

METHODOLOGICAL NOTES

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Spherically symmetric static problem of General Relativity for a continuum

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Contents

Abstract. The paper is an analytical review devoted to the spherically symmetric problem of General Relativity (GR). This problem is of special interest in GR, since it is reducible to ordinary differential equations that allow an analytical solution, used to describe cosmological models. The formulation of the mechanics problem associated with the issues of completeness and solvability of GR equations is analyzed. Solutions to the spherically symmetric static problem for a vacuum, ideal fluid, and elastic medium are considered within the framework of classical mechanics, the linearized system of GR equations, the Schwarzschild metric, and a pseudo-Riemannian space of a special type that is flat with respect to spatial coordinates and curved with respect to time.

Keywords: General Relativity, fluid and gas mechanics, mechanics of a deformable solid, spherically symmetric static problem

1. Introduction

The spherically symmetric problem of General Relativity (GR) occupies a special place in the theory, since it is reducible to ordinary differential equations and for some physical models allows an analytical solution. This problem is the subject of many publications, the most significant of which will be considered below. It is essential that in the present paper the spherically symmetric problem of GR be considered within the framework of the approach characteristic of continuum mechanics. In accordance with this, GR is treated as a phenomenological theory based on the traditional model of the mechanics of a homogeneous isotropic continuum, the real microstructure of which is ignored. Therefore, possible applications of the obtained solutions to problems of cosmology (neutron stars, etc.) are not considered below. Much attention is paid to the correctness of the mathematical formulation of the problem, particularly to the analysis of the completeness of the resolving system of equations. Below, we consider sequentially solutions to the spherically symmetric problem within the framework of General Relativity as applied to the Schwarzschild metric and within the framework of the proposed pseudo-Riemannian space of a special type, which is flat with respect to spatial coordinates and curved with respect to the time coordinate.

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2. General Relativity equations

Within the framework of the model considered, the continuum is characterized by the energy–momentum tensor (according to N.A. Kilchevskii [1], the term 'kinetic stress tensor' is more adequate) $T_i^j (i, j = 1, 2, 3, 4)$, defined in a four-dimensional Riemannian space with the metric form

$$ds^2 = g_{ij}\,dx^i\,dx^j\,. \tag{1}$$

In problems of mechanics, the energy–momentum tensor satisfies four equations of the conservation law,

$$\nabla_k T_i^k = 0 \ (i, k = 1, 2, 3, 4)\,. \tag{2}$$

Here, the mixed components of the tensor are used, since, for the spherically symmetric problem considered below, they coincide with the physical components. If x^1, x^2, x^3 are spatial coordinates, and x^4 is the time coordinate, then the first three equations (2) are the equations of motion, and the last equation ensures the conservation of mass. GR is based on the equation

$$E_i^j = \chi T_i^j \,, \tag{3}$$

establishing the proportionality between the energy–momentum tensor and the Einstein tensor:

$$E_i^j = R_i^j - \frac{1}{2} g_i^j R \,. \tag{4}$$

Here, R_i^j is the Ricci curvature tensor, expressed through the metric tensor g_i^j, and R is its linear invariant. The proportionality coefficient in Eqn (3),

$$\chi = \frac{8\pi\gamma}{c^4} \,, \tag{5}$$

is expressed through the classical gravitational constant γ and the speed of light c and follows from the condition of coincidence of the GR equations with the equations of Newton's gravitational theory in the case of weak gravity. Since the energy–momentum tensor T_i^j must satisfy Eqn (2), the Einstein tensor also satisfies this equation, so that

$$\nabla_k E_i^k = 0 \,. \tag{6}$$

Equations (3) with left-hand sides in the form of Eqns (4) constitute a system of ten nonlinear partial differential equations, including ten components of the metric tensor. However, as is known [2–5], this system is not complete. The fact is that the tensors E_i^j and T_i^j identically satisfy four equations (6) or (2). This means that, out of ten equations, only six are mutually independent. Traditionally, it is proposed to supplement the system with four coordinate conditions. For this purpose, the so-called harmonic conditions [3] can be used, which have the form

$$\frac{\partial}{\partial x^i} \left(\sqrt{G} g^{ij}\right) = 0 \,, \tag{7}$$

where G is the determinant of the metric tensor. Despite proof of the existence of a vacuum solution to the equations of general relativity, supplemented by Eqn (7) [6], and the obtained solution to the spherically symmetric problem for vacuum [7, 8], the coordinate condition (7) has not found wide application. There is no general form of coordinate conditions in general relativity, and all solutions obtained to date use coordinate conditions specified from various considerations.

The situation can be somewhat clarified by addressing the theory of elasticity [9, 10]. For static problems in the absence of gravity, the energy–momentum tensor coincides with the stress tensor σ_{ij}. In Cartesian coordinates x_1, x_2, x_3, the stresses are related by three equilibrium equations,

$$\sigma_{ij,j} = 0 \,(1,2,3) \,. \tag{8}$$

Here, $(\cdot)_{,j} = \partial(\cdot)/\partial x_j$, and the symbol $(1,2,3)$ denotes a circular permutation of indices. Equations (8) are similar to equations (2) of GR. In the second half of the 19th century, the following transformation of stresses was proposed by D. Maxwell and G. Maurer [11]:

$$\begin{aligned} &\sigma_{11} = \varphi_{22,33} + \varphi_{33,22} - 2\varphi_{23,23} \,, \\ &\sigma_{12} = -\varphi_{33,12} - \varphi_{12,33} + \varphi_{13,23} + \varphi_{23,13} \,(1,2,3) \,. \end{aligned} \tag{9}$$

Here, φ_{ij} is the tensor of stress functions [12]. Substituting stresses (9) into the equilibrium equations (8) allows satisfying these equations identically. A similar situation occurs in GR: substituting the energy–momentum tensor from equalities (3) into Eqn (2) identically satisfies these equations. Continuing the analogy with elasticity theory, we write the linearized form of the GR equations. Assume that the components of the metric tensor have the form

$$g_{ij} = 1 + f_{ij}\,(i,j = 1,2,3) \,, \quad g_{i4} = 0 \,, \quad g_{44} = -1 \,,$$

where the amplitude values of functions f_{ij} are much less than unity. Linearizing the equations of General Relativity with respect to these functions, we obtain

$$\begin{aligned} &\chi\sigma_{11} = \frac{1}{2}\,(-f_{22,33} - f_{33,22} + 2f_{23,23}) \,, \\ &\chi\sigma_{12} = \frac{1}{2}\,(f_{33,12} + f_{12,33} - f_{13,23} - f_{23,13})\,(1,2,3) \,. \end{aligned}$$

These equations coincide with Eqn (9) if we take $f_{ij} = -2\varphi_{ij}/\chi$. Thus, the linearized GR equations are like the equations of elasticity theory. Naturally, the analogy between the linear equations of elasticity theory and the nonlinear equations of General Relativity is conditional. However, in elasticity theory, an important property of Eqn (9) is proven: they cannot, in principle, be solved with respect to the tensor of stress functions. A solution to these equations exists only if their right-hand sides, the stresses, satisfy certain compatibility conditions [11]. These conditions have a simple geometric meaning: they require that the stressed space be Euclidean, i.e., that the curvature of this space be zero [13]. In GR, the space is not Euclidean and such conditions cannot exist. However, the analysis carried out allows us to assume that the incompleteness of the GR equations is not related to the missing coordinate conditions. It is natural to assume that gravity generates some pseudo-Riemannian space of a special type, limited by some physical conditions, which should supplement the GR equations. Such a space is proposed below in Section 6.

The incompleteness of the GR system of equations with respect to the metric tensor was discussed above. A similar situation occurs for the energy–momentum tensor. Ten components of this tensor are related by four equations (2), and the system is incomplete. The proposed model of pseudo-Riemannian space allows eliminating this problem as well.

3. Energy–momentum tensor

The structure of the energy–momentum tensor involved in the equations of General Relativity follows from the equations of classical solid mechanics and Newton's gravitational theory.

Let us derive this tensor for a spherically symmetric problem. In spherical coordinates r, θ, φ and in Euler variables, the motion of the continuum in a gravitational field is described by the following equations [1]:

$$\begin{aligned} &\frac{\partial\sigma_{\rm r}}{\partial r} + \frac{2}{r}\,(\sigma_{\rm r} - \sigma_\theta) - \rho\,\frac{\partial\psi}{\partial r} = \rho\left(\frac{\partial v_{\rm r}}{\partial t} + v_{\rm r}\,\frac{\partial v_{\rm r}}{\partial r}\right) , \\ &\frac{\partial(\rho v_{\rm r})}{\partial r} + \frac{2}{r}\,\rho v_{\rm r} + \frac{\partial\rho}{\partial t} = 0 \,. \end{aligned} \tag{10}$$

Here, σ_r and σ_θ are the radial and hoop stress, v_r is the radial velocity, ρ is the density, t is the time, and ψ is the Newtonian gravitational potential. Using identical transformations and the second equation (10), we can reduce the right-hand side of the first equation to the following form [1]:

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\,\frac{\partial v_r}{\partial r}\right)=\frac{\partial(\rho v_r)}{\partial t}+\frac{\partial(\rho v_r^2)}{\partial r}-v_r\left[\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v_r)}{\partial r}\right]$$
$$=\frac{\partial(\rho v_r)}{\partial t}+\frac{\partial(\rho v_r^2)}{\partial r}+\frac{2}{r}\,\rho v_r^2\,.$$

Using this result, we rewrite Eqn (10) as

$$\frac{\partial}{\partial r}\,(\sigma_r-\rho v_r^2)+\frac{2}{r}\left[(\sigma_r-\rho v_r^2)-\sigma_\theta\right]-\rho\,\frac{\partial\psi}{\partial r}-\frac{\partial(\rho v_r)}{\partial t}=0\,,$$
$$\frac{\partial}{\partial r}\,(\rho c v_r)+\frac{2}{r}\,\rho c v_r+\frac{\partial(\rho c^2)}{c\partial t}=0\,. \tag{11}$$

To reduce these equations to Eqns (2) used in GR, we consider a Riemannian space corresponding to the following metric form:

$$\mathrm{d}s^2=\mathrm{d}r^2+r^2\,\mathrm{d}\Omega^2-(1-f)c^2\,\mathrm{d}t^2\,,$$
$$\mathrm{d}\Omega^2=\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2\,, \tag{12}$$

where the amplitude value of the function f is much less than unity. Writing Eqns (2) for the metric form (12) and linearizing with respect to the function f, we obtain

$$\frac{\partial T_1^1}{\partial r}+\frac{2}{r}\,(T_1^1-T_2^2)+\frac{1}{2}\,\frac{\partial f}{\partial r}\,T_4^4+\frac{\partial T_1^4}{c\partial t}=0\,,$$
$$\frac{\partial T_4^1}{\partial r}+\frac{2}{r}\,T_4^1+\frac{\partial T_4^4}{c\partial t}=0\,. \tag{13}$$

Comparing Eqns (11) and (13), we can conclude that

$$T_1^1=\sigma_r-\rho v_r^2\,,\quad T_2^2=\sigma_\theta\,,\quad T_4^4=\rho c^2\,,\quad T_4^1=\rho v_r c\,,$$
$$T_1^4=-\rho v_r c\,,\quad \frac{\partial f}{\partial r}=-\frac{2}{c^2}\,\frac{\partial\psi}{\partial r}\,.$$

Considering the last of these relations, in which f is a component of the metric tensor of pseudo-Riemannian space, and ψ is the gravitational potential of Newton's theory, we can conclude that the analogy between gravity and the curvature of space follows from Newton's theory. In what follows, we will use more general tensor relations:

$$T_1^1=\sigma_1^1-\rho v_1 v^1\,,\quad T_2^2=\sigma_2^2\,,\quad T_4^4=\rho c^2\,,$$
$$T_4^1=\rho c v^1\,,\quad T_1^4=-\rho c v_1\,, \tag{14}$$

where $\sigma_1^1=\sigma_r$, $\sigma_2^2=\sigma_\theta$, since, as already noted, the mixed components of tensors in spherical coordinates are related to physical components. In all equalities (14), the density is the same, which is a consequence of the hypothesis of the equivalence of gravitational and inertial masses.

Complete expressions for the components of the energy–momentum tensor, considering the effects of the special theory of relativity, were obtained by V.A. Fock [7].

4. Classical solution to spherically symmetric static problem

Let us consider the problem of gravity for a solid elastic spherical body with radius R and constant density ρ. In the framework of Newton's theory of gravity, the gravitational potential satisfies the Poisson equation, which for a spherically symmetric gravitational field has the form

$$\varphi''+\frac{2}{r}\,\varphi'=4\pi\gamma\rho\,, \tag{15}$$

where $(\cdot)'=\mathrm{d}\,(\cdot)/\mathrm{d}r$. For the external Euclidean space surrounding the body, $(R\leqslant r<\infty)$ $\rho=0$, and the solution of Eqn (15) tending to zero as $r\to\infty$ has the form

$$\varphi_e=-\frac{\gamma m}{r}\,, \tag{16}$$

where m is the mass of the sphere and the index 'e' refers to the external space. We introduce the so-called gravitational radius, which plays an important role in the problem under consideration:

$$r_g=\frac{2\gamma m}{c^2}\,. \tag{17}$$

Then, solution (16) can be given the following form:

$$\varphi_e=-\frac{r_g c^2}{2r}\,. \tag{18}$$

For the internal space of the sphere ($0\leqslant r\leqslant R$, index 'i'), the solution of Eqn (15), regular at $r=0$, has the form

$$\varphi_i=\frac{2}{3}\,\pi\gamma\rho r^2+C\,. \tag{19}$$

The constant C is determined from the condition of continuity of the potential at the boundary of the sphere, i.e., $\varphi_i(R)=\varphi_e(R)$. As a result, equality (19) takes the following final form:

$$\varphi_i=-\frac{r_g c^2}{4R}\left(3-\frac{r^2}{R^2}\right). \tag{20}$$

The body gravitational forces acting inside the sphere are expressed in Newton's theory through the gravitational potential as follows: $f_g=-\rho\varphi_i'$. Then, the equilibrium equation of an element of the sphere takes the form

$$\sigma_r'+\frac{2}{r}\,(\sigma_r-\sigma_\theta)-\frac{\rho r_g c^2 r}{2R^3}=0\,. \tag{21}$$

The second equation for stresses follows from the condition of compatibility of deformations, which are expressed through the radial displacement as follows:

$$\varepsilon_r=u'\,,\quad \varepsilon_\theta=\frac{u}{r}\,. \tag{22}$$

Excluding displacement from these relations, we obtain the equation of compatibility of deformations

$$(r\varepsilon_\theta)'=\varepsilon_r\,. \tag{23}$$

For a linearly elastic medium, deformations are expressed through stresses using Hooke's law,

$$\varepsilon_r=\frac{1}{E}\,(\sigma_r-2\nu\sigma_\theta)\,,\quad \varepsilon_\theta=\frac{1}{E}\left[(1-\nu)\,\sigma_\theta-\nu\sigma_r\right], \tag{24}$$

where E is the modulus of elasticity, and ν is Poisson's ratio. Substituting deformations (24) into equation (23), we obtain the equation of compatibility of deformations in stresses

$$r\left[(1-\nu)\,\sigma_\theta'-\nu\sigma_r'\right]+(1+\nu)(\sigma_\theta-\sigma_r)=0\,. \tag{25}$$

Thus, we have two equations, (21) and (25), for stresses σ_r and σ_θ. The solution of these equations, regular at $r = 0$ and satisfying the boundary condition at the surface of the sphere $\sigma_r(R) = 0$, is determined by the equalities

$$\bar{\sigma}_r = -k\bar{r}_g\,(1 - \bar{r}^2)\,, \quad \bar{\sigma}_\theta = -k\bar{r}_g\left(1 + \frac{1+\nu}{3-\nu}\,\bar{r}^2\right),$$

$$k = \frac{3-\nu}{20(1-\nu)}\,. \tag{26}$$

Here,

$$\bar{\sigma}_r = \frac{\sigma_r}{\rho c^2}\,, \quad \bar{\sigma}_\theta = \frac{\sigma_\theta}{\rho c^2}\,, \quad \bar{r}_g = \frac{r_g}{R}\,, \quad \bar{r} = \frac{r}{R}\,. \tag{27}$$

Let us consider a special case of a sphere consisting of an ideal incompressible fluid with density ρ. In this case, $\sigma_r = \sigma_\theta = -p(r)$, where p is the pressure in the fluid, which is determined from the equation following from Eqn (21), that is

$$p' + \frac{\rho r_g c^2 r}{2R^3} = 0\,. \tag{28}$$

The solution of this equation, satisfying the boundary condition $p(R) = 0$, has the form

$$\bar{p} = \frac{\bar{r}_g}{4}\,(1 - \bar{r}^2)\,, \quad \bar{p} = \frac{p}{\rho c^2}\,. \tag{29}$$

It is essential that, to determine the pressure, it was not necessary to involve the equation of compatibility of deformations (25), which is satisfied identically.

The level of gravity is determined by the value of the parameter $\bar{r}_g$. At $\bar{r}_g = 0$, gravity is absent. For real objects, the value of $\bar{r}_g$ is small compared to unity. For example, for Earth, $\bar{r}_g = 1.4 \times 10^{-6}$. Considering that for Earth $\rho = 5520$ kg m^{-3}, we obtain a very high pressure at the center of Earth, $p = 1.74 \times 10^5$ GPa. As shown below, GR gives an even greater value for pressure.

Of interest is the geometric interpretation of the results following from Newton's theory of gravitation. The metric form (1) in a Riemannian space with spherical symmetry has the form

$$\mathrm{d}s^2 = g_{11}\,\mathrm{d}r^2 + g_{22}\,\mathrm{d}\Omega^2 - g_{44}c^2\,\mathrm{d}t^2\,. \tag{30}$$

For a static problem, the metric coefficients included here depend only on the radial coordinate. In the classical theory of gravitation, the motion of a body occurs under the action of gravitational and inertial forces. In GR, this motion occurs along a geodesic line curved by gravity. Comparing the equations of motion, we can find the following metric coefficients of space corresponding to Newton's theory of gravitation [2]:

$$g_{11}^n = 1\,, \quad g_{22}^n = r^2\,, \quad g_{44}^n = 1 - \frac{r_g}{r}\,. \tag{31}$$

It follows that the spatial metric coefficients correspond to the Euclidean space and gravity manifests itself only through the time coefficient. This result will be used further in Section 6.

The solution obtained can be verified experimentally. As is known, gravity causes a change in the direction of light rays passing near the Sun. The angle by which the trajectory of a light ray deviates from a straight line is determined by the equality [8]

$$\alpha = 2J - \pi\,, \quad J = \int_R^\infty \sqrt{\frac{g_{11}}{g_{22}\left((g_{22}g_{44}^R/g_{22}^R g_{44}) - 1\right)}}\,\mathrm{d}r\,, \tag{32}$$

where $g_{ii}^R = g_{ii}(R)$. Using equalities (31) and taking into account that for the Sun $\bar{r}_g = 4.3 \times 10^{-6}$, we have

$$J \approx R\int_R^\infty \frac{1}{r\sqrt{r^2 - R^2}}\left(1 - \frac{r_g}{2r}\right)\mathrm{d}r = \frac{1}{2}\,(\pi - \bar{r}_g)\,. \tag{33}$$

As a result, the first equality (32) yields $\alpha = \bar{r}_g = 0.675''$. This result is half the experimental value ($\alpha = 1.75''$ [14]). Thus, Newton's gravitational theory is not confirmed experimentally.

The equation of motion for a particle of unit mass in the radial direction, taking into account Eqn (18), is written as

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\rho\varphi_e' = -\frac{r_g c^2}{2r^2}\,. \tag{34}$$

Its first integral determines the physical velocity,

$$v^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = c^2\,\frac{r_g}{r} + C\,.$$

The escape velocity for a spherical body of radius R is obtained from here if we take $C = 0$ and $r = R$, i.e.,

$$v_e = c\sqrt{\frac{r_g}{R}}\,. \tag{35}$$

Thus, when $R = r_g$, the escape velocity is equal to the speed of light and the body is invisible. This result is discussed in Ref. [15]. It was obtained at the end of the 18th century by J. Michell and P. Laplace, who predicted the existence of so-called dark stars.

The trajectory of a light beam in the equatorial ($\theta = 0$) plane of a spherical body (Fig. 1) is determined by the following equalities [8]:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = c\sqrt{\frac{g_{44}}{g_{11}}\left(1 - k^2\,\frac{g_{44}}{g_{22}}\right)}\,, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = kc\,\frac{g_{44}}{g_{22}}\,,$$

where k is a coefficient depending on the initial condition. Let us find the physical components, i.e., the radial and circumferential velocities:

$$v_r = \sqrt{\frac{g_{11}}{g_{44}}}\,\frac{\mathrm{d}r}{\mathrm{d}t} = c\sqrt{1 - k^2\,\frac{g_{44}}{g_{22}}}\,, \quad v_\varphi = kc\sqrt{\frac{g_{44}}{g_{22}}}\,. \tag{36}$$

For the velocity directed at an angle α to the radius (see Fig. 1), we have $v_r = c\cos\alpha$, $v_\varphi = c\sin\alpha$. Then,

$$k = \sin\alpha\sqrt{\frac{g_{22}^R}{g_{44}^R}}\,,$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = c\sqrt{\frac{g_{44}}{g_{11}}\left(1 - \sin^2\alpha\,\frac{g_{22}^R g_{44}}{g_{44}^R g_{22}}\right)}\,.$$

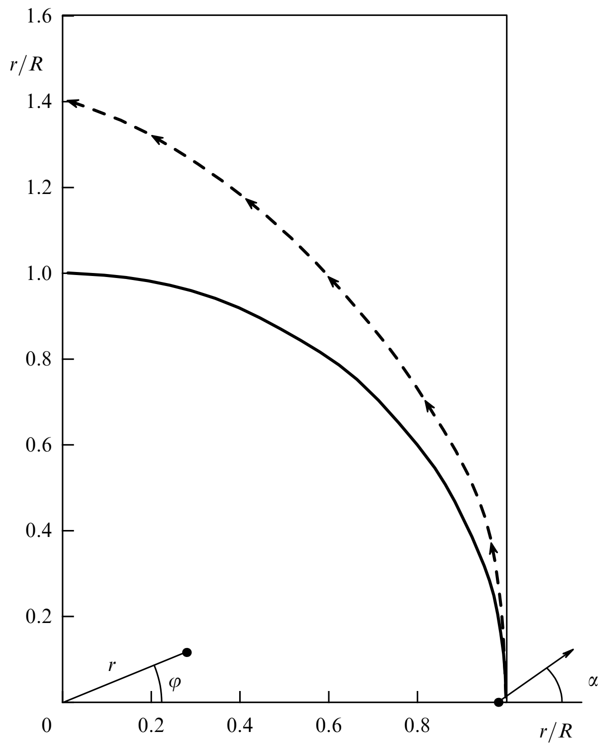

Figure 1. Light propagation from surface of a sphere.

On the surface of the sphere, for $r = R$, we obtain

$$\frac{\mathrm{d}r}{\mathrm{d}t} = c\cos\alpha\sqrt{\frac{g_{44}^R}{g_{11}^R}}\,. \tag{37}$$

Substituting here the metric coefficients (31), corresponding to Newton's theory, we find

$$\frac{\mathrm{d}r}{\mathrm{d}t} = c\cos\alpha\sqrt{1-\frac{r_{\mathrm{g}}}{R}}\,.$$

It follows that, for $R = r_{\mathrm{g}}$, light does not propagate along the radius. Note that a dark star is invisible only from the outside. It follows from the second equality (36) that, on the surface of the body, light propagates along circular trajectories.

5. Linearized General Relativity solution

We assume that the geometry of pseudo-Riemannian space differs little from flat geometry. In this case, the metric coefficients in the form (30) can be written as

$$g_{11} = 1 + f_1\,, \quad g_{22} = r^2(1+f_2)\,, \quad g_{44} = 1 + f_4\,, \tag{38}$$

where the amplitude values of functions f are much less than unity. Then, the equations of GR, linearized with respect to functions f, take the form

$$\frac{1}{r^2}\,(f - rf_4') = \chi\sigma_{\mathrm{r}}\,, \quad \frac{1}{2r}(f - rf_4')' = \chi\sigma_\theta\,, \quad \frac{1}{r^2}(rf)' = \chi\rho c^2\,, \tag{39}$$

where $f = f_1 - (rf_2)'$.

Let us consider the external empty space for which $\rho = 0$ and $\sigma_{\mathrm{r}} = \sigma_\theta = 0$. System (39) then turns out to be homogeneous, i.e.,

$$f - rf_4' = 0\,, \quad (f - rf_4')' = 0\,, \quad (rf)' = 0\,. \tag{40}$$

Note that the second equation is a consequence of the first one. This property of the system is due to the incompleteness of the GR equations, noted in Section 2. Thus, for three unknown functions f_1, f_2, f_4, there are only two independent equations. Let us express the function f from the first equation (40) and substitute it into the second one. As a result, we obtain

$$f_4'' + \frac{2}{r}f_4' = 0\,.$$

For $f_4 = \varphi$, this equation coincides with Eqn (15) for the gravitational potential in empty space ($\rho = 0$) of Newton's theory. Thus, in the linear approximation, General Relativity reduces to Newton's theory in empty space.

To find three functions f from two independent equations of system (40), we extend this system. For a linear approximation, it is natural to take $f_2 = 0$, i.e., $g_{22} = r^2$. This means that the circumference of the sphere is $2\pi R$. However, the space is not Euclidean, since the distance between two points lying on a radius is not equal to the difference between the radial coordinates of these points. For $f_2 = 0$, the solution to Eqns (40) has the form

$$f_1 = \frac{C_1}{r}\,, \quad f_4 = -\frac{C_1}{r} + C_2\,.$$

Since $f_4 \to 0$, for $r \to \infty$, we have $C_2 = 0$ and, according to equalities (38),

$$g_{11} = g^2 \approx 1 + \frac{C_1}{r}\,, \quad g_{22} = r^2\,, \quad g_{44} = h^2 \approx 1 - \frac{C_1}{r}\,.$$

For $r \to \infty$, the solution must asymptotically coincide with expressions (31), corresponding to Newton's theory. We finally obtain $C_1 = r_{\mathrm{g}}$ and

$$g_{11} = 1 + \frac{r_{\mathrm{g}}}{r}\,, \quad g_{22} = r^2\,, \quad g_{44} = 1 - \frac{r_{\mathrm{g}}}{r}\,. \tag{41}$$

These expressions differ from equalities (31), corresponding to Newton's theory. Using formulas (32) and (33), we find the angle of deflection of a light beam in the vicinity of the Sun. Substituting equalities (41), we have

$$J \approx R\int_R^\infty \left[1 + \frac{r_{\mathrm{g}}}{2r} + \frac{r_{\mathrm{g}}r}{2R(R+r)}\right]\frac{\mathrm{d}r}{r\sqrt{r^2 - R^2}} = \frac{\pi}{2} + \frac{2r_{\mathrm{g}}}{R}\,.$$

As a result, we obtain $\alpha = 2\bar{r}_{\mathrm{g}}$, which is twice the angle following from Newton's theory and agrees with the experimental results ($\alpha = 1.75''$). Obtained at the beginning of the 20th century and confirming General Relativity, this result served as the basis for the general recognition of the theory; however, strictly speaking, it confirms only its linear approximation and only for empty space.

Using equalities (37) and (41), we find that, on the surface of the sphere,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = c\sqrt{\frac{1 - r_{\mathrm{g}}/R}{1 + r_{\mathrm{g}}/R}}\cos\alpha\,.$$

Thus, as in Newton's theory, light does not propagate from the surface of a sphere with radius $R = r_{\mathrm{g}}$.

Let us consider the internal problem described by equations (39), which must be supplemented by an equation following from system (2), i.e.,

$$\sigma_r' + \frac{2}{r}\,(\sigma_r - \sigma_\theta) + f_4'\,(\sigma_r - \rho c^2) = 0\,. \tag{42}$$

If we ignore σ_r, in contrast to ρc^2, we obtain the equilibrium equation of elasticity theory (21).

The gravitational constant of GR χ (5) is found, as already noted, by comparing the linearized equations of GR, which are considered in this section, with the equation following from Newton's gravitational theory. We express f from the first equation (39) and f' from the second equation and substitute into the third equation. As a result, we obtain

$$f_4'' + \frac{2f_4'}{r} = \frac{\chi}{2}\,(\rho c^2 - \sigma)\,, \tag{43}$$

where $\sigma = \sigma_r + 2\sigma_\theta$ is the invariant of the stress tensor. Disregarding σ, as opposed to ρc^2, considering that f_4 is expressed through the classical gravitational potential according to the formula $f_t = \pi\varphi/c^2$, and setting $\chi = 8\pi\gamma/c^4$, we arrive at Eqn (15) of Newton's gravitational theory. However, the above derivation is erroneous. The fact is that the two equations (39) include the same combination of functions $(f - rf_t')$ and are generally incompatible. Excluding this combination from the indicated equations, we obtain the following compatibility condition:

$$\sigma_r' + \frac{2}{r}\,(\sigma_r - \sigma_\theta) = 0\,.$$

This equation differs from Eqn (42) by the absence of a term considering gravity. Thus, in the linear approximation, the Einstein tensor does not satisfy the equation of the conservation law for the material tensor, and the linear approximation of GR does not describe gravity in a continuous medium (however, it does describe it in empty space). The reason is that the terms including stresses in Eqn (42) are linear, and the gravitation term is quadratic. Equality (5) for the gravitational constant of GR can be obtained by considering the quadratic approximation of the Einstein equations.

6. Solution in Schwarzschild metric

The complete system of GR equations for a spherically symmetric problem includes three equations (4) [17]:

$$E_1^1 = \frac{1}{g_{22}} - \frac{1}{g_{11}}\left[\frac{1}{4}\left(\frac{g_{22}'}{g_{22}}\right)^2 + \frac{g_{22}'g_{44}'}{2g_{22}g_{44}}\right] = \chi T_1^1\,, \tag{44}$$

$$E_2^2 = -\frac{1}{2g_{11}}\left[\frac{g_{44}''}{g_{44}} - \frac{1}{2}\left(\frac{g_{44}'}{g_{44}}\right)^2 + \frac{g_{22}''}{g_{22}} - \frac{1}{2}\left(\frac{g_{22}'}{g_{22}}\right)^2 + \frac{g_{22}'}{2g_{22}}\left(\frac{g_{44}'}{g_{44}} - \frac{g_{11}'}{g_{11}}\right) - \frac{g_{11}'g_{44}'}{2g_{11}g_{44}}\right] = \chi T_2^2\,, \tag{45}$$

$$E_4^4 = \frac{1}{g_{22}} - \frac{1}{g_{11}}\left[\frac{g_{22}''}{g_{22}} - \frac{1}{4}\left(\frac{g_{22}'}{g_{22}}\right)^2 - \frac{g_{11}'g_{22}'}{2g_{11}g_{22}}\right] = \chi T_4^4 \tag{46}$$

and one equation (6)

$$\frac{dE_1^1}{dr} + \frac{g_{22}'}{g_{22}}\,(E_1^1 - E_2^2) + \frac{g_{44}'}{2g_{44}}\,(E_1^1 - E_4^4) = 0\,, \tag{47}$$

in which $E_i^i = \chi T_i^i$.

6.1 Solution to the external problem

Let us consider the external solution $(\infty > r \geqslant R)$. For empty space, $T_i^i = 0$ and Eqns (44)–(46) are homogeneous. They allow a solution in the general form. Expressing g_{11} and g_{44} through g_{22} by means of Eqns (44) and (46), we obtain [18, 19]

$$g_{11} = \frac{(g_{22}')^2}{4(g_{22} + C_1\sqrt{g_{22}})}\,, \quad g_{44} = C_2\left(1 + \frac{C_1}{g_{22}}\right), \tag{48}$$

where C_1 and C_2 are constants of integration. Substituting these results into Eqn (45), we can conclude that it is satisfied for any function $g_{22}(r)$. As noted in Section 2, Eqns (44)–(46) are not mutually independent. Since the right-hand sides of these equations identically satisfy Eqn (47), equation (45) is a consequence of Eqns (44), (46), and (47).

To obtain a solution, equalities (48) must be supplemented by a certain coordinate condition. The Schwarzschild solution is obtained under the condition $g_{22} = r^2$. Note that Schwarzschild initially used another condition imposed on the determinant of the metric tensor $(G = 1)$, which can be reduced to the condition written above [20, 21]. The justification for Schwarzschild's coordinate condition is known [3, 5, 22], which for the statics problem appears as follows. Let us take $g_{22} = f^2(r)$ and introduce a new variable $r' = f(r)$. Then, $g_{22} = (r')^2$ and, omitting the prime to shorten the notation, we obtain $g_{22} = r^2$.

However, such a transformation is valid for an unbounded space in which $0 \leqslant (r, r') < \infty$ and the difference between r and r' is insignificant. If there is a sphere of radius $r = R$ in space, then it is impossible to determine the corresponding value of r' and formulate the boundary conditions on the surface of the sphere.

The integration constants included in equalities (48) are determined from the asymptotic conditions, according to which, at $r \to \infty$, the resulting solution must degenerate into solution (31), corresponding to Newton's theory. Taking $g_{22} = r^2$, we arrive at the external Schwarzschild solution,

$$g_{11}^e = \frac{1}{1 - r_g/r}\,, \quad g_{44}^e = 1 - \frac{r_g}{r}\,. \tag{49}$$

Here, r_g is defined by equality (17), and the index 'e' corresponds to the external space. From equalities (32) and (33), we obtain $\alpha = 2\bar{r}_g$, which agrees with the experiment. Substituting solution (49) into equality (37), we can conclude that, on the surface of a sphere of radius R,

$$\frac{dr}{dt} = c\cos\alpha\left(1 - \frac{r_g}{R}\right).$$

That is, light does not propagate from the surface of a sphere of radius $R = r_g$ [23, 24]. From equalities (49), it follows that the metric coefficient g_{11}^e becomes infinite on the surface of this sphere, and the coefficient g_{44}^e is equal to zero. Let us assume that solution (49) is valid in all space, i.e., for $0 \leqslant r < \infty$. In this case, solution (49) is singular on a sphere of radius r_g and at the point $r = 0$. The sphere of radius r_g is called the event horizon of a black hole. Inside this sphere, the coefficients g_{11}^e and g_{44}^e change sign. Since metric coefficients cannot be negative, in the metric form (30), we should add a plus sign to the first term and a minus sign to the last term. This means that the radial coordinate behaves like a time coordinate, and, since time is irreversible, after crossing the event horizon, movement is possible only to the center, which is a singularity point. No information can be obtained from under the event horizon. Experimental confirmation of the hypothesis under consideration is, in principle, impossible, since this would require penetrating through the event horizon, showing that there is empty space behind it, and transmitting this information.

Determining the components of the curvature tensor of a pseudo-Riemannian space with metric coefficients (49), we can find that some of them are singular at $r = r_g$, but the invariant of the curvature tensor, which does not depend on the coordinate system, is not singular. From this, it is traditionally concluded that the singularity at $r = r_g$ is coordinate, i.e., it can be eliminated by moving from coordinates r and t to coordinates u and v, using the transformation

$$\mathrm{d}u = c_{11}\,\mathrm{d}r + c_{12}\,\mathrm{d}t\,, \quad \mathrm{d}v = c_{21}\,\mathrm{d}r + c_{22}\,\mathrm{d}t\,. \tag{50}$$

There are several known options for such a transformation. The most common are:

— Lemaitre transformation, for which

$$c_{11} = \sqrt{\frac{r_g}{r}}\,\frac{1}{1 - r_g/r}\,, \quad c_{12} = c_{22} = 1\,,$$
$$c_{21} = \sqrt{\frac{r}{r_g}}\,\frac{1}{1 - r_g/r}\,; \tag{51}$$

— Kruskal transformation, for which

$$c_{11,21} = \frac{rf_\pm}{4r_g^2\sqrt{(r/r_g) - 1}}\,, \quad c_{12,22} = \frac{cf_\mp}{4r_g}\sqrt{\frac{r}{r_g} - 1}\,,$$
$$f_\pm = \exp\left(\frac{(r - ct)}{2r_g}\right) \pm \exp\left(\frac{(r + ct)}{r_g}\right). \tag{52}$$

In coordinates u, v, the coefficients of the metric form are not singular; however, as follows from equalities (51) and (52), the singularity is not eliminated, but is transferred to the transformation, since the coefficients c_{ij} turn out to be singular at $r = r_g$. The transformation of differentials, the coefficients of which are singular, can hardly be called mathematically correct [25].

Using again the analogy with the theory of elasticity, we define the deformations of space as the differences between the metric coefficients of flat and curved space [9]:

$$\varepsilon_{11} = \frac{r_g}{r - r_g}\,, \quad \varepsilon_{22} = 0\,.$$

It follows that the annular deformation at $r = r_g$ is absent, and the radial deformation has a discontinuity. It is quite difficult to explain such a deformation of homogeneous empty space.

In GR, there is the so-called Birkhoff theorem [17], according to which an external spherically symmetric gravitational field is static. From this theorem, in particular, it follows that the pulsation of a spherical body of constant mass does not change the geometry of the space surrounding it. To illustrate this theorem, let us assume that the metric coefficients depend not only on the radial coordinate, but also on time. Then, equations (44) and (46), from which solution (49) is obtained, are written as follows:

$$(E_1^1)_D = E_1^1 + \frac{1}{g_{44}}\left[\frac{\ddot g_{22}}{g_{22}} - \left(\frac{\dot g_{22}}{2g_{22}}\right)^2 - \frac{\dot g_{22}\dot g_{44}}{2g_{22}g_{44}}\right],$$
$$(E_4^4)_D = E_4^4 + \frac{1}{4g_{44}}\left[\left(\frac{\dot g_{22}}{g_{22}}\right)^2 + \frac{2\dot g_{11}\dot g_{22}}{g_{11}g_{22}}\right].$$

Here, E_i^i are determined by equalities (44) and (46), corresponding to the static problem, and $\dot g = \partial g/c\partial t$. When $g_{22} = r^2$, the time derivatives disappear, and the problem reduces to a static one. Thus, Birkhoff's theorem is a consequence of the Schwarzschild coordinate condition. For problems of dynamics, this condition looks even less justified than for problems of statics, since the unknown function of two variables $g_{22}(r, t)$ is equated to one of its arguments. The Schwarzschild coordinate condition is not used in cosmology problems either [2, 26, 27].

The Schwarzschild condition is not the only one used to solve a static spherically symmetric problem. In Refs [7, 8], the harmonic condition (7) is used, which takes the form

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(g_{22}\sqrt{\frac{g_{44}}{g_{11}}}\right) = 2r\sqrt{g_{11}g_{44}}\,.$$

As a result, the following components of the metric tensor are obtained:

$$g_{11}^{\mathrm{e}} = \frac{r + r_g/2}{r - r_g/2}\,, \quad g_{22}^{\mathrm{e}} = \left(r + \frac{r_g}{2}\right)^2, \quad g_{44}^{\mathrm{e}} = \frac{r - r_g/2}{r + r_g/2}\,. \tag{53}$$

For such a metric, equality (37) yields

$$\frac{\mathrm{d}r}{\mathrm{d}t} = c\,\frac{R - r_g/2}{R + r_g/2}\cos\alpha\,.$$

It follows that light does not propagate from the surface of a sphere with a radius $R = r_g/2$. With such a radius, the coefficient g_{11}^{e} (53) turns out to be singular, and $g_{44}^{\mathrm{e}} = 0$.

The correspondence of solution (53) to Schwarzschild solution (49) is discussed in Ref. [28]. There are also other forms of the coordinate condition [29, 30]. In Refs [18, 19], the condition of stationarity of the metric tensor density is proposed to be used as a coordinate condition. This model allows an explanation of gravitational attraction. The optimal space within this model is one with a gravitating mass of spherical shape. If there are several such masses, they, mutually attracted, tend to unite.

6.2 Solution to the internal problem for an ideal incompressible fluid

A more realistic situation is when there is a spherical body with radius R in the center, creating gravity. In this case, solution (49) is valid for $r \geqslant R$ and the singularity of solution (49) for $r = 0$ does not appear. In solution (49) for outer space, the minimum radius is $r = R$. Thus, the Schwarzschild singularity occurs on the surface of the material sphere. This surface, naturally, is not an event horizon, since penetration through it is impossible.

The basic system of equations (44)–(47) for the internal problem for $g_{22} = r^2$ takes the form

$$\frac{1}{r^2} - \frac{1}{g_{11}}\left[\frac{1}{r^2} + \frac{g'_{44}}{rg_{44}}\right] = \chi\sigma_{\mathrm{r}}\,, \tag{54}$$

$$-\frac{1}{2g_{11}}\left[\frac{g''_{44}}{g_{44}} - \frac{1}{2}\left(\frac{g'_{44}}{g_{44}}\right)^2 + \frac{1}{r}\left(\frac{g'_{44}}{g_{44}} - \frac{g'_{11}}{g_{11}}\right) - \frac{g'_{11}g'_{44}}{2g_{11}g_{44}}\right] = \chi\sigma_\theta, \tag{55}$$

$$\frac{1}{r^2} - \frac{1}{g_{11}}\left[\frac{1}{r^2} - \frac{g'_{11}}{rg_{11}}\right] = \chi\rho c^2\,, \tag{56}$$

$$\sigma'_{\mathrm{r}} + \frac{2}{r^2}\,(\sigma_{\mathrm{r}} - \sigma_\theta) + \frac{g'_{44}}{2g_{44}}\,(\sigma_{\mathrm{r}} - \rho c^2) = 0\,. \tag{57}$$

With equation (57), only two of equations (54)–(56) are linearly independent. Thus, for a known density, we have three equations that include four unknown functions $g_{11}, g_{44}, \sigma_r, \sigma_\theta$, i.e., the system is incomplete. The resulting system allows a solution for a sphere consisting of an ideal fluid, for which $\sigma_r = \sigma_\theta = -p$, where p is the pressure in the fluid. From the point of view of elasticity theory, such a problem is statically determinable, i.e., only the equilibrium equation is used to solve it. Equation (57) is simplified as follows:

$$p' + \frac{g'_{44}}{2g_{44}}\,(\rho c^2 + p) = 0\,. \tag{58}$$

The solution to the internal problem was obtained by Schwarzschild for an incompressible fluid, whose density ($\rho = \rho_0$) does not depend on pressure. Equations (54), (56), and (58) were used as the basic ones. The solution to equation (56) has the form

$$g^{\mathrm{i}}_{11} = \frac{1}{1 - (\chi\rho_0 c^2/3)\,r^2 + C_3/r}\,, \tag{59}$$

where the superscript 'i' corresponds to the internal space, and C_3 is the constant of integration. It is necessary to take $C_3 = 0$; otherwise, the coefficient g^{i}_{11} turns out to be singular at the center of a sphere of any radius. Thus,

$$g^{\mathrm{i}}_{11} = \frac{1}{1 - (\chi\rho_0 c^2/3)\,r^2}\,. \tag{60}$$

Expression (60) no longer contains the integration constant. This result is associated with the coordinate condition $g_{22} = r^2$. Substituting this condition into Eqn (46), which is of the second order, leads to Eqn (56), which is of the first order. Solution (59) of this equation contains only one constant, which is determined from the condition of regularity at the center of the sphere. In this case, the constant that would allow the boundary condition on the surface of the sphere to be satisfied,

$$g^{\mathrm{i}}_{11}(r = R) = g^{\mathrm{e}}_{11}(r = R) = \frac{1}{1 - R/r_{\mathrm{g}}}\,, \tag{61}$$

is missing from the solution. Substituting equality (60) into this condition and using relations (5) and (17) for χ and r_{g}, we obtain the following expression for the mass of the sphere:

$$m = \frac{4}{3}\,\pi\rho_0 R^3\,. \tag{62}$$

Expression (62) corresponds to Euclidean space. However, according to equality (60), the space inside the sphere is pseudo-Riemannian, and its mass has the form

$$m = 4\pi\rho_0 \int_0^R \sqrt{g^{\mathrm{i}}_{11}}\,r^2\,\mathrm{d}r \approx \frac{4}{3}\,\pi\rho_0 R^3 \left[1 + \frac{3r_{\mathrm{g}}}{10R} + \frac{9}{56}\left(\frac{r_{\mathrm{g}}}{R}\right)^2 + \ldots\right]. \tag{63}$$

This result coincides with equality (62) for $r_{\mathrm{g}} = 0$, i.e., in the absence of gravity. Since the mass is determined by equality (63), we can conclude that the boundary condition (61) is not satisfied in the Schwarzschild solution. Nevertheless, we will continue the analysis. Using equalities (5), (17), and (62), we find

$$\frac{1}{3}\,\chi\rho_0 c^2 = \frac{r_{\mathrm{g}}}{R^3}\,. \tag{64}$$

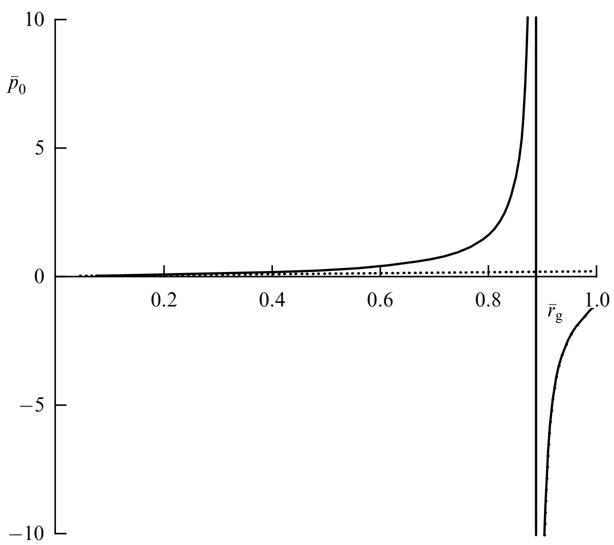

Figure 2. Pressure versus gravitational radius. Solid line — Schwarzschild solution, dotted line — classical solution.

Then, expression (60) takes the form

$$g^{\mathrm{i}}_{11} = \frac{1}{1 - r_{\mathrm{g}} r^2/R^3}\,. \tag{65}$$

Assuming $\sigma_r = -p$ in Eqn (17) and substituting g^{i}_{11} from equality (65), we find

$$\frac{1}{g^{\mathrm{i}}_{44}}\,\frac{\mathrm{d}g^{\mathrm{i}}_{44}}{\mathrm{d}\bar{r}} = \bar{r}_{\mathrm{g}}\bar{r}\,\frac{3\bar{p} + 1}{1 - \bar{r}_{\mathrm{g}}\bar{r}^2}\,, \tag{66}$$

where

$$\bar{r} = \frac{r}{R}\,, \quad \bar{r}_{\mathrm{g}} = \frac{r_{\mathrm{g}}}{R}\,, \quad \bar{p} = \frac{p}{\rho_0 c^2}\,. \tag{67}$$

Substituting expression (66) into equation (58), we obtain

$$\frac{\mathrm{d}\bar{p}}{\mathrm{d}\bar{r}} + \frac{1}{2}\,\bar{r}_{\mathrm{g}}\bar{r}\,\frac{3\bar{p} + 1}{1 - \bar{r}_{\mathrm{g}}\bar{r}^2}\,(\bar{p} + 1) = 0\,. \tag{68}$$

The solution of this equation, satisfying the boundary condition $\bar{p}(\bar{r} = 1) = 0$, has the form

$$\bar{p} = \frac{\sqrt{1 - \bar{r}_{\mathrm{g}}\bar{r}^2} - \sqrt{1 - \bar{r}_{\mathrm{g}}}}{3\sqrt{1 - \bar{r}_{\mathrm{g}}} - \sqrt{1 - \bar{r}_{\mathrm{g}}\bar{r}^2}}\,. \tag{69}$$

Substituting this result into equality (66) and integrating with the boundary condition $g^{\mathrm{i}}_{44}(\bar{r} = 1) = g^{\mathrm{e}}_{44}(\bar{r} = 1)$, we find

$$g^{\mathrm{i}}_{44} = \frac{1}{4}\left(3\sqrt{1 - \bar{r}_{\mathrm{g}}} - \sqrt{1 - \bar{r}_{\mathrm{g}}\bar{r}^2}\right)^2. \tag{70}$$

Assuming that $r = 0$ in Eqn (69), we obtain the pressure at the center of the sphere:

$$\bar{p}_0 = \frac{1 - \sqrt{1 - \bar{r}_{\mathrm{g}}}}{3\sqrt{1 - \bar{r}_{\mathrm{g}}} - 1}\,. \tag{71}$$

The dependence $\bar{p}_0(\bar{r}_{\mathrm{g}})$, constructed using formula (71), is shown in Fig. 2 by a solid line. The dotted line corresponds to the classical solution (29). The denominator of expression (71) vanishes at $\bar{r}_{\mathrm{g}} = \bar{r}_{\mathrm{s}} = 8/9$. Thus, the pressure at the center of the sphere becomes infinite at $R = R_s = 9/8 r_g = 1.125 r_g$. A similar result is traditionally used to physically substantiate the existence of black holes [31]. However, two questions then arise. First, why does the pressure become singular at radius $R = R_s$? This radius is greater than $R = r_g$, at which a singularity is formed on the surface of the sphere. Second, equality (71) determines the pressure at $r > r_s$ as well, but the pressure in this region is negative (see Fig. 2), which has no physical meaning. Thus, we can conclude that the Schwarzschild internal solution for an ideal incompressible fluid is valid at $R > R_s = 1.125 r_g$. In this case, the Schwarzschild singularity, which occurs at $R = r_g$, does not appear.

6.3 Solution to the internal problem for an ideal compressible fluid

The model of an incompressible fluid does not fully correspond to GR, since the speed of sound in such a fluid is infinitely large, and in GR the speed of sound is limited by the speed of light [32]. To solve the problem, we use equations (54)–(57), in which $\sigma_r = \sigma_\theta = -p$ and $\rho = \rho(r)$. As already noted, of the four equations listed, only three are mutually independent. In this case, two approaches are possible. In studies of the spherically symmetric problem of GR, three Einstein equations (54)–(56) for the metric tensor are used for the solution, and the conservation law equation (57) is satisfied automatically. A different approach is typical for the problem of mechanics: the basic equation is (57), and Eqns (54) and (56) are used to determine the coefficients included in this equation. In this case, Eqn (55) is satisfied automatically. In this interpretation, the theory was called relativistic mechanics [33]. Note that system (54), (56), (57) is of lower order than system (54)–(56). In studies where system (54)–(56) is used, it is proposed to obtain the first integral of these equations [34, 35]. When using system (54), (56), (57), this operation is redundant, since these equations are already of lower order. Note that the Schwarzschild solution is obtained based on these equations. Regardless of which equations are used, the system is incomplete and is closed by a physical relationship linking density with pressure.

Note that the approach based on the use of Einstein equations without involving the equations of the conservation law (2) has some methodological features when applied to the continuum. The right-hand sides of the Einstein equations include components of the energy–momentum tensor, which are usually specified. It is essential that the specified components not contradict equations (2). An example is the well-known paper by J. Oppenheimer and H. Snyder [36]. In the Einstein equations, it is assumed that all components of the energy–momentum tensor, except for $T_4^4 = \rho c^2$, are equal to zero. However, Eqns (2) are homogeneous, and it follows from them that $T_4^4 = 0$ and the Einstein equations in the Schwarzschild metric correspond to empty space.

An inverse solution to the GR problem for compressible fluid was obtained by R. Tolman [37]. In Eqns (54)–(56), the function $g_{11}(r)$ or $g_{44}(r)$ was specified and the functions $p(r)$ and $\rho(r)$ were found. Seven possible solutions were found that had relative value, since explicit dependences $\rho(p)$ were not obtained. When the function $\rho(p)$ was specified, it was not possible to obtain an analytical solution. The first numerical solution was constructed by J. Oppenheimer and G. Volkov [38] for the power dependence $\rho(p)$).

Numerical integration of equations (54)–(56) was performed from the center of the sphere, where the desired functions were specified, to the radius at which the pressure vanished. This radius was considered to be the outer radius of the sphere, which was not initially specified.

With the availability of computing technology, solving this problem for a sphere with a given radius does not cause difficulties [39]. Let us assume that the density is a linear function of pressure, i.e.,

$$\rho = \rho_0 (1 + kp) \,. \tag{72}$$

To obtain the solution, we use Eqns (54), (55), and (57) or (58). Let us take into account that $\sigma_r = \sigma_\theta = -p$ and introduce the following notations:

$$\bar{r} = \frac{r}{R}\,, \quad \bar{p} = \frac{p}{\rho_0 c^2}\,, \quad \bar{k} = \frac{k}{\rho_0 c^2}\,, \quad (\cdot)' = \frac{\mathrm{d}(\cdot)}{\mathrm{d}\bar{r}}\,.$$

Here, R is the radius of the outer surface of the sphere, which for the case of a compressible liquid depends on the pressure and is unknown in advance, and ρ_0 is the density of the liquid at $p = 0$. For the problem under consideration, two relative gravitational radii can be introduced: true

$$\bar{r}_g = \frac{2\gamma m}{c^2 R}\,, \quad m = 4\pi\rho_0 \int_0^R (1 + kp)\, r^2\, \mathrm{d}r = 4\pi R^3 \rho_0 \int_0^1 (1 + \bar{k}\bar{p})\, \bar{r}^2\, \mathrm{d}\bar{r}\,,$$

and conditional

$$\bar{r}_g^0 = \frac{2\gamma m}{c^2 R}\,, \quad m = \frac{4}{3}\,\pi\rho_0 R^3\,.$$

When the liquid is compressed, the mass of the sphere does not change, so $\bar{r}_g$ and $\bar{r}_g^0$ are related as

$$\bar{r}_g = 3\bar{r}_g^0 \int_0^1 (1 + \bar{k}\bar{p})\, \bar{r}^2\, \mathrm{d}\bar{r}\,. \tag{73}$$

For a sphere with constant density, equality (64),

$$\frac{1}{3}\,\chi\rho_0 c^2 = \frac{\bar{r}_g^0}{R^2}\,,$$

is valid, with the help of which the coefficient χ can be eliminated from the equations. As a result, equations (54), (55), and (58) take the form

$$\frac{1}{\bar{r}^2} - \frac{1}{g_{11}}\left(\frac{1}{\bar{r}^2} + \frac{g'_{44}}{r g_{44}}\right) = -3\bar{r}_g^0 \bar{p}\,, \quad \frac{\mathrm{d}}{\mathrm{d}\bar{r}}\left(\bar{r} - \frac{\bar{r}}{g_{11}}\right) = 3\bar{r}_g^0 \bar{r}^2 (1 + \bar{k}\bar{p}),$$

$$\bar{p}' + \frac{g'_{44}}{2g_{44}}\left[1 + (1 + \bar{k})\bar{p}\right] = 0\,. \tag{74}$$

These equations are of the third order and are integrated numerically under the boundary conditions

$$g_{11}(\bar{r} = 0) = 1\,, \quad \bar{p}(\bar{r} = 1) = 0\,, \quad g_{44}(\bar{r} = 1) = 1 - \bar{r}_g\,. \tag{75}$$

A special feature of the problem under consideration is that equations (74) include the parameter $\bar{r}_g^0$, and the last boundary condition (75) contains the parameter $\bar{r}_g$. These parameters are related by relationship (73), which includes the unknown pressure $p(\bar{r})$. In this regard, the boundary value problem (74), (75) is solved by the iteration method. At the first stage, $\bar{r}_g^0$ is specified, and it is assumed that $\bar{r}_g = \bar{r}_g^0$. The function $\bar{p}(\bar{r})$ is determined and the value of $\bar{r}_g$

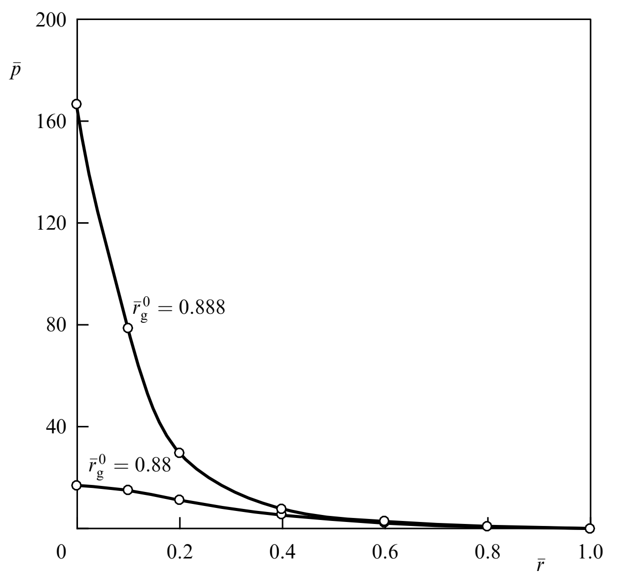

Figure 3. Pressure versus radial coordinate. ○ — numerical solution, solid line — analytical solution.

is found with the aid of (73), which is used as $\bar{r}_g^0$ for the second approximation. The process continues until the third boundary condition is satisfied with a given degree of accuracy.

As a test problem, we consider a sphere consisting of an incompressible fluid, for which $\bar{k} = 0$ and $\bar{r}_g = \bar{r}_g^0$. Figure 3 shows the pressure distributions for $\bar{r}_g^0 = 0.88$ and $\bar{r}_g^0 = 0.888$. The dots correspond to the numerical solution, and the lines to the analytical solution (69). For $\bar{r}_g^0 = 8/9 = 0.888(8)$, the analytically found pressure at the center of the sphere tends to infinity, and the numerical solution diverges. As follows from Fig. 3, the numerical solution coincides well with the analytical one, and its divergence indicates a pressure singularity at the center of the sphere. The Table shows the specified values of the parameter $\bar{r}_g^0$ and the corresponding values of $\bar{r}_g$, at which the numerical solution begins to diverge for different values of the parameter $\bar{k}$.

It can be concluded that, with an increase in the liquid compressibility, the limiting value of $\bar{r}_g$, at which the pressure at the center of the sphere becomes infinite, decreases. The dependences of the relative pressure on the radial coordinate for different values of the parameter $\bar{k}$ (they correspond to the values of the radii given in the Table) are shown in Fig. 4.

Note that the mass of the sphere in equality (73) corresponds to Euclidean space. For mass (63), corresponding to the pseudo-Riemannian internal space, the last boundary condition (75) is not satisfied.

A more complex model of the sphere material is a linearly elastic medium. However, the Schwarzschild metric does not allow solving such a problem, since two stresses are related by one equilibrium equation (57). In the theory of elasticity, the equation of compatibility of deformations (25) is added to this equation, which does not exist in the Riemannian space. A paradoxical situation is created, namely, Newton's theory allows determining the stresses caused by gravity (see Section 3), while the more broad-ranging General Relativity does not. This problem is considered in the next section.

7. Solution in a pseudo-Riemannian space of a special type

Let us turn to equalities (14), which determine the structure of the energy–momentum tensor, and equations (3), (4), which relate this tensor to the curvature of the pseudo-Riemannian space. It follows from these equations that the curvature of space is generated not only by gravity but also by stresses. Let us conduct the following virtual

Table. Gravitational radii corresponding to different values of parameter $\bar{k}$.

$\bar{k}$	0.0001	0.001	0.01	0.1	0.5	1.0	1.5	2.0
$\bar{r}_g^0$	0.8840	0.8820	0.8850	0.7710	0.6110	0.5120	0.4470	0.3990
$\bar{r}_g$	0.8842	0.8836	0.8803	0.8276	0.7197	0.6492	0.5877	0.5359

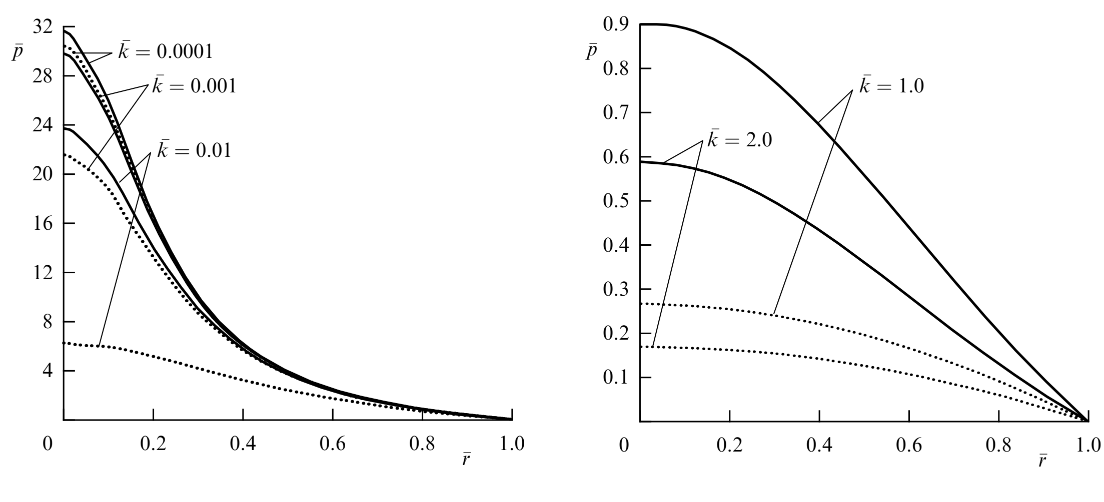

Figure 4. Pressure versus radial coordinate. Solid line — compressible fluid, dotted line — incompressible fluid.

experiment. Assume that a rigid body is in a state of free fall, no forces act on it, and the space is Euclidean both inside and outside the body. Let us apply to the body a system of self-balanced forces that generates a stress state inside the body and does not set it in motion. Then, according to GR, a Riemannian space arises inside the body. However, a three-dimensional Riemannian space is contained in a six-dimensional Euclidean space [22]. The situation turns out to be much simpler and more natural if we assume that there is some pseudo-Riemannian space that is flat with respect to the spatial coordinates and curved only with respect to the time coordinate. Such is the space in Newton's gravitational theory (see Section 3). In such space, processes not related to gravity are described by classical continuum mechanics, and the equations of GR are used only to study gravity. Naturally, we do not mean real space, but its mathematical model — the structure of real space is unknown. If we use the described model of space, then the ten equations of GR for the metric tensor, six of which are, as already noted, mutually independent, include four unknown components of the metric tensor $(g_{14}, g_{24}, g_{34}, g_{44})$ and the system turns out to be overdetermined. Note that, in the theory of partial differential equations, the situation with the number of equations being greater than the number of unknown functions included in them is less critical than that with the number of unknowns exceeding the number of equations. Some of the extra equations can be used to determine arbitrary integration functions included in the solutions of other equations, and some can be satisfied identically. This situation occurs in elasticity theory: the system of equations for solving the problem in stresses contains nine equations (three equilibrium and six strain compatibility), which include six unknown components of the stress tensor. Even though in the general case the system is overdetermined, when solving specific problems, the number of equations always coincides with the number of unknowns. As shown below, such a situation occurs in the spherically symmetric problem of General Relativity, for which the system of GR equations turns out to be complete.

7.1 Equations of General Relativity in a pseudo-Riemannian space of a special type

In spherical coordinates, the general metric form for a spherically symmetric problem is

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 + 2g_{14}c\,\mathrm{d}r\,\mathrm{d}t + 2g_{24}c\,\mathrm{d}\theta\,\mathrm{d}t - g_{44}c^2\,\mathrm{d}t^2\,. \tag{76}$$

The equations of General Relativity for such a form are rather cumbersome and are not given here. The coefficients of these equations include θ and do not correspond to a spherically symmetric problem. Thus, two variants are possible: $g_{14} = 0,\ g_{24} \neq 0$ and $g_{14} \neq 0,\ g_{24} = 0$. For the first case, the solution is the coefficient $g_{24} = Cr^2$, which does not satisfy the asymptotic condition $g_{24}(r \to \infty) = 0$ [40]. Thus, we should take $g_{24} = 0$, and the metric form (76) is expressed as

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 + 2g_{14}c\,\mathrm{d}r\,\mathrm{d}t - g_{44}c^2\,\mathrm{d}t^2\,. \tag{77}$$

We assume that, in the general case, the metric coefficients depend on r and t. Then, the GR equations take the form

$$E_1^1 = \frac{1}{r^2 g^2}\left[g(rg'_{44} - g_{14}^2) + 2rg_{44}\dot{g}_{14} - rg_{14}\dot{g}_{44}\right] = \chi T_1^1\,, \tag{78}$$

$$E_2^2 = \frac{1}{4rg^2}\,(4g_{14}g_{44}g'_{14} - 4g_{14}^2 g'_{44} - 2g_{44}g'_{44} - 2rg_{44}g''_{44} + rg'^2_{44} - 2rg_{14}^2 g''_{44} + 2rg_{14}g'_{14}g'_{44} - 4g_{44}\dot{g}_{14} + 2g_{14}\dot{g}_{44} + 4rg_{14}g'_{14}\dot{g}_{14} + 2rg'_{14}\dot{g}_{44} - 4rg\dot{g}'_{14}) = \chi T_2^2\,, \tag{79}$$

$$E_4^4 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{rg_{14}^2}{g}\right) = \chi T_4^4\,, \tag{80}$$

$$E_1^4 = -\frac{g_{14}}{r}\frac{\partial}{\partial r}\left(\frac{1}{g}\right) = \chi T_1^4\,, \tag{81}$$

$$E_4^1 = -\frac{g_{14}}{rg^2}\,(2g_{44}\dot{g}_{14} - g_{14}\dot{g}_{44}) = \chi T_4^1\,, \quad g = g_{44} + g_{14}^2\,. \tag{82}$$

As before, here, $(\cdot)' = \partial(\cdot)/\partial r$ and $\dot{(\cdot)} = \partial/c\partial t$. The energy–momentum tensor satisfies the equations of the conservation law (2), i.e.,

$$(T_1^1)' + \frac{2}{r}\,(T_1^1 - T_2^2) + \frac{g'_{44}}{2g}\,(T_1^1 - T_4^4) + \frac{g_{14}g'_{44}}{2g}\,T_1^4 + \frac{g'_{14}}{g}\,T_4^1 + \dot{T}_1^4 + \frac{\dot{g}}{2g}\,T_1^4 = 0\,, \tag{83}$$

$$rg'_{44}\big[g_{14}(T_1^1 - T_4^4) - g_{44}T_1^4\big] + 2g\big[r\dot{T}_4^4 + r(T_4^1)' + 2T_4^1\big] + 2rg_{14}g'_{14}T_4^1 - 2rg_{44}\dot{g}_{14}T_1^4 + rg_{14}\dot{g}_{44}T_1^4 = 0\,. \tag{84}$$

The equations of GR are not strictly static even for static problems, since they involve the speed of light directed along the time axis. If the radial and time axes are orthogonal, this speed has no projections on the radial axis or for the static problem in equalities (14) $v_1 = v^1 = 0$. However, with the metric form (77), the axes are not orthogonal and only the contravariant (parallel) projection v^1 is equal to zero, while the covariant (orthogonal) projection v_1 is different from zero. Thus, equalities (14) for the energy-momentum tensor have the form

$$T_1^1 = \sigma_{\mathrm{r}}\,, \quad T_2^2 = \sigma_\theta\,, \quad T_4^4 = \rho c^2\,, \quad T_4^1 = 0\,, \quad T_1^4 = -\rho c v_1\,. \tag{85}$$

7.2 Solution to the external problem

Let us consider the outer empty space surrounding a sphere of radius R. For vacuum $T_i^j = 0$, Eqns (78)–(82) are homogeneous, and Eqns (83), (84) are satisfied identically. Let us find a solution to Eqns (78)–(82), which for empty space take the form

$$rg'_{44} - g_{14}^2 = 0\,, \tag{86}$$

$$4g_{14}(g_{44}g'_{14} - g_{14}g'_{44}) - 2g_{44}(rg'_{44})' + rg'^2_{44} - 2rg_{14}(g''_{44} - g'_{14}g'_{44}) = 0\,, \tag{87}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{rg_{14}^2}{g}\right) = 0\,, \quad \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{g}\right) = 0\,. \tag{88}$$

Due to the existence of two equations (83) and (84), in which $T_i^j = E_i^j/\chi$, only two of the four equations (86)–(88) are mutually independent. Accordingly, there are two unknown functions $g_{14}(r)$ and $g_{44}(r)$, i.e., the system of GR equations is complete. As a result of integrating Eqns (88), we obtain

$$g_{14}^2 = \frac{C_1 g}{r}\,, \quad g = C_2\,, \quad g_{44} = C_2 - \frac{C_1 C_2}{r}\,.$$

This solution should be reduced at $r \to \infty$ to relations (31), corresponding to Newton's theory. Finally, we find

$$g_{14}^{\mathrm{e}} = \pm\sqrt{\frac{r_{\mathrm{g}}}{r}}\,, \quad g_{44}^{\mathrm{e}} = 1 - \frac{r_{\mathrm{g}}}{r}\,. \tag{89}$$

Solution (89) defines two symmetric external spaces corresponding to positive and negative values of g_{14}^{e}. It is obtained from Eqns (88). Substituting solution (89) into Eqns (86) and (87), we can verify that they are satisfied identically. Thus, equalities (89) are a solution of the Einstein equations for the external spherically symmetric problem. Note that this solution, unlike the Schwarzschild solution (49), is not singular. In [40–42], it is proved that light does not propagate from the surface of a sphere with radius $R = r_{\mathrm{g}}$ and that the angle of deviation of the light trajectory from a straight line in the vicinity of the Sun in a space with metric coefficients in the form (89) is equal to $2\bar{r}_{\mathrm{g}}$ and corresponds to the experimental value.

It should be noted that the metric form (89) has been known in GR since the 1920s as the Gullstrand–Painlevé coordinate form [43, 44]. It was obtained by replacing the coordinates in the Schwarzschild metric form and was studied in Refs [45, 46]. In the present paper, this metric is not related to the Schwarzschild metric and follows from the model of pseudo-Riemannian space used.

Performing the identical transformation of the metric form (77), we obtain [40]

$$\mathrm{d}s^2 = \left(1 + \frac{g_{14}^2}{g_{44}}\right)\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 - g_{44}\left(c - \frac{g_{14}}{g_{44}}\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\mathrm{d}t^2\,.$$

For the statics problem, $\mathrm{d}r/\mathrm{d}t = 0$ and

$$\mathrm{d}s^2 = g_{11}^s\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 - g_{44}c^2\,\mathrm{d}t^2\,. \tag{90}$$

This metric form coincides with the form used in the Schwarzschild solution if we accept

$$g_{11}^{\mathrm{e}} = 1 + \frac{g_{14}^2}{g_{44}}\,. \tag{91}$$

Note that, formally, the space turns out to be Riemannian with respect to the radial coordinate, but the corresponding metric coefficient is formed from the time coefficients of the original metric form.

7.3 Solution to the internal problem for an ideal incompressible fluid

For a continuous medium, the equations of general relativity corresponding to the static problem are taken to be

$$\frac{1}{r^2 g}\left(g_{14}^2 - r g_{44}'\right) = \chi\sigma_{\mathrm{r}}\,, \tag{92}$$

$$\frac{1}{4rg^2}\big[4g_{14}(g_{44}g_{14}' - g_{14}g_{44}') - 2g_{44}(rg_{44}')' + r(g_{44}')^2 - 2rg_{14}(g_{44}'' - g_{14}'g_{44}')\big] = \chi\sigma_\theta\,, \tag{93}$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{rg_{14}^2}{g}\right) = \chi\rho c^2\,, \tag{94}$$

$$\frac{g_{14}}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{g}\right) = -\chi\rho c v_1\,, \quad g = g_{44} + g_{14}^2\,. \tag{95}$$

Accordingly, Eqns (83) and (84) take the following form:

$$\sigma_{\mathrm{r}}' + \frac{2}{r}(\sigma_{\mathrm{r}} - \sigma_\theta) + \frac{g_{44}'}{2g}(\sigma_{\mathrm{r}} - \rho c^2) + \frac{g_{14}g_{44}'}{2g}\,T_1^4 = 0\,, \tag{96}$$

$$rg_{44}'\big[g_{14}(\sigma_{\mathrm{r}} - \rho c^2) + g_{44}\rho c v_1\big] = 0\,. \tag{97}$$

For an ideal incompressible fluid, $\sigma_{\mathrm{r}} = \sigma_\theta = -p$ and $\rho = \rho_0$. System (92)–(97) contains four mutually independent equations and includes four unknown functions, g_{14}, g_{44}, p, and v_1, i.e., the system of GR equations is complete. From Eqn (97), we have

$$v_1 = \frac{g_{14}}{\rho_0 c g_{44}}\,(p + \rho_0 c^2)\,. \tag{98}$$

Substituting this result into Eqn (96), we obtain

$$p' + \frac{g_{44}'}{2g_{44}}\,(p + \rho_0 c^2) = 0\,. \tag{99}$$

This equation coincides with Eqn (58), used in the Schwarzschild solution. From Eqn (94), taking into account the condition of regularity of the solution at the center of the sphere, we have

$$g_{14}^2 = \frac{g}{3}\,\chi\rho_0 c^2 r^2\,.$$

Since the space inside the sphere is Euclidean, equality (64) is valid, with the help of which we can eliminate χ. Then,

$$g_{14}^2 = \frac{g}{R^3}\,r_{\mathrm{g}}r^2\,. \tag{100}$$

In accordance with equality (91), we introduce a new function:

$$f = g_{11}^{\mathrm{e}} = 1 + \frac{g_{14}^2}{g_{44}} = \frac{g}{g_{44}}\,. \tag{101}$$

Substituting this result into relation (100), we obtain

$$f = g_{11}^{\mathrm{s}} = \frac{1}{1 - r_{\mathrm{g}}r^2/R^3}\,. \tag{102}$$

This expression coincides with the Schwarzschild solution (65). Transforming equation (92) using equalities (64), (101), and (102), we find

$$\frac{g_{44}'}{g_{44}} = \frac{r_{\mathrm{g}}r}{R^3 - r_{\mathrm{g}}r^2}\left(\frac{3p}{\rho_0 c^2} + 1\right). \tag{103}$$

This result coincides with equality (66), used in the Schwarzschild solution. Since Eqns (99) and (102) coincide with Eqns (58) and (66) from Section 6.2, the functions $p(r)$ and $g_{44}(r)$ turn out to be the same as in the Schwarzschild solution and are determined by relations (69) and (70). Accordingly, the analysis presented in Section 6.2 as applied to the Schwarzschild solution also turns out to be valid. However, there are some differences. First, the contradiction between equalities (62) and (63), which determine the mass of the sphere in Euclidean and Riemannian spaces, that occurs in the Schwarzschild solution does not manifest itself in the obtained solution. The internal space of the sphere in coordinates r, φ, θ is Euclidean, and equality (70) does not exist. Second, relation (100) allows us to find the metric coefficient g_{14}, i.e.,

$$g_{14} = \pm\frac{1}{2}\left(3\sqrt{1-\bar{r}_{\rm g}} - \sqrt{1-\bar{r}_{\rm g}\bar{r}^2}\right)\frac{\bar{r}\sqrt{\bar{r}_{\rm g}}}{\sqrt{1-\bar{r}_{\rm g}\bar{r}^2}}\,.$$

When $\bar{r} = 1$, we have $g_{14} = \pm\sqrt{\bar{r}_{\rm g}}$ and the boundary condition on the surface of the sphere is satisfied. And finally, equality (98) allows us to find the velocity,

$$v_1 = \pm\frac{4c\bar{r}\sqrt{\bar{r}_{\rm g}(1-\bar{r}_{\rm g})}}{\sqrt{1-\bar{r}_{\rm g}\bar{r}^2}\left(3\sqrt{1-\bar{r}_{\rm g}} - \sqrt{1-\bar{r}_{\rm g}\bar{r}^2}\right)^2}\,. \qquad (104)$$

As follows from equation (97), this velocity exists if $g_{44}' \neq 0$. Thus, if the traditional velocity is associated with the dependence of the radial coordinate on time, then the velocity v_1 is associated with the dependence of the time metric coefficient on the radial coordinate. This velocity is studied in Ref. [45]. From relation (104), it can be concluded that any point in space has velocities equal in magnitude and opposite in direction. This allows us to assume that the velocity v_1 is not associated with any real motion.

7.4 Solution to the internal problem for an ideal compressible fluid

Let us assume that the density is related to the pressure by a linear dependence (72). In the notation of Section 6.4, the resolving system of equations, similar to equations (74) for the Schwarzschild metric, can be represented as [42]

$$\bar{r}g_{44}' - g_{14}^2 = 3\bar{r}_{\rm g}^0\, g\bar{p}\bar{r}^2\,, \quad \frac{\rm d}{{\rm d}\bar{r}}\left(\frac{\bar{r}g_{14}^2}{g}\right) = 3\bar{r}_{\rm g}^0\bar{r}^2(1+\bar{k}\bar{p})\,,$$

$$\bar{p}' + \frac{g_{44}'}{2g_{44}}\left[1+(1+\bar{k})\bar{p}\right] = 0\,.$$

This system is integrated partially under the boundary conditions

$$g_{14}(1) = \sqrt{\bar{r}_{\rm g}}\,, \quad \bar{p}(1) = 0\,, \quad g_{44}(1) = 1 - \bar{r}_{\rm g}\,.$$

The solution procedure is described in Section 6.4. The calculation results completely coincide with the results given in Section 6.4, which seems natural in connection with the presence of equalities (90) and (91).

7.5 Solution for a linearly elastic medium

For a spherical body with radius R consisting of a linearly elastic isotropic material, equality (102) is valid, and relation (103) takes the form

$$\frac{g_{44}'}{g_{44}} = \frac{1}{r}\,(f-1) - \chi f r\sigma_{\rm r}\,. \qquad (105)$$

Let us determine the stresses. Substituting v_1 from (97) into (96), we arrive at the equilibrium equation

$$\sigma_{\rm r}' + \frac{2}{r}\,(\sigma_{\rm r} - \sigma_\theta) + \frac{g_{44}'}{2g_{44}}\,(\sigma_{\rm r} - \rho c^2) = 0\,.$$

Substituting equality (102) here and making use of the dimensionless parameters (67), we have

$$\bar{\sigma}_{\rm r}' + \frac{2}{r}\,(\bar{\sigma}_{\rm r} - \bar{\sigma}_\theta) - \frac{\bar{r}_{\rm g}\bar{r}}{2(1-\bar{r}_{\rm g}\bar{r}^2)}\,(1-3\bar{\sigma}_{\rm r})(1-\bar{\sigma}_{\rm r}) = 0\,, \qquad (106)$$

where $\bar{\sigma}_{{\rm r},\theta} = \sigma_{{\rm r},\theta}/\rho_0 c^2$. In the pseudo-Riemannian space traditional for GR, there is no second equation for stresses, and the problem has no solution. However, according to the model used, the pseudo-Riemannian space is flat with respect to coordinates r, θ, φ. In this space, the stresses must satisfy the deformation compatibility equation (25), which at $\nu = 0$ takes the form

$$\bar{\sigma}_{\rm r} = (\bar{r}\bar{\sigma}_\theta)'\,.$$

Using this equation to exclude $\bar{\sigma}_{\rm r}$ from Eqn (106), we finally have

$$\varphi'' + \frac{2\varphi'}{\bar{r}(1-\bar{r}_{\rm g}\bar{r}^2)} - \frac{3\bar{r}_{\rm g}\bar{r}}{2(1-\bar{r}_{\rm g}\bar{r}^2)}\,(\varphi')^2 - \frac{2\varphi}{\bar{r}^2} = \frac{\bar{r}_{\rm g}\bar{r}}{2(1-\bar{r}_{\rm g}\bar{r}^2)}\,, \qquad (107)$$

where $\varphi = \bar{r}\bar{\sigma}_\theta$ and $\bar{\sigma}_{\rm r} = \varphi'$. For small ratios $\bar{r}_{\rm g} = r_{\rm g}/R$, the term $\bar{r}_{\rm g}\bar{r}^2$ can be disregarded compared to unity. Discarding the nonlinear term, we obtain

$$\varphi'' + \frac{2}{r}\,\varphi' - \frac{2}{r^2}\,\varphi = \frac{1}{2}\,\bar{r}_{\rm g}\bar{r}\,.$$

Under the boundary conditions

$$\bar{\sigma}_{\rm r}(0) = \bar{\sigma}_\theta(0)\,, \quad \bar{\sigma}_{\rm r}(1) = 0\,, \qquad (108)$$

the solution of this equation coincides with equalities (26) (for $\nu = 0$).

Equation (107) is solved numerically under the boundary conditions (108) for different values of $\bar{r}_{\rm g}$. The stress dependences $\bar{\sigma}_{\rm r}(\bar{r})$ and $\bar{\sigma}_\theta(\bar{r})$ for $\bar{r}_{\rm g} = 0.05$; 0.1; 0.25; 0.5; 0.75; 0.9 are shown in Figs 5–7 by solid lines. The dotted lines correspond to the classical solution (26). Figure 8 shows the dependence of the relative stress at the center of the sphere on the parameter $\bar{r}_{\rm g}$. This stress is seen to tend to infinity for $\bar{r}_{\rm g} \to 1$.

Integration of Eqn (105) under the boundary condition $g_{44}(1) = 1 - \bar{r}_{\rm g}$ allows finding g_{44}, i.e.,

$$g_{44} = \sqrt{\frac{(1-\bar{r}_{\rm g})^3}{1-\bar{r}_{\rm g}\bar{r}^2}}\,\exp\left(-3\bar{r}\int_1^{\bar{r}}\frac{\bar{r}\bar{\sigma}_{\rm r}\,{\rm d}\bar{r}}{1-\bar{r}_{\rm g}\bar{r}^2}\right).$$

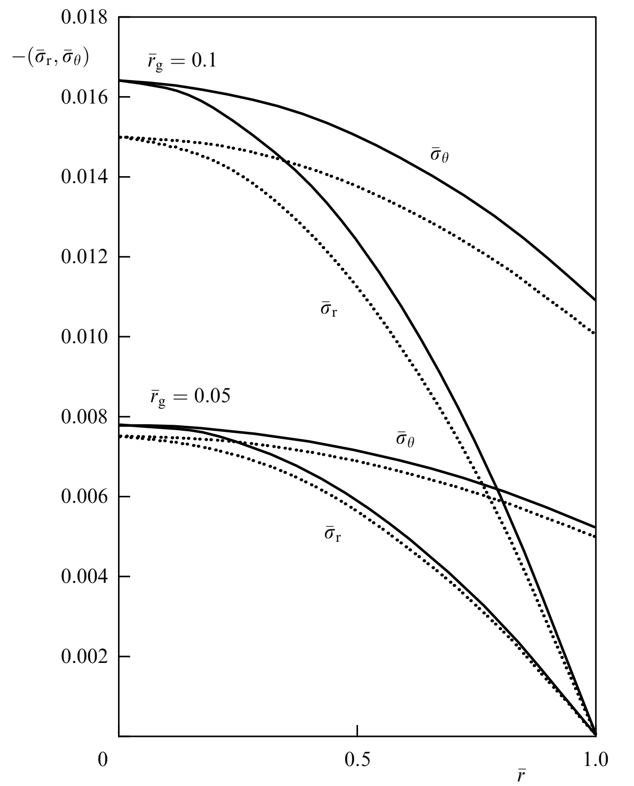

Figure 5. Stress versus radial coordinate for $\bar r_g = 0.05$ and $\bar r_g = 0.1$. Solid line — numerical solution of GR, dotted line — classical solution.

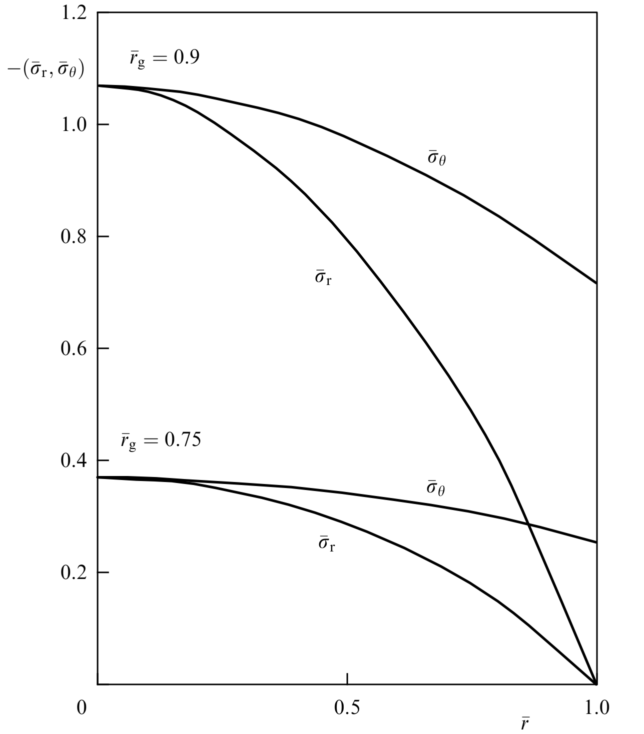

Figure 7. Dependence of stresses on radial coordinate for $\bar r_g = 0.75$ and $\bar r_g = 0.9$.

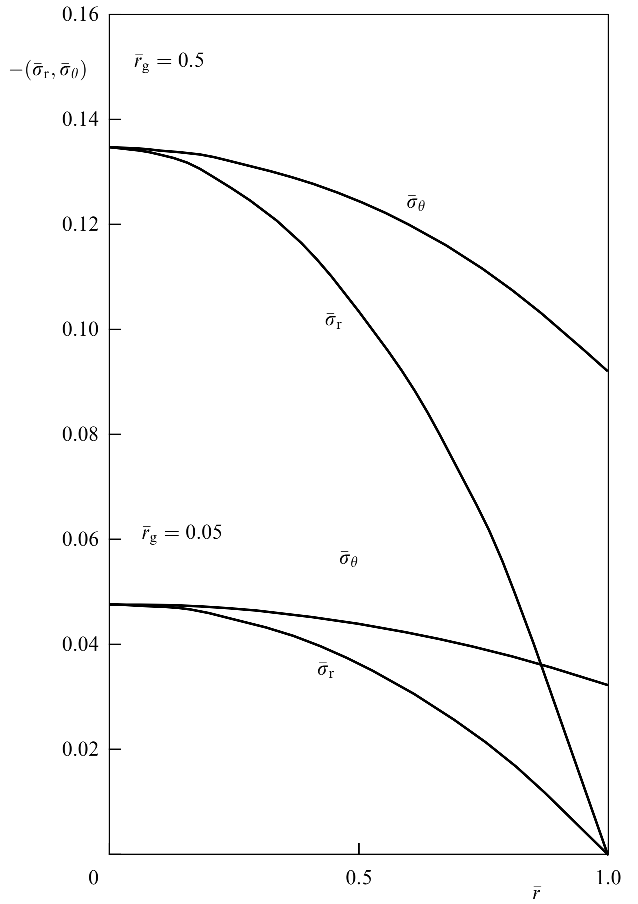

Figure 6. Stress versus radial coordinate for $\bar r_g = 0.25$ and $\bar r_g = 0.5$.

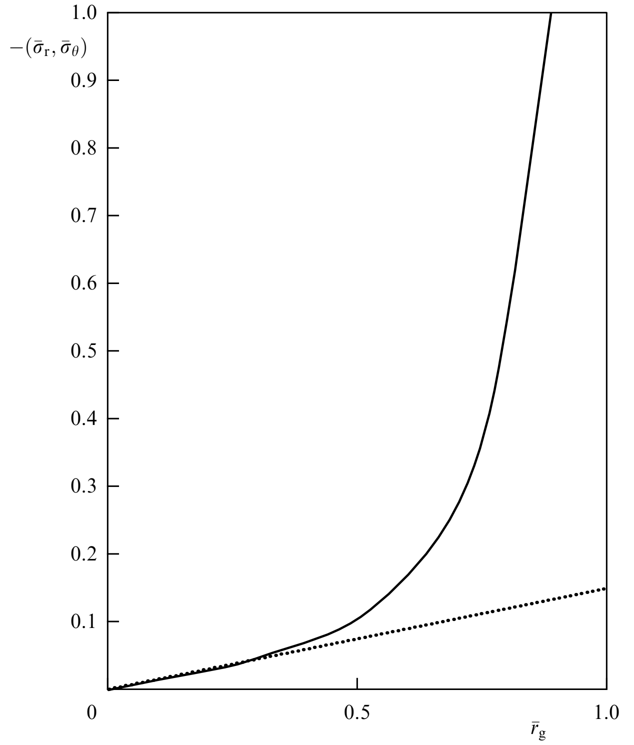

Figure 8. Dependence of stresses at center of sphere on $\bar r_g$. Solid line — numerical solution of GR, dotted line — classical solution.

Then, the coefficient g_{14} can be determined from equality (100). Dependences $g_{44}(\bar r)$ and $g_{14}(\bar r)$ are shown in Fig. 9 and 10, respectively.

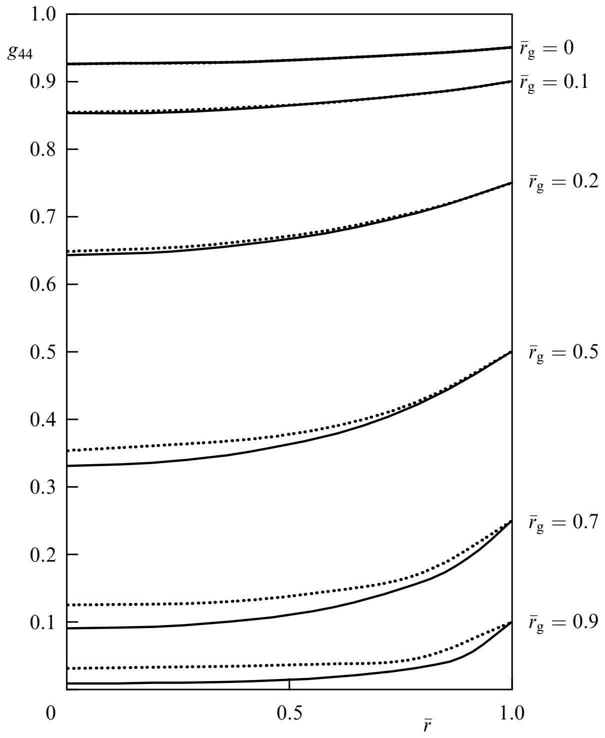

Figure 9. Dependence of g_{44} on radial coordinate.

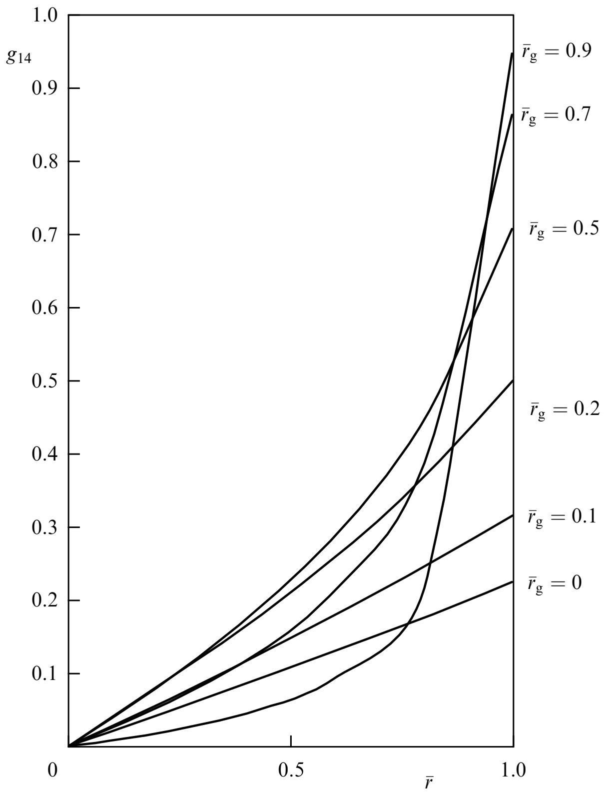

Figure 10. Dependence of g_{14} on radial coordinate.

At $\bar{r}_g = 1$, the spherical body described above has all the essential features of a black hole [31] — light does not propagate from its surface, and the stresses at the center are infinitely large. The difference is the absence of an event horizon and a Schwarzschild singularity. The fundamental possibility of achieving the value $\bar{r}_g = 1$ follows from the relations

$$\bar{r}_g = \frac{2m\gamma}{Rc^2}\,, \quad m = \frac{4}{3}\,\pi\rho_0 R^3\,. \tag{109}$$

Let us assume that the body absorbs matter with a density of ρ_0 from the space surrounding it. It follows from the first equality (109) that $\bar{r}_g$ increases if the mass of the body increases faster than its radius does. The second equality shows that this is precisely the situation that takes place, since the radius changes proportionally to $m^{1/3}$.

Note that the results presented relate to the mechanistic interpretation of GR and do not consider the increase in temperature during compression and other physical effects.

8. Conclusion

An analytical review of the results obtained for a spherically symmetric static problem of relativistic mechanics is presented. Solutions to the external problem for a vacuum and internal problems for an ideal incompressible and compressible fluid and a linearly elastic body are discussed. Along with classical solutions based on the traditional model of pseudo-Riemannian space with a metric form for orthogonal radial and time coordinates, problems in pseudo-Riemannian space, which is flat with respect to spatial coordinates and curved only with respect to time, are considered.

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