

METHODOLOGICAL NOTES

PACS numbers: 75.30.Ds, 75.70.Ak, **75.76.+j**

Elastostatic spin waves — 'fine structure' of magnon polaron spectrum of acoustically subwavelength magnetic layer

S.V. Tarasenko, V.G. Shavrov

DOI: https://doi.org/10.3367/UFNe.2025.03.039878

Abstract. Using the example of a solitary acoustically subwavelength magnetic layer, it is shown that employing the tensor Green's function to solve the elastostatic boundary value problem makes it possible to obtain the spectrum structure of magnon polarons in a form convenient for analysis, associated with the formation of a special class of exchange-free magnons — elastostatic spin waves and resonance anomalies accompanying them. In combination with a similar approach previously developed for calculating the spectrum of dipole-exchange magnons, the method can be an alternative to the traditional method of studying analytically the dispersion features of magnetoacoustic waves propagating in layered structures under conditions of simultaneous magnetoelastic, magnetodipole, and inhomogeneous exchange interactions.

Keywords: acoustically subwavelength magnetic layers, magnon polarons, Green's functions

In recent years, the effects caused by magnon polarons have been actively studied [1–3]. They are based on the idea of a transition from charge currents to spin currents in order to create faster devices with higher energy efficiency. This approach to the problem has led to the incorporation of a number of concepts, previously proposed for a more adequate theoretical description of electron motion in a polarized medium, into the framework used to describe the propagation of spin currents in magnetic media in general and in magnetic dielectrics in particular. This especially concerns the concept of the 'magnon polaron,' introduced as an analogue of the 'polaron,' a term originally proposed by S.I. Pekar [4] to provide a more rigorous description of the lattice influence on the character of electron motion in a polarized medium (the electron dressed in a 'phonon cloud').

According to the definition proposed in [5], magnon polarons are hybrid magnon-phonon states arising because of magnetoelastic (ME) interaction in the region of anticrossing of the magnon and phonon branch spectra, i.e., in the vicinity of magnetoacoustic resonance (MAR). By now, many papers have been published related to the study of various aspects of magnon polaron physics. However, as a rule, the authors of these papers initially assume that in the magnon spectrum the activation energy associated with the contribution of effective fields stabilizing a given equilibrium magnetic state (in particular, a state with magnetic anisotropy) is much greater than the contribution induced by the linear magnon-phonon interaction. As a result, spectral regions of microwave (MW) waves far from the MAR point have so far not been considered a basis for the formation of magnon polarons, since, in this case, the influence of lattice vibrations on the magnon spectrum — and consequently on the hybridization of magnon and phonon states — is small (the dimensionless parameter of the linear magnon-phonon coupling $\xi^2_{\text{mph}} \ll 1$). At the same time, already in a magnet with isotropic elastic and magnetoelastic properties, the magnitude of the ME coupling ξ^2_{mph} for certain magnetoacoustic configurations (MACs) can depend significantly not only on the polarization but also on the direction of propagation of the interacting waves [6]. In magnetic media, the hybridization ME waves, inhomogeneous exchange (IE), and magnetodipole (MD) interactions enable not only single-beam refraction but also various forms of multibeam refraction of bulk MW waves at the interface between magnetic and nonmagnetic materials, occurring in both the elastodynamic regime (see, e.g., [7–10]) and the elastostatic limit [11].

In this case, the magnetoelastic dynamics of the magnetic medium can be described by a system of equations that, along with the equations of magnetostatics and the spin motion, also includes the equations of elastostatics [12]. For an acoustically subwavelength magnetic layer of thickness $2d$, the adequacy of using the equations of elastostatics to describe the influence of the elastic subsystem on the dynamics of magnons with frequency ω and wave vector $\mathbf{k}$ is determined by the possibility of fulfilling the condition $s_t|\mathbf{k}| \gg \omega(\mathbf{k})$ in a sufficiently wide range of frequencies and longitudinal wave numbers h, where $\mathbf{k}^2 = h^2 + (\pi v/2d)^2$, and s_t is the velocity of the shear acoustic wave in a nonmagnetic medium.

Finally, it is also important to emphasize that, as is known [6, 13], in the plane of external parameters $\omega - h$, even in unbounded magnetic media (in particular, in both ferro- and antiferromagnets), the very assumption of a weak linear magnon–phonon interaction ($\xi^2_{\text{mph}} \ll 1$) outside the MAR region can be violated both in the case of weakly anisotropic (cubic or easy-plane) magnets (low activation energy) and near the stability boundary of a given magnetic state (in particular, in the vicinity of proper ferroelastic spin-orientation phase transitions of the second order). Under such conditions, the formation of a strong linear ME coupling between individual branches of the spectrum of magnons and

S.V. Tarasenko [1,*], **V.G. Shavrov** [2]
[1] Galkin Donetsk Institute for Physics and Engineering,
ul. R. Luxemburg 72, 283048 Donetsk, Russian Federation
[2] Kotelnikov Institute of Radioengineering and Electronics,
Russian Academy of Sciences,
ul. Mokhovaya 11, kor. 7, 125009 Moscow, Russian Federation
E-mail: [*] s.v.tarasenko@mail.ru

Received 19 January 2025
Uspekhi Fizicheskikh Nauk **195** (9) 1008 – 1014 (2025)
Translated by V.L. Derbov

acoustic phonons becomes possible (the dimensionless parameter of the linear magnon–phonon coupling can be of the order of unity $\xi^2_{\rm mph} \propto 1$ [6]). In this case, already in the elastostatic limit, even despite the weak coupling between the magnon and phonon partial modes, the nature of the dispersion of the low-frequency branches of the ME-wave spectrum can differ significantly from the spectrum of the corresponding partial excitations. Notably, it was shown in [11] that, if in the elastostatic region of the spectrum of bulk ME waves of the magnetic layer for the selected MAC $0 \leqslant \xi^2_{\rm mph}(\mathbf{k}/|\mathbf{k}|) \leqslant 1$, then the indirect spin-spin exchange through the long-range field of high-frequency virtual phonons leads to the formation of elastostatic spin waves (ESWs) — a special class of exchangeless magnons propagating along the magnetic layer. Quantitatively, the frequency range of ESW formation is comparable in this case to the width of the ME gap in the MAR region.

Taking into account all of the above and following the analogy between the physics of magnetostatic waves (MSWs) and ESWs, one can expect (keeping in mind the results in [14, 15]) that, in analytical calculations of the spin-wave excitation spectrum in the elastostatic limit (magnon polarons by definition [5]), it will be effective to use tensor Green's functions to reduce the boundary value problem of elastostatics to an inhomogeneous integral equation. This particularly will make it possible to exclude both elastic variables and the boundary conditions corresponding to them from subsequent consideration, which will reduce the problem of analytical calculation of the magnon polaron spectrum to solving the tensor integro-differential equation only for variable components of magnetization and considering only exchange boundary conditions.

However, to date, such an approach to the analytical study of the spectrum of magnon polarons in an acoustically subwavelength magnetic layer with simultaneous consideration of ME, IE, and MD interactions has not been implemented. At the same time, in this case, the frequency dependence of the spectrum of magnon polarons can be found in an explicit form, and the solution to the corresponding exchange boundary value problem for the tensor integro-differential equation can be represented as a series expansion in eigenfunctions of the Sturm–Liouville exchange boundary value problem (as was previously done in the analysis of the spectrum of MSWs in a ferromagnetic layer (see, e.g., [15]).

As noted in Ref. [6], it is convenient to study the effects of strong ME coupling in magnets using the model of an orthorhombic ferromagnet (FM) or antiferromagnet (AFM) as an example, and therefore we will consider as an example a two-sublattice model of a uniaxial exchange-collinear ferrimagnet (FIM) with a compensation point and an easy magnetization direction along the OZ-axis (see also [16]). As limiting cases, it contains the cases of an exchange-collinear FM and AFM medium. Below, for simplicity and clarity of calculations, we will assume that the magnetoelastic and elastic properties of such a model magnet are isotropic. As a result, the corresponding density of the thermodynamic potential can be represented as $(\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2,\ \mathbf{L} = \mathbf{M}_1 - \mathbf{M}_2)$:

$$W = \frac{\delta}{2}\,\mathbf{M}^2 + \frac{\bar{\alpha}}{2}(\nabla\mathbf{L})^2 + \frac{b_x}{2}\,L_x^2 + \frac{b_y}{2}\,L_y^2 + \gamma L_i L_k u_{ik} - \mathbf{MH} + \frac{\lambda}{2}\,u_{ll}^2 + \mu u_{ik}^2\,. \tag{1}$$

Here, δ is the constant of the homogeneous exchange interaction, $\bar{\alpha} = \alpha_1 + \alpha_2 - 2\alpha_{12}$, where α_1, α_2, α_{12} are the constants of the intra- and intersublattice inhomogeneous exchange interaction, respectively, $b_{x,y} > 0$ are the constants of the orthorhombic magnetic anisotropy, γ is the constant of the magnetoelastic coupling, u_{ik} is the tensor of elastic deformations, and λ, μ are the Lamé coefficients. The magnetoelastic dynamics of such a model of magnet, as is known, is described by a closed system of equations consisting of the equations of motion for the vectors $\mathbf{M}$ and $\mathbf{L}$, the equations of magnetostatics, and the equations of elastodynamics [7–10, 12]. If, in addition to the notations already introduced above, in accordance with [16], we add $M_0^2 \equiv (M_1^2 + M_2^2)/2$, $\kappa \equiv M_s/2M_0$, $M_s^2 = (M_1^2 - M_2^2)/|\mathbf{M}_1 - \mathbf{M}_2|$, then, for $\mathbf{k} \in YZ$, $\mathbf{L} \parallel OZ$ and arbitrary orientation of the lattice elastic displacement vector, the spectrum of a plane monochromatic magnetoelastic wave in an unlimited rhombic FIM (1) with simultaneous consideration of magnetoelastic, magnetodipole, and inhomogeneous exchange interactions can be conveniently represented in the following form ($\omega_s \equiv (gM_0)$, g is the gyromagnetic ratio, which we will assume to be the same for both sublattices, $c^2 = \omega_s^2\delta\bar{\alpha}$, $\omega_{0\alpha}^2 = \omega_s^2\delta\bar{b}_\alpha$, $\mathbf{k}^2 = k_y^2 + k_z^2$, $\omega \leqslant \delta\omega_s$):

$$D_{\rm SH}(\omega, \mathbf{k}) D_{\rm L}(\omega, \mathbf{k}) D_{\rm m}(\omega, \mathbf{k}) = 0\,, \tag{2}$$

$$D_{\rm m}(\omega, \mathbf{k}) \equiv \Omega_y \left[\Omega_x + \frac{4\pi k_y^2}{\tilde{k}^2}\left(\frac{\omega}{\delta\omega_s}\right)^2\right] + \kappa^2\left[\Omega_x\,\frac{4\pi k_y^2}{\tilde{k}^2} - \omega^2\right],$$

$$\Omega_y \equiv \frac{\bar{\alpha}}{c^2}\left[\omega_{0y}^2 + c^2k^2 - \omega^2 + \omega_{\rm me}^2\left(1 - \frac{\mu}{\rho D_{\rm L}}\left[D_{zz}k_z^2 - 2D_{yz}k_yk_z + D_{yy}k_y^2\right]\right)\right], \tag{3}$$

$$\Omega_x \equiv \frac{\bar{\alpha}}{c^2}\left[\omega_{0x}^2 + c^2k^2 - \omega^2 + \omega_{\rm me}^2\left(1 - \frac{\mu k_z^2}{\rho D_{\rm SH}}\right)\right], \qquad \tilde{k}^2 \equiv \mathbf{k}^2 + \frac{4\pi}{\delta}\,k_y^2\,, \tag{4}$$

$$D_{\rm L}(\omega, \mathbf{k}) \equiv D_{yy}D_{zz} - D_{yz}^2\,, \quad D_{ik}(\omega, \mathbf{k}) \equiv \frac{\Lambda_{ik}(\omega, \mathbf{k})}{\rho} - \omega^2\delta_{ik}\,, \qquad D_{\rm SH}(\omega, \mathbf{k}) \equiv \frac{\Lambda_{xx}(\omega, \mathbf{k})}{\rho} - \omega^2\,, \tag{5}$$

where $\omega_{\rm me}^2 = \omega_s^2\delta\gamma^2M_0^3/\mu$, ρ is the density, Λ_{ik} is the Christoffel tensor [17], and δ_{ik} is the unit tensor.

If we consider Eqns (2)–(5) as a characteristic equation for determining the nature of the localization of an ME wave with given values of frequency and longitudinal wave number near the magnet surface, then it is easy to see that, already in the case of an easy magnetic axis normal to the magnet surface, depending on the frequency of the wave incident on the interface between the magnetic and nonmagnetic media and its angle of inclination in the case of simultaneous consideration of ME, IE, and MD interactions, it is possible to simultaneously implement physically different mechanisms of acoustic multi-beam refraction both with and without changing the cavity of the refraction surface of a normal ME wave. As in the case of an FM [7–10], this can significantly affect the structure of the spectrum of normal bulk ME waves formed as a result of interference in the FIM layer, especially since, according to (2)–(5), even without considering the MD and IE interactions, the effect of birefringence (bireflection) is possible already in the elastostatic limit.

Considering dispersion equation (2) as characteristic, the spatial structure of the z-component of the elastic displacement vector in the FIM layer can, following [9, 10], be represented in the form

$$u_z(y, z, t) = \sum_{j=1}^{6} \big(A_j \cosh(\eta_j z) + B_j \sinh(\eta_j z)\big) \exp(\mathrm{i}\psi)\,, \quad (6)$$

where $\eta^2 \equiv -(\mathbf{kq})^2$, $\psi \equiv hy - \omega t$.

As is well known, waveguide modes propagating along a layer — depending on the wave characteristics of the medium and the interlayer boundary conditions — can exist either as eigenmodes or as radiative modes [18]. To account for both possibilities, we consider a normally magnetized easy-axis (EA) FIM layer (2)–(5) embedded in a symmetric environment, which allows, in principle, the existence of a single open phonon scattering channel (for example, when the nonmagnetic medium surrounding the FIM layer is acoustically coupled to it and has a lower density). To describe the spectrum of magnetoelastic (ME) waves in the magnetic layer, we aim to treat ME, exchange (IE), and MD interactions simultaneously. In order to simplify subsequent calculations and reduce their complexity, we rely on previously obtained results, namely, the spectrum of bulk dipole-exchange magnons in a normally magnetized layer with double-sided metallization by an ideal conductor [19], and the main types of elastic boundary conditions outlined in Ref. [20]. On this basis, we assume that, on both surfaces of the normally magnetized FIM layer ($\mathbf{q} \parallel \mathbf{m}_0 \parallel OZ$, $\mathbf{k} \in YZ$), one of the following mixed boundary-condition systems is satisfied ($\tilde{Z}$ is the surface elastic wave impedance of the environment, φ is the magnetostatic potential, $\mathbf{B}$ is the magnetic induction, $\alpha = x, y$):

$$\sigma_{\alpha z} = 0\,, \quad \sigma_{zz} = \mp\tilde{Z}u_z\,, \quad \varphi = 0\,, \quad L_\alpha = 0\,, \quad z = \pm d \quad (7)$$

or

$$u_\alpha = 0\,, \quad \sigma_{zz} = \mp\tilde{Z}u_z\,, \quad \frac{\partial L_\alpha}{\partial z} = 0\,, \quad B_z = 0\,, \quad z = \pm d\,. \quad (8)$$

Taking into account the symmetry of the ME wave field relative to the median plane of the magnetic layer considered, both in case (7) and (8), it is possible, similar to [18], to exclude from further consideration the unknown amplitudes A_{2-6}, B_{2-6} included in (6) to be determined, expressing them through A_1, B_1, respectively,

$$A_j = F_{j1}\frac{c_{1d}}{c_{jd}}A_1\,, \quad B_j = F_{j1}\frac{s_{1d}}{s_{jd}}B_1\,,$$
$$\begin{pmatrix} u_z \\ \sigma_{zz} \end{pmatrix}_z = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}\begin{pmatrix} A_1 \\ B_1 \end{pmatrix}. \quad (9)$$

In this case, the transition matrix for the considered layer of the easy-axis FIM with ME, IE, and MD interactions taken into account, due to Eqn (9), will have the following structure:

$$\begin{pmatrix} u_z \\ \sigma_{zz} \end{pmatrix}_{z=d} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} u_z \\ \sigma_{zz} \end{pmatrix}_{z=-d}, \quad \overline{\overline{T}}(2d) = \overline{\overline{N}}(d)\,\overline{\overline{N}}^{\,-1}(-d)\,. \quad (10)$$

As a result, according to (9), (10), the spectrum of ME waves with ME, IE, and MD interactions simultaneously considered, both in case (7) and (8), can be represented as

$$\mathrm{i}\tilde{Z}(T_{11} + T_{22}) - T_{21} + T_{12}\tilde{Z}^2 = -2(N_{21} - \mathrm{i}\tilde{Z}N_{11})(N_{22} - \mathrm{i}\tilde{Z}N_{12}) = 0\,. \quad (11)$$

In this case, spectrum of ME eigenmodes (or radiative modes) occur when, in Eqns (7), (8), and (11), either $\mathrm{Im}\,\tilde{Z} \neq 0$, $\mathrm{Re}\,\tilde{Z} = 0$ (eigenmodes) or $\mathrm{Im}\,\tilde{Z} = 0$, $\mathrm{Re}\,\tilde{Z} \neq 0$ (radiative modes). In the special case, when for $\mathbf{q} \parallel \mathbf{L}_0 \parallel OZ$, $\mathbf{k} \in YZ$, the boundary conditions on both surfaces of the FIM layer (7) are such that

$$\sigma_{\alpha z} = 0\,, \quad \sigma_{zz} = 0\,, \quad \varphi = 0\,, \quad L_\alpha = 0\,, \quad \alpha = x, y\,, \quad z = \pm d\,, \quad (12)$$

then, taking into account Eqns (9)–(11), the spectrum of ME waves of the normally magnetized FIM layer under consideration is determined as

$$N_{21}N_{22} = 0\,. \quad (13)$$

If, however, in the same MAC, relations (8) take the form

$$u_\alpha = 0\,, \quad u_z = 0\,, \quad \frac{\partial L_\alpha}{\partial z} = 0\,, \quad B_z = 0\,, \quad \alpha = x, y\,, \quad z = \pm d\,, \quad (14)$$

then, taking into account Eqns (9)–(11), instead of Eqn (13), we obtain

$$N_{11}N_{12} = 0\,. \quad (15)$$

The analysis shows that, in both cases (13) and (15) for the spectrum of normal ME waves propagating along the easy-axis FIM layer (2)–(5), (7), (8), by analogy with the case of the spectrum of Lamb waves for a free elastic layer, using (9)–(11), we can introduce an analogue of the Mindlin grid [21] as

$$\prod_n |c_n||s_n| = 0\,, \quad n = 1, 2, \ldots, \quad (16)$$

where the grid nodes on the plane $\omega - h$ are located at the points $(|c_n| + |c_p|)(|s_n| + |s_p|) = 0$, $p = 1, 2, \ldots$, $n = 1, 2, \ldots$, $v \neq \rho$, or

$$D_n(\omega, h) = D_p(\omega, h)\,, \quad n = 1, 2, \ldots, \quad p = 1, 2, \ldots, \quad v \neq \rho\,, \quad (17)$$

where, taking into account Eqns (2)–(5), $D_n(\omega, h) = D(\omega, k_y = h, k_z = \kappa_n = \pi n/2d)$.

Note that, together with (2)–(5), the relation $D_v(\omega, h) = 0$, $v = 1, 2, \ldots$ corresponds to the spectrum of waveguide ME modes of the considered EA FIM layer, on both surfaces of which simultaneously

$$\sigma_{\alpha z} = 0\,, \quad u_z = 0\,, \quad \varphi = 0\,, \quad L_\alpha = 0\,, \quad \alpha = x, y\,, \quad z = \pm d \quad (18)$$

or

$$u_\alpha = 0\,, \quad \sigma_{zz} = 0\,, \quad \frac{\partial L_\alpha}{\partial z} = 0\,, \quad B_z = 0\,, \quad \alpha = x, y\,, \quad z = \pm d\,. \quad (19)$$

As a result, for both (18) and (19), the spectrum of ME waves (11) propagating along the normally magnetized layer of the considered rhombic FIM, considering (2)–(6), has the following structure:

$$\prod_{n=1}^{\infty} D_n(h, \omega) = 0\,. \quad (20)$$

It should be emphasized that the combinations of frequency and longitudinal wave number corresponding to Eqn (17) are characteristic of the spectrum of proper ME waves of the considered magnetic layer in (11) not only for $\mathrm{Im}\,\tilde{Z} \neq 0$, $\mathrm{Re}\,\tilde{Z} = 0$, but also for $\mathrm{Im}\,\tilde{Z} = 0$, $\mathrm{Re}\,\tilde{Z} \neq 0$. Moreover, if in (11) $\mathrm{Im}\,\tilde{Z} \neq 0$, $\mathrm{Re}\,\tilde{Z} = 0$ simultaneously, then these are degeneracy points in the spectrum of proper ME excitations of the considered magnetic heterostructure, and if in Eqn (11) $\mathrm{Im}\,\tilde{Z} = 0$ and $\mathrm{Re}\,\tilde{Z} \neq 0$ simultaneously, then these are, according to [22], bound states in the continuum (BICs) against the background of a continuous spectrum of radiation of bulk ME waves propagating along the considered FIM layer.

The formation of nodes of the Mindlin grid in the plane of external parameters $\omega - h$ under the considered conditions indicates the existence of dispersion curves of the spectrum of bulk ME waves both at the degeneracy points of the spectrum of waveguide ME modes of the magnetic layer and in their vicinity. In the latter case, their spectrum can be approximately reconstructed using a series expansion in the vicinity of a specific node of the Mindlin grid (for the case of a nonmagnetic layer, see, e.g., [21]). Analysis shows that, in the considered MAC, such degeneracy points as (16), (17) are possible, first of all, in the elastodynamic region ($h < \omega/s_t$), where they (in addition to the points characteristic of the spectrum of elastic waves in the nonmagnetic layer [21]) are the result of degeneracy of the modes of the spectrum of bulk magnon (magnetostatic or exchange) and elastic waves. However, the fact that degeneracy points in the spectrum of ME waves of the considered layer at $\gamma \neq 0$ are, in principle, also possible in the elastostatic region of the external parameters $\omega - h$, $h > \omega/s_t$, is undoubtedly important for the purposes of this paper. As analysis shows, this does not necessarily have to be the result of simultaneous consideration of IE ($\alpha \neq 0$) or MD interactions (formally in Eqns (2)–(5) $4\pi \neq 0$), but is a consequence of spin-spin exchange through the field of virtual high-frequency phonons.

Thus, in the acoustically subwavelength FIM layer in the frequency range $\omega < s_t/(2d)$, a number of features of the magnon polaron spectrum can be studied quite correctly already in the elastostatic limit on the basis of a coupled system of dynamic equations, including the Landau–Lifshitz, magnetostatic, and elastostatic equations. In order to take full advantage of the simplified description of the magnon polaron spectrum in the case of an acoustically subwavelength magnetic layer by going from the elastodynamic equations to the elastostatic equations [12], we use an analogy with calculations in the Coulomb limit of the dipole-exchange magnon spectrum within the framework of the approach developed in Refs [14, 15]. To this end, at the first stage, using the apparatus of tensor Green's functions, we will proceed from the equations of elastostatics and elastic boundary conditions to the corresponding integral equations that determine the nonlocal relationship between the field of elastic deformations and the field of spin oscillations in the considered magnetic layer. As an example of calculating the tensor Green's function for an elastic (in our case, elastically isotropic) layer, we will use the results from [23], with the difference being that in our case we mean the tensor Green's function for an elastic layer and the boundary value problem of elastostatics, rather than elastodynamics, as in Ref. [23]. This means that, unlike [23], the system of fundamental solutions for constructing the corresponding tensor Green's function for an elastically isotropic layer for $\mathbf{k} \in YZ$ will have the form (see, e.g., [24])

$$\begin{gathered}\mathbf{u} \in YZ:\ E_1 \sinh(hz),\ F_1 \cosh(hz),\ zE_2 \cosh(hz),\ zF_2 \sinh(hz),\\ u_x(z):\ E_3 \sinh(hz),\ F_3 \cosh(hz)\,.\end{gathered} \tag{21}$$

The unknown amplitudes E_{1-3}, F_{1-3} in (21) are determined according to the standard procedure (see [23, 25]) based on elastic boundary conditions on the surface of the magnetic layer and the continuity conditions for the components of the tensor Green's function itself and the jump of its first derivative along the coordinate normal to the surface of the layer at an arbitrary internal point of the layer under consideration. When calculating the tensor Green's function for the magnetostatic boundary value problem in a similar manner [25], it is necessary to keep in mind that, for $\omega \ll g\delta M_0$ from the Landau–Lifshitz equations for the ferrite sublattice magnetizations (1), according to [16], in the magnetostatic equations

$$\mathbf{M} = 2M_0\kappa\mathbf{L} + \frac{1}{\omega_s\delta}\left[\frac{\partial\mathbf{L}}{\partial t}\times\mathbf{L}\right] + \frac{\mathbf{H}}{\delta} - \frac{\mathbf{L}(\mathbf{HL})}{\delta L^2}\,. \tag{22}$$

As a result, for a symmetric FIM layer (1) with boundary conditions (7) or (8), after using the tensor Green's functions to solve the boundary magnetostatic ($\overline{\overline{G^{\mathrm{md}}}}$) and elastostatic ($\overline{\overline{G^{\mathrm{e}}}}$) problem, we obtain the following system of tensor integro-differential equations with respect to the Fourier amplitudes of the variable components of the vector $\mathbf{l} \equiv \mathbf{L}/|\mathbf{L}|$ ($\mathbf{k} \in YZ$, $\mathbf{q} \parallel \mathbf{L} \parallel OZ$), which are inhomogeneous across the thickness of the magnetic layer:

$$\begin{gathered}P_{\alpha\beta} l_\beta + S_{\alpha\beta} h_\beta + \gamma R_{\alpha\beta} u_{z\beta} = 0\,, \quad \alpha, \beta = x, y\,,\\ P_{\alpha\beta} = \begin{pmatrix} P_{xx} & P_{xy} \\ P_{yx} & P_{yy} \end{pmatrix}, \quad S_{\alpha\beta} = \begin{pmatrix} \dfrac{-\mathrm{i}\omega}{\delta\omega_s} & -\kappa \\ \kappa & \dfrac{-\mathrm{i}\omega}{\delta\omega_s} \end{pmatrix},\\ R_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\end{gathered} \tag{23}$$

$$\begin{gathered}P_{xx} = \frac{\mathrm{i}\kappa\omega}{\omega_s}\,, \quad P_{xy} = \bar{\alpha}\left(\frac{\partial^2}{\partial z^2} - h^2\right) + \frac{\bar{\alpha}}{c^2}(\omega^2 - \omega_{0y}^2 - \omega_{\mathrm{me}}^2)\,,\\ P_{yx} = \bar{\alpha}\left(\frac{\partial^2}{\partial z^2} - h^2\right) + \frac{\bar{\alpha}}{c^2}(\omega^2 - \omega_{0x}^2 - \omega_{\mathrm{me}}^2)\,, \quad P_{yy} = -\frac{\mathrm{i}\kappa\omega}{\omega_s}\,,\\ u_i(z) = \gamma\int_{-d}^{d} G_{ik}^{\mathrm{e}}(z, z')\, l_k(z')\,\mathrm{d}z'\,,\\ h_i(z) = \frac{1}{2M_0}\int_{-d}^{d} G_{ik}^{\mathrm{md}}(z, z')\, M_k(z')\,\mathrm{d}z'\end{gathered} \tag{24}$$

with exchange boundary conditions of the form

$$\frac{\partial l_\alpha}{\partial z} \pm \varepsilon l_\alpha = 0\,, \quad \alpha = x, y\,, \quad z = \pm d\,. \tag{25}$$

We particularly emphasize that, for the convenience of further analysis and in order for relations (16) to serve as the solution to the boundary value problem (22)–(25) in the elastostatic limit for the considered MAC (($\mathbf{q} \parallel \mathbf{L}_0 \parallel OZ$, $\mathbf{k} \in YZ$) — that is, so that the system of boundary conditions (18) or (19) is simultaneously satisfied — the mixed boundary conditions for the elastic, electromagnetic, and spin subsystems are chosen self-consistently ($\alpha = x, y$) when deriving

(22)–(25):

$$\varepsilon\sigma_{\alpha z} \pm u_\alpha = 0\,, \quad \sigma_{zz} = \pm\varepsilon u_z\,, \quad B_z \pm \varepsilon\varphi = 0\,,$$
$$\frac{\partial L_\alpha}{\partial z} \pm \varepsilon L_\alpha = 0\,, \quad z = \pm d\,, \tag{26}$$

i.e., if in (25), (26) $\varepsilon \to \infty$, then from (26) we obtain (18), and when $\varepsilon \to 0$, (26) transitions to (19).

The solution to the boundary value problem (22)–(25), as in [14, 15], is sought in the form of a series expansion in the system of orthonormal functions $\Phi_{\alpha n}(z)$ (where $\alpha = x, y$) of the exchange boundary value problem (25) of the normally magnetized FIM layer under consideration $((\eta_\alpha^2 - \varepsilon^2)\tan(2\eta_\alpha d) = 2\eta_\alpha\varepsilon)$:

$$L_\alpha(\mathbf{r}) = \sum_{n=0}^{\infty} C_{\alpha n}\Phi_{\alpha n}(z)\exp(\mathrm{i}\psi)\,,$$
$$\Phi_{\alpha n}(z) = \cos\left[\eta_{\alpha n}(z+d)\right] + \frac{\varepsilon}{\eta_{\alpha n}}\sin\left[\eta_{\alpha n}(z+d)\right]. \tag{27}$$

As a result, considering the orthonormality properties of $\Phi_{\alpha n}(z)$ from (22)–(27), we obtain an infinite system of matrix equations for the unknown amplitudes of spin-wave oscillations $C_{\alpha n}$ in Eqn (27):

$$\sum_p D_{\alpha\beta}^{np} C_{\beta p} = 0\,,$$
$$D_{\alpha\beta}^{np} \equiv \int_{-d}^{d} \Phi_{\alpha n}(z')\,\overline{\overline{P}}\left[\Phi_{\beta p}(z')\right]\mathrm{d}z'\,, \quad \alpha = x, y\,, \quad \beta = x, y\,,$$
$$\text{so that } D_{\alpha\beta}^{nn} C_{\beta n} + \sum_{\rho \neq v} D_{\alpha\beta}^{np} C_{\beta p} = 0\,. \tag{28}$$

From (26), taking into account (22)–(25) for $\varepsilon = \infty$ or $\varepsilon = 0$, we obtain in (28) a block-diagonal matrix, since, according to (27), $\eta_{\alpha n} = \pi n/(2d)$, $n = 1, 2, \ldots$. As a result, the corresponding dispersion relation for the spectrum of waveguide magnon polarons propagating along the considered acoustically subwavelength layer of an easy-axis FIM (22)–(25), taking into account the ME, IE, and MD interactions, has a form similar to Eqn (20), but now in the elastostatic limit:

$$\prod_{n=1}^{\infty} D_n(h, \omega) = 0\,, \quad D_n \equiv \begin{vmatrix} D_{xx}^{nn} & D_{xy}^{nn} \\ D_{yx}^{nn} & D_{yy}^{nn} \end{vmatrix}\,, \tag{29}$$

where

$$D_n(h, \omega) \equiv \Omega_{yn}\left[\Omega_{xn} + \frac{4\pi h^2}{\tilde{k}_n^2}\left(\frac{\omega}{g\delta M_0}\right)^2\right] + \kappa^2\left[\Omega_{xn}\,\frac{4\pi h^2}{\tilde{k}_n^2} - \omega^2\right] = 0\,,$$
$$\Omega_{yn} \equiv \frac{\alpha}{c^2}\left[\omega_{0y}^2 + c^2(h^2 + k_n^2) - \omega^2 + 4\omega_{\mathrm{me}}^2\left(1 - \frac{\mu}{\lambda + 2\mu}\right)\frac{k_y^2 k_n^2}{(h^2 + k_n^2)^2}\right],$$
$$\Omega_{xn} \equiv \frac{\alpha}{c^2}\left[\omega_{0x}^2 + c^2(h^2 + k_n^2) - \omega^2 + \omega_{\mathrm{me}}^2\,\frac{h^2}{h^2 + k_n^2}\right],$$
$$\tilde{k}_n^2 \equiv k_n^2 + h^2\left(1 + \frac{4\pi}{\delta}\right). \tag{30}$$

In the special cases of interest, the spectrum of magnon polarons follows from (29), (30), taking into account the ME ($\omega_{\mathrm{me}} \neq 0$), IE ($\alpha \neq 0$), and MD ($\omega_{\mathrm{md}} \neq 0$) interactions in an orthorhombic two-sublattice FIM (1)–(5) with an easy magnetic axis along OZ and at $\mathbf{k} \in YZ$. The spectrum of magnon polarons for an AFM or FM is obtained from (29), (30) by moving to the limit $\kappa = 0$ in the first case and $(|\delta^{-1}| + 1 - |\kappa|) \to 0$ in the second case. If the magnetic medium under consideration is a uniaxial FIM (OZ), then, in (29), (30), $\omega_{0x} = \omega_{0y}$. In the opposite limiting case of an easy-plane magnet with $\mathbf{l} \parallel OZ$ for a given MAC, there are two main options for the relative position of the easy magnetic plane with respect to the given plane of ME wave propagation ($\mathbf{k} \in YZ$ $(k_x = 0)$). For $\omega_{0x} \gg a\mathbf{k}^2$, ω_{me}, ω_{md}, Eqns (29), (30) present the spectrum of magnon polarons of the FIM layers with the easy plane YZ, and for $\omega_{0y} \gg a\mathbf{k}^2$, ω_{me}, ω_{md}, with the easy plane XZ. In accordance with the results from Refs [6, 13], a strong ME coupling is possible both in the case of an easy-plane magnet and near the boundary of loss of stability of the magnetic state under consideration. In the particular case where both IE ($\alpha \to 0$) and MD ($\omega_{\mathrm{md}} \to 0$) interactions are simultaneously ignored, relations (29), (30) correspond to the spectrum of elastostatic spin waves. This is a special class of exchangeless magnons propagating along an acoustically subwavelength magnetic layer. Their dispersion properties follow from indirect spin-spin interaction through the field of high-frequency phonons, taking into account the effects of interference and re-reflection from the boundaries of the magnetic layer.

Thus, in the case of an acoustically subwavelength magnetic layer, the use of the apparatus of tensor Green's functions in solving the elastostatic boundary value problem makes it possible to obtain in explicit form the dependence of the frequency of the magnon polaron spectrum on the longitudinal wave number h. In the particular case of mixed boundary conditions, the solution to the boundary value problem with simultaneous consideration of ME, IE, and MD interactions makes it possible to represent the spectrum of magnon polarons of the FIM layer in the form of roots of a biquadratic linear algebraic equation with coefficients depending on h relative to the frequency of spin oscillations. In particular, from (29), (30) follows the possibility of forming on the dispersion curves of the spectrum of waveguide magnon polarons for a mode with a given number of regions corresponding to a forward or backward wave, as well as the possibility of changing the wave type depending on the value of the longitudinal wave number. Of particular interest are the conditions for the formation of degeneracy points in the spectrum of the magnon polarons under consideration when they are eigenmodes for the spectrum of ME waves of the acoustically subwavelength magnetic layer. It is also important that such points also correspond to eigenstates of the spectrum of ME waves of a symmetric magnetic layer even when the boundary conditions on its surfaces are such that the magnon polarons propagating along it are radiative modes. These effects are also possible in principle both upon and without taking into account the IE interaction, since, in the acoustically subwave magnetic layer under consideration in the elastostatic region of the spectrum of normal ME waves, in addition to the IE and MD interactions, there is also an indirect spin-spin exchange through the field of virtual high-frequency phonons. It is precisely its presence that reflects in (23)–(25) the nonlocal interaction determined by the elastostatic tensor Green's function.

Additional features in the spectrum of both waveguide magnon polarons and ME waveguide modes arise even with a slight difference between boundary conditions (25), (26) and those used in deriving (29), (30). As in the physics of MSWs, this can lead to the removal of degeneracy for a selected point of the ME wave spectrum (in particular, to anticrossing). And, in this case, the use of the tensor Green's functions also turns out to be useful for calculating and analyzing the spectrum of waveguide magnon polarons.

Let us consider only the proper ME waves of the FIM layer (1)–(5), assuming that in Eqn (11) $\mathrm{Re}\,\tilde{Z} = 0$, $\mathrm{Im}\,\tilde{Z} \neq 0$, and the deviation of boundary conditions (25) from (18) or (19) is extremely small. As in Ref. [15], this allows us to consider only the dispersion curves of those modes which, when condition (18) or (19) is satisfied, previously participated in the formation of the degeneracy point of the magnon polaron spectrum (20), (29), (30). If these are modes with numbers n and p, then, taking into account the notations introduced above, the relationship describing such anticrossing in the vicinity of the degeneracy point (20), (29), (30) for $\varepsilon \to \infty$ or $\varepsilon \to 0$ in (26), can be represented as

$$\begin{cases} D^{nn}_{\alpha\beta} C_{\beta n} + \sum\limits_{p\neq n} D^{np}_{\alpha\beta} C_{\beta p} = 0\,, \quad \alpha = x, y\,, \quad \beta = x, y\,, \\ \sum\limits_{p\neq n} D^{pn}_{\alpha\beta} C_{\beta p} + D^{pp}_{\alpha\beta} C_{\beta p} = 0\,, \quad n = 1, 2, \ldots, \quad p = 1, 2, \ldots . \end{cases} \tag{31}$$

As a result, if, in Eqn (26), $\varepsilon \gg 1$ or $\varepsilon \ll 1$, then, for the vicinity of the degeneracy point $D_n(\omega, h) = D_p(\omega, h)$ from (31), we obtain (see also [10, 26])

$$\begin{gathered} \left(\delta h_n - \frac{\delta\omega_n}{v_n}\right)\left(\delta h_p - \frac{\delta\omega_p}{v_p}\right) - \frac{(\Delta\omega_{np})^2}{v_n v_p} = 0\,, \\ v_t \equiv -\frac{\partial D_t/\partial h}{\partial D_t/\partial\omega}\,, \quad \delta h_t \equiv h - h_t\,, \quad \delta\omega_t \equiv \omega - \omega_t\,, \quad t = n, p\,, \end{gathered} \tag{32}$$

and therefore, according to (32), for a fixed value of ω, the corresponding values of the longitudinal wave number are determined as

$$\begin{gathered} h_{\pm}(\omega) = \frac{\bar{h}_n + \bar{h}_p}{2} \pm \sqrt{\frac{(\bar{h}_n - \bar{h}_p)^2}{4} + \frac{(\Delta\omega_{np})^2}{v_n v_p}}\,, \\ \bar{h}_t \equiv h_t + \frac{\delta\omega_t}{v_t}\,, \quad t = n, p\,. \end{gathered} \tag{33}$$

From Eqn (33), notably, it follows that, even in the case of an acoustically subwavelength magnetic layer, the spectrum of propagating magnon polarons contains a frequency range in which the formation of complex-conjugate waves is possible ($\mathrm{Re}\,\{h_{\pm}(\omega)\}$, $\mathrm{Im}\,\{h_{\pm}(\omega)\} \neq 0$), provided that one of the ME modes in this region of ω is a backward wave, while the other is a forward wave ($v_n v_p < 0$) [10, 26]. In the MAC under consideration, such an effect can, in principle, already arise in the elastostatic limit, even without considering IE and MD interactions. As a result, in the anticrossing region of the magnon polaron spectrum of the acoustically subwavelength magnetic layer, a pair of complex-conjugate waves will be formed, whose frequency and longitudinal wavenumber are determined from Eqn (33).

Within the framework of the calculation scheme discussed here, as well as when using Maxwell's equations instead of the magnetostatic equations in the spin-wave electrodynamics of a magnetic layer [27], taking into account the role of the finiteness of the propagation velocity of elastic waves in the dynamics of magnon polarons will also require, when constructing the corresponding tensor Green's function in (22)–(25), replacing the system of fundamental solutions (21) with a system related to the same boundary value problem, but now for the elastodynamics of an elastically isotropic layer (see, e.g., [23]).

Thus, the application of tensor Green's functions for analysis in the elastostatic limit of the spectrum of ME waves in an acoustically subwavelength magnetic layer is an efficient method for the analytical study of additional anomalies in the spectrum of waveguide magnon polarons. In a number of practically interesting cases, such a calculation method allows one to obtain a solution to boundary value problems for the spectrum of such magnons even in the case of simultaneous consideration of ME, MD, and IE interactions, not in transcendental, but in explicit form (frequency as a function of the longitudinal wave number). It is also undoubtedly important that this class of resonant ME effects in an acoustically subwavelength magnetic layer may be of particular interest as a variant of a controlled acoustic metasurface [3], the hybrid resonances of which are caused by the spin-wave degrees of freedom of the magnetic medium that forms the layer. The proposed calculation method may also be useful for analyzing the nonlinear dynamics of magnon polarons in acoustically subwave magnetic layers.

References

1. Maekawa S et al. *J. Appl. Phys.* **133** 020902 (2023)
2. Gidding M et al. *Nat. Commun.* **14** 2208 (2023)
3. Zabihi A, Ellouzi C, Shen C *Front. Mater.* **10** 1132585 (2023)
4. Pekar S I *Research in Electron Theory of Crystals* (Washington, DC: US Atomic Energy Commission. Division of Technical Information, USA, Department of Commerce, 1963); Translated from Russian: *Issledovaniya po Elektronnoi Teorii Kristallov* (Moscow–Leningrad: GITTL, 1951)
5. Kamra A et al. *Phys. Rev. B* **91** 104409 (2015)
6. Turov E A, Shavrov V G *Sov. Phys. Usp.* **26** 593 (1983); *Usp. Fiz. Nauk* **140** 429 (1983)
7. Parekh J P, Bertoni H L *J. Appl. Phys.* **44** 2866 (1973)
8. Lukomskii V P *Sov. Phys. JETP* **57** 297 (1983); *Zh. Eksp. Teor. Fiz.* **84** 513 (1983)
9. Filippov B N, Preprint No. 80/1 (Sverdlovsk: Institute of Metal Physics, Ural Scientific Center of the USSR Academy of Sciences, 1980)
10. Gulyaev Yu V, Zil'berman P E *Sov. Phys. J.* **31** 860 (1988); *Izv. Vyssh. Uchebn. Zaved. Fiz.* **31** (11) 6 (1988)
11. Gulyaev Yu V, Tarasenko S V, Shavrov V G *Phys. Usp.* **54** 573 (2011); *Usp. Fiz. Nauk* **181** 595 (2011)
12. Sirotin Yu I, Shaskolskaya M P *Fundamentals of Crystal Physics* (Moscow: Mir Publ., 1982); Translated from Russian: *Osnovy Kristallofiziki* (Moscow: Nauka, 1979)
13. Bar'yakhtar V G et al. *Sov. Phys. JETP* **60** 587 (1984); *Zh. Eksp. Teor. Fiz.* **87** 1028 (1984)
14. Vendik O G, Chartorizhskii D N *Sov. Phys. Solid State* **12** 1209 (1970); *Fiz. Tverd. Tela* **12** 1538 (1970)
15. Kalinikos B A *Sov. Phys. J.* **24** 718 (1981); *Izv. Vyssh. Uchebn. Zaved. Fiz.* **24** (8) 42 (1981)
16. Zuev A V, Sukstanskii A L *Fiz. Met. Metalloved.* **59** 13 (1985)
17. Tucker J W, Rampton V W *Microwave Ultrasonics in Solid State Physics* (Amsterdam: North-Holland, 1972); Translated into Russian: *Giperzvuk v Fizike Tverdogo Tela* (Moscow: Mir, 1975)
18. Brekhovskikh L M *Waves in Layered Media* (New York: Academic Press, 1960); Translated from Russian: *Volny v Sloistykh Sredakh* (Moscow: Izd. AN SSSR, 1957)
19. Gurevich A G, Melkov G A *Magnetization Oscillations and Waves* (Boca Raton, FL: CRC Press, 1996); Translated from Russian: *Magnitnye Kolebaniya i Volny* (Moscow: Nauka, 1994)
20. Isakovich M A *Obshchaya Akustika* (General Acoustics) (Moscow: Nauka, 1973)

21. Grinchenko V T, Meleshko V V *Garmonicheskie Kolebaniya i Volny v Uprugikh Telakh* (Harmonic Oscillations and Waves in Elastic Bodies) (Kiev: Naukova Dumka, 1981)
22. Azzam S I, Kildishev A V *Adv. Opt. Mater.* **9** 2001469 (2021)
23. Kharitonov A V *Akust. Zh.* **24** 602 (1978)
24. Nowacki V *Teoria Sprężystości* (Warszawa: Państwowe Wydawn. Naukowe, 1970); Translated into Russian: *Teoriya Uprugosti* (Moscow: Mir, 1975)
25. Morse P M, Feshbach H *Methods of Theoretical Physics* Vol. 2 (New York: McGraw-Hill, 1953); Translated into Russian: *Metody Teoreticheskoi Fiziki* Vol. 2 (Moscow: IL, 1960)
26. Vainshtein L A *Elektromagnitnye Volny* (Electromagnetic Waves) (Moscow: Radio i Svyaz', 1988)
27. Demidov V E, Kalinikos B A *Tech. Phys. Lett.* **26** 729 (2000); *Pis'ma Zh. Tekh. Fiz.* **26** (16) 68 (2000)