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*ATMOSPHERIC DIFFUSION**

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IN the theory of atmospheric diffusion, a study is made of the propagation of impurities in the air. One of the most important problems confronting the theory of atmospheric diffusion is the question of the contamination of the air by industrial establishments and transport vehicles, especially the contamination of the air in cities. In the absence of atmospheric diffusion, the contamination would accumulate in the lower atmosphere, which would make the normal activities of human beings difficult. No less important is the problem of the propagation of radioactive particles in the atmosphere, a problem which has been troubling all mankind in recent years. Because of atmospheric diffusion, a certain fraction of the radioactivity generated in atomic explosions comes to us. We are confronted with the phenomenon of atmospheric diffusion even in agriculture, for example, in the dusting of plants by chemical substances in the fight against pests, or in the use of artificial smoke sources for the protection of plants from frost. Because of atmospheric diffusion, sea salt, volcanic dust, bacteria and viruses, pollen and plant seeds are propagated in the air, air masses over the sea are saturated with moisture, and those over the desert, by dust.

Because of the complicated nature of the phenomena investigated, the development of a theory of atmospheric diffusion requires the combined efforts of specialists in a number of branches of science. In first degree, this refers to specialists in hydromechanics and geophysics, into which categories falls the largest number of problems arising in the study of atmospheric diffusion.

1. Factors which affect atmospheric diffusion. Atmospheric diffusion is a complicated phenomenon and depends on many factors. In the first place, it is necessary to know what kinds of contamination enter the air, i.e., what is the nature of the contamination source. The contaminations can enter the air from industrial plants, from the surface of the earth or from artificial sources. The sources can be instantaneously or continuously acting with constant or time-dependent discharge. The sources can be concentrated (on the ground surface or elevated) or distributed along a line, surface or volume. It is also important to know whether the contamination particles acquire a definite velocity at the exit of the source (for example, the velocity of exhaust gases from a smoke stack) and what sort of temperature is possessed by the contaminated air at the source exit (a gas heated with respect to the surrounding air will rise upwards, one cooled will move downwards).

In the second place, it is necessary to know the laws of propagation of the contamination in the air for different meteorological conditions. The contaminations are transported by air currents and diffuse in the air as a result of the action of turbulence. A description of these processes relates to the realm of hydromechanics.

For an account of the transport of the contamination by the wind, one needs to know the kinematics of air currents. In particular, for calculations of the propagation of contaminations in the lower atmosphere, one must have information on the vertical wind profile for various meteorological conditions (in the first place, for different character of thermal stratification of the air). For computation of the mean contamination around a given source after a long time interval, one must assemble statistical data on the direction and force of the wind in the given region. Thus, by grouping the values of the wind force about intervals with centers A_j, and knowing the frequencies p_{ij} of occurrence of the corresponding pairs of values of force and wind direction, it is possible to compute the average distribution of the contamination by the formula

$$\bar{s}(x, y, z) = \sum_{i,j} p_{ij} s(x, y, z \mid u_i, A_j),$$

where $s(x, y, z \mid u_i, A_j)$ is the computed distribution of the contamination for wind force u_i and direction A_j.

For the calculation of the propagation of contamination from an instantaneous source in scales of the terrestrial sphere, it is necessary to know the kinematics of the air currents, which are controlled by synoptic processes, on a large part of the globe after a sufficiently long period of time (measured in weeks).

*The expanded text of a paper given at Oxford, England, in August, 1958 on the International Symposium on Atmospheric Diffusion and Air Contamination.

Besides the regular, macroscopic currents, there are chaotic hydrodynamic motions of different scales, down to very small, of the order of centimeters, which are known as turbulence. The displacement of the air brought about by turbulence is the reason for turbulent diffusion of the contamination. For a description of turbulent diffusion, we must know some statistical characteristics of the turbulent velocity field. These characteristics, generally speaking, are dependent on meteorological conditions, primarily, the field of the mean wind velocity and the thermal stratification of the air. For example, in stable thermal stratification of the air, turbulent diffusion takes place slowly, and the contaminations are transported by the wind almost without scattering. Under convective conditions, on the other hand, the turbulent diffusion takes place intensively, and leads to a rapid scattering of the contamination.

The third group of factors which influence atmospheric diffusion is related to the properties of the contaminations themselves. In the first place, it is necessary to know the effect of the force of gravity on the contamination. For example, gases heavier than air, and with sufficiently large particles, will settle, while the precipitation rate of the particles will depend on their dimensions, specific gravity and shape. One must take into account the possibility of chemical and radioactive transformations of the contamination, and also such physical transformations as coagulation, sublimation and adsorption on aerosols. In particular, interaction can take place between the impurities and atmospheric moisture — water vapor, water droplets in clouds and fogs, and particles of precipitates. Thus, rain can purify the air of contaminations, leading to their precipitation on the surface of the earth.

A fourth group of factors relates to the conditions of interaction of the contamination with the surface of the earth (or water). The contamination can either be retained by this surface, as it were, "adhering" to it or being absorbed by it (the surface of water possesses such a property relative to most contaminations), or be "reflected" from it and returned to the air. Intermediate cases of partial absorption and partial reflection, or "adhesion" at some (random) time, after which the contamination rises again into the air, are also possible. For a mathematical formulation of the boundary conditions for the contamination of the surface of the earth, it is necessary to take into account its degree of roughness and its capacity to absorb the contamination of given form. Obvious complications will be created by the inhomogeneity of the earth's surface — peculiarities of relief, the presence of structures, trees, etc.

Investigations in the theory of atmospheric diffusion are devoted to the development of a standard method of calculation of the propagation of the contaminations in the air under idealized average conditions (usually over a plane homogeneous region under stationary atmospheric conditions), and also to the study of the effect on atmospheric diffusion of one or another of the factors mentioned above (for example, the effect of the thermal stratification of the air).

Without pausing to give any sort of exhaustive review of the literature on the theory of atmospheric diffusion, we recall the names of just a few authors. We note that the principal contributions to the theory of atmospheric diffusion were made by the English scientists L. F. Richardson, G. I. Taylor, O. G. Sutton and G. K. Batchelor.

2. Specific characteristic of turbulent diffusion. The specific characteristic of turbulent diffusion is the multiple scale of the turbulent motion creating the mixing of the air. The character of the turbulent diffusion is determined by the way in which the energy is distributed among the turbulent motions of different scales. The largest of the scales of motion, on whose account almost all the energy flow takes place, is called the scale of the turbulence l. The value of the velocity of motion of the air at points, whose separation does not exceed l, is shown to be statistically random, relative to one another. Therefore, the particles or portions of contamination, whose separation does not exceed l, will not move independently of each other. Generally speaking, this disturbs the analogy between turbulent and molecular diffusion.

In a number of cases, the scale of turbulence l is shown to be small in comparison with the dimensions of the region in which the diffusion mainly takes place (for example, in comparison with the diameter L of the contamination cloud). In these cases, we can say that the diffusion takes place principally because of the small-scale turbulence. Here the particles of the contamination at relative small distances (in comparison with L) will move independently of each other. In similar cases, evidently, the description of turbulent diffusion by analogy with molecular diffusion is justified. Such an approach is usually applied for the description of turbulent diffusion along the vertical in the lower atmosphere.

3. Analogies between diffusion in the field of small-scale turbulence and molecular diffusion. The chaotic molecular motion can be characterized

by the mean velocity v_m of motion of the molecules (which depends on the temperature of the gas) and the length of the mean free path of the molecules l_m. The coefficient of molecular diffusion $k_m \sim v_m l_m$ can be determined by these quantities. The coefficient is understood as the coefficient of proportionality between the diffusion current of the given substance and the gradient of its concentration, ∇s: $S = -\rho k_m \nabla s$ (ρ is the density of the gas). Similarly, chaotic turbulent motion can be characterized by the mean value v of the velocity of turbulent fluctuations (which serves as a measure of the intensity of the turbulence) and the scale of the turbulence, l. (L. Prandtl[1] introduced the "mixing path" as l — a quantity similar to the mean free path length of the molecules.*) Thereafter, the coefficient of turbulent diffusion† $k \sim vl$, understood as the coefficient of proportionality between the mean turbulent flow of a given substance $S = \overline{\rho s' u'}$ and the gradient of its average concentration $\nabla \bar{s}$ could be introduced:

$$S = -\rho k \nabla \bar{s} \qquad (2)$$

(the bar indicates averaging, the prime indicates departure from average value and u = the velocity field). The assumption of the proportionality between S and $\nabla \bar{s}$ was formulated by W. Schmidt.[3] On the basis of this assumption, the so-called semi-empirical theory of turbulent diffusion was developed, in mathematical behavior analogous to the theory of molecular diffusion in an inhomogeneous medium.

The scales of molecular and turbulent motions differ from one another in many ways. Thus, in the lower atmosphere, for molecular motion, $v_m \sim 10^4$ cm/sec, $l_m \sim 10^{-5}$ cm, $k_m \sim 10^{-1}$ cm^2/sec, while for turbulent motion, $v \sim 10$ cm/sec, $l \sim 10^2 - 10^3$ cm, $k \sim 10^3 - 10^4$ cm^2/sec. It is clear from these figures that one can neglect molecular processes in most problems of atmospheric diffusion. At the same time, the difference in the scales no longer leads to a qualitative difference of turbulent and atmospheric diffusion. More important is the difference in the velocities of the motion, since the usefulness of the parabolic equation of diffusion is the more limited, the smaller the real velocities of motion of the diffusing particles (we shall return to this question below). It is also important that, in contrast to molecular diffusion, turbulent exchange in the atmosphere is anisotropic as a rule. However, the corresponding generalization of the theory does not present any difficulties (namely, by l and k, one must understand tensors).

4. The semi-empirical equation of turbulent diffusion. In the semi-empirical theory, the equation of turbulent diffusion for the lower atmosphere is written down in the form

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = \frac{\partial}{\partial x} k_x \frac{\partial s}{\partial x} + \frac{\partial}{\partial y} k_y \frac{\partial s}{\partial y} + \frac{\partial}{\partial z} k_z \frac{\partial s}{\partial z}. \qquad (3)$$

This equation expresses the law of conservation of the matter s in differential form. Here the x axis is directed along the wind direction, the x axis is vertically upward; t = time; u = wind velocity; k_x, k_y, k_z are the coefficients of turbulent diffusion in the direction x, y, z. If it is necessary to consider gravitational settling of the diffusing particles (with velocity w) and the possibility of exponential decay of the amount of the diffusing material (with half life $(\ln 2)/\alpha$), then we must add the quantity $(-w(\partial s/\partial z) + \alpha s)$ to the left hand side of Eq. (2). However, in the construction of the standard method of calculations, this quantity is usually not taken into consideration. Equation (3) was formulated for the halfspace $z > z_0$, where z_0 is the so-called "roughness height" of the surface of the earth, while at the level $z = z_0$ some boundary condition is given for the concentration s.

In the most general form (with consideration of gravitational settling of particles of contamination with velocity w) the boundary condition can be written in the form

$$k_z \frac{\partial s}{\partial z} + ws = \beta s \quad \text{for} \quad z = z_0, \qquad (4)$$

where β = some constant of the dimensions of velocity. Here, $k_z(\partial s/\partial z)$ is the vertical flow of the contamination because of turbulent diffusion, while ws is due to gravitational precipitation. For $\beta = 0$, the condition (4) means that the flow of the contamination on the surface of the earth is equal to zero, i.e., all the contamination remains in the air, being "reflected" from the earth's surface. For $\beta = \infty$, the condition (4) takes the form $s = 0$ $(z = z_0)$ and means that the contamination falling on the surface of the earth "disappears," as it were, being absorbed by this surface of "adhering" to it. In the intermediate cases $(0 < \beta < \infty)$, the contamination is partially "reflected" from the surface of the earth and partially "adheres" to it. Usually, only two limiting possibilities are observed — "reflection" or "absorption."

Typical problems are encountered in the search

*The concept of "mixing path" was suggested still earlier by G. I. Taylor.[2]

†Translator's note: ρk is the "interchange coefficient" of Schmidt.[3]

for the solution of Eq. (3) corresponding to instantaneous or continuous sources of contamination (in the study of continuous sources, the component $(\partial/\partial x)k_x(\partial s/\partial x)$ is usually neglected in comparison with $u(\partial s/\partial x)$).

The coefficients u, k_x, k_y, k_z of Eq. (3) are generally variable. The analytic solution of the equation, suitable for standard mass calculations, is obtained only under certain special assumptions on these coefficients. In the case of constant coefficients, the solutions of Eq. (3) corresponding to the basic type of sources were studied by O. Roberts.[4]

Thus, in the presence at the point $x = y = 0$, $z = h$ of a standard point source of contamination and for a boundary condition of "reflection" at the level $z = 0$, the solution of Eq. (3), under neglect of the component $(\partial/\partial x)k_x(\partial s/\partial x)$, has the form

$$s(x, y, z) = \frac{c}{4\pi x\sqrt{k_y k_z}} e^{-\frac{uy^2}{4k_y x}}\left[e^{-\frac{u(z-h)^2}{4k_z x}} + e^{-\frac{u(z+h)^2}{4k_z x}}\right]. \quad (5)$$

The distribution of contamination across the direction of the wind has here the form of a Gaussian curve with mean square deviation $(2k_y x/u)^{1/2}$, which increases with the distance x from the source. The distribution of contamination on the surface of the earth, in terms of the direction of the wind from the source, has great practical interest:

$$s(x, 0, 0) = \frac{c}{2\pi x\sqrt{k_y k_z}} e^{-\frac{uh^2}{4k_z x}}. \quad (6)$$

The highest concentration $s_m = (2c/\pi e u h^2)(k_z/k_y)^{1/2}$ (inversely proportional to the square of the altitude of the source) is achieved at a distance $x_m = uh^2/4k_z$ from the cource (proportional to h^2). For the boundary condition of "absorption," the contamination distribution in the air is described by a formula which differs from (5) in the replacing of the plus sign between the terms in square brackets by a minus sign. The quantity of practical importance in this case is the settling rate of the contamination on the earth's surface:

$$\sigma(x, y) = \left(k_z\frac{\partial s}{\partial z}\right)_{z=0} = \frac{cuh}{4\pi x^2\sqrt{k_y k_x}} e^{-\frac{uy^2}{4k_y x}-\frac{uh^2}{4k_z x}}. \quad (7)$$

This function characterizes the rate of formation (on the earth's surface) of a "track" of fallen contamination. The track is symmetric with respect to the wind direction. The maximum settling rate of the contamination is achieved on the axis of the track at the point $x_m^* = uh^2/8k_z$ and is equal to $\sigma_m = (4ck_z/\pi e^2 uh^3)(k_z/k_y)^{1/2}$.

The solutions of Roberts give a correct qualitative description of the diffusion processes; however, they are not in quantitative agreement with experimental data (the rate of decay of the concentration of the impurity upon removal from the source is too small). Moreover, as the theory of the turbulent regime in the lower atmosphere shows (and also direct measurements of the turbulent diffusion) these coefficents are not constant, but increase with altitude (for neutral stratification, they are proportional to the altitude). Solutions of Eq. (3) for $u = \text{const}$ and k_y, $k_z \sim z$ were studied by S. Bosanquet and J. Pearson.[5] Finally, the case was considered by a number of authors in which the wind velocity u and the coefficient of turbulent diffusion are given as proportional to some power of the altitude z. Formulas of such a type permit an excellent approximation of the experimental regularity. In the USSR, a similar method was treated in detail in a series of papers by D. L. Laikhtman.[6-7] For example, in one of the variants of the formulas of D. L. Laikhtman there was obtained $u = \text{const}$, $k_y = \text{const}$, $k_z = k_1(z/z_1)^{1-(1/p)}$, where p is a parameter characterizing the degree of thermal stability of the air (for neutral stratification, $p = \infty$, for stability $p > 0$, for convection, $p < 0$), while the value of k_1 for $z_1 = 1$ m can be roughly determined by the formula $k_1 = u/40$ (u is measured in m/sec, while k_1 is measured in m^2/sec). For such coefficients of Eq. (3), there is obtained

$$s(x, y, z) = \frac{p+1}{p}\frac{cL}{2hx\sqrt{\pi u x k_y}} e^{-\frac{uy^2}{4k_y x}-\frac{L}{x}\left[\left(\frac{z}{h}\right)^{1+\frac{1}{p}}+1\right]} \times\left(\frac{z}{h}\right)^{\frac{1}{2p}} I_{\pm\frac{1}{p+1}}\left[\frac{2L}{x}\left(\frac{z}{h}\right)^{\frac{1}{2}+\frac{1}{2p}}\right], \quad (8)$$

in place of the formula of Roberts (5), where $L = (p/(p+1))^2(uh^2/k_z(h))$. In the index of the cylindrical function, the plus sign is to be taken for the boundary condition of "absorption," and the minus for the condition of "reflection." In analogy with Eq. (6), we have

$$s(x, 0, 0) = \frac{c}{2h\Gamma\left(1+\frac{p}{p+1}\right)\sqrt{\pi u x k_y}}\left(\frac{L}{x}\right)^{\frac{p}{p+1}} e^{-\frac{L}{x}}. \quad (9)$$

The maximum of this function is obtained at the point $x_m = ((2p+2)/(3p+1))L$. The abscissa of the maximum is proportional to $h^{1+(1/p)}$, the value of the maximum is proportional to $h^{-3/2-1/2p}$.

In the use of Eq. (3) for the description of the propagation in the air of smoke and gasses given off by smoke stacks, account of the possible floating of the smoke as a result of its warming and the presence of a velocity w_0 of the exhaust smoke

is accomplished in practice by replacing the real altitude of the smoke stack h by some "effective height" $h+\delta h$, where δh is determined, for example, by the formula

$$\delta h = 3.8 R \frac{w_0}{u} \frac{T_1}{T}. \quad (10)$$

Here R is the radius of the mouth of the pipe, u is the wind velocity, T_1 is the temperature of the smoke and T is the temperature of the air at the altitude of the pipe.

5. Statistical approach to turbulent diffusion. The equation of diffusion (3) can be derived from the assumption that each individual diffusing particle is moving randomly, while its coordinates change with time according to the law of a Markov random process. Equation (3) is the equation of A. H. Kolmogorov[8] for this random process. Such a conclusion leads to a statistical interpretation of the coefficients of turbulent diffusion:

$$k_x = \frac{1}{2}\frac{d\sigma_x^2(t)}{dt}, \quad \sigma_x^2(t) = \overline{[x(t) - x(0)]^2}, \quad (11)$$

where s(t) is the abscissa of the diffusing particle at time t (similar formulas are valid for k_y and k_z). It is then evident that the primary concept is not the coefficient of turbulent diffusion, but the dispersion of the coordinate of the diffusing particle (which depends on the diffusion time).

The advantage of following the diffusing particle (i.e., the Lagrangian rather than the Eulerian method of describing the motion of the medium) is the special characteristic of the theory of turbulent diffusion, in contrast with the theory of a number of other phenomena produced by the turbulence. Here, the most convenient characteristic of turbulence is the Lagrangian correlation function of the velocity field:

$$\overline{u_x(t)\,u_x(t+\tau)} = \overline{u_x^2} R_x(\tau), \quad (12)$$

where $u_x(t) = dx(t)/dt$ is the x-component of the velocity of motion of the diffusing particle at the time t, and the bar denotes time averaging. The assumption of the existence of an instantaneous velocity of motion of the particle is the essential difference of the statistical theory of turbulent diffusion thus far advanced from the theory of molecular diffusion (i.e., inertia-free Brownian motion).

We note that in the turbulent atmosphere, an "evolution of the level" generally takes the place in the velocity field: the time average of the value of the characteristics of the velocity field depends essentially on the duration of the averaging interval. Therefore, the determination of the correlation function (13) is valid, strictly speaking, only in cases of small-scale turbulence (in the same sense as was pointed out above). In the most general cases, it is better to start out from the Lagrangian correlation function not for the velocity, but for the acceleration.

The dispersion $\sigma_x^2(t)$ can be represented by the correlation function (12) in the following way:

$$\sigma_x^2(t) = \overline{u_x^2} \int_0^t \int_0^t R_x(|\tau_1 - \tau_2|)\, d\tau_1\, d\tau_2. \quad (13)$$

This important formula was put forward by G. I. Taylor. O. Sutton proposed to approximate the function $R(\tau)$ by the formula $R(\tau) = (1+\tau/T)^{-n}$, where T is some characteristic time, to which corresponds $\sigma^2(t) \approx (c^2/2)(ut)^{2-n}$ at large t, and obtained the corresponding basic types of sources on the basis of the formula for the concentration of the contamination.

In particular, in place of Eqs. (5) and (8), Sutton obtained the formula

$$s(x, y, z) = \frac{C}{\pi c_y c_z u x^{2-n}} e^{-\frac{y^2}{c_y^2 x^{2-n}}} \left[e^{-\frac{(z-h)^2}{c_z^2 x^{2-n}}} + e^{-\frac{(z+h)^2}{c_z^2 x^{2-n}}} \right]. \quad (14)$$

This formula can be obtained as the stationary solution of Eq. (3) for u = const; $k_x = 0$, $k_{y,z} = ((2-n)/4)\, c_{y,z}^2 u x^{1-n}$ (a value between 0 and 0.2 is assumed for the parameter n, while the coefficients c_y and c_z are chosen in dependence on the altitude of the source h, decreasing with increase in h). According to Eq. (14), the maximum of the contamination concentration in the lower atmosphere is obtained on the x axis at the point $x_m = (h/c_z)^{2/2-n}$ (which does not depend on the wind speed); this maximum is equal to $s_m = (C/\pi e u h^2)(c_z/c_y)$.

The formulas of Sutton have been shown to be very suitable for the description of experimental data and have obtained a widespread currency as the basis for calculations of the diffusion of contamination in the air.

It follows from the formula of Taylor (13) that for short diffusion times, $\sigma^2 \sim t^2$ and $k \sim t$, while for long times, $\sigma^2 \sim t$ and $k \to$ const (the latter rule is analogous to the case of molecular diffusion). Making use of these relations, and describing the contamination concentration in the presence of an instantaneous point source by a Gaussian function with dispersion $\sigma_x^2(t)$, $\sigma_y^2(t)$, $\sigma_z^2(t)$, F. Frenkiel[11] worked out a method for the calculation of diffusion of the contamination, with a successful application by him to the description of the contamination in cities.

6. The effect of thermal stratification of the air on turbulent diffusion. As has already been mentioned, the turbulent diffusion in the lower

levels of the atmosphere always depends significantly on the thermal stratification of the air. In the consideration of the effect of the stratification on the turbulent regime, it is appropriate to make use of a dimensionless parameter

$$\mathrm{Ri} = \frac{g}{\theta}\frac{\partial\theta}{\partial z}\left(\frac{\partial u}{\partial z}\right)^{-2}, \qquad (15)$$

where g = acceleration due to gravity, while θ = the so-called temperature potential (in the lower atmosphere, $\theta \approx T + \Gamma z$, where T is the ordinary temperature, while $\Gamma \cong 1°\mathrm{C}/100$ m). This parameter was introduced by L. Richardson.[12] For thermal instability, $\mathrm{Ri} < 0$, while for stable stratification, $\mathrm{Ri} > 0$. With the aid of energy considerations, Richardson established the fact that for $\mathrm{Ri} > \mathrm{Ri}_{cr} > 0$, turbulence dies out, losing its energy in the work against the Archimedes' forces. Recently, S. C. Priestley[13] and E. Deacon[14] in Australia, H. Lettau[15] in the USA (sic), A. M. Obukhov and A. S. Monin[16-19] in the USSR worked out similar methods of calculation of the effect of stratification on the turbulent regime in the lower atmosphere. Obukhov and Monin developed the theory of similarity, according to which the turbulent regime is completely determined by three dimensional quantities: the turbulent pressure of friction, the turbulent flow of heat along the vertical, and the parameter g/T_0 which characterizes the effect of Archimedes' forces. In this case, the effect of stratification on the characteristics of the turbulent regime is described by non-dimensional factors depending on Ri. These factors can be effectively determined in a number of cases. In particular, the velocity profiles of the wind $\bar{u}(z)$ and of the temperature $\bar{T}(z)$ in the lower atmosphere are given by the formulas

$$\left.\begin{aligned} u(z) &= \frac{v_*}{\varkappa}\left[f\left(\frac{z}{L}\right) - f\left(\frac{z_0}{L}\right)\right], \\ \bar{T}(z) &= \bar{T}(z_0) + T_*\left[f\left(\frac{z}{L}\right) - f\left(\frac{z_0}{L}\right)\right], \\ L &= \frac{v_*^3}{\varkappa\frac{g}{T_0}\left(-\frac{q}{c_p\rho}\right)}, \quad T_* = -\frac{1}{\varkappa v_*}\frac{q}{c_p\rho}, \end{aligned}\right| \qquad (16)$$

where $v_* = (\tau/\rho)^{1/2}$ is the so-called "friction" velocity (τ = turbulent force of friction, ρ = density of the air); q = vertical turbulent flow of heat; $\varkappa \cong 0.4$ = von Karman's constant, and $f(\zeta)$ is a universal function, which is connected with the Richardson number (15) by the relation

$$\frac{\mathrm{Ri}(z)}{\mathrm{Ri}_{cr}} = \frac{1}{f'\left(\frac{z}{L}\right)}. \qquad (17)$$

For small ζ, the function $f(\zeta)$ is approximately equal to $\ln|\zeta| + \beta\zeta$ (β is a number of order unity), for large positive ζ (the case of a stable stratification), the function is asymptotically proportional to ζ, while for large negative ζ (case of thermal convection), it is asymptotically approaches a constant according to the law $c_1 + c_2\zeta^{-1/3}$.

7. Account of the limiting velocity of turbulent diffusion. The theory set forth above of small-scale turbulent diffusion has the inadequacy that the actual limitation of velocity of propagation of the contamination in the turbulent atmosphere, connected with the limitation of the magnitude of the fluctuation of the wind velocity (which produced the turbulent displacement), is not taken into account. The parabolic character of the semi-empirical equation of diffusion means that the contamination, upon exit from the source, is instantaneously propagated in all directions and can be discovered quickly, even though in completely negligible quantity, at any large distance from the source. Usually, this inadequacy is acceptable, since the volume inside which the contamination concentration is not too small is always limited, and the concentration distribution inside this volume is generally satisfactorily described by the parabolic diffusion equation. However, in some cases (in particular, close to the real boundary of the contamination cloud), use of the parabolic diffusion equation can lead to significant errors. For example, smoke issuing from a pipe of height h reaches the ground at a distance from the pipe not less than uh/v, where u is the wind velocity, v is the maximum velocity of the propagation of the smoke along the vertical. In this same time, according to the solution of the parabolic equation of diffusion, the smoke is found at the surface of the earth arbitrarily close to the pipe.

A method of calculation of the smoke propagation from the pipe which is free from this limitation was proposed by the Soviet scientist G. Sheleikhovskiĭ.[20] This is based on the use of the theory of free turbulent streaming. According to the formula of Sheleikhovskiĭ, the smoke issuing from the pipe fills a cone whose axis lies in the direction of the wind, and whose angle depends on the intensity of the turbulence. Sheleikhovskiĭ's formula permits one to determine only the mean concentration in different transverse cross sections of the smoke flow.

More radical is the generalization of the diffusion equation, which would give this equation a hyperbolic character. Such a generalization was proposed as early as 1926 by V. A. Fock,[21] and later by B. I. Davydov,[22] E. S. Lyapin,[23-24] S. Goldstein,[25] R. Davies[26] and A. S. Monin.[27-28] For the

derivation of the one-dimensional hyperbolic equation of diffusion, we start out from the following assumptions: (a) each individual diffusing particle moves randomly; (b) the instantaneous velocity of motion of the particle exists almost everywhere and is bounded; (c) the coordinate of the particle and the direction of its motion together form a Markovian random process.

The equation of diffusion is obtained in the form

$$\frac{\partial s}{\partial t}+\frac{\partial S}{\partial z}=0;\quad \frac{1}{2a}\frac{\partial S}{\partial t}+S=-\frac{v^2}{2a}\frac{\partial s}{\partial z}, \tag{18}$$

where s = concentration of particles, S = turbulent flow of particles, v = maximum velocity of motion of the particles, a = characteristic frequency of turbulent pulsations. Eliminating the turbulent flow S, for concentrations of particles s from (18), we can obtain the so-called telegrapher's equation. In the limit, as $a \to \infty$, $v \to \infty$, $v^2/2a \to k$, and the general parabolic equation of diffusion is obtained from (18).

The solution of the hyperbolic equation of vertical turbulent diffusion for the lower atmosphere for neutral stratification (in the case of a terrestrial instantaneous point source of contamination of unit intensity) has the form:[27-28]

$$s(z, t)=\frac{1}{\varkappa v_* t}\frac{(1-z/\lambda v_* t)^{\varepsilon-1}}{(1+z/\lambda v_* t)^{\varepsilon+1}};\quad 0\leqslant z\leqslant \lambda v_* t. \tag{19}$$

Here $\lambda = v/v_*$ is a number close to unity, and $\epsilon = \lambda/2\kappa$. Use of the hyperbolic equation of diffusion in place of Eq. (6) for the terrestrial concentration of contamination in the presence at a height h of a stationary point source leads to the formula

$$s(x, 0, 0)=\frac{c\varepsilon}{uh}\sqrt{\frac{\lambda v_*}{\pi k_y h}}\frac{\xi^{-1/2}(\xi-1)^{\varepsilon}}{(\xi+1)^{\varepsilon+1}};\quad \xi=\frac{\lambda v_* x}{uh}\geqslant 1 \tag{20}$$

(for $\xi < 1$, the sea level concentration is equal to zero). The abscissa of the maximum of the sea level concentration is proportional to the height of the source h; the magnitude of the maximum is proportional to $h^{3/2}$.

8. Diffusion in the field of large-scale turbulence. In the case in which the scale of turbulence l is not small in comparison with the dimensions of the contamination cloud, the rules of turbulent diffusion are significantly different from the case of molecular diffusion. For example, in contrast with the case of molecular diffusion, the rate of change of the distance L between the two individual particles depends on the distance L itself: this rate is in the mean small when L is small, and becomes large when L is large. The small-scale motions change this distance slightly, but large-scale motions, which affect both particles simultaneously, transport them without any essential change in the distance between them. Thus an increase in the dimensions of the contamination cloud leads to an increase in the effective "diffusion coefficient."

L. Richardson[29] was the first to turn his attention to this phenomenon. He proposed to describe this phenomenon with the help of the "distance neighbor function" g(L, t) — the probability density for the distance L between the two diffusing particles. Richardson proposed to describe the change in the function g(L, t) with time by a parabolic diffusion equation with the diffusion coefficient k dependent on L. With the help of empirical data, Richardson established the fact that $k(L) \sim L^{4/3}$. This law is valid for phenomena of very different scales.

The formulas of Sutton, in which it is taken into account that the diffusion coefficient increases with the diffusion time, take the Richardson effect qualitatively into account. However, for diffusion along the vertical in the lower atmosphere, the effect described is possibly not essential, since the vertical diffusion takes place primarily by means of the small-scale turbulence. At the same time, the turbulent motions from small to very large scales take part in the horizontal mixing of the air. For example, experiments on the continuous recording of the wind direction, and observations of the flow of smoke show that the turbulent motions with scales in the hundreds of meters and kilometers, leading to fluctuations of the wind direction with periods of several minutes, exhibit a significant effect on the diffusion of the contaminations. Therefore, in the description of horizontal mixing, it is necessary to take the Richardson effect into account.

Richardson's law $k(L) \sim L^{4/3}$ was explained by A. M. Obukhov[30] as a result of the hypothesis of similarity of A. N. Kolmogorov[31] for turbulence at very large Reynolds' numbers. According to the Kolmogorov hypothesis, there exists the so-called inertial interval of scales of turbulent motion, in which the statistical regime of these motions is completely determined by the action of inertial forces, leading to the transfer of energy from the large-scale motion to motion of smaller scales with a rate ϵ that is constant in time. The quantity ϵ, equal to the rate of dissipation of turbulent energy, is the only parameter, determining the turbulent regime in the inertial interval. If the diffusion takes place as a result of the effect of the turbulent motion with a scale of the inertial interval, then $k(L) = c\epsilon^{1/3} L^{4/3}$, where c is a number, i.e., Richardson's law is obtained. Ap-

plications of similarity theory to turbulent diffusion were considered in the works of G. Batchelor.[32-33]

It is of practical importance to consider the concentration of the contamination s in the case of Richardson diffusion. Knowledge of the "distance neighbor function" g(L, t) for this is insufficient, since s can be determined by g only if the motion of the diffusing particles can be regarded as independent. The theory of similarity[34-35] can give some indications of the concentration s for the presence of various sources of contamination. Thus, in the two-dimensional case, the concentration distribution relative to the center of the contamination could in the presence of an instantaneous source of strength Q must have the form

$$s(r, t) = \frac{Q}{\pi \varepsilon t^3} f_0\left(\frac{r^2}{\varepsilon t^3}\right), \tag{21}$$

r = distance from the center of the cloud, the diameter of the cloud $\sim t^{3/2}$. For a calculation of the general form of the equation describing the Richardson diffusion, we can take advantage of that which in the system under consideration is connected with the average motion of the medium, the homogeneous and isotropic turbulence, in addition to the theory of similarity. In this system, for an initial concentration $s_0(\mathbf{r})$, the concentration at the time t is determined by formula[34-35]

$$\tilde{s}(\mathbf{p}, t) = a(\varepsilon^{1/3} p^{2/3} t)\, \tilde{s}_0(\mathbf{p}), \tag{22}$$

where the sign ~ denotes the Fourier transform in $\mathbf{r}$, $\mathbf{p}$ is the wave vector (p is its modulus); $a(\theta)$ is some dimensionless function, equal to unity for $\theta = 0$ and monotonically decreasing to zero as $\theta \to \infty$. The choice of a concrete form of the function $a(\theta)$ leads to a definite diffusion equation. For example, under the assumption that changes in the concentration distribution with time form a subgroup, we must set $a(\theta) = e^{-c\theta}$. Such an approximation corresponds to a diffusion equation which is differential in time and integral in the coordinate space; all the solutions of this equation are contained in the solution of the equation

$$\frac{\partial^3 s}{\partial t^3} = c\varepsilon \Delta s. \tag{23}$$

Another approach to the description of the Richardson diffusion was suggested by A. M. Obukhov. The diffusion equation is assumed to be written down in six-dimensional coordinate and velocity space, but with the additional requirements of invariance (the turbulence is locally homogeneous and isotropic).

9. Prospects for the development of the theory of turbulent diffusion. The possibilities of a statistical theory of turbulent diffusion are still far from exhausted. This theory will undoubtedly develop vigorously in the coming years as a result of the development of a statistical theory of turbulence. Even now, there is interest in the use in the theory of turbulent diffusion of a number of methods which apply for the description of turbulence. As an example, we shall point to the recent work of P. Roberts,[36] in which, with the aid of the equations of hydrodynamics, equations were obtained for the correlation moments of the concentration of contamination and the velocity field; the hypothesis of M. D. Millionshchikov[37] was employed for completing these equations (i.e., fourth moments are expressed in terms of the second by formulas which are valid for multi-dimensional normal distribution). The method of characteristic functions put forth by E. Hopf is another example.[38]

Finally, the development of methods of Lagrangian description of turbulent flow is an excellent prospect for the theory of turbulent diffusion.

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