Physics – Uspekhi **69** (7) 635 – 662 (2026) ©2026 Uspekhi Fizicheskikh Nauk, Russian Academy of Sciences

REVIEWS OF TOPICAL PROBLEMS PACS numbers: **04.30. – w, 04.50. – h, 04.60. – m**

Gravitational waves in extended gravity theories

S.O. Alexeyev, S.K. Kuzmin, E.A. Pluzhnikov, R.A. Stamov, I.I. Chekh

DOI: https://doi.org/10.3367/UFNe.2026.05.040140

Contents

1. Introduction **635**
2. General relativity **636**
3. Scalar-tensor theories **637**
3.1 Brans–Dicke theory; 3.2 Horndeski theory; 3.3 $f(R)$ gravity; 3.4 Theories with a massive graviton
4. Bimetric modified newtonian dynamics **645**
5. Quantum theories of gravity **646**
5.1 Loop quantum gravity; 5.2 Gravitational waves in string theory; 5.3 Hořava–Lifshitz theory
6. Higher-dimensional theories **651**
6.1 ***D***-dimensional general relativity
7. Theories with nonriemannian geometry **653**
7.1 Metric-affine gravity; 7.2 Metric-affine gravity: the general teleparallel case; 7.3 Metric-affine gravity: Weitzenbck geometry; 7.4 Metric-affine gravity: symmetric teleparallelism; 7.5 Metric-affine gravity: the most general case; 7.6 $f(T)$ gravity; 7.7 $f(T, B)$ gravity; 7.8 $f(T, B)$ gravity with a cosmological constant and $f(T, T_G)$ gravity; 7.9 $f(Q)$ gravity
8. Conclusions **661**
References **661**

Abstract. Any theory proposed to explain dark matter, dark energy, the evolution of the early Universe, and other unsolved problems must correctly reproduce existing observational and experimental data, including the properties of gravitational waves. Therefore, we discuss the main properties of gravitational waves such as the propagation velocity, characteristic amplitudes, polarizations, and other features predicted by extended theories of gravity: scalar-tensor gravity, including the Brans–Dicke models; Horndeski theories; $f(R)$-gravity; theories with a massive graviton; modified Newtonian dynamics; quantum theories of gravity, including loop quantum gravity and string theory IIB; Hořava–Lifshitz theory; higher-dimensional theories, including the general case of *D*-dimensional spacetime and Kaluza–Klein theory; and non-Riemannian geometry, including both the general case of metric affine gravity and various versions of teleparallel gravity ($f(T)$ and $f(Q)$), as well as the extended cases $f(T, B)$ and $f(T, T_G)$. With increasing experimental accuracy, the discussed characteristics of gravitational waves will be discovered, helping in the selection of gravity theories.

Keywords: gravitational waves, general relativity, Brans–Dicke theory, scalar-tensor gravity, Horndeski theory, teleparallel gravity, metric affine gravity, gravitational-wave astronomy

S.O. Alexeyev [(1, 2, a)], **S.K. Kuzmin** [(2, b)], **E.A. Pluzhnikov** [(3, c)], **R.A. Stamov** [(1, 2, d)], **I.I. Chekh** [(2, e)]
[(1)] Lomonosov Moscow State University, Sternberg Astronomical Institute, Universitetskii prosp. 13, 119234 Moscow, Russian Federation
[(2)] Lomonosov Moscow State University, Faculty of Physics, Leninskie gory 1, str. 2, 119991 Moscow, Russian Federation
[(3)] Lomonosov Moscow State University, Faculty of Cosmic Research, Leninskie Gory 1/52, 119991 Moscow, Russian Federation
E-mail: [(a)] alexeyev@sai.msu.ru, [(b)] sergey.kuzmin.jr@gmail.com, [(c)] pluzhnikov.egor@student.msu.ru, [(d)] stamov_roma@mail.ru, [(e)] ilya@spacecyborg.ru

Received 14 October 2025, revised 28 April 2026
Uspekhi Fizicheskikh Nauk **196** (7) 677 – 706 (2026)
Translated by the authors

1. Introduction

One of the consequences of general relativity (GR) is the existence of gravitational waves (GWs). They are solutions of the Einstein equations describe small perturbations of flat spacetime [1]. The GW spectrum covers a wide range of frequencies and can be classified according to the radiation source: from 10^{-17} Hz for fluctuations of the cosmological background up to 10^3 Hz for the formation of neutron stars following supernova explosions [2].

Note that in 1974 two compact massive stars orbiting each other were discovered. One of them was identified as a pulsar. After fifteen years of orbital period measurements it was found that the distance between these stars decreases with the speed predicted by GR [3, 4]. This fact provides an indirect evidence for the GWs emission by binary systems. For this discovery, Russell Hulse and Joseph Taylor were awarded the Nobel Prize in Physics in 1993. [1] In the following decades laser interferometers for the GWs direct detecting were constructed and commissioned. The list includes the Laser Interferometer Gravitational-Wave Observatory (LIGO) in the United

[1] https://www.nobelprize.org/prizes/physics/1993/summary/.

States [5], Virgo in Italy/France [6], GEO 600 in Germany/the United Kingdom [7], and TAMA 300 in Japan [8]. The first gravitational-wave signal from a binary black-hole merger was detected by LIGO in 2015 [9] and was found to be in excellent agreement with the GR predictions. Further GW detections followed, and the Nobel Prize in Physics was awarded for their discovery in 2017.[2]

An important milestone in gravitational-wave astronomy was the detection by the LIGO/Virgo Collaboration of GWs from a neutron-star merger in 2017 [10]. Owing to the relative proximity of the source ($z < 0.01$, where z is the redshift), it was possible to localize the event and detect electromagnetic counterparts with arrival-time differences ranging from 1.6 seconds to approximately 10 hours. These observations included the GRB 170817A event detected by the DECam Collaboration [11]; optical observations by Las Cumbres Observatory, the MASTER robotic telescope network, and the Hubble Space Telescope [12, 14, 16]; observations by the Fermi and INTEGRAL space-based gamma-ray observatories [13, 15]; ultraviolet observations by the Neil Gehrels Swift Observatory [17]; and observations by the Chandra X-ray Observatory [18]. The GW170817 event confirmed that GWs propagate at the speed of light, thereby placing constraints on numerous alternative models and providing an important test of theories of gravity [19, 20]. GW170817 represented a major breakthrough in multimessenger astronomy.

Ground-based laser interferometers possess sufficient displacement sensitivity to detect GWs with frequencies above 10 Hz, such as those produced by mergers of black holes or neutron stars. Following the success of LIGO, several space-based laser interferometers are currently being developed, including LISA [21], TianQin [22], and Taiji [23].

On the other hand, GR continues to face new challenges. For example, only about $4-5\%$ of the content of the Universe is directly described by ordinary matter, whereas the remaining $\approx 95\%$ is attributed to dark matter [24] and dark energy [25]. To explain the origin and physical nature of these phenomena, numerous approaches have been proposed in recent years that either extend GR or provide alternative descriptions of gravitational interaction. In these theories terms added to the GR action could serve as sources of dark energy, dark matter, and cosmological inflation, forming an 'effective' energy–momentum tensor.

One way to extend GR is to introduce higher-order curvature terms, for example R^2 leading to higher-order theories such as $f(R)$ gravity. Another approach is the construction of quantum theories of gravity, including Hořava–Lifshitz gravity, in which the gradient of a scalar field replaces the vector field and spatial and temporal dimensions are assumed to become inequivalent at high energies; string theory, where additional spacetime dimensions are introduced in order to unify fundamental interactions; and loop quantum gravity, where gravity is quantized independently of the other interactions by introducing Ashtekar variables. The addition of new fields, possibly coupled to curvature invariants, leads to Horndeski theory, which represents the most general scalar-tensor theory with second-order field equations. Its extensions and massive gravity theories contain massive modes. A further generalization of the geometric structure leads to metric-affine theories and teleparallel gravity, where the gravitational interaction is described by torsion or nonmetricity rather than spacetime curvature. Finally, there exist models based on different principles, such as holographic models, which establish a correspondence between a manifold and its boundary, and entropic gravity, in which gravity emerges as an effective consequence of the quantum entanglement of spacetime information. This classification is, of course, rather schematic, and the different approaches can be combined with one another. The extension of GR leads to more complicated equations governing gravitational waves. Consequently, GWs may possess theory-specific features whose future detection could help determine which description of gravity is realized in nature. The aim of the present work is therefore to review how alternative theories of gravity describe the properties of gravitational waves in cases where this problem has already been investigated.

The paper is organized as follows. In Section 2, we discuss the derivation of GWs in GR. Section 3 is devoted to gravitational waves in scalar-tensor gravity, including the Brans–Dicke model (Section 3.1.1), Brans–Dicke theory with a potential (Section 3.1.2), Horndeski theory (Section 3.2), $f(R)$ gravity (Section 3.3), and theories with a massive graviton (Section 3.4). Section 4 discusses gravitational waves in modified Newtonian dynamics. Section 5 considers gravitational waves in quantum theories of gravity, including loop quantum gravity (Section 5.1), type IIB string theory (Section 5.2), and Hořava–Lifshitz gravity (Section 5.3). Section 6 addresses gravitational waves in higher-dimensional theories, including the general case of a D-dimensional spacetime (Section 6.1) and Kaluza–Klein theory (Section 6.1.1). Section 7 is devoted to gravitational waves in non-Riemannian geometries, including the general framework of metric-affine gravity (Section 7.1) and various formulations of teleparallel gravity, such as $f(T)$ and $f(Q)$ gravity and the extended cases $f(T, B)$ and $f(T, T_G)$ (Sections 7.2–7.9). Section 8 contains our conclusions.

2. General relativity

In this section we discuss weak gravitational waves propagating on a flat spacetime background within GR and introduce the basic definitions and notations used throughout the paper. The subsequent results are compared with the corresponding GR expressions in order to identify deviations from the standard theory.

The action of GR is

$$S = -\frac{c^3}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}R + \frac{1}{c}\int \mathrm{d}^4x\sqrt{-g}\,L_m\,, \tag{1}$$

where c is the speed of light in vacuum; G is Newton's gravitational constant; g is the determinant of the metric tensor $g_{\mu\nu}$ with signature $(-,+,+,+)$; and the curvature scalar R is obtained by contracting the Ricci tensor $R_{\mu\nu}$, $R = g^{\mu\nu}R_{\mu\nu}$, which in turn is obtained by contracting the Riemann tensor $R^{\alpha}{}_{\mu\beta\nu}$, $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$. Furthermore, $\mathcal{L}_m = \sqrt{-g}\,L_m(\psi_m, g_{\mu\nu})$ is the Lagrangian density for the matter fields ψ_m, representing an arbitrary collection of such fields. Throughout this paper, unless stated otherwise, Greek indices $(\mu, \nu, \ldots)$ take the values $(0, 1, 2, 3)$, whereas Latin indices $(i, j, \ldots)$ take the values $(1, 2, 3)$.

Variation of the action (1) with respect to $g^{\mu\nu}$ yields the Einstein equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R = \frac{8\pi G}{c^4}\,T_{\mu\nu}\,, \tag{2}$$

[2] https://www.nobelprize.org/prizes/physics/2017/summary/.

where the left-hand side $G_{\mu\nu}$ is the Einstein tensor, while the right-hand side contains the energy–momentum tensor of matter, $T_{\mu\nu}$, obtained by varying the matter part of the action:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}} . \tag{3}$$

To describe gravitational waves the metric is decomposed as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} , \tag{4}$$

where the metric is represented as the sum of the flat Minkowski background $\eta_{\mu\nu} = \mathrm{diag}\,(-1, 1, 1, 1)$ and a small perturbation $h_{\mu\nu}$, whose derivatives up to second order are also assumed to be small, $h_{\mu\nu} \ll 1$. Indices are raised and lowered using $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$.

Assuming that the perturbations propagate either in vacuum or sufficiently far from gravitating masses, we neglect the right-hand side of the Einstein Eqn (2), i.e., we set $T_{\mu\nu} = 0$. In this case,

$$G_{\mu\nu} = 0 . \tag{5}$$

To obtain the linear approximation of this equation with respect to the decomposition (4), one consider

$$R^{(1)}_{\mu\nu} = \frac{1}{2} \left(-\eta^{\alpha\beta} \partial_{\alpha\beta} h_{\mu\nu} - \partial_{\mu\nu} h + \partial_{\alpha\mu} h^{\alpha}_{\nu} + \partial_{\alpha\nu} h^{\alpha}_{\mu} \right) , \tag{6}$$

$$R^{(1)} = -\eta^{\alpha\beta} \partial_{\alpha\beta} h + \partial_{\alpha\beta} h^{\alpha\beta} , \tag{7}$$

where $h \equiv h^{\alpha}_{\alpha}$ and $\partial_{\alpha\beta} \equiv \partial_{\alpha} \partial_{\beta}$, with $\partial_{\alpha} \equiv \partial / \partial x^{\alpha}$ and coordinates $x^{\alpha} = (ct, x^i) = (ct, x, y, z)$ correspond to the decomposition (4) with the Minkowski metric as the background. Expressions (6) and (7) are invariant under the transformations

$$h'_{\mu\nu} = h_{\mu\nu} + \mathcal{L}_{\xi} \eta_{\mu\nu} = h_{\mu\nu} + \partial_{\mu} \xi_{\nu} + \partial_{\nu} \xi_{\mu} , \tag{8}$$

which are gauge ones. Here, $\mathcal{L}_{\xi}$ denotes the Lie derivative along the displacement vector ξ^{α}, whose indices in (8) are also lowered using $\eta_{\mu\nu}$. The components ξ^{α} are arbitrary, sufficiently smooth, and small enough to be consistent with the perturbative character of $h_{\mu\nu}$.

In the linear approximation Eqn (2) takes the form

$$G^{(1)}_{\mu\nu} = \frac{1}{2} \left(-\eta^{\alpha\beta} \partial_{\alpha\beta} \theta_{\mu\nu} - \eta_{\mu\nu} \partial_{\alpha\beta} \theta^{\alpha\beta} + \partial_{\alpha\mu} \theta^{\alpha}_{\nu} + \partial_{\alpha\nu} \theta^{\alpha}_{\mu} \right) = 0 , \tag{9}$$

where

$$\theta_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h . \tag{10}$$

Equations (9) are gauge invariant under the transformations

$$\theta'_{\mu\nu} = \theta_{\mu\nu} + \partial_{\mu} \xi_{\nu} + \partial_{\nu} \xi_{\mu} - \eta_{\mu\nu} \partial_{\alpha} \xi^{\alpha} . \tag{11}$$

Since the components ξ_{α} are arbitrary, the components of $\theta_{\mu\nu}$ can be constrained. This is usually achieved by imposing the Lorenz gauge conditions

$$\partial_{\mu} \theta^{\mu}_{\nu} = 0 . \tag{12}$$

Then Eqns (9) reduce to the wave equation

$$\Box_{\eta} \theta_{\mu\nu} = 0 , \tag{13}$$

where the d'Alembert operator is defined as

$$\Box_{\eta} \equiv \eta^{\alpha\beta} \partial_{\alpha\beta} \equiv \Delta - \frac{\partial^2}{c^2 \partial t^2}$$

in the coordinates associated with the decomposition (4), and Δ is the three-dimensional Laplacian. Thus, the field $\theta_{\mu\nu}$ propagates through Minkowski spacetime with the speed of light.

Conditions (12) reduce the number of components of $\theta_{\mu\nu}$ from 10 to 6. There are also residual gauge degrees of freedom in the choice of ξ^{α}, constrained by

$$\Box_{\eta} \xi^{\alpha} = 0 \tag{14}$$

which further reduce the number of independent components of $\theta_{\mu\nu}$ from 6 to 2. Within these transformations, one usually imposes the condition $\theta = \theta^{\alpha}_{\alpha} = 0$, which means the tracelessness of $h_{\mu\nu}$:

$$h = 0 , \tag{15}$$

so that $\theta_{\mu\nu} = h_{\mu\nu}$.

Let us consider a solution of the wave Eqn (13) in the form of plane waves,

$$h_{\mu\nu} = a_{\mu\nu} \exp\left(\mathrm{i} k_{\alpha} x^{\alpha}\right) + b_{\mu\nu} \exp\left(-\mathrm{i} k_{\alpha} x^{\alpha}\right) , \tag{16}$$

where $a_{\mu\nu}$ and $b_{\mu\nu}$ are polarization tensors that do not vanish simultaneously. Substituting (16) into Eqn (13), we obtain the dispersion relation for weak plane gravitational waves in GR:

$$k_{\alpha} k^{\alpha} = 0 . \tag{17}$$

In the coordinates $x^{\alpha} = (ct, x^i)$ associated with the decomposition (4), where the Minkowski metric acts as the background, the four-vector k^{α} is written as $k^{\alpha} = (\omega/c, k^i)$, where ω is the angular frequency and k^i is the wave vector. Therefore, Eqn (17) becomes

$$\omega = ck , \tag{18}$$

where $k = \sqrt{|k_i k^i|}$. Thus, gravitational waves obey a linear dispersion relation and propagate at the speed of light in vacuum.

Using (16) as an example, let us consider the polarization of gravitational waves. Choosing the propagation direction of a plane gravitational wave along the z axis, Eqn (16) becomes

$$h_{\mu\nu} = a_{\mu\nu} \exp\left(-\mathrm{i}(\omega t - kz)\right) + b_{\mu\nu} \exp\left(\mathrm{i}(\omega t - kz)\right) . \tag{19}$$

Physical quantities must be represented by real functions; therefore, $h_{\mu\nu} = h^{*}_{\mu\nu}$, and hence $a_{\mu\nu} = b^{*}_{\mu\nu}$ and vice versa. As a result, the two independent polarizations can be written as

$$\theta^{\mathrm{TT}}_{\mu\nu} = h^{\mathrm{TT}}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{+} & h_{\times} & 0 \\ 0 & h_{\times} & -h_{+} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{20}$$

where $h_{+} = h_{+}(\omega t - kz)$ and $h_{\times} = h_{\times}(\omega t - kz)$.

Thus, weak gravitational waves in GR propagate at the speed of light, possess two polarization states, and are transverse. The choice of polarization components in the form (20) is referred to as the transverse-traceless gauge (TT gauge).

3. Scalar-tensor theories

3.1 Brans–Dicke theory

One of the earliest and most widely used scalar-tensor theories is the Brans–Dicke (BD) model [29, 30], where the gravitational sector contains a scalar field in addition to the metric tensor field. The matter fields ψ_m do not couple directly to the scalar field; they interact with it only indirectly through the metric (tensor) sector.

In some cases, the classical BD model [29, 30] is supplemented by a scalar-field potential $V(\phi)$ [31]; we shall refer to this theory as BDV. The action of the BDV theory is

$$S = -\frac{c^3}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left(\phi R - \frac{\omega_{\mathrm{BD}}}{\phi}\, g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + V(\phi)\right) + \frac{1}{c}\int \mathrm{d}^4x\sqrt{-g}\, L_m(\psi_m, g_{\mu\nu})\,, \tag{21}$$

where ω_{BD} is the dimensionless coupling constant, or the Brans–Dicke parameter, and ∇_μ is the covariant derivative associated with the curved-spacetime metric $g_{\mu\nu}$.

The variation of the action (21) with respect to $g^{\mu\nu}$ yields the equations of motion

$$G_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}V(\phi) - \frac{\omega_{\mathrm{BD}}}{\phi^2}\, A_1 - \frac{1}{\phi}A_2 = \frac{8\pi G}{c^4}\,\frac{T_{\mu\nu}}{\phi}\,, \tag{22}$$

where

$$A_1 = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}\, g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi\,,$$
$$A_2 = \nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\,.$$

Varying the action (21) with respect to ϕ and combining the result with the trace of Eqns (22) gives

$$\Box\phi + \frac{\phi V'(\phi) - 2V(\phi)}{2\omega_{\mathrm{BD}} + 3} = \frac{8\pi G}{c^4}\,\frac{T}{2\omega_{\mathrm{BD}} + 3}\,. \tag{23}$$

Here $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d'Alembert operator in curved spacetime, $V'(\phi) \equiv \partial V/\partial\phi$, and $T \equiv g^{\mu\nu}T_{\mu\nu}$.

To describe gravitational waves, we use the metric decomposition (4) together with a decomposition of the scalar field:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad \phi = \phi_0 + \varphi\,, \tag{24}$$

where ϕ_0 denotes the background value of the scalar field. To make the Minkowski background metric to satisfy the above equations, it is convenient to choose $\phi_0 = \mathrm{const}$. Without loss of generality, we may also assume that $\phi_0 \approx 1$ and that the perturbation $\varphi \ll 1$ is small together with its derivatives up to second order. Assuming that the perturbations propagate either in vacuum or far from gravitating masses, we neglect the right-hand sides of Eqns (22) and (23) and set $T_{\mu\nu} = T = 0$.

Since gravitational waves behave qualitatively differently in the absence and presence of the potential, we consider these two cases separately.

3.1.1 Brans–Dicke theory without a potential: $V(\phi) = 0$. Setting $V(\phi) = 0$, substituting the decompositions (24) into Eqns (22), and retaining terms up to linear order, we obtain

$$G^{(1)}_{\mu\nu} - \frac{1}{\phi_0}\left(\partial_\mu\partial_\nu\varphi - \eta_{\mu\nu}\Box_\eta\varphi\right) = 0\,, \tag{25}$$

where $\Box_\eta \equiv \eta^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d'Alembert operator of flat Minkowski spacetime, while $G^{(1)}_{\mu\nu}$ is defined in (9) using the notation (10). Equation (25) can then be rewritten in a form formally identical to (9):

$$\Box_\eta\theta_{\mu\nu} + \eta_{\mu\nu}\partial_{\alpha\beta}\theta^{\alpha\beta} - \partial_{\alpha\mu}\theta^\alpha_\nu - \partial_{\alpha\nu}\theta^\alpha_\mu = 0\,, \tag{26}$$

where, however, in contrast to (10),

$$\theta_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,h - \eta_{\mu\nu}\frac{\varphi}{\phi_0}\,. \tag{27}$$

Substituting the decompositions (24) into Eqn (23) and retaining terms up to linear order gives

$$\Box_\eta\varphi = 0\,. \tag{28}$$

The system of Eqns (26) and (28) is invariant under the gauge transformations (8), supplemented by

$$\varphi' = \varphi + \pounds_\xi\,\phi_0 = \varphi\,. \tag{29}$$

Thus, the gauge transformations (11) obtain their form, and the left-hand sides of Eqns (26) and (28) remain invariant. This allows one to impose constraints on the components of $h_{\mu\nu}$ and, correspondingly, on $\theta_{\mu\nu}$. As in GR, the Lorenz gauge conditions are now imposed for the quantity defined in (27):

$$\partial_\mu\theta^\mu_\nu = 0\,, \tag{30}$$

which reduces Eqn (26) to

$$\Box_\eta\theta_{\mu\nu} = 0 \tag{31}$$

and also allows one to use the residual gauge freedom constrained by Eqn (14), as in GR.

Equations (28) and (31) are both homogeneous equations involving the d'Alembert operator $\Box_\eta$. Therefore, both the tensor and scalar modes propagate at the speed of light. Consequently, solutions of the form (16) apply to both modes, and the dispersion relation takes the form (17) or (18). There is a close analogy with GR, where obtaining the TT gauge required the equality $\theta_{\mu\nu} = h_{\mu\nu}$. This achieved by imposing condition (15). Here, in order to obtain $\theta_{\mu\nu} = h_{\mu\nu}$, instead of (15) one imposes

$$h = -2\,\frac{\varphi}{\phi_0}\,, \tag{32}$$

while the remaining seven conditions $h_{0\alpha} = h_{13} = h_{23} = h_{33} = 0$ are retained if propagation along the z direction is assumed. Consequently, instead of (20) one obtains

$$\theta^{\mathrm{T}}_{\mu\nu} = h^{\mathrm{T}}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ - \dfrac{\varphi}{\phi_0} & h_\times & 0 \\ 0 & h_\times & -h_+ - \dfrac{\varphi}{\phi_0} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{33}$$

where

$$h_+ = h_+(\omega t - kz)\,,$$
$$h_\times = h_\times(\omega t - kz)\,,$$
$$\varphi = \varphi(\omega t - kz)\,,$$

and the superscript T denotes 'transverse.'

Thus, as in GR, the GW is transverse and propagates at the speed of light in vacuum. The difference from GR is the presence of an additional scalar component, which is also transverse, massless, and propagates at the speed of light. This third mode is commonly referred to as the 'scalar breathing' mode, and its presence makes the solution traceful.

3.1.2 Brans–Dicke theory with a potential: $V(\phi)\neq 0$. For $V(\phi)\neq 0$, the background value ϕ_0 in the decomposition (24) must be specified when constructing the linearized equations. It is usually chosen to correspond to a minimum of the potential, with $V(\phi_0)=0$. Since this is a minimum, a sufficiently smooth potential also satisfies $V'(\phi_0)=0$, whereas in general $V''(\phi_0)\neq 0$. With this choice, Minkowski spacetime constitutes the zeroth-order solution of Eqns (22) and (23).

In the linear approximation, Eqns (22) become

$$\Box_\eta \theta_{\mu\nu} + \eta_{\mu\nu}\partial_{\alpha\beta}\theta^{\alpha\beta} - \partial_{\alpha\mu}\theta^{\alpha}_{\nu} - \partial_{\alpha\nu}\theta^{\alpha}_{\mu} = 0\,, \tag{34}$$

which are formally identical to (26), where, as in (27),

$$\theta_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h - \eta_{\mu\nu}\frac{\varphi}{\phi_0}\,. \tag{35}$$

However, in contrast to (28),

$$\left(\Box_\eta - m_s^2\right)\varphi = 0\,, \tag{36}$$

where the squared effective mass is defined as

$$m_s^2 \equiv -\frac{\phi_0 V''(\phi_0)}{3+2\omega_{BD}}\,. \tag{37}$$

A physically interesting situation requires $m_s^2 \geqslant 0$. Since $V''(\phi_0)\geqslant 0$ at the minimum $V(\phi_0)$ of the potential, one must take $\phi_0 \leqslant 0$.

The system of Eqns (34) and (36) is formally invariant under the same gauge transformations (8) of the tensor component, while φ remains unchanged, $\varphi' = \varphi$. This makes it possible to impose the Lorenz gauge conditions (30), reducing Eqns (34) to

$$\Box_\eta \theta_{\mu\nu} = 0\,. \tag{38}$$

The essential difference from the BD theory without a potential is the presence of an effective mass in the scalar-field equation. Nevertheless, since the fields $\theta_{\mu\nu}$ and ϕ in Eqns (38) and (36) are independent, Eqn (38) can formally be analyzed in the same way as Eqn (13) in GR. As a result, the dispersion relations (17) or (18) can be derived for $\theta_{\mu\nu}$, while for a wave propagating along the z direction one may introduce the TT gauge:

$$\theta^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \theta_+ & \theta_\times & 0 \\ 0 & \theta_\times & -\theta_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{39}$$

where

$$\theta_+ = \theta_+(\omega t - kz)\,,$$
$$\theta_\times = \theta_\times(\omega t - kz)\,.$$

Let us now consider Eqn (36). We assume that the scalar component is a plane wave propagating along the z direction. Its solution is taken in the form

$$\begin{aligned}\varphi &= A\exp\left(\mathrm{i}q^\alpha x_\alpha\right) + B\exp\left(-\mathrm{i}q^\alpha x_\alpha\right)\\ &= A\exp\left(-\mathrm{i}(\omega_s t - qz)\right) + B\exp\left(\mathrm{i}(\omega_s t - qz)\right),\end{aligned} \tag{40}$$

where $q^\alpha = (\omega_s/c, q^i)$, with angular frequency ω_s and wave vector q^i. For the solution to be real, one assumes $A = B^*$. The dispersion relation becomes

$$q^\alpha q_\alpha + m_s^2 = 0\,, \tag{41}$$

which yields

$$\omega_s = c\sqrt{q^2 + m_s^2}\,. \tag{42}$$

The standard definition of the group velocity then gives

$$v_s = \frac{\mathrm{d}\omega_s}{\mathrm{d}q} = c\sqrt{1 - \frac{m_s^2 c^2}{\omega_s^2}}\,. \tag{43}$$

Thus, the propagation speed of the scalar component is less than the speed of light.

From (35) and (39), we obtain

$$h_{\mu\nu} = \theta^{TT}_{\mu\nu} - \eta_{\mu\nu}\frac{\varphi}{\phi_0}\,, \tag{44}$$

or, in matrix form,

$$h_{\mu\nu} = \begin{pmatrix} \dfrac{\varphi}{\phi_0} & 0 & 0 & 0 \\ 0 & \theta_+ - \dfrac{\varphi}{\phi_0} & \theta_\times & 0 \\ 0 & \theta_\times & -\theta_+ - \dfrac{\varphi}{\phi_0} & 0 \\ 0 & 0 & 0 & -\dfrac{\varphi}{\phi_0} \end{pmatrix}, \tag{45}$$

where

$$\theta_+ = \theta_+(\omega t - kz)\,,$$
$$\theta_\times = \theta_\times(\omega t - kz)\,,$$
$$\varphi = \varphi(\omega_s t - qz)\,.$$

As in the BD theory without a potential, the scalar contribution to the components h_{11} and h_{22} gives rise to the 'scalar breathing' mode. The difference from BD theory without a potential is that the components h_{00} and h_{33} describe a longitudinal polarization. In addition, in BDV theory the scalar component of the gravitational wave is massive and has a group velocity less than the speed of light.

3.2 Horndeski theory

When describing GWs in the Horndeski theory, we consider perturbations on a vacuum background and therefore restrict ourselves to the action without the matter contribution [32, 33]:

$$S = -\frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\,(L_2 + L_3 + L_4 + L_5)\,, \tag{46}$$

where

$$L_2 = K(\phi, X)\,, \qquad L_3 = -G_3(\phi, X)\Box\,\phi\,,$$
$$L_4 = G_4(\phi, X)R + G_{4,X}\left[(\Box\,\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)\right],$$
$$L_5 = G_4(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}\,G_{5,X}\left[(\Box\,\phi)^3 - 3(\Box\,\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) + 2(\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla_\mu\phi)\right]. \tag{47}$$

Here

$$X = -\frac{1}{2}\,g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\,, \qquad \Box\,\phi = \nabla_\mu\nabla^\mu\phi\,,$$

where K, G_3, G_4, and G_5 are arbitrary functions of ϕ and X. The Horndeski theory, being the most general scalar-tensor theory with second-order field equations, reduces to a number of well-known models for particular choices of these functions. For example, GR corresponds to

$$K = G_3 = G_5 = 0\,, \qquad G_4 = 1\,.$$

The case of the BD theory without a potential corresponds to

$$G_3 = G_5 = 0\,, \qquad K = \frac{2\omega_{\rm BD}X}{\phi}\,, \qquad G_4 = \phi\,,$$

whereas the BDV theory differs only in the function K:

$$K = \frac{2\omega_{\rm BD}X}{\phi} + V(\phi)\,.$$

The variation of the action (46) yields the field equations [33]. To obtain gravitational-wave solutions, we again consider metric perturbations about flat spacetime and scalar-field perturbations about the background value according to (24). Thus, to linear order in $h_{\mu\nu}$ and φ, the equations take the form

$$-\frac{1}{2}\,K(0) + G_4(0)G^{(1)}_{\mu\nu} - G_{4,\phi}(0)(\varphi_{\mu\nu} - \eta_{\mu\nu}\Box_\eta\varphi) = 0\,, \tag{48}$$

$$K_{,\phi}(0) + \left[K_{,X}(0) - 2G_{3,\phi}(0)\right]\Box_\eta\varphi + K_{,\phi\phi}(0)\varphi + G_{4,\phi}(0)R^{(1)} = 0\,, \tag{49}$$

where

$$G_4(0) = G_4(\phi_0, 0), \qquad K(0) = K_4(\phi_0, 0)\,,$$

and $G^{(1)}_{\mu\nu}$ and $R^{(1)}$ are the linearized Einstein tensor (9) and Ricci scalar (7), respectively. Furthermore, $K_{,X} = \partial K/\partial X$ is evaluated at $\phi = \phi_0$ and $X = X_0 = 0$; the remaining quantities are understood analogously.

In order to obtain gravitational-wave solutions on a flat background, $\eta_{\mu\nu}$ and ϕ_0 must satisfy Eqns (48) and (49). Therefore,

$$K(0) = 0\,, \qquad K_{,\phi}(0) = 0\,. \tag{51}$$

Substitution into Eqns (48) and (49) gives

$$G^{(1)}_{\mu\nu} - \sigma(\partial_\mu\partial_\nu\varphi - \eta_{\mu\nu}\Box_\eta\varphi) = 0\,, \tag{51}$$

$$(\Box_\eta - m_{\rm s}^2)\,\varphi = 0\,, \tag{52}$$

where $\sigma = G_{4,\phi}(0)/G_4(0)$ and $G_4(0) \neq 0$.

For both the BD theory without a potential and the BDV theory, $\sigma = 1/\phi_0$. In this case, Eqn (52) reduces to Eqn (25). The scalar Eqn (51), similarly to (36), also contains the squared effective mass of the scalar field:

$$m_{\rm s}^2 = -\frac{K_{,\phi\phi}(0)}{K_{,X}(0) - 2G_{3,\phi}(0) + 3G^2_{4,\phi}(0)/G_4(0)}\,. \tag{53}$$

In the BD theory without a potential, $K_{,\phi\phi}(0) = 0$, and hence $m_{\rm s} = 0$. In the BDV case, substitution of the corresponding functions K, G_3, G_4, and G_5 reduces expression (53) exactly to Eqn (37). Thus, in the BDV theory the squared effective mass is determined only by the local form of the potential, whereas in the Horndeski theory it depends on the field gradients and geometry through derivatives of the functions G_i and K.

Introducing the new variables

$$\theta_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,h - \eta_{\mu\nu}\sigma\varphi\,, \tag{54}$$

the original metric perturbation can be recovered by inverting the above relation:

$$h_{\mu\nu} = \theta_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\theta - \eta_{\mu\nu}\sigma\varphi\,. \tag{55}$$

Consequently, Eqn (51) becomes

$$\partial_\rho\partial_{(\mu}\theta^\rho_{\nu)} - \frac{1}{2}\Box_\eta\theta_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\partial_\rho\partial_\sigma\theta^{\rho\sigma} = 0\,. \tag{56}$$

After imposing the gauge conditions, Eqns (51) and (53) reduce to wave equations [34]:

$$\Box_\eta\theta_{\mu\nu} = 0\,, \tag{57}$$

$$(\Box_\eta - m_{\rm s}^2)\varphi = 0\,. \tag{58}$$

Equation (57) shows that the tensor mode is again a massless graviton with two polarization states, '+' and '×.' The scalar field ϕ is massive and decoupled from the massless tensor field $\theta_{\mu\nu}$. The subsequent analysis is analogous to that in the BDV theory: the dispersion relation, angular frequency, and group velocity are given by expressions (41), (42), and (43), respectively, while the polarization matrix takes the form

$$h_{\mu\nu} = \begin{pmatrix} \sigma & 0 & 0 & 0 \\ 0 & \theta_+ - \sigma & \theta_\times & 0 \\ 0 & \theta_\times & -\theta_+ - \sigma & 0 \\ 0 & 0 & 0 & -\sigma \end{pmatrix}, \tag{59}$$

where

$$\theta_+ = \theta_+(\omega t - kz)\,,$$
$$\theta_\times = \theta_\times(\omega t - kz)\,,$$
$$\sigma = \sigma(\omega_{\rm s}t - qz)\,.$$

Thus, in the Horndeski theory the propagation of gravitational waves is completely analogous to that in the Brans–Dicke theory with a potential: the scalar field gives rise to an additional transverse mode responsible for the 'scalar breathing' behavior, as well as a longitudinal polarization mode. The scalar mode propagates at a speed less than the speed of light. If the scalar component is massless, the theory reduces to the BD theory without a potential, leaving only the transverse scalar mode, which propagates at the speed of light [34].

3.3 $f(R)$ gravity

The vacuum action of $f(R)$ gravity is

$$S = -\frac{c^3}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\, f(R)\,, \tag{60}$$

where f is a sufficiently smooth function. Variation of this action with respect to $g^{\mu\nu}$ yields

$$f' R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} + g_{\mu\nu}\Box f' - \nabla_\mu\nabla_\nu f' = 0\,, \tag{61}$$

where $f' = \partial f(R)/\partial R$. Thus, the field equations in this theory are of fourth order.

We include this theory in the scalar-tensor section because it is equivalent to a model with the action

$$S = -\frac{c^3}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\,\left[\phi R - 2U(\phi)\right]. \tag{62}$$

The equivalence is established through a Legendre transformation by introducing

$$\phi \equiv f'(\chi)\,, \tag{63}$$

which is assumed to be invertible, $\chi = \chi(\phi)$, followed by setting $\chi = R$. The potential in Eqn (62) is

$$U(\phi) = \frac{1}{2}\left[\phi\chi(\phi) - f(\chi(\phi))\right]. \tag{64}$$

Varying the action (62) independently with respect to $g^{\mu\nu}$ and ϕ, one obtains

$$\phi G_{\mu\nu} + g_{\mu\nu}\Box\,\phi + U(\phi) g_{\mu\nu} - \nabla_\mu\nabla_\nu\phi = 0\,, \tag{65}$$

$$\Phi \equiv R - 2U_\phi = 0\,, \tag{66}$$

where $U_\phi = \mathrm{d}U/\mathrm{d}\phi$. Combining the trace of Eqn (65) with Eqn (66), we obtain the scalar equation

$$\Box\phi + 4U(\phi) - 2\phi U_\phi = 0\,. \tag{67}$$

Equations (65)–(67) are used to perform the linearization defined in (24), as in scalar-tensor theories. In $f(R)$ gravity, Minkowski spacetime is assumed to be the background solution. Using gauge conditions analogous to those employed in the BD and BDV theories, the linearized equations reduce to the familiar form

$$\Box_\eta\theta_{\mu\nu} = 0\,, \tag{68}$$

$$(\Box_\eta - m_\mathrm{s}^2)\,\varphi = 0\,. \tag{69}$$

Formally, these equations coincide with (38) and (36). The only difference is that now

$$\theta_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,h - \eta_{\mu\nu}\,\frac{\varphi}{f'(0)}\,, \tag{70}$$

while the squared effective mass is given by

$$m_\mathrm{s}^2 \equiv \frac{1}{3}\frac{f'(0)}{f''(0)}\,, \tag{71}$$

with the requirement $m_\mathrm{s}^2 \geqslant 0$. Thus, the reasoning and constructions from (39) to (59) can be repeated, with the same subsequent interpretation of the gravitational-wave properties. This close analogy is not surprising, since the scalar-tensor representation of $f(R)$ gravity differs from the BDV theory only by the absence of a kinetic term in the Lagrangian.

Finally, returning to a purely tensorial representation is achieved by applying the inverse linearized Legendre transformation, in which case the scalar mode is expressed in terms of the linearized curvature scalar as $\varphi = f''(0)R^{(1)}$.

3.4 Theories with a massive graviton

In models involving scalar fields, spontaneous symmetry breaking may occur, giving rise to massive modes. At present, one of the key experimental constraints on the graviton mass is provided by the GW170817 event [10]. The corresponding distance scale is $z < 0.01$, where z denotes the redshift. Since massive modes can act, for example, as sources of inflation, models with a massive graviton continue to attract a lot of interest. At the same time, agreement between the GW propagation speed and the speed of light at $z < 0.01$ can be achieved through screening mechanisms [33].

We consider the de Rham–Gabadadze–Tolley (dRGT) model, which is free from ghost degrees of freedom. In the absence of matter, its action is [36]:

$$S = -\frac{c^3}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\,\big(R + V(g,f)\big)\,, \tag{72}$$

where $V(g,f)$ is the graviton potential. It depends on the physical metrics g and auxiliary (reference) metric f, which makes it possible to avoid unphysical degrees of freedom while providing a source for the cosmological expansion. The potential $V(g,f)$ is defined as

$$V(g,f) = m^2\sum_{i=0}^{4} a_i U_i(g,f)\,, \tag{73}$$

$$m^2 = \frac{m_\mathrm{g}^2 c^2}{\hbar^2}\,,$$

where m is a mass parameter, m_g is the graviton mass, and a_i are dimensionless coupling constants. A key feature of this theory is that the graviton mass m_g is introduced explicitly at the level of the action rather than arising as an effective quantity. The potential can be expressed in terms of symmetric polynomials constructed from the tensor $\mathcal{K}^\mu_\nu$:

$$\begin{aligned} &U_0 = 1\,, \quad U_1 = [\mathcal{K}]\,, \quad U_2 = [\mathcal{K}]^2 - [\mathcal{K}^2]\,,\\ &U_3 = [\mathcal{K}]^3 - 3[\mathcal{K}]\,[\mathcal{K}^2] + 2[\mathcal{K}^3]\,,\\ &U_4 = [\mathcal{K}]^4 - 6[\mathcal{K}^2][\mathcal{K}]^2 + 8[\mathcal{K}^3][\mathcal{K}] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4]\,, \end{aligned} \tag{73a}$$

where $\mathcal{K}^\mu_\nu$ is defined by

$$\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \sqrt{g^{\mu\alpha}f_{\alpha\nu}}\,,$$

and δ^μ_ν is the Kronecker delta. The square brackets in (73a) denote traces: $[\mathcal{K}] = \mathcal{K}^\mu_\nu$, $[\mathcal{K}^2] = \mathcal{K}^\mu_\nu\mathcal{K}^\nu_\mu$, $[\mathcal{K}^3] = \mathcal{K}^\mu_\alpha\mathcal{K}^\alpha_\beta\mathcal{K}^\beta_\mu$, and so on. The number of such polynomials follows from the fact that, in a D-dimensional spacetime, there are exactly $D+1$ independent elementary symmetric polynomials. The term U_0 is trivial, while U_4 can be expressed through U_3, U_2, and U_1 using the field equations.

Let us discuss the auxiliary metric in more detail. To preserve general covariance when introducing it, the Stueckelberg-field formalism is employed, with fields ϕ^a, where the index $a = 0, 1, 2, 3$ labels the scalar fields. These fields map spacetime onto another manifold equipped with the Minkowski metric $\eta_{ab} = \mathrm{diag}\,(-1, 1, 1, 1)$. The auxiliary metric then takes the form

$$f_{\alpha\nu} = \partial_\alpha \phi^a \partial_\nu \phi^b \eta_{ab}\,, \tag{74}$$

where the Latin indices refer to the internal degrees of freedom of the fields and are contracted with η_{ab}, thereby preserving Lorentz invariance in the internal space. In the unitary gauge, $\phi^a = x^a$, the metric $f_{\mu\nu}$ reduces to $\eta_{\mu\nu}$, and the theory describes a massive graviton on a Minkowski background.

The equations of motion of the model can be written in the general form [37]

$$G_{\mu\nu} = -\frac{m^2}{\sqrt{-g}}\,\frac{\delta}{\delta g_{\mu\nu}} \sum_{i=0}^{4} \left(a_i \sqrt{-g}\, U_i\right), \tag{75}$$

where $i = 0, 1, 2, 3, 4$.

3.4.1 Plane gravitational waves. To obtain a plane-wave gravitational-wave solution, we consider the Stueckelberg fields in the unitary gauge, where the auxiliary metric reduces to the Minkowski metric, $f_{\mu\nu} = \eta_{\mu\nu}$. Since the interaction is constructed from scalars formed from both metrics, we introduce a mixed-index tensor. It can be defined by raising an index of $\eta_{\alpha\nu}$ with the inverse metric $g^{\mu\alpha}$:

$$\eta^\mu_\nu = g^{\mu\alpha}\eta_{\alpha\nu}\,. \tag{76}$$

The tensor $\mathcal{K}_{\mu\nu}$ can then be written as

$$\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \beta^\mu_\nu\,, \tag{77}$$

where β^μ_ν satisfies $\beta^\mu_\lambda \beta^\lambda_\nu = \eta^\mu_\nu$. Within this approach, it is convenient to write separately the terms entering the equation of motion (75):

$$\frac{\delta U_2}{\delta g_{\mu\nu}} = \mathcal{K}^{\mu\nu} - H^{\mu\nu} + [\mathcal{K}](g^{\mu\nu} - \mathcal{K}^{\mu\nu})\,, \tag{78}$$

$$\begin{aligned}\frac{\delta U_3}{\delta g_{\mu\nu}} = {}& \frac{3}{2}\,\mathcal{K}^\mu_\alpha H^{\alpha\nu} + \frac{3}{2}\,H^\mu_\alpha \mathcal{K}^{\alpha\nu} + 3H^{\mu\nu} - 6\mathcal{K}^{\mu\nu} \\ & + \frac{3}{2}\,U_2\,(g^{\mu\nu} - \mathcal{K}^{\mu\nu}) - 3[\mathcal{K}](H^{\mu\nu} - \mathcal{K}^{\mu\nu})\,,\end{aligned} \tag{79}$$

$$\begin{aligned}\frac{\delta U_4}{\delta g_{\mu\nu}} = {}& 12H^\mu_\alpha H^{\alpha\nu} - 24H^{\mu\nu} + 48\mathcal{K}^{\mu\nu} \\ & - 18\mathcal{K}^\mu_\alpha H^{\alpha\nu} - 18H^\mu_\alpha \mathcal{K}^{\alpha\nu} + 2U_3(g^{\mu\nu} - \mathcal{K}^{\mu\nu}) \\ & - 6U_2(H^{\mu\nu} - \mathcal{K}^{\mu\nu}) + 2[\mathcal{K}](3\mathcal{K}^\mu_\alpha H^{\alpha\nu} \\ & + 3H^\mu_\alpha \mathcal{K}^{\alpha\nu} + 6H^{\mu\nu} - 12\mathcal{K}^{\mu\nu})\,,\end{aligned} \tag{80}$$

where $H^\mu_\nu = \delta^\mu_\nu - \eta^\mu_\nu$.

Suppose that a plane wave is described by the metric

$$\mathrm{d}s^2 = -\,\mathrm{d}u\,\mathrm{d}v - F(u, x, y)\,\mathrm{d}u^2 + \mathrm{d}x^2 + \mathrm{d}y^2 \tag{81}$$

which satisfies the dRGT Eqn (75) [37], where $u = ct - z$ and $v = ct + z$ are null coordinates, and $F(u, x, y)$ is a solution of the equations of motion. Within this approach, the gravitational-wave solution of GR is also recovered when $F(u, x, y)$ is represented as a product of arbitrary functions of u and solutions of the Laplace equation in the variables x and y. One then determines successively $\eta_{\mu\nu}$, η^μ_ν, β^μ_ν and $\mathcal{K}^\mu_\nu$ in matrix form. Upon substitution into Eqns (78)–(80), only the expression (78) remains nonzero:

$$\frac{\delta U_2}{\delta g_{\mu\nu}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2F & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{82}$$

The equation of motion (75) then takes the form

$$G_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\nabla^2 F(u, x, y) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{83}$$

Thus, all components of the Einstein tensor in (83) vanish except

$$G_{11} = -2\nabla^2 F(u, x, y) \tag{84}$$

where ∇^2 is the Laplacian in the transverse plane [37]:

$$\nabla^2 F(u, x, y) = a_2 m^2 F(u, x, y)\,. \tag{85}$$

Hence, a plane-wave solution is obtained provided that its profile satisfies, depending on the sign of a_2, either the two-dimensional Helmholtz equation or the modified Helmholtz equation. Therefore,

$$F(u, x, y) = f_1(u)F_1(x, y) + f_2(u)F_2(x, y)\,, \tag{86}$$

where $f_1(u)$ and $f_2(u)$ are arbitrary functions of u, while $F_1(x, y)$ and $F_2(x, y)$ are solutions of the two-dimensional (modified) Helmholtz equation. Thus, $F(u, x, y)$ describes a plane wave propagating along the z direction at the speed of light.

The solution remains valid independently of the values of the coefficients a_i in the Lagrangian and reproduces the GR gravitational-wave solution in the massless limit.

3.4.2 Spherical gravitational waves. To obtain a spherical GW solution, it is convenient to write the equation of motion (75) explicitly as

$$R_{\mu\nu} - \frac{1}{2}\,R g_{\mu\nu} + m^2 \chi_{\mu\nu} = 0\,, \tag{87}$$

where $\chi_{\mu\nu}$ is the potential contribution to the gravitational field:

$$\begin{aligned}\chi_{\mu\nu} = {}& -\frac{a_1}{2}(U_1 g_{\mu\nu} - \mathcal{K}_{\mu\nu}) \\ & - \frac{a_2}{2}(U_2 g_{\mu\nu} - 2U_1 \mathcal{K}_{\mu\nu} + 2\mathcal{K}^2_{\mu\nu}) \\ & - \frac{a_3}{2}(U_3 g_{\mu\nu} - 3U_2 \mathcal{K}_{\mu\nu} + 6U_1 \mathcal{K}^2_{\mu\nu} - 6\mathcal{K}^3_{\mu\nu}) \\ & - \frac{a_4}{2}(U_4 g_{\mu\nu} - 4U_3 \mathcal{K}_{\mu\nu} + 12U_2 \mathcal{K}^2_{\mu\nu} \\ & - 24U_1 \mathcal{K}^3_{\mu\nu} + 24\mathcal{K}^4_{\mu\nu})\,.\end{aligned} \tag{88}$$

Consider a solution in Eddington–Finkelstein coordinates where the Minkowski metric is

$$\mathrm{d}s^2 = -\,\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,, \tag{89}$$

r is the areal coordinate, θ and ϕ are spherical angular coordinates, and u is the retarded Eddington coordinate. For a gravitational wave, we consider the metric tensor:

$$g_{\mu\nu} = \eta_{\mu\nu} + S_{\mu\nu}\,, \tag{90}$$

where $S_{\mu\nu}$ is a Brinkmann pp-wave [38], which in Eddington–Finkelstein coordinates has only one nonzero component, S_{uu}:

$$S_{uu} = \frac{2\mathcal{U}(u)P(\theta,\phi)}{r}\,\mathrm{d}u^2\,. \tag{91}$$

Here $\mathcal{U}(u)$ is the gravitational-wave profile, while $P(\theta,\phi)$ is a spherical harmonic of degree $l=0$ (i.e., the monopole term). If $\mathcal{U}$ and P are constant, the metric reduces to the Schwarzschild spacetime.

We next specify the auxiliary metric. In general, its choice depends on the problem under consideration. For plane waves, it is more convenient to use the Minkowski metric, whereas for spherical waves a singular reference metric is preferable [39]:

$$f_{uu} = \frac{\mathcal{U}^2}{r^6}\left[\left(\frac{\partial P}{\partial\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial P}{\partial\phi}\right)^2\right]\mathrm{d}u^2\,. \tag{92}$$

Taking Eqns (90), (91) and (92) into account, the expressions for $\mathcal{K}^{\mu}_{\nu}$ take the form

$$\begin{aligned}\mathcal{K}^{u}_{u} &= \frac{2}{r^2}\left(P\,\mathrm{d}\mathcal{U} - \mathcal{U}\,\mathrm{d}P\right)\mathrm{d}u\\ &= \frac{2\mathcal{U}}{r^2}\left(P\frac{\dot{\mathcal{U}}}{\mathcal{U}}\,\mathrm{d}u^2 - \frac{\partial P}{\partial\theta}\,\mathrm{d}u\,\mathrm{d}\theta - \frac{\partial P}{\partial\phi}\,\mathrm{d}u\,\mathrm{d}\phi\right).\end{aligned} \tag{93}$$

When constructing the spherical solution, U_4 can be expressed through U_3, U_2, and U_1 using the field equations, so that $a_4 = 0$. We then set $a_1 = 2/m^2$ and $a_2 = 0$ [38], in which case Eqn (87) reduces to

$$\mathcal{U}(u)\left[\frac{\partial^2 P}{\partial\phi^2} + \sin\theta\left(\cos\theta\,\frac{\partial P}{\partial\theta} + \sin\theta\,\frac{\partial^2 P}{\partial\theta^2}\right)\right] = 0\,. \tag{94}$$

The condition $\mathcal{U}=0$ corresponds to Minkowski spacetime, whereas the expression in square brackets describes the gravitational wave. Since P is a spherical harmonic of degree $l=0$ depending on θ and ϕ, there is a unique solution, $P=\mathrm{const}$, while $\mathcal{U}$ remains an arbitrary function of u. Substituting this solution into Eqn (87) yields precisely the Vaidya metric in Eddington–Finkelstein coordinates. In GR, the Vaidya metric solves the Einstein equations with null dust as the source. The function $\mathcal{U}(u)$ is interpreted as the Misner–Sharp mass, whose variation is associated with the energy flux carried by radiation. For example, the mass of a black hole may change due to outgoing or absorbed radiation. In dRGT gravity, since the coupling constant a_2 is dimensionless, one may state that $\mathcal{U}(u)$ has the dimensions of mass and does not depend directly on the parameter m. Thus, $\mathcal{U}(u)$ is interpreted as the Misner–Sharp mass. Comparing the phenomenology of GR and dRGT gravity, the resulting solution can be interpreted as describing energy transport by gravitational waves.

To derive the dispersion relation, let us consider a quasi-spherical solution with $P\neq\mathrm{const}$ and linearize the field Eqn (87). For this purpose, Eqn (91) must represent a small perturbation of the Minkowski background:

$$\left|\frac{2\mathcal{U}(u)P(\theta,\phi)}{r}\right| \ll 1 \Rightarrow \mathcal{U}(u) \ll r, P(\theta,\phi) \ll r\,. \tag{95}$$

Taking into account the singular auxiliary metric (92), the field equation becomes

$$\Box h_{ab} + \frac{m^2 a_1}{2}\,\mathcal{K}_{ab} = 0\,. \tag{96}$$

This equation can be interpreted as a wave equation for h_{ab} with a coordinate-dependent effective graviton mass:

$$m_{\mathrm{s}}^2 = -\frac{a_1 m^2\,\dot{\mathcal{U}}P}{r^2}\,, \tag{97}$$

Importantly, this effective mass is not constant but decreases as $1/r^2$ and depends on the coordinates u, r, θ, ϕ. At large distances, as $r\to\infty$, the mass tends to zero, which corresponds to massless behavior far from the source.

Note that u, appearing in (89), is the retarded Eddington coordinate and depends on r. To consider the wave in Minkowski spacetime and derive the dispersion relation for radial gravitons, the radial dependence must be restored in $\mathcal{U}(u)$:

$$S_{uu} = \frac{2\mathcal{U}\left(t - r_*(r)\right)P}{r}\,\mathrm{d}u^2\,, \tag{98}$$

where the tortoise coordinate is

$$r_* = r + 2\mathcal{U}P\ln\frac{r-2\mathcal{U}P}{2\mathcal{U}P}\,. \tag{99}$$

The latter relation is a recursive equation for r_* because $\mathcal{U}$ is itself a function of r_*. In the linear approximation, assuming that $\mathcal{U}$ varies slowly,

$$\mathcal{U}P \ll r\,. \tag{100}$$

Then

$$\frac{\mathrm{d}r_*}{\mathrm{d}r} = 1 + \frac{2\mathcal{U}P}{r}\,. \tag{101}$$

Thus, the resulting dispersion relation involving the coordinate r is

$$\begin{aligned}&\frac{c}{v_g} = \frac{kc}{\omega} = 1 + \frac{2\mathcal{U}P}{r}\ln\left(\frac{r}{2\mathcal{U}P} - 1\right) \Rightarrow\\ &v_{\mathrm{g}} = c\left(\frac{1}{1 + (2\mathcal{U}P/r)\ln\left((r/2\mathcal{U}P) - 1\right)}\right).\end{aligned} \tag{102}$$

This relation implies that v_{g} is always less than the speed of light outside the horizon, $r > \mathcal{U}P$, which is always satisfied in the linear approximation according to (95). In other words, the GW propagation speed appears to be less than the speed of light. However, according to [38], this is only an apparent effect arising because the linearized description neglects the backreaction of the wave on the geometry. This can be seen more clearly from the relation between the coordinates, $\mathrm{d}r_*/\mathrm{d}r > 0$. Thus, over the same coordinate-time interval, a graviton undergoes a smaller change in the areal coordinate than in the tortoise coordinate. Hence, the velocity measured with respect to the physical distance r is less than that

measured with respect to r_*, the latter being equal to the speed of light. We also note that, unlike in the theories considered above, the expression for the group velocity depends only on the coordinates. The effective mass can be written as

$$m_s^2 = \frac{\omega^2}{c^2} - k^2 = \omega^2\left(\frac{1}{c^2} - \frac{1}{v_g^2}\right). \tag{103}$$

First, m_g depends on the parameter m and on the coordinates u, r, θ, ϕ according to (97). Second, outside the horizon $m_s^2 < 0$, i.e., m the corresponding mass parameter becomes imaginary. In dRGT gravity, such a result is unphysical and arises because a linear approximation is being applied to an intrinsically nonlinear theory [38].

3.4.3 Gravitational-wave polarizations in dRGT gravity. To analyze polarizations, the most informative approach is to consider perturbations about a Minkowski background with the auxiliary metric f

$$f_{\mu\nu} = \eta_{\mu\nu}\,. \tag{104}$$

Expanding the dRGT action to second order and then varying it with respect to $h_{\mu\nu}$ yields the Fierz–Pauli equations [40]:

$$\begin{aligned} m^2(h_{\mu\nu} - \eta_{\mu\nu}h) = {} & \Box_\eta\, h_{\mu\nu} - \eta_{\mu\nu}\Box_\eta\, h + \partial_\mu\partial_\nu h \\ & + \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} - \partial_\mu\partial_\alpha h_\nu^\alpha - \partial_\nu\partial_\alpha h_\mu^\alpha\,. \end{aligned} \tag{105}$$

Thus, the remainder of this subsection concerns gravitational waves in the Fierz–Pauli model. To determine their polarization states, it is convenient to work in momentum space and consider a plane-wave solution

$$h_{\mu\nu} = a_{\mu\nu}\exp\left(\mathrm{i}p_\alpha x^\alpha\right). \tag{106}$$

Here $a_{\mu\nu}$ is the polarization tensor and $p^\mu = (\omega, \mathbf{p})$.

The tensor $a_{\mu\nu}$ is symmetric and therefore has 10 independent components. The explicit presence of the mass parameter m in Eqn (105) limits the number of conditions that can be imposed on the perturbation components compared with the linearized equations of GR. Nevertheless, the following constraints can be imposed. The transversality conditions $p^\mu a_{\mu\nu} = 0$ impose four constraints, while the tracelessness condition $a_\mu^\mu = 0$ provides one additional constraint. Thus, five degrees of freedom remain. We choose the graviton rest frame, in which $p^\mu = (m, 0, 0, 0)$. In this frame, the transversality condition becomes

$$p^\mu a_{\mu\nu} = m a_{0\nu} = 0\,,$$

so that $a_{0\nu} = 0$ for all ν. Thus, only the spatial components a_{ij} are nonzero in this frame. The tracelessness condition gives $a_{ii} = 0$. Consequently, there are two tensor modes identical to those of GR:

$$a_+ = a_{xx} = -a_{yy}\,, \tag{107}$$

$$a_\times = a_{xy} = a_{yx}\,. \tag{108}$$

Two vector modes a_{ij}^V correspond to oscillations in planes involving the direction of propagation:

$$a_{xz} = a_{zx}\,, \tag{109}$$

$$a_{yz} = a_{zy}\,. \tag{110}$$

A single scalar mode a_{ij}^S corresponds to longitudinal oscillations. From the tracelessness condition, one obtains

$$a_{zz} = -(a_{xx} + a_{yy})\,. \tag{111}$$

The metric perturbation $h_{\mu\nu}$ can be written as

$$h_{\mu\nu} = \sum_A a_{\mu\nu}^A F_A(t, z)\,, \tag{112}$$

where A labels the oscillation mode and the function $F_A(t, z)$ depends on the type of source. Taking normalization into account, $h_{\mu\nu}$ can be written explicitly as

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \dfrac{1}{\sqrt{2}}F_+ + \dfrac{1}{\sqrt{6}}F_s & \dfrac{1}{\sqrt{2}}F_\times & \dfrac{1}{\sqrt{2}}F_x \\ 0 & \dfrac{1}{\sqrt{2}}F_\times & -\dfrac{1}{\sqrt{2}}F_+ + \dfrac{1}{\sqrt{6}}F_s & \dfrac{1}{\sqrt{2}}F_y \\ 0 & \dfrac{1}{\sqrt{2}}F_x & \dfrac{1}{\sqrt{2}}F_y & -\dfrac{2}{\sqrt{6}}F_s \end{pmatrix}. \tag{113}$$

The polarization states in Fierz–Pauli theory coincide with those in dRGT gravity. In this problem, the linear approximation does not alter the physical polarization content. Let us additionally show how the dispersion relations differ between Fierz–Pauli theory and dRGT gravity. Taking into account the transversality condition $p^\mu a_{\mu\nu} = 0$ and the tracelessness condition $a_\mu^\mu = 0$, substitution of the solution (106) into Eqn (105) yields

$$(-p^2 + m^2)a_{\mu\nu} = 0\,, \tag{114}$$

and therefore, in terms of the angular frequency ω and wave vector $\mathbf{k}$, the expression is identical to that discussed above for scalar-tensor theories in the coordinates $x^i = (t, \mathbf{r})$:

$$\omega^2 = \sqrt{k^2 + m^2} \Rightarrow v_g = c\sqrt{1 - \frac{m^2c^2}{\omega^2}}\,. \tag{115}$$

Comparing (115) with (102), we see that in Fierz–Pauli theory, unlike in dRGT gravity, the GW propagation speed depends on both the parameter m and the wave frequency, and is less than the speed of light. We recall that the main drawback of Fierz–Pauli theory is the emergence of ghost degrees of freedom.

To summarize, dRGT gravity differs from the theories discussed above in several respects. First, the mass is introduced explicitly at the level of the action rather than appearing as an effective quantity in the field equations. Second, the presence of the original mass parameter leads to the spherical-wave solution (94), which is independent of this parameter. Considering a quasi-spherical wave with a singular auxiliary metric gives rise to the effective mass (97), which depends both on the original mass parameter and on the coordinates. Third, the dispersion relation (102) shows that the GW propagation speed in the curved metric equals the speed of light, whereas in the linearized approximation on a flat background it is lower and depends on the coordinates and on the function $\mathcal{U}(u)$, but not on the frequency. Hence, no frequency dispersion occurs. Fourth, five polarization states are possible: two transverse tensor modes, two longitudinal vector modes, and one scalar mode containing both transverse and longitudinal components, Eqns (107)–(111). Fifth, unlike in the scalar-tensor theories considered above, all modes are massive rather than only the scalar mode.

4. Bimetric modified newtonian dynamics

Bimetric Modified Newtonian Dynamics (BIMOND) is a relativistic generalization of Modified Newtonian Dynamics (MOND), which is intended to account for the effects usually attributed to dark matter by modifying Newtonian gravity at large distances. In BIMOND, gravity is described by two metrics, $g_{\mu\nu}$ and $\hat{g}_{\mu\nu}$. The action is given by [27]:

$$S = -\frac{c^3}{16\pi G}\int\left[\beta\sqrt{-g}\,R + \alpha\sqrt{-\hat{g}}\,\hat{R} + 2\,\frac{v_{g\hat{g}}}{l_M^2}\,\mathcal{M}\right]\mathrm{d}^4x + S_m[g_{\mu\nu},\psi_m] + \hat{S}_m[\hat{g}_{\mu\nu},\hat{\psi}_m]\,, \tag{116}$$

where R and $\hat{R}$ are the Ricci scalars associated with the metrics $g_{\mu\nu}$ and $\hat{g}_{\mu\nu}$, respectively, g and $\hat{g}$ are their determinants, and $v_{g\hat{g}} \equiv (g\hat{g})^{1/4}$ is the combined volume form. The length $l_M \equiv c^2/a_0$ sets the coupling strength, where a_0 is the MOND constant with dimensions of acceleration. The dimensionless interaction $\mathcal{M}$ is a function of the 'relative acceleration' tensors $l_M C^{\alpha}_{\beta\gamma}$, where

$$C^{\alpha}_{\beta\gamma} \equiv \Gamma^{\alpha}_{\beta\gamma} - \hat{\Gamma}^{\alpha}_{\beta\gamma}\,, \tag{117}$$

and $\Gamma^{\alpha}_{\beta\gamma}$ and $\hat{\Gamma}^{\alpha}_{\beta\gamma}$ are the Levi–Civita connections associated with the metrics $g_{\mu\nu}$ and $\hat{g}_{\mu\nu}$. The actions S_m and $\hat{S}_m$ correspond to ordinary matter and twin matter, respectively, each being a functional of the corresponding metric and matter field ψ_m or $\hat{\psi}_m$. The constants α and β are parameters of the model.

If, in (116), R and $\hat{R}$ are replaced by

$$-\Gamma^{(2)} \equiv g^{\mu\nu}\left(\Gamma^{\gamma}_{\mu\lambda}\Gamma^{\lambda}_{\nu\gamma} - \Gamma^{\gamma}_{\mu\nu}\Gamma^{\lambda}_{\lambda\gamma}\right), \tag{118}$$

$$-\hat{\Gamma}^{(2)} \equiv \hat{g}^{\mu\nu}\left(\hat{\Gamma}^{\gamma}_{\mu\lambda}\hat{\Gamma}^{\lambda}_{\nu\gamma} - \hat{\Gamma}^{\gamma}_{\mu\nu}\hat{\Gamma}^{\lambda}_{\lambda\gamma}\right), \tag{119}$$

the equations of motion remain unchanged because R and $\hat{R}$ differ from (118) and (119), respectively, only by total divergences. We can therefore rewrite the action as

$$S = -\frac{c^3}{16\pi G}\int\mathrm{d}^4x\left[-\beta\Gamma^{(2)}\sqrt{-g} - \alpha\hat{\Gamma}^{(2)}\sqrt{-\hat{g}} + 2\,\frac{v_{g\hat{g}}}{l_M^2}\,\mathcal{M}\right] + S_m[g_{\mu\nu},\psi_m] + \hat{S}_m[\hat{g}_{\mu\nu},\hat{\psi}_m]\,. \tag{120}$$

Consider a universe in which ordinary matter and twin matter are distributed symmetrically with respect to one another, and take the weak–field limit. Each metric tensor can then be represented as the sum of the Minkowski metric and a small perturbation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad h_{\mu\nu} \ll 1\,,$$
$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} + \hat{h}_{\mu\nu}, \qquad \hat{h}_{\mu\nu} \ll 1\,. \tag{121}$$

The metric determinants are then approximated by

$$g \approx \hat{g} \approx \det(\eta_{\mu\nu}) = -1\,. \tag{122}$$

We now rewrite the action (120) in the weak-field limit. In what follows, instead of $\Gamma^{(2)}$ and $\hat{\Gamma}^{(2)}$, we use their second-order approximations, $\Gamma^{(2)}(h_{\mu\nu})$ and $\Gamma^{(2)}(\hat{h}_{\mu\nu})$, respectively. Taking (121) into account, we obtain

$$C^{\alpha}_{\beta\gamma} = \frac{1}{2}\,\eta^{\alpha\rho}\left(\partial_\beta h^-_{\rho\gamma} + \partial_\gamma h^-_{\rho\beta} - \partial_\rho h^-_{\beta\gamma}\right), \tag{123}$$

where $h^-_{\mu\nu} \equiv h_{\mu\nu} - \hat{h}_{\mu\nu}$. As a consequence of (123), the dimensionless interaction $\mathcal{M}$ becomes a functional $\mathcal{M}_1(q^-)$, where $q^- \equiv ((1/a_0)\partial_\sigma h^-_{\mu\nu})^2$. For convenience in the subsequent derivation, we also introduce

$$\mathcal{M}_2(q^-) \equiv \mathcal{M}_1(q^-) - \frac{\alpha\beta}{2a_0^2(\alpha+\beta)}\,\Gamma^{(2)}(h^-_{\mu\nu})\,. \tag{124}$$

At this stage, we consider only the case $\alpha + \beta \neq 0$.

Let us examine the first two terms in the integrand of (120) in the weak-field approximation:

$$\beta\Gamma^{(2)}\sqrt{-g} + \alpha\hat{\Gamma}^{(2)}\sqrt{-\hat{g}} \approx \beta\Gamma^{(2)}(h_{\mu\nu}) + \alpha\Gamma^{(2)}(\hat{h}_{\mu\nu}) = \frac{\Gamma^{(2)}(h^+_{\mu\nu})}{\alpha+\beta}, \tag{125}$$

where we have introduced $h^+_{\mu\nu} \equiv \beta h_{\mu\nu} + \alpha\hat{h}_{\mu\nu}$. Note that the expression on the right-hand side is of second order in the first derivatives of the metric perturbation $h^+_{\mu\nu}$.

The matter part of the action (120) can be written as

$$S_m + \hat{S}_m = \frac{1}{2c}\int\left(h_{\mu\nu}\mathcal{T}^{\mu\nu} + \hat{h}_{\mu\nu}\hat{\mathcal{T}}^{\mu\nu}\right)\mathrm{d}^4x\,, \tag{126}$$

where the matter and twin–matter actions are represented as

$$S_m = \frac{1}{2c}\int h_{\mu\nu}\mathcal{T}^{\mu\nu}\,\mathrm{d}^4x\,, \tag{127}$$

$$\hat{S}_m = \frac{1}{2c}\int \hat{h}_{\mu\nu}\hat{\mathcal{T}}^{\mu\nu}\,\mathrm{d}^4x\,, \tag{128}$$

with $\mathcal{T}^{\mu\nu}$ and $\hat{\mathcal{T}}^{\mu\nu}$ denoting the corresponding energy–momentum tensors in Minkowski spacetime.

Multiplying (126) by $\alpha+\beta$, we can rewrite the integrand as follows:

$$(\alpha+\beta)\left(h_{\mu\nu}\mathcal{T}^{\mu\nu} + \hat{h}_{\mu\nu}\hat{\mathcal{T}}^{\mu\nu}\right) = (\beta h_{\mu\nu} + \alpha\hat{h}_{\mu\nu})\left(\mathcal{T}^{\mu\nu} + \hat{\mathcal{T}}^{\mu\nu}\right) + (h_{\mu\nu} - \hat{h}_{\mu\nu})\left(\alpha\mathcal{T}^{\mu\nu} - \beta\hat{\mathcal{T}}^{\mu\nu}\right) = h^+_{\mu\nu}\mathcal{T}^{\mu\nu}_+ + h^-_{\mu\nu}\mathcal{T}^{\mu\nu}_-\,, \tag{129}$$

where $\mathcal{T}^{\mu\nu}_+ \equiv \mathcal{T}^{\mu\nu} + \hat{\mathcal{T}}^{\mu\nu}$ and $\mathcal{T}^{\mu\nu}_- \equiv \alpha\mathcal{T}^{\mu\nu} - \beta\hat{\mathcal{T}}^{\mu\nu}$. Equation (126) can therefore be rewritten as

$$S_m + \hat{S}_m = \frac{c^3}{16\pi G(\alpha+\beta)}\int\frac{8\pi G}{c^4}\left(h^+_{\mu\nu}\mathcal{T}^{\mu\nu}_+ + h^-_{\mu\nu}\mathcal{T}^{\mu\nu}_-\right)\mathrm{d}^4x\,. \tag{130}$$

Substituting these intermediate results into (120) and multiplying by $\alpha+\beta$, we obtain

$$(\alpha+\beta)S = -\frac{c^3}{16\pi G}\int\left[-\Gamma^{(2)}(h^+_{\mu\nu}) + \frac{8\pi G}{c^4}h^+_{\mu\nu}\mathcal{T}^{\mu\nu}_+ + \frac{8\pi G}{c^4}\,h^-_{\mu\nu}\mathcal{T}^{\mu\nu}_- + 2(\alpha+\beta)a_0^2\mathcal{M}_2(q^-)\right]\mathrm{d}^4x\,. \tag{131}$$

Thus, after taking the weak-field approximation [28], the gravitational action separates into two mutually decoupled sectors associated with $h^+_{\mu\nu}$, represented by the first and second terms in (131), and $h^-_{\mu\nu}$, represented by the third and fourth terms. The field equations for $h^+_{\mu\nu}$ are identical to those of GR, but their source is the Minkowski-spacetime energy–momentum tensor $\mathcal{T}^+_{\mu\nu}$. The equations for $h^-_{\mu\nu}$ contain $\mathcal{T}^-_{\mu\nu}$ as a source and describe the MOND modification.

Let us discuss the choice of scalar arguments of $\mathcal{M}$. Define

$$\Upsilon_{\mu\nu} \equiv C^{\gamma}_{\mu\lambda} C^{\lambda}_{\nu\gamma} - C^{\gamma}_{\lambda} C^{\mu\nu}_{\lambda\gamma}\,,$$
$$\Upsilon \equiv g^{\mu\nu}\Upsilon_{\mu\nu}, \quad \hat{\Upsilon} \equiv \hat{g}^{\mu\nu}\Upsilon_{\mu\nu}\,. \tag{132}$$

The scalars Υ and $\hat{\Upsilon}$ have the same structure as $\Gamma^{(2)}$, $\hat{\Gamma}^{(2)}$. In the weak-field regime, $\Upsilon \approx \hat{\Upsilon} \approx \Gamma^{(2)}(h^{-}_{\mu\nu})$. For this choice of variables, $\mathcal{M}_2(q^{-})$ depends on the variable $\tilde{q}$:

$$\tilde{q} \equiv -\frac{1}{2a_0^2}\,\Upsilon = \frac{1}{8a_0^2}\left(\partial^{\gamma} h^{-\nu\rho}\left(\partial_{\gamma} h^{-}_{\nu\rho} - 2\partial_{\rho} h^{-}_{\nu\gamma}\right)\right.$$
$$\left. - \partial^{\gamma} h^{-}\left(\partial_{\gamma} h^{-} - 2\partial_{\rho} h^{-\rho}_{\gamma}\right)\right). \tag{133}$$

The field equations for $h^{-}_{\mu\nu}$ can then be written as

$$\Sigma_{\mu\nu} = \frac{8\pi G}{c^4(\alpha+\beta)}\,\mathcal{T}^{-}_{\mu\nu}\,, \tag{134}$$

where

$$\Sigma_{\mu\nu} \equiv \partial_{\lambda}\left(\mathcal{M}_2'(\tilde{q}) S^{\lambda}_{\mu\nu}\right),$$
$$S^{\lambda}_{\mu\nu} \equiv C^{\lambda}_{\mu\nu} - \frac{1}{2}\delta^{\lambda}_{\mu} C_{\nu} - \frac{1}{2}\delta^{\lambda}_{\nu} C_{\mu} + \frac{1}{2}\eta_{\mu\nu}\left(C^{\lambda} - \overline{C}^{\lambda}\right),$$
$$C_{\nu} \equiv C^{\sigma}_{\nu\sigma}, \quad \overline{C}^{\lambda} \equiv \eta^{\mu\nu} C^{\lambda}_{\mu\nu}\,. \tag{135}$$

In this case, $-\partial_{\lambda} S^{\lambda}_{\mu\nu} = G_{\mu\nu}(h^{-}_{\alpha\beta})$ is the Einstein tensor associated with the metric $\eta_{\mu\nu} + h^{-}_{\mu\nu}$ in the weak-field limit.

Let us now consider the case $\alpha + \beta = 0$, setting $\beta = 1$ for definiteness. The field equations then take the form

$$G_{\mu\nu}(h^{-}_{\lambda\rho}) = -\frac{8\pi G}{c^4}\left(\mathcal{T}_{\mu\nu} + \hat{\mathcal{T}}_{\mu\nu}\right), \tag{136}$$

$$G_{\mu\nu}(h_{\lambda\rho}) = \Sigma_{\mu\nu}(h^{-}_{\lambda\rho}) - \frac{8\pi G}{c^4}\,\mathcal{T}_{\mu\nu}\,. \tag{137}$$

The divergence of the field Eqn (136) vanishes identically, but $h^{-}_{\mu\nu}$ possesses no gauge freedom. Equation (137) implies four equations

$$\partial_{\mu}\Sigma^{\mu\nu}(h^{-}_{\lambda\rho}) = 0\,, \tag{138}$$

which, together with the six independent conditions following from (136), determine the Cauchy problem for $h^{-}_{\mu\nu}$. Once the equations for $h^{-}_{\mu\nu}$ have been solved, the resulting solution is substituted into (137), yielding the weak-field GR approximation for $h_{\mu\nu}$ with an additional source

$$-\frac{c^4}{8\pi G}\,\Sigma_{\mu\nu}(h^{-}_{\lambda\rho})\,, \tag{139}$$

which plays the role of phantom ('dark') matter for $h_{\mu\nu}$.

BIMOND involves two metrics: one describes gravity in the weak-field regime and the other in the strong-field regime. The acceleration scale a_0 and related quantities characterize the transition between these regimes. In strong gravitational fields, corresponding to accelerations greater than a_0, GWs propagate as in GR [27]. In weak gravitational fields, where accelerations are below a_0, GW properties such as frequency and amplitude may be modified, which could affect how such waves are detected and interpreted.

Any GWs within the Solar System can be represented as combinations of the waves $h_{\mu\nu}$ and $\hat{h}_{\mu\nu}$, each of which possesses two polarization modes, as in GR. BIMOND predicts no additional polarization modes. Studying GWs within BIMOND may therefore provide a way to test the predictions of the theory and to identify deviations from the standard gravitational framework. For example, GWs produced by mergers of compact objects in low-surface-brightness galaxies may carry information about the MOND regime of gravity. Propagation on the $\hat{g}_{\mu\nu}$ background may also lead to a difference between graviton and photon propagation speeds: a GW can respond to both the $g_{\mu\nu}$ and $\hat{g}_{\mu\nu}$ backgrounds, whereas photons are affected only by $g_{\mu\nu}$.

5. Quantum theories of gravity

5.1 Loop quantum gravity

Loop quantum gravity (LQG) [41–44] is a quantum theory of gravity formulated on a differentiable manifold, with GR recovered as its classical limit. Observable geometric quantities, such as areas and volumes, acquire discrete spectra [45–48], with the corresponding discreteness scale being of the order of the Planck scale. LQG allows the initial singularity to be avoided, replacing it with a bounce of the scale factor [49–52].

The properties of gravitational waves in LQG [53] are determined by the fact that spacetime discreteness in the semiclassical regime[3] leads to quantum corrections to the effective wave equation. Deviations from GR manifest themselves through a modification of the dispersion relation $\omega(k)$.

A cubic spatial lattice is considered. The index v runs over all lattice vertices, whose coordinates are given by $z = n\epsilon$ with $n = 0, \pm 1, \ldots, \pm N/2$, where ϵ is the discretization parameter, i.e., the edge length of a lattice. A semiclassical state Ψ is considered, represented as a tensor product of states associated with individual lattice edges. The quantum dynamics is approximated by the expectation value of the Hamiltonian $\hat{H} = \sum_v \hat{C}(v)$ in the semiclassical state Ψ. The effective Hamiltonian is constructed as the expectation value of the Hamiltonian-constraint operator $\hat{C}$, summed over all vertices v of the spatial lattice, in the full coherent state Ψ describing the discretized geometry corresponding to gravitational waves propagating on a Minkowski background:

$$H_{\text{eff}} = H_{\text{eff}}(h_i, p_i) = \left\langle \Psi, \sum_v \hat{C}(v)\Psi \right\rangle, \tag{140}$$

where $h_+ = h_1$ and $h_\times = h_2$ correspond to the components of the metric perturbations in the TT gauge, while $p_+ = p_1$ and $p_\times = p_2$ are the corresponding canonically conjugate momenta, i.e., $\{h_i(z), p_j(z')\} = \kappa\delta^i_j \delta(z - z')/L^2$. Here L is the linear size of the region under consideration, related to the lattice spacing ϵ and the number of nodes N by $N = L/\epsilon$. In what follows, the value of any variable f shifted by m lattice sites relative to the site z is denoted by $f^{\pm m} = f(z \pm m\epsilon)$; this notation is unrelated to GW polarizations.

The corresponding Hamilton equations

$$\dot{f} = \{f, H_{\text{eff}}\}$$

[3] The semiclassical regime refers to an approximation in which the quantum state of the system (an LQG coherent state) is sharply peaked around the value of the corresponding classical variable. In this regime, expectation values of the fundamental operators reproduce the classical quantities up to small quantum corrections.

are obtained by applying the Poisson bracket to any observable $f = f(h_i, p_i)$, resulting in the system

$$\dot{p}_i = (-)^{i+1}\frac{\beta}{2}\frac{p_{i+1}^{+} - p_{i+1}^{-}}{\epsilon} + \frac{1+\beta^2}{2\beta\epsilon} \times \left(\epsilon\mathbb{B}_i + (-)^i \frac{p_{i+1}^{+2} + 2(p_{i+1}^{+} - p_{i+1}^{-}) - p_{i+1}^{-2}}{4}\right), \quad (141)$$

$$\dot{\mathbb{B}}_i = -\frac{\beta}{2}\left(\frac{p_i^{+2} - 2p_i + p_i^{-2}}{\epsilon^2} - (-)^i \frac{\mathbb{B}_{i+1}^{+} - \mathbb{B}_{i+1}^{-}}{\epsilon}\right) + \frac{1+\beta^2}{8\beta}\left(\Delta_i - (-)^i \frac{\mathbb{B}_{i+1}^{+2} + 2\left(\mathbb{B}_{i+1}^{+} - \mathbb{B}_{i+1}^{-}\right) - \mathbb{B}_{i+1}^{-2}}{\epsilon}\right), \quad (142)$$

where

$$\mathbb{B}_i = \frac{B_i^{+2} + B_i^{-2}}{\epsilon^2\beta}$$

is a discrete auxiliary quantity constructed from the averaged values B_i^{+2} and B_i^{-2} of the metric perturbations h_i over the discretization scale ϵ, as well as the Immirzi parameter $\beta > 0$, which is a fundamental constant of the theory. Here Δ_i is another discrete quantity of the same type as $\mathbb{B}_i$, constructed analogously from the canonical momenta p_i; see Eqn (21) in [53].

Solving Eqns (141), (142) for $\mathbb{B}_i$ and differentiating the resulting expression with respect to time, one obtains a finite-difference equation for $\ddot{p}_i$ governing the dynamics of the system:

$$\ddot{p}_i = \frac{\beta^2}{8\epsilon^2}(sp_i^{+3} - 2p_i^{+2} - sp_i^{+} + 4p_i - sp_i^{-} - 2p_i^{-2} + sp_i^{-3}). \quad (143)$$

Given the periodic boundary conditions on the torus and the discrete translational invariance of the lattice, it is convenient to seek a solution in the form of a plane wave ansatz. To this end, we perform a discrete Fourier transform, representing the field at the lattice sites as an integral over plane waves in the first Brillouin zone:

$$p(z) = \frac{\epsilon}{2\pi}\int_{\mathrm{B}} \mathrm{d}k \exp(\mathrm{i}kz)\, u(k)\,.$$

Substitution of this ansatz into the dynamical equation (143) leads to the wave equation for the Fourier coefficients $u(k)$:

$$\ddot{u}(k) = -\omega^2(k)u(k)\,. \quad (144)$$

This yields the modified dispersion relation

$$\omega^2(k) = \frac{\sin^2(k\epsilon)}{\epsilon^2}\left((1+\beta^2)\cos(k\epsilon) - \beta^2\right).$$

In the continuum limit $\epsilon \to 0$, this expression reduces to the classical GR relation $\omega^2(k) = k^2$. Thus, the modification originates from the discrete structure of spacetime at Planckian scales. For sufficiently fine discretization, the changes induced by the modification of $\omega(k)$ are expected to be relatively small and difficult to detect. Conversely, gravitational waves with sufficiently large momentum k may exhibit appreciable deviations from the classical predictions. Unlike GR, where GWs propagate at the speed of light, LQG predicts the following GW velocity:

$$v_k = \left.\frac{\mathrm{d}|\omega(k)|}{\mathrm{d}k}\right|_{k=0} = 1 - \frac{5+3\beta^2}{4}k^2\epsilon^2 + \mathcal{O}(k^4\epsilon^4)\,, \quad (145)$$

which implies that short-wavelength gravitons propagate more slowly than light. No additional modes arise, so the number of polarizations remains equal to two, as in GR.

5.2 Gravitational waves in string theory

In string theory, fundamental interactions are described not in terms of point particles but in terms of one-dimensional extended objects — quantum strings propagating on a curved background. Within these theories, all elementary particles and their interactions arise from string excitations at scales of the order of the Planck length. Consistent quantum string theories can exist only in spacetimes of dimension greater than four, and one of the central problems is to describe the reduction from these higher dimensions to low-energy four-dimensional physics. Despite the absence of direct experimental confirmation, string theories are actively developed because they provide a powerful mathematical framework for studying gravity and quantum fields.

Gravitational-wave solutions in string theories follow from exact solutions of the string equations of motion [54, 55] to all orders in the inverse string tension. In this case, the classical background receives no additional world-sheet corrections. A special role among such solutions is played by the so-called Hpp waves [56–58] — plane-fronted pp waves in the presence of a homogeneous Ramond–Ramond 5-form flux. These solutions are remarkable because string theory on such a background is exactly solvable. In addition, such configurations arise in the 'Penrose limit' of the $\mathrm{AdS}^5 \times S^5$ string background [59–61] and are related to four-dimensional gauge theories with $\mathcal{N} = 4$ supersymmetry [62, 63].

In this section, we restrict our attention to explicit gravitational-wave solutions on an AdS background within type IIB theory. Type IIB theory is a ten-dimensional superstring theory with two Majorana–Weyl supercharges of the same chirality. The theory is divided into two sectors. The NS–NS (Neveu–Schwarz–Neveu–Schwarz) sector contains the metric $g_{\mu\nu}$, the dilaton ϕ, and the Kalb–Ramond 2-form B_2. The R–R (Ramond–Ramond) sector contains differential forms of even degree: the 0-form C_0 (the axion), the 2-form C_2, and the 4-form C_4. The field strengths associated with these potentials have degrees 1, 3, and 5. We consider the effective low-energy limit of this theory — ten-dimensional type IIB supergravity — on the $\mathrm{AdS}_3 \times S^3 \times \mathbb{R}^4$ background [112].

The metric is parametrized by a Siklos spacetime [64], which is an exact solution of the Einstein equations with a negative cosmological constant describing a plane GW propagating in anti-de Sitter spacetime. Siklos solutions are conformally equivalent to pp waves and constitute the only nontrivial Einstein spaces conformal to nondegenerate pp waves. In the present case, the metric takes the form

$$\mathrm{d}s^2 = q\left[+\frac{\mathrm{d}u^2}{u^2} + \frac{1}{u^2}\left(2\,\mathrm{d}x^+\,\mathrm{d}x^- + H(u, x^+)\,\mathrm{d}x^{+2}\right) + \mathrm{d}\Omega_3^2\right] + \sum_{i=1}^{4}\mathrm{d}x^{i\,2}\,. \quad (146)$$

Here x^+, x^-, and u are Poincará coordinates on AdS_3: $x^\pm$ are light-cone coordinates, while u is a spacelike 'radial' coordinate, with $u \to 0$ corresponding to the conformal boundary of AdS_3. For $H = 0$, the metric inside the square brackets reproduces $AdS_3 \times S^3$. The element $d\Omega_3^2$ is the metric of the 3-sphere parametrized by the angles θ, ϕ, ψ. The final term $\sum_{i=1}^{4}$ represents the flat Cartesian part of the four additional dimensions $\mathbb{R}^4$, which do not participate in the wave dynamics but are required to satisfy the critical dimensionality $d = 10$ of type IIB theory. Finally, the constant $q > 0$ determines the curvature radius of the AdS_3 and S^3 spaces.

The function $H(u, x^+)$ characterizes the gravitational-wave profile. The dependence of H on u in addition to x^+ means that the GW polarization is localized only within part of AdS_3, while for $H = 0$ the metric (146) reduces to the pure $AdS_3 \times S^3 \times \mathbb{R}^4$ background. For the gravitational-wave solution considered here, the only nonzero fields are the metric (146) and the R–R 3-form field strength $F^{(3)}$, which supports the curvature of the background. All other fields, including the dilaton, which is a scalar field corresponding to the first string excitation mode, may be set to zero when $H = 0$. Otherwise, as shown below, nonzero components of the NS–NS 3-form field must also be included.

The nonzero components of the Ricci tensor $(R_{\mu\nu})$ along the $x^\pm$ and u directions in Eqn (146) are

$$R_{++} = -\frac{H_{,uu}}{2} + \frac{1}{2}\frac{h_{,u}}{u} - \frac{2H}{u^2}\,,$$
$$R_{+-} = -\frac{2}{u^2}\,, \qquad R_{uu} = -\frac{2}{u^2}\,. \tag{147}$$

Along the S^3 directions, the nonzero components of the Ricci tensor are

$$R_{\theta\theta} = 2, \quad R_{\phi\phi} = 2\cos^2\theta, \quad R_{\psi\psi} = 2\sin^2\theta\,. \tag{148}$$

The components (148) reflect the curvature of S^3 and are independent of the wave profile H. The component R_{++} in (147) contains all information about the wave.

The field equations of type IIB theory on the background metric (146) are satisfied if the function $H(u, x^+)$ and the additional components of the NS–NS field, described by the functions $A_a(x^a, x^+)$ and $B_{ab}(x^a, x^+)$ on S^3, satisfy the following system. The field equation for the metric component g_{++} gives

$$-\frac{H_{,uu}}{2} + \frac{1}{2}\frac{H_{,u}}{u} = \frac{q^2}{u^4}\left(\frac{1}{2}A_a A^a + \frac{1}{4}B_{ab}B^{ab}\right), \tag{149}$$

where the indices of A_a and B_{ab} are raised and lowered with the S^3 metric. It is noteworthy that Eqn (149) resembles the equations for anti-de Sitter gravitational waves in four-dimensional gravity [64] when A_a and B_{ab} vanish. The equations for the 3-forms further imply

$$\nabla^a A_a = 0\,, \qquad A_a + \frac{1}{2}\nabla^b B_{ab} = 0\,, \tag{150}$$

where ∇^a is the covariant derivative defined by the metric on S^3. Using the Bianchi identity for the field $H^{(3)}$, one obtains

$$\partial_a A_b - \partial_b A_a = -2B_{ab}\,. \tag{151}$$

Consequently, B_{ab} is the field strength associated with the potential A_a. Thus, the general structure of GWs in type IIB string theory on the $AdS_3 \times S^3 \times R^4$ background is determined by the three functions H, A_a, and B_{ab} satisfying the system (149)–(151) [112].

The first solution is obtained for $A_a = B_{ab} = 0$, i.e., when the NS–NS field is trivial. Equation (149) with a vanishing right-hand side becomes

$$-\frac{1}{2}H_{,uu} + \frac{1}{2u}H_{,u} = 0 \tag{152}$$

whose general solution, i.e., the general wave profile, is

$$H(u, x^+) = f(x^+)u^2 + g(x^+)\,, \tag{153}$$

where f and g are arbitrary functions of x^+ only. In this case, the GW is described entirely by the metric, while the R–R 3-form merely compensates for the curvature of the $AdS_3 \times S^3$ background. This solution is directly analogous to gravitational waves in GR: the wave carries two polarizations encoded in the profile H, and no additional fields beyond the purely gravitational sector are required.

The second solution is obtained as a limiting case of the known $D1-D5$ brane solution on a pp-wave background [65]. In this case, the NS–NS field is nontrivial, while the wave profile becomes independent of x^+. Direct substitution using the S^3 metric shows that the functions

$$A_\theta = 0\,, \quad A_\phi = 2\mu\cos^2\theta\,, \quad A_\psi = 2\mu\sin^2\theta\,, \tag{154}$$

$$B_{\theta\phi} = \mu\sin 2\theta\,, \quad B_{\theta\psi} = \mu\sin 2\theta\,. \tag{155}$$

satisfy Eqns (149)–(151). Here the gravitational wave is characterized by the parameter μ. Substitution of (154), (155) into Eqn (149) gives the wave profile

$$H = -\frac{\mu^2 q^2}{u^2}\,. \tag{156}$$

For $\mu = 0$, the functions A_a and B_{ab} vanish, $H \to 0$, and the metric (146) reduces to the pure $R_3 \times S^3$ background.

Thus, type IIB theory admits at least two distinct classes of gravitational-wave solutions. The first, (153), is fully analogous to GR solutions: the wave is determined entirely by the metric and no additional fields are involved. The second, (153), (154), is a specifically string-theoretic solution and requires the simultaneous presence of the NS–NS field together with the metric perturbation.

Both solutions exist in ten-dimensional spacetime and are not directly observable: they do not describe gravitational waves in the four-dimensional Universe detected by instruments such as LIGO. Nevertheless, they demonstrate that string theory does in principle admit gravitational-wave solutions and establish their structure, in particular the relation between metric perturbations and the tensor fields of string theory. This structure determines which degrees of freedom emerge after compactification of the additional dimensions to four dimensions. The transition from exact ten-dimensional solutions to observational predictions requires, in particular, a choice of compactification scheme for the moduli fields.

The paradigm of string compactifications is discussed in [114], where possible observable consequences of string corrections to GR with additional compact dimensions are considered. A D-dimensional spacetime $M = \mathbb{R}^4 \times \mathcal{M}_{\text{int}}$ is considered, where $\mathcal{M}_{\text{int}}$ is a compact $(D-4)$-dimensional Ricci-flat manifold referred to as the *internal space*. In

addition to the two standard tensor GW modes, a scalar breathing mode appears, as well as vector modes associated with massless gauge fields (graviphotons) generated by compactification. The number of the latter is equal to the first Betti number of the manifold $\mathcal{M}_{\rm int}$. Additional scalar modes correspond to moduli fields describing volume-preserving deformations of the internal space. Standard detectors are not directly sensitive to this channel, since the moduli do not produce geodesic deviation in the noncompact directions.

5.3 Hořava–Lifshitz theory

The Hořava–Lifshitz (HL) model [66, 67] is an example of a renormalizable theory of gravity [68, 69]. At low energies, the theory is characterized by two coupling constants, β and α, while in the limit $\beta = 1$ and $\alpha = 0$ its field equations reduce to those of GR.

HL gravity violates Lorentz invariance. An additional scalar field, known as the *chronon*, is introduced into the GR action. The direction of its variation in spacetime defines a preferred time direction. Hypersurfaces of constant chronon correspond to hypersurfaces of constant time, which is why the theory is also referred to as chronometric. This implies the existence of preferred reference frames, a feature absent in GR. By foliating spacetime into hypersurfaces, the chronon is responsible for the violation of Lorentz invariance in the UV regime, which renders the theory power-counting renormalizable. We now consider the classical manifestations of the theory.

5.3.1 The $(3+1)$ representation of Hořava–Lifshitz theory. In the $(3+1)$ decomposition of HL theory, four-dimensional spacetime is foliated into three-dimensional spacelike hypersurfaces Σ_t parametrized by time t. On these hypersurfaces one defines the three-dimensional metric g_{ij}, whose determinant is denoted by g, the lapse function N, determining the proper-time separation between neighboring hypersurfaces, and the shift vector N^i, entering not as a generalized coordinate, but as a Lagrange multiplier. Unlike in GR, the lapse function N enters the potential rather than serving as a Lagrange multiplier. This is a consequence of the reduction of the symmetry group from the full diffeomorphism group Diff(M) to the foliation-preserving diffeomorphism group FDiff(M). In the Hamiltonian formulation, the generalized coordinates are the metric components g_{ij} and the lapse function N, with conjugate momenta π^{ij} and P_N, respectively.

The general action of HL theory in the $(3+1)$ representation is

$$S = \frac{2}{\kappa^2}\int \mathrm{d}t\, \mathrm{d}^3x\, N\sqrt{g}\big(K_{ij}K^{ij} - \lambda K^2 + \mathcal{V}(g_{ij}, N)\big)\,, \quad (157)$$

where the second fundamental form, or extrinsic-curvature tensor, is defined as

$$K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right), \quad (158)$$

and has trace K. The part of the action containing terms quadratic in K_{ij} is called the kinetic term, $K_{ij}K^{ij} - \lambda K^2$. The value $\lambda = 1/3$ of the dimensionless coupling constant λ is known as the *kinetic-conformal point*. At this point, the kinetic term acquires an additional conformal symmetry, although this symmetry does not apply to the full action; the physical degrees of freedom coincide exactly with the transverse-traceless tensor modes of GR, and no additional scalar modes arise [119].

The Hamiltonian of the $(3+1)$ formulation of HL theory at the kinetic-conformal point [68, 69, 116–118] is

$$H\int_{\Sigma_t}\mathrm{d}^3x\Big[+N\sqrt{g}\Big(\frac{\pi^{ij}\pi_{ij}}{g} - \mathcal{V}(g_{ij},N)\Big) - N_j H^j - \sigma P_N - \mu\pi\Big] + \beta E_{\rm ADM}\,, \quad (159)$$

where (N_j, σ, μ) are Lagrange multipliers and (H^j, P_N, π) are primary constraints:

$$\pi \equiv g_{ij}\pi^{ij} = 0, \qquad P_N = 0, \qquad H^j \equiv 2\nabla_i \pi^{ij} = 0\,.$$

The constraints $\pi = 0$ and $P_N = 0$ are second-class constraints and arise because N is not a Lagrange multiplier. The constraint $H^j = 0$ is a first-class constraint analogous to that in GR [116]. The potential

$$\mathcal{V} = \mathcal{V}^{(1)} + \mathcal{V}^{(2)} + \mathcal{V}^{(3)}$$

is the most general scalar quantity constructed from the three-dimensional spatial metric and the lapse function N. The potential does not depend on the conjugate momenta π^{ij} or on the functions $N_i \equiv g_{ij}N^j$. Its three contributions correspond to terms of second, fourth, and sixth order in spatial derivatives, respectively; the higher-order terms improve the ultraviolet behavior. The quantity

$$E_{\rm ADM} \equiv \oint_{\partial\Sigma_t}(\partial_j g_{ij} - \partial_i g_{jj})\,\mathrm{d}S^i$$

is the Arnowitt–Deser–Misner energy [117, 118], defined for asymptotically flat spacetimes describing isolated gravitating systems.

The dynamical field equations following from the Hamiltonian formalism [70] form the system

$$\dot{g}_{ij} = \frac{2N}{\sqrt{g}}\,\pi_{ij} + 2\nabla_{(i}N_{j)} - \mu g_{ij}\,, \quad (160)$$

$$\begin{aligned}\dot{\pi}^{ij} = {}& \frac{N}{2}\frac{g^{ij}}{\sqrt{g}}\,\pi^{kl}\pi_{kl} - \frac{2N}{\sqrt{g}}\,\pi^{ik}\pi^j_k + N\sqrt{g}\beta\Big(\frac{R}{2}\,g^{ij} - R^{ij}\Big)\\ & - \alpha N\sqrt{g}\Big(a^i a^j - \frac{1}{2}\,g^{ij}a_k a^k\Big)\\ & - \nabla_k\big[2\pi^{k(i}N^{j)} - \pi^{ij}N^k\big]\\ & + \beta\sqrt{g}\big[\nabla^{(i}\nabla^{j)}N - g^{ij}\nabla^2 N\big]\\ & + \mu\pi^{ij} + \{\pi^{ij},\mathcal{U}\}_{\rm PB}\,, \end{aligned} \quad (161)$$

where $a_i \equiv \partial_i N/N$ is a three-dimensional vector associated with spatial variations of the lapse function, while the subscript PB denotes the Poisson bracket in order to distinguish it from the standard anticommutator. Compared with the canonical GR system, Eqns (160), (161) contain additional terms: the contribution $-\mu g_{ij}$ in (160), associated with the constraint $\pi = 0$, and the term $\{\pi^{ij},\mathcal{U}\}_{\rm PB}$ in (161). These terms generate contributions involving spatial derivatives of order higher than two. Here

$$\mathcal{U} \equiv -\int_{\Sigma_t}\mathrm{d}^3x\, N\sqrt{g}\big(\mathcal{V}^{(2)} + \mathcal{V}^{(3)}\big) \quad (162)$$

is the part of the Hamiltonian corresponding to the higher-derivative contributions to the potential. The terms involving $\mathcal{V}^{(2)}$ and $\mathcal{V}^{(3)}$ generate fourth- and sixth-order spatial derivatives. The variables N^i, σ, and μ are nondynamical and are determined by the constraints. At the same time, unlike in GR, the lapse function N is a dynamical variable and enters the potential.

To evaluate the Hamiltonian, we consider the Lagrangian evaluated on the constraint surface [71]:

$$L = \int \mathrm{d}t\,\mathrm{d}^3x(\pi^{ij}\dot{g}_{ij} - \mathcal{H}) - E_{\mathrm{ADM}}\,, \tag{163}$$

where

$$\mathcal{H} \equiv -2\alpha\sqrt{g}\,\nabla_i\nabla^i N + \text{t.d.}$$

is the Hamiltonian density, consisting of the covariant Laplacian $\Delta = \nabla_i\nabla^i$ of the lapse function and a total divergence. The latter does not contribute to the Lagrangian owing to the asymptotic-flatness conditions. The coordinate condition

$$g_i = x^i + \left(\frac{1}{4\Delta}\right)g^{\mathrm{T}}_{,i}, \qquad \Delta \equiv \delta^{ij}\partial_i\partial_j\,, \tag{164}$$

is imposed, where g_i is the longitudinal part of the metric and Δ is the Laplacian in flat three-dimensional space. The inverse operator $1/\Delta$ is the Green operator of the Laplacian acting on functions that vanish at infinity: $(\Delta\varphi = \rho) \Rightarrow (\varphi = (1/\Delta)\rho)$. Following [71], the quantity

$$\mathcal{H} = -\beta\Delta g^{\mathrm{T}} - 2\alpha\sqrt{g}\,\nabla_i(g^{ij}\partial_j N) \tag{165}$$

is the Hamiltonian density.

5.3.2 Definition of the wave zone. The wave zone is a region of spacetime located at a distance from the gravitational-wave source that is much larger than the emitted wavelength, i.e., at $r \gg \lambda$ or $kr \gg 1$, where the physical degrees of freedom propagate as free radiation [70]. In other words, to leading order the solution satisfies a wave equation and describes spherical waves propagating to infinity without interaction with the source. In nonlinear theories, the existence of a wave zone is nontrivial because the free-propagation regime may be disrupted by field self-interactions. The direct detection of GWs by the LIGO-Virgo Collaboration makes the question of the existence of a wave zone particularly relevant.

In the most general case, a purely spatial tensor S_{ij} can be decomposed into three parts, S^{TT}_{ij}, S^{T}_{ij} and S^{L}_{ij}, referred to as the 'transverse-traceless,' 'transverse,' and 'longitudinal' parts, respectively. The transverse-traceless part is defined by the transversality condition $S^{\mathrm{TT}}_{ij,j} = 0$ and the tracelessness condition $S^{\mathrm{TT}}_{ii} = 0$ and contains the physical, observable degrees of freedom of the gravitational field. Thus, the physical degrees of freedom of the gravitational field are determined by the TT components of the metric g^{TT}_{ij} and of the conjugate momentum $\pi^{ij\,\mathrm{TT}}$.

In GR, the wave zone is characterized by the fact that the background curvature does not affect the propagation of the TT modes, which behave as gravitational waves in vacuum. This means that the physical degrees of freedom have the asymptotic behavior

$$g^{\mathrm{TT}}_{ij} \sim \pi^{ij\mathrm{TT}} = \mathcal{O}\left(\frac{1}{r}\right),$$

whereas the transverse part contains oscillatory and static contributions:

$$g^{\mathrm{T}}_{ij}\big|_{\mathrm{osc}} \sim \mathcal{O}\left(\frac{1}{r^2}\right), \quad g^{\mathrm{T}}_{ij}\big|_{\mathrm{st}} = \mathcal{O}\left(\frac{1}{r}\right), \quad \pi^{ijL} = \mathcal{O}\left(\frac{1}{r^2}\right).$$

The remaining parts of g_{ij} and π_{ij}, as well as the variables (N, N^i), are gauge dependent and therefore do not affect the propagation of the canonical modes [72]. We emphasize once again that (N, N^i) are not canonical variables but auxiliary fields determined from the constraint equations.

In $(3+1)$-dimensional HL theory at the kinetic-conformal point, there exists a wave zone analogous to that of GR [70]. Namely, the TT modes

$$\left(g^{\mathrm{TT}}_{ij},\, \pi^{ij\,\mathrm{TT}}\right) \sim f(\theta,\phi)\,\frac{\exp\left(\mathrm{i}(kr - \omega(k)t)\right)}{r}\,,$$
$$f(\theta,\phi) = \mathcal{O}(1)\,,$$

propagate without being affected by the static parts, just as in the linearized theory. The difference from GR lies in the dispersion relation and the GW propagation speed in the UV and IR regimes. We now discuss these differences.

5.3.3 Dynamical equations in the wave zone. The equations for the canonical transverse-traceless modes describe freely propagating fields [70]. The components of Eqn (160) in the gauge $g_{ij,j} = 0$ take the form

$$\dot{g}^{\mathrm{TT}}_{ij} - 2\pi^{\mathrm{TT}}_{ij} + (\mu g_{ij})^{\mathrm{TT}} \lesssim \frac{B}{r^2} + k^2\,\frac{\hat{A}\exp(\mathrm{i}kr)}{r^2}\,, \tag{166}$$

$$-2N_{(i,j)} + (\mu g_{ij})^{L} \lesssim \frac{B}{r^2} + k^2\,\frac{\hat{A}\exp(\mathrm{i}kr)}{r^2}\,, \tag{167}$$

$$\dot{g}^{\mathrm{T}}_{ij} + (\mu g_{ij})^{\mathrm{T}} \lesssim \frac{B}{r^2} + k^2\,\frac{\hat{A}\exp(\mathrm{i}kr)}{r^2}\,, \tag{168}$$

where $B = B(t,\theta,\phi)$ is the static contribution and $\hat{A} = \hat{A}(t,\theta,\phi)$ is the amplitude of the wave contribution; both functions and all their derivatives are bounded. The symbol $\lesssim$ indicates a falloff no slower than that of the corresponding right-hand side. The quantity μg_{ij} is decomposed as

$$\mu g_{ij} = (\mu g_{ij})^{\mathrm{TT}} + (\mu g_{ij})^{\mathrm{T}} + (\mu g_{ij})^{L}\,,$$
$$(\mu g_{ij})^{\mathrm{T}} = \mu\delta_{ij} - \frac{1}{\Delta}\mu_{,ij} + \mathcal{O}\left(\frac{1}{r^2}\right),$$
$$(\mu g_{ij})^{L} = \frac{1}{\Delta}\mu_{,ij} + \mathcal{O}\left(\frac{1}{r^2}\right), \quad (\mu g_{ij})^{\mathrm{TT}} = \mathcal{O}\left(\frac{1}{r^2}\right).$$

Equation (168) implies that $\mu \lesssim B/r^2 + k^2\hat{A}\exp(\mathrm{i}kr)/r^2$, and hence

$$\dot{g}^{\mathrm{TT}}_{ij} - 2\pi^{\mathrm{TT}}_{ij} \lesssim \frac{B}{r^2} + k^2\,\frac{\hat{A}\exp(\mathrm{i}kr)}{r^2}\,, \tag{169}$$

$$N_{(i,j)} \lesssim \frac{B}{r^2} + k^2\,\frac{\hat{A}\exp(\mathrm{i}kr)}{r^2}\,. \tag{170}$$

Evaluating the contributions to (161) gives

$$\dot{\pi}^{\mathrm{TT}}_{ij} = \frac{1}{2}\left(\beta\Delta + \beta_1\Delta^2 - \beta_3\Delta^3\right)g^{\mathrm{TT}}_{ij} + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{171}$$

Equations (169) and (171) show that in the wave zone the TT parts of the fields, up to terms of order $\mathcal{O}(1/r^2)$, satisfy a

system of differential equations containing spatial derivatives up to sixth order:

$$\ddot{g}_{ij}^{\mathrm{TT}} - \left(\beta\Delta + \beta_1\Delta^2 - \beta_3\Delta^3\right) g_{ij}^{\mathrm{TT}} = 0\,, \tag{172}$$

$$\ddot{\pi}_{ij}^{\mathrm{TT}} - \left(\beta\Delta + \beta_1\Delta^2 - \beta_3\Delta^3\right) \pi_{ij}^{\mathrm{TT}} = 0\,. \tag{173}$$

The higher-order Laplacian terms arise from the contributions of the potentials $\mathcal{V}^{(2)}$ and $\mathcal{V}^{(3)}$ and are precisely those responsible for the renormalizability of the theory in the UV regime.

A solution of (172) is constructed using the spherical Hankel functions $h_l^{(1)}(kr)$ and $h_l^{(2)}(kr)$. Since $-\Delta u = k^2 u$, the function

$$u(r,k) = \sum_{l,m} \left[A_{lm}^{(1)}\, h_l^{(1)}(kr) + A_{lm}^{(2)}\, h_l^{(2)}(kr)\right] Y_{lm}(\theta,\phi)$$

satisfies Eqn (172), provided that the dispersion relation

$$\omega^2 = \beta k^2 - \beta_1 k^4 - \beta_3 k^6 \tag{174}$$

holds. In the wave zone, the asymptotic behavior of the Hankel functions is

$$h_l^{(1)}(kr) \approx (-\mathrm{i})^{l+1}\,\frac{\exp(\mathrm{i}kr)}{r}\,, \quad h_l^{(2)}(kr) \approx (\mathrm{i})^{l+1}\,\frac{\exp(-\mathrm{i}kr)}{r}\,, \tag{175}$$

and the solution takes the form of an outgoing wave,

$$\Psi = \frac{\exp(-\mathrm{i}\omega t + \mathrm{i}kr)}{r} \sum_{l,m} A_{lm}^{(1)} (-\mathrm{i})^{l+1}\, Y_{lm}(\theta,\phi)\,, \tag{176}$$

where k is determined by (174). Thus, the phase velocity following from (174),

$$v_{\mathrm{ph}}^2 = \frac{\omega^2}{k^2} = \beta - \beta_1 k^2 - \beta_3 k^4\,, \tag{177}$$

depends on the wave number, meaning that the theory predicts gravitational-wave dispersion. In the IR limit $k \to 0$, the relation (174) reproduces the nondispersive wave equation of GR up to the value of the phase velocity, $v_{\mathrm{ph}} \to \sqrt{\beta}$. The low-energy limit of Eqn (174) for $k_{\max} \ll K_{\mathrm{UV}}$ is

$$\ddot{g}_{ij}^{\mathrm{TT}} - \beta\Delta g_{ij}^{\mathrm{TT}} = 0\,,$$

which coincides with the corresponding GR equation for $\beta = 1$.

The character of the UV corrections is determined by the signs of β_1 and β_3. For the right-hand side of (174) to remain positive for all $k^2 > 0$, i.e., for the theory to be free of instabilities such as ghosts and tachyons, one requires [66]

$$\alpha \neq 2\beta\,, \quad \beta > 0\,, \quad \beta_3 < 0\,, \quad \beta_1 < 2\sqrt{\beta|\beta_3|}\,. \tag{178}$$

The last three conditions guarantee the positivity of ω^2. When (178) is satisfied, each specified value of ω^2 corresponds to a unique value of k^2 satisfying (174).

The sign of β_3 is fixed by condition (178). Therefore, the contribution $-\beta_3 k^4 > 0$ to v_{ph}^2 is positive and increases monotonically with k. In the limit $k \to \infty$, this term dominates and the phase velocity grows without bound. Thus, the theory directly predicts superluminal GW propagation in the UV regime. The sign of β_1, however, is not fixed by (178). For $\beta_1 > 0$, the contribution $-\beta_1 k^2 < 0$ decreases v_{ph}^2, allowing for a subluminal GW propagation regime. For $\beta_1 \leqslant 0$, both high-energy contributions are positive, and $v_{\mathrm{ph}}^2 \geqslant \beta$ increases monotonically with k. Thus, both subluminal and superluminal GW propagation are possible within the theory.

The modified dispersion relation is a consequence of Lorentz-invariance violation in HL theory. Observations of the GW170817 event constrain the deviation of the GW velocity from the speed of light to $-3\times10^{-15} \leqslant (v_{\mathrm{GW}} - c)/c \leqslant 7\times10^{-16}$, which for $\beta = 1$ is automatically satisfied in the IR limit of the theory.

Solutions predicting additional polarization modes in HL theory are currently being discussed. One example contains an additional mixed scalar mode combining breathing and longitudinal polarizations [115]. The issue remains open.

6. Higher-dimensional theories

6.1 *D*-dimensional general relativity

In this section, we consider GW propagation in an empty D-dimensional spacetime within GR. The notation for scalar quantities remains unchanged. Tensor quantities carry indices $M, N = 0, 1, \ldots, D$. Constants acquire the subscript D. The GR action in dimension $D \geqslant 4$ with cosmological constant Λ_D is [92]

$$S = -\frac{1}{2\kappa_D}\int \mathrm{d}^D x \sqrt{|\hat{g}|}\,(R - 2\Lambda_D)\,, \tag{179}$$

where $\hat{g}_{MN}$ is the D-dimensional metric, R_{MN} and R are the Ricci tensor and curvature scalar of the D-dimensional spacetime, respectively, and κ_D is the coupling constant, i.e., the D-dimensional Einstein constant. The Einstein equations take the form

$$R_{MN} - \frac{\hat{g}_{MN}}{2}(R - 2\Lambda_D) = 0. \tag{180}$$

The metric $\hat{g}_{MN}$ is decomposed into a background metric g_{MN} and an infinitesimal perturbation h_{MN}:

$$\hat{g}_{MN} = g_{MN} + h_{MN}\,. \tag{181}$$

We consider the gauge transformation

$$h_{MN} \mapsto h_{MN} + \delta h_{MN}\,, \tag{182}$$

where $\delta h_{MN} = 2\nabla^{(0)}_{(M}\xi_{N)}$. Here $\nabla^{(0)}_M$ denotes the covariant derivative with respect to the background metric g_{MN}, which is also used to raise and lower indices, while ξ^M is an arbitrary sufficiently smooth D-vector with infinitesimal components of the same order as h_{MN}. The linearized equations of motion obtained from (180) are invariant under the transformations (182) on any background satisfying the background equations of motion, as follows from the diffeomorphism invariance of GR. In particular, this holds for the Ricci-flat background, $R^{(0)}_{MN} = 0$ and $\Lambda_D = 0$, used below in the analysis of four-dimensional modes.

To *simplify* the linearized Eqn (180), the linearized de Donder, or harmonic, gauge condition is imposed [92]:

$$\nabla_P^{(0)} g^{PQ} h_{QN} - \frac{1}{2}\nabla_N^{(0)} h_D = 0\,, \tag{183}$$

where $h_D = h_{QP} g^{PQ}$ is the trace of the metric perturbation. After imposing the gauge condition (183), the linearized Eqn (180) becomes

$$-\frac{1}{2}\Box_D^{(0)} h_{MN} + R^{(0)S}{}_{MNP}\, g^{PQ} h_{QS} = 0\,, \tag{184}$$

where $\Box_D^{(0)} \equiv g^{PQ}\nabla_P^{(0)}\nabla_Q^{(0)}$ is the d'Alembert operator associated with the background metric g_{MN} and $R^{(0)S}{}_{MNP}$ is the background higher-dimensional Riemann tensor. This equation is a generalization of the Lichnerowicz equation on a Ricci-flat background [120].

A D-dimensional spacetime is divided into four standard dimensions and N additional spatial dimensions. Following [92], we consider the most general maximally symmetric metric [93, 94], represented as a warped sum of metrics on the standard spacetime with coordinates x^μ, where $\mu, \nu = 0, \ldots, 3$, and on the internal space[4] with coordinates y^m, with $m, n = 0, \ldots, N$ and $N = D - 4$:

$$\mathrm{d}s^2 = \exp\big(2A(y)\big)\tilde{g}_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + g_{mn}(y)\,\mathrm{d}y^m\,\mathrm{d}y^n\,. \tag{185}$$

Here $\exp(2A(y))$ is the warp factor, and in what follows $g_{\mu\nu} = \exp(2A(y))\tilde{g}_{\mu\nu}$. The perturbation h_{MN} can contain three types of index structure: $h_{\mu\nu}$, $h_{\mu m}$, and h_{mn}. The latter two types are referred to in [92] as 'vectors' and 'scalars,' respectively. Substitution of this decomposition into (184) yields equations for each type of component:

$$\begin{aligned}
\mu\nu:\ & \exp(-2A)\tilde{\Box}_4 h_{\mu\nu} + \Delta_{\mathcal{M}} h_{\mu\nu} - h_{\mu\nu}\Delta_{\mathcal{M}} \ln\exp(2A) \\
& - 2\tilde{R}^{\pi}_{\mu\nu\sigma}\, g^{\sigma\rho} h_{\rho\pi} - \frac{1}{2}\exp(-2A) g^{pq}\partial_p \exp(2A)\partial_q \exp(2A) \\
& \times \big(\tilde{g}_{\nu\mu}\, h_4 + 2h_{\nu\mu}\exp(-2A)\big) + 2\exp(-2A)\tilde{\nabla}_{(\mu} h_{\nu)m}\, g^{mn}\partial_n \\
& \times \exp(2A) - \tilde{g}_{\mu\nu} h_{mn}\, g^{mr} g^{np}(\nabla_r \partial_p \exp(2A) \\
& - \exp(-2A)\partial_r \exp(2A)\partial_p \exp(2A)) = 0\,,
\end{aligned} \tag{186}$$

next,

$$\begin{aligned}
\mu n:\ & \exp(-2A)\tilde{\Box}_4 h_{\mu n} + \Delta_{\mathcal{M}} h_{\mu n} + \exp(-2A) g^{pq}\nabla_p h_{\mu n}\partial_q \\
& \times \exp(2A + \exp(-2A)\, h_{\mu m}\, g^{mp}\nabla_n \partial_p \exp(2A) \\
& - 2\exp(-4A) h_{\mu m}\, g^{mp}\partial_p \exp(2A)\partial_n \exp(2A) \\
& - \exp(-4A)\, h_{\mu n}\, g^{pq}\partial_p \exp(2A)\partial_q \exp(2A) \\
& - \frac{1}{2}\, h_{\mu n}\Delta_{\mathcal{M}} \ln\exp(2A) - \exp(-4A)\tilde{g}^{\pi\rho}\tilde{\nabla}_\pi\, h_{\mu\rho}\partial_n \exp(2A) \\
& + \exp(-2A) g^{pq}\partial_\mu h_{np}\partial_q \exp(2A) = 0\,,
\end{aligned} \tag{187}$$

and finally

$$\begin{aligned}
mn:\ & \exp(-2A)\tilde{\Box}_4\, h_{mn} + \Delta_{\mathcal{M}}\, h_{mn} + 2\exp(-2A)\, g^{pq}\partial_p \\
& \times \exp(2A)\nabla_q\, h_{mn} - 2\exp(-4A)\, g^{pq}\partial_p \exp(2A)\, h_{q(m}\partial_{n)} \\
& \times \exp(2A) - 2R^{s}_{mnp} g^{pq} h_{qs} - 2\exp(-4A)\tilde{g}^{\pi\rho}\tilde{\nabla}_\pi\, h_{\rho(m}\partial_{n)} \\
& \times \exp(2A) - h_4\nabla_n(\exp(-2A)\partial_m \exp(2A)) = 0\,.
\end{aligned} \tag{188}$$

[4] At this stage, the topology of the internal space is not assumed to be compact.

In these equations, $\tilde{\Box}_4$ and $\Delta_{\mathcal{M}}$ denote the four-dimensional wave operator and the N-dimensional Laplacian on the extra-dimensional space, respectively, while $h_4 = h_{\mu\nu} g^{\nu\mu}$, $\tilde{h}_4 = h_{\mu\nu}\tilde{g}^{\nu\mu}$ and $h_N = h_{mn} g^{nm}$ are the corresponding traces.

It should be noted that D-dimensional GR does not allow attenuation of the amplitude of a four-dimensional gravitational wave. Such attenuation would require a dissipative term of the form $\tilde{\nabla}_\mu h_{\pi\rho}$ in Eqn (186). However, no such term is present: it can be shown that, owing to linearization, preservation of diffeomorphism invariance, and the chosen form of the background metric (185), such a contribution cannot appear in Eqn (186) [92]. In particular, this distinguishes D-dimensional GR from theories of gravity such as the Dvali–Gabadadze–Porrati model, in which the 'leakage' of gravity into the extra dimensions does indeed lead to GW attenuation [92].

6.1.1 Kaluza–Klein theory. The Kaluza–Klein model is D-dimensional GR in which the $N = D - 4$ additional spatial dimensions are compactified, i.e., have a finite size small compared with observable scales. If the additional space $\mathcal{M}$ is compact and has no boundary, the Laplacian $\Delta_{\mathcal{M}}$ appearing in Eqns (186)–(188) has a discrete spectrum, with the lowest eigenvalue equal to zero:

$$\Delta_{\mathcal{M}}\omega_{\mathbf{k}} = -m_{\mathbf{k}}^2\omega_{\mathbf{k}}\,, \qquad m_0 = 0\,, \tag{189}$$

here $\omega_{\mathbf{k}}$ form an orthonormal basis of eigenfunctions of the Laplacian on the internal space, with eigenvalues $m_{\mathbf{k}}^2$. Therefore, perturbations of the four-dimensional part of the metric are expanded into a Kaluza–Klein 'tower of modes:'

$$h_{\mu\nu}(x, y) = \sum_{\mathbf{k}} h_{\mu\nu}^{\mathbf{k}}(x)\ \omega_{\mathbf{k}}(y)\,. \tag{190}$$

We recall that a Ricci-flat background, $R_{MN}^{(0)} = 0$, is assumed. The expansions of $h_{\mu\nu}$, $h_{\mu m}$ and h_{mn} are substituted into the linearized Eqns (186)–(188). For perturbations of the four-dimensional components on a flat background and with a constant warp factor $A(y) = A = \mathrm{const}$, the system diagonalizes, and each mode satisfies the Klein–Gordon equation:

$$\tilde{\Box}_4\, h_{\mu\nu}^{\mathbf{k}} - m_{\mathbf{k}}^2\, h_{\mu\nu}^{\mathbf{k}} = 0\,. \tag{191}$$

Consequently, the quantities $m_{\mathbf{k}}$ can be interpreted as the masses of the corresponding KK modes.

The zero mode $h_{\mu\nu}^0$ satisfies the equation for a free massless graviton, $\tilde{\Box}_4\, h_{\mu\nu}^0 = 0$, but the de Donder condition takes a nonstandard form:

$$g^{\pi\rho}\nabla_\pi\, h_{\rho\nu}^0 - \frac{1}{2}\nabla_\nu\, h_4^0 = \frac{1}{2}\nabla_\nu\, h_N^0\,, \tag{192}$$

where the right-hand side contains the generally nonzero trace h_N^0 over the extra-dimensional indices. Note that h_N^0 is invariant under gauge transformations, and therefore it cannot be removed from condition (192) by a gauge transformation. Nevertheless, the zero mode $h_{\mu\nu}^0$ still carries *two* degrees of freedom owing to the equivalence of the residual gauge freedom to that in GR.

Substituting plane-wave expansions of $h_{\mu\nu}^0$ and h_N^0 with an isotropic wave vector into condition (192) and fixing the residual gauge freedom by setting the components $h_{0\nu}$ to zero,

one obtains the polarization matrix [92]:

$$h^0_{ab}(t,x^3)=\begin{pmatrix} h^+-\frac{1}{2}f_N & h^\times \\ h^\times & -h^+-\frac{1}{2}f_N \end{pmatrix}_{ab}\cos\left(\omega\left(t-\frac{x^3}{c}\right)\right) \equiv h^\times_{ab}+h^+_{ab}+h^\circ_{ab}\,. \tag{193}$$

In addition to the standard h^+ and $h^\times$ modes, the polarization matrix contains the scalar breathing mode h°, associated with the trace h^0_N, which in turn represents an independent scalar field.

The equations for the higher modes also possess residual gauge freedom. Fixing this freedom allows the transversality and tracelessness conditions, i.e., the TT gauge, to be imposed on the higher KK modes $h^{\mathbf{k}}_{\mu\nu}(x)$, $\mathbf{k}\neq 0$ [99]:

$$\eta^{\kappa\lambda}\partial_\kappa h^{\mathbf{k}}_{\lambda\mu}=0\,,\qquad \eta^{\mu\nu}h^{\mathbf{k}}_{\mu\nu}=0\,.$$

In this case, $h^{\mathbf{k}}_{\mu\nu}$ satisfies the Pauli–Fierz equation for a massive spin-2 field of mass $m_{\mathbf{k}}$:

$$(\eta^{\kappa\lambda}\partial_\kappa\partial_\lambda - m^2_{\mathbf{k}})h^{\mathbf{k}}_{\mu\nu}=0\,. \tag{194}$$

A massive graviton in four dimensions possesses five independent polarizations. Although the polarization matrix contains six nonzero components, one of them is expressed through the others by the tracelessness condition. The same applies to Kaluza–Klein gravitons satisfying Eqn (194). In addition, the massive character of the modes modifies the dispersion relation:

$$\omega^2_{\mathbf{k}}=m^2_{\mathbf{k}}+\mathbf{p}^2_{\mathbf{k}}\,. \tag{195}$$

Thus, analysis of the perturbation h_{MN} of the metric in D dimensions reveals three types of deviations from the standard four-dimensional case in the resulting four-dimensional wave equations. The first is the coupling between different components: the vectors $h_{\mu n}$ and scalars h_{mn} enter the equations describing four-dimensional GWs. The second is associated with the dependence of the extra-dimensional geometry on the coordinates y^m, which results in a 'tower' of mutually related Kaluza–Klein modes $h^{\mathbf{k}}_{\mu\nu}(x^\pi)$ rather than a single four-dimensional wave. One of these modes is massless, whereas the others are massive and may correspond to Kaluza–Klein gravitons [95, 96]. The third effect originates from the presence of the conformal factor, i.e., the nontrivial background geometry. In the general case, terms depending on the warp factor $A(y)$ and its derivatives exhibit a complicated mixing between $h_{\mu\nu}$ and the scalar components.

The following general, and potentially model-independent, signatures of additional dimensions can be identified [92, 99]:

1. *Breathing* polarization mode: owing to the additional scalar degrees of freedom, a massless scalar breathing mode is present in addition to the usual '+' and '×' GW polarizations. It is characterized by a uniform deformation of the two transverse directions. It should be noted that analogous breathing modes also arise in other theories of gravity, in particular in scalar-tensor theories; therefore, if such a mode is detected, identifying the underlying theory becomes a separate problem.

2. *High-frequency* signals: in addition to the modified massless wave, there are additional four-dimensional signals corresponding to higher Kaluza–Klein modes $h^{k\neq 0}_{\mu\nu}$. They form a tower of massive modes and satisfy the massive-field dispersion relation (195), characterized by a discrete set of high frequencies fixed by the discrete Kaluza–Klein masses $m_{\mathbf{k}}$. Each massive mode possesses all six theoretically possible polarization components, although only five of them are independent.

The characteristic frequency scale of KK signals is determined by the relation $\nu\approx c/r_N$, where r_N is the characteristic size of an extra dimension. Current experimental constraints give $r_N\lesssim 10^{-4}$ m, corresponding to a lower estimate $\nu\gtrsim 10^{12}$ Hz. In string compactification models with $N=6$ and $r_N\sim 10^{-13}$ m, the characteristic frequencies can reach $\nu\sim 10^{21}$ Hz [92]. Both values exceed the upper sensitivity limit of LIGO, $\sim 10^3$–10^4 Hz, by many orders of magnitude, making *direct* detection of KK modes with current ground-based detectors unlikely. At the same time, primordial high-energy GWs subjected to cosmological redshift may potentially become observable with space-based interferometers.

It is also possible to consider GWs on a warped Minkowski background[5] with additional dimensions and a nonconstant warp factor [99], as motivated by string configurations. In this case, the eigenfunctions of the operator appearing in the equation for the KK spectrum are determined not by the standard Laplacian on $\mathcal{M}$ but by its warped modification. Numerical estimates for compact dimensions with toroidal topology and warp factors generated by string solutions such as p-branes show that, for physically reasonable parameters, $L/l_s\gtrsim 10$, where L is the torus radius, the shift in the KK spectrum relative to the standard case is small: the mass of the first KK mode decreases by no more than a factor of two. Although this effect is still too weak to bring the corresponding frequencies into the range of modern ground-based GW detectors, the spectral shift makes primordial high-energy GWs subjected to cosmological redshift potentially observable with space-based interferometers.

7. Theories with nonriemannian geometry

The theories considered in this section are formulated using differential forms, including a coframe basis,[6] i.e., a set of matrix-valued differential 1-forms $e^{(a)}=e^{(a)}_\mu\,\mathrm{d}x^\mu\in\Omega^1(M,\mathrm{Lie}(T_4))$ taking values in the Lie algebra of the translation group T_4. The coframes are basis sections of the cotangent bundle T^*M over spacetime M. Greek indices $\mu=0,\ldots,3$ denote spacetime indices, whereas Latin indices $a,b=0,1,\ldots,n$ denote matrix indices, i.e., indices in the Lie algebra. The coefficients $e^{(a)}_\mu$ are related to the tetrads $e_a=e^\mu_a\partial_\mu$ through the Kronecker delta according to $e^a e_b=\delta^a_b$. Integration over the manifold M is performed using the volume form

$$\mathrm{d\,vol}=\frac{1}{d!}\,\varepsilon_{abcd}\,e^a\wedge e^b\wedge e^c\wedge e^d\,, \tag{196}$$

where $\varepsilon_{abcd}=\sqrt{-g}\epsilon_{abcd}$ is the Levi–Civita tensor, related to the corresponding symbol through the determinant of the metric tensor. A detailed presentation of the formalism and its applications can be found in [116].

[5] The metric $\mathrm{d}s^2=\exp\big(2A(y)\big)\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu+\hat{g}_{pq}(y)\,\mathrm{d}y^p\,\mathrm{d}y^q$ is assumed, whose four-dimensional part is conformally flat while the metric in the remaining dimensions is arbitrary.

[6] Also referred to as the cotetrad.

A linear connection $\Gamma_a^b = \Gamma_{a\mu}^b \, \mathrm{d}x^\mu$, $\Gamma_a^b \in \Omega^1(M, gl_4(\mathbb{R}))$, is introduced, transforming as a gauge field under local linear frame transformations. The general construction of theories with non-Riemannian geometry consists in considering a sufficiently large gauge group and introducing gauge potentials associated with its direct or semidirect factors. The most general theory of this type for which GW solutions have been studied is metric-affine gravity (MAG). Its gauge potentials are the metric, the coframe, and the linear connection. It contains most classical teleparallel theories as special cases [101].

7.1 Metric-affine gravity

Metric-affine gravity is a gauge theory for the real affine group $\mathrm{Aff} = T_4 \rtimes GL(4, \mathbb{R})$ [100], i.e., the semi-direct product of the translation group T_4 and the general linear group $GL(4, \mathbb{R})$. The gaude potentials of the theory are: metric g_{ab}, coframe $e^a = e_\mu^a \, \mathrm{d}x^\mu$ and the linear connection $\Gamma_a^b = \Gamma_{\mu a}^b \, \mathrm{d}x^\mu$. The corresponding field strengths are: nonmetricity Q_{ab}, torsion T^a and curvature R^{ab}:

$$Q_{ab} = -Dg_{ab}\,, \tag{197}$$

$$T^a = De^a\,, \tag{198}$$

$$R_a^b = \mathrm{d}\Gamma_a^b - \Gamma_a^\mu \wedge \Gamma_\mu^b\,. \tag{199}$$

With each gauge potential one associates a Noether current: for the metric, the symmetric energy–momentum current σ^{ab}; for the coframe, the canonical energy–momentum current Σ_a; and for the linear connection, the hypermomentum Δ_b^a [100]. These currents act as sources in the equations of motion. The action is required to be invariant under transformations belonging to Aff. As a consequence, at any spacetime point one can choose a coframe e^a such that the components of the connection vanish, $\Gamma_a^b = 0$. Therefore, the most general gauge-invariant MAG Lagrangian can depend on Γ_a^b only indirectly [100], i.e., through the exterior covariant derivative D, the nonmetricity Q_{ab}, the torsion T^a, or the curvature R_a^b:

$$\mathcal{L}_{\mathrm{mag}} = \mathcal{L}_{\mathrm{mag}}(g_{ab}, e^a, Q_{ab}, T^a, R_a^b)\,, \tag{200}$$

The MAG equations of motion are obtained by varying the action with respect to the coframe and the linear connection. They are written in terms of generalized field momenta defined by

$$M_{ab} = -2\,\frac{\partial \mathcal{L}_{\mathrm{mag}}}{\partial Q_{ab}}\,, \tag{201a}$$

$$H_a = -\frac{\partial \mathcal{L}_{\mathrm{mag}}}{\partial T^a}\,, \quad H_b^a = -\frac{\partial \mathcal{L}_{\mathrm{mag}}}{\partial R_a^b} \tag{201b}$$

as

$$\frac{\delta \mathcal{L}_{\mathrm{mag}}}{\delta e^a} = -DH_a + E_a = \Sigma_a\,, \tag{202}$$

$$\frac{\delta \mathcal{L}_{\mathrm{mag}}}{\delta \Gamma_a^b} = -DH_b^a + E_b^a = \Delta_b^a\,. \tag{203}$$

Here E_a and E_b^a are the canonical gauge currents of energy–momentum and hypermomentum, respectively [26]. It should be noted that the MAG equation for nonmetricity, the so-called 'zeroth MAG equation,' is satisfied trivially as a consequence of the Noether identities [100].

The model considered below is referred to as the complete quadratic MAG model. Gravity at near-Planckian scales is assumed to be described by quadratic terms in the Lagrangian. The most general complete quadratic Lagrangian that does not depend explicitly on the linear connection Γ and depends on the remaining field strengths and potentials is

$$\begin{aligned} V = \frac{1}{2\kappa c}\Bigg[& a_0\, \eta_{ab} \wedge R^{ab} - T^a \wedge \sum_{I=1}^{3} a_I \star \left({}^{(I)}T_a\right) \\ & - Q_{ab} \wedge \sum_{I=1}^{4} b_I \star \left({}^{(I)}Q^{ab}\right) \\ & - 2b_5\left({}^{(3)}Q_{ac} \wedge e^a\right) \wedge *\left({}^{(4)}Q^{bc} \wedge e_b\right) \\ & - 2e^a \wedge \star T^b \wedge \sum_{I=1}^{3} c_I\, {}^{(I+1)}Q_{ab}\Bigg] - \frac{1}{2\rho} R_{ab} \wedge \star\Bigg[\sum_{I=1}^{6} w_I\, {}^{(I)}W^{ab} \\ & + \sum_{I=1}^{5} z_I\, {}^{(I)}Z^{ab} + v_1 e^a \wedge \left(e_c \lrcorner\, {}^{(5)}W^c{}_b\right) \\ & + v_2 e^c \wedge \left(e_a \lrcorner\, {}^{(2)}Z_{cb}\right) + \sum_{I=3}^{5} v_I e^a \wedge \left(e_c \lrcorner\, {}^{(I)}Z^c{}_b\right)\Bigg]\,, \end{aligned}$$

where $\star$ is the Hodge duality operator and $\lrcorner$ denotes the interior product, i.e., the contraction of a vector with a differential form. The Lagrangian is constructed as a gauge-invariant linear combination of 4-forms involving curvature R, torsion T, nonmetricity Q, the coframe e, and the antisymmetric $W_{ab} = R_{[ab]}$ and symmetric $Z_{ab} = R_{(ab)}$ parts of the curvature. In addition, a scale parameter $l_\rho^2 = \kappa c/\rho$ is introduced, where κ is the Einstein gravitational constant and ρ is a coupling constant with dimensions inverse to action, $[\rho] = [1/\hbar]$. Hence $[l_\rho^2] = [L^2]$ has dimensions of area and determines the characteristic scale at which post-Riemannian effects, i.e., the influence of quadratic terms in the Lagrangian, become relevant.

To describe plane waves, an ansatz is introduced for the gauge potentials g_{ab}, e^a, Γ_a^b, with the flat background $g_{ab} = \eta_{ab} = \mathrm{diag}\,(-,+,+,+)$. A coframe is constructed by dividing the local coordinates into two groups, $x^\mu = (x^i, x^I)$, where $i = 0, 3$ are longitudinal coordinates and $I = 1, 2$ are transverse coordinates. The first group is denoted by $x^i = (x^0 = \sigma, x^3 = \rho)$. In these coordinates,

$$e^0 = \frac{1}{2}\,(U-1)\,\mathrm{d}\sigma - \frac{1}{2}\,\mathrm{d}\rho\,, \tag{204}$$

$$e^3 = \frac{1}{2}\,(U+1)\,\mathrm{d}\sigma - \frac{1}{2}\,\mathrm{d}\rho\,, \tag{205}$$

$$e^I = \mathrm{d}x^I\,, \tag{206}$$

where $U = U(\sigma, x^I)$. The interval $\mathrm{d}s^2 = \eta_{ab} e^a e^b$ is then

$$(e^0)^2 = \frac{(U-1)^2}{4}\,\mathrm{d}\sigma^2 + \frac{1}{4}\,\mathrm{d}\rho^2 - \frac{U-1}{2}\,\mathrm{d}\sigma\,\mathrm{d}\rho\,,$$

$$(e^3)^2 = \frac{(U+1)^2}{4}\,\mathrm{d}\sigma^2 + \frac{1}{4}\,\mathrm{d}\rho^2 - \frac{U+1}{2}\,\mathrm{d}\sigma\,\mathrm{d}\rho\,.$$

Hence,

$$-(e^0)^2+(e^3)^2=\frac{(U+1)^2-(U-1)^2}{4}\,\mathrm{d}\sigma^2 + \left(\frac{U-1}{2}-\frac{U+1}{2}\right)\mathrm{d}\sigma\,\mathrm{d}\rho = U\,\mathrm{d}\sigma^2-\mathrm{d}\sigma\,\mathrm{d}\rho\,.$$

Taking (206) into account, we obtain the plane wave metric

$$\mathrm{d}s^2=-\mathrm{d}\sigma\,\mathrm{d}\rho+U\,\mathrm{d}\sigma^2+\delta_{IJ}\,\mathrm{d}x^I\,\mathrm{d}x^J\,. \tag{207}$$

It remains to construct the linear connection. By analogy with electromagnetism, a wave 1-form $k=\mathrm{d}\sigma$ is introduced, with tetrad and cotetrad components $k_a=(1,0,0,-1)$ and $k^a=(-1,0,0,-1)$, together with vector quantities $V^a=V^a(\sigma,X^I)$ and $W^a=W^a(\sigma,X^I)$ orthogonal to k_a:

$$k_aV^a=0\,,\quad k_aW^a=0\,, \tag{208}$$

as well as a 1-form $u=u_a(\sigma,x^I)e^a$ orthogonal to k^a:

$$k\wedge\star u=0\,,\quad k^au_a=0\,. \tag{209}$$

The linear-connection 1-form can then be written as

$$\Gamma^b_{\ a}=-k\big(k_aV^b+k^bW_a\big)+k_ak^bu\,. \tag{210}$$

The ansatz for the gauge potentials describing the gravitational-wave solution therefore consists of the coframe $e^a=(e^0,e^I,e^3)$ and the connection $\Gamma^b_{\ a}$. To ensure orthogonality of the introduced quantities U, V, and W to the covector k^a, only projections of the following form are considered:

$$W^a=\begin{cases}W^i=0,\\ W^I=W^I(\sigma,x^J),\end{cases} \tag{211}$$

with analogous assumptions for $V=V^I(\sigma,x^J)$, $u_I=u_I(\sigma,x^J)$ and $U=U(\sigma,X^J)$.

Thus, the potential ansatz contains seven functions determining the wave profile. Their explicit form is obtained from the field equations, whose component representation is given below. The following quantities are used:

$$\underline{W}_I=\delta_{IJ}W^J\,,\quad \underline{V}_I=\delta_{IJ}V^J\,,\quad \underline{u}^A=\delta^{IJ}u_J\,, \tag{212}$$

together with the differential operator $\underline{\partial}^I=\delta^{IJ}\partial_J$ and the two-dimensional Levi–Civita tensor ε_{IJ}.

The first equation of the resulting system corresponds to the longitudinal (ij) component of the symmetric part of (203) and reads

$$\frac{2c_1-a_1}{2}\,\partial_IU+\left[\frac{a_0+2a_1}{2}-3c_1-\frac{4b_1-4b_2}{3}\right]\underline{W}_I + \left[\frac{a_0}{2}-c_1-\frac{4b_1-4b_2}{3}\right]\underline{V}_I + \left[4c_1-a_1-\frac{4b_1+8b_2}{3}\right]u_I+\frac{l_\rho^2}{4}\varepsilon_{IJ}\underline{\partial}^J \times\Big[(z_1-z_2+2v_2)\varepsilon^{KL}\partial_K\underline{W}_L+(z_1-z_2)\varepsilon^{KL}\partial_K\underline{V}_L + 2(z_1+z_2-v_2)\varepsilon^{KL}\partial_Ku_L\Big]=0\,. \tag{213}$$

The second equation, obtained from the same part, is

$$\partial_\sigma\big[(z_4-z_1+3v_4)\partial_IW^I+(z_4-z_1+v_4)\partial_IV^I-4z_1\partial_I\underline{u}^I\big]=0. \tag{214}$$

The third equation follows from the transverse-longitudinal $[Ij]$ component of the antisymmetric part of (203):

$$\frac{a_0+a_1}{2}\,\partial_IU+\frac{2c_1-a_0-2a_1}{2}\,\underline{W}_I+\frac{a_0+2c_1}{2}\,\underline{V}_I + (a_1-2c_1)u_I-\frac{l_p^2}{4}\Big[2w_1\Delta\underline{W}_I-2w_1\Delta\underline{V}_I + (2w_4+v_4)\,\partial_I\partial_JW^J+(-2w_4+v_4)\,\partial_I\partial_JV^J\Big] - \frac{l_p^2}{4}\,\varepsilon_{IJ}\underline{\partial}^J\Big[(-2w_2+v_2)\varepsilon^{KL}\partial_K\underline{W}_L+ + (2w_2+v_2)\varepsilon^{KL}\partial_K\underline{V}_L-2v_2\varepsilon^{KL}\partial_Ku_L\Big]=0\,. \tag{215}$$

The fourth equation corresponds to the transverse-longitudinal (Ij) component of the symmetric part of (203):

$$\frac{a_1-2c_1}{2}\partial_IU+\left[\frac{a_0}{2}-a_1+3c_1-\frac{8b_1+4b_2}{3}\right]\underline{W}_I + \left[\frac{a_0}{2}+c_1-\frac{8b_1+4b_2}{3}\right]\underline{V}_I+\left[a_0+a_1-4c_1 - \frac{8b_1-8b_2}{3}\right]u_A-\frac{l_p^2}{4}\Big[2z_1\Delta\underline{W}_I+2z_1\Delta_I\underline{V}_I + (z_1+z_4+3v_4)\partial_I\partial_JW^J + (z_1+z_4+v_4)\partial_I\partial_JV^J\Big]-\frac{l_\rho^2}{4}\varepsilon_{IJ}\underline{\partial}^J\Big[(-z_1-z_2 + 2v_3)\varepsilon^{KL}\partial_K\underline{W}_L-(z_1+z_2)\varepsilon^{KL}\partial_K\underline{V}_L - 2(z_1-z_2+v_2)\varepsilon^{KL}\partial_Ku_L\Big]=0\,. \tag{216}$$

Finally, the system is supplemented by the MAG field Eqn (202) written explicitly as

$$-\frac{a_1}{2}\Delta U+\left[\frac{a_0}{2}-c_1+a_1\right]\partial_IW^I - \left[\frac{a_0}{2}+c_1\right]\partial_IV^I-\left[a_1-2c_1\right]\partial_I\underline{u}^A=0\,. \tag{217}$$

The GR plane-wave solution is also a solution of MAG because the *Riemannian geometry* underlying GR is characterized by vanishing torsion and nonmetricity. This is achieved by choosing $u=0$, $W^I=-V^I=1/2\delta^{IJ}\partial_JU$ and $\Theta=0$. With this substitution into Eqns (213)–(217), the equations reduce to

$$\frac{a_0}{2}\Delta U=0\,, \tag{218a}$$

$$v_4\partial_\sigma\Delta U=0\,, \tag{218b}$$

$$-2(w_1+w_4)\,\frac{l_\rho^2}{4}\,\partial_A\Delta U=0\,, \tag{218c}$$

$$v_4\,\frac{l_\rho^2}{4}\,\partial_A\Delta U=0\,. \tag{218d}$$

Equations (218a–d) are satisfied provided that the function U describing the wave profile obeys the Laplace equation in

the transverse space:

$$\Delta U \equiv \delta^{IJ}\partial_I\partial_J U = 0\,. \tag{219}$$

This equation contains no derivatives with respect to σ. Consequently, first, there is no dispersion, and second, the GWs propagate at the speed of light. The general solution of (219) can be decomposed into two independent harmonic functions:

$$U = h_+(\sigma, x^1, x^2) + h_\times(\sigma, x^1, x^2),$$
$$\Delta h_+ = \Delta h_\times = 0\,, \tag{220}$$

where h_+ and $h_\times$ are the standard tensor polarizations of GR, $+$ and $\times$.

It should be emphasized that the system (213)–(217) is a general result. We now turn to its particular realizations. In the most general case, GWs in MAG may possess up to six possible polarization modes, one of which corresponds to a massless graviton analogous to that of GR, while the remaining modes are massive torsion and nonmetricity modes [102].

7.2 Metric-affine gravity: the general teleparallel case

General teleparallel geometry is a particular case of metric-affine geometry characterized by vanishing spacetime curvature, $R_a^b = 0$, with the choice $\underline{\mathrm{d}}W^a = 0$ and $\underline{\mathrm{d}}V^a = 0$, where $\underline{\mathrm{d}} = \mathrm{d}x^I\partial_I$ is the transverse differential. With this substitution, Eqn (214) becomes an identity, while the remaining equations of the system can be rewritten as

$$(a_0 - 4b_1)\,\Phi_I = 0\,, \tag{221}$$

$$3(a_0 + 2c_1)\Theta_I + 2(a_0 + 2b_2)\Psi_A = 0\,, \tag{222}$$

$$2(a_0 + a_1)\,\Theta_I + (a_0 + 2c_1)\,\Psi_A = 0\,, \tag{223}$$

where the functions Θ, Φ, and Ψ, describing the wave profile, are defined as

$$\Theta_I = \frac{1}{2}\,\partial_I U - \underline{W}_I + u_I\,, \tag{224}$$

$$\Phi_I = \underline{W}_I + \underline{V}_I + u_I\,, \tag{225}$$

$$\Psi_I = \underline{W}_I + \underline{V}_I - 2u_I\,. \tag{226}$$

Equation (217) reduces to

$$a_1\Delta U + (a_1 - 2c_1)\Delta\mathcal{U} = 0\,, \tag{227}$$

where $\mathcal{U}$ is a potential such that $u_I = 1/2\partial_I\mathcal{U}$. The system (221)–(223) can be written in matrix form:

$$\begin{pmatrix} 0 & (a_0 - 4b_1) & 0 \\ 2(a_0 + a_1) & 0 & (a_0 + 2c_1) \\ 3(a_0 + 2c_1) & 0 & 2(a_0 + 2b_2) \end{pmatrix} \begin{pmatrix} \Theta_I \\ \Phi_I \\ \Psi_I \end{pmatrix} = 0\,, \tag{228}$$

and admits nontrivial solutions provided that the determinant vanishes:

$$(a_0 - 4b_1)\left[3(a_0 + 2c_1)^2 - 4(a_0 + a_1)(a_0 + 2b_2)\right] = 0\,. \tag{229}$$

This condition specifies the class of MAG models that admit gravitational-wave solutions compatible with teleparallel gravity and, correspondingly, with GR.

Note that if the determinant of the system is nonzero, i.e., if the constraints on the coupling constants are not satisfied, the only admissible solution is flat Minkowski spacetime, corresponding to vanishing Q, T, and R. Thus, the constraint ttll1 on the coupling constants ensures the existence of nontrivial solutions.

7.3 Metric-affine gravity: Weitzenbck geometry

Weitzenbck geometry, i.e., classical teleparallelism, corresponds to MAG with vanishing nonmetricity. This is achieved by choosing $u = 0$ and $W^a = -V^a$ in the metric-affine equations, while the functions Φ_I and Ψ_I, defined in (224)–(226), satisfy $\Phi_I = \Psi_I = 0$. These conditions reduce the equations to

$$\frac{a_1}{2}\,\Delta U = 0\,. \tag{230}$$

and impose the following constraints on the coupling constants:

$$a_0 + a_1 = 0\,, \qquad c_1 + \frac{a_0}{2} = 0\,. \tag{231}$$

Thus, Eqn (230) is analogous to Eqn (218a) for GWs in GR. Consequently, the structure of gravitational waves in teleparallel gravity with vanishing nonmetricity is identical to that of gravitational waves in GR.

7.4 Metric-affine gravity: symmetric teleparallelism

Symmetric teleparallel geometry corresponds to vanishing curvature $R_a^b = 0$ and torsion $T^a = 0$, with nontrivial nonmetricity Q_{ab}. For this purpose, one chooses

$$W^a = W^a(\sigma),\ V^a = V^a(\sigma),\ \partial_I(U + \mathcal{U}) = 2\underline{W}_I\,. \tag{232}$$

As a result, the system (228) and Eqn (217) reduce to

$$(a_0 - 4b_1)\,\Phi_I = 0\,, \tag{233}$$

$$(a_0 + 2b_2)\,\Psi_I = 0\,, \tag{234}$$

$$(a_0 + 2c_1)\,\Psi_I = 0\,, \tag{235}$$

$$-c_1\Delta U = 0\,, \tag{236}$$

which yields the following constraint on the coupling constants:

$$a_0 - 4b_1 = 0,\ a_0 + 2b_2 = 0,\ 2c_1 + a_0 = 0\,. \tag{237}$$

As in the case of classical teleparallel geometry, when the constraint (237) is satisfied, the wave equation becomes identical to (218), i.e., to the corresponding GR equation.

7.5 Metric-affine gravity: the most general case

We now consider the most general ansatz for the transverse vector quantities. Six scalar potentials are introduced: three even-parity potentials W, V, and U, and three odd-parity potentials $\bar{W}$, $\bar{V}$, $\bar{U}$, in terms of which the components W^I, V^I and u_I are expressed as

$$W^I = \frac{1}{2}\left(\delta^{IJ}\partial_J W + \varepsilon^{IJ}\partial_J\bar{W}\right), \tag{238}$$

$$V^I = \frac{1}{2}\left(\delta^{IJ}\partial_J V + \varepsilon^{IJ}\partial_J\bar{V}\right), \tag{239}$$

$$u_I = \frac{1}{2}\left(\partial_I U + \varepsilon_{IJ}\delta^{JK}\partial_K\bar{U}\right). \tag{240}$$

The six introduced potentials are analogues of the classical *Hertz potentials* in electrodynamics. The even potentials W, V, and U are associated with even-parity gravitational modes, while the odd potentials $\bar{W}$, $\bar{V}$, and $\bar{U}$ are associated with odd-parity modes.

For convenience, new potentials are introduced:

$$X_0 = W - V, \quad X_2 = W + V + U,$$
$$X_1 = U - W + U, \quad X_3 = W + V - 2U, \tag{241}$$

and, analogously, for the odd sector:

$$\bar{X}_1 = -\bar{W} + \bar{U}, \quad \bar{X}_2 = \bar{W} + \bar{V} + \bar{U}, \quad \bar{X}_3 = \bar{W} + \bar{V} - 2\bar{U}. \tag{242}$$

The inverse relations are

$$W = \frac{X_0}{2} + \frac{X_2}{3} + \frac{X_3}{6}, \quad V = -\frac{X_0}{2} + \frac{X_2}{3} + \frac{X_3}{6}, \quad U = \frac{X_2}{3} - \frac{X_3}{3}, \tag{243}$$

while the metric-profile function is expressed as $U = X_0/2 + X_1 + X_3/2$.

Substitution of (238)–(240) into the system (213)–(217) leads to a complete separation of the even and odd sectors, which follows from the absence of parity-odd terms in the Lagrangian (200).

Let us first consider the **even sector**. The equations for the potentials X_0, X_1, X_2, and X_3 form a second-order system:

$$(2c_1 - a_1)X_1 + \frac{a_0 - 4b_1}{3} X_2 + \left(-\frac{a_0}{2} - c_1 + \frac{2(a_0 + 2b_2)}{3}\right) X_3 = 0, \tag{244a}$$

$$(a_0 + a_1)X_1 + \left(\frac{a_0}{2} + c_1\right) X_3 - \frac{l_\rho^2}{4}\left[2(w_1 + w_4)\,\Delta X_0 + \frac{2v_4}{3}\,\Delta X_2 + \frac{v_4}{3}\Delta X_3\right] = 0, \tag{244b}$$

$$(a_1 - 2c_1)X_1 + \frac{2(a_0 - 4b_1)}{3} X_2 + \left(\frac{a_0}{2} + c_1 - \frac{2(a_0 + 2b_2)}{3}\right) X_3 - \frac{l_\rho^2}{4}\left[v_4 \Delta X_0 + \frac{2\Lambda_0}{3}\Delta X_2 + \frac{\Lambda_0}{3}\Delta X_3\right] = 0, \tag{244c}$$

$$\frac{a_0}{2}\Delta X_0 - a_1 \Delta X_1 - c_1 \Delta X_3 = 0, \tag{244d}$$

where $\Lambda_0 = 3z_1 + z_4 + 2v_4$. An additional equation determines the dependence on the longitudinal coordinate:

$$\partial_\sigma\left[v_4 \Delta X_0 + \frac{2(z_4 - 3z_1 + 2v_4)}{3}\,\Delta X_2 + \frac{z_4 + 3z_1 + 2v_4}{3}\,\Delta X_3\right] = 0. \tag{245}$$

A wave solution is sought in the form

$$X_n = X_n^{(0)}(\sigma)\,\exp\left(\mathrm{i} q_I x^I\right), \quad n = 0, 1, 2, 3. \tag{246}$$

After substitution of (246), the system (244a–d) reduces to an algebraic problem for the amplitudes $X_n^{(0)}$ in matrix form:

$$\mathcal{M}(Q^2)\begin{pmatrix} X_0^{(0)} \\ X_1^{(0)} \\ X_2^{(0)} \\ X_3^{(0)} \end{pmatrix} = 0, \tag{247}$$

where $Q^2 = l_\rho^2/4\, q_I q_J \delta^{IJ}$ is unrelated to the nonmetricity scalar, which will be introduced below. The matrix $\mathcal{M}$ is of size 4×4; however, the condition $\det \mathcal{M} = 0$ reduces to a quadratic equation in Q^2, determining the masses of two massive modes.

The system (244a–d) admits a distinguished particular solution for $X_2 = X_3 = 0$, i.e., for $U = 0$ and $W = -V$. The remaining equations for X_0 and X_1, for $v_4 \neq 0$ and $a_1 \neq 2c_1$, give $\Delta X_0 = 0$ and $X_1 = 0$, from which

$$\Delta U = 0, \quad W = -V = U, \quad U = 0. \tag{248}$$

This solution is identical to the GR solution and describes a massless graviton.

We now turn to the **odd sector**. In this case, the variable $\bar{X}_2$ is independent of the pair $\bar{X}_1, \bar{X}_3$. The equation for $\bar{X}_2$ is

$$(a_0 - 4b_1)\bar{X}_2 - l_\rho^2 z_1 \Delta \bar{X}_2 = 0, \tag{249}$$

while the pair $\bar{X}_1, \bar{X}_3$ satisfies a coupled system. After substituting $\bar{X}_J = \bar{X}_J^{(0)}(\sigma)\,\exp\left(\mathrm{i}\bar{q}_I x^I\right)$, the algebraic problem for the odd amplitudes can be written as

$$\mathcal{M}(\bar{Q}^2)\begin{pmatrix} \bar{X}_1^{(0)} \\ \bar{X}_2^{(0)} \\ \bar{X}_3^{(0)} \end{pmatrix} = 0. \tag{250}$$

The matrix is

$$\mathcal{M}(\bar{Q}^2) = \begin{pmatrix} 0 & a_0 - 4b_1 + 4z_1\bar{Q}^2 & 0 \\ a_0 + 2c_1 - 2\bar{Q}^2\Lambda_2 & 0 & \frac{2(a_0 + 2b_2)}{3} - \bar{Q}^2(\Lambda_2 + \Lambda_3) \\ a_0 + a_1 - \bar{Q}^2\Lambda_1 & 0 & \frac{a_0}{2} + c_1 - \bar{Q}^2\Lambda_2 \end{pmatrix}, \tag{251}$$

where $\bar{Q}^2 = l_\rho^2/4\,\bar{q}_I \bar{q}_J \delta^{IJ}$ and

$$\Lambda_1 = 4(w_1 + w_2), \quad \Lambda_2 = 2(w_1 + w_2) + v_2, \quad \Lambda_3 = \frac{z_1 + 3z_2}{3}. \tag{252}$$

Nontrivial solutions exist when the following two conditions are satisfied:

$$a_0 - 4b_1 + 4z_1\bar{Q}^2 = 0, \qquad A\bar{Q}^4 + B\bar{Q}^2 + C = 0, \tag{253}$$

where the combinations of coupling constants are defined as

$$A = 2\Lambda_2^2 + \Lambda_1(\Lambda_2 + \Lambda_3), \tag{254}$$

$$B = -4\left(\frac{a_0}{2} + c_1\right)\Lambda_2 + (a_0 + a_1)(\Lambda_2 + \Lambda_3) - \frac{2(a_0 + 2b_2)}{3}\Lambda_1, \tag{255}$$

$$C = 2\left(\frac{a_0}{2} + c_1\right)^2 - \frac{2(a_0 + 2b_2)(a_0 + a_1)}{3}. \tag{256}$$

Thus, the odd sector contains three propagating odd-parity modes determined by the relations (253).

Therefore, in the most general case, gravitational waves in MAG are described by a superposition of six modes: one massless mode, corresponding to the GR graviton (248); two massive even-parity modes, determined by the quadratic equation for Q^2 obtained from the matrix (247); and three massive odd-parity modes, determined by (248). The appearance of odd-parity modes within a parity-even Lagrangian is explained by the fact that the odd potentials $\bar{W}$, $\bar{V}$, and $\bar{U}$ enter the ansatz (238)–(240) through the parity-odd Levi-Civita tensor ε_{IJ} rather than through parity-odd terms in the Lagrangian [102]. The complete separation into even and odd sectors no longer holds in MAG models containing parity-odd coupling constants, which constitutes a separate problem that has not yet been fully investigated [103].

7.6 $f(T)$ gravity

Gravitational waves in modified teleparallel theories were first studied within $f(T)$ gravity [104, 105]. It was shown that no additional modes appear at first order and that the form of the gravitational-wave signal coincides with that in GR. At the perturbative level, $f(T)$ gravity effectively reduces to the teleparallel equivalent of GR, and therefore GWs possess the same two standard polarizations as in GR. To search for further deviations from GR, we consider this situation in more detail.

The Lagrangian of $f(T)$ gravity is a function of the torsion scalar. The coefficients $T^a_{\mu\nu}$ of the torsion form (see (198)),

$$T^a = T^a_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$$

are expressed in terms of the coefficients e^a_μ and $\omega^b_{a\mu}$ of the tetrads $e^a = e^a_\mu\,\mathrm{d}x^\mu$ and the Lorentz connection $\omega^a_b = \omega^a_{b\mu}\,\mathrm{d}x^\mu$ as

$$T^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^a_{b\mu} e^b_\nu - \omega^a_{b\nu} e^b_\mu \,. \tag{257}$$

We introduce the *superpotential tensor*, which carries two spacetime indices and one tetrad index:

$$S^{\mu\nu}_a = \frac{1}{2}\left(K^{\mu\nu}_a + e^\mu_a T^{\alpha\nu}_\alpha - e^\nu_a T^{\alpha\mu}_\alpha\right), \tag{258}$$

where K is the *contorsion tensor*

$$K_{\mu\nu\rho} = \frac{1}{2}\left(T_{\mu\nu\rho} + T_{\rho\nu\mu} - T_{\nu\rho\nu}\right). \tag{259}$$

Coordinate indices are raised by the metric, and the last index is contracted with the tetrad:

$$K^{\mu\nu}_a = g^{\mu\alpha} g^{\nu\eta} e^\rho_a K_{\alpha\beta\rho} \,.$$

These two tensors allow the torsion scalar T to be defined as the contraction of the superpotential and the torsion tensor:

$$T = S^{\mu\nu}_a T^a_{\mu\nu} \,. \tag{260}$$

The action of the $f(T)$ modification of teleparallel gravity is

$$S = \frac{1}{16\pi G}\int \mathrm{d}^4 x e f(T) + \int \mathrm{d}^4 x e \mathcal{L}_m \,, \tag{261}$$

where $\mathcal{L}_m$ is the matter Lagrangian density and $e = \det(e^a_\mu) = \sqrt{-g}$.

The field equations in tetrad form are given by [107]

$$e^{-1} f_{\mathrm{T}}\,\partial_\nu(e S^{\mu\nu}_a) + f_{\mathrm{TT}}\, S^{\mu\nu}_a\,\partial_\nu T - f_{\mathrm{T}} T^b_{\nu a} S^{\nu\mu}_b + \frac{1}{4} f(T) e^a_\mu = 0 \,, \tag{262}$$

where

$$f_{\mathrm{T}} = \frac{\partial f(T)}{\partial T} \,, \qquad f_{\mathrm{TT}} = \frac{\partial^2 f(T)}{\partial T^2} \,,$$

and $\mathcal{L}_m = 0$ is assumed for simplicity. The tetrad is expanded perturbatively as

$$e^a_\mu = \gamma^{(0)a}_\mu + \gamma^{(1)a}_\mu + \mathcal{O}\left(\gamma^{(2)a}_\mu\right), \tag{263}$$

where $|\gamma^{(i)a}_\mu| \ll 1$ for $i > 0$, and each successive order is assumed to be much smaller than the preceding one, i.e., $|\gamma^{(2)a}_\mu| \ll |\gamma^{(1)a}_\mu|$. The zeroth-order background is expressed through the Lorentz metric η_{ab} (see Remark 7 in Section 7.1):

$$\eta_{\mu\nu} = \eta_{ab}\gamma^{(0)a}_\mu \gamma^{(0)b}_\nu \,. \tag{264}$$

After the necessary transformations and taking the boundary conditions into account, the field Eqn (262) become [107]

$$\partial_\nu S^{(1)\mu\nu}_a = 0 \,. \tag{265}$$

Like the Einstein equations, this equation has no general closed-form solution; however, the standard GR gauge conditions can be imposed on the perturbed metric $h^{(1)}_{\mu\nu}$. Note that substituting the expansion $e^a_\mu = \gamma^{(0)a}_\mu + \gamma^{(1)a}_\mu$ into the metric tensor $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\mu$ gives the first-order metric perturbation

$$h^{(1)}_{\mu\nu} = \eta_{ab}\left(\gamma^{(0)a}_\mu \gamma^{(1)b}_\nu + \gamma^{(1)a}_\mu \gamma^{(0)b}_\nu\right). \tag{266}$$

We impose the traceless condition

$$h^{(1)\mu}_\mu = 2\eta^{\mu\nu}\eta_{ab}\gamma^{(0)a}_\mu \gamma^{(1)b}_\nu = 0 \,, \tag{267}$$

and the Lorenz gauge:

$$0 = \partial^\mu h^{(1)}_{\mu\nu} = \partial_b \gamma^{(0)b}_\nu + \eta_{ab}\left[\gamma^{(1)b}_\nu \partial^\mu \gamma^{(0)b}_\nu + \gamma^{(1)a}_\mu \partial^\mu \gamma^{(0)a}_\mu + \gamma^{(1)a}_\mu \partial^\mu \gamma^{(0)b}_\nu\right]. \tag{268}$$

Using the relation $\gamma^{(0)d}_\nu = \eta^{cd}\eta_{\mu\nu}\gamma^{(0)\mu}$ and considering the simplified case $\gamma^{(0)a}_\mu = \delta^a_\mu$, Eqn (265) can be expressed in terms of $\gamma^{(1)a}_\mu$ as

$$A^\mu_d = \eta^{\mu\alpha}\eta_{df}\Box\gamma^{(1)f}_\alpha + \delta^\mu_a \delta^\rho_d \Box\gamma^{(1)a}_\rho = 0 \,, \tag{269}$$

which leads to the following system of equations [107]:

$$A^0_0 : \Box\gamma^{(1)0}_0 = 0 \,, \tag{270}$$

$$A^0_i = -A^i_0 : \Box(\gamma^{(1)0}_i - \gamma^{(1)i}_0) = 0 \,, \tag{271}$$

$$A^i_j (i \neq j) : \Box(\gamma^{(1)j}_i + \gamma^{(1)i}_j) = 0 \,, \tag{272}$$

$$A^i_m (i = m) : \Box\gamma^{(1)i}_i = 0 \,. \tag{273}$$

Here $i, j = 1, 2, 3$. The transverse gauge condition $h^{(1)}_{0\mu} = 0$ is then imposed, which in terms of the components means $\gamma^{(1)0}_0 = 0$ and $\gamma^{(1)i}_0 = \gamma^{(1)0}_i$. Combining the above conditions gives

1. The trace vanishes:

$$\gamma^{(1)i}_i = 0 \,, \tag{274}$$

2. The Lorenz gauge:

$$\partial_j\,(\gamma_i^{(1)j} - \gamma_j^{(1)i}) = 0\,, \tag{275}$$

3. The conditions on A_j^i for $i \neq j$ and A_m^i for $i = m$:

$$\Box(\gamma_i^{(1)j} + \gamma_j^{(1)i}) = 0\,, \quad \Box\gamma_i^{(1)i} = 0\,. \tag{276}$$

Without loss of generality, we assume that the gravitational wave propagates along the z direction. The wave equations then imply

$$\gamma_i^{(1)i} = A_i^i \exp(\mathrm{i}k_\mu x^\mu)\,, \tag{277}$$

where $i = 1, 2, 3$ is a fixed index with no summation implied, and

$$\gamma_i^{(1)j} + \gamma_j^{(1)i} = B_i^j \exp(\mathrm{i}p_\mu x^\mu)\,, \tag{278}$$

where $i, j = 1, \ldots, 3$, while k_μ and p_μ are wave vectors satisfying $k_\mu k^\mu = p_\mu p^\mu = 0$. The coefficients A_i^i and B_i^j satisfy $A_1^1 = -A_2^2$, $A_3^3 = 0$, $B_1^3 = B_2^3 = 0$. The perturbed metric therefore takes the form

$$h_{\mu\nu}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\gamma_1^{(1)1} & B_1^2 \exp(\mathrm{i}p_\mu x^\mu) & 0 \\ 0 & B_1^2 \exp(\mathrm{i}p_\mu x^\mu) & -2\gamma_1^{(1)1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{279}$$

The standard + and × GW polarizations of GR are immediately apparent in the solution (279). Defining $h_+ \equiv 2\gamma_1^{(1)1}$ and $h_\times \equiv B_1^2 \exp(\mathrm{i}p_\mu x^\mu)$, we conclude that the perturbed tetrad has two physical degrees of freedom, h_+ and $h_\times$. Explicit solutions therefore confirm that no new polarization modes arise at first order in $f(T)$ gravity.

Higher-order perturbations are considered as perturbations of a flat Minkowski background [107]. For the metric tensor, this means

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{(1)} + h_{\mu\nu}^{(2)} + h_{\mu\nu}^{(3)} + \mathcal{O}(h_{\mu\nu}^{(4)})\,. \tag{280}$$

Since the tetrad is the fundamental variable in the teleparallel formulation, the above metric perturbation can be related to a tetrad perturbation:

$$e_\mu^a = \delta_\mu^a + \gamma_\mu^{(1)a} + \gamma_\mu^{(2)a} + \gamma_\mu^{(3)a} + \mathcal{O}(\gamma_\mu^{(4)a})\,. \tag{281}$$

For convenience, the field Eqn (262) can be rewritten in the conventional form

$$-f_{\mathrm{T}}G_{\mu\nu} + \frac{1}{2}\,g_{\mu\nu}(f_{\mathrm{T}}T - f) + 2S_{\mu\nu}^{\alpha}\,\partial_\alpha f_{\mathrm{T}} = 0\,. \tag{282}$$

Substituting the perturbed tetrad and metric, considering the nontrivial case $f_{\mathrm{T}}(0) \neq 0$, and expanding $f(T)$ in a Taylor series, one obtains the perturbation equations

$$\eta_{\mu\nu} f(0) = 0\,, \tag{283}$$

$$G_{\mu\nu}^{(1)} = 0\,, \tag{284}$$

$$G_{\mu\nu}^{(2)} = 0\,, \tag{285}$$

$$G_{\mu\nu}^{(3)} = 2\,\frac{f_{\mathrm{TT}}(0)}{f_{\mathrm{T}}(0)}\,S_{\nu\mu}^{(1)\alpha}\partial_\alpha T^{(2)}\,, \tag{286}$$

The zeroth-order equation implies the absence of a cosmological constant, since perturbations about Minkowski spacetime are being considered. The first- and second-order equations coincide with the standard perturbative vacuum equations of GR. However, the third-order Eqn (286) exhibits a deviation from the GR perturbation equations owing to the presence of the f_{TT} term. Thus, within $f(T)$ theory, deviations from GR in the perturbation equations arise only at orders higher than the second. The reason is that the torsion scalar is quadratic in the torsion tensor; consequently, in the GR limit $f_{\mathrm{TT}}(0) = 0$, the standard GR result is recovered.

Nevertheless, signatures of $f(T)$ gravity may be sought through modifications of the dispersion relation [111], which takes the form

$$\left|\frac{\mathrm{d}\omega}{\mathrm{d}k}\right| = \frac{1}{a}\left[1 - \frac{9a^2}{4k^2}\,H^2(1-\beta_{\mathrm{T}})^2\right]^{-1/2}, \tag{287}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter.

7.7 $f(\boldsymbol{T}, \boldsymbol{B})$ gravity

It has recently been shown [106, 107] that new polarization modes arise when $f(T)$ gravity is extended by introducing scalar fields or higher-derivative torsion terms, as in $f(T, B)$ and $f(T, T_{\mathrm{G}})$ theories [109, 110]. Here $B = R + T$ is the boundary term and T_{G} is the teleparallel equivalent of the Gauss–Bonnet term.

The action of $f(T, B)$ gravity extends (261):

$$S = \frac{1}{16\pi G}\int \mathrm{d}^4x e f(T, B) + \int \mathrm{d}^4x e\mathcal{L}_m\,. \tag{288}$$

The field equations can then be written locally as [107]

$$\begin{aligned} &-f_{\mathrm{T}}\,G_{\mu\nu} + \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)f_{\mathrm{B}} + \frac{1}{2}g_{\mu\nu}\left(f_{\mathrm{B}}B + f_{\mathrm{T}}T - f\right) \\ &+ 2S_{\nu\mu}^{\alpha}\,\partial_\alpha\left(f_{\mathrm{T}} + f_{\mathrm{B}}\right) = 8\pi G\Theta_{\mu\nu}\,, \end{aligned} \tag{289}$$

where $G_{\mu\nu}$ is the Einstein tensor, $\Theta_a^\mu = \delta\mathcal{L}_m/\delta e_\mu^a$ is the energy–momentum tensor in local coordinates, and $S_a^{\mu\nu}$ is the superpotential tensor (258). We consider linear perturbations of the potentials about a flat background [107]:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{(1)} + \mathcal{O}\left(h_{\mu\nu}^{(2)}\right), \quad \left|h_{\mu\nu}^{(i)}\right| \ll 1\,. \tag{290}$$

$$e_\mu^a = \delta_\mu^a + \gamma_\mu^{(1)a} + \mathcal{O}\left(\gamma_\mu^{(2)a}\right). \tag{291}$$

Since the torsion scalar T is quadratic in the torsion tensor $T_{\mu\nu}^a$, it vanishes at linear order. Therefore, because $R = -T + B$, at first perturbative order

$$R^{(1)} = B^{(1)}\,. \tag{292}$$

Since we work in the gauge with vanishing spin connection, i.e., in Weitzenbck geometry, substitution of the tetrad expansion (263) into the torsion tensor (257) gives, at first order,

$$T_{\mu\nu}^{(1)a} = \partial_\mu\gamma_\nu^{(1)a} - \partial_\nu\gamma_\mu^{(1)a}\,. \tag{293}$$

Lowering the index gives

$$T_{\rho\mu\nu}^{(1)} = \eta_{\rho a}T_{\mu\nu}^{(1)a}(\partial_\mu\gamma_\nu^{(1)a} - \partial_\nu\gamma_\mu^{(1)a})\,. \tag{294}$$

Contracting the indices ρ and v with the full metric $g^{v\rho}$, which at first order is equivalent to the Minkowski metric, $g^{v\rho} \approx \eta^{v\rho} + \mathcal{O}(\gamma^2)$, gives

$$T^{(1)v}_{\mu\nu} = \eta^{v\rho} T^{(1)}_{\rho\mu\nu} + \mathcal{O}(\gamma^2)\,. \tag{295}$$

Consequently,

$$T^{(1)av}_{\mu\nu} = \eta^{v\rho}\eta_{\rho a}(\partial_\mu \gamma^{(1)a}_{v} - \partial_v \gamma^{(1)a}_{\mu})\,, \tag{296}$$

where $\eta^{v\rho}\eta_{\rho a} = \delta^{v}_{a}$ is the mixed Kronecker delta. For the Weitzenbck connection, the curvature scalar is $R = -T - 2\nabla^{\mu} T^{v}_{\mu v}$, and the boundary term is identified as

$$B = -2\nabla^{\mu} T^{v}_{\mu v}\,. \tag{297}$$

At first order on a Minkowski background, the covariant derivative linearizes to the partial derivative, so that

$$(\nabla^{\mu} T^{v}_{\mu v})^{(1)} = \partial^{\mu} T^{(1)v}_{\mu v}\,. \tag{298}$$

Substituting (296) into the relation between the boundary term and torsion (297) gives

$$B^{(1)} = -2\partial^{\mu} T^{(1)v}_{\mu v} = -2\eta^{\mu\rho}\partial_\rho T^{(1)v}_{\mu v}$$

$$= -2\eta^{\mu\rho}\partial_\rho\left[\delta^{v}_{a}(\partial_\mu \gamma^{(1)a}_{v} - \partial_v \gamma^{(1)a}_{\mu})\right]\,.$$

Thus,

$$R^{(1)} = B^{(1)} = 2\delta^{\rho}_{b}\left(\eta^{\mu v}\partial_v \partial_\rho \gamma^{(1)b}_{\mu} - \Box \gamma^{(1)b}_{\rho}\right)\,. \tag{299}$$

Expanding $f(T, B)$ in a Taylor series gives the following first-order equations:

$$\eta_{\mu\nu} f(0,0) = 0\,, \tag{300}$$

$$-f_{\rm T}(0,0) G^{(1)}_{\mu\nu} + f_{\rm BB}(0,0)(\eta_{\mu\nu}\Box - \partial_\mu\partial_v) R^{(1)} = 0\,. \tag{301}$$

Taking the trace yields

$$f_{\rm T}(0,0) R^{(1)} + 3 f_{\rm BB}(0,0)\Box R^{(1)} = 0\,, \tag{302}$$

which can be rewritten as

$$(\Box - m^2) R^{(1)} = 0\,, \tag{303}$$

with the effective mass

$$m^2 \equiv -\frac{f_{\rm T}(0,0)}{3 f_{\rm BB}(0,0)}\,. \tag{304}$$

In the limit $|m^2| \to \infty$, the linearized $f(T, B)$ theory reduces to linearized GR.

In the Lorenz gauge for the tensor $\bar{h}^{(1)}_{\mu\nu}$ defined by

$$h^{(1)}_{\mu\nu} = \bar{h}^{(1)}_{\mu\nu} - \frac{1}{2}\bar{h}^{(1)}\eta_{\mu\nu} + \frac{f_{\rm BB}(0,0)}{f_{\rm T}(0,0)}\,\eta_{\mu\nu} R^{(1)}\,, \tag{305}$$

where $f_{\rm BB}/f_{\rm T} = -1/(3m^2)$, and imposing the tracelessness of $\bar{h}^{(1)}_{\mu\nu}$, the wave equation reduces to

$$\Box\,\bar{h}^{(1)}_{\mu\nu} = 0\,, \tag{306}$$

and has the familiar plane-wave solution depending on the wave four-vector k_ρ and amplitude $A_{\mu\nu}$:

$$\bar{h}^{(1)}_{\mu\nu} = A_{\mu\nu}\exp(\mathrm{i}k_\rho x^\rho)\,. \tag{307}$$

Another solution of the wave Eqn (303) is the exponential solution for the curvature,

$$R^{(1)} = F\exp(\mathrm{i}p_\mu x^\mu)\,. \tag{308}$$

where F is its amplitude and p_μ is the corresponding wave four-vector. The $R^{(1)}$ mode represents a massive scalar mode propagating at a subluminal velocity, since the wave vector satisfies $p_\mu p^\mu = -m^2$, and therefore $E^2 = |p|^2 + m^2$. Thus, the group velocity again depends on frequency.

The complete solution is therefore the sum

$$h^{(1)}_{\mu\nu} = A_{\mu\nu}\exp(\mathrm{i}k_\rho x^\rho) - \frac{1}{3m^2}\,\eta_{\mu\nu} F\exp(\mathrm{i}p_\rho x^\rho)\,. \tag{309}$$

Substituting the solution (308) of the scalar-curvature Eqn (303) into the linearized field equation (301) and using the definition of the effective mass (304), the scalar curvature can be expressed as

$$R^{(1)}_{\mu\nu} = \left(\frac{1}{6}\,\eta_{\mu\nu} - \frac{1}{3m^2}\,p_\mu p_v\right) R^{(1)}\,. \tag{310}$$

Analysis of the geodesic-deviation equation $\ddot{x}^i = -R^{i}_{0j0}x^j$, where

$$R_{i0j0} = \frac{1}{2}k_0^2\bar{h}^{(1)}_{ij} - \frac{1}{6m^2}\left(\eta_{ij}p_0^2 R^{(1)} + p_i p_j R^{(1)}\right)\,, \tag{311}$$

gives the following deviations from GR:

$$\ddot{x} = \left[\frac{1}{2}\,k_0^2\bar{h}^{(1)}_{+} + \frac{1}{6m^2}\,p_0^2 R^{(1)}\right]x + \frac{1}{2}\,k_0^2\bar{h}^{(1)}_{\times}\,y\,, \tag{312}$$

$$\ddot{y} = \left[-\frac{1}{2}\,k_0^2\bar{h}^{(1)}_{+} + \frac{1}{6m^2}\,p_0^2 R^{(1)}\right]y + \frac{1}{2}\,k_0^2\bar{h}^{(1)}_{\times}\,x\,, \tag{313}$$

$$\ddot{z} = \frac{1}{6m^2}\,(p_0^2 - p_3^2)R^{(1)}z = -\frac{1}{6}\,R^{(1)}z\,. \tag{314}$$

Here $\hat{h}^{(1)}_{+}$ and $\hat{h}^{(1)}_{\times}$ are the standard + and × polarizations. Note that the GR limit corresponds to $f_{\rm BB} \to 0$, i.e., to the transition to a linear theory with the boundary term B. By definition, the effective mass (304) then tends to infinity, and hence the classical limit is reached for $|m^2| \to \infty$. Thus, in the GR limit the additional polarizations become infinitely massive and consequently cease to propagate.

The main result is that gravitational waves in $f(T, B)$ gravity, in addition to the expected two polarizations, possess additional longitudinal and scalar breathing modes [107]. The theory itself is also of interest because modern models of cosmological inflation employ actions containing terms quadratic in curvature in order to reproduce the tensor-to-scalar perturbation ratio more accurately [108].

7.8 $f(T, B)$ gravity with a cosmological constant and $f(T, T_{\rm G})$ gravity

In $f(T, B)$ gravity, perturbations can be considered not only about a Minkowski background but also in the presence of a cosmological constant Λ [107]. The term $f(0,0)$ in the Taylor expansion of $f(T, B)$ is identified with 2Λ, since it enters the equations of motion in the same way. TEGR[7] with a

[7] The teleparallel equivalent of GR $g_{ab} = \eta_{ab}$, i.e., the case of general teleparallel geometry.

cosmological constant, $f(T,B) = T + 2\Lambda$, corresponds to the standard formulation of GR, for which $R = 4\Lambda$ after taking the trace. The equations of motion remain equivalent to (301); therefore, the additional modes persist in the presence of a cosmological constant [107].

Another theory of interest is $f(T, T_{\rm G})$ gravity, where $T_{\rm G}$ is the teleparallel equivalent of the Gauss–Bonnet invariant $\mathcal{G}_{\rm GB}$. At first order in perturbation theory, the properties of gravitational waves are analogous to those in $f(T)$ gravity and hence to those in GR. The Gauss–Bonnet term is an invariant quadratic in curvature, and therefore additional polarizations arise only at higher orders in perturbation theory [107].

7.9 $f(Q)$ gravity

The action of $f(Q)$ gravity, which generalizes symmetric teleparallelism, is [85, 86]

$$S = \int_\Omega \mathrm{d}^4x \left[\frac{1}{2\kappa^2}\sqrt{-g} f(Q) + \sqrt{-g}\,\mathcal{L}_m(g) \right], \tag{315}$$

where $\kappa^2 = 8\pi G/c^4$. To impose the conditions of symmetric teleparallelism, one sets $R^b_a = 0$, $T^a = 0$, with Lagrange multipliers included in the action [90]. The quantity Q is the nonmetricity scalar, defined in terms of the traces $Q_\alpha \equiv Q^\mu_{\alpha\mu}$ and $\tilde{Q}^\alpha \equiv Q^{\mu\alpha}_\mu$ as

$$Q \equiv \frac{1}{4} Q_{\alpha\beta\gamma} Q^{\alpha\beta\gamma} - \frac{1}{2} Q_{\alpha\beta\gamma} Q^{\gamma\beta\alpha} - \frac{1}{4} Q_\alpha Q^\alpha + \frac{1}{2} Q_\alpha \tilde{Q}^\alpha . \tag{316}$$

Variation with respect to the metric $g_{\mu\nu}$ yields the equations of motion [87–90]

$$\frac{2}{\sqrt{-g}} \nabla_\alpha \left(\sqrt{-g} f_Q P^\alpha_{\mu\nu} \right) - \frac{1}{2} g_{\mu\nu} f + \\ + f_Q \left(P_{\mu\alpha\beta} Q_\nu^{\alpha\beta} 2 Q_{\alpha\beta\mu} P^{\alpha\beta}{}_\nu \right) = \kappa^2 T_{\mu\nu} , \tag{317}$$

where $f_Q = \partial f/\partial Q$. The nonmetricity superpotential is denoted by P [90]:

$$P^\alpha_{\mu\nu} = \frac{1}{2} \frac{\partial Q}{\partial Q^{\mu\nu}_\alpha} .$$

The metric equation can also be rewritten as

$$\frac{1}{2} g_{\mu\nu} [Q f_Q - f] + f_Q G_{\mu\nu} + 2 P^\alpha{}_{\mu\nu} \nabla_\alpha f_Q = T_{\mu\nu} . \tag{318}$$

The connection equation reduces to the condition [90]

$$\nabla_\mu \nabla_\nu (\sqrt{-g} f_Q P^{\mu\nu}_\alpha) = 0 . \tag{319}$$

To study gravitational waves, metric perturbations about a flat Minkowski background are considered:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} , \qquad |h_{\mu\nu}| \ll 1 . \tag{320}$$

The GW analysis is performed in the coincident gauge with vanishing connection, i.e., within symmetric teleparallel geometry. In this gauge, Eqn (319) is satisfied trivially. Expanding $f(Q)$ in a Taylor series around $Q = 0$ and substituting the perturbation into Eqn (318) gives, at zeroth order,

$$\eta_{\mu\nu} f(0) = 0 . \tag{321}$$

Thus, the cosmological constant vanishes on a flat background, i.e., $f(0) = 0$. Provided that $f_Q(0) \neq 0$, one obtains [90, 91, 113]

$$f_Q(0) G^{(1)}_{\mu\nu} = 0 . \tag{322}$$

Equation (322) is therefore fully equivalent to the linearized vacuum equations of GR. Consequently, at linear order the GW polarizations and propagation speed in $f(Q)$ gravity coincide exactly with those in GR: there are two tensor polarizations, $+$ and $\times$, propagating at the speed of light.

Thus, deviations from GR arise either at higher orders in perturbation theory, in the nonlinear regime, or after the introduction of additional fields. For example, analogously to $f(T,B)$ gravity and using a similar calculation, the addition of a boundary term in $f(Q,B)$ gravity produces an additional massive scalar breathing mode propagating at a subluminal velocity [113]. Note that no longitudinal mode appears in the work under discussion, which distinguishes the $f(Q,B)$ case from $f(T,B)$ gravity.

8. Conclusions

As a result of gravitational-wave astronomy, the propagation law of gravitational waves in the most realistic theories of gravity currently agrees with that of GR (at least to leading order). These waves are transverse and possess two polarization states, '+' and '×'. Already at the level of Brans–Dicke theory, a third mode — the 'scalar breathing' mode — appears and becomes massive when a potential is introduced. With further extensions of the theory, the number of polarization modes increases. This is precisely why experimental searches for the gravitational-wave polarizations are so important.

In extended theories of gravity, existing GW observations are often interpreted as a particular case corresponding to propagation in flat four-dimensional Minkowski spacetime. Therefore, as experimental accuracy improves, new characteristics of gravitational waves — such as additional polarization modes and high-frequency signals — may become detectable, and the theoretical predictions discussed here may help determine which theory of gravity is realized in our Universe.

Acknowledgments. The work of I.I. Chekh on the initial versions of Sections 4, 5, and 7 was supported by Russian Science Foundation Grant No. 23-42-00055 within the framework of studies of the possible quantum nature of primordial gravitational waves. The first stage of S.O. Alexeyev's work on the remaining sections was supported by Russian Science Foundation Grant No. 23-22-00073 within the framework of studies of extensions of general relativity and the search for improved agreement with astrophysical observational data. At subsequent stages, S.O. Alexeyev, S.K. Kuzmin, E.A. Pluzhnikov, and R.A. Stamov substantially expanded and supplemented the manuscript in close collaboration.

References

1. Alexeyev S O, Pamyatnykh E A, Ursulov A V, Tret'yakova D A, Latosh B N *Obshchaya Teoriya Otnositel'nosti: Vvedenie. Sovremennoe Razvitie i Prilozheniya* (General Relativity: Introduction, Modern Developments and Applications) (Moscow: URSS, 2022)
2. Rowan S, Hough J, in *Proc. of the 1998 European School of High-Energy Physics, St. Andrews, Scotland, 23 Aug.–5 Sep. 1998* (CERN School of Physics, 37, Eds M Ellis, J March-Russell) (Geneva: CERN, 1999) pp. 301–311
3. Hulse R A *Rev. Mod. Phys.* **66** 699 (1994)
4. Taylor J H (Jr.) *Rev. Mod. Phys.* **66** 711 (1994)
5. Hoak D, LIGO Scientific Collab. *Nuovo Cimento C* **40** 119 (2017) DOI:10.1393/ncc/i2017-17119-1

6. Accadia T et al. *Class. Quantum Grav.* **28** 114002 (2011)
7. Dooley K L, LIGO Scientific Collab. *J. Phys. Conf. Ser.* **610** 012015 (2015)
8. Tatsumi D, TAMA Collab. *J. Phys. Conf. Ser.* **120** 032011 (2008)
9. Abbott B P et al. (LIGO Scientific Collab. and Virgo Collab.) *Phys. Rev. Lett.* **116** 241102 (2016)
10. Abbott B P et al. *Astrophys. J. Lett.* **848** L13 (2017)
11. Soares-Santos M et al. *Astrophys. J. Lett.* **848** L16 (2017)
12. Coulter D A et al. *Science* **358** 1556 (2017)
13. Valenti S et al. *Astrophys. J. Lett.* **848** L24 (2017)
14. Arcavi I et al. *Nature* **551** 64 (2017)
15. Tanvir N R et al. *Astrophys. J. Lett.* **848** L27 (2017)
16. Lipunov V M et al. *Astrophys. J. Lett.* **850** L1 (2017)
17. Evans P A et al. *Science* **358** 1565 (2017); arXiv:1710.05437v1
18. Margutti R et al. *Astrophys. J. Lett.* **848** L20 (2017); arXiv:1710.05431v1
19. Alexeyev S O et al. *Mnogokanal'naya Astronomiya* (Multimessenger Astronomy) (Ed. A M Cherepashchuk) (Moscow: DMK Press, 2022)
20. Alexeyev S, Prokopov V *Universe* **8** (5) 283 (2022)
21. Amaro-Seoane P et al. *Living Rev. Relativ.* **26** 2 (2023)
22. Luo J et al. *Class. Quantum Grav.* **33** 035010 (2016)
23. Hu W-R, Wu Y-L *Natl. Sci. Rev.* **4** 685 (2017)
24. Sofue Y, Rubin V *Annu. Rev. Astron. Astrophys.* **39** 137 (2001)
25. Yadav M et al. *Eur. Phys. J. C* **85** 1013 (2025)
26. Baker J G et al., arXiv:0708.4202
27. Milgrom M *Phys. Rev. D* **89** 024027 (2014)
28. Milgrom M *Phys. Rev. D* **82** 043523 (2010)
29. Brans C, Dicke R H *Phys. Rev.* **124** 925 (1961)
30. Brans C H *Phys. Rev.* **125** 2194 (1962)
31. Alsing J et al. *Phys. Rev. D* **85** 064041 (2012)
32. Horndeski G W *Int. J. Theor. Phys.* **10** 363 (1974)
33. Kobayashi T, Yamaguchi M, Yokoyama J *Prog. Theor. Phys.* **126** 511 (2011)
34. Hou S, Gong Y, Liu Y *Eur. Phys. J. C* **78** 378 (2018)
35. Dyadina P I, Avdeev N A, Alexeyev S O *Mon. Not. R. Astron. Soc.* **483** 947 (2019)
36. de Rham C, Gabadadze G, Tolley A J *Phys. Rev. Lett.* **106** 231101 (2011)
37. Mohseni M *Phys. Rev. D* **84** 064026 (2011)
38. Zhang H, Huang Y *J. High Energ. Phys.* **2021** 56 (2021); arXiv:2109.01391v2
39. Cai R-G et al. *Phys. Rev. D* **91** 024032 (2015)
40. Fierz M, Pauli W E *Proc. R. Soc. London A* **173** 211 (1939)
41. Gambini R, Pullin J *Loops, Knots, Gauge Theories and Quantum Gravity* (Cambridge: Cambridge Univ. Press, 2000)
42. Rovelli C *Quantum Gravity* (Cambridge: Cambridge Univ. Press, 2004)
43. Ashtekar A, Lewandowski J *Class. Quantum Grav.* **21** R53 (2004)
44. Thiemann T *Modern Canonical Quantum General Relativity* (Cambridge: Cambridge Univ. Press, 2008)
45. Ashtekar A *Phys. Rev. Lett.* **57** 2244 (1986)
46. Rovelli C, Smolin L *Nucl. Phys. B* **442** 593 (1995)
47. Ashtekar A, Lewandowski J *Class. Quantum Grav.* **14** A55 (1997)
48. Ashtekar A, Lewandowski J *Adv. Theor. Math. Phys.* **1** 388 (1998)
49. Bojowald M *Living Rev. Relativ.* **11** 4 (2008)
50. Ashtekar A, Pawlowski T, Singh P *Phys. Rev. Lett.* **96** 141301 (2006)
51. Ashtekar A, Pawlowski T, Singh P *Phys. Rev. D* **74** 084003 (2006)
52. Assanioussi M et al. *Phys. Rev. Lett.* **121** 081303 (2018)
53. Dapor A, Liegener K *Eur. Phys. J. C* **80** 741 (2020)
54. Horowitz G T, Steif A R *Phys. Rev. D* **42** 1950 (1990)
55. Tseytlin A A *Class. Quantum Grav.* **12** 2365 (1995)
56. Blau M et al. *J. High Energy Phys.* **2002** (01) 047 (2002)
57. Metsaev R R *Nucl. Phys. B* **625** 70 (2002)
58. Metsaev R R, Tseytlin A A *Phys. Rev. D* **65** 126004 (2002)
59. Penrose R, in *Differential Geometry and Relativity* (Ser. Mathematical Physics and Applied Mathematics, Vol. 3, Eds M Cahen, M Flato) (Dordrecht: Kluwer Acad. Publ., 1976) p. 271, DOI:10.1007/978-94-010-1508-0_23
60. Güven R *Phys. Lett. B* **482** 255 (2000)
61. Blau M, Figueroa-O'Farrill J, Papadopoulos G *Class. Quantum Grav.* **19** 4753 (2002)
62. Berenstein D, Maldacena J, Nastase H *J. High Energy Phys.* **2002** (04) 013 (2002)
63. Sadri D, Sheikh-Jabbari M M *Rev. Mod. Phys.* **76** 853 (2004)
64. Podolský J *Class. Quantum Grav.* **15** 719 (1998)
65. Biswas A, Kumar A, Panigrahi K L *Phys. Rev. D* **66** 126002 (2002)
66. Hořava P *Phys. Rev. D* **79** 084008 (2009)
67. Blas D, Pujolàs O, Sibiryakov S *Phys. Rev. Lett.* **104** 181302 (2010)
68. Charmousis C et al. *J. High Energy Phys.* **2009** (08) 070 (2009)
69. Pospelov M, Shang Y *Phys. Rev. D* **85** 105001 (2012)
70. Mestra-Páez J, Peña J M, Restuccia A *Phys. Rev. D* **104** 124073 (2021)
71. Mestra-Páez J, Peña J M, Restuccia A *Eur. Phys. J. C* **81** 923 (2021)
72. Arnowitt R, Deser S, Misner C W *Phys. Rev.* **121** 1556 (1961)
73. Ramos O, Barausse E *Phys. Rev. D* **99** 024034 (2019)
74. Gümrükçüoğlu A E, Saravani M, Sotiriou T P *Phys. Rev. D* **97** 024032 (2018)
75. Abbott B P et al. *Astrophys. J.* **848** L13 (2017)
76. Abbott B P et al. (LIGO Scientific Collab. and Virgo Collab.) *Phys. Rev. Lett.* **123** 011102 (2019)
77. Bellorín J, Restuccia A *Phys. Rev. D* **94** 064041 (2016)
78. De Felice A, Tsujikawa S *Living Rev. Relativ.* **13** 3 (2010)
79. Kalita S, Mukhopadhyay B *Astrophys. J.* **909** 65 (2021)
80. Liang D et al. *Phys. Rev. D* **95** 104034 (2017)
81. Prasia P, Kuriakose V C *Int. J. Mod. Phys. D* **23** 1450037 (2014)
82. Burrage C, Sakstein J *Living Rev. Relativ.* **21** 1 (2018)
83. Gong Y, Hou S *Universe* **4** (8) 85 (2018)
84. Yang L, Lee C-C, Geng C-Q *J. Cosmol. Astropart. Phys.* **2011** (08) 029 (2011)
85. Jiménez J B et al. *Phys. Rev. D* **101** 103507 (2020)
86. Zhao D *Eur. Phys. J. C* **82** 303 (2022)
87. Dialektopoulos K F, Koivisto T S, Capozziell S *Eur. Phys. J. C* **79** 606 (2019)
88. Capozziello S et al. *Eur. Phys. J. C* **81** 1141 (2021)
89. Vignolo S *Class. Quantum Grav.* **39** 015009 (2022)
90. Capozziello S, Capriolo M, arXiv:2405.16163
91. Hohmann M et al. *Phys. Rev. D* **99** 024009 (2019)
92. Andriot D, Gómez G L *J. Cosmol. Astropart. Phys.* **2017** (06) 048 (2017); arXiv:1704.07392
93. van Nieuwenhuizen P, in *General Relativity and Gravitation. Invited Papers and Discussion Reports of the 10th Intern. Conf. on General Relativity and Gravitation, Padua, July 3–8, 1983* (Fundamental Theories of Physics, Vol. 9, Eds B Bertotti, F Felice, A Pascolini) (Dordrecht: D. Reidel Publ. Co., 1984) p. 471, DOI:10.1007/978-94-009-6469-3_29
94. Van Nieuwenhuizen P, in *Relativity, Groups and Topology*: *Proc., 40th Summer School of Theoretical Physics, Session 40, Les Houches, France, June 27 – August 4, 1983* Vol. 2 (Eds B S DeWitt, R Stora) (Amsterdam: North-Holland, 1984) p. 823
95. Giddings S B, Zhang H *Phys. Rev. D* **93** 115002 (2016)
96. Agashe K et al. *Phys. Rev. D* **76** 036006 (2007)
97. Bachas C, Estes J *J. High Energy Phys.* **2011** (06) 05 (2011)
98. Csáki C et al. *Nucl. Phys. B* **581** 309 (2000)
99. Andriot D, Tsimpis D *J. High Energy Phys.* **2020** 100 (2020); arXiv:1911.01444
100. Hehl F W et al. *Phys. Rep.* **258** 1 (1995)
101. Jiménez-Cano A, Obukhov Yu N *Phys. Rev. D* **103** 024018 (2021)
102. Hohmann M, Pfeifer C *Phys. Lett. B* **834** 137437 (2022)
103. Rao H, Liu C, Geng C-Q *Phys. Lett. B* **850** 138497 (2024)
104. Bamba K et al. *Phys. Lett. B* **727** 194 (2013)
105. Cai Y-F et al. *Phys. Rev. D* **97** 103513 (2018)
106. Abedi H, Capozziello S *Eur. Phys. J. C* **78** 474 (2018)
107. Farrugia G et al. *Phys. Rev. D* **97** 124064 (2018)
108. Pozdeeva E O et al. *J. Cosmol. Astropart. Phys.* **2025** 081 (2025)
109. Bahamonde S, Böhmer C G, Wright M *Phys. Rev. D* **92** 104042 (2015)
110. Kofinas G, Saridakis E N *Phys. Rev. D* **90** 084044 (2014)
111. Cai Y-F et al., arXiv:1801.05827
112. Kumar A *Phys. Lett. B* **594** 368 (2004)
113. Soudi I et al. *Phys. Rev. D* **100** 044008 (2019)
114. Ferko C, Satishchandran G, Sethi S *Phys. Rev. D* **105** 024072 (2022)
115. Gong Y et al. *Phys. Rev. D* **98** 104017 (2018)
116. Misner C W, Thorne K S, Wheeler J A *Gravitation* Vol. 1 (San Francisco, CA: W.H. Freeman, 1973)
117. Arnowitt R, Deser S, Misner C W "Dynamical structure and definition of energy in general relativity" *Phys. Rev.* **116** 1322 (1959)
118. Arnowitt R, Deser S, Misner C W "The dynamics of general relativity," in *Gravitation*: *An Introduction to Current Research* (Ed. L Witten) (New York: John Wiley and Sons, 1962) p. 227
119. Bellorín J, Restuccia A, Sotomayor A *Phys. Rev. D* **87** 084020 (2013)
120. Lichnerowicz A "Propagateurs et commutateurs en relativité générale" *Publ. Math. IHES* **10** 5 (1961) DOI:10.1007/BF02684612