METHODOLOGICAL NOTES

PACS numbers: 03.50.De, 04.20.Fy, 41.20.Cv

Interaction energy of point electric multipoles

V.P. Kazantsev, O.A. Zolotov, O.P. Zolotova, V.E. Zalizniak

DOI: https://doi.org/10.3367/UFNe.2024.12.039830

Contents

Abstract. Based on various representations of the potential of point basis sources of an electrostatic field, we report the solution to the problem of their interaction energy. The obtained results can be used to approximate the electric field by point multipole fields using the variational principle.

Keywords: point multipole, interaction energy, variational principle

1. Introduction

The authors of this paper have demonstrated the efficiency of approximating the electric field of conductors by fields of point charges [1] and point multipoles [2]. This approximation is based on a variational principle associated with minimizing the electrostatic energy by the values of charges and multipoles, as well as by the coordinates of their localization points.

In implementing this principle, it was necessary to find, in particular, the interaction energy of point multipoles. Note that the potentials of point multipoles of all orders are present in the expansions of potentials in characteristic multipoles of a sphere [3–6]. Representing these potentials in a spherical coordinate system markedly complicates the calculation of the interaction energy of point multipoles. It is also worth noting that the concept of point multipoles was introduced by Maxwell [7].

A complex representation has proven convenient for describing electric potentials and fields on a plane [8]. The magnitude of a point electric charge on a plane (the linear charge density of a uniformly charged straight line) is determined by the real number λ_0, while the nth-order multipole moment of a point multipole is determined by a complex number $\lambda_n = \lambda_{nr} + \mathrm{i}\lambda_{ni}$ ($\lambda_{nr} = \operatorname{Re}\lambda_n; \lambda_{ni} = \operatorname{Im}\lambda_n$).

If the charge and multipoles are located at points with complex coordinates z_0 and z_n, then their complex potentials and electric fields will be defined by the formulas

$$\Pi_0(z, z_0) = -\frac{\lambda_0}{2\pi\varepsilon_0} \ln\left(\frac{z - z_0}{R}\right),$$

$$\Pi_n(z, z_n) = -\frac{\lambda_n}{2\pi\varepsilon_0 n} \frac{1}{(z - z_n)^n},$$

$$E_0^*(z, z_0) = -\Pi_0'(z, z_0) = E_{0x} - \mathrm{i}E_{0y} = \frac{\lambda_0}{2\pi\varepsilon_0(z - z_0)},$$

$$E_n^*(z, z_n) = -\Pi_n'(z, z_n) = E_{nx} - \mathrm{i}E_{ny} = \frac{\lambda_n}{2\pi\varepsilon_0(z - z_0)^{n+1}},$$

where the subscripts x and y denote the projections of the electric field strength onto the corresponding axes, and the prime denotes the derivative with respect to z. The choice of the positive normalization constant R is described in monograph [8].

Complex potential energies of interaction of a point charge $\lambda_0^{(p)}$ with a point charge $\lambda_0^{(q)}$ and point multipole $\lambda_m^{(q)}$ can be found from the relations

$$U_{00}\left(z_0^{(p)}, z_0^{(q)}\right) = -\frac{\lambda_0^{(p)}\lambda_0^{(q)}}{2\pi\varepsilon_0} \ln\left(\frac{z_0^{(p)} - z_0^{(q)}}{R}\right),$$

$$U_{0n}\left(z_0^{(p)}, z_n^{(q)}\right) = \frac{\lambda_0^{(p)}\lambda_n^{(q)}}{2\pi\varepsilon_0 n\left(z_0^{(p)} - z_n^{(q)}\right)^n}.$$

The complex potential energy of interaction of two point multipoles is described by the formula [8]

$$U_{km}\left(z_k^{(p)}, z_m^{(q)}\right) = \frac{(-1)^k}{2\pi\varepsilon_0} \frac{(k + m - 1)!}{k!m!} \frac{1}{\left(z_k^{(p)} - z_m^{(q)}\right)^{k+m}}.$$

V.P. Kazantsev [1], **O.A. Zolotov** [1,*], **O.P. Zolotova** [2], **V.E. Zalizniak** [1]
[1] Siberian Federal University,
prosp. Svobodnyi 79, 660041 Krasnoyarsk, Russian Federation
[2] Reshetnev Siberian State University of Science and Technology,
prosp. imeni gazety Krasnoyarskii Rabochii 31,
660037 Krasnoyarsk, Russian Federation
E-mail: [*] ozolot_@mail.ru

Received 5 July 2024, revised 11 December 2024
Uspekhi Fizicheskikh Nauk **195** (8) 897 – 904 (2025)
Translated by I.A. Ulitkin

The actual potential energies of interaction will coincide with the real parts of the corresponding complex energies.

Note that the relative simplicity of the formulas for interaction energies of point multipoles is explained by their complex representation.

In this paper, based on the representation of the potentials of spatial point multipoles in various forms, we will derive formulas for their interaction energies.

2. Point electric multipoles and their representations in various forms

The concepts of point multipoles were introduced by Maxwell [7]. He defined the electric potential of a multipole of the order k, located at the origin of coordinates, as

$$\varphi^{(k)}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} A_k \frac{(-1)^k}{k!} \Pi_{m=1}^{k}(\mathbf{n}_m\nabla)\frac{1}{r}\,, \quad \mathbf{n}_m^2 = 1\,. \tag{1}$$

According to Maxwell, A_k is the magnitude of the multipole moment, and the direct product $\mathbf{n}_1\mathbf{n}_2\ldots\mathbf{n}_k$ is its 'direction.'

The advantage of such a definition is its invariance with respect to rotations of the coordinate system, as well as the fact that the subsequent multipole moment, the mth one, is constructed as two previous moments, identical in magnitude, but opposite in 'direction,' spaced a small distance in the direction $\mathbf{n}_m$, with a subsequent limiting transition to a point multipole. This method of constructing a point multipole allows an expression to be immediately written for its energy in an external electric field with a potential $\varphi_{\text{out}}(\mathbf{r})$:

$$W^{(k)} = \frac{A_k}{k!}\,\Pi_{m=1}^{k}\,(\mathbf{n}_m\nabla)\varphi_{\text{out}}(\mathbf{r})\Big|_{\mathbf{r}=0}\,. \tag{2}$$

Expression (1) for the electric potential of a point multipole can be reduced to the form

$$\varphi^{(k)}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q_k(x,y,z)}{r^{2k+1}} = \frac{1}{4\pi\varepsilon_0}\frac{Y_k(\mathbf{r}/r)}{r^{k+1}}\,. \tag{3}$$

Here, $Q_k(x,y,z)$ is a homogeneous harmonic polynomial of the degree k of the Cartesian coordinates of point $\mathbf{r}$, and $Y_k(\mathbf{r}/r) = Y_k(\theta,\alpha)$ is the direction function of the radius vector determined, for example, by the angular spherical coordinates. Maxwell called the direction function $Y_k(\theta,\alpha)$ a spherical harmonic of the order k and showed that the number of linearly independent polynomials $Q_k(x,y,z)$ and spherical harmonics $Y_k(\theta,\alpha)$ is equal to $2k+1$; i.e., they form a linear space of dimension $2k+1$. Note that spherical coordinates are related to Cartesian coordinates by the formula

$$\mathbf{r} = (x; y; z) = (r\sin\theta\cos\alpha; r\sin\theta\sin\alpha; r\cos\theta)\,.$$

As basis functions in the space of spherical harmonics $Y_k(\theta,\alpha)$, it is convenient to choose functions

$$y_0^{(n)}(\theta,\alpha) = P_n(\cos\theta)\,,$$
$$y_{mr}^{(n)}(\theta,\alpha) = (-1)^m\sqrt{2\,\frac{(n-m)!}{(n+m)!}}\,P_n^{(m)}(\cos\theta)\cos(m\alpha)\,, \tag{4}$$
$$y_{mi}^{(n)}(\theta,\alpha) = (-1)^m\sqrt{2\,\frac{(n-m)!}{(n+m)!}}\,P_n^{(m)}(\cos\theta)\sin(m\alpha)\,,$$

where

$$P_n(x) = \frac{(-1)^n}{2^n n!}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,(1-x^2)^n\,,$$
$$P_n^{(m)}(x) = (1-x^2)^{m/2}\,\frac{(-1)^{n+m}}{2^n n!}\,\frac{\mathrm{d}^{n+m}}{\mathrm{d}x^{n+m}}\,(1-x^2)^n$$

are polynomials and associated Legendre functions.

Basic spherical harmonics are orthogonal in the sense of the scalar product of functions of points of the sphere:

$$(f,g) = \int fg\,\mathrm{d}\Omega = \int_0^{\pi}\int_0^{2\pi} f(\theta,\alpha)\,g(\theta,\alpha)\sin\theta\,\mathrm{d}\alpha\,\mathrm{d}\theta\,;$$

moreover,

$$\int\left(y_0^{(n)}(\theta,\alpha)\right)^2\mathrm{d}\Omega = \int\left(y_{mr}^{(n)}(\theta,\alpha)\right)^2\mathrm{d}\Omega = \int\left(y_{mi}^{(n)}(\theta,\alpha)\right)^2\mathrm{d}\Omega = \frac{4\pi}{2n+1}\,. \tag{5}$$

Electric potentials of point multipoles located at the origin of coordinates are defined by the functions

$$q_0^{(n)}\varphi_0^{(n)} = \frac{q_0^{(n)}}{4\pi\varepsilon_0}\,\frac{y_0^{(n)}(\theta,\alpha)}{r^{n+1}}\,, \quad q_{mr}^{(n)}\varphi_{mr}^{(n)} = \frac{q_{mr}^{(n)}}{4\pi\varepsilon_0}\,\frac{y_{mr}^{(n)}(\theta,\alpha)}{r^{n+1}}\,,$$
$$q_{mi}^{(n)}\varphi_{mi}^{(n)} = \frac{q_{mi}^{(n)}}{4\pi\varepsilon_0}\,\frac{y_{mi}^{(n)}(\theta,\alpha)}{r^{n+1}}\,. \tag{6}$$

The quantities $q_0^{(n)}$, $q_{mr}^{(n)}$, and $q_{mi}^{(n)}$ are the values of the moments of point multipoles of the corresponding orders.

Note that the charge distributions over the sphere $r = a$ with surface densities

$$q_0^{(n)}\sigma_0^{(n)}(\theta,\alpha) = q_0^{(n)}\,\frac{2n+1}{4\pi a^{n+2}}\,y_0^{(n)}(\theta,\alpha)\,,$$
$$q_{mr}^{(n)}\sigma_{m\gamma}^{(n)}(\theta,\alpha) = q_{mr}^{(n)}\,\frac{2n+1}{4\pi a^{n+2}}\,y_{m\gamma}^{(n)}(\theta,\alpha)\;(\gamma = r, i) \tag{7}$$

will serve as sources of potentials (6) both outside the sphere and

$$q_0^{(n)}\pi_0^{(n)} = \frac{q_0^{(n)}}{4\pi\varepsilon_0}\,\frac{r^n y_0^{(n)}(\theta,\alpha)}{a^{2n+1}}\,, \quad q_0^{(n)}\pi_{m\gamma}^{(n)} = \frac{q_0^{(n)}}{4\pi\varepsilon_0}\,\frac{r^n y_{m\gamma}^{(n)}(\theta,\alpha)}{a^{2n+1}}$$

inside it. Kazantsev [9] proposed calling the set of distributions with basic charge densities $\sigma_0^{(n)}(\theta,\alpha)$ and $\sigma_{m\gamma}^{(n)}(\theta,\alpha)$ the characteristic multipoles of the sphere. In this case, point multipoles can be considered to arise by 'sweeping' charges from the surface of the sphere into the region of a ball bounded by the sphere.

The energies of interaction can be conveniently calculated by representing the basic homogeneous harmonic polynomials in a Cartesian coordinate system:

$$r^n y_0^{(n)}(\theta,\alpha) = \sum_{j=0}^{[n/2]} a_j^{(n0)}(\eta\bar{\eta})^j z^{n-2j}\,,$$
$$r^n y_{mr}^{(n)}(\theta,\alpha) = \frac{1}{2}\,(\eta^m + \bar{\eta}^m)\sum_{j=0}^{[(n-m)/2]} a_j^{(nm)}(\eta\bar{\eta})^j z^{n-m-2j}\,, \tag{8}$$
$$r^n y_{mi}^{(n)}(\theta,\alpha) = \frac{1}{2\mathrm{i}}\,(\eta^m - \bar{\eta}^m)\sum_{j=0}^{[(n-m)/2]} a_j^{(nm)}(\eta\bar{\eta})^j z^{n-m-2j}\,,$$

where $\eta = x + \mathrm{i}y$; $\bar{\eta} = x - \mathrm{i}y$;

$$a_j^{(n0)} = \frac{(-1)^j n!}{2^{2j}(j!)^2(n-2j)!}\,,\ 0 \leqslant j \leqslant \left[\frac{n}{2}\right],$$

$$a_j^{(nm)} = \frac{(-1)^j\sqrt{2(n+m)!(n-m)!)}}{2^{2j+m}j!\,(m+j)!(n-m-2j)!}\,,\quad m \geqslant 1\,,\quad 0 \leqslant j \leqslant \left[\frac{n-m}{2}\right]. \tag{9}$$

We will now determine the electric potentials of point multipoles using differential operators

$$\hat{T}_0^{(n)}(\mathbf{r}') = \frac{\partial_{z'}^n}{n!}\,,\ \hat{T}_{mr}^{(n)}(\mathbf{r}') = \sqrt{\frac{2^{2m-1}}{(n+m)!(n-m)!}}\,\partial_{z'}^{n-m}(\partial_{\eta'}^m + \partial_{\bar{\eta}'}^m), \tag{10}$$

$$\hat{T}_{mi}^{(n)}(\mathbf{r}') = \sqrt{\frac{2^{2m-1}}{(n+m)!(n-m)!}}\,\partial_{z'}^{n-m}\,\frac{1}{\mathrm{i}}\,(\partial_{\bar{\eta}'}^m - \partial_{\eta'}^m)$$

from the formulas

$$q_0^{(n)}\varphi_0^{(n)}(\mathbf{r},\mathbf{r}') = \frac{q_0^{(n)}}{4\pi\varepsilon_0}\,\hat{T}_0^{(n)}(\mathbf{r}')\,\frac{1}{\sqrt{(z-z')^2+(\eta-\eta')(\bar{\eta}-\bar{\eta}')}},$$

$$q_{mr}^{(n)}\varphi_{mr}^{(n)}(\mathbf{r},\mathbf{r}') = \frac{q_{mr}^{(n)}}{4\pi\varepsilon_0}\,\hat{T}_{mr}^{(n)}(\mathbf{r}')\,\frac{1}{\sqrt{(z-z')^2+(\eta-\eta')(\bar{\eta}-\bar{\eta}')}},$$

$$q_{mi}^{(n)}\varphi_{mi}^{(n)}(\mathbf{r},\mathbf{r}') = \frac{q_{mi}^{(n)}}{4\pi\varepsilon_0}\,\hat{T}_{mi}^{(n)}(\mathbf{r}')\,\frac{1}{\sqrt{(z-z')^2+(\eta-\eta')(\bar{\eta}-\bar{\eta}')}}\,. \tag{11}$$

It is assumed here that point multipoles are located at a point with radius vector $\mathbf{r}'$. This form of derivation is due to the fact that the Laplace operator has the form

$$\Delta = 4\partial_\eta\partial_{\bar{\eta}} + \partial_z^2\,;$$

as a result, the harmonic functions vanish. This follows from the equalities

$$\partial_x = \partial_\eta + \partial_{\bar{\eta}} = \hat{T}_{1r}^{(1)}(\mathbf{r})\,,\quad \partial_y = \mathrm{i}\,(\partial_\eta - \partial_{\bar{\eta}}) = \hat{T}_{1i}^{(1)}(\mathbf{r})\,,$$

$$\partial_z = \hat{T}_0^{(1)}(\mathbf{r})\,. \tag{12}$$

The electric field strengths of point multipoles can be found by the action of operators (11) on potentials (10). For example,

$$E_{0x}^{(n)} = -\hat{T}_{1r}^{(1)}(\mathbf{r})\,\varphi_0^{(n)}(\mathbf{r},\mathbf{r}') = \hat{T}_{1r}^{(1)}(\mathbf{r}')\hat{T}_0^{(n)}(\mathbf{r}')\,\varphi_0^{(0)}(\mathbf{r},\mathbf{r}') = \sqrt{\frac{(n+1)}{2n}}\,\hat{T}_{1r}^{(n+1)}(\mathbf{r}')\varphi_0^{(0)}(\mathbf{r},\mathbf{r}') = \sqrt{\frac{(n+1)}{2n}}\,\varphi_{1r}^{(n+1)}(\mathbf{r},\mathbf{r}')\,,$$

$$E_{0y}^{(n)} = -\hat{T}_{1i}^{(1)}(\mathbf{r})\,\varphi_0^{(n)}(\mathbf{r},\mathbf{r}') = \hat{T}_{1i}^{(1)}(\mathbf{r}')\hat{T}_0^{(n)}(\mathbf{r}')\,\varphi_0^{(0)}(\mathbf{r},\mathbf{r}') = \sqrt{\frac{(n+1)}{2n}}\,\hat{T}_{1i}^{(n+1)}(\mathbf{r}')\,\varphi_0^{(0)}(\mathbf{r},\mathbf{r}') = \sqrt{\frac{(n+1)}{2n}}\,\varphi_{1i}^{(n+1)}(\mathbf{r},\mathbf{r}')\,,$$

$$E_{0z}^{(n)} = -\hat{T}_0^{(1)}(\mathbf{r})\,\varphi_0^{(n)}(\mathbf{r},\mathbf{r}') = \hat{T}_0^{(1)}(\mathbf{r}')\hat{T}_0^{(n)}(\mathbf{r}')\,\varphi_0^{(0)}(\mathbf{r},\mathbf{r}') = (n+1)\hat{T}_0^{(n+1)}(\mathbf{r}')\,\varphi_0^{(0)}(\mathbf{r},\mathbf{r}') = (n+1)\,\varphi_0^{(n+1)}(\mathbf{r},\mathbf{r}')\,.$$

Note that the calculation of the energy of a point multipole in an external electric field will be based on relations (8)–(11).

3. Energy of point multipoles in external electric field

Let us find an expression for the energy of interaction of a point multipole with an external electric field. We assume that a point multipole with moment $q_{m\gamma}^{(n)}$ is located at a point with radius vector $\mathbf{r}'$; then, the electric potential of this multipole at a point with radius vector $\mathbf{r}$ can be found using formulas (9) and (10). The interaction energy of a multipole with a charge distribution $\mathrm{d}q(\mathbf{r})$ can be found using the formula

$$W = \int \varphi_{m\gamma}^{(n)}(\mathbf{r},\mathbf{r}')\,\mathrm{d}q\,(\mathbf{r}) = \frac{q_{m\gamma}^{(n)}}{4\pi\varepsilon_0}\,T_{m\gamma}^{(n)}(\mathbf{r}')\int\frac{\mathrm{d}q\,(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|}\,. \tag{13}$$

Taking into account the fact that the potential of an external electric field is defined as

$$\varphi_{\text{out}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\mathrm{d}q\,(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,,$$

we obtain

$$W_0^{(n)} = \int \varphi_0^{(n)}(\mathbf{r})\,\mathrm{d}q\,(\mathbf{r}) = q_0^{(n)}\,\hat{T}_0^{(n)}(\mathbf{r})\,\varphi_{\text{out}}(\mathbf{r})\big|_{\mathbf{r}=\mathbf{r}'} = q_0^{(n)}\,\frac{1}{n!}\,\partial_z^n\,\varphi_{\text{out}}(\mathbf{r})\big|_{\mathbf{r}=\mathbf{r}'}\,,$$

$$W_{mr}^{(n)} = \int \varphi_{mr}^{(n)}(\mathbf{r})\,\mathrm{d}q\,(\mathbf{r}) = q_{mr}^{(n)}\,\hat{T}_{mr}^{(n)}(\mathbf{r})\,\varphi_{\text{out}}(\mathbf{r})\big|_{\mathbf{r}=\mathbf{r}'} = q_{mr}^{(n)}\sqrt{\frac{2^{2m-1}}{(n+m)!(n-m)!}}\,\partial_z^{n-m}(\partial_\eta^m + \partial_{\bar{\eta}}^m)\,\varphi_{\text{out}}(\mathbf{r})\big|_{\mathbf{r}=\mathbf{r}'}\,, \tag{14}$$

$$W_{mi}^{(n)} = \int \varphi_{mi}^{(n)}(\mathbf{r})\,\mathrm{d}q\,(\mathbf{r}) = q_{mi}^{(n)}\hat{T}_{mi}^{(n)}(\mathbf{r})\,\varphi_{\text{out}}(\mathbf{r})\big|_{\mathbf{r}=\mathbf{r}'} = \frac{q_{mi}^{(n)}}{\mathrm{i}}\sqrt{\frac{2^{2m-1}}{(n+m)!(n-m)!}}\,\partial_z^{n-m}\,(\partial_{\bar{\eta}}^m - \partial_\eta^m)\,\varphi_{\text{out}}(\mathbf{r})\big|_{\mathbf{r}=\mathbf{r}'}$$

for the energies of point multipoles in an external electric field with a potential $\varphi_{\text{out}}(\mathbf{r})$. Note that relations (14) for calculating the energy of point multipoles in an external electric field, in fact, extend Maxwell's formula (2) to basic point multipoles.

It is assumed in formulas (14) that $\varphi_{\text{out}}(\mathbf{r})$ should be represented as a function of $z, \eta, \bar{\eta}$:

$$\varphi_{\text{out}}(\mathbf{r}) = \varphi_{\text{out}}(z,\eta,\bar{\eta})\,.$$

For example, the harmonic function

$$\varphi_{\text{out}}(\mathbf{r}) = A\exp(\mathbf{pr})\cos(\mathbf{qr})\,,$$

where A is a constant value, and $\mathbf{p}$ and $\mathbf{q}$ are constant orthogonal vectors with the same modulus, should first be written in the form

$$\varphi_{\text{out}}(\mathbf{r}) = A\,\exp(p_+\bar{\eta} + p_-\eta + p_z z)\,\cos(q_+\bar{\eta} + q_-\eta + q_z z),$$

$$p_+ = \frac{p_x + \mathrm{i}p_y}{2}\,,\quad p_- = \frac{p_x - \mathrm{i}p_y}{2}\,,$$

$$q_+ = \frac{q_x + \mathrm{i}q_y}{2}\,,\quad q_- = \frac{q_x - \mathrm{i}q_y}{2}\,.$$

Then, formulas (14) should be used to calculate the energy of a point multipole in an external electric field.

4. Interaction energy of point multipoles

Let us find the interaction energy of two basic point multipoles using relations (8)–(11) and (14). We choose a coordinate system so that its z-axis connects the first multipole with the second one. We also assume that the axes of this coordinate system will be parallel to the corresponding axes of the accompanying coordinate systems for these multipoles. Then, for the electric potentials of such multipoles of possible orders, we can write

$$\begin{aligned}
\varphi_0^{(1n_1)}(\mathbf{r}) &= \frac{(-1)^{n_1}}{4\pi\varepsilon_0}\, q_0^{(1n_1)}\, \hat{T}_0^{(n_1)}(\mathbf{r})\, \frac{1}{\sqrt{(z-z_1)^2+\eta\bar{\eta}}}\,,\\
\varphi_{m_1\gamma}^{(1n_1)}(\mathbf{r}) &= \frac{(-1)^{n_1}}{4\pi\varepsilon_0}\, q_{m_1\gamma}^{(1n_1)}\, \hat{T}_{m_1\gamma}^{(n_1)}(\mathbf{r})\, \frac{1}{\sqrt{(z-z_1)^2+\eta\bar{\eta}}}\,,\\
\varphi_0^{(2n_2)}(\mathbf{r}) &= \frac{(-1)^{n_2}}{4\pi\varepsilon_0}\, q_0^{(2n_2)}\, \hat{T}_0^{(n_2)}(\mathbf{r})\, \frac{1}{\sqrt{(z-z_2)^2+\eta\bar{\eta}}}\,,\\
\varphi_{m_2\gamma}^{(2n_2)}(\mathbf{r}) &= \frac{(-1)^{n_2}}{4\pi\varepsilon_0}\, q_{m_2\gamma}^{(2n_2)}\, \hat{T}_{m_2\gamma}^{(n_2)}(\mathbf{r})\, \frac{1}{\sqrt{(z-z_2)^2+\eta\bar{\eta}}}\,.
\end{aligned} \tag{15}$$

This takes into account that the coordinates of the point multipoles are $\eta_1 = \eta_2 = 0$.

One can see from relations (8)–(11) and (14) that the interaction energies of the point multipoles in question will be different from zero for multipoles with the same values of γ and $m_2 = m_1 = m \leqslant \max(n_1, n_2)$. In particular, for $m = 0$, we find

$$\begin{aligned}
W_{00}^{(2n_2 1n_1)} &= \frac{(-1)^{n_1}}{4\pi\varepsilon_0}\, q_0^{(n_1)} q_0^{(n_2)}\, \hat{T}_0^{(n_1)}(\mathbf{r})\, \hat{T}_0^{(n_2)}(\mathbf{r})\\
&\times \left.\frac{1}{\sqrt{(z-z_1)^2+\eta\bar{\eta}}}\right|_{\mathbf{r}=\mathbf{r}_2} = \frac{(-1)^{n_1}}{4\pi\varepsilon_0}\, q_0^{(1n_1)} q_0^{(2n_2)}\\
&\times \frac{1}{n_1! n_2!}\, \partial_z^{n_1+n_2} \left.\frac{1}{\sqrt{(z-z_1)^2+\eta\bar{\eta}}}\right|_{\mathbf{r}=\mathbf{r}_2}\\
&= \frac{q_0^{(1n_1)} q_0^{(2n_2)}}{4\pi\eta_0}\, \frac{(-1)^{n_2}(n_1+n_2)!}{n_1! n_2! (z_2-z_1)^{n_1+n_2}|z_2-z_1|}\\
&= \frac{q_0^{(1n_1)} q_0^{(2n_2)}}{4\pi\varepsilon_0}\, Q_0^{(1n_1 2n_2)}(z_2-z_1)\,.
\end{aligned} \tag{16}$$

At $\max(n_1, n_2) \geqslant m > 0$, for the interaction energy of point multipoles we have

$$\begin{aligned}
W_{m\gamma m\gamma}^{(2n_2 1n_1)} &= \frac{(-1)^{n_1}}{4\pi\varepsilon_0}\, q_m^{(n_1)} q_m^{(n_2)}\, \hat{T}_{m\gamma}^{(n_1)}(\mathbf{r})\, \hat{T}_{m\gamma}^{(n_2)}(\mathbf{r})\\
&\times \left.\frac{1}{\sqrt{(z-z_1)^2+\eta\bar{\eta}}}\right|_{\mathbf{r}=\mathbf{r}_2}.
\end{aligned}$$

Noting that

$$\begin{aligned}
\hat{T}_{m\gamma}^{(n_1)}(\mathbf{r})\, \hat{T}_{m\gamma}^{(n_2)}(\mathbf{r}) &= A\partial_z^{n_1+n_2-2m}\left(\pm(\partial_\eta^m \pm \partial_{\bar{\eta}}^m)^2\right)\\
&= A\partial_z^{n_1+n_2-2m}\left(\pm(\partial_\eta^{2m} + \partial_{\bar{\eta}}^{2m} \pm 2\partial_\eta^m \partial_{\bar{\eta}}^m)\right)\\
&= A\partial_z^{n_1+n_2-2m}\left(\pm\left(\partial_\eta^{2m} + \partial_{\bar{\eta}}^{2m} \pm \frac{(-1)^m}{2^{2m-1}}\partial_z^{2m}\right)\right)\\
&= \pm B\hat{T}_{2mr}^{(n_1+n_2)} + \frac{(-1)^m A}{2^{2m-1}}\, \partial_z^{n_1+n_2}\,,
\end{aligned}$$

and taking into account the equalities

$$A = \frac{2^{2m-1}}{\sqrt{(n_1+m)!(n_1-m)!(n_2+m)!(n_2-m)!}}\,, \quad B = \text{const}\,,$$

$$\hat{T}_{2mr}^{(n_1+n_2)} \left.\frac{1}{\sqrt{(z-z_1)^2+\eta\bar{\eta}}}\right|_{\mathbf{r}=\mathbf{r}_2} = 0\,,$$

$$\partial_z^{n_1+n_2} \left.\frac{1}{\sqrt{(z-z_1)^2+\eta\bar{\eta}}}\right|_{\mathbf{r}=\mathbf{r}_2} = \frac{(-1)^{n_1+n_2}(n_1+n_2)!}{(z_2-z_1)^{n_1+n_2}|z_2-z_1|}\,,$$

we obtain

$$\begin{aligned}
W_{m\gamma m\gamma}^{(2n_2 1n_1)} &= \frac{q_{m\gamma}^{(1n_1)} q_{m\gamma}^{(2n_2)}}{4\pi\varepsilon_0}\, Q_m^{(1n_1 2n_2)}(z_2-z_1)\,,\\
Q_m^{(1n_1 2n_2)}(z_2-z_1) &= \frac{(-1)^{n_2+m}(n_1+n_2)!}{\sqrt{(n_1+m)!(n_1-m)!(n_2+m)!(n_2-m)!)}}\\
&\times \frac{1}{(z_2-z_1)^{n_1+n_2}|z_2-z_1|}\,.
\end{aligned} \tag{17}$$

Now, let each of the multipoles be basic in the accompanying coordinate system. The orientation of the axes of the accompanying coordinate systems relative to the systems introduced at the beginning of this section is characterized by the Euler angles $(\alpha_1; \beta_1; \gamma_1)$ and $(\alpha_2; \beta_2; \gamma_2)$ for the first and second multipoles, respectively. Note that the choice of Euler angles here is the same as those in book [11]; therefore, the unit vectors directed from the points of location of the multipoles along the third axes of the accompanying coordinate systems will be

$$\mathbf{k}_p = (\sin\beta_p \cos\alpha_p; \sin\beta_p \sin\alpha_p; \cos\beta_p)\,, \quad p = 1, 2\,.$$

The angles formed by the radius vectors drawn from the point multipoles to the observation point with the z-axis will be denoted by $\theta^{(p)}$, and the angles formed by the projections of these radius vectors on the xy plane with the x-axis will be denoted by ψ.

In the accompanying coordinate systems, the electric potentials of point multipoles of different orders will have the form

$$\varphi_0^{(np)} = \frac{q_0^{(np)}}{4\pi\varepsilon_0}\, \frac{y_0^{(n)}(\theta_p, \psi_p)}{r_p^{n+1}}\,, \quad \varphi_{mr}^{(np)} = \frac{q_{mr}^{(np)}}{4\pi\varepsilon_0}\, \frac{y_{m\gamma}^{(n)}(\theta_p, \psi_p)}{r_p^{n+1}}\,,$$
$$\gamma = r, i\,, \quad p = 1, 2\,. \tag{18}$$

The spherical functions $y_0^{(n)}(\theta_p, \psi_p)$ and $y_{m\gamma}^{(n)}(\theta_p, \psi_p)$ can be expressed through the spherical functions $y_0^{(n)}(\theta^{(p)}, \psi)$ and $y_{m\gamma}^{(n)}(\theta^{(p)}, \psi)$ by means of linear relations

$$\begin{aligned}
y_0^{(n)}(\theta_p, \psi_p) &= y_0^{(n)}(\beta_p, \alpha_p)\, y_0^{(n)}(\theta^{(p)}, \psi)\\
&+ \sum_{m=1}^{n}\sum_{\gamma=r}^{i} y_{m\gamma}^{(n)}(\beta_p, \alpha_p)\, y_{m\gamma}^{(n)}(\theta^{(p)}, \psi)\,,\\
y_{m\delta}^{(n)}(\theta_p, \psi_p) &= H_{m\delta 0}^{(n)}(\alpha_p, \beta_p, \gamma_p)\, y_0^{(n)}(\theta^{(p)}, \psi)\\
&+ \sum_{l=1}^{n}\sum_{\varepsilon=r}^{i} H_{m\delta l\varepsilon}^{(n)}(\alpha_p, \beta_p, \gamma_p)\, y_{l\varepsilon}^{(n)}(\theta^{(p)}, \psi)\,.
\end{aligned} \tag{19}$$

The first equality here expresses the theorem of addition of spherical functions in a form convenient for electrostatics. The functions $H^{(n)}_{m\delta 0}(\alpha_p, \beta_p, \gamma_p)$ and $H^{(n)}_{m\delta l\varepsilon}(\alpha_p, \beta_p, \gamma_p)$ can be expressed through the Wigner functions [11]. The corresponding formulas are given below.

It is evident from equality (18) and the first equality (19) that the problem of the interaction energy of two arbitrarily oriented point multipoles with moments $q_0^{(n1)}$ and $q_0^{(k2)}$ is equivalent to the problem of the interaction of two systems of point multipoles with moments

$$\tilde{q}_0^{(n1)} = q_0^{(n1)} y_0^{(n)}(\beta_1, \alpha_1)\,,\ \ \tilde{q}_{m\gamma}^{(n1)} = q_0^{(n1)} y_{m\gamma}^{(n)}(\beta_1, \alpha_1)\,,\ \ m = 1, \dots n\,, \tag{20}$$

$$\tilde{q}_0^{(k2)} = q_0^{(k2)} y_0^{(k)}(\beta_2, \alpha_2)\,,\ \ \tilde{q}_{m\gamma}^{(k2)} = q_0^{(k2)} y_{m\gamma}^{(k)}(\beta_2, \alpha_2);\ m = 1, \dots k\,.$$

The first and second systems of point multipoles are located, respectively, at points z_1 and z_2 of the z-axis. Moreover, the axes of the accompanying coordinate systems for all multipoles are parallel, and one of these axes is the z-axis.

It follows from relations (16), (17), and (20) that the interaction energy of point multipoles with different subscripts m will be equal to zero. Therefore, the interaction energy of the studied two point multipoles with moments $q_0^{(n1)}$ and $q_0^{(k2)}$ can be written in the form

$$W_{00}^{n1k2}(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, z_2 - z_1) = \frac{q_0^{(n1)} q_0^{(k2)}}{4\pi\varepsilon_0} \times \Bigg(Q_0^{n1k2}(z_2 - z_1) y_0^{(n)}(\beta_1, \alpha_1) y_0^{(k)}(\beta_2, \alpha_2) + \sum_{m=1}^{\min(n,k)} Q_m^{n1k2}(z_2 - z_1) \sum_{\gamma=r}^{i} y_{m\gamma}^{(n)}(\beta_1, \alpha_1) y_{m\gamma}^{(k)}(\beta_2, \alpha_2) \Bigg). \tag{21}$$

Similarly, we find the interaction energy of point multipoles with moments $q_0^{(n1)}$ and $q_{m\gamma}^{(k2)}$:

$$W_{0m\gamma}^{n1k2}(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, z_2 - z_1) = \frac{q_0^{(n1)} q_{m\gamma}^{(k2)}}{4\pi\varepsilon_0} \times \Bigg(Q_0^{n1k2}(z_2 - z_1) y_0^{(n)}(\beta_1, \alpha_1) H_{m\gamma 0}^{(k)}(\alpha_2, \beta_2, \gamma_2) + \sum_{l=1}^{\min(n,k)} Q_l^{n1k2}(z_2 - z_1) \sum_{\varepsilon=r}^{i} y_{l\varepsilon}^{(n)}(\beta_1, \alpha_1) H_{m\gamma l\varepsilon}^{(k)}(\alpha_2, \beta_2, \gamma_2) \Bigg). \tag{22}$$

Finally, for the interaction energy of point multipoles with moments $q_{m\delta}^{(n1)}$ and $q_{p\tau}^{(k2)}$, we have

$$W_{m\delta\, p\tau}^{nk}(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, z_2 - z_1) = \frac{q_{m\delta}^{(n1)} q_{p\tau}^{(k2)}}{4\pi\varepsilon_0} \times \Bigg(Q_0^{n1k2}(z_2 - z_1) H_{m\delta 0}^{(n)}(\alpha_1, \beta_1, \gamma_1) H_{p\tau 0}^{(k)}(\alpha_2, \beta_2, \gamma_2) + \sum_{l=1}^{\min(n,k)} Q_l^{n1k2}(z_2 - z_1) \sum_{\varepsilon=r}^{i} H_{m\delta l\varepsilon}^{(n)}(\alpha_1, \beta_1, \gamma_1) H_{p\tau l\varepsilon}^{(k)}(\alpha_2, \beta_2, \gamma_2) \Bigg). \tag{23}$$

Formulas (22) and (23) take into account that the interaction energy of two point multipoles with moments $q_{m\delta}^{(n1)}$ and $q_{p\tau}^{(k2)}$ in the accompanying coordinate systems will be equal to the interaction energy of two systems of point multipoles located at points z_1 and z_2 of the z-axis in the accompanying coordinate system, possessing moments

$$\begin{gathered} q_{m\delta}^{(n1)} H_{m\delta 0}^{(k)}(\alpha_1, \beta_1, \gamma_1)\,,\quad q_{m\delta}^{(n1)} H_{m\delta l\varepsilon}^{(n)}(\alpha_1, \beta_1, \gamma_1)\,,\\ l = 1, \dots n\,,\quad \varepsilon = r, i\,,\\ q_{p\tau}^{(k2)} H_{p\tau 0}^{(k)}(\alpha_2, \beta_2, \gamma_2)\,,\quad q_{p\tau}^{(k2)} H_{p\tau l\varepsilon}^{(k)}(\alpha_2, \beta_2, \gamma_2)\,,\\ l = 1, \dots k\,,\quad \varepsilon = r, i\,. \end{gathered} \tag{24}$$

We now present expressions for the Euler angle functions $H_{mr0}^{(n)}(\alpha, \beta, \gamma)$ and $H_{m\gamma l\delta}^{(n)}(\alpha, \beta, \gamma)$:

$$\begin{aligned} &H_{mr0}^{(n)}(\alpha, \beta, \gamma) = \sqrt{2}\cos(m\gamma)\, d_{m0}^{n}(\beta)\,,\\ &H_{mi0}^{(n)}(\alpha, \beta, \gamma) = -\sqrt{2}\sin(m\gamma)\, d_{m0}^{n}(\beta)\,,\\ &H_{mr\,lr}^{(n)}(\alpha, \beta, \gamma) = (-1)^l \cos(m\gamma - l\alpha)\, d_{m-l}^{n}(\beta) + \cos(m\gamma + l\alpha)\, d_{ml}^{n}(\beta),\\ &H_{mr\,li}^{(n)}(\alpha, \beta, \gamma) = (-1)^l \sin(l\alpha - m\gamma)\, d_{m-l}^{n}(\beta) + \sin(m\gamma + l\alpha)\, d_{ml}^{n}(\beta),\\ &H_{mi\,lr}^{(n)}(\alpha, \beta, \gamma) = (-1)^l \sin(l\alpha - m\gamma)\, d_{m-l}^{n}(\beta) - \sin(m\gamma + l\alpha)\, d_{ml}^{n}(\beta),\\ &H_{mi\,li}^{(n)}(\alpha, \beta, \gamma) = \cos(m\gamma + l\alpha)\, d_{ml}^{n}(\beta) - (-1)^l \cos(m\gamma - l\alpha)\, d_{m-l}^{n}(\beta). \end{aligned} \tag{25}$$

In monograph [11], several methods are proposed for calculating the coefficients $d_{ml}^{n}(\beta)$. In particular, they can be expressed through the Jacobi polynomials $P_s^{(\mu,\nu)}(\cos\beta)$:

$$d_{ml}^{n}(\beta) = \xi_{ml} \left(\frac{s!(s+\mu+\nu)!}{(s+\mu)!(s+\nu)!} \right)^{1/2} \sin^{\mu}\left(\frac{\beta}{2}\right) \cos^{\nu}\left(\frac{\beta}{2}\right) P_s^{(\mu,\nu)}(\cos\beta), \tag{26}$$

$$\mu = |m - l|\,,\quad \nu = |m + l|\,,\quad s = n - \frac{1}{2}(\mu + \nu)\,,$$

$$\xi_{ml} = \begin{cases} 1 & \text{at} \quad l \geqslant m\,,\\ (-1)^{l-m} & \text{at} \quad l < m\,. \end{cases}$$

Note also that

$$P_s^{(\mu,\nu)}(x) = \frac{(-1)^s}{2^s s!} (1-x)^{-\mu}(1+x)^{-\nu} \frac{\mathrm{d}^s}{\mathrm{d}x^s} \left[(1-x)^{s+\mu}(1+x)^{s+\nu} \right].$$

Here, the parameters μ and ν take nonnegative integer values. In this case, when calculating $P_s^{(\mu,\nu)}(x)$, it is convenient to use the formula

$$P_s^{(\mu,\nu)}(x) = \frac{(s+\mu)!(s+\nu)!}{s!2^s} \sum_{k=0}^{s} \frac{C_s^k (x-1)^{s-k}(1+x)^k}{(s-k+\mu)!(k+\nu)!}\,. \tag{27}$$

Specific expressions for the transformation coefficients (25) are rather cumbersome. Let us give as an example the formulas that determine these coefficients for $n = 0, 1, 2, 3$:

$$\begin{aligned} &H_{00}^{(0)}(\alpha, \beta, \gamma) = 1\,,\\ &H_{00}^{(1)}(\alpha, \beta, \gamma) = \cos\beta\,,\quad H_{01r}^{(1)}(\alpha, \beta, \gamma) = \sin\beta\cos\alpha\,,\\ &H_{01i}^{(1)}(\alpha, \beta, \gamma) = \sin\beta\sin\alpha\,,\\ &H_{1r0}^{(1)}(\alpha, \beta, \gamma) = -\sin\beta\cos\gamma\,,\\ &H_{1r1r}^{(1)}(\alpha, \beta, \gamma) = \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma\,,\\ &H_{1r1i}^{(1)}(\alpha, \beta, \gamma) = \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma\,,\\ &H_{1i0}^{(1)}(\alpha, \beta, \gamma) = \sin\beta\sin\gamma\,,\\ &H_{1i1r}^{(1)}(\alpha, \beta, \gamma) = -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma\,,\\ &H_{1i1i}^{(1)}(\alpha, \beta, \gamma) = \cos\alpha\cos\gamma - \sin\alpha\cos\beta\sin\gamma\,. \end{aligned}$$

5. Interaction energy of charge systems distributed over two nonintersecting balls

As an application of the obtained results, we calculate the interaction energy of two charge distributions over the regions of two nonintersecting balls.

We find the interaction energy of charges $dq(\mathbf{r})$, distributed over the region of the ball $r < a$, with an external electric field with a potential $\varphi_{\text{out}}(\mathbf{r})$, represented in the region of the ball by a harmonic function. It can be written inside the ball as a series of harmonic polynomials

$$\varphi_{\text{out}}(\mathbf{r}) = \sum_{n=0}^{\infty} r^n \left(\delta_0^{(n)} y_0^{(n)}(\theta, \alpha) + \sum_{m=1}^{n} \left(\delta_{mr}^{(n)} y_{mr}^{(n)}(\theta, \alpha) + \delta_{mn}^{(n)} y_{mi}^{(n)}(\theta, \alpha) \right) \right)$$

with coefficients

$$\delta_0^{(n)} = \frac{\left(\varphi_{\text{out}}(\mathbf{r}), r^n y_0^{(n)}(\theta, \alpha)\right)}{\left(r^n y_0^{(n)}(\theta, \alpha), r^n y_0^{(n)}(\theta, \alpha)\right)},$$

$$\delta_{m\gamma}^{(n)} = \frac{\left(\varphi_{\text{out}}(\mathbf{r}), r^n y_{m\gamma}^{(n)}(\theta, \alpha)\right)}{\left(r^n y_{m\gamma}^{(n)}(\theta, \alpha), r^n y_{m\gamma}^{(n)}(\theta, \alpha)\right)}.$$

Here, we use the notation

$$\left(f(\mathbf{r}), g(\mathbf{r})\right) = \int_{r<a} f(\mathbf{r})\, g(\mathbf{r})\, dV.$$

Then, for the interaction energy of the charges of the ball with the external field, we will have

$$W = \int \varphi_{\text{out}}(\mathbf{r})\, dq(\mathbf{r}) = \sum_{n=0}^{\infty} \left(\delta_0^{(n)} q_0^{(n)} + \sum_{m=1}^{n} \left(\delta_{mr}^{(n)} q_{mr}^{(n)} + \delta_{mi}^{(n)} q_{mi}^{(n)} \right) \right).$$

To calculate the coefficients $\delta_0^{(n)}$ and $q_{m\gamma}^{(n)}$, we note that the quantities

$$\rho_0^{(n)} = \frac{r^n y_0^{(n)}(\theta, \alpha)}{\left(r^n y_0^{(n)}(\theta, \alpha), r^n y_0^{(n)}(\theta, \alpha)\right)},$$

$$\rho_{m\gamma}^{(n)} = \frac{r^n y_{m\gamma}^{(n)}(\theta, \alpha)}{\left(r^n y_{m\gamma}^{(n)}(\theta, \alpha), r^n y_{m\gamma}^{(n)}(\theta, \alpha)\right)}$$

can be interpreted as the densities of charges that generate potentials of point multipoles outside the ball, localized in the center of the ball with unit moments, which interact with the external field. When calculating their interaction energies, relations (14) should be used.

If $\varphi_{\text{out}}(\mathbf{r})$ is the potential of the charges of another ball, represented outside the ball by the sum of the potentials of point multipoles localized at its center, then the interaction energy of the charges of two nonintersecting balls will be equal to the sum of the interaction energies of point multipoles localized at the centers of the balls with moments determined by the conventional method.

6. Example

As an example of using the interaction energies of point multipoles, we present a variational solution to the problem of two conducting balls with identical potentials in a uniform electric field directed along the line connecting their centers.

Let the centers of the balls lie on the z-axis. The center of the first ball of radius a_1 coincides with the origin of coordinates, and the center of the second ball of radius a_2 is shifted along the z-axis by a distance $b \geqslant a_1 + a_2$. We consider each ball in spherical coordinate systems with polar and azimuthal angles $(\theta_k; \varphi)\, k = 1, 2$.

Let E be the value of the external electric field directed along the z-axis. The potentials of charges induced on the balls by an external field are approximated by a superposition of the potentials of the characteristic multipoles of the balls:

$$\varphi_N(\mathbf{r}) = \sum_{k=1}^{2} \sum_{n=0}^{N} q_{k0}^{(n)}\, \pi_0^{(n)}(\theta_k). \tag{28}$$

For the intrinsic energies of the charges of the balls, we have

$$W_k = \frac{1}{8\pi\varepsilon_0 a_k} \sum_{n=0}^{N} \left(\frac{q_{k0}^{(n)}}{a_k^n} \right)^{2n}. \tag{29}$$

For the interaction energy of the charges of the balls, we use relation (17) to find that

$$W_{12} = \frac{1}{4\pi\varepsilon_0 b} \sum_{n_1=0}^{N} \sum_{n_2=0}^{N} (-1)^{n_2} \frac{(n_1+n_2)!}{n_1! n_2!} \frac{a_1^{n_1} a_2^{n_2}}{b^{n_1+n_2}} \frac{q_{10}^{(n_1)}}{a_1^{n_1}} \frac{q_{20}^{(n_2)}}{a_2^{n_2}}. \tag{30}$$

The interaction energy of the charges of the balls with the external field will be equal to

$$W_e = -\left(q_{10}^{(1)} + q_{20}^{(1)}\right) E - q_{20}^{(0)}\, bE. \tag{31}$$

Taking into account the law of conservation of charge, i.e., $q_{10}^{(0)} = -q_{20}^{(0)} = -q$, it is convenient to represent the energy of the induced charges in the form

$$W = W_1^0 + W_2^0 + W_{12}^0 + W_1 + W_2 + W_{12} + W_e.$$

In turn, we represent the energy of the induced charges in the matrix form, introducing the notation

$$\begin{aligned}
&\frac{q_{k0}^{(n)}}{a_k^n} = Q_k^{(n)}, \quad \mathbf{Q}_k = \left(Q_k^{(1)}, Q_k^{(2)}, \ldots, Q_k^{(N)}\right), \quad \mathbf{c}_k = \left(\frac{a_k}{b}, 0, \ldots, 0\right), \\
&A_{ij} = (-1)^j \frac{(i+j)!}{i!\, j!} \frac{a_1^i a_2^j}{b^{i+j}}, \\
&(\mathbf{a}_1)_i = A_{i0}, \quad (\mathbf{a}_2)_i = A_{0i}, \\
&W_1^0 + W_2^0 = \frac{1}{4\pi\varepsilon_0 b} \left(\frac{b}{2a_1} + \frac{b}{2a_2} \right) q^2, \\
&W_{12}^0 = -\frac{1}{4\pi\varepsilon_0 b} q^2 + \frac{q}{4\pi\varepsilon_0 b} \mathbf{a}_1 \mathbf{Q}_1 - \frac{q}{4\pi\varepsilon_0 b} \mathbf{a}_2 \mathbf{Q}_2, \\
&W_k = \frac{1}{4\pi\varepsilon_0 b} \frac{b}{2a_k} \mathbf{Q}_k^{\mathrm{T}} \hat{e} \mathbf{Q}_k, \quad W_{12} = \frac{1}{4\pi\varepsilon_0 b} \mathbf{Q}_1^{\mathrm{T}} \hat{A} \mathbf{Q}_2, \\
&W_e = -(\mathbf{c}_1 \mathbf{Q}_1 + \mathbf{c}_2 \mathbf{Q}_2)\, bE - qbE,
\end{aligned} \tag{32}$$

where $k = 1, 2$, $\hat{e}$ is the identity matrix of the order N.

Then,

$$W = \frac{1}{4\pi\varepsilon_0 b}\left(\frac{b}{2a_1} + \frac{b}{2a_2} - 1\right)q^2 - qbE$$
$$+ \frac{q}{4\pi\varepsilon_0 b}\left(\mathbf{a}_1\mathbf{Q}_1 - \mathbf{a}_2\mathbf{Q}_2\right)$$
$$+ \frac{1}{4\pi\varepsilon_0 b}\left(\frac{b}{2a_1}\,\mathbf{Q}_1^{\mathrm{T}}\hat{e}\,\mathbf{Q}_1 + \frac{b}{2a_2}\,\mathbf{Q}_2^{\mathrm{T}}\hat{e}\,\mathbf{Q}_2 + \mathbf{Q}_1^{\mathrm{T}}\hat{A}\mathbf{Q}_2\right)$$
$$- \left(\mathbf{c}_1\mathbf{Q}_1 + \mathbf{c}_2\mathbf{Q}_2\right)bE\,.$$

Minimizing the obtained expression for the energy with respect to q, $\mathbf{Q}_1$, and $\mathbf{Q}_2$, we finally obtain

$$W = -\frac{2\pi\varepsilon_0 b^3 E^2}{(b/a_1 + b/a_2 - 2)}\,(1 + \mathbf{Z}\,\mathbf{X})\,, \tag{33}$$

where $\mathbf{Z}$ and $\mathbf{X}$ are vectors of the order $2N$ and

$$\mathbf{Z} = \begin{pmatrix} \left(\dfrac{b}{a_1} + \dfrac{b}{a_2} - 2\right)\mathbf{c}_1 - \mathbf{a}_1 \\ \left(\dfrac{b}{a_1} + \dfrac{b}{a_2} - 2\right)\mathbf{c}_2 + \mathbf{a}_2 \end{pmatrix},$$

$\mathbf{X}$ is the solution to the equation

$$\hat{B}\mathbf{X} = \mathbf{Z}\,,$$

and $\hat{B}$ is a square matrix of the order $2N$,

$$\hat{B} = \begin{pmatrix} \left(\dfrac{b}{a_1}+\dfrac{b}{a_2}-2\right)\dfrac{b}{a_1}\,\hat{e} - \mathbf{a}_1\otimes\mathbf{a}_1^{\mathrm{T}} & \left(\dfrac{b}{a_1}+\dfrac{b}{a_2}-2\right)\hat{A} + \mathbf{a}_1\otimes\mathbf{a}_2^{\mathrm{T}} \\ \left(\dfrac{b}{a_1}+\dfrac{b}{a_2}-2\right)\hat{A}^{\mathrm{T}} + \mathbf{a}_2\otimes\mathbf{a}_1^{\mathrm{T}} & \left(\dfrac{b}{a_1}+\dfrac{b}{a_2}-2\right)\dfrac{b}{a_2}\,\hat{e} - \mathbf{a}_2\otimes\mathbf{a}_2^{\mathrm{T}} \end{pmatrix},$$

with the $\otimes$ sign denoting the tensor product.

As a numerical example, let us consider the case of contact between two balls of the same radius R. The exact solution to this problem is given, for example, in [12]. The results of calculations using the above formulas are presented in the table. The error Δ with respect to the exact solution, which is expressed through the Riemann zeta function,

$$-\frac{W}{8\pi\varepsilon_0 R^3 E^2} = \sum_{n=1}^{\infty}\frac{1}{n^3} = \zeta(3) \approx 1.20205690\,,$$

is also given in [12].

The rapid convergence of the method is clearly visible from the table.

Table. Results of numerical calculations of energy of two identical touching conducting balls in external field directed along line connecting their centers.

N	$-\dfrac{W}{8\pi\varepsilon_0 R^3 E^2}$	Δ, %
0	1	17
1	1.200000000	0.17
2	1.201834862	0.02
3	1.202041429	0.001
4	1.202056396	0.00004
5	1.202056654	0.00002

7. Conclusions

The results obtained in this paper allow us to find the interaction energy of two basic point multipoles with their arbitrary orientation relative to each other. The proposed work can also be considered the basis for the section "Point sources of the electric field" in textbooks covering such topics as electricity and magnetism.

References

1. Kazantsev V P *Phys. Usp.* **45** 325 (2002); *Usp. Fiz. Nauk* **172** 357 (2002)
2. Kazantsev V P, Zolotov O A, Dolgopolova M V *Phys. Usp.* **49** 517 (2006); *Usp. Fiz Nauk* **176** 537 (2006)
3. Landau L D, Lifshitz E M *The Classical Theory of Fields* (Oxford: Pergamon Press, 1975); Translated from Russian: *Teoriya Polya* (Moscow: Nauka, 1967)
4. Batygin V V, Toptygin I N *Problems in Electrodynamics* (London: Academic Press, 1978); Translated from Russian: *Sbornik Zadach po Elektrodinamike* (Moscow: Nauka, 1970)
5. Jackson J D *Classical Electrodynamics* (New York: J. Willey, 1962); Translated into Russian: *Klassicheskaya Elektrodinamika* (Moscow: Mir, 1965)
6. Panofsky W K H, Phillips M *Classical Electricity and Magnetism* (Cambridge, MA: Addison-Wesley Publ. Co., 1955); Translated into Russian: *Klassicheskaya Elektrodinamika* (Moscow: Fizmatgiz, 1963)
7. Maxwell J C A *Treatise on Electricity and Magnetism* Vol. 1 (London: Oxford Univ. Press, 1873); Translated into Russian: *Traktat ob Elektrichestve i Magnetizme* Vol. 1 (Moscow: Nauka, 1989)
8. Kazantsev V P *Analiticheskaya Elektrostatika na Ploskosti* (Analytical Electrostatics on a Plane) Pt. 1 (Krasnoyarsk: SFU, 2008)
9. Kazantsev V P *Theor. Math. Phys.* **119** 750 (1999); *Teor. Mat. Fiz.* **119** 441 (1999)
10. Sobolev S L *Uravneniya Matematicheskoi Fiziki* (Equations of Mathematical Physics) (Moscow: Nauka, 1966) p. 417
11. Varshalovich D A, Moskalev A N, Khersonskii V K *Quantum Theory of Angular Momentum* (Singapore: World Scientific, 1988); Translated from Russian: *Kvantovaya Teoriya Uglovogo Momenta* (Leningrad: Nauka, 1975)
12. Kazantsev V P *J. Commun. Technol. Electron.* **47** 48 (2002); *Radiotekh. Elektron.* **47** 55 (2002)