

Old quantum mechanics by Bohr and Sommerfeld from modern perspective

K. Barley, A. Ruffing, S.K. Suslov

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Abstract. We review Bohr’s atomic model and its extension by Sommerfeld from a mathematical perspective of wave mechanics. The derivation of quantization rules and energy levels is revisited using semiclassical methods. Sommerfeld-type integrals are evaluated by elementary techniques, and connections with the Schrödinger and Dirac equations are established. Historical developments and key transitions from classical to quantum theory are discussed to clarify the structure and significance of the old quantum mechanics.

Keywords: Bohr’s atomic model, Bohr–Sommerfeld quantization rule, Schrödinger equation, Dirac equation, WKB method, fine structure formula, Sommerfeld puzzle, computer algebra system: Mathematica, transuranic elements

Prediction is very difficult, especially if it’s about the future!
Niels Bohr

*If you want to be a physicist, you must do three things — first, study mathematics, second, study more mathematics, and third, do the same.*¹

Arnold Sommerfeld

1. Introduction

Exactly a century ago, wave quantum mechanics was born in Arosa, Switzerland. Erwin Schrödinger was vacationing in this classic Swiss Alps town at Christmas 1925 when he made his breakthrough discovery of the wave equation [1].

The study of blackbody radiation and the quantum theory that emerged from it laid the foundation for Bohr’s atomic model, the first major step in understanding atomic structure developed about a century ago. By recognizing the quantum nature of energy and the discrete energy levels of electrons, Planck [2], Einstein [3], Rutherford [4], and Bohr [5] helped to explain the behavior of light and matter at the atomic scale, thereby paving the way for the development of quantum mechanics. Among the primary sources on the so-called ‘old quantum mechanics’ of Bohr and Sommerfeld are the classic publications [6–20], the references therein, and several educational videos [21].

Brief history. The fine structure of hydrogen atom spectral lines was first observed by Albert A. Michelson in 1887 [22, 23]. After the failure of his ether-wind experiments, he turned to spectroscopy and discovered that the prominent H_α line of

¹ Original: “Wenn man Physiker sein will, muss man drei Dinge tun: erstens Mathematik studieren, zweitens mehr Mathematik studieren und drittens dasselbe tun.”

K. Barley^(1,a), A. Ruffing^(2,b), S.K. Suslov^(3,c)

⁽¹⁾ Department of Mathematics, College of Arts and Sciences, Howard University, 2141 6th St. NW, Washington, DC 20059, USA

⁽²⁾ Landeshauptstadt München, Seminar TMG Universität, Referat für Bildung und Sport, Marienplatz 8, 80331 München, Germany

⁽³⁾ School of Mathematical and Statistical Sciences, Arizona State University, Tempe, AZ 85287–1804, USA

E-mail: ^(a) Kamal.Barley@howard.edu,

^(b) alruffing@web.de,

^(c) sks@asu.edu

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the Balmer series was, in fact, a doublet [16, 24, 25]. The electron was discovered by J.J. Thomson in 1897 [26], and Lord Rutherford proposed the planetary model of the atom in 1911. Niels Bohr introduced his theory of hydrogen-like systems in 1913 [6, 7, 9, 12, 15, 16], and, in 1916, Arnold Sommerfeld extended Bohr's quantization rules to the relativistic hydrogen atom [18] (see also [25, 27, 28]). An exact solution was finally achieved in 1928 by C.G. Darwin [29] and W. Gordon [30], following the discovery of the Dirac equation [31, 32]. Remarkably, their results matched precisely the 'old' Sommerfeld formula, an outcome known as the 'Sommerfeld puzzle' [24], discussed further in [8, pp. 426–429]. In this review article, we aim to explore the following topics from a mathematical perspective:

(i) The Bohr model: circular orbits of electrons in hydrogen-like atoms and the derivation of the Bohr formula (Nobel Prize in Physics, 1922 [5]).

(ii) Wilson and Sommerfeld: quantization rules for multi-dimensional periodic systems via classical action; derivation of Sommerfeld's relativistic formula for elliptical orbits in classical and wave mechanics.

(iii) Elementary evaluation of Sommerfeld-type integrals.

(iv) Additional examples and a resolution of the 'Sommerfeld puzzle'; a mistake that Schrödinger never made.

(v) Appendix A: Vector calculus tools for circular motion.

(vi) Appendix B: Instability of the hydrogen atom in classical physics, arising from the electron's predicted collapse into the nucleus in Rutherford's model.

(vii) Appendix C: Independent evaluation of Sommerfeld-type integrals via parameter differentiation.

(viii) Appendix D: Letter from Schrödinger to Sommerfeld, dated January 29, 1926.

(ix) Appendix E: Use of the Mathematica computer algebra system.

(x) Appendix F: Uranium vs oganesson and beyond in 'old quantum mechanics'.

Traditional physics textbooks never discuss the semiclassical derivation of the Sommerfeld fine structure formula due to its complexity and the fact that an accurate and elegant solution exists within relativistic quantum mechanics. While the semiclassical approach offers conceptual insight, it entails a rigorous and often challenging analysis, rendering it less suitable for introductory courses.

Our goal and motivation. These notes are intended as a supplement to traditional textbooks [33–38] and our recent article [39], providing original explanations, historical context, and extended discussion on selected topics. They may serve as a valuable resource in teaching and learning quantum physics and can support honors projects at any level—from introductory to graduate. To this end, the presentation is as self-contained as possible. This work is motivated by a course in the mathematics of quantum mechanics, taught for more than two decades at Arizona State University by one of the authors (SKS) [39–46] (see also the references therein and [47]).

2. Bohr's atomic model

Newton's second law for the uniform circular motion of a charged particle, such as an electron in the static Coulomb field of a heavy ion with positive charge Ze , states

$$ma = F = \frac{Ze^2}{r^2}, \quad a = \frac{v^2}{r} \quad (2.1)$$

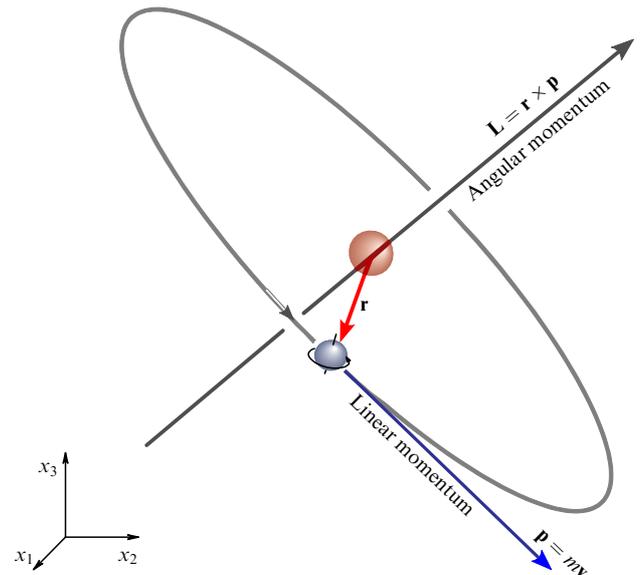


Figure 1. Bohr's atom.

by (A.1) from Appendix A. (Here, $m \approx 9.1094 \times 10^{-28}$ g and $e \approx 4.8032 \times 10^{-10}$ statcoulombs are the electron mass and the absolute value of its electric charge in centimeter-gram-second (CGS) units, respectively.) For an electron's linear momentum, $p = mv$, we obtain

$$p^2 = \frac{mZe^2}{r}, \quad (2.2)$$

and the total energy is given by

$$E = \frac{p^2}{2m} - \frac{Ze^2}{r} = -\frac{Ze^2}{2r}, \quad (2.3)$$

which is exactly one half of the potential energy, as stated by the *virial theorem*.

Niels Bohr [6, 7, 12], following the experiments of Nicholson [48], proposed to quantize the corresponding electron angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \quad pr = \hbar n \quad (n = 1, 2, \dots) \quad (2.4)$$

in terms of the *reduced Planck constant* $\hbar \approx 1.0546 \times 10^{-27}$ cm² g s⁻¹ in CGS units. For uniform circular motion, the vectors \mathbf{r} and \mathbf{p} are perpendicular to each other (A.3) (see Fig. 1). As a result, he derived the so-called Bohr orbits:²

$$r = r_n = \frac{\hbar^2 n^2}{mZe^2} \quad (2.5)$$

and the corresponding discrete energy levels of the electron:

$$E_n = -\frac{mZ^2e^4}{2\hbar^2 n^2}, \quad (2.6)$$

where $n = 1, 2, 3, \dots$ is the principal quantum number.

Indeed, using (2.2) and (2.4),

$$\frac{mZe^2}{r} = p^2 = \left(\frac{\hbar n}{r}\right)^2, \quad (2.7)$$

² In terms of the de Broglie wavelength λ , the quantization rule states that the length of the orbit equals $2\pi r_n = n\lambda$, where $\lambda = h/p = 2\pi\hbar/p$.

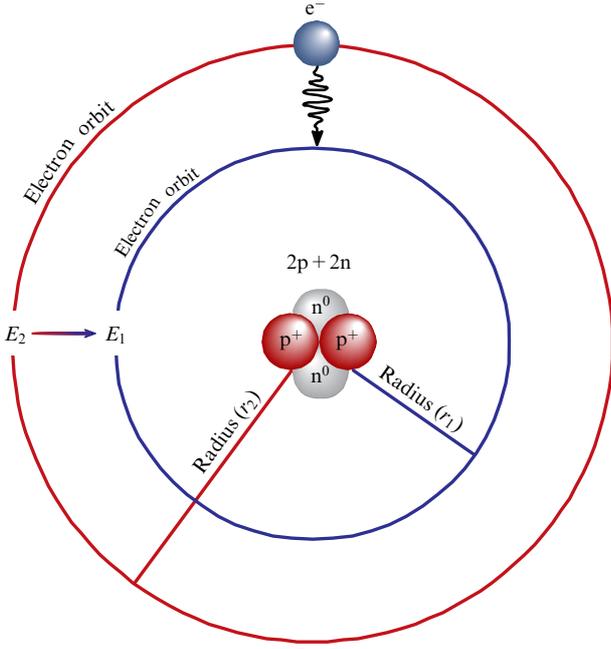


Figure 2. Change in electron orbits and energies, $r_2 \rightarrow r_1$ and $E_2 \rightarrow E_1$, in helium ion He^+ upon emission of photon in Bohr's model. For helium, $Z = 2$; therefore, by (2.5), (2.6): $r_1 = .2646 \times 10^{-8}$ cm, $r_2 = 4r_1 = 1.0584 \times 10^{-8}$ cm and $E_1 = -8.719 \times 10^{-11}$ erg = -54.424 eV, $E_2 = E_1/4 = -2.1798 \times 10^{-11}$ erg = -13.606 eV, respectively. Emitted photon has wavelength of $\lambda \approx 30.379$ nm, within ultraviolet region of electromagnetic spectrum.

leads to (2.5). Similarly, combining (2.3) and (2.5),

$$E_n = -\frac{Ze^2}{2r_n} = -\frac{mZ^2e^4}{2\hbar^2n^2}, \quad (2.8)$$

which completes the derivation of Bohr's discrete energy formula (2.6).

In Bohr's atomic model, electrons in the orbits (2.5) are stable and do not radiate energy. (Instability in the original Rutherford atom is discussed in Appendix B.) Electrons can transition between energy levels (2.6) by absorbing or emitting photons (light quanta) with specific energies. The energy of the photon corresponds to the difference between the initial and final energy levels [12] (see, for example, Fig. 2). Albert Einstein described this model as "the highest form of musicality in the realm of thoughts" [49].

Mendeleev's Periodic Table and the Bohr model are two significant developments in the understanding of the structure of atoms and elements. Mendeleev's table, introduced in 1869, organized elements by increasing atomic weight and recurring chemical properties [17, pp. 2, 3]. The Bohr model, proposed in 1913 [7], offered a theoretical framework for atomic structure by depicting electrons orbiting the nucleus in fixed, quantized energy levels. This simplified representation of electronic structure directly underpins the periodic arrangement of elements in the modern table [28].

3. Wilson and Sommerfeld quantization rules in wave mechanics

Topics to review: Kepler problems in classical mechanics [24; 28, pp. 84–90, 109–119, 251–258; 50, pp. 146–148; 51, pp. 92–102, 466–477, 481, 482]; spherical harmonics [36, 40,

52, 53]; the Schrödinger equation [35, 36, 38, 54] and relativistic Schrödinger and Dirac equations [33–35, 38, 39, 46]; the spinor spherical harmonics [33, 34, 37, 45, 46, 53] and separation of variables for the Dirac equation in a central field; semiclassical approximation [38, 52, 55, 56].

Brief history: As an extension of Bohr's rules, Wilson [20] and Sommerfeld [18] independently proposed a method for quantizing action integrals in classical mechanics for a multidimensional periodic system over one period of motion (see also [28, footnote on p. 111]).

3.1 Sommerfeld fine structure formula

We follow [24; 28, pp. 251–258] on the basis of the laws of conservation of energy and angular momentum, with somewhat different details. The classical relativistic Hamiltonian, or total energy E , of hydrogen-like systems under an attractive Coulomb potential, has the quadratic form

$$\left(E + \frac{Ze^2}{r}\right)^2 = \mathbf{p}^2c^2 + m^2c^4. \quad (3.1)$$

In polar coordinates,

$$\mathbf{p}^2 = (p_r)^2 + \frac{1}{r^2}(p_\theta)^2, \quad (3.2)$$

with $p_r = \gamma m \dot{r} = \gamma m (dr/dt)$ (radial momentum), and $p_\theta = \gamma m r^2 \dot{\theta} = \gamma m r^2 (d\theta/dt)$ (angular momentum). In the relativistic case, $\gamma = (1 - v^2/c^2)^{-1/2}$ is the familiar Lorentz factor (see Appendix A for more detail).

Due to conservation of angular momentum, p_θ is constant. Introducing the new variable $s = 1/r$, we note that

$$\frac{ds}{d\theta} = -\frac{p_r}{p_\theta}. \quad (3.3)$$

In this notation, Eqn (3.1) becomes

$$\left(\frac{E}{mc^2} + \frac{Ze^2}{mc^2} s\right)^2 = 1 + \left(\frac{p_\theta}{mc}\right)^2 \left[\left(\frac{ds}{d\theta}\right)^2 + s^2\right]. \quad (3.4)$$

Differentiation with respect to θ yields an ordinary differential equation:

$$\frac{d^2s}{d\theta^2} + \omega^2(s - D) = 0. \quad (3.5)$$

Here, by definition,

$$\omega^2 = 1 - \frac{Z^2e^4}{c^2p_\theta^2}, \quad D = \frac{Ze^2E}{\omega^2c^2p_\theta^2}. \quad (3.6)$$

Solving (3.5) yields the relativistic Kepler orbits in the form

$$s = \frac{1}{r} = C_1 \cos(\omega\theta) + C_2 \sin(\omega\theta) + D, \quad (3.7)$$

where C_1 and C_2 are constants. If the point of closest approach (perihelion) occurs at $\theta = 0$, then $C_2 = 0$, and

$$\frac{1}{r} = C_1 \cos(\omega\theta) + D. \quad (3.8)$$

Classical relativistic Kepler orbits have the form of conic sections, as in the nonrelativistic case [24, 28], but with a new angular variable $\phi = \omega\theta$. Thus, for elliptical orbits (bound states), the motion from one perihelion ($\phi = 0$) to the next ($\phi = 2\pi$) requires $\theta = 2\pi/\omega$, with a per-revolution shift of

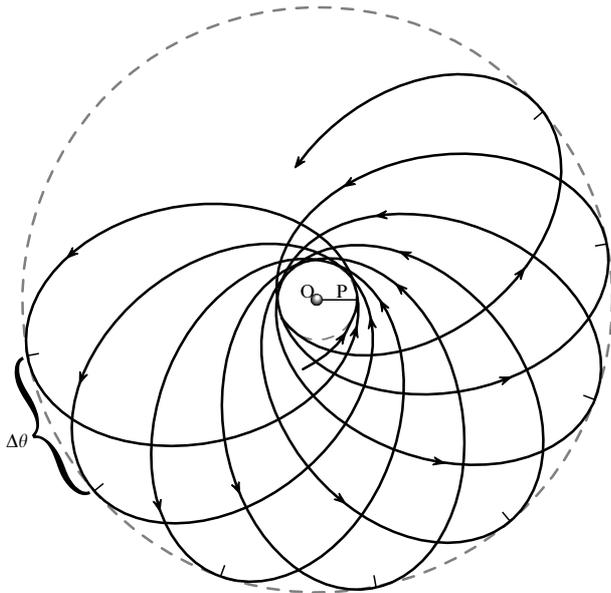


Figure 3. Relativistic Kepler motion [28, p. 254]. (Here, O is fixed focus at which nucleus is situated; P is initial position of perihelion.) Perihelion and aphelion move along two concentric circles around nucleus at O.

$\Delta\theta = 2\pi/\omega - 2\pi$ ³ (see Fig. 3 and Mathematica notebook, EllipsesAnimateAu.nb [57] for animation).

Introducing the eccentricity ϵ , we have, for $\phi = 0$, the perihelion distance $r_{\min} = a(1 - \epsilon)$, and for $\phi = \pi$, the aphelion distance $r_{\max} = a(1 + \epsilon)$. In standard geometrical terms, the orbit equation is

$$\frac{1}{r} = \frac{1 + \epsilon \cos(\omega\theta)}{a(1 - \epsilon^2)}. \quad (3.9)$$

We can now apply the original Wilson–Sommerfeld quantization rules:

$$\int_{\theta=0}^{\theta=2\pi} p_{\theta} d\theta = h n_{\theta} \quad (\text{which gives } p_{\theta} = \hbar n_{\theta}), \quad (3.10)$$

$$\int_{\theta=0}^{\theta=2\pi/\omega} p_r dr = h n_r. \quad (3.11)$$

To evaluate the latter integral, we transform the radial momentum as follows:

$$p_r = \gamma m \dot{r} = \gamma m \left(\frac{dr}{d\theta} \right) \dot{\theta} = \frac{p_{\theta}}{r^2} \left(\frac{dr}{d\theta} \right). \quad (3.12)$$

Thus,

$$p_r dr = p_{\theta} \left(\frac{1}{r} \frac{dr}{d\theta} \right)^2 d\theta = p_{\theta} \epsilon^2 \omega \frac{\sin^2 \phi}{(1 + \epsilon \cos \phi)^2} d\phi, \quad (3.13)$$

upon using the orbit equation (3.9). The radial quantization condition (3.11) then becomes

$$\frac{1}{2\pi} \int_{\phi=0}^{\phi=2\pi} \frac{\epsilon^2 \sin^2 \phi d\phi}{(1 + \epsilon \cos \phi)^2} = \frac{n_r}{\omega n_{\theta}}. \quad (3.14)$$

³ In the terminology of classical work [28], where only sufficiently small $\Delta\theta$ are discussed. Throughout this article, we shall keep the original name ‘elliptical’ curve, although for certain ω ’s a different topological structure may occur, such as orbits with multiple points of self-intersections (see Figs 10 and 11 below).

For evaluation of the integral,

$$\frac{1}{2\pi} \int_{\phi=0}^{\phi=2\pi} \frac{\epsilon^2 \sin^2 \phi d\phi}{(1 + \epsilon \cos \phi)^2} = (1 - \epsilon^2)^{-1/2} - 1, \quad (3.15)$$

see [28, pp. 476, 477, German edn.] and our complementary Mathematica notebook, BohrAtomMathematica.nb [57].

As a result, we obtain

$$\frac{1}{1 - \epsilon^2} = \left(1 + \frac{n_r}{\omega n_{\theta}} \right)^2, \quad (3.16)$$

$$D = \frac{\alpha Z E}{n_{\theta}^2 \omega^2 \hbar c} = \frac{1}{a(1 - \epsilon^2)}, \quad (3.17)$$

where $\alpha = e^2/(\hbar c)$ is the fine-structure constant. Finally, using the last two equations together with the orbit (3.9) and energy (3.4) equations, after tedious but straightforward calculations, we arrive at the original Sommerfeld formula:

$$\frac{E_{n_r, n_{\theta}}}{mc^2} = \left[1 + \frac{\alpha^2 Z^2}{(n_r + (n_{\theta}^2 - \alpha^2 Z^2)^{1/2})^2} \right]^{-1/2}, \quad (3.18)$$

where n_r (the *radial quantum number*) and n_{θ} (the *azimuthal quantum number*) are positive integers. This result made it possible to explain, for the first time, the fine structure of spectral lines. (For further details, see [24; 28, pp. 251–258] and Appendix E with a supplementary Mathematica notebook.)

Note. Equations (3.6), (3.10), and (3.16)–(3.18) allow us to determine the quantized values of parameters of the electron’s elliptical orbits (3.9) as follows:

$$\omega n_{\theta} n_{\theta} = (n_{\theta}^2 - \alpha^2 Z^2)^{1/2}, \quad (3.19)$$

$$\epsilon_{n_r, n_{\theta}} = \sqrt{n_r} \frac{(n_r + 2\sqrt{n_{\theta}^2 - \alpha^2 Z^2})^{1/2}}{n_r + \sqrt{n_{\theta}^2 - \alpha^2 Z^2}}, \quad (3.20)$$

$$a_{n_r, n_{\theta}} = \frac{a_0}{Z} \left(n_r + \sqrt{n_{\theta}^2 - \alpha^2 Z^2} \right) \times \sqrt{\alpha^2 Z^2 + \left(n_r + \sqrt{n_{\theta}^2 - \alpha^2 Z^2} \right)^2}, \quad (3.21)$$

where $a_0 = \hbar^2/(me^2)$ is the familiar Bohr radius. These formulas generalize the circular orbits. (Further details can be found in the complementary Mathematica notebook [57] and in Appendix E.)⁴

In the nonrelativistic limit, we obtain

$$\omega n_{\theta} = 1 - \frac{\alpha^2 Z^2}{2n_{\theta}^2} - \frac{\alpha^4 Z^4}{8n_{\theta}^4} + \mathcal{O}(\alpha^6), \quad (3.22)$$

$$\epsilon_{n_r, n_{\theta}} = \frac{\sqrt{n_r}(n_r + 2n_{\theta})^{1/2}}{n_r + n_{\theta}} + \frac{\sqrt{n_r} \alpha^2 Z^2}{2(n_r + n_{\theta})^2 (n_r + 2n_{\theta})^{1/2}} + \frac{\sqrt{n_r}(3n_r + 5n_{\theta}) \alpha^4 Z^4}{8n_{\theta}(n_r + n_{\theta})^3 (n_r + 2n_{\theta})^{3/2}} + \mathcal{O}(\alpha^6), \quad (3.23)$$

$$\frac{Z a_{n_r, n_{\theta}}}{a_0} = (n_r + n_{\theta})^2 - \frac{\alpha^2 Z^2 (2n_r + n_{\theta})}{2n_{\theta}} - \frac{\alpha^4 Z^4}{8} \left(\frac{2n_r}{n_{\theta}^3} + \frac{1}{(n_r + n_{\theta})^2} \right) + \mathcal{O}(\alpha^6), \quad (3.24)$$

⁴ Classical solutions to the relativistic Kepler problem are also discussed in [51, pp. 481, 482] and [58, pp. 100–102].

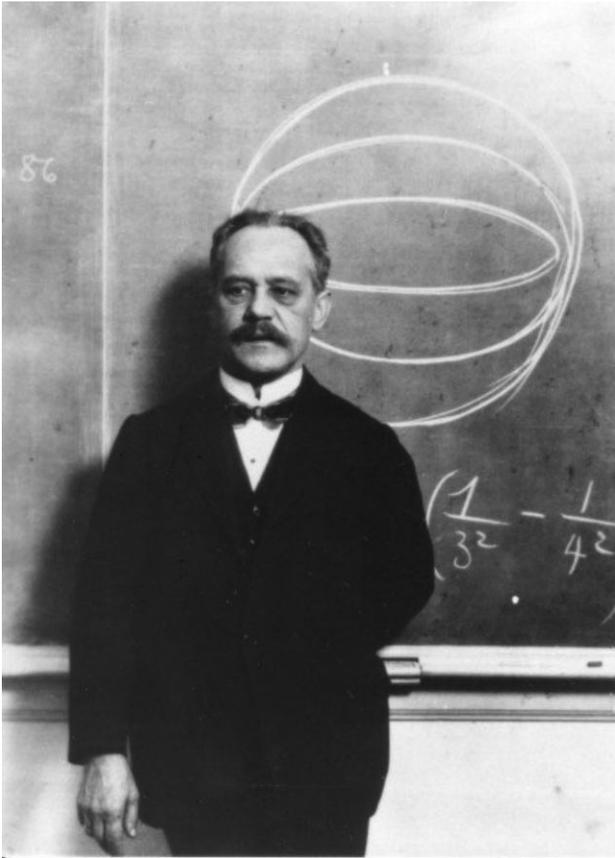


Figure 4. Sommerfeld's elliptic orbits for hydrogen atom [28, p. 116].

$$\frac{E_{n_r, n_\theta}}{mc^2} = 1 - \frac{\alpha^2 Z^2}{2(n_r + n_\theta)^2} - \frac{\alpha^4 Z^4 (4n_r + n_\theta)}{8n_\theta (n_r + n_\theta)^4} + \mathcal{O}(\alpha^6), \quad (3.25)$$

where $\alpha = e^2/(\hbar c)$ and $c \rightarrow \infty$ (see our Mathematica file, BohrAtomMathematica.nb [57]). (This asymptotic expansion will be analyzed below (3.49).) In this limit, we recover Sommerfeld's elliptic orbits for hydrogen-like systems [28, pp. 109–119] (Fig. 4). As follows from (3.20) and (3.23), Bohr's circular orbits occur only when $n_r = 0$.

In his classical book, a “Bible” of atomic theory, Sommerfeld concludes [28, p. 258]: *The above-calculated energy levels and the line complexes to be derived from them also retain their validity in wave mechanics. The way in which wave mechanics derives them is not only far less picturesque (anschaulich), but also much more laborious than the way described above. Hence it was necessary to carry out the calculation as far as possible according to the method of the older quantum theory; the inferences drawn can then later be taken over directly into wave mechanics.*

Indeed, in his introductory book on wave mechanics, Sommerfeld also discussed the quantization of Kepler's problems for the relativistic Schrödinger and Dirac equations [59, pp. 100–104, 112–118, and 282–286]. (See also [19].)

3.2 Fine structure formula in wave mechanics

In the development of quantum theory, the Bohr–(Wilson)–Sommerfeld quantization rule served as an original ‘bridge’ between classical and quantum mechanics (for historical details, see [13, 15, 16, 18, 20, 28, 48]). Today, the Schrödinger [60] and Dirac [61] wave equations are used to analyze the

corresponding Kepler problems. How did Schrödinger derive his celebrated equation and subsequently apply it to the hydrogen atom? According to his own testimony [1, 62, 63] and [64, 030† pp. 141–143],⁵ de Broglie's seminal work on the wave theory of matter (1923–1924) [65] and Einstein's studies on ideal Bose gases (1924–1925) laid the foundation for the discovery of wave mechanics (see also [39, 66, 67]).

The phenomenological quantization rules of the ‘old’ quantum theory [18, 20] are, in modern physics, derived from the corresponding wave equations via the so-called semiclassical approximation — also known as the Wentzel–Kramers–Brillouin (WKB) method — as developed in [36, 52, 55, 68–70].

This approximation refines the Bohr–Sommerfeld quantization rule within the framework of wave mechanics. The WKB method, which yields approximate solutions to wave equations, leads to a quantization condition similar to the Bohr–Sommerfeld rule, but with a crucial phase correction.

During separation of variables in spherical coordinates, the quantization of angular momentum and spin is exact, since the concept of spin is inherently built into the structure of the wave equation itself [42]. After this, the analysis reduces to solving radial equations.

We follow [39, pp. 95, 96] with somewhat different details. Let us recall the one-dimensional stationary Schrödinger equation:

$$u'' + \frac{2m}{\hbar^2} [E - U(x)]u = 0. \quad (3.26)$$

For a particle in a central field, the corresponding 3D wave equations can be separated in spherical coordinates, yielding a radial equation of the form

$$u''(x) + q(x)u(x) = 0, \quad (3.27)$$

where $x^2 q(x)$ is continuous along with its first and second derivatives for $0 \leq x \leq b < \infty$. These equations can be approximately solved by the WKB method.

More to review: The WKB wave functions, their relation to Airy functions [71], associated quantization rules, and further technical details are discussed in [38, 52] and elsewhere. We recommend reviewing sections §19, pp. 235–251, and §28, pp. 178–188, on the semiclassical approximation in [52, 38], as well as [72, pp. 380–390] and Chapter 9 of [71] on Airy functions.

It is well known that the traditional semiclassical approximation breaks down near $x = 0$ for central fields. However, using the change of variables $x = e^z$ and $u = e^{z/2}v(z)$ transforms the equation into a new form:

$$v''(z) + q_1(z)v(z) = 0, \quad (3.28)$$

where

$$q_1(z) = -\frac{1}{4} + (x^2 q(x))_{x=e^z}. \quad (3.29)$$

This is known as Langer's modification [55, 73, 74]. As $z \rightarrow -\infty$ (i.e., $x \rightarrow 0$), the function $q_1(z)$ approaches a constant:

$$-\frac{1}{4} + \lim_{x \rightarrow 0} x^2 q(x) \quad \text{and} \quad \lim_{z \rightarrow -\infty} q_1^{(k)}(z) = 0 \quad (k = 1, 2).$$

⁵ In his letter to Einstein dated November 3, 1925, he writes: “A few days ago, I read with great interest the ingenious theses of Louis de Broglie, which I finally got hold of ...”

Thus, $q_1(z)$ and its derivatives vary slowly for large negative z [72, p. 387].

The WKB method can then be applied to this transformed equation, and, in the original equation, $q(x)$ is replaced with

$$q(x) = \frac{1}{4x^2} = p_{\text{effective}}^2(x) \tag{3.30}$$

(see [52, 55, 73, 74] for further details).

The Bohr–Sommerfeld quantization rule, derived, for example, in [38] and [52], takes the form

$$\int_{r_1}^{r_2} p(r) dr = \pi \left(n_r + \frac{1}{2} \right) \tag{3.31}$$

provided $p(r_1) = p(r_2) = 0$. (Here, $n_r = 0, 1, 2, \dots$ is the radial quantum number.)

For all Coulomb problems under consideration, we utilize a generic integral originally evaluated by Sommerfeld [28, pp. 611–612] using complex analysis: if

$$p(r) = \sqrt{-A + \frac{B}{r} - \frac{C}{r^2}} \quad (A, C > 0), \tag{3.32}$$

then

$$\int_{r_1}^{r_2} p(r) dr = \pi \left(\frac{B}{2\sqrt{A}} - \sqrt{C} \right) \tag{3.33}$$

with $p(r_1) = p(r_2) = 0$ (see also [51, pp. 468–470]). In Section 4 and Appendix C, we present two independent elementary evaluations of this integral.

As a result, for the discrete energy levels, we obtain the following generic equation:

$$\frac{B}{2\sqrt{A}} - \sqrt{C} = n_r + \frac{1}{2}, \tag{3.34}$$

which is valid for all Coulomb-type problems under consideration and beyond [44, 59].

Kepler problems in wave mechanics. For the well-known case of the *nonrelativistic Coulomb problem*, the radial equation in dimensionless units reads [1, 36, 38]

$$u'' + \left[2 \left(\varepsilon_0 + \frac{Z}{x} \right) - \frac{l(l+1)}{x^2} \right] u = 0 \tag{3.35}$$

$$\left(\varepsilon_0 = \frac{E}{E_0}, E_0 = \frac{e^2}{a_0}, a_0 = \frac{\hbar^2}{me^2}, x = \frac{r}{a_0} \right),$$

where $l = 0, 1, 2, \dots$ is the quantized orbital angular momentum.

Applying the Bohr–Sommerfeld quantization rule, we must, in accordance with (3.30), use

$$p(r) = \left[2 \left(\varepsilon_0 + \frac{Z}{r} \right) - \frac{(l+1/2)^2}{r^2} \right]^{1/2}, \quad p(r_1) = p(r_2) = 0, \tag{3.36}$$

as corrected by Langer’s substitution, discussed, for instance, in [39, 44, 52, 55].

Identifying parameters in the generic integral (3.32), we have

$$A = -2\varepsilon_0, \quad B = 2Z, \quad C = \left(l + \frac{1}{2} \right)^2.$$

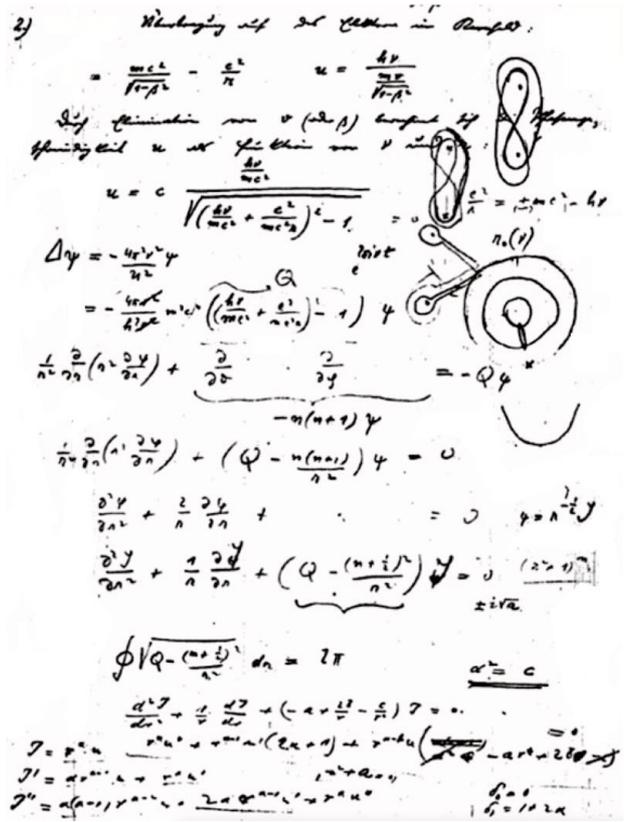


Figure 5. Page from Notebook N1, showing first record of wave equation [17, 67, 75, 76] (likely written around Christmas 1925).

Substituting into the quantization rule (3.34), we find

$$\frac{Z}{\sqrt{-2\varepsilon_0}} - l - \frac{1}{2} = n_r + \frac{1}{2}. \tag{3.37}$$

Solving for ε_0 yields the exact energy levels for a nonrelativistic hydrogen-like atom:

$$\varepsilon_0 = \frac{E}{E_0} = - \frac{Z^2}{2(n_r + l + 1)^2}. \tag{3.38}$$

Here, $n = n_r + l + 1$ is the *principal quantum number*, recovering Bohr’s formula for discrete energy levels, as presented in Eqn (2.6) via the semiclassical approximation. (The WKB method is typically introduced in quantum mechanics only after the exact solution has been established [36].)

Our main goal is to analyze the corresponding relativistic problems. For the *relativistic Schrödinger equation*, we write [39]

$$u'' + \left[\left(\varepsilon + \frac{\mu}{x} \right)^2 - 1 - \frac{l(l+1)}{x^2} \right] u = 0, \tag{3.39}$$

(see Fig. 5 for the original version⁶), and apply Langer’s transformation to define the effective momentum:

$$p(x) = \left[\left(\varepsilon + \frac{\mu}{x} \right)^2 - 1 - \frac{(l+1/2)^2}{x^2} \right]^{1/2}. \tag{3.40}$$

⁶ Schrödinger’s notebooks are reproduced in the Archive for the History of Quantum Physics (AHQP); for further details, see [66, 77].

We identify the parameters:

$$A = 1 - \varepsilon^2, \quad B = 2\mu\varepsilon, \quad C = \left(l + \frac{1}{2}\right)^2 - \mu^2.$$

Applying the Bohr–Sommerfeld quantization condition (3.34) yields

$$\frac{\mu\varepsilon}{\sqrt{1 - \varepsilon^2}} = n_r + v + 1, \quad \varepsilon = \frac{E}{mc^2}, \quad (3.41)$$

which gives the exact relativistic energy levels:

$$E = E_{n_r} = \frac{mc^2}{\sqrt{1 + (\mu/(n_r + v + 1))^2}} \quad (n_r = 0, 1, 2, \dots), \quad (3.42)$$

where

$$\mu = \frac{Ze^2}{\hbar c}, \quad v = v_{\text{Schrödinger}} = -\frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 - \mu^2}. \quad (3.43)$$

In the nonrelativistic limit $c \rightarrow \infty$ (or $\mu \rightarrow 0$), we obtain [35, 38]

$$\begin{aligned} \frac{E_{n_r, l}}{mc^2} &= \frac{1}{\sqrt{1 + \frac{\mu^2}{\left[n_r + \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 - \mu^2}\right]^2}}} \\ &= 1 - \frac{\mu^2}{2n^2} - \frac{\mu^4}{2n^4} \left(\frac{n}{l + 1/2} - \frac{3}{4}\right) + \mathcal{O}(\mu^6), \quad \mu \rightarrow 0, \end{aligned} \quad (3.44)$$

where $n = n_r + l + 1$ is the corresponding non-relativistic principal quantum number (see also [57] for a complementary Mathematica notebook, BohrAtomMathematica.nb).

In this Taylor expansion:

- the first term corresponds to the rest energy $E_0 = mc^2$;
- the second term yields the nonrelativistic Schrödinger energy eigenvalue;
- the third term represents the *fine structure*, lifting the degeneracy among states with the same n but different l .

Sommerfeld's fine structure formula for the relativistic Coulomb problem represents one of the most significant achievements of the 'old' quantum mechanics [28, pp. 251–258]. Here, we derive this result in the semiclassical approximation using the *radial Dirac equations* (separation of variables in spherical coordinates is discussed in detail in Refs [37, 45, 46, 52, 78]).

In dimensionless units, one of the second-order differential equations for the Dirac spinor component takes the form

$$v_1'' + \frac{(\varepsilon^2 - 1)x^2 + 2\varepsilon\mu x - v(v + 1)}{x^2} v_1 = 0, \quad (3.45)$$

while the second equation is obtained via the substitution $v \rightarrow -v$ (see Eqns (6.58), (6.59) in Ref. [45] and/or Eqns (3.81), (3.82) in Ref. [46]).

Note. It should be emphasized that the above (Schrödinger-type) form of the radial equations—verifiable using a computer algebra system [40]—is critical for the successful application of the WKB approximation to the relativistic

Coulomb problem [39]. (For an alternative, see, for example, [79, 80].)

Applying Langer's transformation leads to the effective momentum function

$$p(x) = \left[\left(\varepsilon + \frac{\mu}{x}\right)^2 - 1 - \frac{(v + 1/2)^2 + \mu^2}{x^2} \right]^{1/2}. \quad (3.46)$$

Thus, for the Dirac equation, we identify

$$A = 1 - \varepsilon^2, \quad B = 2\mu\varepsilon, \quad C = \left(v + \frac{1}{2}\right)^2.$$

Applying the Bohr–Sommerfeld quantization rule (3.34), we again arrive at Eqn (3.41), yielding the corresponding energy spectrum:

$$E = E_{n_r, j} = \frac{mc^2}{\sqrt{1 + \mu^2/(n_r + v)^2}} \quad (n_r = 0, 1, 2, \dots) \quad (3.47)$$

with the adjustment $n_r \rightarrow n_r - 1$ as discussed in [45, 46, 52]. Here, once again, $\mu = Ze^2/(\hbar c)$, and in Dirac theory,

$$v = v_{\text{Dirac}} = \sqrt{\left(j + \frac{1}{2}\right)^2 - \mu^2}, \quad (3.48)$$

where $j = 1/2, 3/2, 5/2, \dots$ is the total angular momentum (including spin). (Observe that only at this stage, in the 'old' formula (3.18), can we identify Sommerfeld's *azimuthal quantum number*, n_θ , as $n_\theta = j + 1/2$.)

In the nonrelativistic limit ($\mu \rightarrow 0$), the Dirac–Sommerfeld formula yields [35, 38, 46, 78]

$$\frac{E_{n_r, j}}{mc^2} = 1 - \frac{\mu^2}{2n^2} - \frac{\mu^4}{2n^4} \left(\frac{n}{j + 1/2} - \frac{3}{4}\right) + \mathcal{O}(\mu^6), \quad (3.49)$$

where $n = n_r + j + 1/2$ is the principal quantum number for hydrogen-like atoms (see also [57]).

In this expansion:

- the first term is the rest mass energy of the electron, $E_0 = mc^2$;
- the second term recovers the nonrelativistic Schrödinger energy;
- the third term provides the fine-structure correction, originating from spin-orbit interaction in the Pauli approximation.

This prediction agrees with experimental data on fine-structure splitting in hydrogen-like systems. In contrast, Schrödinger's relativistic formulation fails to accurately describe the fine structure of hydrogen-like atoms (e.g., hydrogen, ionized helium, doubly ionized lithium). For example, the total fine-structure splitting at $n = 2$ is overestimated by a factor of 8/3 relative to Sommerfeld's prediction, which is consistent with experimental measurements.

Indeed, the maximum spread in the fine-structure levels occurs for $l = 0$ and $l = n - 1$, with total angular momentum $j = 1/2$ and $j = n - 1/2$, respectively, as seen in Eqns (3.44) and (3.49) [35, 38]. The ratio of these spreads is

$$\frac{\Delta E_{\text{Schrödinger}}}{\Delta E_{\text{Sommerfeld}}} = \frac{4n}{2n - 1} \quad (n = 2, 3, \dots). \quad (3.50)$$

When $n = 2$, $\Delta E_{\text{Schrödinger}} = (8/3) \Delta E_{\text{Sommerfeld}}$ is obtained.

Note. With the help of Mathematica, we derived the next two terms in expansion (3.49) as follows (BohrAtomMathematica.nb) [28, 57]:

$$-\frac{\mu^6}{4n^6} \left[\frac{5}{4} - \frac{3n}{j+1/2} + \frac{3n^2}{2(j+1/2)^2} + \frac{n^3}{2(j+1/2)^3} \right], \quad (3.51)$$

$$\frac{\mu^8}{16n^8} \left[\frac{35}{8} - \frac{15n}{j+1/2} + \frac{15n^2}{(j+1/2)^2} - \frac{n^3}{(j+1/2)^3} - \frac{3n^4}{(j+1/2)^4} - \frac{n^5}{(j+1/2)^5} \right]. \quad (3.52)$$

Hence, Sommerfeld’s fine structure formula can now be derived semiclassically from the radial Dirac equations. This derivation not only recovers the quantization rules of Bohr and Sommerfeld [28, 59]—introduced nearly a decade before the concept of spin⁷—but also links them to modern quantum theory. Indeed, the classical relativistic Hamiltonian does not include spin, creating an inherent ambiguity in Bohr–Sommerfeld quantization (see [27, 83]). *De facto*, we have completed Sommerfeld’s original WKB arguments [59, pp. 134–143] from a modern mathematical perspective.

For a full analytical treatment, including the non-relativistic limit, see Refs [40, 45, 46, 52] (based on the Nikiforov–Uvarov method), as well as standard texts [33–35, 37, 38].

Summary: The principal result of Sommerfeld’s fine structure theory, namely, formulas (3.47), (3.48), remains the correct expression for energy levels in hydrogen-like systems within wave mechanics. By a remarkable twist of historical fate, Sommerfeld managed in 1916 to derive the correct formula from what ultimately proved to be an inadequate theoretical framework. It must be recalled that, at the time, both quantum mechanics and the concept of spin were still nearly a decade away. Thus, “perhaps the most remarkable numerical coincidence in the history of physics,” as Kronig remarked, illustrates the curious fact that flawed physical models can nevertheless yield correct quantitative predictions (see [25, pp. 84, 85] and references therein).

4. Evaluation of Sommerfeld-type integrals

The teaching of mathematics and calculus in the United States has developed a ‘modern tradition’ of favoring so-called ‘real world problems’—and this is indeed the ‘one’ remarkable example: the integral required for the semiclassical derivation of one of the most remarkable formulas of the last century—the Sommerfeld fine structure formula.

For all problems under consideration, we employ the generic integral (3.32), (3.33), originally evaluated by Sommerfeld using complex analysis [28, pp. 611, 612] (Fig. 6). In contrast, an elementary derivation of this integral was presented in [39]. By integrating by parts on the left-hand

⁷ The concept of electron spin was introduced by G.E. Uhlenbeck and S. Goudsmit in a letter published in *Die Naturwissenschaften*, issue of 20 November 1925; see [81] for details. The spin concept explained why the famous Stern–Gerlach experiment [28, pp. 124–126, 505–507] ends up with two separated beams of silver atoms, in contrast to the prediction of ‘old’ quantum theory without spin, which would result in the prediction of three separated beams of silver atoms [82].

gezogen. Die r -Ebene ist zwischen r_{\min} und r_{\max} aufgeschlitzt zu denken und stellt das obere Blatt einer zweiblättrigen Riemannschen Fläche dar. Wegen des positiven Charakters der Phasenintegrale ist bei positivem dr (unteres Ufer des Schlitzes) das Vorzeichen der Quadratwurzel positiv, bei negativem dr (oberes Ufer desselben) negativ zu nehmen, wie in der Figur angedeutet ist. Daraus folgt zugleich, daß die Quadratwurzel außerhalb des Schlitzes auf der reellen Achse der r -Ebene imaginär ist, und zwar positiv imaginär für $r > r_{\max}$, negativ imaginär für $0 < r < r_{\min}$, wie ebenfalls in der Figur angedeutet ist. Man erkennt dies, wenn man von dem positiven oder negativen Ufer des Verzweigungsschnittes aus je einen halben Umlauf um die Verzweigungspunkte $r = r_{\max}$ oder $r = r_{\min}$ macht.

Wir fahren mit der Erweiterung des Integrationsweges fort und ziehen diesen auf die Pole des Integranden zusammen. Es sind dies die Stellen

$$r = 0 \quad \text{und} \quad r = \infty.$$

An der Stelle $r = 0$ verhält sich J_z wie

$$\sqrt{C} \int \frac{dr}{r} \left(1 + \frac{B}{C} r + \dots \right).$$

Figure 6. Half-page from Sommerfeld’s book showing contour of integration. <https://archive.org/details/atombauundspekt00sommgoog/page/478/mode/2up> (see [51, pp. 468–470] and [28, pp. 611, 612] for more details.)

side of (3.33), we obtain

$$\begin{aligned} \int_{r_1}^{r_2} p(r) dr &= rp(r) \Big|_{r_1}^{r_2} - \int_{r_1}^{r_2} \frac{r[-(B/r^2) + 2(C/r^3)]}{2p(r)} dr \\ &= \frac{B}{2} \int_{r_1}^{r_2} \frac{dr}{\sqrt{-Ar^2 + Br - C}} - \int_{r_1}^{r_2} \frac{(C/r^2) dr}{\sqrt{-A + (B/r) - (C/r^2)}}. \end{aligned} \quad (4.1)$$

For the penultimate integral, we write

$$\begin{aligned} \int_{r_1}^{r_2} \frac{dr}{\sqrt{(B^2/(4A) - C) - (r\sqrt{A} - B/(2\sqrt{A}))^2}} \\ = \frac{1}{\sqrt{A}} \arcsin \frac{r\sqrt{A} - B/(2\sqrt{A})}{\sqrt{B^2/(4A) - C}} \Big|_{r_1}^{r_2} = \frac{\pi}{\sqrt{A}}. \end{aligned} \quad (4.2)$$

Next, applying the substitution $r = 1/x$ in the final integral of (4.1), we obtain

$$\begin{aligned} - \int_{x_1=1/r_1}^{x_2=1/r_2} \frac{C dx}{\sqrt{-A + Bx - Cx^2}} \\ = -\sqrt{C} \int_{x_1}^{x_2} \frac{\sqrt{C} dx}{\sqrt{(B^2/(4C) - A) - (x\sqrt{C} - B/(2\sqrt{C}))^2}} \\ = \pi\sqrt{C}, \end{aligned} \quad (4.3)$$

where we complete the square and evaluate a standard definite integral. (Alternatively, we may interchange A and C and follow a similar route.) Combining the results from the last two integrals completes the proof.

An additional independent evaluation of the Sommerfeld-type integral is presented in Appendix C.

5. Further examples and resolution of ‘Sommerfeld’s puzzle’

As is well known, Bohr introduced semiclassical quantization rules for hydrogen-like atoms based on classical circular

motion, while Sommerfeld extended these ideas to relativistic elliptical orbits [28]. Measurements of fine-structure splitting by Paschen were interpreted as experimental tests of the special theory of relativity [19, 24, 25, 28]. The exact solution was obtained for the first time by C.G. Darwin [29] and W. Gordon [30], but only after the discovery of the Dirac equation [31, 32] — *and remarkably, the new result reproduced the ‘old’ Sommerfeld formula (3.47)!*

Werner Heisenberg [84] referred to this agreement as a ‘miracle’ and wrote: “It would be intriguing to explore whether this is about a miracle or it is the group-theoretical approach which leads to this formula” [85]. In a 1956 letter, Erwin Schrödinger commented: “This is a fortuitous coincidence” [86]. As demonstrated in [44], Schrödinger appears to have been correct — the ‘Sommerfeld puzzle’ [24] has now been resolved and extended to a class of multi-dimensional problems involving different symmetry groups.

Topic to review: The Nikiforov–Uvarov approach [39, pp. 97, 98; 40, pp. 44–47, 72, pp. 339–347], and [52].

For exact solutions, the generalized equation of hypergeometric type [52] is given by

$$u'' + \frac{\tilde{\tau}(x)}{\sigma(x)} u' + \frac{\tilde{\sigma}(x)}{\sigma^2(x)} u = 0, \quad (5.1)$$

with the parameter choices

$$\begin{aligned} \sigma(x) &= x, & \tilde{\tau}(x) &= 0, \\ \tilde{\sigma}(x) &= -ax^2 + bx - c + \frac{1}{4}. \end{aligned} \quad (5.2)$$

Then,

$$\begin{aligned} \pi(x) &= \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma} \\ &= \frac{1}{2} \pm \sqrt{ax^2 + (k-b)x + c} \end{aligned} \quad (5.3)$$

must reduce to a linear function [52]. When $k = b \pm 2\sqrt{ac}$, one can complete the square to obtain

$$\pi = \frac{1}{2} \pm (\sqrt{ax} \pm \sqrt{c}), \quad \tau = \tilde{\tau} + 2\pi = 1 \pm 2(\sqrt{ax} \pm \sqrt{c}). \quad (5.4)$$

We then choose

$$\tau' = -2\sqrt{a} < 0 \quad \text{and} \quad \lambda = k + \pi' = b - 2\sqrt{ac} - \sqrt{a}. \quad (5.5)$$

As a result, for all Sommerfeld-type potentials, the Nikiforov–Uvarov quantization rule [52],

$$\lambda + n\tau' + \frac{1}{2} n(n-1)\sigma'' = 0 \quad (n = 0, 1, 2, \dots), \quad (5.6)$$

yields

$$\frac{b}{2\sqrt{a}} - \sqrt{c} = n + \frac{1}{2} \quad (5.7)$$

as a generic formula for the exact energy levels with $n = n_r$. (It is worth noting that Sommerfeld had already obtained a similar relation in special cases [59].)

The puzzle resolution: By comparing (3.34) and (5.7), we arrive at the following result [44]:

Theorem

$$a = A, \quad b = B, \quad c = C. \quad (5.8)$$

Indeed, the generic (WKB-based) rule (3.34) is also valid for the exact energy levels (5.7) obtained via the Nikiforov–Uvarov approach [40, 44] for all Coulomb problems under consideration. Other examples include quantum harmonic oscillators and systems with Kratzer and Pöschl–Teller potentials [40]. (See also [43] for an extension of Schrödinger’s coherent states [62].)

In connection with Sommerfeld’s fine-structure formula, Erwin Schrödinger testified, *inter alia*, in a letter dated February 29, 1956 [86]: ... *you are naturally aware of the fact that Sommerfeld derivation of the fine-structure formula provides only fortuitously the result demanded by the experiment. One may notice then from this particular example that newer form of quantum theory (i.e., quantum mechanics) is by no means such an inventible continuation of the older theory as is commonly supposed. Admittedly the Schrödinger theory, relativistically framed (without spin), gives a formal expression of the fine-structure formula of Sommerfeld, but it is incorrect owing to the appearance of half-integers instead of integers. My paper in which this is shown has ... never been published; it was withdrawn by me and replaced by non-relativistic treatment.... The computation [by the relativistic method] is far too little known. It shows in one respect how necessary Dirac’s improvement was, and on the other hand it is wrong to assume that the older form of quantum theory is ‘broadly’ in accordance with the newer form.* (See also [39, Appendix D, p. 100] and [87] about the discovery of the relativistic Schrödinger equation.)

Methodological note: It should now be clear that only after the ‘two quantum revolutions’ could the ambiguity in the quantization of the Kepler problem in the ‘old quantum mechanics’ [27, 83] be resolved — namely, when spherical symmetry is explicitly taken into account through separation of variables, and the corresponding radial equations are derived exactly, without approximation, and then subjected to the WKB method under Langer’s correction.

Schrödinger appears to have been the first to follow this route in 1925 (or was close to doing so but avoided a misstep by switching to the exact solution?) in his unpublished notes. He used equation (3.39), which corresponds to a spin-zero particle (see [39], Fig. 5, for his original notebook, and [42] for further discussion on the concept of spin and the wave equations). As later attested, due to the discrepancy with experimental results, Schrödinger never published this work (see also [39, Appendix D] for his letter to Weyl).

6. Mistake that Schrödinger never made

Interestingly, in Fig. 5, the lower left-hand portion displays the ‘old’ Bohr–Sommerfeld quantization rule with what is now recognized as the Langer correction [55], namely,⁸ $n(n+1) \rightarrow (n+1/2)^2$, but with l rather than $l+1/2$ on the right-hand side, as it should appear in the WKB approximation (3.31) (in Schrödinger’s notation, you may wish to interchange $n \leftrightarrow l$, as is customary today; see also footnotes^{16, 17} below). As we now understand well, this form would yield an incorrect spectrum for his relativistic equation — in particular, the principal quantum number in the nonrelativistic Kepler problem would acquire half-integer values rather than integers (see our supplementary Mathematics notebook for further details [57]).

⁸ In this case, one obtains $\sqrt{C} = n + 1/2$.

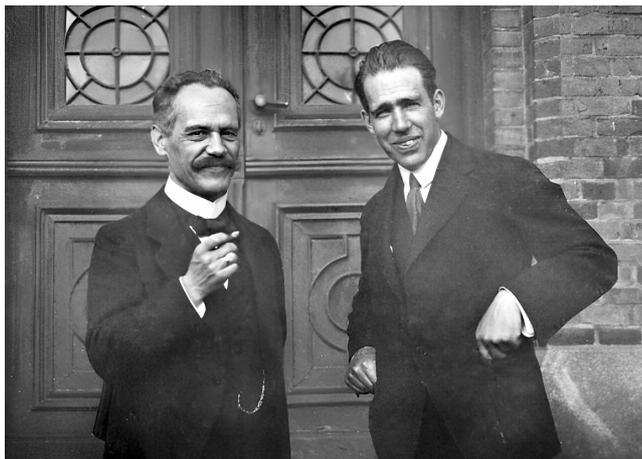


Figure 7. Arnold Sommerfeld and Niels Bohr in Lund, September 1919, during his impressive lecture tour of Scandinavia after World War I: “...a sudden return to the good times before the war” [8, pp. 240–246]. Courtesy of Deutsches Museum, Munich.

However, Schrödinger never made this mistake. Instead, he outlined, “...(although having used just written words)...” [39, Appendix D, p. 111]; see also [64, 184† pp. 484, 485], the exact solution via the Laplace method, expressed as a contour integral [39, Appendix C]!

Schrödinger knew the Sommerfeld-type integral intimately. For example, it is explicitly referenced in his letter to Sommerfeld dated January 29, 1926 [66, p. 462]; see also [64, 041† pp. 170–172]: ...*Finally, I still wish to add that the discovery of the whole connection [between the wave equation and the quantization of the hydrogen atom] goes back to your beautiful quantization method for evaluating the radial quantum integral. It was the characteristic $-\frac{B}{\sqrt{A}} + \sqrt{C}$, which suddenly shone out from the exponents α_1 and α_2 like a Holy Grail.*

In this letter, Schrödinger reported for the first time the success of the wave theory in solving the quantum oscillator, rotator, nonrelativistic (and partially relativistic) hydrogen atom (Kepler problems), and free motion of a point mass in infinite space and in a box, prior to the formal publications [1, 88]. He also formulated a program for future research. For the reader’s benefit, the complete letter has been translated from German to English in Appendix D.

In a letter dated February 3, 1926 [64, 042† pp. 173–175], Sommerfeld responded enthusiastically: *What you write, in your essay and letter, is terribly interesting. My personal opinion on the mysticism of integers must remain silent, as must my personal convenience... . My impression is this: Your*

method is a substitute for the new quantum mechanics of Heisenberg, Born, Dirac... . Because your results are completely consistent with theirs... . This marked the beginning of the triumph of Schrödinger’s wave mechanics [66, pp. 617–636] (see also [89] and [90]).

Timetable: The exact dates of Schrödinger’s foundational discoveries, leading to his first publications [1, 88], are not precisely recorded [75, 76, 91, and 66, pp. 459–465]. However, one can estimate the timeline based on his letter to Einstein [64, 030† pp. 141–143], dated November 3, 1925; Bloch’s recollection of two colloquia in Zürich [92], presumably held in late November and/or early December 1925 [66, pp. 419–423]; a letter to Wien [64, 037† pp. 162–165] from Arosa on December 27, 1925; and a letter to Sommerfeld [64, 041† pp. 170–172] from Zürich on January 29, 1926. This yields a reasonable estimate spanning from early November 1925 to the end of January 1926.

At the same time, in January 1926, Bohr reflected on the development of the ‘old’ theory in a letter to his friend, the Swedish physicist Carl Oseen [25, p. 85] (quoted from [93, p. 73]): *At the present stage of the development of quantum theory, we can hardly say whether it was good or bad luck that the properties of the Kepler motion could be brought into such simple connection with the hydrogen spectrum, as was believed possible at one time. If this connection had merely had that asymptotic character which one might expect from the correspondence principle, then we should not have been tempted to apply mechanics as crudely as we believed possible for some time. On the other hand, it was just these mechanical considerations that were helpful in building up the analysis of the optical phenomena which gradually led to quantum mechanics. — It was, indeed, hard to predict!*

7. Conclusion

Traditional textbooks [33–36, 38, 54] do not address the derivation of the Sommerfeld fine structure formula via the semiclassical approximation—and now the reader can appreciate why. Indeed, *de facto*, there are three distinct levels of complexity, reflecting the historical “three quantum revolutions” in the development of quantum physics:

- the elementary level of classical mechanics [50, 51], in the Bohr model of the atom (Fig. 1);
- introductory quantum mechanics [36, 54], covering the non-relativistic (and unsuccessfully relativistic) hydrogen atom;
- and finally, relativistic quantum theory [33–35, 37, 38], involving the Dirac equation and the Sommerfeld fine structure formula.

This history line is laid out in Table 1.

Table 1

Time period (bottom to top)	Main results
Dirac’s wave mechanics: 1928	Relativistic Dirac equation; Fine structure formula (3.47), (3.48); we derive this result in WKB approximation.
Schrödinger’s wave mechanics: 1925–1926	Nonrelativistic and relativistic Schrödinger equations; Bohr’s formula (3.38), Schrödinger’s fine structure formula (3.42)–(3.44); we derive (3.38) and (3.42)–(3.44) in WKB approximation.
‘Old’ quantum mechanics: 1911–1916	Rutherford’s planetary model of atom; Bohr–(Wilson)–Sommerfeld quantization rules; Bohr’s formula (2.6) and fine structure formula (3.18).

The WKB method becomes applicable only after a careful study of the basic properties of the Dirac equation, including the construction of spinor spherical harmonics [33, 34, 37, 45, 46, 53], and a nontrivial separation of variables in spherical coordinates. Naturally, this poses a ‘pedagogical challenge’ and requires significant time in both teaching and learning quantum mechanics. These notes—originally motivated by introductory courses on the mathematics of quantum mechanics at Arizona State University [39–46, 94] and at Technische Universität München [47]—though not exhaustive, may help readers bridge this gap.

The resolution of the ‘Sommerfeld puzzle’ remains of theoretical and pedagogical interest [44]. Only in Dirac’s theory, which gave a fully relativistic quantum-mechanical description of the hydrogen atom and accurately predicted its fine structure, can this question be formulated unambiguously. And only at this level can we arrive, not at a ‘puzzle,’ but at a mathematically well-defined statement, such as a hypothesis or a theorem.

Computer algebra systems such as Maple and Mathematica are valuable tools for teaching and learning quantum mechanics, particularly for beginners. These systems can handle complex mathematical calculations, allowing students to focus on understanding the underlying concepts and problem-solving strategies rather than getting bogged down in tedious computations. We should admit nonetheless that each of the quantum mechanical problems under consideration, with the exception of a few trivial ones, usually requires a separate analysis with lots of specific details that are far away from a formal application of a given computer algebra system, as it might seem at first glance (see, for example, [40, 41, 94]).

Ultimately, the practical use of Mathematica in this article [57] can also aid students in tackling complex computations and better understanding quantum physics. Artificial Intelligence calculation with its linear functions and simple activation functions clearly cannot, even remotely, achieve the level of correct mathematical derivation of, say, equations (3.18)–(3.21) provided in Appendix E.

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8. Appendices

A. Velocity, acceleration, and angular momentum for circular motion

For uniform circular motion, we get

$$a = \frac{v^2}{r}. \quad (\text{A.1})$$

Indeed, if

$$\mathbf{r} = \mathbf{r}(t) = r \cos(\omega t) \mathbf{e}_1 + r \sin(\omega t) \mathbf{e}_2, \quad \mathbf{r}(0) = r \mathbf{e}_1, \quad (\text{A.2})$$

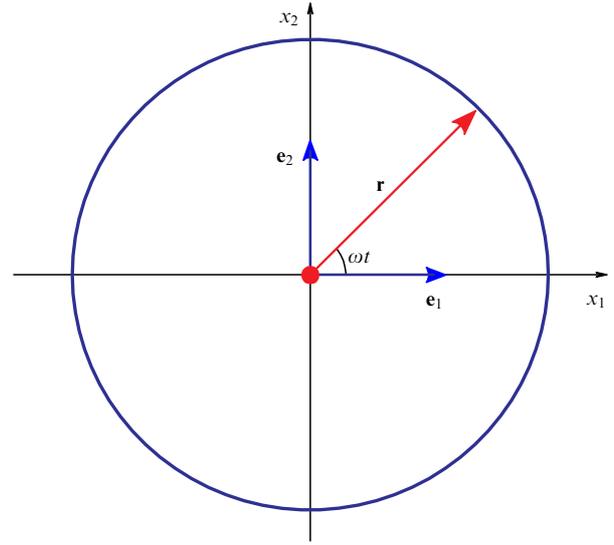


Figure 8. Uniform circular motion.

where \mathbf{e}_1 and \mathbf{e}_2 are two orthonormal vectors (Fig. 8), then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = r\omega(-\sin(\omega t)\mathbf{e}_1 + \cos(\omega t)\mathbf{e}_2), \quad \mathbf{r}\mathbf{v} = 0, \quad (\text{A.3})$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -r\omega^2(\cos(\omega t)\mathbf{e}_1 + \sin(\omega t)\mathbf{e}_2) = -\omega^2\mathbf{r}. \quad (\text{A.4})$$

Thus,

$$\mathbf{v}^2 = \mathbf{v}\mathbf{v} = r^2\omega^2(\sin^2(\omega t) + \cos^2(\omega t)) = r^2\omega^2 = v^2, \quad v = \omega r. \quad (\text{A.5})$$

In a similar fashion,

$$\mathbf{a}^2 = \mathbf{a}\mathbf{a} = \omega^4 r^2, \quad a = \omega^2 r. \quad (\text{A.6})$$

Relation (A.1) follows from the last two expressions: $a = \omega^2 r$ and $\omega = v/r$.

Moreover,

$$\begin{aligned} \mathbf{r} \times \mathbf{v} &= r^2\omega \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \cos(\omega t) & \sin(\omega t) & 0 \\ -\sin(\omega t) & \cos(\omega t) & 0 \end{vmatrix} \\ &= r^2\omega(\cos^2(\omega t) + \sin^2(\omega t))\mathbf{e}_3 = r\mathbf{v}\mathbf{e}_3 \end{aligned} \quad (\text{A.7})$$

(see Fig. 1).

Equation (3.2) can be derived in a similar way. Indeed, in polar coordinates

$$\mathbf{r} = r \cos \theta \mathbf{e}_1 + r \sin \theta \mathbf{e}_2, \quad (\text{A.8})$$

we get

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} = (\dot{r} \cos \theta - r \sin \theta \dot{\theta}) \mathbf{e}_1 + (\dot{r} \sin \theta + r \cos \theta \dot{\theta}) \mathbf{e}_2, \\ \mathbf{v}^2 &= (\dot{r})^2 + r^2(\dot{\theta})^2, \end{aligned} \quad (\text{A.9})$$

which implies (3.2).

For a relativistic particle

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \quad \mathbf{p} = \gamma m \mathbf{v} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} \quad (\text{A.10})$$

in a central field $U = U(r)$,

$$\mathbf{F} = -\nabla U = -\frac{dU}{dr} \frac{\mathbf{r}}{r}. \quad (\text{A.11})$$

The corresponding angular momentum, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, is the integral of motion:

$$\begin{aligned} \frac{d}{dt} \mathbf{L} &= \frac{d}{dt} (\mathbf{r} \times \mathbf{p}) = \mathbf{v} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} \\ &= m\gamma(\mathbf{v} \times \mathbf{v}) - \frac{dU}{dr} \frac{1}{r} (\mathbf{r} \times \mathbf{r}) = \mathbf{0}, \end{aligned} \quad (\text{A.12})$$

or $\mathbf{L} = \text{const.}$

In polar coordinates (A.8), we have

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \gamma m r^2 \dot{\theta} \mathbf{e}_3. \quad (\text{A.13})$$

B. Instability of hydrogen atom in classical physics

As is known [15–17, 21], a rotating electron in Rutherford's planetary model must fall into the nucleus—a spiral-in—according to the laws of classical mechanics and electrodynamics. Indeed, the total instantaneous power emitted over all solid angles is given by the well-known result of Larmor [95]:

$$\frac{dE}{dt} = -\frac{2e^2 a^2}{3c^3}, \quad (\text{B.1})$$

where $c \approx 2.9979 \times 10^{10}$ cm s⁻¹ is the speed of light in CGS units. Here, according to (2.1)–(2.3),

$$a = \frac{v^2}{r} = \frac{e^2}{mr^2}, \quad (\text{B.2})$$

and thus,

$$\frac{dE}{dt} = -\frac{2e^2}{3c^3} \left(\frac{e^2}{mr^2} \right)^2 = -\frac{2e^6}{3c^3 m^2 r^4}. \quad (\text{B.3})$$

On the other hand, from the virial theorem (2.3), we get

$$\frac{dE}{dt} = \frac{e^2}{2r^2} \frac{dr}{dt}. \quad (\text{B.4})$$

Equating (B.3) and (B.4), we obtain

$$r^2 \frac{dr}{dt} = -\frac{4e^4}{3m^2 c^3} \quad \text{or} \quad 3r^2 dr = -\frac{4e^4}{m^2 c^3} dt. \quad (\text{B.5})$$

Integrating both sides yields

$$-r_1^3 = r^3 \Big|_{r_1}^0 = \int_{r=r_1}^0 3r^2 dr = -\frac{4e^4}{m^2 c^3} \int_{t=0}^{\tau} dt = -\frac{4e^4}{m^2 c^3} \tau, \quad (\text{B.6})$$

where $r_1 \approx 0.52921 \times 10^{-8}$ cm is the first Bohr radius of a hydrogen atom, as given in (2.5). Therefore, an electron in Rutherford's model would fall into the nucleus in less than a nanosecond:

$$\begin{aligned} \tau &= \frac{m^2 c^3}{4e^4} r_1^3 \approx \frac{(9.1094 \times 10^{-28})^2 (2.9979 \times 10^{10})^3}{4(4.8032 \times 10^{-10})^4} \\ &\times (0.52921 \times 10^{-8})^3 \approx 1.5564 \times 10^{-11} \text{ s} \end{aligned} \quad (\text{B.7})$$

(according to [15], this estimate was obtained in 1904 by G.A. Schott).

The electron velocity on the first Bohr orbit can be estimated as follows:

$$\begin{aligned} v_1 &= \frac{e}{\sqrt{mr_1}} \approx \frac{4.8032 \times 10^{-10}}{(0.52921 \times 10^{-8} \times 9.1094 \times 10^{-28})^{1/2}} \\ &\approx 2.1876 \times 10^8 \text{ cm s}^{-1} \approx 0.7297 \times 10^{-2} c \end{aligned} \quad (\text{B.8})$$

(nonrelativistic motion), and for the time of one revolution, we arrive at

$$t_{\text{rot}} = \frac{2\pi r_1}{v_1} \approx 2 \times 3.1415 \frac{0.52921 \times 10^{-8}}{2.1876 \times 10^8} \approx 1.5199 \times 10^{-16} \text{ s}. \quad (\text{B.9})$$

Therefore, the total number of rotations before falling into the center can be estimated as

$$N_{\text{total}} = \frac{\tau}{t_{\text{rot}}} \approx \frac{1.5564 \times 10^{-11}}{1.5199 \times 10^{-16}} \approx 102,400. \quad (\text{B.10})$$

The spiral-in time, τ , is much longer than the orbital time, t_{rot} , so treating the spiral-in as a succession of circular orbits in a hydrogen atom is plausible (see [16, 17] for more details).

C. Independent evaluation of Sommerfeld-type integrals

On the contrary, one can use the technique of differentiation with respect to parameters for the familiar integrals related to the Bohr–Sommerfeld quantization rule [28, 59]. As is well known, if

$$J(x) = \int_{f(x)}^{g(x)} F(x, y) dy, \quad (\text{C.1})$$

then

$$\frac{dJ}{dx} = \int_{f(x)}^{g(x)} \frac{\partial F(x, y)}{\partial x} dy + F(x, g(x)) \frac{dg}{dx} - F(x, f(x)) \frac{df}{dx}. \quad (\text{C.2})$$

In the WKB case, the last two terms vanish, because the limits are turning points where the integrand vanishes [56].

We now apply this procedure for an independent evaluation of the Sommerfeld-type integrals discussed in this note [44]. Indeed, writing

$$I = \int_{r_1}^{r_2} p(r) dr, \quad p(r) = \sqrt{-A + \frac{B}{r} - \frac{C}{r^2}} \quad (A, C > 0), \quad (\text{C.3})$$

provided $p(r_1) = p(r_2) = 0$, we find

$$\begin{aligned} \frac{dI}{dB} &= \frac{1}{2} \int_{r_1}^{r_2} \frac{dr}{\sqrt{-Ar^2 + Br - C}} \\ &= \frac{1}{2\sqrt{A}} \int_{r_1}^{r_2} \frac{dr}{\sqrt{(B^2 - 4AC)/(4A^2) - (r - B/(2A))^2}} \\ &= \frac{1}{2\sqrt{A}} \arcsin \left(\frac{2Ar - B}{\sqrt{B^2 - 4AC}} \right) \Big|_{r_1}^{r_2} = \frac{\pi}{2\sqrt{A}}. \end{aligned} \quad (\text{C.4})$$

As a result,

$$\frac{dI}{dB} = \frac{\pi}{2\sqrt{A}}, \quad I(B_0 = 2\sqrt{AC}) = 0, \quad (\text{C.5})$$

and, by integration,

$$I = \pi \left(\frac{B}{2\sqrt{A}} - \sqrt{C} \right). \quad (\text{C.6})$$

This is perhaps the simplest way to calculate this integral.

D. Letter from Schrödinger to Sommerfeld⁹

Zurich, January 29, 1926

Most honorable Herr Professor,
for a long time I didn't let hear anything from me,¹⁰ and hence I am writing now to you in a quick manner, namely to avoid that you are going to write to me earlier — on the following behalf: you may have looked already to my quantum article which I have sent to Mr. Geheimrat Wien,¹¹ for the annals, having kindly asked him to show it to you, before publishing it in the journal. Of course, I am most curious to hear no other opinion — first — but yours, namely: whether you are sharing the ambitious hope I have, which I am coupling to the derivation of quantum equations, being derived from a Hamiltonian principle.¹²

So far, I have transferred some more mechanical problems into the new perception's world. As far as my mathematics is sufficient, all evolves in a most beautiful manner — it is no primitive copy of old quantization rules, but differs from them at some characteristic points.¹³

The harmonic oscillator is to be treated with the same analytic aids like the Kepler problem* (in the equation of vibration which one obtains to determine the function ψ , one has to introduce the square of the abscissa as an independent variable).¹⁴ Again, there appears the remarkable case that an equation, which is not integrable by ordinary quadratures, becomes precisely integrable in case of the eigenvalues, namely by elementary functions — this shows how nature, in some kind of lovely way, is interested in making its exploration easier for us. As for the eigenvalues (energy levels), there are: $\frac{2n+1}{2} h\nu$, i.e. the so-called half-integer quantization.¹⁵ Although the quantum differences are unchanged here, I see a great meaning behind it, since $\frac{2n+1}{2}$ is the arithmetic mean of n and $n + 1$.

⁹ Translated from [64, pp. 170–172]). This letter is also printed in Arnold Sommerfeld, *Wissenschaftlicher Briefwechsel* Band 2, S. 236–238.

¹⁰ Schrödinger's last (extant) letter [64, pp. 132–135] to Sommerfeld was written in July 1925.

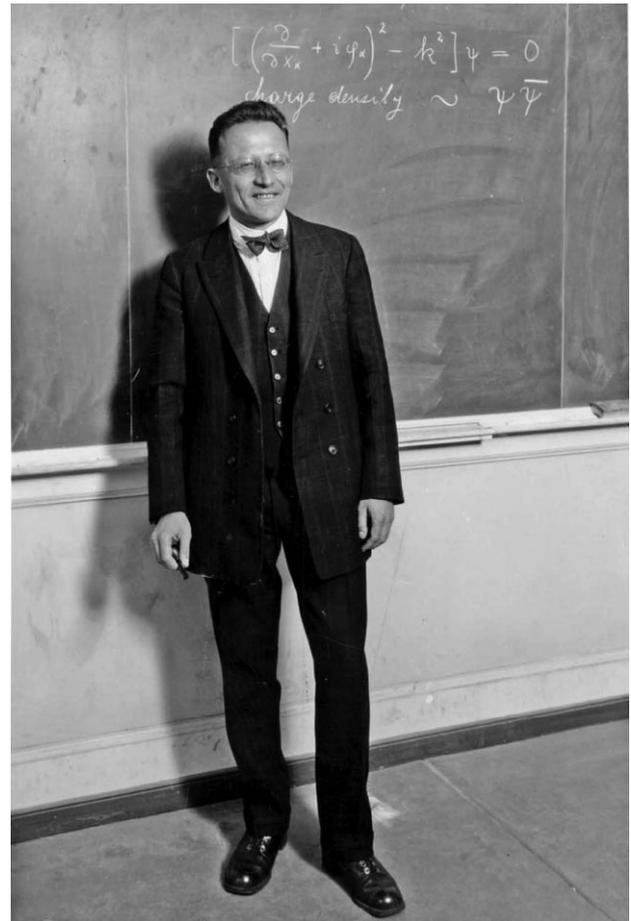
¹¹ This was Schrödinger's first communication on wave mechanics [1], which he had sent a few days earlier to one of the editors of *Annalen der Physik*, Wilhelm Wien (see also Wien's reply [64, p. 177]).

¹² Cf. [50, 51]. See also Sommerfeld's reply [64, pp. 173–175].

¹³ According to Schrödinger's former colleague Peter Paul Ewald [96, p. 385], Courant and Hilbert's seminal 1924 work on methods of mathematical physics [97] "enabled physicists to grasp the spirit of a unified mathematical method in an especially important field, marked by keywords: eigenvalues, eigenfunctions. These problems... arise in physics through oscillations of all kinds." It is almost certain that this book was not available to Schrödinger in Arosa [39]. Only in the second article on wave mechanics does Schrödinger thank his assistant E. Fues for pointing out a connection with the Hermite polynomials for the harmonic oscillator problem and acknowledge the relation of his wave function in the 'Kepler problem' with the 'polynomials of Laguerre' [88].

¹⁴ The 'Planck oscillator' is treated in Schrödinger's second paper [88] as the first example.

¹⁵ Cf. [88]. Half-integer quantum numbers had also been introduced in Heisenberg's early attempts to describe the anomalous Zeeman effect.



Wording on reverse side of photograph:
Professor Erwin Schrödinger, December 1926
(see <https://search.library.wisc.edu/digital/AHDIU5YGAIKZOW8N>)

The rotator (dumbbell) in three dimensions (i.e., with two variables ϑ, φ) is quite simple, the eigenfunctions are ordinary spherical harmonics, the eigenvalues (energy levels) are $n(n+1) \frac{h^2}{8\pi^2 J}$. This characteristic expression $n(n+1)$ stems from the differential equation of the spherical harmonics. Again, I find it most delighting, not because of the considered case, but since we may wish that for future investigations, that one might obtain $n(n+1)$ instead of n^2 ,¹⁶ where necessary (regarding your formulas on intensity and the formulas for anomalous Zeeman splitting).¹⁷ At the contrary, in case of the dumbbell, one has first to evaluate whether the explanation of the band spectra will be damaged. But I don't think so. The effect is, as one easily recognizes, given by a very slight difference between the linear term of the positive and the negative branch. And this difference, if I remember right, indeed appears (or anyhow at least in a similar way — I calculated all that yesterday and did not have the chance to compare with the experimental values in detail.)

For the free motion of the mass point one receives that every energy value may appear if the mass point is located in the infinite space. If it is in a box, which one has to consider as

¹⁶ In the introduction to his supplement on wave mechanics [59, p. 2], Sommerfeld pointed out that these difficulties with the old quantum theory had now found their natural explanation through the new wave mechanics.

¹⁷ As Sommerfeld noted in [28, pp. 333, 476], the appearance of $j(j+1)$ instead of j^2 in the Landé g -factor "suggests that not one state j but two adjacent quantum states j and $j+1$ are physically relevant."

a boundary condition for the ψ function, one will obtain the same energy values as for the quantization of the zig zag motion. The eigenfunctions correspond in case of the free mass point — presumed one calculates in a relativistic way — to the phase waves of de Broglie.¹⁸ For the mass point in the box, these are standing eigenvibrations of the box volume being related to the dispersion law of the de Broglie phase waves.

The next important task appears to be — apart from the calculation of important special cases, like Stark effect, Zeeman effect and relativistic Kepler motion — the formulation of a rule for intensity and polarization which has to replace the correspondence principle.¹⁹ Since I believe that the ψ function really describes all those processes in the atom which are the reason for light emission, the function has to give insight into these processes. One has to investigate the intensity beats between two simultaneously excited eigenvibrations and their (i.e. the beats') spatial distribution. On this way, the rich mathematical theory (orthogonality of eigenfunctions, meaning of the eigenvalues as extremal values of the Hamiltonian integral etc.) will surely lead to simple results.

I am worried about the relativistic Kepler problem.²⁰ I am not sure whether it is true — what I claim in the manuscript — that the nuclear co-motion is that essential for the new method of description. I would even rather withdraw my remark that this would already be the case in the old method of description, and I would kindly ask you to drop this remark if you think that is nonsense.²¹

But as regards my perception, it is only the nuclear's motion which will provide help, otherwise one would obtain half partial quanta — in contrast to the experience. And these half partial quanta stem exactly from those $n(n+1)$ of the spherical harmonics, what is on the other hand so commendable.

Finally, I would like to say that the discovery of the whole connection of the structures goes back — even if it is not easy to recognize from outside — to your beautiful integration method for evaluating the radial quantum integral. It was the characteristic and familiar $-\frac{B}{\sqrt{A}} + \sqrt{C}$, which suddenly, like a Holy Grail, was shining from the exponents α_1 and α_2 .²²

I hope that you, most honorable Herr Professor, and all yours are doing well. With the best and most sincere greetings from house to house, I always remain

Your faithful and grateful sincere

E. Schrödinger

P.S. Innsbruck has not yet been officially decided.²³ But I think I'll stay here. It is mainly Schweidler's departure to

¹⁸ As we learn from his letter to Einstein [64, pp. 141–143], Schrödinger was inspired in this work by the 'brilliant theses' (1925) of Louis de Broglie on the phase waves of electrons, which he had first encountered in early November 1925.

¹⁹ See also the remarks in his letters to Lorentz [64, pp. 203–205] and Wentzel [64, pp. 226–228].

²⁰ Schrödinger, as shown in his notes (Fig. 5), had first proposed a relativistic wave equation before arriving at the well-known nonrelativistic one [39]. See also his letters to Sommerfeld [64, pp. 178–184] and Lorentz [64, pp. 252–261], together with historical studies [66, 76, 77].

²¹ Sommerfeld addressed this point in his next letter [64, pp. 173–175].

²² Cf. [28, pp. 611, 612].

²³ Schrödinger had also received an offer from Innsbruck (cf. also his letter to Sommerfeld [64, pp. 132–135]), which he declined in mid-March (cf. his letter [64, pp. 197–200] of March 17, 1926, to Thirring).

Vienna²⁴ that decides for me. Herzfeld wrote to me after speaking with you (I don't know if it was directly your opinion) that we should try to support Smekal.²⁵ I think it will be difficult because March is named *àque loco* and has been supplying for quite some time. But Thirring wants to do it anyway and wrote to me at the time, asking me to inform him after an official decision has been made. It would certainly be more appropriate from a factual perspective.

Please don't assume my decision is final during the discussion. It would be unpleasant for both ministries. And on the other hand, I quite appreciate the delay because, after much effort, I'll finally get a little something, namely a new blackboard in the lecture hall and, hopefully, a little more funding for the seminar library.

E. Mathematica derivation of equations (3.18)–(3.21)

This section of our article is singled out into a separate document, AppendixE.pdf, which can be found on the webpage of *Physics–Uspekhi* www.ufn.ru in Russian (DOI: <https://doi.org/10.3367/UFNr.2025.08.040014>) and English (DOI: <https://doi.org/10.3367/UFNe.2025.08.040014>) versions together with supplementary programs. From our complementary Mathematica notebook, AppendixE.nb: “The reader may copy all Mathematica *In[#]*:= sections below into a new notebook and execute them one by one in order to repeat our calculations.”

More Mathematica support: In addition to AppendixE.nb presented above, our complimentary Mathematica notebooks also include BohrAtomMathematica.nb and EllipsesAnimateAu.nb, both at introductory level.

While EllipsesAnimateSofisticated.nb is designed to create advanced animations of the relativistic Keplerian motion of hydrogen-like ions in the periodic table of elements (more details can be found in [57]).

F. Uranium ($Z = 92$) versus hypothetical oganesson ($Z = 118$) relativistic hydrogen-like ions and beyond ($Z < 137$)

We compare, within the framework of the old Bohr–Sommerfeld atomic model, uranium ($Z = 92$) versus hypothetical oganesson ($Z = 118$) relativistic hydrogen-like ions and beyond. The existence of a self-intersecting orbit in the super strong Coulomb field of oganesson is demonstrated with the aid of the Mathematica computer algebra system. Another multi self-intersecting trajectory is found in a hypothetical unbibium, an element with $Z = 122$.

We believe that utilizing the Bohr–Sommerfeld model to contrast U^{91+} , Og^{117+} , and even hypothetical Ubb^{121+} hydrogen-like ions gives a fresh pedagogical and historical perspective — it links early quantum theory [18, 28] to modern superheavy transuranium relativistic systems [98] (in the quasi-classical approximation, using Dirac's equation as the next step [99]).

²⁴ The previous Innsbruck chair, Egon von Schweidler (1873–1948), had just been appointed to the University of Vienna Physics Institute. His successor in Innsbruck, after the chair was downgraded to a lectureship (cf. [64, pp. 178–184]), became Schrödinger's friend Arthur March (1891–1957).

²⁵ Adolf Smekal (1895–1959) did not receive an extraordinary professorship in Vienna until 1927, and in 1928 he accepted a call to Halle. Herzfeld was at that time preparing to accept a professorship at Johns Hopkins University in Baltimore.

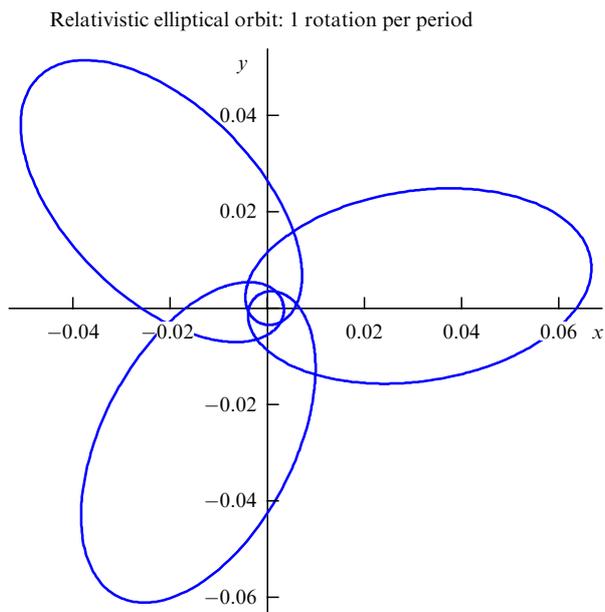


Figure 9. Kepler’s elliptical counterclockwise motion in uranium hydrogen-like ion U^{91+} with $n_\theta = n_r = 1$. ‘Winding’ number is one.

F.1 Uranium: $Z = 92$

Uranium (U) is a naturally occurring, radioactive, heavy metal with atomic number $Z = 92$, known for its ability to release concentrated energy through nuclear fission, which powers nuclear reactors and weapons. The quantum electrodynamics of the hydrogen-like uranium ion was studied in [100, 101] (see also [98, 102–106] and references therein). We use Mathematica for the following animations [57].

In the Bohr–Sommerfeld atomic model under consideration, even for the strong electric field of the uranium nucleus, a single electron stays on classical relativistic elliptical orbits with advancing perihelion, in the form of open ‘rosettes’ [24, 28, 58] (see Figs 3 and 9 with the standard orbit ‘winding number’ one).

In the above animation of the relativistic Kepler motion of an electron in the hydrogen-like uranium ion U^{91+} , when $Z = 92$, the nucleus is situated in the fixed focus at the origin. For this quantum state, we choose $n_r = n_\theta = 1$ in the fine structure formula (3.18). Using (3.19)–(3.21), we get $\omega = 0.741135$, $\epsilon = 0.904882$, and $a = 0.0353163$, in Bohr’s atomic units. The perihelion and aphelion move along two concentric circles around the nucleus with radii $r_{\min} = a(1 - \epsilon) = 0.003359209$ and $r_{\max} = a(1 + \epsilon) = 0.067273472$, respectively. In the animation, the geometrical loci of the successive perihelia and aphelia, the outer and inner envelopes of the orbit, are not shown for simplicity. Counterclockwise rotation of the ellipse over one period is given by $\Delta\theta = 2\pi(1/\omega - 1) = 2.19461$.

F.2 Oganesson: $Z = 118$

Oganesson (Og) is the heaviest synthetic element in the periodic table, with atomic number $Z = 118$. Discovered by scientists at the Joint Institute for Nuclear Research in Russia and Lawrence Livermore National Laboratory in 2006 [98], it is named after physicist Yuri Oganessian (see [107–112] and references therein).

Our Mathematica animations show that, in a super strong electric field of the oganesson nucleus, the classical

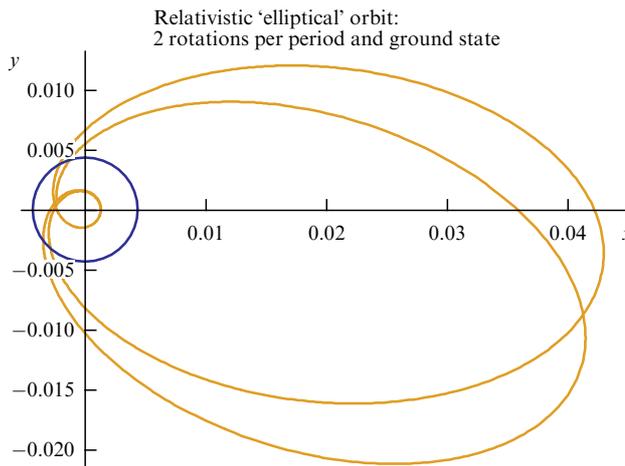


Figure 10. Kepler’s ‘elliptical’ clockwise motion with self-intersection in hypothetical oganesson hydrogen-like ion Og^{117+} with $n_\theta = n_r = 1$ versus ground state, when $n_\theta = 1$ and $n_r = 0$. ‘Winding’ number is two.

relativistic finite Kepler orbits of a single electron may change its topology (see Fig. 10 with the orbit ‘winding number’ two²⁶).

In the above animation of the relativistic Kepler motion of an electron in the hydrogen-like oganesson ion, Og^{117+} , when $Z = 118$, the nucleus is situated in the fixed focus at the origin. For this quantum state, we choose $n_r = n_\theta = 1$ in the fine structure formula (3.18). With (3.19)–(3.21), we get $\omega = 0.508457$, $\epsilon = 0.941479$, and $a = 0.0222041$, in Bohr’s atomic units. The perihelion and aphelion move along two concentric circles around the nucleus with radii $r_{\min} = a(1 - \epsilon) = 0.0012994$ and $r_{\max} = a(1 + \epsilon) = 0.0431087$, respectively. Total rotation of the self-intersecting ‘ellipse’ over one period is given by $\Delta\theta = 2\pi(1/\omega - 1) = 6.07418$.

At the point of self-intersection, with the help of Mathematica, we get approximately

$$r(3.074037420536865) = r(9.283329709624294) = 0.0025044,$$

with

$$\begin{aligned} & r(9.283329709624294) - r(3.074037420536865) \\ &= 4.33681 \times 10^{-19}, \\ & \omega \times \frac{9.283329709624294 + 3.074037420536865}{2} \\ &= 3.141592653589793 = \pi. \end{aligned}$$

F.3 Unbibium: $Z = 122$

As another example, let us discuss a hypothetical element with $Z = 122$, which has a temporary name, unbibium (Ubb), one of the yet-to-be-discovered superheavy elements at the end of the periodic table.

In the above animation of the relativistic Kepler motion of an electron in a hypothetical hydrogen-like unbibium ion, Ubb^{121+} , when $Z = 122$, the nucleus is situated in the fixed focus at the origin. For this quantum state, we choose $n_r = n_\theta = 1$ in the fine structure formula (3.18). Using

²⁶ The so-called ‘chemical orbit’ or ‘double necklace’ in terminology of the American cosmologist John Archibald Wheeler [113–115].

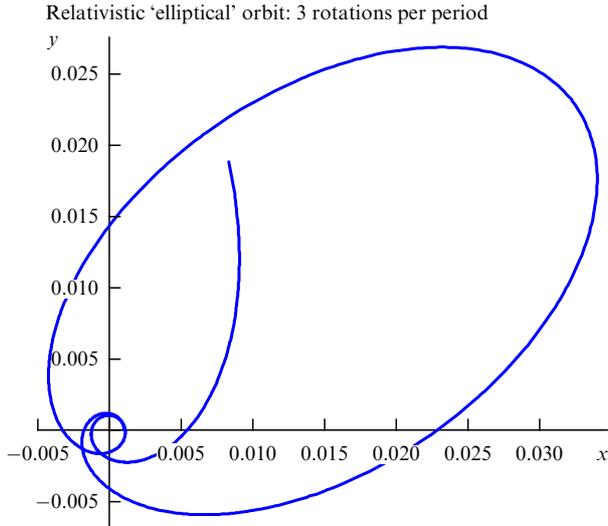


Figure 11. Kepler's 'elliptical' counterclockwise motion with four self-intersections in hypothetical unbibium hydrogen-like ion Ubb^{121+} with $n_\theta = n_r = 1$. 'Winding' number is three.

(3.19)–(3.21), we obtain $\omega = 0.455419$, $\epsilon = 0.949782$, and $a = 0.0203534$, in Bohr's atomic units.

The perihelion and aphelion move along two concentric circles around the nucleus with radii $r_{\min} = a(1 - \epsilon) = 0.00102211$ and $r_{\max} = a(1 + \epsilon) = 0.0396847$, respectively. Rotation of the ellipse over one period is given by $\Delta\theta = 2\pi(1/\omega - 1) = 7.5133$. The 'winding' number is three.

At the point of the first self-intersection, we get approximately

$$\begin{aligned} r(10.002448573515926) &= r(3.7940322175405243) \\ &= 0.00234067, \end{aligned}$$

with

$$r(10.002448573515926) - r(3.7940322175405243) = 0.$$

Here,

$$\begin{aligned} \omega \times \frac{10.002448573515926 + 3.794032217540524}{2} \\ = 3.14159 = \pi. \end{aligned}$$

At the second self-intersection, we find approximately

$$\begin{aligned} r(13.18142570270781) &= r(0.6150550883486369) \\ &= 0.00104189 \end{aligned}$$

and

$$\begin{aligned} r(13.18142570270781) - r(0.6150550883486369) \\ = 4.33681 \times 10^{-19}. \end{aligned}$$

Here,

$$\begin{aligned} \omega \times \frac{13.18142570270781 + 0.6150550883486369}{2} \\ = 3.14159 = \pi. \end{aligned}$$

There are two more consecutive self-intersections:

$$r(16.9329615821129) = r(10.66) = 0.0017562,$$

with

$$\begin{aligned} r(16.9329615821129) - r(10.660000000000002) \\ = -8.67362 \times 10^{-19}. \end{aligned}$$

Here,

$$\omega \times \frac{16.9329615821129 + 10.66}{2} = 6.28319 = 2\pi.$$

And,

$$r(20.07966609823604) = r(7.513295483876864) = 0.022845,$$

with

$$\begin{aligned} r(20.07966609823604) - r(7.513295483876864) \\ = 2.77556 \times 10^{-17}. \end{aligned}$$

Here,

$$\begin{aligned} \omega \times \frac{20.07966609823604 + 7.513295483876864}{2} \\ = 6.28319 = 2\pi. \end{aligned}$$

Orbit parameters under consideration are compared in Table 2 for the reader's convenience. Further animation details can be found in our complementary Mathematica notebooks [57].

Table 2

Parameters	U^{91+} ion	Og^{117+} ion	Ubb^{121+} ion
ω	0.741135	0.508457	0.455419
ϵ	0.904882	0.941479	0.949782
a/a_0	0.0353163	0.0222041	0.0203534
r_{\min}	0.003359209	0.0012994	0.00102211
r_{\max}	0.067273472	0.0431087	0.0396847
$\Delta\theta$	2.19461	6.07418	7.5133
'Winding' number	1	2	3

We have demonstrated, in terms of the 'old quantum mechanics,' that, in a super strong static electric field, similar to the one in a hypothetical oganesson hydrogen-like ion, there is a self-intersecting trajectory, which is different from the standard classical elliptic orbits. Following Mendeleev's approach—getting "Mach out of Little" [115]—a possible rotation of an incoming electron around the oganesson nucleus during the scattering should also be investigated [18].

It would be interesting to analyze the possibility of a similar, 'oganesson-type,' effect in the theory of gravitation [58, 115] and to study nonstandard consequences in quantum electrodynamics in the strong, U, versus super strong, Og, Coulomb fields [33, 98, 100–105, 108, 109, 116].

For example, in the creation of an electron-positron pair, the electron may be captured by the nucleus directly from the internal orbit. Further nuclear reaction with one of oganesson's protons may result in a neutron and neutrino, thus contributing to the instability of this super heavy element. In a hypothetical unbibium element with $Z = 122$, a multiple self-intersecting trajectory occurs, which should contribute even more to an orbit electron capture and the corresponding Ubb transformation.

References

- Schrödinger E “Quantisierung als Eigenwertproblem” *Ann. Physik* **79** 361 (1926) Vol. 384 of the new volume numbering of the Wiley Online Library, DOI:10.1002/andp.19263840404; Translated into English: Schrödinger E “Quantisation as a problem of proper values (Part I),” in *Collected Papers on Wave Mechanics* (AMS Chelsea Publishing, Vol. 302) (Providence, RI: Chelsea Publ. Co., 1982) pp. 1–12; <https://bookstore.ams.org/chel-302>
- Planck M, The Nobel Prize in Physics 1918 was awarded to Max Karl Ernst Ludwig Planck “in recognition of the services he rendered to the advancement of Physics by his discovery of energy quanta,” <https://www.nobelprize.org/prizes/physics/1918/summary/>
- Einstein A, The Nobel Prize in Physics 1921 was awarded to Albert Einstein “for his services to Theoretical Physics, and especially for his discovery of the law of the photoelectric effect,” <https://www.nobelprize.org/prizes/physics/1921/summary/>
- Rutherford E, The Nobel Prize in Chemistry 1908 was awarded to Ernest Rutherford “for his investigations into the disintegration of the elements, and the chemistry of radioactive substances,” <https://www.nobelprize.org/prizes/chemistry/1908/summary/>
- Bohr N, The Nobel Prize in Physics 1922 was awarded to Niels Henrik David Bohr “for his services in the investigation of the structure of atoms and of the radiation emanating from them,” <https://www.nobelprize.org/prizes/physics/1922/summary/>
- Aaserud F, Heilbron J L *Love, Literature, and the Quantum Atom: Niels Bohr's 1913 Trilogy Revisited* (Oxford: Oxford Univ. Press, 2013) DOI:10.1093/acprof:oso/9780199680283.001.0001; <https://academic.oup.com/book/6847>
- Bohr N *Niels Bohr Collected Works* (Ed. L Rosenfeld) Vol. 2 *Work on Atomic Physics (1912–1917)* (Ed. U Hoyer) (Amsterdam: North-Holland Publ. Co., 1981) DOI:10.1016/S1876-0503(08)70031-8
- Eckert M *Arnold Sommerfeld: Science, Life and Turbulent Times. 1868–1951* (New York: Springer, 2013) DOI:10.1007/978-1-4614-7461-6
- El'yashevich M A “Niels Bohr’s development of the quantum theory of the atom and the correspondence principle (his 1912–1923 work in atomic physics and its significance)” *Sov. Phys. Usp.* **28** 879–909 (1985); “Razvitie Nil'som Borom kvantovoi teorii atoma i printsipa sootvetstviya (Raboty N. Bora 1912–1923 gg. po atomnoi fizike i ikh znachenie)” *Usp. Fiz. Nauk* **147** 253–301 (1985)
- Firme C L *Quantum Mechanics: Detailed Historical, Mathematical and Computational Approaches* (Boca Raton, FL: CRC Press, Taylor and Francis Group, 2022)
- Kragh H *Niels Bohr and the Quantum Atom: The Bohr Model of Atomic Structure 1913–1925* (Oxford: Oxford Univ. Press, 2012) DOI:10.1093/acprof:oso/9780199654987.001.0001
- Kragh H (Ed.) *Niels Bohr: On the Constitution of Atoms and Molecules* (Classic Texts in the Sciences) (Cham: Birkhäuser, Springer Nature Switzerland AG, 2022) DOI:10.1007/978-3-030-97664-4
- Mehra J, Rechenberg H *The Quantum Theory of Planck, Einstein, Bohr and Sommerfeld: Its Foundation and the Rise of Its Difficulties 1900–1925* 1st ed. (The Historical Development of Quantum Theory, Vol. 1, Pt. 2) (New York: Springer-Verlag, 1982); <https://link.springer.com/book/9780387951751>
- Migdal A B “Niels Bohr and quantum physics” *Sov. Phys. Usp.* **28** 910–934 (1985); “Niels Bohr i kvantovaya fizika” *Usp. Fiz. Nauk* **147** 303–342 (1985)
- Milant'ev V P “Creation and development of Bohr’s theory (on the 90th anniversary of the Bohr theory of the atom)” *Phys. Usp.* **47** 197 (2004); “Sozdanie teorii Bora i ee razvitie (k 90-letiyu borovskoi teorii atoma)” *Usp. Fiz. Nauk* **174** 209–215 (2004)
- Reed B C *The Bohr Atom: A Guide* (Bristol: IOP Publ., 2020) DOI:10.1088/978-0-7503-3611-6
- Steinhaus M O *Quantenmechanik für Naturwissenschaftler: Ein Lehr- und Übungsbuch mit zahlreichen Aufgaben und Lösungen* (Berlin: Springer, 2022) DOI:10.1007/978-3-662-62610-8
- Sommerfeld A “Zur Quantentheorie der Spektrallinien” *Ann. Physik* **356** 1–94 (1916) DOI:10.1002/andp.19163561702
- Sommerfeld A “Zur Feinstruktur der Wasserstofflinien. Geschichte und gegenwärtiger Stand der Theorie” *Naturwissenschaften* **28** 417–423 (1940) DOI:10.1007/BF01490583
- Wilson W “LXXXIII. The quantum-theory of radiation and line spectra” *London Edinburgh Dublin Philos. Mag. J. Sci.* **29** 795–802 (1915) DOI:10.1080/14786440608635362
- Various Educators Bohr’s atom videos: “Bohr Model in Brief: The planetary model, its connection to emission spectra and quantized electrons,” https://www.youtube.com/watch?v=xytRf3fs_gY; “What is the Bohr model of the atom?,” https://www.youtube.com/watch?v=_Gt7mo8SNkA; “How Niels Bohr created the quantum atom,” <https://www.youtube.com/watch?v=xINR4MoqYVc>; “This star almost broke Bohr’s atomic model,” <https://www.youtube.com/watch?v=BcX1aYrLct4>; “Sommerfeld and the fine structure of the atom,” <https://www.youtube.com/watch?v=H4ZnVpiQTDw>; educational YouTube videos on Bohr’s atomic model and related concepts: “Why don’t electrons fall onto the nucleus?,” <https://www.youtube.com/watch?v=cf7t-tZnNuE>
- Michelson A A, The Nobel Prize in Physics 1907 was awarded to Albert Abraham Michelson “for his optical precision instruments and the spectroscopic and metrological investigations carried out with their aid,” <https://www.nobelprize.org/prizes/physics/1907/summary/>
- Albert A. Michelson, online accessed 8 February 2026, https://en.wikipedia.org/wiki/Albert_A._Michelson#Early_measurements
- Biedenharn L C “The ‘Sommerfeld Puzzle’ revisited and resolved” *Found. Phys.* **13** 13–34 (1983) DOI:10.1007/BF01889408
- Kragh H “The fine structure of hydrogen and the gross structure of the physics community, 1916–26” *Hist. Stud. Phys. Sci.* **15** 67–125 (1985) DOI:10.2307/27757550
- Thomson J J, The Nobel Prize in Physics 1906 was awarded to Joseph John Thomson “in recognition of the great merits of his theoretical and experimental investigations on the conduction of electricity by gases,” <https://www.nobelprize.org/prizes/physics/1906/summary/>
- Granovskii Ya I “Sommerfeld formula and Dirac’s theory” *Phys. Usp.* **47** 523 (2004); “Formula Zommerfel’da i teoriya Diraka” *Usp. Fiz. Nauk* **174** 577 (2004)
- Sommerfeld A *Atomic Structure and Spectral Lines* (New York: Dutton, 1923) translated from the 3rd German ed.; <https://archive.org/details/atomicstructures0000somm/page/n7/mode/2-up>; in German: Sommerfeld A *Atombau und Spektrallinien* Bd. 1, 2. Aufl. (Braunschweig: F. Vieweg and Sons, 1921); <https://archive.org/details/atombauundspekt00sommgoog/page/478/mode/2up>
- Darwin C G “The wave equations of the electron” *Proc. R. Soc. London A* **118** 654–680 (1928) DOI:10.1098/rspa.1928.0076
- Gordon W “Die Energieniveaus des Wasserstoffatoms nach der Diracschen Quantentheorie des Elektrons” *Z. Phys.* **48** 11–14 (1928) DOI:10.1007/BF01351570
- Dirac P A M “The quantum theory of the electron” *Proc. R. Soc. London A* **117** 610 (1928) DOI:10.1098/rspa.1928.0023
- Dirac P A M “The evolution of the physicist’s picture of nature” *Sci. Am.* **208** (5) 45 (1963) DOI:10.1038/scientificamerican0563-45
- Akhiezer A I, Berestetskii V B *Quantum Electrodynamics* (Interscience Monographs and Texts in Physics and Astronomy, Vol. 11) (New York: Interscience Publ., 1965); Translated from Russian: Akhiezer A I, Berestetskii V B *Kvantovaya Elektrodinamika* (Moscow: Fizmatgiz, 1959)
- Berestetskii V B, Lifshitz E M, Pitaevskii L P *Relativistic Quantum Theory* (Oxford: Pergamon Press, 1971); Translated from Russian: *Relyativistskaya Kvantovaya Teoriya* (Moscow: Fizmatgiz, 1968)
- Davydov A S *Quantum Mechanics* (Oxford: Pergamon Press, 1965); Translated from Russian: *Kvantovaya Mekhanika* (Moscow: Fizmatgiz, 1963)
- Landau L D, Lifshitz E M *Quantum Mechanics: Non-Relativistic Theory* 3rd ed. (Oxford: Pergamon Press, 1977) DOI:10.1016/C2013-0-02793-4; Translated from Russian: *Kvantovaya Mekhanika. Nerelevativistskaya Teoriya* 3rd ed., rev. and enl. (Moscow: Nauka, 1974)

37. Rose M E *Relativistic Electron Theory* (New York: John Wiley and Sons, 1961); <https://ia801405.us.archive.org/22/items/in.ernet.dli.2015.148057/2015.148057.Relativistic-Electron-Theory.pdf>
38. Schiff L I *Quantum Mechanics* 3rd ed. (International Ser. in Pure and Applied Physics) (New York: McGraw-Hill Book Co., 1968); <https://physicsg.files.wordpress.com/2021/01/schiff-li.-quantum-mechanics-mgh-1949t417s.pdf>
39. Barley K, Vega-Guzmán J, Ruffing A, Suslov S K “Discovery of the relativistic Schrödinger equation” *Phys. Usp.* **65** 90 (2022); “Otkrytie relyativistskogo uravneniya Shchredingera” *Usp. Fiz. Nauk* **192** 100 (2022)
40. Ellis L, Ellis I, Koutschan C, Suslov S K “On potentials integrated by the Nikiforov–Uvarov method,” in *Applications and q-Extensions of Hypergeometric Functions* (Contemporary Mathematics, Vol. 819, Eds H S Cohl, R S Costas-Santos, R S Maier) (Providence, RI: American Mathematical Society, 2025) p. 43; <https://bookstore.ams.org/view?ProductCode=CONM/819>
41. Goren B, Barley K, Suslov S K “Matrix approach to helicity states of Dirac free particles,” in *Proc. of the Second Intern. Workshop on Quantum Nonstationary Systems* (Eds A Dodonov, C C H Ribeiro) (São Paulo: LF Editorial, 2024) pp. 333–384, Ch. 19; <http://www.cif.unb.br/eventos/proceedings-qns2>
42. Kryuchkov S I, Lanfear N A, Suslov S K “The role of the Pauli–Lubański vector for the Dirac, Weyl, Proca, Maxwell and Fierz–Pauli equations” *Phys. Scr.* **91** 035301 (2016) DOI:10.1088/0031-8949/91/3/035301; arXiv:1510.05164
43. Kryuchkov S I, Suslov S K, Vega-Guzmán J M “The minimum-uncertainty squeezed states for atoms and photons in a cavity” *J. Phys. B* **46** 104007 (2013) DOI:10.1088/0953-4075/46/10/104007
44. Suslov S K “The ‘Sommerfeld’s Puzzle’ and its extensions,” in *Proc. of the Second Intern. Workshop on Quantum Nonstationary Systems* (Eds A Dodonov, C C H Ribeiro) (São Paulo: LF Editorial, 2024) pp. 43–59; <https://lfeeditorial.com.br/produto/proceedings-of-the-second-international-workshop-on-quantum-nonstationary-systems/>
45. Suslov S K, Trey B “The Hahn polynomials in the nonrelativistic and relativistic Coulomb problems” *J. Math. Phys.* **49** 012104 (2008) DOI:10.1063/1.2830804
46. Suslov S K, Vega-Guzmán J M, Barley K “An introduction to special functions with some applications to quantum mechanics,” in *Orthogonal Polynomials. 2nd AIMS–Volkswagen Stiftung Workshop, Douala, Cameroon, 5–12 October, 2018. Conf. Proc.* (Tutorials, Schools, and Workshops in the Mathematical Sciences, Eds M Foupouagnigni, W Koepf) (Cham: Birkhäuser, 2020) pp. 517–628, DOI:10.1007/978-3-030-36744-2_21
47. Ruffing A “Mathematische Methoden der Quantenmechanik,” Lectures given at Technische Univ. München, 2001–2014
48. Mehra J, Rechenberg H *The Historical Development of Quantum Theory* Vol. 1, Pt. 1 *The Quantum Theory of Planck, Einstein, Bohr and Sommerfeld: Its Foundation and the Rise of Its Difficulties 1900–1925* (New York: Springer-Verlag, 1982); <https://link.springer.com/book/9780387951744>
49. Einstein A “Autobiographical note”, in *Albert Einstein: Philosopher-Scientist* (Library of Living Philosophers, Vol. 7) (Ed. P A Schilpp) Vol. 1 (La Salle, IL: Open Court Publ. Co., 1970) pp. 46–47
50. Gantmacher F *Lectures in Analytical Mechanics* (Moscow: Mir Publ., 1975); Translated from Russian: *Lektsii po Analiticheskoi Mekhanike* (Moscow: Nauka, 1966)
51. Goldstein H, Poole C, Saffo J *Classical Mechanics* (San Francisco, CA: Addison Wesley, 2002)
52. Nikiforov A F, Uvarov V B *Special Functions of Mathematical Physics. A Unified Introduction with Applications* (Basel: Birkhäuser, 1988)
53. Varshalovich D A, Moskalev A N, Khersonskii V K *Quantum Theory of Angular Momentum* (Singapore: World Scientific, 1988) DOI:10.1142/0270
54. Merzbacher E *Quantum Mechanics* 3rd ed. (New York: John Wiley and Sons, 1998)
55. Berry M V, Mount K E “Semiclassical approximations in wave mechanics” *Rep. Prog. Phys.* **35** 315 (1972) DOI:10.1088/0034-4885/35/1/306
56. Ghatak A K, Gallawa R L, Goyal I C *Modified Airy Function and WKB Solutions to the Wave Equation* (NIST Monograph, Vol. 176) (Boulder, CO: National Institute of Standards and Technology, 1991)
57. Barley K, Ruffing A, Suslov S K “Old quantum mechanics by Bohr and Sommerfeld from modern perspective,” Supplementary files, DOI: <https://doi.org/10.3367/UFNe.2025.08.040014>
58. Landau L D, Lifshitz E M *The Classical Theory of Fields* 4th ed. (Oxford: Pergamon Press, 1975) DOI:10.1016/B978-0-08-025072-4.50001-0; Translated from Russian: *Teoriya Polya* (Moscow: Nauka, 1973)
59. Sommerfeld A *Wave-Mechanics* 2nd ed. (New York: E.P. Dutton and Co. Inc., 1938) Suppl. Vol. to “Atomic Structure and Spectral Lines”
60. Schrödinger E, The Nobel Prize in Physics 1933 was awarded jointly to Erwin Schrödinger and Paul Adrien Maurice Dirac “for the discovery of new productive forms of atomic theory,” <https://www.nobelprize.org/prizes/physics/1933/summary/>
61. Dirac P A M, The Nobel Prize in Physics 1933 was awarded jointly to Erwin Schrödinger and Paul Adrien Maurice Dirac “for the discovery of new productive forms of atomic theory,” <https://www.nobelprize.org/prizes/physics/1933/dirac/facts/>
62. Schrödinger E “The continuous transition from micro- to macro-mechanics,” in *Collected Papers on Wave Mechanics* (Providence, RI: Chelsea Publ. Co.: AMS Chelsea Publ., 2010) pp. 41–44; Translated from German: *Naturwissenschaften* **28** 664–666 (1926) DOI:10.1007/BF01507634
63. Schrödinger E “An undulatory theory of the mechanics of atoms and molecules” *Phys. Rev.* **28** 1049–1070 (1926)²⁷ DOI:10.1103/PhysRev.28.1049
64. von Meyenn K (Ed.) *Eine Entdeckung von ganz außerordentlicher Tragweite: Schrödingers Briefwechsel zur Wellenmechanik und zum Katzenparadoxon* (Berlin: Springer, 2011) DOI:10.1007/978-3-642-04335-2
65. de Broglie L, The Nobel Prize in Physics 1929 was awarded to Prince Louis-Victor Pierre Raymond de Broglie “for his discovery of the wave nature of electrons,” <https://www.nobelprize.org/prizes/physics/1929/summary/>
66. Mehra J, Rechenberg H *Erwin Schrödinger and the Rise of Wave Mechanics Pt. 2 The Creation of Wave Mechanics; Early Response and Applications 1925–1926* (The Historical Development of Quantum Theory, Vol. 5/2) 1st ed. (New York: Springer-Verlag, 1987); <https://link.springer.com/book/9780387951805>
67. Moore W *Schrödinger: Life and Thought* (Ser. Canto Classics) (Cambridge: Cambridge Univ. Press, 2015) DOI:10.1017/CBO9781316424056
68. Brillouin L “La mécanique ondulatoire de Schrödinger une méthode générale de résolution par approximations successives” *C. R. Acad. Sci.* **183** 24–26 (1926); <https://gallica.bnf.fr/ark:/12148/bpt6k3136h/f24>
69. Kramers H A “Wellenmechanik und halbzahlige Quantisierung” *Z. Phys.* **39** 828–840 (1926) DOI:10.1007/BF01451751
70. Wentzel G “Eine Verallgemeinerung der Quantenbedingungen für die Zwecke der Wellenmechanik” *Z. Phys.* **38** 518–529 (1926) DOI:10.1007/BF01397171
71. Olver F, Daalhuis A, Lozier D, Schneider B, Boisvert R, Clark C, Mille B, Cohl H, McClain M, NIST Digital Library of Mathematical Functions (2025), release 1.2.4 of 2025-03-15, <https://dlmf.nist.gov/>
72. Nikiforov A F, Novikov V G, Uvarov V B *Quantum-Statistical Models of Hot Dense Matter. Methods for Computation Opacity and Equation of State* (Progress in Mathematical Physics, Vol. 37) (Basel: Birkhäuser, 2005) DOI:10.1007/b137687
73. Langer R E “On the asymptotic solutions of ordinary differential equations, with an application to the Bessel functions of large

²⁷ Schrödinger’s last article in 1926.

- order” *Trans. Am. Math. Soc.* **33** 23–64 (1931) DOI:10.1090/S0002-9947-1931-1501574-0
74. Langer R E “On the connection formulas and the solutions of the wave equation” *Phys. Rev.* **51** 669–676 (1937) DOI:10.1103/PhysRev.51.669
 75. Kragh H “Erwin Schrödinger and the wave equation: The crucial phase” *Centaurus* **26** 154–197 (1982) DOI:10.1111/j.1600-0498.1982.tb00659.x
 76. Kragh H “Equation with the many fathers. The Klein–Gordon equation in 1926” *Am. J. Phys.* **52** 1024–1033 (1984) DOI:10.1119/1.13782
 77. Joas C, Lehner C “The classical roots of wave mechanics: Schrödinger’s transformations of the optical-mechanical analogy” *Stud. Hist. Phil. Sci. B* **40** 338–351 (2009) DOI:10.1016/j.shpsb.2009.06.007
 78. Bethe H A, Salpeter E *Quantum Mechanics of One- and Two-Electron Atoms* (Mineola, NY: Dover Publ., 2008)
 79. Good R H (Jr.) “The generalization of the WKB method to radial wave equations” *Phys. Rev.* **90** 131–137 (1953) DOI:10.1103/PhysRev.90.131
 80. Lu P “Derivation of Sommerfeld–Dirac fine-structure formula by WKB method” *Phys. Rev. A* **1** 1283–1285 (1970) DOI:10.1103/PhysRevA.1.1283
 81. Mehra J (Ed.) *The Discovery of Electron Spin* Vol. 1 (Singapore: World Scientific, 2001) Ch. 17, pp. 585–611 DOI:10.1142/9789812810588_0017
 82. Stern O, The Nobel Prize in Physics 1943 was awarded to Otto Stern “for his contribution to the development of the molecular ray method and his discovery of the magnetic moment of the proton,” <https://www.nobelprize.org/prizes/physics/1943/summary/>
 83. Petrov S V “Was Sommerfeld wrong? (To the history of the appearance of spin in relativistic wave equations)” *Phys. Usp.* **63** 721 (2020); “Oshibalsya li Sommerfeld? (K istorii poyavleniya spina v relyativistskikh volnovykh uravneniyakh)” *Usp. Fiz. Nauk* **190** 777 (2020)
 84. Heisenberg W, The Nobel Prize in Physics 1932 was awarded to Werner Karl Heisenberg “for the creation of quantum mechanics, the application of which has, inter alia, led to the discovery of the allotropic forms of hydrogen,” <https://www.nobelprize.org/prizes/physics/1932/summary/>
 85. Heisenberg W “Ausstrahlung von Sommerfelds Werk in der Gegenwart” *Phys. Blätter* **24** 530–537 (1968) DOI:10.1002/phbl.19680241203
 86. Yourgrau W, Mandelstam S *Variational Principles in Dynamics and Quantum Theory* 3rd ed. (Philadelphia, PA: W.B. Saunders Co., 1968)
 87. Schrödinger E “Preface,” in *Collected Papers on Wave Mechanics* 3rd ed. (Montréal: Minkowski Institute Press, 2021) pp. ix–xiii
 88. Schrödinger E “Quantisation as a problem of proper values,” in *Collected Papers on Wave Mechanics* (AMS Chelsea Publishing Ser., Vol. 302) (Providence, RI: Chelsea Publ. Co., 2010) Pt. II, pp. 13–40; <https://bookstore.ams.org/chel-302>
 89. Einstein A *Letters on Wave Mechanics: Correspondence with H.A. Lorentz, Max Planck, and Erwin Schrödinger* (New York: Philosophical Library, Open Road, 2011)
 90. Fock V “Zur Schrödingerschen Wellenmechanik” *Z. Phys.* **38** 242–250 (1926) DOI:10.1007/BF01399113
 91. Wessels L “Schrödinger’s route to wave mechanics” *Stud. Hist. Phil. Sci. A* **10** 311–340 (1979) DOI:10.1016/0039-3681(79)90018-9
 92. Bloch F “Heisenberg and the early days of quantum mechanics” *Phys. Today* **29** (12) 23–27 (1976) DOI:10.1063/1.3024633
 93. Rozentel S (Ed.) *Niels Bohr: His Life and Work as Seen by His Friends and Colleagues* (Amsterdam: North-Holland Publ. Co., 1968) p. 73
 94. Koutschan C, Suazo E, Suslov S K “Fundamental laser modes in paraxial optics: from computer algebra and simulations to experimental observation” *Appl. Phys. B* **121** 315–336 (2015) DOI:10.1007/s00340-015-6231-9
 95. Jackson J D *Classical Electrodynamics* 2nd ed. (New York: John Wiley and Sons, 1975)
 96. Ewald P P “Ein Buch über mathematische Physik: Courant–Hilbert” *Naturwissenschaften* **13** 384–387 (1925) DOI:10.1007/BF01559098
 97. Courant R, Hilbert D *Methoden der Mathematischen Physik* (Grundlehren der mathematischen Wissenschaften, Vol. 12) (Berlin: Springer, 1924) DOI:10.1007/978-3-662-36445-1
 98. Oganessian Y “Discovery of the island of stability for super heavy elements,” in *IPAC2017: Proc. of the 8th Intern. Particle Accelerator Conf., Copenhagen, Denmark, 14–19 May, 2017* (Eds G Arduini, M Lindroos, J Pranke, V R W Schaa, M Seidel) (Copenhagen, Denmark: Eur. Spallation Source (ESS), MAXIV and Aarhus Univ., 2017) pp. 4848–4851; <https://proceedings.jacow.org/ipac2017/>
 99. Barley K, Ruffing A, Suslov S K “Oganesson versus uranium hydrogen-like ions from the viewpoint of old quantum mechanics,” arXiv:2509.06249, DOI:10.48550/arXiv.2509.06249
 100. Gumberidze A, Stöhlker Th, Banaš D, Beckert K, Beller P, Beyer H F, Bosch F, Hagmann S, Kozhuharov C, Liesen D, Nolden F, Ma X, Mokler P H, Steck M, Sierpowski D, Tashenov S “Quantum electrodynamics in strong electric fields: The ground-state Lamb shift in hydrogenlike uranium” *Phys. Rev. Lett.* **94** 223001 (2005) DOI:10.1103/PhysRevLett.94.223001
 101. Gumberidze A, Stöhlker Th, Banaš D, Beckert K, Beller P, Beyer H F, Bosch F, Cai X, Hagmann S, Kozhuharov C, Liesen D, Nolden F, Ma X, Mokler P H, Steck M, Sierpowski D, Tashenov S, Warczak A, Zou Y “Precision tests of QED in strong fields: experiments on hydrogen- and helium-like uranium” *J. Phys. Conf. Ser.* **58** 87–92 (2007) DOI:10.1088/1742-6596/58/1/013
 102. Flerov G N, Karamian S A “The search for superheavy elements in nature: foundations and prospects”, in *Atti del Convegno Mendeleeviano, Periodicità e simmetrie nella struttura elementare della materia. Torino-Roma, 15–21 settembre, 1969, Convegno Mendeleeviano. Torino-Roma, 1969* (Ed. M Verde) (Turin, Italy: Vincenzo Bona, 1971) pp. 73–96
 103. Mohr P J, Plunien G, Soff G “QED corrections in heavy atoms” *Phys. Rep.* **293** 227–369 (1998) DOI:10.1016/S0370-1573(97)00046-X
 104. Shabaev V M “Two-time Green’s function method in quantum electrodynamics of high-Z few-electron atoms” *Phys. Rep.* **356** 119–228 (2002) DOI:10.1016/S0370-1573(01)00024-2; physics/0009018, DOI:10.48550/arXiv.physics/0009018
 105. Shabaev V M “Quantum electrodynamics of heavy ions and atoms: current status and prospects” *Phys. Usp.* **51** 1175 (2008); “Kvantovaya elektrodinamika tyazhelykh ionov i atomov: status i perspektivy” *Usp. Fiz. Nauk* **178** 1220 (2008) DOI:10.1070/PU2008v051n11ABEH006801
 106. Suslov S K “Expectation values in relativistic Coulomb problems” *J. Phys. B* **42** 185003 (2009) DOI:10.1088/0953-4075/42/18/185003
 107. Chapman K “The oganesson odyssey” *Nature Chem.* **10** 796 (2018) DOI:10.1038/s41557-018-0098-4
 108. Giuliani S A, Matheson Z, Nazarewicz W, Olsen E, Reinhard P-G, Sadhukhan J, Schuettrumpf B, Schunck N, Schwerdtfeger P “Colloquium: Superheavy elements: Oganesson and beyond” *Rev. Mod. Phys.* **91** 011001 (2019) DOI:10.1103/RevModPhys.91.011001
 109. Kaygorodov M Y, Skripnikov L V, Tupitsyn I I, Eliav E, Kozhedub Y S, Malyshev A V, Oleynichenko A V, Shabaev V M, Titov A V, Zaitsevskii A V “Electron affinity of oganesson” *Phys. Rev. A* **104** 012819 (2021) DOI:10.1103/PhysRevA.104.012819
 110. Kragh H *From Transuranic to Superheavy Elements: A Story of Dispute and Creation* (SpringerBriefs in History of Science and Technology) (Cham: Springer, 2018) DOI:10.1007/978-3-319-75813-8
 111. Oganessian Yu Ts, Utyonkov V K, Lobanov Yu V, Abdullin F Sh, Polyakov A N, Sagaidak R N, Shirokovsky I V, Tsyganov Yu S, Voinov A A, Moody K J, Patin J B, Shaughnessy D A, Stoyer M A, Stoyer N J, Wilk P A, Kenneally J M, Landrum J H, Wild J F, Gulbekian G G, Bogomolov S L, Gikal B N, Mezentsev A N, Iliev S, Subbotin V G, Sukhov A M, Subotic K, Zagrebaev V I, Vostokin G K, Itkis M G, Loughheed R W “Synthesis of the isotopes of elements 118 and 116 in the ^{249}Cf and $^{245}\text{Cm} + ^{48}\text{Ca}$ fusion reactions” *Phys. Rev. C* **74** 44602 (2006) DOI:10.1103/PhysRevC.74.044602

112. Oganessian Yu Ts, Abdullin F Sh, Alexander C, Binder J, Boll R A, Dmitriev S N, Ezold J, Felker K, Gostic J M, Grzywacz R K, Hamilton J H, Henderson R A, Itkis M G, Miernik K, Miller D, Moody K J, Polyakov A N, Ramayya A V, Roberto J B, Ryabinkin M A, Rykaczewski K P, Sagaidak R N, Shaughnessy D A, Shirokovsky I V, Shumeiko M V, Stoyer M A, Stoyer N J, Subbotin V G, Sukhov A M, Tsyganov Yu S, Utyonkov V K, Voinov A A, Vostokin G K “Production and decay of the heaviest nuclei ²⁹³294117 and ²⁹⁴118” *Phys. Rev. Lett.* **109** 162501 (2012) DOI:10.1103/PhysRevLett.109.162501
113. Ceulemans A, Thyssen P “The ‘chemical mechanics’ of the periodic table,” in *Mendeleev to Oganesson: A Multidisciplinary Perspective on the Periodic Table* (Eds E Scerri, G Restrepo) (New York: Oxford Univ. Press, 2018) pp. 104–121, DOI:10.1093/oso/9780190668532.003.0008
114. Powers R T “Frequencies of Radial Oscillation and Revolution as Affected by Features of a Central Potential,” in *Atti del Convegno Mendeleeviano, Periodicità e simmetrie nella struttura elementare della materia. Torino–Roma, 15–21 settembre, 1969, Convegno Mendeleeviano. Torino–Roma, 1969* (Ed. M Verde) (Turin, Italy: Vincenzo Bona, 1971) pp. 235–242
115. Wheeler J A “From MendeléeV’s atom to the collapsing star,” in *Philosophical Foundations of Science: Proc. of Section L, 1969, American Association for the Advancement of Science* (Boston Studies in the Philosophy of Science, Vol. 11, Eds R J Seeger, R S Cohen) (Dordrecht: Springer, 1974) Ch. 15, pp. 257–301, DOI:10.1007/978-94-010-2126-5_15
116. Pomeranchuk I, Smorodinsky J “On the energy levels of systems with $Z > 137$ ” *J. Phys. USSR* **9** 97–100 (1945); https://ufn.ru/ufn13/ufn13_4/Russian/pomeranchuk.pdf