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The exclusion principle and the evolution of matter in statistics of particles with hierarchical structure *

M.Yu. Romanovsky

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Abstract. Statistics of the most probable energy distributions are constructed and investigated in systems with many particles, some of them are contained (or delimited) within other particles, as well as in systems with particles delimited within other particles, which, in turn, are delimited within third particles. The statistics of the distributions of such particles with respect to the maximum and minimum energy values turn out to coincide with those obtained in the initial combinatorial solution. The results give a basis for proving the Pauli exclusion principle, the minimality of energy release in various processes, and the absence of particles with negative energy, as well as describe the complication of the structure of matter.

Keywords: most probable distributions, distributions of extremes, hierarchical structures, delimitation, Pauli exclusion principle, matter complication

M.Yu. Romanovsky
(1) PE Science and Innovation,
ul. Bol'shaya Ordynka 24, 119017 Moscow, Russian Federation
(1) ANO National Center for Physics and Mathematics,
ul. Parkovaya 1, str. 3, 607182 Sarov, Nizhny Novgorod region, Russian Federation
(3) Pirogov Russian National Research Medical University,
ul. Ostrovityanova 1, 117997 Moscow, Russian Federation
E-mail: MYRomanovsky@rosatom.ru

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1. Introduction

The principle to exclude some particles to be in the same energy state, as well as the possibility for another type of particles to condense in low-energy states were obtained in statistical physics due to *a priori* assumptions [1]. These principles made it possible to construct a phenomenology of the evolution of matter as well as the laws of its interaction without answering the questions of where these exclusions and possibilities physically come from. It is reasonable to consider, from the point of view of statistical physics, the simplest laws to construct such 'protostructures' beginning from systems of unstructured and indistinguishable particles.

It is known [1, 2] that a system of identical indistinguishable particles with a discrete energy spectrum gives, under natural assumptions about a finite number of all particles and the existence of an average particle energy as the most probable, an exponential probability density of the distribution of a particle in a certain ith energy state ϵ_i of the form $W(\epsilon_i) = 1/T_{\text{eff}} \exp(-\epsilon_i/T_{\text{eff}})$. Here T_{eff} is a normalization coefficient that makes sense of some average energy or effective temperature of the particle system. Let's call this result 'problem 0.' The thermodynamic values and laws in such a system of structureless particles are well known.

More complex systems consisting of indistinguishable particles with an internal structure, firstly, modify thermodynamic relationships, as, for example, it happens for plasma

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[3], and lead to new effects such as 'negative friction' [4] (such particles are often named as 'active'). Therefore, the question naturally arises what will happen with energy distributions of particles in the most general case — certain groups of particles are delimited from others, or if some particles (let's call them upper-level particles) contain others (lower-level, also indistinguishable).

This problem was considered earlier also (see, for example, [1, 2]). The most probable distribution is given by a large canonical ensemble if we use the additional assumption of *a priori* different quantities of lower-level particles inside the upper-level particles. Here, the condition of the identity of the upper-level particles is not established. Let's call this 'problem 0′.' The approach is not necessarily limited to particles and energy distributions — consider, for example, systems that have certain structures inside that affect the external manifestations of the system, in this case the role of the 'energy distribution' of lower-level particles is played by the distribution of some property (or feature) of this lower-level system.

It is natural to go to a more general formulation of the problem, taking into account the observed indistinguishability of upper-level particles. Let's call it 'problem 1'. Here we need to find the most probable energy distributions of lower-level particles. It cannot exclude that there may be several such distributions. First, these are the assumed 'true' energy distributions of the lower-level particles in one upper-level particle. It should be clarified here whether they are observable, and if yes, in what experiments. Secondly, what distributions will be observable, and what is their difference from the true ones. The natural limit of problem 1, if there is only one lower-level particle inside the upper-level particle (under certain conditions), is problem 0.

The problem can be more complicated when the particles have an internal structure in the form of other particles, which, in turn, contain third particles inside themselves, this can be named as problem 2. Here it is also necessary to determine the true and observed most probable distributions of lower-level particles over energy. The idea of particles located in boxes is quite common (see, for example, [1]). The latter can be delimited in some external boxes, then problem 2 is obtained. With certain assumptions, atomic nuclei consisting of nucleons, which, in turn, consist of quarks, can also be as an example of problem 2.

The solution of problem 1 suggests a combinatorial problem about the number of combinations when distributing identical items into groups. We also show that consideration of the distribution of maximum and minimum energy values of lower-level particles in upper-level particles from a sampling of upper-level particles will lead to the same results. The solution of problem 2 will be given below. In the following, we will refer to problem 1 and the particles in it as two-level or two-layer, in problem 2, respectively, as three-level or three-layer.

Results of both problems can be interpreted as the separation of lower-level particles into kinds of pseudo-particles with different statistical and mechanical behavior observed in different experiments. In combinatorial problem 1, there are 2 kinds of such pseudo-particles, in problem 2, there are already 4. In both problems for maximum and minimum energy distribution, it turns out to be possible to associate constants arising in combinatorial problems that make sense of certain chemical potentials with the possible radius of interaction of upper-level particles.

For pseudoparticles appearing in single-particle observations in both formulations — combinatorial and for the distribution of maximum and minimum values — the Pauli exclusion principle arises naturally. Section 2 presents the main facts of the 'operation' of the Pauli *a priori* principle.

It is possible to carry out a combinatorial procedure to determine the most probable distribution of energy release using some natural assumptions. They are the different number of particles (molecules) of a substance that can enter into an energy release reaction (burning, explosion, etc.) due to their location on the surface/inside the structure as well as the dynamics of access to reacting molecules and the obvious probabilistic nature of the process. In some simple approximation, the energy release will be determined by a single parameter of the combinatorial scheme. The most probable distribution will be the same as the distribution of the undoubtedly minimum energy release.

If the chemical potentials (the energy to add/remove of one lower-level particle from a higher-level particle) are constant, the probability density of the distributions are determined by the T_{eff} value only, and when it decreases (under the natural cooling down of the particle system), they become the same in shape, which will be interpreted as stabilization of the matter. In different energy ranges, different particles turn out to be the most probable, this will be discussed in Section 5.

It also turns out that a number of the distributions obtained particles that lack negative energy states that may be interpreted as the absence of antiparticles.

2. To the connection of particle statistics with the axioms of modern physics

In addition to the simple Boltzmann statistics of indistinguishable particles, others arise in quantum mechanics (in its section 'quantum statistics') — Fermi–Dirac and Bose–Einstein statistics. Simple relations for the number of particles in the corresponding energy state are obtained under the assumption of Fermi–Dirac statistics in the case that there can be no more than one particle in this energy state, under the assumption of Bose–Einstein statistics — that any number of (indistinguishable) particles can be in this energy state. The derivation of this fact is primitive enough [1, 2].

Thus, for particles obeying Fermi–Dirac statistics (fermions), the Pauli exclusion principle holds *a priori*. W. Pauli had started from spectroscopic experiments in a magnetic field, and formulated his principle as follows [5]: "There cannot be two or more equivalent electrons in an atom that have the values of all four quantum numbers the same. If there is an electron in an atom that has a certain value for all these numbers, then this state is 'occupied'." The hypothesis of the presence of spin in an electron, which appeared and was quickly proved [6], required W. Pauli to provide a physical justification for this ban, which he had done in his work [7] for the connection of spin with statistics. In this theorem, W. Pauli obtained the classic result that particle field operators with integer spin cannot obey fermionic commutation relations, since this leads to a violation of the causality principle. The field operators of particles with half-integer spin cannot obey bosonic switching relations, as this leads to negative values of the total energy of the system. Hence, it was concluded that particles with an integer spin obey Bose–Einstein statistics, while particles with a half-integer obey Fermi–Dirac statistics.

Further, the consideration of the problem particle statistics took place definitely with respect to the spin. The topic was surely not closed in [7], and its discussion continues to this day [8] due to its comprehensive nature. There were attempts to derive the Pauli principle [9], but the most interesting attempts have been to introduce 'attractive forces' for Bose–Einstein particles (bosons) and 'repulsive forces' for fermions [10] for clear classical explanation of both the Pauli exclusion and the 'condensation' of bosons in the lowest energy states. Alternatively, an additional 'pressure' of fermions was introduced [11], which turned out to be different from the pressure of bosons (although the Wentzel–Kramers–Brillouin method was used for necessary quantum calculations, the convergence of this method to the real result still needs to be proved). The relationship between quantum probabilities and statistical probabilities for different statistics was also investigated [12].

There have been attempts to calculate their thermodynamic values through statistics for more complex structured and/or hierarchically organized systems. For example, the so-called 'soft matter' is such a structured medium [13]. Various properties (not only energy distributions) can arise due to the complex internal structure of particles. There are systems with two levels (or layers) of hierarchy that are studied both experimentally and theoretically. One of them is an example of a soft matter — concentrated colloids of star-shaped polymer mixtures [14, 15]. In these media, the interaction potential between star-shaped and linear polymers has the character of a double exponential dependence due to the two-level structure of the core and shell in star-shaped polymers. The interaction potential of linear particles and 'bottle brush' particles [16] has the same character.

Another example of a similar two-layer system is considered in [17]. There are alternative examples of problems leading to the appearance of 'multilayer' hierarchical systems with the implementation of the principle of exclusion of states, for example, in chemistry [18]. Thus, it becomes necessary to describe the above systems in order to generalize the known approaches of statistical physics.

3. The combinatorial problem of determining the most probable energy distributions of particles delimited inside other particles

3.1 The energy distribution of lower-level particles delimited in upper-level particles is a two-level problem

The combinatorial formulation of problem 1 was first considered in [19]. So, let's have a set of $\{m_i\}$ upper-level particles such that m_1 indistinguishable particles contain p_1 lower-level particles,[1] m_2 contains p_2, etc. up to the N-th lower-level particle. Here n is the total number of different arrangements of lower-level particles by upper-level particles. The total number of N lower-level particles

$$N = \sum_{i=1}^{n} m_i p_i \tag{1}$$

are distributed over M upper-level particles

$$M = \sum_{i=1}^{n} m_i \,. \tag{2}$$

In this case, the total number of permutations of all lower-level particles over all upper-level particles is W_{10} [19]:

$$W_{10} = \frac{N!}{\prod_{i=1}^{n} (p_i!)^{m_i}} \,, \tag{3a}$$

and under the condition of the mentioned indistinguishability of upper-level particles

$$W_1' = \frac{N!}{\prod_{i=1}^{n} (p_i!)^{m_i} \prod_{i=1}^{n} (m_i!)} \,. \tag{3b}$$

Typically, there use the expression of $\ln W_1'$:

$$\ln W_1' = \ln (N!) - \sum_{i=1}^{n} m_i \ln (p_i!) - \sum_{i=1}^{n} \ln (m_i!) \,. \tag{4a}$$

Note that if we put $m_i \equiv 1$ here, we will have a formulation to get the most probable distribution that determined by conditions (1), (2), (3b) similar to the formulation of problem 0.[2]

Let each particle of the lower level be in a state with some energy ϵ_i. To find the most probable distribution over these energies for an arbitrary m_i, it should be taken into account that, in addition to expressions (3b) or (4a) and relations (1) and (2), there are also conditions for the conservation of the total energy of the entire particle system

$$E = \sum_{i=1}^{n} p_i m_i \epsilon_i \,, \tag{5}$$

which practically means that there is an average 'energy' of one particle of the lower level $\langle \epsilon_i \rangle$.

Let's solve problem 1. To do this, we first form a variation of expression (4a) taking into account conditions (1), (2), and (5). Then, we add the terms $\alpha N + \beta M + \gamma E$ to the right-hand side of equation (4a), with Lagrangian multipliers α, β and γ:

$$\ln W_1 = \ln W_1' + \alpha N + \beta M + \gamma E = \ln (N!) - \sum_{i=1}^{n} m_i \ln (p_i!) - \sum_{i=1}^{n} \ln (m_i!) + \alpha \sum_{i=1}^{n} m_i p_i + \beta \sum_{i=1}^{n} m_i + \gamma \sum_{i=1}^{n} p_i m_i \epsilon_i \,. \tag{4b}$$

We also use the first approximation of the Stirling formula for $m!$ and $p!$, and vary δp_i and δm_i. Then we obtain the variation of the most probable distribution $\ln W_1$:

$$\delta \ln W_1 = \sum_{i=1}^{n} \big[-(\delta m_i) p_i (\ln p_i - 1) - (\delta p_i) m_i \ln p_i - (\delta m_i) \ln m_i + \alpha (\delta m_i) p_i + \alpha m_i (\delta p_i) + \beta (\delta m_i) + \gamma (\delta m_i) p_i \epsilon_i + \gamma m_i (\delta p_i) \epsilon_i \big] = 0 \,. \tag{6}$$

[1] If you read 'bundles' instead of 'upper-level particles,' you read 'mushrooms' instead of 'lower-level particles,' then you arrive to the formulation of the problem of drying mushrooms [20], or the problem of cyclic permutations [21].

[2] If N is allowed to vary in the problem to obtain the large canonical ensemble, it can be reduced to problem 1. This was not done (see, for example, [1]). Note that formula (3b) was discussed both in [1] and in the well-known textbook [22]. Why this was done this way (and the plain further steps were not taken) is unclear.

Collect the terms with independent variations δp_i and δm_i. For the first variation on δp_i by reducing all terms to nonzero values δp_i and m_i, we get the first equation of the system:

$$\ln p_i = \alpha + \gamma\epsilon_i\,. \tag{7}$$

For the second variation on δm_i, after reducing to a nonzero value δm_i and using equation (7), we obtain:

$$\ln m_i = \beta + p_i\,. \tag{8}$$

Equations (7) and (8) give the solution of problem 1.

The solution of equation (7) corresponds exactly to the usual Boltzmann–Gibbs representation (see Introduction, [1, 2]) and to the solution of problem 0. The differences from the solution of problem $0'$ are that the variation over the number of upper-level particles is not carried out in problem $0'$. The solutions of the system (7) and (8) are clear:

$$p_i = \exp\left(\alpha + \gamma\epsilon_i\right) \tag{9}$$

and, in turn,

$$m_i = \exp\left(\beta + \exp\left(\alpha + \gamma\epsilon_i\right)\right). \tag{10}$$

Thus, the genuine function of the most probable distribution of lower-level particles over energy p_i turns out to be exactly the same as the most probable distribution function in problem 0 — this is the probability density of finding lower-level particles in the energy state ϵ_i.

However, this function is not directly observable and directly measurable. In order to measure this distribution function, it is necessary to have a device that measures p_i as a function of ϵ_i. It is impossible to do in a single upper-level particle due to the indistinguishability of these m_i particles.

What functions will be observable? To find out, we rewrite the system (7) and (8) by forming the simplest linear combinations of the equations:

$$\ln p_i \pm \ln m_i = \alpha \pm \beta + \gamma\epsilon_i \pm p_i = \alpha \pm \beta + \gamma\epsilon_i \pm \exp\left(\alpha + \gamma\epsilon_i\right), \tag{11}$$

where we used the expression (10). Thus, we should consider two functions, $p_i m_i$ and p_i/m_i, that are the solutions of the system (7) and (8).

For further analysis, we need to determine the constants in (7) and (8). Let's define the constant β in problem 1. The value of m_i changes from a positive value of $\exp\left(\beta + \exp\left(\alpha\right)\right)$ when $\epsilon_i = 0$ to $\exp\left(\beta\right)$ when $\epsilon_i \to \infty$. When is large ϵ_i, due to clear negativity of γ, one finds

$$\exp\left(\exp\left(\alpha + \gamma\epsilon_i\right)\right) \approx 1 + \exp\left(\alpha + \gamma\epsilon_i\right).$$

According to the physical meaning of the problem, there cannot be many upper-level particles containing such an energetic lower-level particle at such high energies of the lower-level particle. Thus, in the first consideration of problem 1, we can assume that $\beta = 0$ when $\epsilon_i \to \infty$, and m_i tends to 1 when $\epsilon_i \to \infty$, i.e., as already mentioned, $\beta = 0$.

It is clear that the $p_i m_i$ function should potentially be observable, because it describes the distribution of particle numbers (1). This function differs significantly from p_i (Fig. 1,

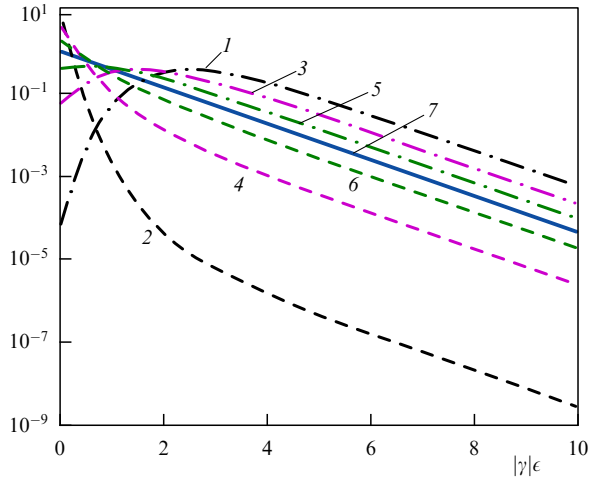

Figure 1. Normalized distribution functions $p_i m_i$ (dashed curves *2*, *4*, and *6*) and p_i/m_i (dash-dotted curves *1*, *3*, and *5*) for different parameters α. For curves *1* and *2*, $\alpha = 2.5$, for curves *3* and *4*, $\alpha = 1.5$, for curves *5* and *6*, $\alpha = 0.5$, and $p_i = |\gamma| \exp\left(\gamma\epsilon_i\right)$ (solid line *7*). It is assumed that $|\gamma| = 1$. Semi-logarithmic scale.

definitions of functions 1–7 in this figure see below):

$$p_i m_i = \exp\left(\alpha + \gamma\epsilon_i\right) \exp\left(\exp\left(\alpha + \gamma\epsilon_i\right)\right). \tag{12}$$

The normalized on 1 function of this distribution, which describes a some probability density

$$\left(p_i m_i\right)_{\text{norm}} = |\gamma| \frac{\exp\left(\alpha + \gamma\epsilon_i\right) \exp\left(\exp\left(\alpha + \gamma\epsilon_i\right)\right)}{\exp\left(\exp\left(\alpha\right)\right) - 1}\,, \tag{12a}$$

is shown on Fig. 1 for $\alpha = 0.5$, 1.5 and 2.5 by dashed curves. As α increases, i.e., as the temperature decreases while $\alpha/|\gamma|$ remains constant, the value of the function (12a) at zero increases sharply compared to line *7*, and the asymptotic value at moderate and large ϵ_i decreases very significantly (see Fig. 1). Thus, the probability density $p_i m_i$ (or the normalized probability density $\left(p_i m_i\right)_{\text{norm}}$ describes the condensation of lower-level particles at low energies, which becomes very significant at low temperatures. The observability of this function is best realized, apparently, in multiparticle effects like phase transitions, when 'all' particles (of the upper level) participate in the process. Here the situation does not depend qualitatively on the value of α.

The second function, which is a solution of (11), is the p_i/m_i function, which is the distribution of p_i per one upper-level particle from the m_i set, it is presented on Fig. 1 also. The conditions for observing this function are clearly different from those for $p_i m_i$. It looks that single-particle measurements of energy spectra (in a wide sense of spectroscopic observations) would be the most physically meaningful.

Consider the distribution of lower-level particles by energy, averaged over one upper-level particle, p_i/m_i:

$$\frac{p_i}{m_i} = \exp\left(\alpha + \gamma\epsilon_i\right) \exp\left(-\exp\left(\alpha + \gamma\epsilon_i\right)\right). \tag{13}$$

It is clear that $|\gamma|$ has the meaning of a certain 'inverse effective temperature' of the system; the normalization to 1 of the distribution (13) gives a factor in front of (13) $|\gamma|/(1 - \exp\left(-\exp\left(\alpha\right)\right)$, which is practically equal to $|\gamma|$ for

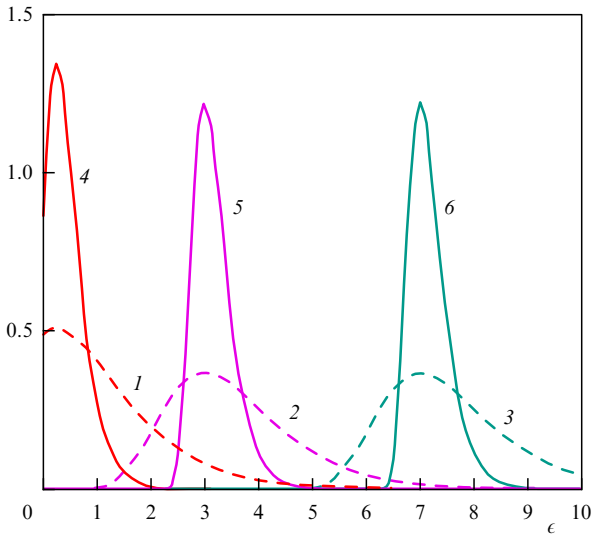

Figure 2. Normalized to 1 probability density distribution functions $(p_i/m_i)_{\text{norm}}$ over the ϵ_i (13a) for $\alpha = 0.5$, 3, 7. $|\gamma| = 0.3$ for the dashed curves *1–3*, and $|\gamma| = 1$ for the solid curves *4–6*. Uniform scale.

moderate and especially large α. The function (13) arrives to a maximum at $\epsilon_i = \alpha/|\gamma|$. Since α is a Lagrange multiplier for the total number of particles (1), the physical meaning of α (or more precisely, $\alpha/|\gamma|$) may be that it is the chemical potential of the lower-level particles μ, with $\alpha = |\gamma|\mu$. Note that the probability density (13) coincides with the probability density of the Gumbel distribution (type I) [23].

We present the probability density distribution normalized to 1, p_i/m_i for a convenience of calculations:

$$\left(\frac{p_i}{m_i}\right)_{\text{norm}} = \frac{|\gamma|}{1-\exp\left(-\exp\left(\alpha\right)\right)}\,\frac{\exp\left(\alpha+\gamma\epsilon_i\right)}{\exp\left(\exp\left(\alpha+\gamma\epsilon_i\right)\right)}\,. \tag{13a}$$

Figure 2 shows the function (13a) for several values of α with different effective temperatures $1/|\gamma|$. When α increases (curves *4* to 6, respectively), the normalization factor in (13a) gradually approaches to $|\gamma|$. For small effective temperatures (large $|\gamma|$), this occurs practically immediately, and the distribution $(p_i/m_i)_{\text{norm}}$ is simply translated along the X-axis without any changes. Figure 2 shows a significant 'narrowing' of the maxima (13a) as $|\gamma|$ increases, i.e., as the effective temperature decreases. The pronounced maximum of the distribution coincides with the value of the chemical potential at $\epsilon_i = \mu = \alpha/|\gamma|$. This suggests that when measuring are doing on the 'single particle' of the upper level, the lower-level particles can only attach to (or detach from) the upper-level particle in energy states close to μ. As the temperature decreases, the width of the distribution function decreases, and at low temperatures, the distribution function can become so narrow that only one lower-level particle can fit into this state.

Thus, a two-stage hierarchical system of particles, where the lower-level particles are delimited in the upper-level particles, breaks down into two types of lower-level particles in the combinatorial analysis. It looks that these types have to be observed in different experiments, focusing on collective and single-particle behavior. The first lower-level particles may condense into states with low energies. The second is to occupy states characterized by the energy of attachment (or detachment) to (from) the system (systems) of these particles. In a naive approximation, these are states near the chemical potential of the lower-level particles inside these 'upper' particles, and only these states can be occupied due to the narrowness of the energy distribution density function.

3.2 The energy distribution of lower-level particles delimited in the intermediate-level particles delimited in the upper-level particles: a three-level problem

Let's consider problem 2 in the combinatorial formulation [19]. As it was mentioned in the Introduction, we can propose the problem of energy distribution of lower-level particles contained in a set of middle-level particles, which in turn are contained in a set of upper-level particles. At all three levels, the particles are indistinguishable.[3] Let's denote the number of lower-level particles contained in a middle-level particle by p_i, the number of middle-level particles with this number of lower-level particles in the middle-level particles by m_i, and the number of upper-level particles that contain m_i middle-level particles by t_i. In this case, the total number of combinations of lower-level particle distributions among all middle-level particles, and the latter among upper-level particles, is given by

$$W_2' = \frac{N!}{\prod_{i=1}^{n}(p_i!)^{m_i t_i}\prod_{i=1}^{n}(m_i!)^{t_i}\prod_{i=1}^{n}(t_i!)}\,. \tag{14}$$

Here, n is again the total number of different arrangements of lower-level particles into middle-level particles, and those into upper-level particles. Naturally, the total number of lower-level particles is now determined as

$$N = \sum_{i=1}^{n} m_i p_i t_i \tag{15}$$

the total number of middle-level particles as

$$M = \sum_{i=1}^{n} m_i t_i \tag{16}$$

and the total number of upper-level particles as

$$T = \sum_{i=1}^{n} t_i\,. \tag{17}$$

The expression for the total energy is now

$$E = \sum_{i=1}^{n} p_i m_i t_i \epsilon_i\,. \tag{18}$$

According to the general rule, we form a logarithm of this number of arrangements

$$\ln W_2' = \ln(N!) - \sum_{i=1}^{n} m_i t_i \ln(p_i!) - \sum_{i=1}^{n} t_i \ln(m_i!) - \sum_{i=1}^{n} \ln(t_i!). \tag{19}$$

Accordingly, the expression (19) should vary with the superimposed connections (15)–(18) and the required Lagrange

[3] In the language of the generalized 'mushroom drying problem,' this would be a count of the distribution of lengths of mushrooms bunches, which in turn are combined into bundles of different sizes. Not only the bunches, but also the bundles themselves, not to mention the mushrooms themselves, are now indistinguishable.

multipliers α, β, γ, Δ, similar to (4b):

$$\ln W_2 = \ln W_2' + \alpha N + \beta M + \gamma E + \Delta T . \tag{20}$$

The variables of variation are now p_i, m_i, and t_i. By varying p_i, we obtain the first equation, which exactly coincides with equation (7) (and the solution of problem 0):

$$\ln p_i = \alpha + \gamma\epsilon_i . \tag{21}$$

Varying (20) with respect to m_i and using (21), we obtain an equation that exactly coincides with equation (8):

$$\ln m_i = \beta + p_i . \tag{22}$$

Finally, by varying (20) with respect to t_i and using (21) and (22), we have the third equation of problem 2:

$$\ln t_i = \Delta + m_i . \tag{23}$$

Let us make some remarks about the values of β and Δ. The latter enters in all expressions as a simple factor, i.e., it is needed only in the case of some special normalization, in the first approximation it can be considered equal to zero, although it would be natural to have $\Delta = -1$, then t_i would tend to 1 as $\epsilon_i \to \infty$ (see above). The value of β in problem 2 can be considered coinciding with β from problem 1 at the beginning, i.e., also equal to zero. The case of a nonzero value of β will be considered below.

Let us create, as in problem 1, the simplest linear combination of equations (21)–(23):

$$\ln p_i \pm \ln m_i \pm \ln t_i = \alpha \pm \beta \pm \Delta + \gamma\epsilon_i \pm p_i \pm m_i$$

and use solutions (22) and (23). Then

$$\ln p_i \pm \ln m_i \pm \ln t_i = \alpha \pm \beta \pm \Delta + \gamma\epsilon_i \pm \exp(\alpha + \gamma\epsilon_i) \pm \exp\bigl(\beta + \exp(\alpha + \gamma\epsilon_i)\bigr) . \tag{24}$$

By analogy with problem 1, the solutions (24) and distribution functions are now the expressions $p_i m_i t_i$, $p_i t_i / m_i$, $p_i / m_i t_i$, and $p_i m_i / t_i$. The first function describes the considered multiparticle measurements with the effect of particle condensation at low energies. It is even stronger than the analogous condensation in problem 1 (Fig. 3). The second function also describes the effect of condensation of lower-level particles, but it is less pronounced than for the first function — compare curves *3* and *4* with curves *1* and *2* in Fig. 3. The third function describes a function measured for one middle-level particle and one upper-level particle, which has a pronounced maximum. Now, however, this maximum does not coincide with the value of α even at zero β, but exceeds it: for $p_i / m_i t_i$, approximately on 1 under the conditions of Fig. 3 (corresponding to approximately one unit of effective temperature), see curves *5* and *6*.

The last function, $p_i m_i / t_i$, is the probability density of the distribution $\exp(\exp(\beta))/t_i$, if the argument of the latter, the energy, is extended to the entire negative half-axis. The normalized distribution of this probability density for positive values of the argument is

$$\left(\frac{p_i m_i}{t_i}\right)_{\text{norm}} = |\gamma| \frac{\exp(\alpha+\gamma\epsilon_i)\exp\bigl(\beta+\exp(\alpha+\gamma\epsilon_i)\bigr)\exp\bigl(-\exp(\beta+\exp(\alpha+\gamma\epsilon_i))\bigr)}{\exp\bigl(-\exp(\beta)\bigr) - \exp\bigl(-\exp(\beta+\exp(\alpha))\bigr)} . \tag{25}$$

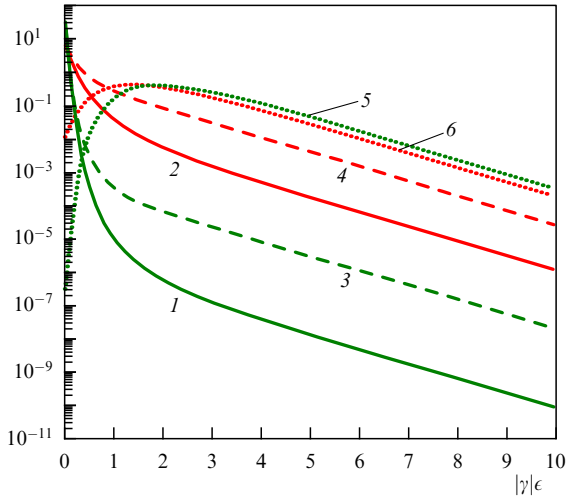

Figure 3. Normalized to 1 functions of ϵ_i probability density distribution $p_i m_i t_i$ (curve *1* for $\alpha = 1$ and curve *2* for $\alpha = 0.5$), $p_i t_i / m_i$ (curve *3* for $\alpha = 1$ and curve *4* for $\alpha = 0.5$), $p_i / m_i t_i$ (curve *5* for $\alpha = 1$ and curve *6* for $\alpha = 0.5$). Semi-logarithmic scale, $|\gamma|$ is equal 1.

As in the case of problem 1, the function (25) is a certain 'average' function measured on a single top-level particle.

Figure 4 shows the behavior of the normalized probability density distribution (25) as a function of the Lagrangian multipliers α and β. The maximum of the function $p_i m_i / t_i$ for $\alpha \geqslant 1$ coincides with the value $|\gamma|\epsilon_i = \alpha + \beta$, so the value β can be 'naively' interpreted as the chemical potential of the intermediate-level particle in the upper-level particle (for α, the interpretation follows from problem 1). With this interpretation, it is clear that the total chemical potential of the lower-level particle in the upper-level particle in problem 2 is the sum of the two chemical potentials of the lower-level particle in the intermediate-level particle and the intermediate-level particle in the upper-level particle. This fact is immediately visible from Fig. 4 — the maxima of the probability densities of curves *2*, *3*, *5*, *6*, *8*, and *9* correspond to the sum $\alpha + \beta$.

For $0 \leqslant \alpha < 1$, the maximum of the function $p_i m_i / t_i$ is only slightly greater than $\alpha + \beta$, see Fig. 4, curves *1*, *4*, and *7*. For the remaining curves, this coincidence is already accurate to $\exp(-2(\alpha + \beta))/2$. It also follows from Fig. 4 that different chemical potentials (proportional to α and β) that give the same sum determine probability densities that are very similar to each other (curves *3* and *5*, as well as *6* and *8* in Fig. 4).

We can obtain only one of the four probability density functions, $p_i m_i / t_i$, from the probability distribution function $\exp(\exp(\beta))/t_i$, which we will discuss in the next section.

Thus, problem 2, which describes a three-level hierarchical system of particles, splits the behavior of lower-level particles into four types. Among them two types describe particles that condense into states with low energy values. One type of particles, described by the probability density function $p_i m_i / t_i$, is very similar to the type of particles described by the p_i / m_i, function in problem 1. In a naive representation, this type (as well as its counterpart in problem 1) may produce 'normal' bound particles.

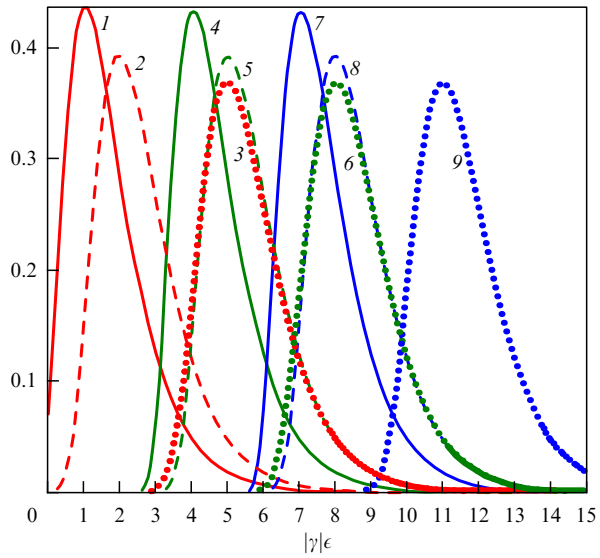

Figure 4. Normalized by 1 functions of ϵ_i the probability density of the distribution $(p_i m_i/t_i)_{\text{norm}}$ (25). Pairs of values $\alpha = 0$, $\beta = 1$; $\alpha = 3$, $\beta = 1$; $\alpha = 6$, $\beta = 1$, are for solid curves *1*, *4* and *7* respectively. Pairs of values $\alpha = 0$, $\beta = 2$; $\alpha = 3$, $\beta = 2$; $\alpha = 6$, $\beta = 2$, are for dashed curves *2*, *5*, and *8* respectively. Pairs of values $\alpha = 0$, $\beta = 5$; $\alpha = 3$, $\beta = 5$; $\alpha = 6$, $\beta = 5$, are for point curves *3*, *6*, and *9* respectively. For all curves, $|\gamma| = 1$. Uniform scale.

4. The problem to determine energy distributions of the minimum and maximum values

4.1 Distributions of maximum values in a two-level problem

Problem 1 allows for a different formulation. In it, the results obtained, firstly, coincide with those demonstrated above, and, secondly, clarify the meaning of the obtained former results [24].

Let's assume that problem 0 is now generalized in a way that there is a probability density of the energy distribution p_i of the lower-level particles inside each upper-level particle (or the distribution inside the box, or the distribution of bundles of mushrooms along the length) from the set m_i. Now considering this set of m_i as a *sample*, we determine the distribution function (and then the probability density) of the maximum value of p_i for all samples of m_i. We will follow the procedure [24].

Let's denote the variable of the probability density function of the distribution p_i as $|\gamma|\epsilon_i = x$, the value of x varies from 0 to $+\infty$.[4] This distribution function $F(x)$ is, of course, $F(x) = 1 - \exp(-x)$. The probability to find the maximum value in the m_i sample is determined by the expression [24]

$$m_i\left[1 - F(x_{i\text{m}})\right] = 1\,, \tag{26}$$

where $x_{i\text{m}}$ is the characteristic maximum value. It follows from (26) that

$$x_{i\text{m}} = \ln m_i \quad \text{and/or} \quad \frac{\exp(x_{i\text{m}})}{m_i} = 1\,. \tag{27}$$

The distribution function $F_{\text{m}}(x)$ of the largest value in the entire m_i samples will be surely the function $F(x)$ raised to the power of m_i [24]. Then, using the second relation (27) and the limit transition, we obtain

$$F_{\text{m}}(x) = \left(1 - \frac{\exp\left(-(x - x_{i\text{m}})\right)}{m_i}\right)^{m_i} \underset{m_i \to \infty}{\longrightarrow} \exp\left(-\exp\left(-(x - x_{i\text{m}})\right)\right). \tag{28}$$

Differentiating the limit distribution function (28), we obtain the probability density p_i/m_i, where $x_{i\text{m}} = \alpha$ that corresponds to the normalization of the Boltzmann–Gibbs distribution

$$m_i \exp(-x) = \exp\left(-(x - \ln m_i)\right)$$

on the number of upper-level particles m_i.

Thus, the problem of determining the probability density of the distribution of the maximum values of functions p_i over samples $\{m_i\}$ in a two-level problem gives exactly the same result as problem 1 of determining the most probable probability density p_i/m_i. The Lagrangian constant α in the latter coincides with the value $x_{i\text{m}} = \ln m_i$, which allows us to make some physical generalizations.

The coincidence of the chemical potential α normalized on the effective temperature from the combinatorial problem with the characteristic mean [23, 24] of the maximum value distribution problem limits the value of m_i, showing the 'sphere of interaction' of upper-level particles, i.e., the sphere outside of which upper-level particles from this sphere no longer interact. This contains the beginnings of dividing interactions into short- and long-range ones, apparently, there is no long-range interaction at high (initial) T_{eff} temperatures.

4.2 Distributions of 'true' minimum values in a two-level problem

Let us now consider the $p_i m_i$ function in a similar way. The m_i function, unlike $1/m_i$, is not a distribution function of any random variable.

We assume that the combinatorial problem is now generalized in such a way that there is a probability density of the distribution of groups p_i of lower-level particles inside each body (upper-level particles) from the set m_i. Now considering this set of m_i as a sample, we determine the distribution function (and then the probability density) of the minimum value of p_i for all samples of m_i. We will follow the procedure [23, 24].

Consider the $p_i m_i$ function. It is known [23] that the probability density function of the distribution of the minimum values of p_i in the m_i sample (or the energy distribution of lower-level particles in upper-level particles) is $m_i \exp(m_i(\alpha - x))$ (we introduced the variable $|\gamma|\epsilon_i = x$).

Let's construct a new probability density function according to the following scheme: first, we take the above distribution of the minimum value, it can be realized in any specific case over the entire m_i sample (in any of the upper-level particles). Since we don't care in which realization this

[4] Note that the transfer of the 'ground state' of the probability density p_i from zero to the negative value $-\epsilon_0$ leads only to a simple shift of the variable ϵ_i, which now changes from $-\epsilon_0$ to $+\infty$. Probability density obtained in problem 1, can now be written as $|\gamma|(\pm\exp(\exp(\pm\alpha)) \mp 1)^{-1} \times \exp(\alpha + \gamma(\epsilon_i + \epsilon_0)) \exp(\pm\exp(\alpha + \gamma(\epsilon_i + \epsilon_0)))$ for $(p_i m_i)_{\text{norm}}$ (upper sign) and $(p_i/m_i)_{\text{norm}}$ (lower sign), respectively. The values of α can also take negative values here, which, in comparison with those discussed in the previous section, are simply shifted by $-\epsilon_0$ to obtain the same probability density functions. All the results obtained below remain the same.

happens (in which upper-level particle it will be), we will take this probability density with the statistical weight of combinatorial permutations with repetition, i.e., with $1/(m_i!)$. This will give the term, $[(m_i-1)!]^{-1}\exp(m_i(\alpha-x)) = \exp(\alpha-x)[(m_i-1)!]^{-1}\exp((m_i-1)(\alpha-x))$, and it will be the first 'participant' of our future sum, the final form of some probability density function.

In m_i-1 realizations incomplete per one body (from upper-level particles), the probability density of the minimum value distribution will be $(m_i-1)\times\exp((m_i-1)(\alpha-x))$. We take this function with a weight, for the same reasons, $1/[(m_i-1)!]$. We get a term of the future sum and a contribution to some probability density function $[(m_i-2)!]^{-1}\exp((m_i-1)(\alpha-x)) = \exp(\alpha-x)[(m_i-2)!]^{-1}\times\exp((m_i-2)(\alpha-x))$, and we will do this procedure a total of m_i-1 times. All obtained probability density functions under the limiting transition

$$\exp(\alpha-x)\left[(m_i-k')!\right]^{-1}\ \exp\left((m_i-k')(\alpha-x)\right)\ \underset{m_i-k'=k}{\longrightarrow}\ \exp(\alpha-x)(k!)^{-1}\exp\left(k(\alpha-x)\right)$$

give in the sum a some required probability density function

$$f(x) = \lim_{m_i\to\infty}\left[\exp(\alpha-x)\sum_{k=0}^{m_i-1}\frac{\exp\left(k(\alpha-x)\right)}{k!}\right] \to \exp(\alpha-x)\exp\left(\exp(\alpha-x)\right). \tag{29}$$

The p_im_i function exactly coincides with $f(x)$ at $\alpha=0$ (Fig. 5). In practice, this function $f(x)$ represents the probability density of the minimum values of p_i obtained in each of the realizations of the m_i sample: the energy of the lower-level particle takes on a minimum value in the upper-level particle, provided that in other realizations, the minimum value is also realized. It looks that this is direct evidence of the fact that $f(x)$ (or p_im_i) describes some collective behavior of upper-level particles, as well as lower-level particles in the mentioned upper-level particles.

Note the important difference between $f(x)$ and p_i/m_i. The latter describes the probability distribution density for a 'normal' lower-level particle to have a certain energy, which is the maximum in a set of upper-level particles. In each of the upper-level particles (in each realization from the sample), the result — the distribution of the maximum value — is the same, although, of course, it is determined by the entire sample. For the function $f(x)$ (or, generally speaking, for the p_i/m_i function), the result for each implementation is fundamentally different. This may mean that the system of upper-level particles with such lower-level particles is some kind of collective system, where the lower-level particles in the upper-level particle somehow 'feel' each other. Thus, in this case, the separation also determines some kind of (collective?) interactions between these particles. This is a stronger statement than that required to describe 'normal' particles given by the p_i/m_i function. Here, the separation leads only to the observation of maximum values of the p_i distribution. Note that in the two-layer problem of determining the 'true' minimum values, it is impossible to introduce a new statistical characteristic such as the characteristic average of the previous problem. This proves in favor of the distribution of 'true' minimum values without the appearance of new constants in the problem.

The calculation of probability densities of lower-level particles energy distributions from a sample of upper-level particles leads to exactly the same result as the combinatorial calculation gives. From a statistical mechanics point of view, the most probable (observable) states can be either the states with the maximum energy or with the absolutely (see above) minimum energy of the lower-level particles.

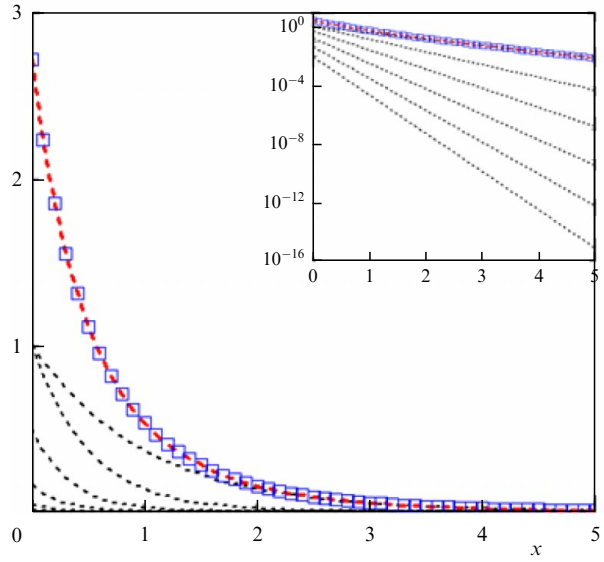

Figure 5. Function $\exp(-x)\exp(\exp(-x))$ (the curve of open squares) in comparison with the sum of functions $\exp(-x)\sum_{k=0}^{5}\exp(-kx)/k!$ (the bold upper dashed curve). The six lower dashed curves are functions $\exp(-x)\exp(-kx)/k!$, where k varies from 0 to 5. Main figure, uniform coordinates. The inset shows the same in a semi-logarithmic scale.

4.3 Distributions of extremes in a three-level problem

Similarly to problem 1, problem 2 can be solved by defining the probability distribution of maximum and minimum values. The corresponding calculations of the probability distribution functions of the maximum and minimum values, which would serve as a confirmation and generalization of combinatorial calculations, can be performed in the case of problem 2 only for the functions p_im_i/t_i and $p_im_it_i$. Indeed, let us consider how the procedure of Section 3 can be applied to the lowest-level particles, which are delimited in the middle-level particles from the m_i sample, which, in turn, are delimited in the highest-level particles from the t_i set. Otherwise, we look for the distribution of p_i in the boxes of the number of m_i, which are located in other boxes in quantity t_i, or the distribution m_i bunches of mushrooms collected in t_i bundles.

We need to determine the distribution functions for the four probability density functions of problem 2. Then we need to form the distribution function of the maximum and minimum values and proceed to the final probability distribution of particles in definite energy states of the above mentioned hierarchical systems. The distribution function for the probability density $(p_im_i)_{\text{norm}}$ is

$$\Phi_{10} = \frac{1-\exp\left(\exp\left((\alpha+\gamma\epsilon_i)-\exp(\alpha)\right)\right)}{1-\exp\left(-\exp(\alpha)\right)}. \tag{30}$$

To simplify the following calculations, since at $\alpha\geqslant 2$ Φ_{10} changes to Φ_1 with good accuracy:

$$\Phi_{10}\cong\Phi_1 = 1-\exp\left(\exp(\alpha+\gamma\epsilon_i)-\exp(\alpha)\right). \tag{30a}$$

In the same approximation, the distribution function Φ_2 for the probability density $(p_i/m_i)_{\text{norm}}$ is

$$\Phi_2 = \exp\left(-\exp\left(\alpha+\gamma\epsilon_i\right)\right) \sim \frac{1}{m_i}\,. \tag{30b}$$

Recall that if F is the distribution function of a random variable, then the distribution function of its maximum value in the sample t_i is F^{t_i}, while the distribution function of the minimum value is $1-(1-F)^{t_i}$. Therefore, it is clear what distribution functions from the two functions Φ_1 and Φ_2 are obvious to use to form the distribution of maximum values: this is Φ_1. In addition, we will use the equation for the maximum characteristic value, as it was done in Section 4.1. Then the distribution function of the maximum values of Φ_1 over the sample t_i has the form

$$\Phi_{\max} = \exp\left(\exp\left(\beta\right) - \left(\exp\left(\beta+\exp\left(\alpha+\gamma\epsilon_i\right)\right)\right)\right), \tag{31}$$

where $\beta = \ln(t_i) - \exp(\alpha)$. It is easy to see that the probability density function for $\Phi_{\max}$ (it derivative over ϵ_i) coincides with the function $p_i m_i/t_i$.

This result is physically quite extraordinary. Indeed, we have determined the distribution of maximum values for the 'three-layer' probability distribution function: the particles of the lower level should be in a state of energy ϵ_i and contained in m_i particles of the middle level, which, in turn, are contained in t_i particles of the upper level. But the 'three-layer' distribution function of the maximum values is obtained from the 'two-layer' distribution functions of the minimum values. In a naive view, it looks like pseudobosons add up to a pseudofermion.

To determine the distribution of the 'true' minimum values of $\Phi_{\min}$, and in this case the distribution is for the probability density function $p_i m_i t_i$, one needs to do the same trick as in Section 3. We use Φ_1 again, i.e., the 'two-layer' distribution of minimum values. After doing the above trick, we find

$$\frac{\partial\Phi_{\min}}{\gamma\partial\epsilon_i} = \exp\left(\alpha+\gamma\epsilon_i\right)\exp\left(\beta+\exp\left(\alpha+\gamma\epsilon_i\right)\right) \times \exp\left(\exp\left(\beta+\exp\left(\alpha+\gamma\epsilon_i\right)\right)\right) = p_i m_i t_i$$

that is the nonnormalized $p_i m_i t_i$ function from Section 3 with $\beta = -\exp(\alpha)$ as in expression (31).

Thus, the application of the distribution function Φ_1 to determine the distribution of both extremes gives the same functions that were obtained in the combinatorial approaches of Section 3. At first glance, it turns out that 'three-layered' particles, both pseudobozones with a probability distribution density of $p_i m_i t_i$ and pseudofermions with a probability distribution density of $p_i m_i/t_i$, can be 'constructed' only from 'two-layer' pseudobosons, since the probability distribution Φ_2 has not yet been used.

It would seem that obtaining the distribution of the minimum value using the function Φ_2 is elementary. Really,

$$\Phi^r_{\min} = 1-(1-\Phi_2)^{t_i} = 1-\left(1-\frac{\Phi_2\exp\left(\ln t_i\right)}{t_i}\right)^{t_i} \to 1-\exp\left(\exp\left(\ln t_i\right)-\exp\left(-\exp\left(\alpha+\gamma\epsilon_i\right)\right)\right) \tag{32}$$

and this $\Phi^r_{\min}$ is the usual distribution of the minimum value for the distribution of Φ_2 over the sample t_i. However, the probability density of this distribution does not coincide with any of the combinatorial probability density functions from Section 3. Similarly, by doing the trick of determining the true minimum value of Φ_2, we will not arrive to any combinatorial functions from Section 3. Does it mean that 'two-layered' pseudofermions cannot be folded into either 'three-layered' pseudobosons or three-layered pseudofermions, as 'two-layered' pseudobosons 'successfully' do?

5. Applications

5.1 Proof (origin) of the Pauli principle

It was shown that the calculation of the probability densities of energy distributions of lower-level particles based on a sample of upper-level particles in two-level problem 1 leads to exactly the same result as the combinatorial calculation. From a statistical and mechanical point of view, the most probable (observable) states can be either the states with the maximum energy or with the absolutely (see above) minimum energy of the lower-level particles. For the former particles, this leads to the Pauli principle. Consider the nature of the Pauli exclusion principle.

Several facts point to an explanation of the nature of the Pauli exclusion principle in the above calculations. First, in the formulation of the maximum value problem, the probability density function p_i/m_i describes the distribution of the maximum value of p_i in a sample of m_i, or, what is the same in terms of particles, the probability density to find a lower-level particle in a state with maximum energy in the entire sample of upper-level particles. Thus, we are talking about only one lower-level particle in a definite energy state, i.e., the realization of the Pauli exclusion principle.

Secondly, both types of calculations of the probability density distribution of p_i/m_i with decreasing temperature give the drop in the width of the function (see Fig. 2), and at low temperatures the probability density distribution function can become so narrow that no more than one particle of the lower level can 'fit' into this state. At the same time, the question is natural, into which states the remaining particles of the lower level go.

Figure 6 shows a model situation where there is an ensemble of upper-level particles with different α (different chemical potentials for lower-level particles). The remaining particles of the lower level will be placed in the energy states of the corresponding distributions (dashed curves in Fig. 6) with the same maxima in this case. The total probability density of the distribution (the solid curve in Fig. 6) over energy will look like a constant up to $\epsilon < \alpha_{\max}/|\gamma|$ in this case, and then decrease exponentially (see inset in Fig. 6). Such a total distribution will coincide well with the Fermi distribution with $\mu \approx \mu_{\max} = \alpha_{\max}/|\gamma|$. Thus, it turns out that the Fermi distribution density for all delimited particles can be represented as the sum of the probability densities of the distributions of groups of particles, each of which has its own chemical potential. The delimitation is determined by a certain energy — the Fermi energy, which is close in value to the chemical potential of this system of delimited particles. This may also be a key interpretation of the Pauli exclusion principle.

We can limit to the consideration that during single-particle measurements of the state of a particle system, some energy of the lower-level particle will always be detected, and at low particle temperatures this energy may be the only one. For this purpose, it is not necessary to restore to complex constructions in Fig. 6. The remaining values may simply not

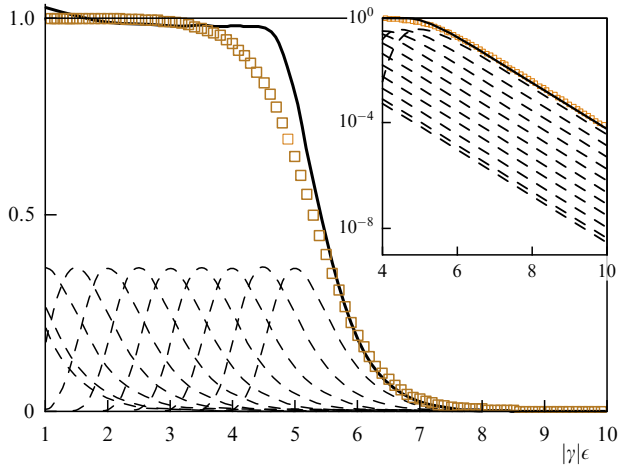

Figure 6. p_i/m_i functions (dashed curves) with α from 0.5 to 5 (through 0.5), as well as a solid curve representing the sum of all these functions. The open squares are the Fermi function $1/[\exp[(\gamma\epsilon - \gamma\alpha) + 1]$ for $\alpha = 5.3$. These functions for large arguments are shown in the inset on a semi-logarithmic scale — the exponential asymptotics of the total function is visible, $|\gamma| = 1$.

fall 'inside' curves *4–6* in Fig. 2 due to the distance between the levels of the discrete energy spectrum. Thus, only one particle will correspond to one energy state: this can be interpreted as the realization of the Pauli principle.

This principle can now be formulated as follows. The particles of the lower level that are delimited in the particles of the upper level have a narrow bell curve as the most probable of the energy distribution probability density. In addition to the fact that at low temperatures only one particle 'fits' into this distribution, it is also a distribution of lower-level particles with maximum energy in an upper-level particle. Thus, only one delimited particle can be in one (maximum) energy state, and it is observed.

5.2 On the most probable energy release in structured media

The process of energy release in any reaction — chemical, nuclear, thermonuclear — has a statistical character. Molecules/atoms/nuclei/particles located in a certain volume of matter may react — practically always upon impact — or they may not react until the process is complete. It is clear that the number of molecules, etc. in this volume that took part in the reaction is a random variable. This random variable determines the energy release. All reacting substances can be divided into groups of molecules, decaying nuclei, etc., the energy release in the first group can be ϵ_1 with the number of particles in it p_1, in the second — ϵ_2 with the number of particles in it p_2, etc. Thus the distribution function (or probability density of distribution) of energy release can be determined. The observed values will be some averages — per molecule, per nucleus, per unit volume, etc. These averages can be the values of the energy dimension, then we can talk about the characteristic energy releases in the reactions under consideration.

The desired distribution function and their mean values may be (and will be) determined using combinatorial schemes considered in Section 3. The simplest combinatorial scheme in case of assumption of equal probability for all particle to participate in the reaction was considered in Section 1 (problem 0).

While the coming of particles in a reaction is not equivalent: some particles are located on the surface of matter, others in the bulk one, we can expect the difference in the energy release. As well, the energy release in similar process in such situations and, moreover, in different processes as burning and explosion, is different. This is discussed in the present Section.

The come in reaction depends on various parameters such as matter density, structure, specific forms of protection, temperature, etc. One can show that all these parameters may reduce to one: the proposed energy of particle exit from the bulk of matter to the follow participation in the reaction. Indeed, the particles on the solid matter surface can burn only in a chemical reaction, nucleus that reached neutrons may only react, particles of the front of shock wave react in an explosion, etc. Our consideration in general can be applied to quasi stationary such as matter burning with gradually separation of reacting molecules from the body. In the rough approximation all these processes can be calculated using one parameter of decoupling energy.

If energy release reactions (interactions of molecules, including complex chain reactions [25], etc.) occur quickly, then the resulting distribution function will have the meaning of the current one, i.e., defined at a given specific time. Accordingly, the decoupling work should also be taken at the same time. If the decoupling work changes much more slowly than the rate of the energy release reaction, it can enter to the result as a (slow) time-dependent value. Then this parameter, the decoupling work, can be used in a combinatorial scheme. The next step is the use of another parameter, the corresponding complication of the combinatorial scheme, etc.

A combinatorial scheme with one of the above-mentioned additional parameters will be used [26]. It describes more complex systems that consist of bodies indistinguishable from the point of view of energy release. These bodies have an internal structure consisting, in turn, also from indistinguishable particles. This indistinguishability is understood here not in the sense of Bose and Fermi particles, but in the sense that we do not care which particle reacted — we are only interested in the released energy. Therefore, the question naturally arises what will happen to the energy distribution in the most general case — if definite groups of particles are separated from others, or if some particle bodies (let's call them upper-level particles) contain others (lower-level, also indistinguishable).

The solution of the problem is given by consideration the combinatorial problem of the number of combinations in the distribution of objects into groups (see Section 3). The same results are obtained by considering a certain special distribution of the minimum energy values of lower-level particles in upper-level particles based on a sample of upper-level particles (Section 4).

The observed values of the problem of the probability density of the energy distribution in the hierarchical system under consideration are averages. To determine the average of the $p_i m_i$ function, you have to first select its normalization. For simplicity, it is advisable to do this per molecule (per nucleus). Then the average (mathematical expectation) $S(\alpha)$ of the normalized distribution function

$$S(\alpha) = \int_0^\infty \epsilon_i (p_i m_i)_{\text{norm}} \, \mathrm{d}\epsilon_i = \frac{\mathrm{Ei}(\exp(\alpha) - C - \alpha}{|\gamma|\big(\exp\big(\exp(\alpha)\big) - 1\big)}. \quad (33)$$

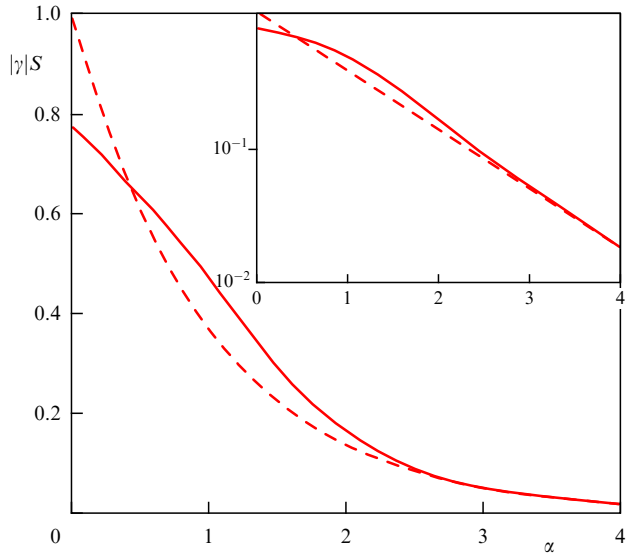

Figure 7. Dependence of the mean S of the normalized distribution function $(p_i m_i)_{\text{norm}}$ on the parameter α, solid curve. To compare, the dashed curve of the function $\exp(-\alpha)$ is given. The inset shows the same relationship in semi-logarithmic coordinates.

It will represent a characteristic energy release in the corresponding process — burning, explosion — calculated per one molecule (core) of the reactant. In (33), $\text{Ei}(x)$ is the integral exponential function [27], $C = 0.577$ is the Euler constant. The dependence of S on the parameter α is shown in Fig.7.

Note that even at $\alpha = 0$, i.e., at zero work of the separation of the molecule from the body to enter into the energy release reaction, $S(0)|\gamma| \approx 0.77 < 1$. We remind you that $|\gamma|^{-1}$ is, by definition, some average energy, which, when calculated for one a particle, and it will be the theoretical energy release during the burning. It looks that this means that the average energy release per particle of the lower level is always less than the theoretical value, even with zero work of separation of the molecule, since there is still some dynamics of the burning/explosion process. For $\alpha \to \infty$, using the asymptotics of $\text{Ei}(x)$, we obtain $S \to \exp(-\alpha)|\gamma|^{-1}$. Qualitatively, this means that if very large numbers of molecules are combined into bodies m_i of the upper level, and their separation work is therefore large, the most likely energy release will be very weak. This is probably typical for nuclear reactors, where a lot of energy is taken from decelerated neutrons, i.e., their decoupling work is high. It lies in the range of the asymptote $S(\alpha)$ in Fig. 7. The strongest energy release is observed at $\alpha = 0$, i.e., in homogeneous gases.

Thus, the solution of the combinatorial problem gives the value of the average energy release per lower-level particle in variously structured media. With an increase in the decoupling work from the corresponding body of the reacting particle (or the complex dynamics of the latter) this energy release decreases exponentially rapidly. Therefore, the simplest thing from a practical point of view is to split the bodies of the upper mi level so that they contain the minimum possible number of particles[5] (from the point of view of the technological process where energy release takes place). Then, the energy release per reacted particle will be significant (although less than the corresponding characteristic energy release in the process). In addition, the macroscopic dynamics of reacting molecules in bodies should not lead to their rapid withdrawal from the reaction zone, i.e., in general, it is necessary not to lose slow-motion neutrons, to deal with linear dynamic formations such as shock waves, but to organize various (micro)vortex movements for better involvement of molecules in reactions.

It looks that a decrease in the average energy release (33) with an increase of the decoupling work $\sim \alpha$ is an obvious fundamental fact in the physics of burning and explosion. This average energy release is definitely the effective energy release of the corresponding process, which is always less than $|\gamma|^{-1}$. This value is compared with $|\gamma|^{-1}$ only when the output operation tends to an infinite negative value, which probably means that α physically takes into account not only the 'direct' work of separating the particle from the body for energy release, but also the accompanying dynamics.

Another fundamental fact is that the probability density of the distribution of the absolutely minimum energy release value (see above) coincides with the most probable one. Apparently, this means that the energy release in the process, which is necessary to maintain the process, is always the least.

Another result may be a nonforbidden slow dependence of α on time. Thus, the characteristic energy release in the process can change over time in accordance with (33) and Fig. 7, where $\alpha = \alpha(t)$ and, respectively, $S = S(t)$.

5.3 The complication of the hierarchy of structures in process of the particle system cools down

In all the results obtained for two-layer particles, only one parameter appears — the Lagrange multiplier α (or some characteristic mean). This parameter is interpreted as the ratio of the chemical potential μ in the system — the energy of attachment/separation of the lower-level particle in problem 1 and the lower and intermediate-level particles in problem 2 — to some effective temperature T_{eff} (measured in energy terms units, see also the Introduction). Then the latter is changed, the resulting distribution functions will change (see Section 3 of Fig. 2). Similarly, the probability density changes in three-layer particles. While we compare the probability densities of the distributions, it turns out that as α increases for the $(p_i/m_i)_{\text{norm}}$ function, as well as the sums $\alpha + \beta$ for the $(p_i m_i/t_i)_{\text{norm}}$ and $(p_i/m_i t_i)_{\text{norm}}$ functions, these functions become the same and represent the same function, translated to the right along the energy axis (Fig. 8).

The function $(p_i/m_i)_{\text{norm}}$ reaches maximum, as we have already seen, at $|\gamma|\epsilon_i = \alpha$, the expectation of this function

$$\langle \epsilon_i \rangle = \frac{1}{|\gamma|}(\alpha - C)\,,$$

i.e., it is somewhat (by the Euler constant multiplied by $1/|\gamma| = T_{\text{eff}}$) less than the energy of the maximum probability density of the distribution. The variance of the probability density of the $p_i/m_i)_{\text{norm}}$ distribution is

$$\sigma_{\epsilon_i} = \sqrt{\langle \epsilon_i^2 \rangle - \langle \epsilon_i \rangle^2} = \frac{\pi}{\sqrt{6}|\gamma|}\,,$$

in other words, it does not depend on any parameters of problem 1 at all, except for the effective temperature. This fact

[5] For nuclear reactors, this probably means that it is better to use reactors with as many fuel assemblies as possible.

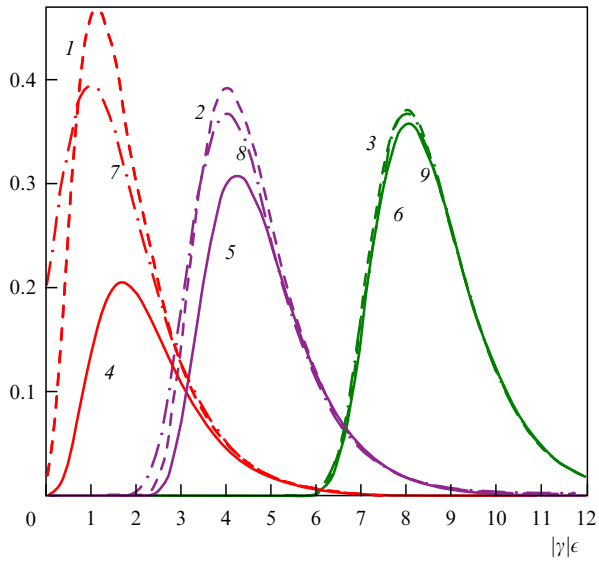

Figure 8. Normalized distributions of $(p_i m_i/t_i)_{\text{norm}}$ (curves *1–3*) and $(p_i/m_i t_i)_{\text{norm}}$ (curves *4–6*). For curves *1* and *4* $\alpha = \beta = 0.5$, for curves *2* and *5* $\alpha = \beta = 2$, for curves *3* and *6* $\alpha = \beta = 4$. The probabilities of the normalized distribution $(p_i/m_i)_{\text{norm}}$ (curves *7–9*) for (*7*) $\alpha = 1$, (*8*) $\alpha = 4$, (*9*) $\alpha = 8$. For all curves, $|\gamma| = 1$ is assumed.

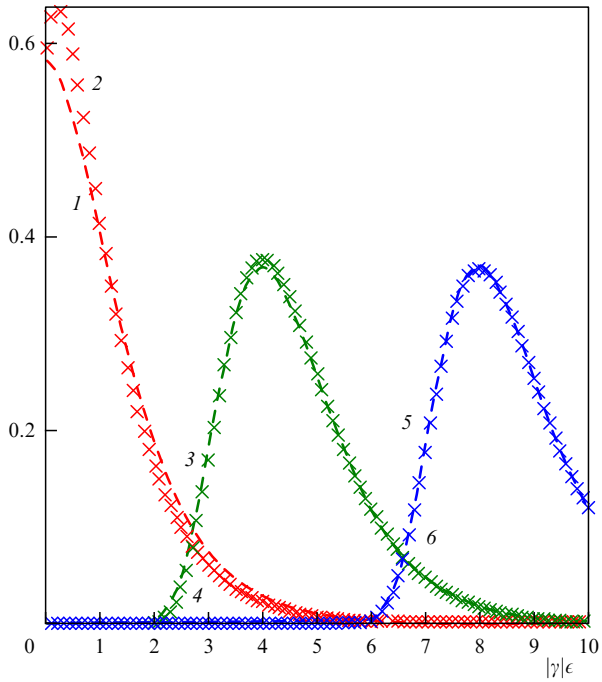

Figure 9. Functions $(p_i/m_i)_{\text{norm}}$ (dashed curves *1*, *3*, *5*) and $(p_i m_i/t_i)_{\text{norm}}$ (cross curves *2*, *4*, *6*) depending on the parameters α and β ($\mu, \mu_1, T_{\text{eff}}$). For curve *1*, α is assumed to be 0, for the curve *3* $\alpha = 4$, and for the curve *5* $\alpha = 8$. For curve *2* $\alpha = \beta = 0$, for the curve *4* $\alpha = 1$, $\beta = 3$, for the curve *6* $\alpha = 2$, $\beta = 6$. It is practically visible complete coincidence of 'two-layer' functions 5 and 'three-layer' functions 6.

indicates the presence of a basic, one might say, 'proto-physical' (πρωτού φύσική) constant, which still needs to be determined. The same simple relations for the $(p_i m_i/t_i)_{\text{norm}}$ function cannot be analytically given.

Thus, only two quantities play a key role in the realization of physical effects — the chemical potential $\mu = \alpha/|\gamma|$ (for the $(p_i m_i/t_i)_{\text{norm}}$ function, also $\mu_1 = \beta/|\gamma|$), and the effective temperature $T_{\text{eff}} = 1/|\gamma|$. The $(p_i/m_i)_{\text{norm}}$ and $(p_i m_i/t_i)_{\text{norm}}$ functions are shown in Fig. 9 and 10. The difference between the figures is as follows: in Fig. 9, $\mu_1 \gg \mu$, and the total chemical potential of $\mu_1 + \mu$ for the function $(p_i m_i/t_i)_{\text{norm}}$ is equal to the chemical potential of the function $(p_i/m_i)_{\text{norm}}$. In turn, Figure 10 shows the opposite situation, $\mu_1 \ll \mu$. The equality of integral chemical potentials is also fulfilled.

It can be seen that in the first case, the probability density curves of distributions already at $\alpha = \mu/T_{\text{eff}} = 4$ for $(p_i/m_i)_{\text{norm}}$, and $\alpha + \beta = \mu + \mu_1/T_{\text{eff}} = 4$ for $(p_i m_i/t_i)_{\text{norm}}$ practically coincide, and a further increase in the chemical potential leads to an even better a coincidence.

This is not the case if $\mu_1 \ll \mu$, i.e., the energy of attachment/separation of a lower-level particle to a middle-level particle is much greater than the energy of attachment/separation of a middle-level particle to an upper-level particle, as shown in Fig. 10. In this case, the probability densities vary up to the integral $\alpha + \beta = 8$, and with further magnification, they almost the same. It seems that Fig. 10 corresponds to the majority of physically realizable situations.

This figure illustrates another important physical result: the invariance of the shape of the probability densities of the distributions $(p_i/m_i)_{\text{norm}}$ and $(p_i m_i/t_i)_{\text{norm}}$ from the magnitude of α and $\alpha + \beta$ the functions are only translated to the right as α and $\alpha + \beta$ increase. At the same time the 'translation to the right' with unchanged μ and μ_1 means the following. First, the system 'cools down:' T_{eff} decreases with unchanged μ and μ_1. It has to also be taken into account that in addition to the shift to the right, the 'width of the bell' of the probability density will decrease, and its

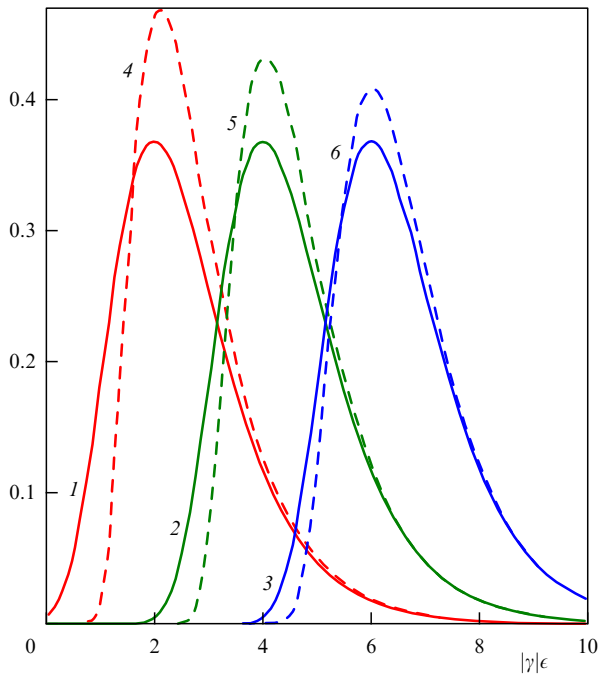

Figure 10. The functions $(p_i/m_i)_{\text{norm}}$ (curves numbered *1–3*) and $(p_i m_i/t_i)_{\text{norm}}$ (curves *4–6*) depending on the parameters α and β (μ, μ_1, T_{eff}). For curve *1*, α is assumed to be equal to 2, for the curve *2* $\alpha = 4$, and for the curve *3* $\alpha = 4$. For curve *4*, $\alpha = 1.5$, $\beta = 0.5$, for the curve *5* $\alpha = 3$, $\beta = 1$, for the curve *6* $\alpha = 4.5$, $\beta = 1.5$.

maximum value will increase; for simplicity, this is not shown in the figures of the article, except in specially specified cases. Secondly, the particle system (and the

particles themselves) stop changing. Note that the almost complete coincidence of the functions $(p_i/m_i)_{\text{norm}}$ and $(p_i m_i/t_i)_{\text{norm}}$ for large values α and $\alpha+\beta$ gives the right to set the value of the first two moments of the second exactly the same as the first. It means that, again, the variance will not depend on anything except T_{eff} and we should expect the appearance of a physical constant.

What physical phenomena can be expected when the system cools down? A decrease in T_{eff} without changes in μ (and μ along with μ_1) will be accompanied by a shift in the energy probability density distributions of particles to the right along the energy axis. This shift will be reduced to a simple quick translation — for the function $(p_i/m_i)_{\text{norm}}$ starting almost from $\alpha=\mu/T_{\text{eff}}=4$, and for the function $(p_i m_i/t_i)_{\text{norm}}$ in the physically interesting case $\mu_1 \ll \mu$ — starting from about $\alpha+\beta=(\mu+\mu_1)/T_{\text{eff}}=10$. The simple translation that has begun apparently means that the particles stop changing their behavior (the probability density of their distribution remains unchanged), or, which is a stronger statement, the particles of all two (or three) levels stop changing. A further decrease in temperature will occur already within the framework of 'normal' physics — and before that (at $\alpha<4$ and $\alpha+\beta<10$) processes took place within the framework of 'protophysics' — and we are dealing with 'modern' particles.

As T_{eff} decreases, another important phenomenon occurs — the complication of particles. Indeed, let us consider Fig. 10. Recall that Fig. 10 corresponds to the case of $\mu_1 \ll \mu$, i.e., the work of separation a lower-level particle from a middle-level particle is much more than the work of separation a middle-level particle from an upper-level particle for all the model cases shown on it. Then the magnitudes of the probability density function of the distribution $(p_i/m_i)_{\text{norm}}$ up to a definite energy value is (much) greater than the magnitudes of the probability density function of the distribution $(p_i m_i/t_i)_{\text{norm}}$, i.e., the probability to find a 'two-layered' particle is (much) higher than a 'three-layered' one. After this value, on the contrary, the probability to find a 'two-layer' particle is (much) less than a 'three-layer' one. Thus, as T_{eff} decreases, first two-level ('two-layer') particles arise from the stochastics of indistinguishable particles of a single ('Boltzmann') level, and then 'three-layer' particles arise from 'two-layer' particles. This is the most rough description of the cosmological evolution of matter.

5.4 On the negligible probability to detect particles with negative energy

Let's pay attention to the fact that if we extend the argument x to the range from $-\infty$ to $+\infty$, then the function $1/m_i=\exp(-\exp(-(x-x_{\text{im}})))$ will be the exact probability distribution of some random variable. For x, we keep the ratio $x=|\gamma|\epsilon_i$. It is interesting that for a definite, not very large value of $x_{\text{im}} \cong \alpha=|\gamma|\mu \geqslant 6$, there are no lower-level particles in states with negative energy at all (the probability of detecting at least one is monstrously small $\sim 10^{-180}$), as illustrated in Fig. 11.

Since the separation of lower-level particles from upper-level particles is achieved due to some interaction, it assumes nonzero values of the chemical potential. It looks that this presence of interaction leads to the absence of particles in states with negative energy.

We note only the following for the functions that arise in problem 2. The exact distribution of some random variable is

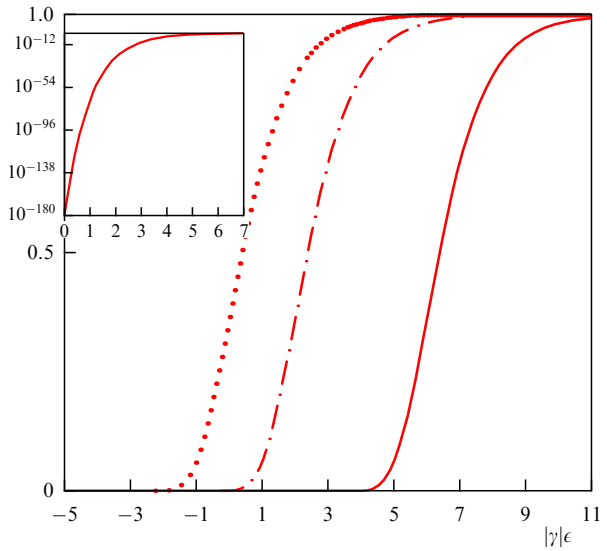

Figure 11. Distribution function $1/m_i$ as a function of energy ϵ_i for values $\alpha=|\gamma|\mu$ equal to 0 for a dotted curve, 2 for a dashed curve and 6.025 for a solid one. For $\alpha=|\gamma|\mu=6.025$, the value of the function at zero is $\sim 10^{-180}$. Uniform coordinates, $|\gamma|$ is assumed to be 1. The inset shows a $1/m_i$ curve with $\alpha=6.025$ on a semi-logarithmic scale.

the function

$$\frac{\exp\left(\exp(\beta)\right)}{t_i}=\exp\left(\exp(\beta)\right)\exp\left(-\exp\left(\beta+\exp(\alpha-x)\right)\right) \tag{35}$$

when the argument $x=|\gamma|\epsilon_i$ is changed from $-\infty$ to $+\infty$ (Fig. 12). The value of this function in zero is less than $<10^{-180}$ already at $\alpha=2$ and $\beta \geqslant 3.4$. If the argument starts from zero (or from some value corresponding to the energy of the ground state of the lower-level particle), then, by simply differentiating over a variable, the probability density function $p_i m_i/t_i$ is obtained (Fig. 12).

This probability density function also describes lower-level particles that cannot be in negative energy states. To do this, it is enough to have at least one of the two nonzero chemical potentials (see above).

From a comparison of Fig. 11 and Fig. 12, it is clear that the 'complete' disappearance of particles in states with negative energy occurs at lower values of the total chemical potential in the three-layer problem compared with the value of the chemical potential of the two-layer problem. Thus, the disappearance of particles with negative energy is affected not only by the general cooling of the system (an increase in $\alpha=\mu/T_{\text{eff}}$ and $\alpha+\beta=(\mu+\mu_1)/T_{\text{eff}}$ with unchanged μ and μ_1), but also by the structuring of matter — a domination of the probability to find a three-layered particle compared to a two-layered one.

6. Discussion of results

Thus, the generalization of the statistical Boltzmann approach by complicating indistinguishable particles using the introduction a hierarchy of the structure of such particles opens up new possibilities to describe their emerging statistical properties. Hierarchical two- and three-level systems are responded to both simple combinatorial calculation and determination of their distribution statistics of maximum and minimum values.

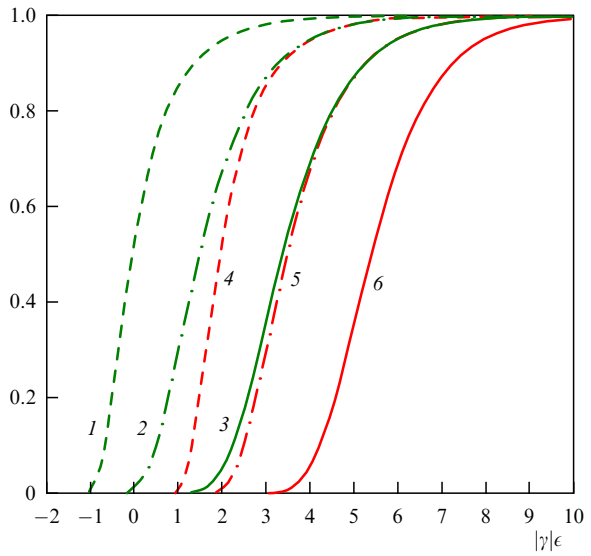

Figure 12. Probability distribution functions (35) for different values of α and β: for curve *1* $\alpha = 0$ $\beta = -1$, for curve 2 $\alpha = 0$ $\beta = 1$, for curve 3 $\alpha = 0$ $\beta = 3$, for curve *4* $\alpha = 2$ $\beta = -1$, for curve *5* $\alpha = 2$ $\beta = 1$, for curve *6* $\alpha = 2$ $\beta = 3$. Uniform coordinates, $|\gamma|$ is equal to 1.

The main result for both problems 1 and 2 is the coincidence of the most probable energy distributions with the distribution of maximum and minimum values.

The results of problems 1 and 2 should be interpreted as depending on the method of observation. All observations in spectroscopic experiments (not only optical ones, in general, most measurements of energy spectra) give some distribution functions related to one particle of the upper level. The main result of problem 1 is to detect the maximum of the probability density function of the energy distribution of particles during single-particle observation. In the simplest approach to the description, this maximum coincides with the chemical potential of the lower-level particles in the upper-level particles — the energy of attachment/separation of the lower-level particle from the rest of such particles in the upper-level particle, of a (sufficiently) constant value. In this simple approach, all lower-level particles with energies less than the chemical potential should be interpreted as bound.

As it was mentioned, at low effective temperatures, the probability density function to find a lower-level particle in a single-particle dimension in the state ϵ_i has a very narrow maximum. This means that at a definitely low (and even lower) temperature, this state occupies by no more than one particle. This result can be interpreted as the realization of the Pauli exclusion principle, when one energy state is occupied by no more than one particle of a certain kind, corresponding to the '−' sign on the left side (11). Some modification of problem 1, in which a set of $\{p_i/m_i\}$ functions with different α arises, leads, when summing these functions, to a function that looks like the Fermi distribution. This is another possible direct realization of the Pauli principle resulting from the solution of problem 1.

Additionally, it was found that the probability density function p_i/m_i describes the distribution of the maximum value of p_i in the m_i sample, or, what is the same in terms of particles, the probability density to find a lower-level particle in a state with maximum energy in the entire sample of upper-level particles. Since it is clear that only one lower-level particle can be in a state with maximum energy, this again indicates a direct realization of the Pauli exclusion principle.

Even at moderate values of $\alpha \sim 6$, the proportion of particles of the normalized probability density of the distribution (13a) with energies greater than the corresponding value of the chemical potential is $\sim 10^{-180}$, i.e., there are probably no such (free?) particles in the entire universe.

A naive interpretation of the results of problem 1 with two distribution functions — $p_i m_i$ and p_i/m_i — is that we are dealing with particles of two kinds. There are pseudobosons for the first function (since the probability density of the energy distribution of particles describes condensation in low-energy states at low temperatures), and pseudofermions (since at low temperatures one energy state, as it turned out, with maximum energy, is occupied by only one such particle).

The same interpretation of the results of problem 2 leads to 4 kinds of particles. Pseudobosons with an even more pronounced condensation effect are described by the $p_i m_i t_i$ function, and there are no particles with energies higher than the chemical potential. The $p_i t_i/m_i$ function gives similar results, but the condensation effect is less pronounced in it. On the contrary, the upper-level particles described by the $p_i/m_i t_i$ function contain lower-level particles in energy states above the chemical potential. Finally, the $p_i/m_i t_i$ function apparently describes some 'normal' particles, since the total chemical potential of the lower-level particle in the middle-level particle and the latter in the upper-level particle is the sum of the chemical potentials of a lower-level particle in a medium-level particle and a medium-level particle in an upper-level particle.

When particle systems cool down (reducing the effective temperature, which is generally accepted, for example, in cosmology) — both two-layer and three-layer — at first it is much more likely that two-level particles, pseudofermions, will appear, and with an even greater decrease in T_{eff}, the same three-layer particles will appear compared to the unstructured 'ocean' of lower-level particles. Here, only one fact is enough: the cooling of some initial chaos of indistinguishable unstructured particles, and their separation (not yet interaction) into particles of a higher level. Thus the description of the particle system is obtained not even physically, but protophysically.

A combinatorial procedure was carried out to determine the most probable distribution of energy release under natural assumptions about the different number of particles of matter that can enter into an energy release reaction (burning, explosion, etc.). It occurs due to their location on the surface/inside the structure, the dynamics of access to reacting particles and the obvious probabilistic nature of the process. In some simple approximation, the energy release is determined by a single parameter of the combinatorial scheme. The most probable distribution turns out to coincide with the distribution of the absolutely minimum energy release values. The result can be used for quantitative interpretation of differences in energy release during burning, explosion and in other processes under different conditions.

The distributions obtained in problems 1 and 2 naturally generalize to states with negative energies. The most naive interpretation for extremely small probabilities of distribution functions $1/m_i$ and $\exp(\exp(\beta))/t_i$ that lower-level particles with negative energy (or with energy less than a certain negative value of ϵ_0, see Footnote 3) cannot be formed into upper-level particles, or that there cannot be particles in a state with energy less than the energy of the ground state.[6] The reason for this, i.e., the reason for such small values of the

[6] That seems to be interpreted as the absence of antiparticles.

distribution functions with a negative value of the argument is the delimitation of some particles in others, or the nonevident assumption of interaction.

Thus, statistics of the most probable energy distributions in systems of hierarchically organized particles are obtained, which coincide with the statistics of distributions of extreme values. The results make it possible to prove the Pauli exclusion principle, the condensation of particles with minimal energies, the minimality of energy release in various processes, the absence of particles with negative energy, and also describe the complication of the structure of protomatter — the appearance of particles with an internal structure and its complication as it cools.

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