(\alpha-\beta)2(q_1+q_2)}{4(\mu+\alpha)(\mu+\beta)+(q_1+q_2)^2} \right\},$$

$$q_{1,2} = \frac{1}{2}|\mathbf{k}-\mathbf{k}'| \pm p.$$

The differential cross section for deuteron formation is given by the square modulus of (9.15) integrated over all possible values of $\mathbf{k}'$. Integration of the square modulus of the last factor in (9.15) gives

$$\int \frac{d\mathbf{k}'}{(2\pi)^3} \left| V^{-\frac{1}{2}} \int e^{i(\mathbf{k}-\mathbf{k}'-\mathbf{k}_d+2\mathbf{P})\mathbf{r}_d} d\mathbf{r}_d \right|^2 = \int d\mathbf{k}'\delta(\mathbf{k}-\mathbf{k}'-\mathbf{k}_d+2\mathbf{P}) = 1.$$

So the cross section for formation of a deuteron with wave vector $\mathbf{k}_d$, for fixed values of $\mathbf{p}$ and $\mathbf{P}$ in the initial state, is given by

$$\sigma_{pP} = \frac{k_d}{2k} 4\{\sigma_{np}(\mathbf{k}-\mathbf{k}') + \sigma_{nn}(\mathbf{k}-\mathbf{k}')\} w G_p^2(\mathbf{k}-\mathbf{k}'),$$

$$\mathbf{k}' = \mathbf{k}-\mathbf{k}_d + 2\mathbf{P}. \qquad (9.16)$$

Here w is the number of neutron-proton pairs in the nucleus which are in the triplet state ($w = \frac{3}{8}Z(A-Z)$). In (9.16) we have taken the sum of the cross sections σ_{np} and σ_{nn}, since the deuteron can be formed in the scattering of the incident neutron by either a proton or a neutron in the nucleus.

The energy of the emitted deuteron is determined from energy conservation:

$$\frac{\hbar^2}{2M}(k^2+p_1^2+p_2^2) - 2U_0 = \frac{\hbar^2 k_d^2}{4M} - \varepsilon + \frac{\hbar^2}{2M}(\mathbf{k}-\mathbf{k}'+\mathbf{p}_1+\mathbf{p}_2)^2. \qquad (9.17)$$

The cross section (9.16) must be averaged over all possible values of the vectors $\mathbf{p}$ and $\mathbf{P}$ in the region p_1 and $p_2 < L$. This averaging can only be done numerically. In doing this we use the experimental values of σ_{np} and σ_{nn} at the appropriate energies.

The total cross section for formation of deuterons by indirect capture is proportional to the square of the mass number A, unlike the direct capture, for which the cross section is proportional to A.

The following values were found for the indirect capture cross section:

$$\sigma = 3.6\cdot A^2 10^{-28}\ \text{cm}^2,\quad E_n = 100\ \text{Mev};$$

$$\sigma = 5.5\cdot A^2\cdot 10^{-28}\ \text{cm}^2,\quad E_n = 200\ \text{Mev};$$

$$\sigma = 4{,}5\cdot A^2\cdot 10^{-28}\ \text{cm}^2,\quad E_n = 300\ \text{Mev}.$$

Assuming that the cross sections σ_{np} and σ_{nn} change very little with energy, one can obtain the asymptotic variation of the indirect capture cross section with energy,

$$\sigma \sim \frac{1}{E_n}.$$

Though the indirect capture is only 11% of the direct capture cross section at E_n = 100 Mev, it is already twice as large as the direct capture at E_n = 300 Mev. Thus the indirect capture plays the principal role at high energies.

The energy spectrum and angular distribution from indirect capture differ from those for direct capture. In indirect capture the maximum in the deuteron energy spectrum is shifted toward lower energies from the maximum in direct capture. The differential cross section for indirect capture has a much weaker angular dependence than the cross section for direct capture. In particular the indirect capture mechanism explains the large number of energetic deuterons which were observed to be emitted at large angles from collision of fast nucleons with nuclei.[83,79,94]

III. APPENDIX

1. Integral of a product of Coulomb functions. The integrals occurring in Secs. 2 and 6, which contain the product of two Coulomb functions, are special cases of the following integral

$$\int d\mathbf{r}\,\frac{e^{-\lambda r}}{r}\,e^{i\mathbf{qr}}F(in_1,\,1,\,i(k_1r-\mathbf{k}_1\mathbf{r}))\,F(in_2,\,1,\,i(k_2r-\mathbf{k}_2\mathbf{r}))$$
$$=\frac{2\pi}{\alpha}\,e^{-\pi n_1}\left(\frac{\alpha}{\gamma}\right)^{in_1}\left(\frac{\gamma+\delta}{\gamma}\right)^{-in_2}$$
$$\times F\left(1-in_1,\,in_2,\,1,\,\frac{\alpha\delta-\beta\gamma}{\alpha(\gamma+\delta)}\right).$$
$$\alpha=\frac{1}{2}(q^2+\lambda^2),\quad \beta=\mathbf{k}_2\mathbf{q}-i\lambda k_2,$$
$$\gamma=\mathbf{k}_1\mathbf{q}+i\lambda k_1-\alpha,\quad \delta=\mathbf{k}_1\mathbf{k}_2+k_1k_2-\beta,$$
$$\mathbf{q}=\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3,\qquad \lambda>0,\ Im\,\lambda=0. \tag{1}$$

The derivation of this integral is given in reference 100.

2. Pseudopotential. At low and medium energies, the interaction between neutron and proton manifests itself mainly in the S state. Since the detailed character of the nuclear interaction is then unimportant, we can describe the neutron-proton interaction by using a pseudopotential corresponding to zero range of the nuclear forces. The pseudopotential can obviously be introduced if the wave length of the relative motion of the neutron and proton is large compared to the range of nuclear forces. This condition is satisfied if the energy of relative motion of neutron and proton is less than 20 Mev.

Let us denote the wave function describing the motion of the neutron and proton by $\Psi(\mathbf{r}_n, \mathbf{r}_p)$. If the neutron and proton are not at the same point, this function satisfies the Schrödinger equation

$$H_0\Psi=E\Psi,\qquad \mathbf{r}_n\neq\mathbf{r}_p, \tag{2}$$

where H_0 is the Hamiltonian for the non-interacting neutron and proton in the external field. For $r\to 0$, this function must satisfy a boundary condition which describes the presence of interaction between the neutron and proton. If we denote the wave vector of the relative motion of neutron and proton at the instant of collision by $\mathbf{k}$, the boundary condition can be expressed as

$$\Psi\to\left(e^{i\mathbf{kr}}+a\,\frac{e^{ikr}}{r}\right)\psi(\mathbf{r}_d),\quad r\to 0, \tag{3}$$

where the function $\psi(\mathbf{r}_d)$ is determined by the external field, and the expression in parentheses is the wave function of the relative motion. If there were no interaction between neutron and proton, the relative motion would be described by a plane wave. The quantity a appearing in the boundary condition is the scattering length, for which we can use the expression $a=-1/(\alpha+ik)$, which holds for the scattering of free neutrons by protons

The boundary condition (3) and Eq. (2) can be written as a single equation[4]

$$(H_0-E)\Psi=-V(\mathbf{r})\,\frac{\partial(re^{-ikr}\Psi)}{\partial r}, \tag{4}$$

where $V(\mathbf{r})=\dfrac{4\pi\hbar^2}{M}\,\dfrac{\delta(\mathbf{r})}{\alpha+ik}$ is the pseudopotential describing the neutron-proton interaction.

It can be shown that (4) admits a solution describing the bound state of the neutron-proton system. From (4), we get the following equation for the relative motion of a free neutron and proton:

$$\left\{\frac{\hbar^2}{M}\Delta_r+\varepsilon\right\}\varphi(\mathbf{r})=V(\mathbf{r})\,\frac{\partial(re^{-ikr}\Psi)}{\partial r}, \tag{5}$$

where $\epsilon=\hbar^2k^2/M$ is the energy of relative motion. It is easy to show that $\varphi_0(\mathbf{r})=\sqrt{\alpha/2\pi}\,(e^{-\alpha r}/r)$, the ground state wave function of the deuteron, satisfies (5) if $\epsilon=-\hbar^2\alpha^2/M$. In fact, using the form of φ_0, the right side of (5) can be transformed as follows:

$$-V(\mathbf{r})\,\frac{\partial(re^{-ikr}\varphi_0)}{\partial r}$$
$$=V(\mathbf{r})\sqrt{\frac{\alpha}{2\pi}}\,(ik+\alpha)\,e^{-ik+\alpha)r}=\sqrt{\frac{\alpha}{2\pi}}\,\frac{4\pi\hbar^2}{M}\,\delta(\mathbf{r}).$$

We thus get from (5) the equation

$$(\Delta_r-\alpha^2)\,\varphi_0(\mathbf{r})=-4\pi\sqrt{\frac{\alpha}{2\pi}}\,\delta(\mathbf{r}),$$

which is identically satisfied.

Using an equation like (4) to describe the interaction of a deuteron with a nucleus, we can get the following exact expression for the stripping amplitude:

$$f=-\frac{M}{2\pi\hbar^2}\int\psi^*_{\mathbf{k}_p}(\mathbf{r}_p)\,F^*(\mathbf{r}_n)\,V(\mathbf{r})\,\frac{\partial(re^{-ikr}\Psi)(\mathbf{r},\mathbf{r}_d)}{\partial r}\,d\mathbf{r}_n\,d\mathbf{r}_p.$$

Substituting the incident wave $\varphi_0\psi_{\mathbf{k}_d}$ for Ψ in this equation, and using the explicit expression for the pseudopotential, we find

$$f=-\frac{M}{2\pi\hbar^2}\int\psi^*_{\mathbf{k}_p}(\mathbf{r}_p)\,F^*(\mathbf{r}_n)$$
$$\left\{-\frac{4\pi\hbar^2}{M}\sqrt{\frac{\alpha}{2\pi}}\,\delta(\mathbf{r}_n-\mathbf{r}_p)\right\}\psi_{\mathbf{k}_d}(\mathbf{r}_d)\,d\mathbf{r}_n\,d\mathbf{r}_p.$$

Comparing this expression with (3.10), we see that for zero range of the nuclear forces, we get the relation

$$V_{np}\varphi_0=-\frac{4\pi\hbar^2}{M}\sqrt{\frac{\alpha}{2\pi}}\,\delta(\mathbf{r}). \tag{6}$$

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