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Intermittency in random flows and stochastic integrals of motion*

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Contents

Abstract. This review presents recent advances in the study of frozen-in material lines and surfaces evolving in random flows. A remarkable feature of this process is the formation of long-lived coherent structures on surfaces, which are associated with certain stochastic integrals of motion. While the exact form of these integrals depends on flow properties, they become universal under the condition of local isotropy. We derive these integrals explicitly and discuss the elegant mathematics underlying this phenomenon.

Keywords: turbulence, advection, dynamo theory, random matrices

1. Introduction

Consider a space filled with a continuous fluid medium. Assume that the particles in this medium move along random but smooth trajectories that do not intersect, and that the velocities of the 'fluid particles' represent a smooth random vector field. This construction is known as a smooth stochastic flow.

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The concept of stochastic flow initially appeared in the theory of turbulence, the rigorous formulation of which is, according to numerous authors, the last unsolved problem of classical physics [1]. In this case, the properties of the flow are defined by the transfer (cascade) of energy from the integral scale to smaller scales, down to the dissipation scale. A full solution to the problem would entail finding the probability measure in the space of velocity field realizations, i.e., a full statistical description of the flow.

Another important example in which the concept of stochastic flow plays a key role is the turbulent transport theory. Here, the velocity field is considered a given stochastic field, and the evolution of auxiliary passive (i.e., not affecting the flow) fields and structures, transported by the flow, is studied. The significance of such models for the theory of turbulence is due to the following, for example. They have helped us to obtain theoretical proofs of the existence of anomalous scaling for the structure functions of the transported fields, showing that this is not connected at all to the turbulent flow cascade [2, 3].

However, it is important not to think of turbulent transport as merely a testing ground for turbulence theory. The theory of turbulent transport is used to study tracer spreading in the atmosphere, and it has important applications in chemistry [4, 5], biology [6–8], and astrophysics [9–11], where it is used to explain the nature of the generation of small-scale magnetic fields in stars, galaxies, their clusters, etc. [12–22].

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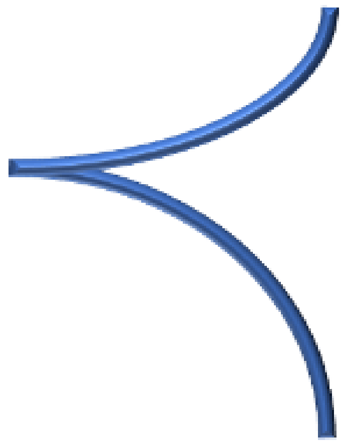

Figure 1. Typical fold on one-dimensional line.

1.1 Evolution of frozen-in structures

As is well known, these objects are composed of plasma, which can be described within the framework of magnetohydrodynamics [23]. At the initial stage of evolution, it can be assumed that the magnetic field is weak and does not act on the medium. The evolution of the magnetic field in these conditions is known as a kinematic dynamo.

In the case of infinite conductivity (zero diffusivity) and a smooth velocity field, the magnetic field evolves in accordance with the Friedman theorem [24]: [1] the lines of magnetic induction are frozen in the flow, and the length of the frozen-in line increases exponentially, as does on average the tangent vector of magnetic induction. The distance between individual pairs of points beyond the viscous sublayer grows only according to a power law [26]. Each line therefore bends multiple times as it is stretched, becoming 'crumpled,' and developing folds (Fig. 1).

This is apparently not only related to magnetic field lines, but to any line frozen in the flow. The same can be said about frozen-in material surfaces (films): their mean area grows exponentially over time, whereas the volume they occupy grows only according to a power law. [2] In the most general terms, this is valid for hyper-surfaces of any dimensionality in the d-dimensional space, $d > 1$. Repeated bending and 'crumpling' of hyper-surfaces, accompanied by folding, is, in the words of the authors of Ref. [27] "a kind of geometrical window into the complex picture of stochastic mixing in random media." Many studies have been devoted to the properties of this process and the structures that evolve within it [28–38], including this review.

As we are dealing with a stochastic flow, the formation of folds must be considered within a probabilistic framework. However, equilibrium thermodynamic fluctuation theory is not applicable here. To demonstrate this, we observe that classical thermodynamics deals with additive systems, in which, by virtue of the central limit theorem, the fluctuations are Gaussian and small in the thermodynamic limit. However, mixing is a multiplicative process in the sense that the motion of frozen-in material structures results from a sequence of random mappings. It should be noted that such processes are not rare; many natural, economic, and sociological phenomena are multiplicative: their observables are not the sums of independent random values, but their products [39]. A characteristic feature of such products is that their statistical moments are defined by fluctuations that are exponentially rare, but relatively long-lived. These fluctuations are beyond the framework of the central limit theorem (CLT). Therefore, they cannot be described by the traditional thermodynamic theory. Since classical review [40], such fluctuations have been referred to as 'coherent structures,' and the phenomenon as 'intermittency.'

In the mixing problem we consider, rare fluctuations are realized, first, in folds, where the surface density increases locally, and second, in rare regions where stretching is much stronger than average. Therefore, the dynamics of stretching and folding are also the 'geometrical window' into the mechanism responsible for the formation of coherent structures in multiplicative systems.

1.2 Mathematical apparatus

In the context of probability theory, the phenomenon of intermittency is described by Cramér's theorem of 'large deviations' [41], which is the generalization of the standard law of large numbers (LLN) and the CLT. The main question in the theory of large deviations [43, 44] is the connection between the ergodic mean (i.e., time mean) $\bar{\xi}(T) = 1/T\int_0^T \xi(t)\,dt$ and the mathematical expectation $\langle\xi\rangle$ (i.e., ensemble mean) for a given stationary random process $\xi(t)$. Cramér showed that, under certain rather natural conditions (essentially, the process correlation time must be finite), the probability density of the ergodic mean is

$$\mathscr{P}(\bar{\xi}) \sim \exp\left(-TJ(\bar{\xi})\right),$$

where J is the Cramér function (rate function), which is convex with a regular minimum at the point $\bar{\xi} = \langle\xi\rangle$.

This fact implies ergodicity, i.e., the convergence in probability of the ergodic mean to the expectation as $T \to \infty$ (an analog of the LLN) and the Gaussianity of small deviations of $\bar{\xi}(T)$ from $\langle\xi\rangle$ (the CLT). However, for us, it is important that Cramér's function also enables the estimation of the probability of exponentially rare non-Gaussian fluctuations when, for a large but finite T, the difference $\bar{\xi}(T) - \langle\xi\rangle$ is not small. We reiterate that, in systems with multiplicative noise, it is these fluctuations that define the observed quantities. [3]

Returning to the local description of the transport of frozen-in structures, we note that this can be reduced to the study of the evolution of an infinitesimal frozen-in parallelepiped subjected to random linear transformations as it moves in a flow. Due to the noncommutativity of these transformations, the description is far from trivial and relies on such fields of mathematics as the theory of random matrices [46] and stochastic matrix equations [47, 48].

The motion of an infinitesimal frozen-in parallelepiped located at the point $\mathbf{x}$ at the initial time t_0 is described by a

[1] Alfvén later showed how this theorem is realized in ideal MHD [25].

[2] For example, interfaces between different phases, thin layers of a nondiffusive passive tracer, and other material structures can play the role of such films.

[3] It is worth mentioning that the large deviation theory in its classical form is a literal repetition of the effective action formalism in quantum field theory (QFT) [45], but applied to a $(1+0)$-dimensional random processes. In this case, the Cramér function appears to be the effective potential.

linear evolution operator $Q(t, t_0; \mathbf{x})$. This operator acts on the tangent spaces of the d-dimensional Euclidian space $\mathbb{E}^d$ (flow differential), and its matrix is the usual Jacobi matrix of the mapping generated by the flow.

Due to multiplicativity, the evolution operator satisfies the standard group relationships

$$Q(t, t_0, \mathbf{x}) = Q(t, t_1, \mathbf{x})\, Q(t_1, t_0, \mathbf{x}), \quad \forall t, t_1, t_0.$$

Therefore, it obeys the linear matrix equation (to be derived further)

$$\frac{\partial}{\partial t} Q(t, t_0, \mathbf{x}) = \mathscr{A}(t, t_0, \mathbf{x})\, Q(t, t_0, \mathbf{x}), \quad Q(t_0, t_0, \mathbf{x}) = 1\,, \tag{1}$$

where $\mathscr{A}(t, t_0, \mathbf{x})$ is the tensor of flow velocity gradients taken at the trajectory of the frozen-in particle (the Lagrangian tensor of gradients) that starts at x. As is well known, $\mathscr{A}(t, \mathbf{x})$ is a stationary isotropic matrix process for incompressible, isotropic, and stationary flows.

A formal solution to problem (1) is well known: the continual product of matrices, known as the multiplicative Volterra integral [49], or T-exponent,[4]

$$Q(t, t_0, \mathbf{x}) = \mathscr{T} \exp\left\{\int_{t_0}^{t} \mathscr{A}(\tau, t_0, \mathbf{x})\, d\tau\right\} = \prod_{\tau=t_0}^{t} \big(1 + \mathscr{A}(\tau, t_0, \mathbf{x})\, d\tau\big).$$

In the context of a description of film transport, we will be interested in the joint evolution of the edges, faces, and higher-dimensional hyperfaces of the frozen-in parallelepiped, each contained in the next. In mathematics, this sequence is referred to as a flag. The squares of the areas of the flag elements s_k^2 are the leading principal minors of the Gram matrix $Q^T Q$.

The main quantities needed to describe the stochastic evolution of a frozen flag are its correlators

$$\left\langle s_1^{m_1}(t) \ldots s_d^{m_d}(t) \right\rangle.$$

Since matrix products are noncommutative, deriving a closed expression for the characteristic correlators is a rather complicated task, even if the statistics of the tensor of gradients are given.

A simple consideration enables us to make an informed judgement about the character of continuous flag evolution. If the evolution time is divided into a large number N of equal, nonintersecting intervals that are, nevertheless, larger than the characteristic correlation time of the flow, the multiplicative Volterra integral of the stationary random matrix process $\mathscr{A}(t)$ can be written approximately as a product of independent identically distributed random matrices, equal to the Volterra integral over each interval,

$$Q(t) = \prod_{k=1}^{N} Q_k\,, \quad \text{where} \quad Q_k = \prod_{\tau=t_{k-1}}^{t_k} \big(1 + \mathscr{A}(\tau)\, d\tau\big).$$

The description of the limit properties of such products is a classical mathematical problem, as considered by G. Ferstenberg [50] and V. Tubatulin [51], for example. In particular, they proved a 'noncommutative law of large numbers,' which asserts that, with probability one, there

[4] When introducing products using the symbol $\prod$, we will always assume that the noncommuting factors are ordered from right to left.

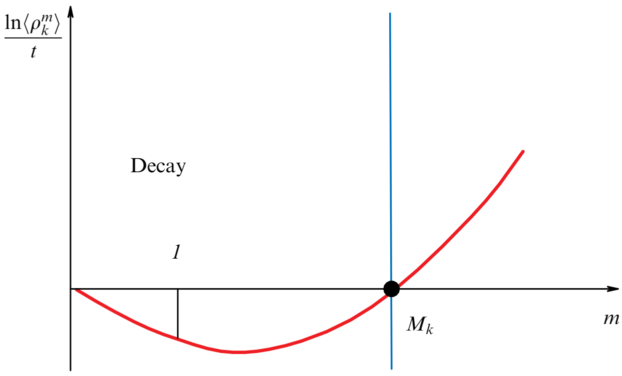

Figure 2. Growth rates of moments of k-dimensional areas.

exists a set of nonrandom numbers

$$\lambda_k = \lim_{N\to\infty} \frac{1}{N} \ln \frac{s_k(N)}{s_{k-1}(N)}\,,$$

called the Lyapunov spectrum. Thus, the following asymptotic relationships are valid for the hyper-surfaces:

$$\lim_{N\to\infty} \frac{1}{N} \ln s_k(N) = \lambda_1 + \ldots + \lambda_k\,.$$

We will further demonstrate that, for incompressible flows,

$$\lambda_1 + \ldots + \lambda_k > 0\,, \quad k < d\,, \quad \lambda_1 + \ldots + \lambda_d = 0\,,$$

so that the mean logarithms of the face areas of the frozen-in flag increase. However, there are always rare trajectories along which they decay relatively slowly. In geometrical terms, this implies that folds continuously evolve on the film. The Lyapunov spectrum does not carry information about them; obtaining it requires knowledge of the details of hyper-area statistics.

Thus, against the background of regular exponential growth in the total area, the formation of folds is accompanied by an exponential decrease in the area of the frozen surface (or the line length) where they are located. This effect can be conveniently visualized by 'painting' the initial film and observing how the paint surface density ρ_k changes at each point, $\rho_k = 1/s_k$, where k is the film dimensionality[5] [52, 53].

Due to exponential stretching, the paint density decreases in most places (the surface becomes pale). In contrast, the density increases at the folds. This implies that sufficiently high moments are 'seeded' in these folds and grow there. Speaking more precisely, there are always numbers $M_k > 0$ such that

$\langle\rho_k^m\rangle$ decrease exponentially for $0 < m < M_k$,

$$\langle\rho_k^{M_k}\rangle = \text{const}\,,$$

$\langle\rho_k^m\rangle$ increase exponentially for $m > M_k$.

Moments $\langle\rho_k^{M_k}\rangle$ that stay constant are called stochastic motion integrals; the value of M_k characterizes the rate of fold formation (Fig. 2). The absence of intermittency and folds corresponds to the case $M_k = \infty$.

[5] Note that, if the hyper-surface dimension k equals the space dimension d, the 'paint' density coincides with the 'fluid' density and remains unchanged in incompressible flows. For hyper-surfaces of smaller dimensions, the density can change even in incompressible flows.

We stress that, in a general case, the statistical integrals are governed by the flow statistics, and their rigorous calculation is practically unattainable. Fortunately, additional flow symmetries allow significant progress to be made towards solving this problem, and in this review we present an explicit and universal answer for isotropic flows.

After this brief introduction, we will proceed with rigorous definitions and a detailed description of the evolution of material films in smooth flows.

2. Basic definitions

2.1 Smooth flow

Let E^d be a real d-dimensional Euclidian space, and $\mathbf{u}(t,\mathbf{r})$ be a differentiable time-dependent vector field on E^d.

Differentiability means that continuous derivatives exist at all times and at all space points

$$A_{ij}(t,\mathbf{r}) = \frac{\partial}{\partial r_j}\, u_i\,(t,\mathbf{r})$$

which form the field of velocity gradients.

Nondivergent fields are an important particular case; for them,

$$\partial_i\, u_i(t,\mathbf{r}) = \operatorname{tr} A\,(t,\mathbf{r}) = 0\,.$$

The field $\mathbf{u}(t,\mathbf{r})$ spawns in a natural way a two-parametric family $\mathscr{L}_{t,s}$ of diffeomorphisms of E^d in itself,

$$\mathscr{L}_{t,t_0}: E^d \to E^d\,,\ \mathbf{x} \mapsto \mathscr{L}_{t,t_0}\,\mathbf{x} = \mathbf{r}(t,t_0;\mathbf{x})\,, \tag{2}$$

such that $\mathbf{r}(t,t_0;\mathbf{x})$ is defined by the Cauchy problem

$$\partial_t \mathbf{r}(t,t_0;\mathbf{x}) = \mathbf{u}\big(\mathbf{r}(t,t_0;\mathbf{x}),t\big)\,,\ \mathbf{r}(t_0,t_0;\mathbf{x}) = \mathbf{x}\,. \tag{3}$$

The mappings $\mathscr{L}_{t,t_0}$ can be interpreted as the motion t_0 by their initial coordinate $\mathbf{x}$ and moving along their trajectories $\mathbf{r}(t,t_0;\mathbf{x})$.

The family of diffeomorphisms $\mathscr{L}_{t,t_0}$ is called a smooth flow, and $\mathbf{u}(t,\mathbf{r})$ is called its generator.

Obviously,

$$\mathscr{L}_{t,t} = I^d\,,\quad \mathscr{L}_{t,t_1}\circ\mathscr{L}_{t_1,t_0} = \mathscr{L}_{t,t_0}\,,\quad \mathscr{L}^{-1}_{t,t_0} = \mathscr{L}_{t_0,t}\,.$$

Note that a nondivergent velocity field generates an incompressible flow, which preserves the volume of any measurable set of points $M \subset E^d$,

$$V(\mathscr{L}_{t,t_0}M) = V(M)\,.$$

2.2 Stochastic flow

A natural generalization of the concept of flow is the concept of stochastic flow [54], in which case the velocity field is a random field $\mathbf{u}_\omega(t,\mathbf{r})$, i.e., it also depends on ω, where ω belongs to the space of elementary outcomes of some probabilistic space Ω. In this case, the flow and its trajectories are also random, and their properties must be described in terms of probabilities. Sometimes, it is convenient to consider flows with additional properties.

(1) A random flow is called stationary if, under an arbitrary time shift, i.e., under a field transformation such that $\mathbf{u}(t,\mathbf{r})$ goes into $\mathbf{u}'(t,\mathbf{r}) = \mathbf{u}(t-\tau,\mathbf{r})$, and for arbitrary measurable functional $F[\mathbf{u}]$, it holds that

$$\big\langle F[\mathbf{u}']\big\rangle = \big\langle F[\mathbf{u}]\big\rangle\,.$$

When studying the statistical properties of stationary flows, we can fix the parameter $t_0 = 0$ once and for all (as we will do subsequently) and consider a one-parametric family of diffeomorphisms $\mathscr{L}_{t,0} \equiv \mathscr{L}_t$. The trajectories of this family $\mathbf{r}(t;\mathbf{x}) = \mathscr{L}_t\,\mathbf{x}$ are the solutions to problem (3) with the condition $\mathbf{r}(0) = \mathbf{x}$.

(2) A random flow is called homogeneous if, for any spatial shift such that $\mathbf{u}(t,\mathbf{r})$ transforms into $\mathbf{u}'(t,\mathbf{r}) = \mathbf{u}(t,\mathbf{r}-\mathbf{y}),\, \mathbf{y} \in E^d$, it holds that

$$\big\langle F[\mathbf{u}']\big\rangle = \big\langle F[\mathbf{u}]\big\rangle\,.$$

(3) A random flow is called isotropic if, for any rotation $O \in SO(d)$ around an arbitrary point $\mathbf{a}$, such that $\mathbf{u}(t,\mathbf{r})$ transforms into

$$\mathbf{u}'(t,\mathbf{r}) = O\mathbf{u}\left(t, O^{-1}(\mathbf{r}-\mathbf{a})+\mathbf{a}\right),$$

it holds that

$$\big\langle F[\mathbf{u}']\big\rangle = \big\langle F[\mathbf{u}]\big\rangle\,.$$

(4) We shall also assume that the flow has finite correlation length and time: the field values at points in space and time that are separated by distances and times exceeding the correlation scales are statistically independent.

2.3 Films in a flow

We are interested in the evolution of a k-dimensional hypersurface (film) carried by a flow. In mathematics and theoretical physics, there are many examples of how a formal increase in the complexity of a problem, and its formulation in a more general form, can help us to find an efficient solution. This is fully applicable to our problem: it is more convenient to consider a complete family of surfaces with dimensions ranging from 1 to d, each one contained in the next (the flag of hyper-surfaces). For definiteness, let us assume that, at the initial moment, these surfaces are hyperplanes spanned by vectors of the standard orthonormal basis $\{\mathbf{e}_1\}, \{\mathbf{e}_1,\mathbf{e}_2\}, \{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}, \ldots, \{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_d\}$. To describe their local evolution, consider an infinitesimal cube initially located at an arbitrary point x contained in all hypersurfaces of the flag and formed by the vectors

$$\varepsilon\mathbf{e}_1, \varepsilon\mathbf{e}_2, \ldots, \varepsilon\mathbf{e}_d\,.$$

The vertices of this cube are frozen in the flow; therefore, its evolution is defined by the divergence of infinitesimally close trajectories. This evolution is always linear; therefore, the cube will evolve into an infinitesimal parallelepiped formed by the vectors

$$\varepsilon Q(t,x)\,\mathbf{e}_1, \varepsilon Q(t,x)\,\mathbf{e}_2, \ldots, \varepsilon Q(t,x)\,\mathbf{e}_d\,.$$

In stochastic transport theory, the linear operator $Q(t,s;\mathbf{x})$ is called the evolution operator. Its matrix is the Jacobi matrix (differential) of diffeomorphism (2),

$$Q^i_j(t,\mathbf{x}) = \frac{\partial r^i(t,\mathbf{x})}{\partial x^j}\,. \tag{4}$$

In what follows, we discard the infinitely small factor ε and study the evolution of the parallelepiped in the space tangent to E^d,

$$Q(t,x)\,\mathbf{e}_1, Q(t,x)\,\mathbf{e}_2, \ldots, Q(t,x)\,\mathbf{e}_d\,.$$

The sequence of edge, face, etc., one contained in the next, up to the hyper-parallelepiped itself, is called its flag. Each element of this flag touches its respective hyper-surface. Consider the sequence of the edge length, face area, and so on,

$$s_1 = |Q\mathbf{e}_1|,\;\; s_2 = |Q\mathbf{e}_1 \wedge Q\mathbf{e}_2|\,, \qquad s_d = |Q\mathbf{e}_1 \wedge \ldots \wedge Q\mathbf{e}_d| = \det Q\,. \tag{5}$$

It should be clear that the local 'paint' density on the hyper-surface k is precisely defined by the magnitude of s_k,

$$\rho_k = \frac{1}{s_k}\,. \tag{6}$$

The change in the volume s_d defines the change in the d-dimensional density of the fluid itself. It does not change in an incompressible flow,

$$s_d = \det Q = 1\,.$$

'The maximum task' is studying the joint probability density of s_k or, which is equivalent, finding the so-called characteristic correlators

$$\left\langle s_1^{m_1}(t)\, s_2^{m_2}(t) \ldots s_d^{m_d}(t) \right\rangle$$

as a function of real indices $m_1, \ldots, m_d$.

To solve this problem, we first need to derive an equation that is satisfied by the Jacobi matrix. This can be achieved by differentiating the trajectory equation (3) with respect to x as a composition of functions,

$$\frac{\partial}{\partial x^j}\frac{\partial r^i(t;\mathbf{x})}{\partial t} = \frac{\partial u^i\big(t,\mathbf{r}(t;\mathbf{x})\big)}{\partial x^j} = \frac{\partial u^i\big(t,\mathbf{r}(t;\mathbf{x})\big)}{\partial r^k}\,\frac{\partial r^k(t;\mathbf{x})}{\partial x^j}\,,$$

which gives

$$\frac{\partial}{\partial t}\,Q^i_j(t,\mathbf{x}) = \mathscr{A}^i_k(t,\mathbf{x})\,Q^k_j(t,\mathbf{x})\,,$$

where $\mathscr{A}^i_k(t;\mathbf{x}) \equiv A^i_k(t,\mathbf{r}\,(t;\mathbf{x}))$ is the tensor of velocity gradients on flow trajectories (the Lagrangian gradient tensor). In random flows, it is a random matrix process. In incompressible stationary flows with a finite correlation length, it is usually stationary. Additionally it will be assumed that its correlation time is also finite.

The argument x will be omitted in what follows, as it is not important due to flow homogeneity:

$$\frac{\partial}{\partial t}\,Q(t) = \mathscr{A}(t)\,Q(t)\,. \tag{7}$$

Equation (7) is completed by the initial condition $Q(0) = 1$. Its solution is known as the T-exponent,

$$Q(t) = \mathscr{T}\exp\left\{\int_0^t \mathscr{A}(\tau)\,\mathrm{d}\tau\right\}. \tag{8}$$

By expressing the evolution operator in the form of (8), we can reduce the continuous time problem to the discrete product problem: we subdivide the evolution time t into finite intervals $\Delta t_k = t_k - t_{k-1}$, rewriting the evolution operator as

$$Q(t) = Q_N \ldots Q_1\,,$$

where

$$Q_k = \mathscr{T}\exp\left\{\int_{t_{k-1}}^{t_k} \mathscr{A}(\tau)\,\mathrm{d}\tau\right\}.$$

If we are interested in the long-term asymptotic behavior $t \to \infty$, we can take intervals Δt_k to be sufficiently large to make the matrices Q_k statistically independent. (This would imply that $\Delta t_k \gg \tau_{\mathrm{c}}$, where τ_{c} is the correlation time of the process $\mathscr{A}(t)$.) Furthermore, if $\mathscr{A}(t)$ is stationary, the matrices Q_k are identically distributed.

The problem is now reduced to the classical problem concerning the behavior of the product of independent and identically distributed random matrices in the limit $N \to \infty$. We stress that the noncommutativity of the matrix product is a serious obstacle to solving this problem. Therefore, it is worthwhile to begin with an analogous 'one-dimensional' problem for the product of random numbers and then demonstrate its connection with the large deviation theory.

3. Product of random numbers

We start with the simplest, yet important, 'toy' model, which illustrates intermittency in multiplicative systems. A similar model of intermittency was first proposed in [27].

3.1 Toy model (casino)

Consider a game in which players are given 'unfair' coins that come up as heads with probability $1/4$ and as tails with probability $3/4$. The players then start tossing these coins. If a player gets heads, their assets are doubled, and if a player gets tails, the assets are nullified, and the player is out of the game.

This can be described by the following model: there is a sequence of independent random numbers q_k, and the following outcomes can be realized:

$$q_k = 2,\;\; p_q(2) = \frac{1}{4}\,,\;\; q_k = 0,\;\; p_q(0) = \frac{3}{4}\,.$$

The assets of each player at the Nth step are a random value

$$q(N) = q_N \ldots q_1\,.$$

It can take only two values,

$$q(N) = 2^N,\;\; p = \frac{1}{4^N}\,,\;\; q(N) = 0,\;\; p = 1 - \frac{1}{4^N}\,.$$

Let us calculate the moments of mth order:

$$\left\langle q^m(N)\right\rangle = 2^{Nm}\,\frac{1}{4^N} + 0^m\left(1 - \frac{1}{4^N}\right) = 2^{N(m-2)}\,.$$

Although the probability of winning in the process of N tosses tends to zero exponentially, $N \to \infty$, the win also grows exponentially. Therefore, the rare events in which a player is not ruined determine all the moments $m > 0$.

Thus, while the expectation $\langle q(N)\rangle = 4^{-N}$ decays exponentially, the moment $\langle q^2(N)\rangle = 1$ remains constant and is a stochastic integral; meanwhile, the moments $\langle q^m(N)\rangle$ for $m > 2$ grow exponentially. From a common-sense point of view, this means that the overwhelming majority of players become ruined, but there are increasingly rare 'lucky ones' with super-assets 2^N who define all the positive moments.

This simple example shows that, unlike the fluctuations in traditional additive thermodynamics, which play an insignificant role due in the central limit theorem, fluctuations in multiplicative (intermittent) systems are extremely important.

3.2 General case

Despite its simplicity, the coin example shows all the characteristic features of intermittency, including the existence of stochastic integrals. We now consider a general case of such products and introduce convenient notions from the large deviation theory.

Let $q_1, q_2, \ldots$ be independent and identically distributed positive random variables. They can be conveniently represented as $q_k = \exp(a_k)$. Rather than specifying the probability density distribution, it is more convenient to define the statistics of a using the cumulant generating function:

$$\omega_a(m) = \ln \left\langle \exp(ma) \right\rangle . \tag{9}$$

This is a convex upward function with the normalization $\omega_a(0) = 0$. Its first derivative at zero is equal to the mathematical expectation $(\mathrm{d}\omega_a/\mathrm{d}m)|_0 = \langle a \rangle$, while the higher derivatives at zero are equal to the so-called cumulants (coupled moments) of the random variable a.

Consider the product

$$q(N) = \prod_{k=1}^{N} q_k = \exp(N\bar{a}) ,$$

where $\bar{a} = 1/N \sum_{k=1}^{N} a_k$ is the arithmetic mean of the sequence $a_1, \ldots, a_N$. In the context of our problem, these sequences will be interpreted as evolution in discrete time, and further, $\bar{a}$ will be referred to as the time mean. From the LLN, it follows that with probability 1 there is a limit

$$\lambda = \lim_{N\to\infty} \frac{\ln q(N)}{N} = \lim_{N\to\infty} \bar{a} = \langle a \rangle = \langle \ln q \rangle . \tag{10}$$

This is the so-called multiplicative form of the LLN.

Consider now the mth moment of the product $\langle q^m(N) \rangle$ as a function of m and N. Since the factors are commutative and independent,

$$\left\langle q^m(N) \right\rangle = \langle q^m \rangle^N = \left\langle q^m(1) \right\rangle^N . \tag{11}$$

The moments $\langle q^m \rangle$ can be expressed trivially through the cumulant generating function $\omega_a(m)$,

$$\langle q^m \rangle = \langle \exp(ma) \rangle = \exp\left(\omega_a(m)\right) ;$$

therefore,

$$\left\langle q^m(N) \right\rangle = \exp\left(N\omega_a(m)\right) . \tag{12}$$

Thus, the mth moment of the product increases exponentially with N if $\omega_a(m) > 0$ and decays exponentially if $\omega_a(m) < 0$. The case $\omega_a(m) = 0$ corresponds to a stochastic integral.

If $\omega_a(m) = \langle a \rangle m$ is not random, the cumulant function is linear, $\omega_a(m) = \langle a \rangle m$, and there are no stochastic integrals except for the trivial one. This corresponds to the obvious fact that the growth or decay of the mth power of the product of nonrandom identical $q_k = \exp(a_k) = \exp(\langle a_k \rangle)$ depends only on m and $\langle a \rangle$: $q^m(N) = \exp(N\langle a \rangle m)$.

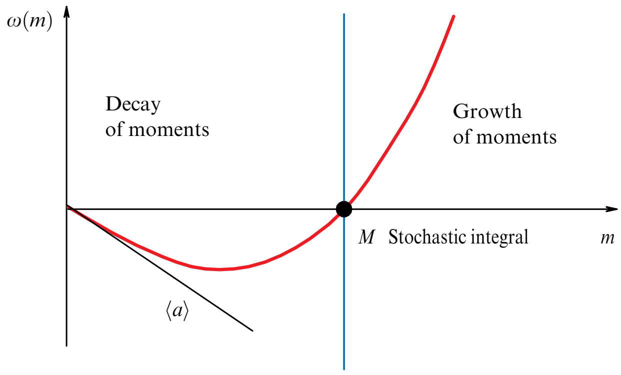

Figure 3. Typical form of cumulant generating function of logarithm of one-dimensional factors.

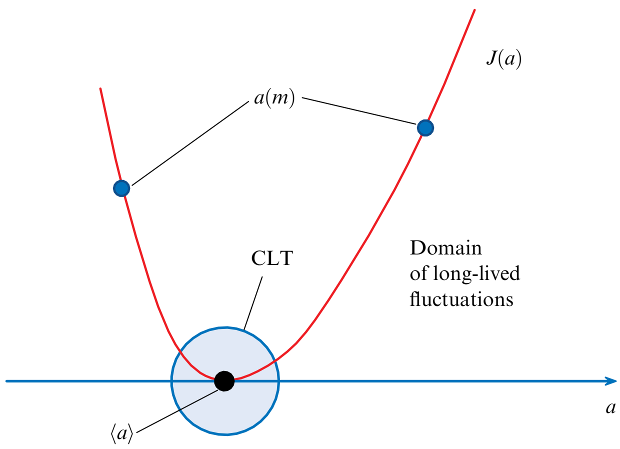

Figure 4. Typical form of Cramér function and its general properties.

However, if a is slightly 'spread' around $\langle a \rangle$, the moments of the product

$$\left\langle q^m(N) \right\rangle = \left\langle \exp(N\bar{a}m) \right\rangle$$

become determined by rare sequences $a_1, \ldots, a_N$, such that $\bar{a} \neq \langle a \rangle$.

To demonstrate this, we need certain results of the large deviation theory. It is known that the probability density of the time mean $\bar{a}$ at large N takes the form

$$\rho_{\bar{a}}(x) \sim \exp\left(-NJ_a(x)\right) ,$$

where J_a is the Cramér function of the random variable a, which is necessarily a convex function. It has a minimum at $x = \langle a \rangle$, which immediately leads to the LLN. It is commonly normalized so that $J_a(\langle a \rangle) = 0$. We also note that, since the Cramér function is quadratic in the vicinity of the minimum $x = \langle a \rangle$, this leads to the CLT.

The moments $\langle q^m(N) \rangle$ (which we have already calculated exactly; see (12)) are defined by the integrals

$$\left\langle q^m(N) \right\rangle = \int \mathrm{d}x \exp(mxN)\rho_{\bar{a}}(x) \sim \int \mathrm{d}x \exp\left(N\left(mx - J_a(x)\right)\right) . \tag{13}$$

Since N is large, these integrals can be estimated using the steepest descent method. The point at which $mx - J_a(x)$ is maximum will be denoted as $a(m)$; it is defined by the equation

$$J'_a\big(a(m)\big) = m\,. \tag{14}$$

Inserting this value into integral (13), we obtain

$$\big\langle q^m(N)\big\rangle \sim \exp\big(ma(m)N\big)\exp\big(-NJ_a(a(m))\big)\,. \tag{15}$$

By comparing (15) and (12), we can easily conclude that the functions $J_a(x)$ and $\omega_a(m)$ are connected by the Legendre transform

$$\sup_x\big(mx - J_a(x)\big) = \omega_a(m)\,.$$

Relationship (15) can be interpreted as follows: for $N \to \infty$, the main contribution to $\langle q^m(N)\rangle$ for $m > 0$ comes from 'nonergodic' sequences $a_1, \ldots, a_N$, such that their time means are far from the mathematical expectation,

$$\bar{a} = a(m) \neq \langle a\rangle\,.$$

The probability of such sequences is $\sim \exp\left(-NJ_a(a(m))\right)$. This probability is exponentially small at large N because $J_a(a(m)) > 0$. This fact also agrees with the LLN.

Thus, we have shown that the product of a large number of independent, random factors exhibits intermittent statistics: the moments of this product are defined by exponentially rare, long-lived fluctuations, while the nontrivial zero of the cumulant function of the logarithm of the factors defines a stochastic integral. For a product of factors, it can be seen that stochastic integrals are defined by the specific form of factor distribution. A significant result that will be obtained later is that these integrals are universal for products of isotropically distributed random matrices, independent of the distribution. This is a manifestation of the symmetry property of matrix probability measures, which are invariant under the action of an orthogonal group.

4. Product of random matrices

Let $Q_1, Q_2 \ldots$ be a sequence of random matrices. Consider their product

$$Q(N) = \prod_{k=1}^{N} Q_k = Q_N \ldots Q_1\,.$$

Studying the behavior of the moments of the matrix elements of this product is difficult due to the impossibility of regrouping the factors,

$$\left(\prod_{k=1}^{N} Q_k\right)^m \neq \prod_{k=1}^{N} (Q_k)^m\,.$$

Consequently, in a general case, even for independent matrices, the mean of the product's powers is not necessarily equal to the product of powers of the mean:

$$\left\langle\left(\prod_{k=1}^{N} Q_k\right)^m\right\rangle \neq \prod_{k=1}^{N}\big\langle (Q_k)^m\big\rangle\,.$$

Additionally,

$$\ln\prod_{k=1}^{N} Q_k \neq \sum_{k=1}^{N} \ln\,Q_k\,,$$

which rules out direct formulation of the LLN for matrix elements. Accordingly, obtaining the moments of the matrix elements of the product of random matrices is difficult in general. However, as previously mentioned, we need the moments of hyper-areas rather than the moments of matrix elements. This is also a complicated task, but it is strongly simplified for isotropic distributions. We will further demonstrate that, despite matrix products being noncommutative, the behavior of the hyper-areas of flag elements is the same as in a commutative case. [6]

4.1 Flag of matrix product

In isotropic flows, the evolution operator is an isotropically distributed random matrix. By definition, a random matrix Q is called isotropically distributed if, for any measurable function of its matrix elements $F(Q)$ and any orthogonal matrix O, it holds that

$$\big\langle F(Q)\big\rangle = \big\langle F(O^{-1}QO)\big\rangle\,. \tag{16}$$

This is equivalent to an analogous property of the probability density

$$\rho(Q) = \rho(O^{-1}QO)\,.$$

Theorem [56]. *Let $Q(N) = Q_N \ldots Q_1$ be the product of identically and isotropically distributed independent matrices. Consider the area of a k-dimensional face of the related parallelepiped*

$$s_k(N) = \big|Q(N)\mathbf{e}_1 \wedge \ldots \wedge Q(N)\mathbf{e}_k\big|\,.$$

Then,

$$\big\langle s_1^{m_1}(N)\ldots s_d^{m_d}(N)\big\rangle = \big\langle s_1^{m_1}(1)\ldots s_d^{m_d}(1)\big\rangle^N\,. \tag{17}$$

(It is worthwhile to compare this with the one-dimensional commutative case (11)).

Proof. We first prove this theorem for monomials $\langle s_k^m(N)\rangle$.

Additionally, let $N = 2$, i.e., we are only dealing with the product of two matrices Q_2Q_1.

We have

$$s_k(2) = |Q_2Q_1\mathbf{e}_1 \wedge \ldots \wedge Q_2Q_1\mathbf{e}_k|\,.$$

For an arbitrary matrix Q and an arbitrary set of vectors $\mathbf{g}_1, \ldots, \mathbf{g}_k$, we define

$$\alpha(Q, \mathbf{g}) = \frac{|Q\mathbf{g}_1 \wedge \ldots \wedge Q\mathbf{g}_k|}{|\mathbf{g}_1 \wedge \ldots \wedge \mathbf{g}_k|}\,.$$

This is simply the factor by which the area $|\mathbf{g}_1 \wedge \ldots \wedge \mathbf{g}_k|$ is multiplied when the operator Q is applied. It can be easily seen that

$$s_k(2) = \alpha(Q_2, \mathbf{g})\,\alpha(Q_1, \mathbf{e})\,, \tag{18}$$

[6] Newman [55] paid attention to this fact in the context of Lyapunov spectrum calculations.

where

$$\mathbf{g}_1 = Q_1\mathbf{e}_1, \ldots, \mathbf{g}_k = Q_1\mathbf{e}_k\,.$$

This representation allows the power to be carried over each factor:

$$\left\langle s_k^m(2)\right\rangle_{Q_2,Q_1} = \left\langle \alpha^m(Q_2,\mathbf{g})\,\alpha^m(Q_1,\mathbf{e})\right\rangle_{Q_2,Q_1}. \tag{19}$$

This expression is averaged over independent Q_1 and Q_2, but, unfortunately, $\mathbf{g}_1, \ldots, \mathbf{g}_k$ depend on Q_1.

Therefore, the only thing we can do when factorizing this mean is to move one bracket and write

$$\left\langle s_k^m(2)\right\rangle_{Q_2,Q_1} = \left\langle \left\langle \alpha^m(Q_2,\mathbf{g})\right\rangle_{Q_2}\alpha^m(Q_1,\mathbf{e})\right\rangle_{Q_1}. \tag{20}$$

Note now that the factor $\alpha(Q,\mathbf{g})$ will not change if the Gramm–Schmidt orthogonalization procedure is applied to $\mathbf{g}_1, \ldots, \mathbf{g}_k$; therefore,

$$\alpha(Q,\mathbf{g}) = \alpha(Q,\mathbf{n})\,, \tag{21}$$

where $\mathbf{n}_1, \ldots, \mathbf{n}_k$ is now some orthonormal set of vectors. Due to isotropy, expression (21) does not depend on the direction $\mathbf{n}_1, \ldots, \mathbf{n}_k$; in particular, we can take it along the standard basis $\mathbf{e}$ and write

$$\left\langle \alpha^m(Q_2,\mathbf{n})\right\rangle_{Q_2} = \left\langle \alpha^m(Q_2,\mathbf{e})\right\rangle_{Q_2}.$$

We have thus obtained a factorized expression for (19):

$$\left\langle s_k^m(2)\right\rangle = \left\langle \alpha^m(Q_2,\mathbf{e})\right\rangle\left\langle \alpha^m(Q_1,\mathbf{e})\right\rangle = \left\langle s_k^m(1)\right\rangle^2.$$

Repeating this procedure for the product of N matrices yields

$$\left\langle s_k^m(N)\right\rangle = \left\langle s_k^m(1)\right\rangle^N,$$

which completes the proof of the theorem for monomials $\langle s_k^m\rangle$. The same procedure can be used to generalize to (17).

4.2 Lyapunov spectrum

We have shown that the moments of the hyper-areas $s_k(N)$ for the flag of a product of random matrices evolve exponentially with respect to the number of factors N. As in the one-dimensional case, it is convenient to take the logarithms

$$a_k = \ln s_k\,, \quad k = 1, \ldots, d\,. \tag{22}$$

Consider their cumulant functions

$$\omega_{a_k}(m) = \ln\left\langle \exp(ma_k)\right\rangle = \ln\langle s_k^m\rangle\,. \tag{23}$$

Taking the logarithm of both sides of equality (17), we obtain a link between the cumulant functions of the logarithms of the product flag and those of the logarithms of the flag of each factor,

$$\omega_{a_k(N)}(m) = N\omega_{a_k(1)}(m)\,. \tag{24}$$

From (24), it can be seen that the cumulant function of $\ln s_k(N)/N = a_k(N)/N$ is

$$\omega_{a_k(N)/N}(m) = N\omega_{a_k(1)}\left(\frac{m}{N}\right),$$

from where it follows that, for $N \to \infty$, it tends to a linear function

$$\omega_{a_k(N)/N}(m) \overset{N\to\infty}{\longrightarrow} \left.\frac{\mathrm{d}\omega_{a_k(1)}}{\mathrm{d}m}\right|_0 m = \langle a_k(1)\rangle m\,.$$

This is the cumulant function of a nonrandom quantity, equal to $\langle a_k(1)\rangle = \langle \ln s_k(1)\rangle$.

Thus, for a product of independent and identically distributed random matrices, the ratio of the logarithms of the product flag hyper-areas to the number of factors has a limit (by probability),

$$\lim_{N\to\infty}\frac{\ln s_k(N)}{N} = \left\langle \ln s_k(1)\right\rangle = \left\langle a_k(1)\right\rangle. \tag{25}$$

Traditionally, the limits (25) are denoted as

$$\lim_{N\to\infty}\frac{\ln s_k(N)}{N} = \lambda_1 + \ldots + \lambda_k\,, \tag{26}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_d$ is the so-called Lyapunov spectrum (LS). For incompressible flows, it is clear that $\lambda_1 + \ldots + \lambda_d = 0$.

Thus, we have rigorously proved the LLN for the product of isotropically distributed matrices. In the general case, it is also possible to prove the existence theorem for the LS as the limit (26). Unfortunately, this is only the existence theorem, and the analytical expressions (24) and (25), which enable us to calculate the moments of flag hyper-areas and the LS of the product of matrices using the known statistics of the factors, are only valid in the isotropic case.

We provide a sketch of the proof for the noncommutative LLN for the general case of nonisotropic matrices; a rigorous proof can be found, e.g., in Ref. [57]. Consider a sequence of sets of k vectors (in reality, a Markov chain), beginning with the first vectors of the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_k$:

$$\mathbf{g}^{(0)} = \mathbf{e}\,,\ \mathbf{g}^{(1)} = Q_1\mathbf{g}^{(0)}, \ldots, \mathbf{g}^{(N)} = Q_N\mathbf{g}^{(N-1)}\,. \tag{27}$$

Then, as before, we write

$$s_k(N) = \alpha(Q_N,\mathbf{g}^{(N-1)})\ldots\alpha(Q_1,\mathbf{g}^{(0)})\,. \tag{28}$$

This is the product of random values, but dependent random values. The idea is to subdivide this product into large 'pieces' that will 'forget the initial vector' and correlate only 'weakly.' Taking the logarithm of the product gives an expression for the sum $\lambda_1 + \ldots + \lambda_k$:

$$\lim_{N\to\infty}\frac{1}{N}\left(\ln\alpha(Q_N,\mathbf{g}^{(N-1)}) + \ldots + \ln\alpha(Q_1,\mathbf{g}^{(0)})\right).$$

Subdividing this sum into 'big independent' pieces gives us the usual LLN. Unfortunately, now we do not know the statistics of these pieces (i.e., we do not know how they can be obtained from the statistics of Q_k).

The isotropic case is notable in this respect because, although the 'elementary pieces' $\ln\alpha(Q_n,\mathbf{g}^{(n-1)})$ are dependent, this dependence is eliminated by averaging over isotropic matrices.

4.3 Structure of characteristic correlators

Theorem (17) and formulas (26) reduce the calculation of characteristic correlators and the LS of the product of isotropically distributed matrices to the calculation of related values for a single matrix only. Therefore, in this section, we will study the properties of a flag of one isotropically distributed matrix.

It will be convenient here to use algebraic expressions for the squares of areas:

$$s_1^2 = (Q^TQ)_{11}\,,$$
$$s_2^2 = (Q^TQ)_{11}(Q^TQ)_{22} - (Q^TQ)_{12}^2\,, \dots\,,$$
$$s_d^2 = \det Q^TQ\,.$$

It turns out that the characteristic correlators of this flag possess one rather important property [52, 53].

Theorem. *For an arbitrary set of numbers $m_1, \dots, m_d$, consider a characteristic correlator of the form $\langle S(m,Q)\rangle$, where*

$$S(m,Q) = s_1^{m_1-m_2-1}(Q)s_2^{m_2-m_3-1}(Q)\dots s_d^{m_d+d}(Q)\,. \tag{29}$$

Then, $\langle S(m,Q)\rangle$ is a symmetric function of $m_1, \dots, m_d$ for any isotropic distribution Q.

Proof. It is sufficient to prove the symmetry for a pair permutation of neighboring arguments

$$m_p \longleftrightarrow m_{p+1} \quad \text{и} \quad p = 1, \dots, d-1\,.$$

Let us fix p and consider orthogonal transformations $O_p(\phi)$ that perform rotation in planes $\mathbf{e}_p, \mathbf{e}_{p+1}$ with the help of 2×2 blocks:

$$o(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

From isotropy, it follows that, for every $O_p(\phi)$,

$$\langle S(m,Q)\rangle = \langle S(m, O_p^{-1}(\phi)QO_p(\phi))\rangle$$

for any ϕ; therefore, the same is valid for the integral

$$\langle S(m,Q)\rangle = \left\langle \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, S\big(m, O_p^{-1}(\phi)QO_p(\phi)\big)\right\rangle.$$

Thus, to prove the theorem, it is sufficient to prove that the value of the integral

$$\int_0^{2\pi} \mathrm{d}\phi\, S\big(m, O_p^{-1}(\phi)QO_p(\phi)\big)$$

does not change when m_p and m_{p+1} are permuted for any fixed (nonrandom) Q. Furthermore, since $s_k(O_p^{-1}QO_p) = s_k(Q)$ does not depend on ϕ for $p \neq k$, it is sufficient to prove the symmetry of integrals

$$s_{p-1}^{-m_p}\, s_{p+1}^{m_{p+1}} \int_0^{2\pi} \mathrm{d}\phi\, s_p^{m_p - m_{p+1} - 1}(\phi)\,,$$

where $s_p(\phi) = s_p(O_p^{-1}(\phi)QO_p(\phi))$ (to make this formula meaningful for $p = 1$, we formally assume $s_0 = 1$). Denote $m_p - m_{p+1} = n$, $m_p + m_{p+1} = N$. Then

$$s_{p-1}^{-m_p}\, s_{p+1}^{m_{p+1}} \int_0^{2\pi} \mathrm{d}\phi\, s_p^{m_p - m_{p+1} - 1}(\phi) = s_{p-1}^{-(N+n)/2}\, s_{p+1}^{(N-n)/2} \int_0^{2\pi} \mathrm{d}\phi\, s_p^{n-1}(\phi)\,,$$

and it is sufficient to prove that

$$s_{p-1}^{-n/2} s_{p+1}^{-n/2} \int_0^{2\pi} \mathrm{d}\phi\, s_p^{n-1}(\phi) \quad \text{is an even function of } n\,.$$

Now we need to determine the form of $s_p(\phi)$. The following lemma is valid [53]:

Lemma.

$$s_p^2(\phi) = u\cos^2(\phi - \phi_0) + v\sin^2(\phi - \phi_0)\,, \tag{30}$$

where $uv = s_{p-1}^2 s_{p+1}^2$, ϕ_0 is an angle dependent on Q.

The proof is cumbersome in the general case, but straightforward; it relies on the Frobenius formula [49].

Using (30), it can easily be checked directly that

$$s_{p-1}^{-n/2}\, s_{p+1}^{-n/2} \int_0^{2\pi} \mathrm{d}\phi\, s_p^{n-1}(\phi) \quad \text{is an even function of } n\,. \tag{31}$$

Indeed, by changing the variable $y = u\cos^2(\phi - \phi_0) + v\sin^2(\phi - \phi_0)$, this integral is reduced to an elliptic integral

$$s_{p-1}^{-n/2}\, s_{p+1}^{-n/2} \int_v^u \mathrm{d}y\, \frac{y^{(n-1)/2}}{\sqrt{(u-y)(y-v)}}\,.$$

After one more variable change $z = uv/y$, we obtain

$$s_{p-1}^{-n/2}\, s_{p+1}^{-n/2} \int_v^u \mathrm{d}y\, \frac{y^{(n-1)/2}}{\sqrt{(u-y)(y-v)}}$$
$$= s_{p-1}^{n/2}\, s_{p+1}^{n/2} \int_v^u \mathrm{d}z\, \frac{z^{-(n+1)/2}}{\sqrt{(u-z)(z-v)}}\,,$$

and (31) is indeed an even function of n. Symmetry (29) with respect to permutations $m_p \longleftrightarrow m_{p+1}$ follows from this, and, as its consequence, follows the symmetry with respect to any permutation.

4.4 Universal stochastic integrals

At first glance, it seems that we have digressed significantly from finding the stochastic integrals of hyper-surface motion. However, we will now demonstrate that the theorem we proved enables us to solve the problem in an unexpected way. Indeed, this theorem implies that

$$\left\langle s_1^{m_1-m_2-1} s_2^{m_2-m_3-1} \dots s_d^{m_d+d}\right\rangle = \left\langle s_1^{m'_1-m'_2-1} s_2^{m'_2-m'_3-1} \dots s_d^{m'_d+d}\right\rangle$$

if $m'_k = m_{\pi(k)}$, where $\pi(k)$ is an arbitrary permutation. Taking $m_k = -k$, we find that, on the one hand, the left-hand side of this equality contains $\langle s_1^{-1+2-1} s_2^{-2+3-1} \dots s_d^{-d+d}\rangle = \langle s_1^0 s_2^0 \dots s_d^0\rangle = 1$ and, on the other hand, the right-hand side should be equal to it:

$$\left\langle s_1^{\pi(2)-\pi(1)-1} s_2^{\pi(3)-\pi(2)-1} \dots s_d^{d-\pi(d)}\right\rangle = 1\,. \tag{32}$$

We see that each nontrivial permutation $\pi(k)$ spawns a 'stochastic identity' [58]. We write them in the table for the case $d = 3$.

Permutation	Stochastic integral
132	$\langle s_1 s_2^{-2} s_3\rangle = 1$
213	$\langle s_1^{-2} s_2\rangle = 1$
231	$\langle s_2^{-3} s_3^2\rangle = 1$
312	$\langle s_1^{-3} s_3\rangle = 1$
321	$\langle s_1^{-2} s_2^{-2} s_3^2\rangle = 1$

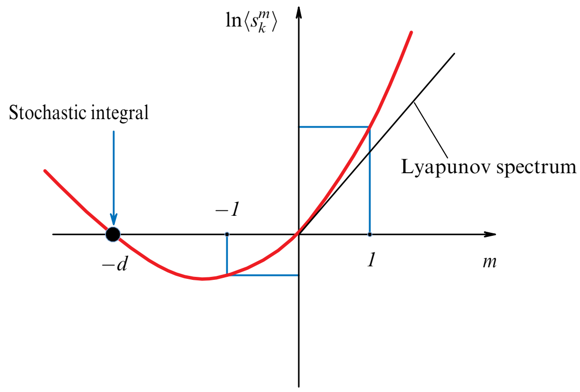

Figure 5. Cumulant function of logarithm of flag element's hyper-area for isotropically distributed matrix.

These equalities hold for any isotropic matrix, including Q_k and the full evolution matrix $Q(N) = Q_N \ldots Q_1$. They are preserved for the entire evolution time and are therefore universal stochastic integrals of the flag.

In the incompressible case, $s_d = 1$ and, as can be easily shown, cyclical permutations are related to the integrals

$$\langle s_k^{-d}\rangle = 1, \ k = 1, \ldots, d. \tag{33}$$

We note that, for a line ($k = 1$) in the space $d = 3$, this result was obtained by Ya.B. Zel'dovich, A.A. Ruzmaykin, S.A. Molchanov, and D.D. Sokoloff in Ref. [59] as an asymptotic relationship for $t \to \infty$. Subsequently, G. Fal'kovich and A. Frishman [60] showed that this integral is not merely asymptotic, but exact. However, extending their proof to the case $k > 1$ is difficult, and even more so to the case of mixed correlators.

We now return to the cumulant functions of the logarithms of the flag hyper-areas $a_k(Q) = \ln s_k(Q)$, $k = 1, \ldots, d-1$:

$$\omega_{a_k}(m) = \ln \big\langle \exp(m a_k)\big\rangle = \ln \langle s_k^m\rangle .$$

From the condition of convexity, it follows that their plots look like that shown in Fig. 5.

Recalling now the connection between the cumulant function of the product and the cumulant function of the individual factors (24), and examining Fig. 5, the following conclusions can be drawn about the character of the evolution of hyper-surfaces in incompressible isotropic flows.

• The sum of the first k Lyapunov exponents is always positive:

$$\lambda_1 + \lambda_2 + \ldots + \lambda_k = \big\langle \ln s_k(Q)\big\rangle = \frac{\mathrm{d}\omega_{a_k}}{\mathrm{d}m}\bigg|_0 > 0, \ k < d.$$

• The mean hyper-surface area of any dimension grows exponentially:

$$\big\langle s_k(N)\big\rangle = \exp\big(N\omega_{a_k}(1)\big), \quad \omega_{a_k}(1) > 0, \quad k < d.$$

• The mean kth surface density exponentially decays (but, generally speaking, with a different exponent):

$$\big\langle \rho_k(N)\big\rangle = \Big\langle \big(s_k(N)\big)^{-1}\Big\rangle = \exp\big(N\omega_{a_k}(-1)\big),$$
$$\omega_{a_k}(-1) < 0, \quad k < d.$$

• The critical exponents of density moments are all equal and coincide with the flow dimension $M_k = d$:

$$\langle \rho_k^{M_k}\rangle = \exp\big(N\omega_{a_k}(-d)\big) = 1 .$$

Note that, in the formal limit of infinite space dimension $d \to \infty$, the critical exponents also go to infinity. In this sense there is 'no intermittency' in an infinitely-dimensional space [26]. Geometrically, this is because, in an infinite number of dimensions, the exponential growth of hyper-areas can occur without fold formation, unlike in a finite-dimensional case, in which the surface, according to the Richardson law [26], is 'clamped' in a slowly expanding finite-dimensional ball. Therefore, it is reasonable that the minimum critical index is achieved in two dimensions, $M_1 = 2$; in this case, the folds form most efficiently.

It is worth recalling that all our conclusions are valid for isotropic flows; however, there are examples of nonisotropic finite-dimensional flows that also lack intermittency.

Consider, for example, a two-dimensional shear flow

$$u_x = u_x(y, t), \ u_y = 0$$

and a one-dimensional line carried by it. The matrix of the gradient tensor for this flow is

$$\mathscr{A} = \begin{pmatrix} 0 & \partial_y u_x \\ 0 & 0 \end{pmatrix}.$$

If its power is greater than 1, it is equal to zero; therefore, the multiplicative solution (8) degenerates into the additive one

$$Q(t) = 1 + \int_0^t \mathscr{A}(t)\,\mathrm{d}t .$$

It can easily be shown that, in this case, all positive density moments decay according to power laws, and all negative moments increase, i.e., the 'paint' is spread without forming coherent structures.

5. Applications

When proving the limit theorems above, we assumed that the flow was stationary, which warranted the existence of the LS. However, the form of stochastic integrals (32) is in no way related to flow stationarity or the Lyapunov spectrum; it is fully determined by the flow spatial isotropy. This fact is important for describing nonstationary flows in MHD and Eulerian turbulence.

5.1 MHD of ideally conducting fluid

The main objective of this review was to demonstrate the mechanism that leads to the formation of coherent structures using the example of the transport of material films of various dimensionality in a d-dimensional stochastic flow, and to highlight the deep connection of this mechanism with the theory of random matrices, invariant measures, and large deviations.

Another important example of intermittency, where the same mechanisms govern the formation of coherent structures, is the magnetohydrodynamics of ideally conducting fluid. As is well known, the magnetic field induction in an ideally conducting medium satisfies a system of equations which generalizes the Navier–Stokes equations:

$$(\partial_t + u^k \partial_k) B^p = A^p_k B^k \,, \tag{34}$$

$$(\partial_t + u^k \partial_k)\, u^p = \nu \Delta u^p - \partial^p p - \frac{1}{4\pi} \left(\mathbf{B} \times (\partial \times \mathbf{B})\right)^p + f^p \,,$$
$$\partial_p u^p = 0 \,, \tag{35}$$

where $\mathbf{B}$ is the induction vector, $\mathbf{u}$ is the flow velocity field, p is the pressure, ν is the fluid kinematic viscosity, and $\mathbf{f}$ is the pumping force. This system describes the joint evolution of the magnetic field and fluid velocities.

As with traditional stochastic fluid dynamics [61], this formulation allows the probability space to be introduced in a mathematically rigorous way, assuming that this space is formed from the set of all possible realizations of the random force $\mathbf{f}$. In this case, both the velocity and magnetic fields become complicated, nonlinear, and nonlocal functionals of the pumping force $B[\mathbf{f}]$, $u[\mathbf{f}]$. Rewriting equation (34) as

$$D_t B^p = A^p_k B^k \,,$$

where D_t is the Galilean-covariant derivative, the evolution of the magnetic field along a Lagrangian flow trajectory can be expressed as the result of the action of evolution operator (4) on the magnetic field vector at the initial point x of this trajectory,

$$B^k(t, \mathbf{x}) = Q^k_n(t, \mathbf{x})\, B^n(0, \mathbf{x}) \,. \tag{36}$$

In this formulation, the Lagrangian trajectories and the evolution operator are themselves complicated functions of the random pumping force $Q[\mathbf{f}, t]$.

During the evolution process, even if the magnetic field feeds back on the velocity field, the lines of induction remain frozen in the flow and are stretched exponentially during the linear (kinematic) evolution stage. During the nonlinear stage, the magnetic field begins to feed back on the flow due to the Lorenz force; accordingly, there is energy exchange due to magnetic field braking. In the case of isotropic pumping force, it is reasonable to assume that there will be no phase transition in the system with a spontaneous isotropy violation and that the evolution operator will preserve isotropy throughout the nonlinear phase. Then, it follows from (36) that an exact stochastic motion integral is available at all stages of the evolution (linear and nonlinear),

$$\left\langle B^{-3}(t) \right\rangle = \text{const} \,. \tag{37}$$

This integral was obtained in Ref. [20] as an asymptotic integral for the kinematic stage of evolution in a stationary flow at times exceeding the correlation time. In reality, it is exact and valid even for nonstationary flows, which, generally speaking, include MHD flows with feedback.

As we now know, the existence of this motion integral is linked to the tendency for rare but strong folds to form in the induction lines. The magnetic field is exponentially small at such locations. Violation of this exact law can only be caused by magnetic viscosity, which can lead to the reconnection of induction lines and the violation of the assumption that they are frozen in the flow.

5.2 Eulerian turbulence

Another important example is Eulerian turbulence. As is well known, the Euler equation describing the motion of 'dry' (inviscid) fluid is

$$(\partial_t + u^k \partial_k) u^p = -\partial^p p,\; \partial_p u^p = 0 \,,$$

giving for the vorticity $\omega^p = (\text{rot}\, \mathbf{u})^p$:

$$D_t \omega^p = A^p_k \omega^k \,, \tag{38}$$

which has the same form as (34) but is now a nonlinear equation at all stages of evolution, because the vorticity is a function of the velocity field of the flow. However, since vortical lines are frozen in the flow, it can be expected that the vorticity will also behave as

$$\left\langle \omega^{-3}(t) \right\rangle = \text{const} \,,$$

under the assumption that the evolution operator is isotropic. This behavior can only be violated by kinematic viscosity or external forces.

6. Conclusions

In this review, we discussed in detail the phenomenon of intermittency, which accompanies the transport of material films of arbitrary dimensionality in d-dimensional flows. We demonstrated that the exponential stretching of such films is often accompanied by the formation of folds, where the surface density remains high despite its overall decrease. These folds are the coherent structures in the sense they were originally defined by the authors of review [40]. These structures arise due to the multiplicative nature of the transport process and cannot be explained by thermodynamic fluctuation theory. To study them, we need the theory of large deviations, which is a generalization of the law of large numbers and the central limit theorem. We demonstrated that the formation of such coherent structures is accompanied by the maintenance of certain exact conservation laws. These laws were calculated for isotropic or locally isotropic flows (according to Kolmogorov's definition [62]) and are based on deep and nontrivial properties of matrix measures. This is the most important case for Kolmogorov's turbulence theory and the small-scale dynamo theory. For incompressible, locally isotropic flows, the simplest of the aforementioned conservation laws takes the form

$$\left\langle \rho^d(t) \right\rangle = \text{const} \,,$$

where ρ is the local surface density on the hyper-surface, and d is the space dimension with averaging (integration) carried out over the map of the surface parameterized by the initial Lagrangian coordinates. When averaging is introduced as integration over the surface itself in its own metrics, a correction is needed to take into account changes in the surface metrics due to surface stretching and 'crumpling.' This can easily be done [52, 53]; the motion integrals for averaging understood in this way take the form

$$\left\langle \rho^{d+1}(t) \right\rangle_{\text{surf}} = \text{const} \,.$$

It should be noted that the formal requirement for the flows to be locally isotropic could perhaps be replaced with a less stringent condition. This isotropy may hold for a wider class of flows than those that are locally isotropic, for which the isotropy of the velocity gradient tensor is required at fixed spatial points. It seems reasonable that motion along a chaotic Lagrangian trajectory will induce additional isotropization of the velocity gradient tensor and evolution operator.

Finally, we consider one more interesting question. In this review, we only studied coherent structures related to the formation of folds, where the surface density increases, as rather general statements can be formulated for these structures. In the context of dynamo theory, this implies that we were interested in regions with an anomalously weak magnetic field on the lines of induction. The same applies to vorticity in the context of Eulerian turbulence. However, as we mentioned in the Introduction, the regions of extremely strong stretching of frozen-in lines (on the mean background) are also coherent structures. In the phenomenon of a dynamo, this corresponds to the formation of rare bunches [40], where the magnetic field energy is much higher than average, and in the Eulerian turbulence, it corresponds to the formation of 'vorticity filaments' [63, 64]. Unlike fold dynamics, filament dynamics depend on flow statistics even when local isotropy is assumed. In Ref. [65], the authors showed notably that the formation mechanism of such filaments is essentially influenced by the spatial 'immobility' (with logarithmic accuracy) of their ends. But the ends of filaments are simply the points of folds!

Thus, the process of filament stretching is connected to the process of fold formation. This fact corresponds to a general property of fluctuations in multiplicative systems. There should always be large deviations on both sides, which, from a mathematical point of view, is a simple consequence of the convexity of the respective cumulant generating function.

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