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100th ANNIVERSARY OF QUANTUM MECHANICS

Uniqueness of spin 1/2

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Abstract. This article is dedicated to the 100th anniversary of quantum mechanics, and in particular to electron spin — a phenomenon that has no counterpart in classical physics. The hypothesis of electron spin was proposed by Uhlenbeck and Goudsmit in 1925–1926 to explain numerous experimental data. Two years later, Dirac discovered the existence of electron spin when constructing the relativistic wave equation for a free particle. This came as a great surprise to Dirac, since, as he later claimed (in 1977), his goal was to obtain a relativistic wave equation for the simplest particle, which he considered to be a particle with zero spin. But it turned out that the simplest particle is one with spin 1/2.

Keywords: electron spin, Dirac wave equation, Pauli matrices, generalized angular momentum, spectral fine structure

Quantum mechanics is on the verge of its 100th anniversary. In 1925, the pioneering work of Heisenberg, Born, and Jordan were published, in which the phrase 'quantum mechanics' first appeared, and in 1926, Schrödinger's articles, which became the basis of wave mechanics, were published. Over time, the definition 'wave mechanics' transformed — the mechanics born from Schrödinger's work began to be called 'quantum.'[1] The methods of quantum mechanics, which led (and very quickly!) to the successful solution to a wide variety of physical problems, allowed Wigner to characterize quantum mechanics as "inscrutably effective."

Soon after its birth, a problem arose (the beginning of which was laid by the famous Bohr–Einstein debate), which remains unresolved to this day — the ability to use quantum mechanics to solve specific problems does not yet guarantee that it is understood.[2] This primarily concerned the interpretation of the wave function and the process of measuring quantum objects. The situation provoked categorical statements from Nobel laureates: "...Everyone knows how to use it (i.e., quantum mechanics — *SVP*), but nobody really understands it" (Gell-Mann); "I think I can responsibly say that nobody understands quantum mechanics" (Feynman). Later areas of quantum mechanics (e.g., quantum entanglement, quantum computing, etc.) did not change the situation — the gap between the ability to use quantum mechanics and understanding it remains wide. Thus, the question arises: if quantum mechanics is so 'inscrutably effective,' then perhaps it is worth limiting ourselves to this and abandoning attempts to 'understand' it? There is no universal answer to this question, and each researcher is free to answer it 'yes' or 'no.' However, the stream of articles that continues to this day testifies to the relevancy of attempts to solve the problem that arose shortly after the birth of quantum mechanics.

The present article does not claim to analyze the concepts related to quantum mechanics as a whole and is devoted to a particular phenomenon of quantum mechanics — electron spin. The importance of this phenomenon is due not only to the absence of a classical analogue of spin, but also to the fundamental nature of the electron itself. Like quantum mechanics as a whole, electron spin is also 100 years old — the hypothesis of electron spin was proposed by Goudsmit and Uhlenbeck [1, 2] in 1925–1926. The history of the discovery of electron spin is very short (by historical standards), but rich in events.[3]

The discovery of spin was initially associated with the anomalous Zeeman effect, and especially with the study of the fine structure of complex atoms. The success of the Bohr–Sommerfeld theory in describing the spectrum of states of the hydrogen atom gave hope that a similar approach could be applied to multi-electron atoms. In such an atom, each electron moves along its own allowed orbit, determined by a set of quantum numbers, among which half-integer values were also allowed (Lande in 1922 introduced a half-integer quantum number to describe X-ray spectra). The only criterion for the correctness of this empirical approach was agreement with experiment.

In 1925, Pauli formulated his famous exclusion principle, which allowed an explanation of the shell structure of the atom and the periodicity of the periodic table. For each allowed orbit, Pauli introduced four quantum numbers, including half-integer values. Goudsmit and Uhlenbeck, studying Pauli's work, concluded that, since each quantum number corresponds to one degree of freedom, and a point electron obviously has three degrees of freedom, the fourth

[1] This is because first Schrödinger and then von Neumann demonstrated the equivalence of both approaches. In modern quantum mechanical terminology, they are known as the Schrödinger picture and the Heisenberg picture.

[2] It should be noted that, in classical mechanics and electrodynamics, which preceded quantum mechanics, such a problem does not exist, since the concepts used there seem obvious to us. And although the theory of relativity radically changed the concept of time, it did not become poorly understood.

[3] Here, we will mention only a few facts from this extensive history. A more detailed account can be found in monograph [3].

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degree of freedom should be associated with some internal motion of the electron. Since the planetary model of the atom was not questioned at that time (it is interesting to note that, in 1925, articles by Heisenberg [4] and Born and Jordan [5] had already been published, which marked the beginning of a complete rejection of classical concepts), it was natural to assume (in addition to orbital motion) the rotation of the electron around its axis. Kronig (1925) expressed the same idea, and, as early as 1921, Compton spoke of an electron rotating like a miniature gyroscope. However, the idea of an electron rotating around its axis contradicted the theory of relativity, since, as calculations showed, a point on the surface of a rotating electron with a classical radius e^2/mc^2 and an intrinsic angular momentum of $\hbar/2$ would have to move at a speed many times greater than the speed of light. Thus, there was a contradiction between the unquestionable 'internal behavior of each electron' [1, 2] and its classical interpretation. This contradiction was resolved by Wolfgang Pauli (who, at the very beginning of 1926, in a conversation with Niels Bohr, called the spin hypothesis heresy), who proposed considering spin to be an essentially quantum property of the electron. Accordingly, he wrote down the wave equation [6] for an electron in an external electromagnetic field with the Hamiltonian [7]

$$H = \frac{1}{2m}\left[\left(\hat{p}_x + \frac{e}{c}A_x\right)^2 + \left(\hat{p}_y + \frac{e}{c}A_y\right)^2 + \left(\hat{p}_z + \frac{e}{c}A_z\right)^2\right] - e\Phi + \boldsymbol{\mu}\mathbf{B}\,, \quad (1)$$

where the internal magnetic moment of the electron is $\boldsymbol{\mu} = \mu_0\hat{\mathbf{S}}$. Here, $\mu_0 = e\hbar/2mc$ is the Bohr magneton, introduced into scientific use by Pauli in 1920, and $\hat{\mathbf{S}} = \hbar/2\boldsymbol{\sigma}$ is the three-dimensional spin operator, the components of which are defined by the matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2)$$

which were called Pauli matrices.[4] Φ and $\mathbf{A}$ are the scalar and vector potential of the electromagnetic field, $\mathbf{B}$ is the magnetic field strength ($\mathbf{B} = \mathrm{rot}\,\mathbf{A}$).

Hamiltonian (1) is obtained in a standard way. The first two terms represent the classical Hamiltonian function for a charge in an external electromagnetic field, in which the momentum components are replaced by the corresponding operators. The third term, introduced by Pauli, is the interaction energy of the electron's intrinsic magnetic moment with the external magnetic field. Although the nonrelativistic Pauli Hamiltonian (1) explicitly contains spin dynamical variables (in the form of $\boldsymbol{\sigma}$ matrices), a necessary condition for this is the presence of an external magnetic field. Otherwise ($\mathbf{B} = 0$), the spin disappears.

A kind of milestone in the history of the discovery and explanation of the electron spin phenomenon was reached in 1928, when Dirac published the wave equation for a relativistic electron [8, 9][5] in the Schrödinger form

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} = H_\mathrm{D}\Psi\,, \quad (3)$$

with the Hamiltonian (in this article we restrict ourselves to the case of a free electron)

$$H_\mathrm{D} = c\boldsymbol{\alpha}\hat{\mathbf{p}} + mc^2\beta = c\,(\alpha_1\hat{p}_x + \alpha_2\hat{p}_y + \alpha_3\hat{p}_z) + mc^2\beta\,, \quad (4)$$

obtained by linearizing the Hamiltonian[6]

$$H = \sqrt{c^2\hat{\mathbf{p}}^2 + m^2c^4}\,. \quad (5)$$

Dirac was convinced that the wave equation should be linear in the time derivative ("linearity in $\partial/\partial t$ was absolutely necessary for me" [10]). On the other hand, the operator $\partial/\partial t$, up to a factor, is the zeroth component of the 4-momentum operator, and from the requirements of relativistic invariance, it follows that the desired equation must also be linear in the spatial components of this operator. Ultimately, this led Dirac to Hamiltonian (4), where the quantities $\alpha_k (k = 1, 2, 3)$ and β (Dirac called them "dynamical variables or operators" [9]) satisfy the relations

$$\left.\begin{aligned} &\alpha_k^2 = \beta^2 = 1\,, \\ &\alpha_k\alpha_m + \alpha_m\alpha_k = \alpha_k\beta + \beta\alpha_k = 0, \quad k \neq m \end{aligned}\right\} \quad (6)$$

and, therefore, cannot be numbers. As such quantities, Dirac chose 4×4 Hermitian matrices of the form

$$\alpha_k = \begin{pmatrix} \mathbb{O} & \sigma_k \\ \sigma_k & \mathbb{O} \end{pmatrix},\ \beta = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix}, \quad (7)$$

where the two-dimensional blocks σ_k are the Pauli matrices, $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, and $\sigma_3 = \sigma_z$, and satisfy the same relations (6).

Thus, when describing the motion of a free relativistic electron, its spin appears automatically. Nevertheless, the natural appearance of the dynamical variables responsible for spin in the problem of a free electron was unexpected for Dirac. He believed that the concept of an electron as a spinless particle was more general, and that spin should be introduced into the theory as needed, as, for example, in the problem of the hydrogen atom.[7] Half a century later, Dirac recalled [10]: "At that time, I only wanted a quantum theory that would satisfy general requirements, such as the possibility of applying transformation theory[8] and the principles of relativity." Thus, striving to obtain a wave equation for the 'simplest' (i.e., spinless) particle, corresponding to the canons of quantum mechanics and relativity theory, Dirac proved (although he did not pursue this goal) that spin 1/2 is an inherent attribute of the electron and that the electron is indeed the 'simplest' particle. Although the three Pauli matrices represent (up to a factor) the three Cartesian components of the spin operator ($\hat{\mathbf{S}} = (1/2)\boldsymbol{\sigma}$), at first glance it seems that, when constructing Hamiltonian (4), one can manage with only these three matrices. However, in that case, as Dirac recalled [10] (see also [11]), it is not possible to obtain

[4] Below, we assume $\hbar = 1$.

[5] Both of these papers were published one after the other with a one-month interval and, in essence, represent two parts of the same article.

[6] Hamiltonian (5) is obtained by replacing the components of the relativistic momentum in the classical relativistic Hamiltonian function with the corresponding operators $\hat{p}_\tau = (\hbar/\mathrm{i})\,\partial/\partial\tau, \tau = x, y, z$.

[7] His remark on this matter is characteristic [8]: "The question remains as to why *Nature* should have chosen this particular model for the electron instead of being satisfied with the point-charge."

[8] The term 'transformation theory' is now hardly used. In Ref. [3], it is stated that transformation theory "made it possible to present nonrelativistic quantum mechanics of systems with a finite number of degrees of freedom as a logically consistent, compact and unified system of views."

Hamiltonian (4). The desired result is achieved by replacing the Pauli σ_k matrices with four-dimensional matrices α_k (7) and adding the matrix β.[9] The spin operator $\hat{\mathbf{S}}$ in this case can be written as a four-dimensional diagonal matrix with two-dimensional blocks $1/2\boldsymbol{\sigma}$. Since the dynamic quantities responsible for spin are present in the Dirac Hamiltonian (4), the solution of wave equation (3) must explicitly demonstrate the presence of spin in the electron. Dirac [8, 9] does not provide a solution to the wave equation for a free electron,[10] but notes that the wave function, in addition to depending on spatial coordinates and time, must also depend on some spin variable. What should such a wave function be if the dynamical quantities responsible for spin are represented in the Dirac Hamiltonian (4) by numerical matrices? The answer to this question is contained in the solution of the stationary Dirac equation (time is separated almost in the same way (see [12]) as in nonrelativistic quantum mechanics, so that $\Psi(\mathbf{r}, t) = \exp[-(\mathrm{i}/\hbar)Et]\,\psi(\mathbf{r})$)

$$H_{\mathrm{D}}\psi = E\psi\,, \tag{8}$$

where the Hamiltonian H_{D} can be rewritten as

$$H_{\mathrm{D}} = c\begin{pmatrix} \mathbb{O} & \boldsymbol{\sigma}\hat{\mathbf{p}} \\ \boldsymbol{\sigma}\hat{\mathbf{p}} & \mathbb{O} \end{pmatrix} + mc^2\begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix} \tag{9}$$

and, taking the explicit form of Pauli matrices (2) into account,

$$\boldsymbol{\sigma}\hat{\mathbf{p}} = \begin{pmatrix} \hat{p}_z & \hat{p}_x - \mathrm{i}\,\hat{p}_y \\ \hat{p}_x + \mathrm{i}\,\hat{p}_y & -\hat{p}_z \end{pmatrix}. \tag{10}$$

It is easy to see that the desired wave function is proportional to the eigenfunction of the momentum operator

$$\psi = \exp\left[\left(\frac{\mathrm{i}}{\hbar}\right)\mathbf{pr}\right]\mathrm{u}\,, \tag{11}$$

where u is a numerical four-dimensional column that allows considering wave function (11) as a bispinor. At first glance, the relativistic wave function (11) resembles the wave function of a free nonrelativistic particle, which has the same form (11), but with the vector u replaced by an arbitrary constant. However, the difference is much more significant, since the vector u is not arbitrary, but is a solution to a system of four linear homogeneous algebraic equations

$$\mathbb{H}_{\mathbb{D}}\mathrm{u} = E\mathrm{u}\,. \tag{12}$$

The matrix $\mathbb{H}_{\mathbb{D}}$ has exactly the same structure as the Dirac Hamiltonian (9), if in the latter the momentum operator $\hat{\mathbf{p}}$ is replaced by its classical analogue $\mathbf{p}$. The presence of Pauli matrices in $\mathbb{H}_{\mathbb{D}}$ allows it to be interpreted as a kind of spin operator, parameterized by the components of the classical momentum vector. At the same time, the presence or absence of spin dynamic quantities does not affect the energy eigenvalues E. Indeed, the roots of the characteristic polynomial of system of equations (12) are

$$E = \pm\sqrt{m^2c^4 + c^2p^2}\,, \tag{13}$$

which corresponds to the relativistic invariant — the scalar product of the 4-vector of energy–momentum $\mathbf{P} = (E/c, \mathbf{p})$ with itself, regardless of whether the electron has spin or not.[11] Obviously, the spin state of a freely moving relativistic electron is related to the vector $\mathbf{u}$ — the eigenvector of the matrix $\mathbb{H}_{\mathbb{D}}$. By analogy with the orbital angular momentum, spin is spoken of as a vector quantity. This analogy is due to the same commutation relations between the Cartesian components of the spin operator as for the components of the orbital momentum operator. As a consequence of this similarity of commutation relations, the projection of the spin operator onto some chosen direction in space (usually the z-axis [13]) is quantized.

Of course, the analogy between spin and orbital angular momentum is not complete. First, spin 1/2 has no classical analogue.[12] Second, orbital angular momentum is a characteristic of the electron's motion, while spin angular momentum is a characteristic of the electron itself.[13] According to the theory of generalized angular momentum [15], the eigenvalues of the square of the angular momentum operator are $-N/2(N/2+1)$ (N being an integer), and the eigenvalues of the projection operator onto a chosen direction are between $-N/2$ and $+N/2$. In the case of orbital angular momentum, $N/2 = l = 0, 1, 2, \ldots$, and different values of the orbital quantum number l (and the corresponding eigenvalues of the projection operator) characterize different states of the same particle. In the case of spin angular momentum, the number N is fixed ($N = 1$ corresponds to the electron spin), and different values of N correspond to different spins, as characteristics of different particles. The number $(2N+1)$ determines the dimensionality of the abstract space in which the projection operators of the generalized angular momentum are defined. The operators $\hat{S}_\alpha(\alpha = x, y, z)$ are defined in a two-dimensional space of spin states.[14] Usually, the eigenvectors of the operator $\hat{S}_z$ are chosen as the basis of this space: $|1/2, 1/2\rangle$ and $|1/2, -1/2\rangle$ (the first number in each vector is $N/2$ for $N = 1$, so the eigenvalue of the square of the spin operator is 3/4, and the second number is one of the two possible eigenvalues of the spin projection operator S_z), so that any spin state has the form

$$a\left|\frac{1}{2}, \frac{1}{2}\right\rangle + b\left|\frac{1}{2}, -\frac{1}{2}\right\rangle, \tag{14}$$

where the coefficients a and b satisfy the normalization condition

$$|a|^2 + |b|^2 = 1\,. \tag{15}$$

In the matrix representation of operators $\hat{S}_\alpha$, it is possible to use the columns $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as the basis vectors $|1/2, 1/2\rangle$ and $|1/2, -1/2\rangle$. Taking Eqn (15) into account, the coefficients a and b can be written in the form

[9] The appearance of matrix β has a more hidden, but no less important, meaning, since the existence of matrices α_k in the form of (5) is closely related to matrix β.

[10] The main goal that Dirac pursued was to obtain a wave equation that fully satisfies the requirements of both quantum mechanics and the theory of relativity. The solution of such an equation is the next step.

[11] The problem of negative energy values is not discussed here, as it is outside the scope of this article (see [12]).

[12] Unlike spin 1, which has such an analogue (see [14]).

[13] Here, it is appropriate to return to the beginning of our article to emphasize the uniqueness of spin. It manifests itself (unlike, for example, charge) already in the wave equation for a free electron.

[14] Here, $\hat{\mathbf{S}} = (1/2)\boldsymbol{\sigma}$.

$a = \exp(\mathrm{i}\alpha)\cos\delta$ and $b = \exp(\mathrm{i}\beta)\sin\delta$, and an arbitrary spin state can be written as

$$\begin{pmatrix} \cos\delta \\ \exp(\mathrm{i}\gamma)\sin\delta \end{pmatrix} \tag{16}$$

($\gamma = \beta - \alpha$, and the unimportant phase factor $\exp(\mathrm{i}\alpha)$ is discarded). The pair of numbers δ and γ determine any spin state. Moreover, any vector (16) is an eigenvector of the operator $\hat{S}^2$ with the same eigenvalue 3/4. It can be shown that each spin state corresponds to a well-defined direction in space [16]. Let the unit vector $\mathbf{n}$ define such a direction, determined by a pair of spherical angles θ and φ:

$$\mathbf{n} = (n_x = \sin\theta\cos\varphi,\ n_y = \sin\theta\sin\varphi,\ n_z = \cos\theta)\,. \tag{17}$$

We define the operator of spin projection onto a given direction as the scalar product[15] $\hat{S}_\mathbf{n} = (\hat{\mathbf{S}}\mathbf{n})$. Considering the explicit form of the Pauli matrices, we obtain

$$\hat{S}_\mathbf{n} = \frac{1}{2}\begin{pmatrix} \cos\theta & \exp(-\mathrm{i}\varphi)\sin\theta \\ \exp(\mathrm{i}\varphi)\sin\theta & -\cos\theta \end{pmatrix}. \tag{18}$$

The vector

$$\begin{pmatrix} \cos\left(\dfrac{\theta}{2}\right) \\ \exp(\mathrm{i}\phi)\sin\left(\dfrac{\theta}{2}\right) \end{pmatrix} \tag{19}$$

is, as is easy to verify, an eigenvector of the matrix $\hat{S}_\mathbf{n}$ with the maximum eigenvalue 1/2. When changing the direction (17) to the opposite one, θ should be replaced by $\pi - \theta$, and ϕ by $\pi + \phi$, while $\hat{S}_{-\mathbf{n}} = -\hat{S}_\mathbf{n}$, and the eigenvector of the operator $\hat{S}_{-\mathbf{n}}$, instead of (19), will take the form

$$\begin{pmatrix} \sin\left(\dfrac{\theta}{2}\right) \\ -\exp(\mathrm{i}\phi)\cos\left(\dfrac{\theta}{2}\right) \end{pmatrix}. \tag{20}$$

The above-mentioned relationship between the spin state and the direction in real three-dimensional space becomes obvious if vectors (16) and (19) are equated to each other (i.e., $\theta = 2\delta$ and $\phi = \gamma$). The statement formulated above (it is called the theorem on the maximum projection of spin 1/2 [16]) is one-to-one: a given spin state (16) corresponds to one and only one direction $\mathbf{n}$, and vice versa. Thus, for example, the x-axis corresponds to the state $\frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, the y-axis corresponds to the state $\frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$, and the z-axis corresponds to the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. With a certain degree of conventionality, we can say about these states that the spin is directed along the x-axis, or along the y-axis, or along the z-axis. If the spin state and the direction in space correspond to each other in the above sense, then the average value of the operator of the spin projection onto a given direction reaches the maximum value 1/2, whereas the average value (in the same spin state) of the operator of the spin projection onto any orthogonal direction is zero. The convention of the expression 'the spin is directed along...' can easily be demonstrated with a simple example. Let the $\mathbf{n}$-axis be chosen as z ($\theta = 2\delta = 0$ and, consequently, $\hat{S}_\mathbf{n} = (1/2)\sigma_z$), which corresponds to the spin state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. In this case, the average values $\overline{S_x} = \overline{S_y} = 0$, $\overline{S_z} = 1/2$, and the variance $\Delta S_z = \sqrt{\overline{S_z^2}} = 0$. However, the variances of the spin projections onto the axes x and y in this state are nonzero: $\Delta S_x = \Delta S_y = 1/2$. It can be said that a relativistic free electron (in contrast to a nonrelativistic one) is cramped in one-dimensional space. Of course, the result obtained agrees with the uncertainty relation

$$\Delta S_x\,\Delta S_y \geqslant \frac{1}{2}\,\overline{S_z}\,. \tag{21}$$

Based on the maximum spin projection theorem, we can deduce the direction of the free electron's spin. The wave equation allows a solution of the form (11), where the momentum $\mathbf{p}$ can be arbitrary in both magnitude and direction. Thus, the motion of a free electron defines a fixed direction in space $\mathbf{n} = \mathbf{p}/p$. Obviously, the direction of the spin vector is determined by the vector $\mathbf{u}$, an eigenvector of the matrix $\mathbb{H}_\mathbb{D}$ (12), and since this matrix is parameterized by the components of the momentum vector, we can expect that the spin direction is related to the direction of the free electron motion. Note that the matrix $\mathbb{H}_\mathbb{D}$ commutes with the matrix

$$\Sigma_\mathbf{p} = \begin{pmatrix} \boldsymbol{\sigma}\mathbf{p} & \mathbb{O} \\ \mathbb{O} & \boldsymbol{\sigma}\mathbf{p} \end{pmatrix}. \tag{22}$$

Therefore, the vector u is a common eigenvector of both matrices. It can be represented in the form $\begin{pmatrix} \mathrm{v} \\ \mathrm{w} \end{pmatrix}$, where v and w are two-dimensional vectors, each being an eigenvector of the matrix $\boldsymbol{\sigma}\mathbf{p}$.[16] Since the matrix $\boldsymbol{\sigma}\mathbf{p}$ coincides with the matrix $\hat{S}_\mathbf{n}$ up to a factor, where $\mathbf{n}$ is determined by the direction of the momentum vector $\mathbf{p}$, then, based on the above regarding the spin direction, we can assume that the spin vector of a free electron is collinear with the momentum vector.

To conclude, I would like to note that the uniqueness of electron spin was evident even before the concept of 'spin' was introduced into scientific circulation, first as a hypothesis and then as a proven (experimentally and theoretically) physical phenomenon. In 1916, Sommerfeld published a paper [17] (see also [11]) in which he used Bohr's quantization rules to calculate the energy levels of the hydrogen atom, assuming the electron to be relativistic. Sommerfeld discovered that the Bohr levels are split, and this splitting is completely consistent with the fine structure of the Balmer lines. It was later discovered that Sommerfeld's results coincided with the solution of the Dirac equation (containing spin operators!) for the hydrogen atom. However, in 1916, no one even suspected the existence of spin! The mystery of this coincidence remains unsolved to this day.

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[15] Despite the conventionality of the term 'scalar product,' since $\hat{\mathbf{S}}$ is a matrix vector.

[16] Vectors v and w differ from each other only by a numerical factor, which can be determined by considering (12) to be a system of two linear equations for v and w [12].

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