

LETTERS TO THE EDITORS

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Probability distributions over energy and areas in processes described by Kolmogorov's 1934 random motion equation

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Abstract. **The Kolmogorov equation is described in detail in the paper by G.S. Golitsyn [*Phys. Usp.* 67 80 (2024); *Usp. Fiz. Nauk* 194 86 (2024)], which is a summary of the book by G.S. Golitsyn, *Probability structures of the macrocosm: earthquakes, hurricanes, floods* (Moscow: Fizmatlit, 2022), but insufficient attention is paid to probability distributions, integral and differential, for the processes described by this equation. The integral distribution for cosmic rays is derived only by the method of similarity and dimensionality, and the distribution of horizontal lengths L of clouds and their perimeters together with the ratio of the cloud perimeter to its size are not explained at all. Here, these shortcomings are corrected, and the results and methods can be useful for other purposes.**

Keywords: distribution function, random motions, second moments of probability distribution, statistical laws of nature

Consider the three second moments of the probability distribution from Refs [1, 3]:

$$\langle u_i^2(t)\rangle = \varepsilon t \equiv E\,, \tag{1}$$

$$\langle u_i x_i(t)\rangle = \varepsilon t^2 \cong K\,, \tag{2}$$

$$\langle x_i^2(t)\rangle = \varepsilon t^3 \equiv r^2\,, \tag{3}$$

where ε is the energy generation rate per unit mass, E is the energy per unit mass in the process under study, K is the diffusivity (turbulent mixing) coefficient, and S is the area of a random process. Expressing $t = (r^2/\varepsilon)^{1/3}$ from (3) and inserting it into (1) and (2), we obtain the 1941 turbulence laws and the Richardson–Obukhov law [3] for turbulent mixing. The probability distribution is their occurrence frequency. Distributions can be integral, also called histograms or cumulative distributions [1, 2], or differential (see expressions (4)–(8) below). If an event is caused by two independent processes, the full probability of the event frequency is equal to the product of the individual probabilities [4]. Here, the probabilities will follow power laws; in this case, the exponents of the full process will be the sum of the exponents in the probabilities of these independent processes. Thus, using (1), for the sequence of events we can introduce the histogram

$$N(\geqslant E) = \int_E^\infty N(E)\,\mathrm{d}E = \frac{\varepsilon}{E} \tag{4}$$

and the differential distribution

$$N(E) = -\frac{\mathrm{d}N(\geqslant E)}{\mathrm{d}E} = \frac{\varepsilon}{E^2}\,. \tag{5}$$

I would clarify this using concrete examples. First, consider the distribution of the number N of cities with population P [5]. According to (1), the histogram in this case is described by the expression $N(\geqslant P) = \varepsilon/P$, and the differential distribution (probability density) is described by the expression $N(P) \sim P^{-2}$. Instead of E, we could consider any positive definite quantity, for example, the masses M of snow avalanches in the mountains. In this case, $N(\geqslant M) \sim M^{-1}$. My discussions with geographers have shown that they find such empirical laws obvious, but they are unaware of how to derive them. On the other hand, the reasoning presented in this work enables us to formulate the following empirical law of probability theory: strong events occur much less frequently than weak ones, provided their causes are homogeneous.

Similarly, using (3), we can write for the distributions over areas

$$N(\geqslant S) = \left(\frac{\varepsilon}{S}\right)^{1/3}, \text{ where possibly } \; S = L^2\,, \tag{6}$$

$$N(S) = \frac{1}{S}\left(\frac{\varepsilon}{S}\right)^{1/3}, \tag{7}$$

and for the distributions with a random diffusivity coefficient $K = K(E)$ in (2),

$$N(\geqslant K) = (\varepsilon/K)^{1/2}\,, \quad N(K) = \varepsilon^{1/2} K^{-3/2}\,. \tag{8}$$

Note that Newman [5] provides semi-integer exponents in his table of the exponents of fractals (for example, 2.5).

In a general case, all these formulas should include dimensionless coefficients, which can be determined by comparing them with observational data [1].

We illustrate these formulas for the cases mentioned above. For cosmic rays (CRs; see [2]), formula (4) should be used, bearing in mind that their density is determined not only per unit time, but also per unit area. This unit can only be determined if their volume energy density is known, which is estimated as $w = 0.5$ eV cm^{-3} $\sim 10^{-11}$ J m^{-3} [6]. CRs are

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formed in supernova explosions. In our galaxy, such explosions occur two or three times per century, with energy ranging from 10^{48} to 10^{49} J, giving the mean input power of $G \approx 10^{33}$ kW. The volume energy density is equivalent to pressure, i.e., $w = nE$, where n is the concentration of CR particles. Thus, the unit of area (its inverse) should be $(w/E)^{2/3}$, and the total integral flux will be

$$I(\geqslant E) = \frac{G}{E}\left(\frac{w}{E}\right)^{2/3} \sim E^{-5/3}\,, \tag{9}$$

while

$$I(E) \sim \frac{G}{E^2}\left(\frac{w}{E}\right)^{2/3} \sim E^{-8/3}\,. \tag{10}$$

The differential flux measured in the PAMELA international experiment on the Russian Resurs-5 satellite gave the value [7]

$$I(E) \sim E^{-n}, \quad n = 2.67 \pm 0.02!$$

There is a surprisingly accurate agreement between the theory and multi-year cosmic measurements. The list of the authors in [6] includes more than 60 Russian and European researchers. Had these authors presented their data in integral form, as is commonly done [6], the error would be an order of magnitude smaller: $I(\geqslant E) \sim E^{-n+1}$, i.e., $n = 1.67 \pm \delta$, $\delta \ll 0.02$.

Two stochastic elements of cloud fields have recently been described in the literature: 1) the ratio of cloud perimeter length to cloud area, and 2) the differential distributions of the lengths of clouds and the lengths of blue sky strips between them (see [1]). All of this can be understood with an accuracy of several percent within the framework of Kolmogorov's 1934 theory of random motions, based on the differential distribution of horizontal lengths.

Clouds are objects with area S and length scale L, defined as $L = S^{1/2}$ (Mandelbrot [8]). Their perimeters P are apparently larger than L, while the ratio P/L was found empirically: 1.35 for rain radar clouds [9] and based on satellite data in the interval $1 < L < 1500$ km for silver icy clouds at 80–90 km altitude [10], and in numerical models (see [1, 2]). However, nobody provides estimates of error bounds around 1.35. We performed fractal calculations and numerical estimates of the data [9], finding that, with a 95% probability, the uncertainties are ± 0.03, i.e., the exponent is 1.35 ± 0.03 [11, 12]. The value 1.35 is close to 4/3. According to (6), the perimeter bounds the cloud area, and the fractal estimate of the exponent should contain 1/3. However, we are interested in its length in units of L, so 1/3 should be increased by 1, i.e., the net exponent of the fractal should be $1 + 1/3 = 4/3$. In [2], it was speculated that the exponent of 1.35 should be explained by turbulence, but at that time turbulence was understood in the light of the work from 1941, while A.N. Kolmogorov's theory of random motions had not been used by anyone except the authors of [3]. It was not used in [9], but the role of pre-fractal multipliers, which carry useful information, was explicitly demonstrated there: for example, they help to form dimensionless similarity combinations.

The similarity of the lengths L of individual clouds can be explained in much the same way. The power-law indices in the cloud distribution in formula (6) and the additional parameter of the cloud length L give $1 + 2/3 = 5/3$ together, which differs by only 1/150 from the empirical value of 1.66 ± 0.00. This uncertainty is caused by the imprecision involved in determining the boundaries of the clouds due to differences in their optical thickness. We note that it was the empirical exponents that prompted the idea of using histograms and different probability distributions. Reference [13] provides the magnitude of the annual mean cloudiness: 0.66 ± 0.03%. It also provides detailed monthly estimates for latitudinal belts.

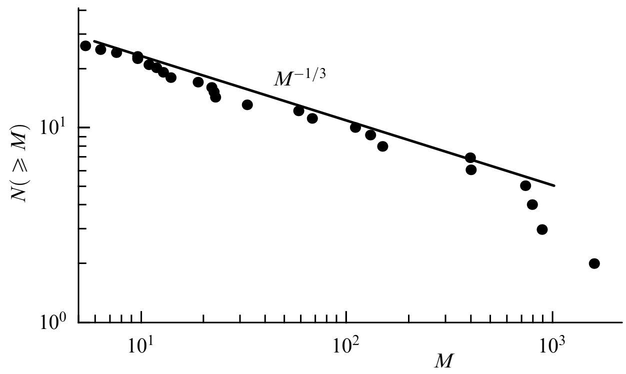

Figure. Cumulative distribution over masses of nearest galaxies.

Formulas (5)–(8) can be used to estimate the probability distributions of integral (histogram, cumulant) or differential distributions by applying the formulas in Feller's book [4], where the former distributions are proportional to t^{-1}, and the latter are proportional to t^{-2}, or $(t_s t)^{-1}$, where $1/t_s$ is the supplementary frequency of the process being studied.

Formula (6) produced remarkably accurate results, demonstrating agreement with empirical distributions of lithospheric plates over areas S [14] and the distribution of spiral galaxies over masses (see [2], § 11.7). Reference [14] shows that

$$N(\geqslant S) = 7S^{-n}, \quad \text{where} \quad n = 0.33\,, \tag{11}$$

and presents a table of all area values expressed in solid angles, ranging from 0.0012 to 2.7. Small plates are very difficult to trace on Earth's surface, and the six large plates do not fit the dependence (11). However, South America agrees reasonably with the remaining 43 plates, which have sizes ranging from 0.002 to 1 sr. Using data on areas within this range, we reproduced the dependences, but obtained 6.54 instead of 7, and $n = 0.33\ldots \pm \delta$, where $\delta \approx 0.03$, with a probability of 95%.

For spiral galaxies, the ratio of their radius to thickness is O(100), and therefore, three years ago, I assumed that their masses were proportional to their areas. I plotted the dependence $N(\geqslant M) \sim M^{-1/3}$. This plot, taken from Ref. [2], is shown in the Figure, which has not yet been published in the scientific literature on astronomy.

A.M. Obukhov's results [3] did not attract attention for more than half a century, and I myself only appreciated their significance in 2017. In the middle of the 20th century, data was sparse; in addition, the work was published in a relatively unknown publication. References [1, 2], as well as this note, describe already about a dozen cases based on using the ideas of A.N. Kolmogorov from 1934, advancing them to the level of useful practical tools of probability theory. The method remains simple and precise. These cases include the Kolmogorov–Obukhov inertial range theory; the theory of the general circulation of planetary atmospheres by the present author, in which the kinetic energy of the entire atmosphere is estimated as $K = Q_0\tau$, where Q_0 is the net solar energy flux, $\tau = r/c_e$ is the time it takes for thermodynamic equilibrium to be established, r is the planetary radius, and c_e is the speed of sound (see [2], § 12); and the Gutenberg–Richter law for the frequency and magnitude of earthquakes, hurricanes and tornadoes, clouds, etc. (see [2]).

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