

REVIEWS OF TOPICAL PROBLEMS

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Nonsingular cosmological scenarios in scalar-tensor theories and their stability

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Contents

Abstract. This article gives a concise overview of the development and current status of studies on models of the early Universe without an initial singularity, namely the cosmological bounce and genesis scenarios, constructed within a broad class of scalar-tensor theories, specifically Horndeski theories and their generalizations. The review focuses on topics related to linear stability at the perturbation level against the background of nonsingular cosmological solutions: (1) the no-go theorem, valid for nonsingular cosmologies within the Horndeski theory, (2) updates on possible approaches to evade the no-go theorem, (3) the role of disformal transformations relating the Horndeski subclasses in generalized theories like DHOST, (4) effects on stability caused by additional matter coupling and the potential emergence of superluminal perturbation modes in the multi-component setting.

Keywords: scalar-tensor theories, nonsingular cosmology, stability, superluminality, cosmological bounce, genesis

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1. Introduction

There is much that is known and much that is unknown about the physics of the early Universe, and the further back in time we go, the less we can say about the Universe with any degree of confidence. While our knowledge of the hot big bang stages, starting with the fractions of a second from the beginning of the Universe's existence, is quite solid from both a theoretical and an observational point of view, we delve into a yet purely theoretical domain when the temperature climbs up to hundreds MeV.

The common standpoint today is that the conventional hot big bang cosmological stages were preceded by the epoch(s) that has to resolve the standard set of big bang problems, namely, the horizon puzzle, the spatial flatness problem, the entropy dilemma, and the issue of primordial density fluctuations. The concept of inflation [1, 2] elegantly resolves these problems, but the inflationary scenario itself faces, among other issues [3, 4], a serious challenge of geodesic incompleteness in the past, i.e., it cannot solve the initial singularity problem [5, 6]. Hence, the inflationary epoch cannot provide a complete discription of the very early Universe.[1]

One of the alternative or possibly complementary cosmological models that addresses the initial singularity problem is the nonsingular bouncing scenario, where the Universe initially undergoes a contraction phase, and then transits

[1] However, the inflationary concept may be considered from a somewhat broader perspective. In particular, Refs [7–9] consider inflationary models which are in a way similar to the Genesis scenario and feature an asymptotically vanishing Hubble parameter at early times. These kinds of cosmological dynamics resolve the standard problem of geodesic incompleteness in the past. Importantly, this type of inflationary model requires violation of the Null Energy Conditions (see the discussion in Section 1.1), just as the bouncing and Genesis models do.

into the currently expanding phase through a moment of a 'bounce' characterized by a finite value of the scale factor (see [10–13] for reviews). Another nonsingular model is the Universe starting off with the cosmological Genesis, where the Universe starts its expansion from the asymptotically empty Minkowski space [14, 15]. Both these scenarios share the feature of relying on nontrivial dynamics, which requires the Hubble parameter to grow. Indeed, the bouncing concept implies that the Hubble rate transits from negative to positive values at the moment of the bounce; the Genesis scenario implies that at the onset the Hubble parameter is vanishing and starts growing when the expansion launches. Such behavior of the Hubble rate is impossible within the framework of General Relativity with conventional matter, respecting all the energy conditions. Hence, the nonstandard dynamics of both scenarios comes at the price of introducing an exotic matter component, which would violate the Null Energy Condition (NEC) and allow avoiding the singularity without abandoning General Relativity or making use of space curvature. Violation of the NEC often leads to the appearance of pathological degrees of freedom among perturbations over a homogeneous background (see [16] for review), which makes it challenging to construct a viable model with a bouncing or Genesis stage, where the instabilities are absent.

In this mini-review, we focus on Horndeski theories and their generalizations as a suitable setting for a safe NEC-violation, which makes this wide class of scalar-tensor theories a promising framework for constructing viable nonsingular cosmologies. The main emphasis is put on analyzing and verifying the stability of nonsingular solutions at the linearized level: we discuss in detail the necessary criteria which enable one to ensure that all types of instabilities are absent over the background throughout the entire evolution of the model. We discuss the potential trouble with stability related to coupling additional matter components to the scalar-tensor setup. We also address some recent developments in the field and, e.g., briefly discuss the new classes of theories which further generalize the Horndeski family.

This review has the following structure. In Section 1.1, we discuss in detail the NEC and its implications for nonsingular cosmologies, and in Section 1.2, we briefly review the structure of Horndeski theory and its generalizations. An explicit analysis of perturbation behavior over the cosmological background in beyond Horndeski (or, equivalently, GLPV) theory is given in Section 2 and provides the base for the stability no-go theorem proved for Horndeski theories in Section 3.1. In Sections 3.2–3.5, we discuss different ways to evade the no-go theorem and analyze the role of disformal transformations in this context in Section 3.6. We address the issues of whether matter coupling causes trouble for stability in the generalized Horndeski theory in Section 4 and also separately analyze the potential problem with emergent superluminal modes upon coupling of the extra scalar field in Section 4.2. We provide a brief conclusion in Section 5.

1.1 Null energy condition and its violation

Energy conditions play a significant role in General Relativity (GR), as they provide natural restrictions on the energy–momentum tensor $T_{\mu\nu}$ of a physical system. Among other energy conditions, the NEC, which states that

$$T_{\mu\nu} n^\mu n^\nu \geqslant 0 \qquad (1)$$

for any null vector n^μ satisfying $g_{\mu\nu} n^\mu n^\nu = 0$, plays a special role. Despite being the weakest of all, the NEC is particularly robust and fundamental, as it cannot be violated by adjusting the vacuum energy contribution and, hence, unambiguously refers to the properties of matter. For an isotropic fluid with pressure p and positive energy density ρ, the NEC amounts to

$$\omega = \frac{p}{\rho} \geqslant -1 \, . \qquad (2)$$

Moreover, violating the NEC has been closely associated with catastrophic consequences from the stability point of view, since NEC violation gives rise to various pathologies, including ghosts, gradient instabilities (exponentially growing modes with arbitrarily short wavelengths), and superluminality[2] [17].

Another aspect which makes the NEC stand out is the validity of the Penrose singularity theorem [18], where the NEC is assumed to be satisfied. In particular, the Penrose theorem almost precludes the nonsingular bouncing scenario within GR, which can be seen immediately from the linear combination of (00)- and (ij)-components of Einstein equations written for a homogeneous, isotropic Universe with a Friedmann–Lemaître–Robertson–Walker (FLRW) metric [16]:

$$\dot{H} = -4\pi G(\rho + p) + \frac{\kappa}{a^2} \, , \qquad (3)$$

where a is a scale factor, $H \equiv \dot{a}/a$ is a Hubble parameter, and κ is a curvature parameter ($\kappa = +1$ for the closed Universe, $\kappa = -1$ for the open Universe, and $\kappa = 0$ for a flat Euclidean 3D space). Indeed, $\dot{H} \leqslant 0$ in a spatially-flat Universe ($\kappa = 0$) or the open Universe ($\kappa = -1$), since the NEC implies that $\rho + p \geqslant 0$ (see Eqn (2)). Hence, the contraction ($H < 0$) cannot evolve into expansion ($H > 0$) unless the NEC is violated. There is a loophole, however, for the case of the closed Universe ($\kappa = +1$), where the bounce is possible provided ρ and p grow more slowly than a^{-2} during the contraction phase (i.e., $\omega = p/\rho < -1/3$) [19–22].

As for the Genesis scenario, it is also ruled out in GR by the Penrose theorem and the NEC. Indeed, the expansion starting from the asymptotically Minkowski space implies the evolution from $H = 0$ to positive values; hence, $\dot{H} > 0$, which, according to Eqn (3), requires NEC violation. Alternatively, the same conclusion follows from the covariant conservation of the energy–momentum tensor $\nabla_\mu T^{\mu\nu} = 0$ written for a homogeneous, isotropic Universe:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -3H(\rho + p) \, . \qquad (4)$$

While the NEC is satisfied, Eqn (4) implies that the phase with the increasing energy density ($\dot{\rho} > 0$), which is crucial for the Genesis stage, is impossible in the expanding Universe. Hence, the Penrose theorem states that the cosmological expansion started from a singularity with an infinite energy density and infinite expansion rate.

Therefore, violating the NEC (1) in a safe way is a primary task for any nonsingular cosmological scenario, which involves either a contraction phase before the Big Bang or an expansion phase starting from the asymptotically static empty space. While different approaches exist to invoke NEC violation, in this review, we focus on the case when it is

[2] For brevity, here and hereafter, we call perturbations superluminal when their propagation speed exceeds the speed of light.

achieved by modifying GR, which is realized within the scalar-tensor theories with higher-derivative Lagrangians, namely, Horndeski theories [23] and their extensions.

1.2 Horndeski theories as suitable NEC-violating framework

Scalar-tensor theories embody probably the simplest way to extend GR, which amounts to introducing an additional scalar degree of freedom on top of the two tensor modes already present within the original GR. Even though the ultraviolet (UV) behavior of the scalar-tensor theories is not much better than that of GR, the scalar-tensor family is of particular significance, since most infrared (IR) modifications of gravity can be described by scalar-tensor theories.

The most general scalar-tensor theory with the second order equations of motion was formulated by Horndeski [23]. The Horndeski theories are parametrized by four arbitrary functions F, K, G_4, and G_5 [3] of the scalar field π and its kinetic term $X = g^{\mu\nu}\partial_\mu\pi\,\partial_\nu\pi$ and have the following form of the Lagrangian (metric signature $(+,-,-,-)$):

$$S_{\rm H} = \int \mathrm{d}^4x\,\sqrt{-g}\big(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5\big)\,, \tag{5}$$

$$\mathcal{L}_2 = F(\pi, X)\,, \tag{6}$$

$$\mathcal{L}_3 = K(\pi, X)\Box\pi\,, \tag{7}$$

$$\mathcal{L}_4 = -G_4(\pi, X)R + 2G_{4X}(\pi, X)\big[(\Box\pi)^2 - \pi_{;\mu\nu}\pi^{;\mu\nu}\big]\,, \tag{8}$$

$$\mathcal{L}_5 = G_5(\pi, X)G^{\mu\nu}\pi_{;\mu\nu} + \frac{1}{3}\,G_{5X}\big[(\Box\pi)^3 - 3\Box\pi\pi_{;\mu\nu}\pi^{;\mu\nu} + 2\pi_{;\mu\nu}\pi^{;\mu\rho}\pi_{;\rho}^{\ \ \nu}\big]\,, \tag{9}$$

where R in (8) and $G^{\mu\nu}$ in (9) stand for the Ricci scalar and Einstein tensor, respectively, and $\pi_{;\mu\nu} = \nabla_\nu\nabla_\mu\pi$, $\Box\pi = g^{\mu\nu}\nabla_\nu\nabla_\mu\pi$, $G_{iX} = \partial G_i/\partial X$. In fact, the form of the action (5) is different from the original one formulated by Horndeski in [23] and was independly derived within a context of Galileon theory [24] studies and its generalizations [25–27]. Both forms are equivalent though and describe the same theory [66]. One of the main advantages of Horndeski theories is that, by definition, every scalar-tensor theory, whose equations of motion are second order, belongs to the Horndeski group and can be cast into the form (5). In particular, given a specific choice of scalar potentials F, K, G_4, and G_5, one can reproduce, for example, Einstein–Hilbert action, Brans–Dicke theory [29], $f(R)$-gravity theories [30, 31], k-essence [32, 33], kinetic gravity braiding [35, 36], Fab Four [37, 38] (which involve couplings to the Gauss–Bonnet term), Dirac–Born–Infeld (DBI) galileons [39, 40], any inflationary theory based on a scalar field, e.g., k-inflation [41] and G-inflation [42], and many more (see, e.g., [43, 44] for a thorough review).

Higher-derivative terms in Lagrangian (5) are the hallmark of Horndeski theories. Since the corresponding equations of motion are still of the second order in derivatives, Horndeski theories are manifestly free of Ostrogradsky ghosts. However, the second order field equations provide a sufficient but not necessary condition for avoiding the Ostrogradsky ghost. Indeed, if the kinetic matrix of the system is degenerate, then, despite having equations of motions with time-derivatives of order higher than two, the extra degree of freedom (DOF) associated with the additional initial conditions gets eliminated. This is the observation which underlies the formulation of Degenerate Higher-Order Scalar-Tensor (DHOST) theories [45–48], which generalize Horndeski theories. The general form of the corresponding Lagrangian for both the quadratic and cubic subclasses [4] of DHOST theories can be found, e.g., in review [49].

The important difference between the DHOST theories and the Horndeski subclass is that the former are described by the third order equations of motion while still propagating $2+1$ DOFs. Another peculiar and quite useful feature is that various physically interesting subclasses of DHOST theories are related by disformal transformations [47, 49, 50]. A conformal-disformal transformation [51] of a metric generally depends on two arbitrary functions $\Omega(\pi, X)$ and $\Gamma(\pi, X)$:

$$g_{\mu\nu} \to \Omega^2(\pi, X)\,g_{\mu\nu} + \Gamma(\pi, X)\partial_\mu\pi\partial_\nu\pi\,. \tag{10}$$

Upon applying the transformation to the metric in Horndeski theory (5), one can obtain different resulting theories based on the specific choice of $\Omega(\pi, X)$ and $\Gamma(\pi, X)$. In particular, if both functions depend only on π, i.e., $\Omega = \Omega(\pi)$ and $\Gamma = \Gamma(\pi)$, the resulting theory is still a Horndeski theory [52, 53]. The next option is to allow $\Gamma = \Gamma(\pi, X)$ with still X-independent conformal factor $\Omega = \Omega(\pi)$. Then, the resulting theory is the so-called 'beyond Horndeski theory' or GLPV theory — the first example of DHOST theories discovered by applying the disformal transformation (10) to the Einstein–Hilbert action [56]. Beyond Horndeski theories were also formulated independently in [57–59]. The general form of the beyond Horndeski Lagrangian involves additional contributions with two new functions $F_4(\pi, X)$ and $F_5(\pi, X)$ along with those in Eqn (5):

$$S_{\rm BH} = \int \mathrm{d}^4x\,\sqrt{-g}\big(F_4(\pi, X)\epsilon^{\mu\nu\rho}{}_{\sigma}\epsilon^{\mu'\nu'\rho'\sigma}\pi_{,\mu}\pi_{,\mu'}\pi_{;\nu\nu'}\pi_{;\rho\rho'} + F_5(\pi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\pi_{,\mu}\pi_{,\mu'}\pi_{;\nu\nu'}\pi_{;\rho\rho'}\pi_{;\sigma\sigma'}\big)\,, \tag{11}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita tensor. It should be noted that not all combinations of terms in Eqns (5) and (11) are allowed, since in some cases an additional propagating mode does arise in fact, which signals that the corresponding kinetic matrix is nondegenerate. The degeneracy requirement for the resulting theory imposes the following constraint on the Lagrangian functions F_4 and F_5:

$$F_4\,G_{5X}X = -3F_5\left[G_4 - 2XG_{4X} + \frac{1}{2}\,G_{5\pi}X\right]\,, \tag{12}$$

which was discussed in [45, 48, 53], given without derivation, e.g., in [54], and explicitly derived, e.g., in [55]. This relation for F_4 and F_5 can be viewed equivalently as follows: upon the disformal transformation of Horndeski theory with $\Omega(\pi)$ and $\Gamma(\pi, X)$, the resulting beyond Horndeski theory features the contributions of F_4 and F_5, which are compliant with constraint (12).

Finally, upon applying the disformal transformation with both conformal and disformal factors depending on X, i.e.,

[3] There is no complete agreement on notations in the literature and, hence, the functions F and K can be alternatively denoted as G_2 and G_3, respectively.

[4] Here, we adopt the standard naming of the subclasses based on the highest power of $\partial^2\pi$ involved.

$\Omega(\pi, X)$ and $\Gamma(\pi, X)$, the Horndeski theory gets converted into the so-called Ia subclass of DHOST theory, which we specify below. The general form of the action for the quadratic DHOST theory reads[5]

$$S_{\rm DHOST} = \int d^4x\,\sqrt{-g}\bigg(F(\pi, X) + K(\pi, X)\Box\pi + F_2(\pi, X)R + \sum_{i=1}^{5} A_i(\pi, X)L_i\bigg)\,, \tag{13}$$

with

$$L_1 = \pi_{;\mu\nu}\pi^{;\mu\nu}\,,\quad L_2 = (\Box\pi)^2\,,\quad L_3 = \pi^{;\mu}\pi_{;\mu\nu}\pi^{;\nu}\Box\pi\,,$$
$$L_4 = \pi^{;\mu}\pi_{;\mu\nu}\pi^{;\nu\rho}\pi_{,\rho}\,,\quad L_5 = (\pi^{;\mu}\pi_{;\mu\nu}\pi^{;\nu})^2\,, \tag{14}$$

where F, K, $F_2(\pi, X)$, and $A_i(\pi, X)$ are functions analogous to G_i in Eqn (5), although interrelated by three equations, which implement the degeneracy constraints to preserve $2+1$ DOFs in the system (see, e.g., [45, 47]). DHOST theories are stable WRT general conformal-disformal transformations (10), meaning that the degeneracy of the kinetic matrix is preserved as a result of such metric transformations [50]. In what follows, we will concentrate on the DHOST Ia subclass, which is characterized by the following relations between the Lagrangian functions $F_2(\pi, X)$ and $A_i(\pi, X)$ (see, e.g., [49]):

$$A_2 = -A_1\,, \tag{15a}$$

$$A_4 = \frac{1}{8(F_2 - XA_1)^2}\big[-16XA_1^3 + 4(3F_2 + 16XF_{2X})A_1^2 - (16X^2F_{2X} - 12XF_2)A_3A_1 - X^2F_2A_3^2 - 16F_{2X}(3F_2 + 4XF_{2X})A_1 + 8F_2(XF_{2X} - F_2)A_3 + 48F_2F_{2X}^2\big]\,, \tag{15b}$$

$$A_5 = \frac{(4F_{2X} - 2A_1 + XA_3)(-2A_1^2 - 3XA_1A_3 + 4F_{2X}A_1 + 4F_2A_3)}{8(F_2 - XA_1)^2}\,. \tag{15c}$$

The DHOST Ia subclass involves both Horndeski and beyond Horndeski theories as special cases, which is supported by the fact that the result of the disformal transformation of Horndeski theory with $\Omega(\pi, X)$ and $\Gamma(\pi, X)$ corresponds to the DHOST Ia subclass. This can be viewed alternatively as a characteristic feature of DHOST Ia theories: Lagrangian (15) can be mapped into the Horndeski form via the conformal-disformal transformation [50].

The DHOST Ia subclass is of particular significance for cosmology, since the rest of the subclasses are phenomenologically disfavored (they either suffer from instabilities or feature nondynamical tensor modes [60, 61]). Therefore, only those DHOST theories which are disformally related to Horndeski are viable. Let us note that the two theories related by conformal-disformal transformation (10) are manifestly equivalent as long as the transformation is invertible and there are no additional matter components present in the system. We address the subtleties related to preserving the degeneracy upon coupling additional matter to DHOST theories and other related issues in Section 4.

As discussed and explicitly shown, e.g., in [16, 43], Horndeski theories and their extensions are capable of violating the NEC[6] and preserving the stability at the level of perturbations at the same time, provided one considers $\mathcal{L}_3$ subclass (7) or higher. However, as we will see in Section 3 not all subclasses of Horndeski theories (5) are suitable for constructing nonsingular cosmological models that are free from pathological degrees of freedom throughout the entire evolution of the Universe.

[5] We refrain from writing down the cubic DHOST action for brevity. As mentioned before, the complete action can be found, e.g., in [49].

2. Stability of cosmological background in Horndeski theories and beyond

Let us start with a brief revision of the linearized beyond Horndeski theory in a cosmological setting, which provides the necessary tools for an analysis of perturbation behavior. Our primary objective is to derive the stability criteria for a homogeneous and isotropic cosmological model. In what follows, we adhere to the covariant formalism, while the results of this section were also formulated within the effective field theory (EFT) approach [63, 64].

Linear perturbations over the cosmological FLRW background can be classified in a standard way according to their behavior WRT rotations into scalar, vector, and tensor modes (see, e.g., [65]). In what follows, we focus on the scalar and tensor perturbations, since the vector modes are nondynamical in scalar-tensor theories.

We adopt the following parametrization for the metric perturbations:

$$ds^2 = N^2\,dt^2 - \gamma_{ij}(dx^i + N^i\,dt)(dx^j + N^j\,dt)\,, \tag{16}$$

where

$$N = 1 + \alpha\,,\quad N_i = \partial_i\beta\,,\quad \gamma_{ij} = a^2(t)e^{2\zeta}\bigg(\delta_{ij} + h_{ij}^T + \frac{1}{2}\,h_{ik}^T h_j^{k\,T}\bigg). \tag{17}$$

In Eqn (17), α, β, and ζ belong to the scalar sector of perturbations, while h_{ik}^T denote transverse traceless tensor modes ($h_{ii}^T = 0$, $\partial_i h_{ij}^T = 0$). Another contribution to the scalar sector comes from the perturbations of the homogeneous scalar field $\pi(t)$, which are denoted as $\delta\pi$. Note that in Eqn (17) we have already removed the longitudinal component $\partial_i\partial_j E$ in γ_{ij} by partially fixing the gauge. The residual gauge freedom is given by the following transformations:

$$\alpha \to \alpha + \dot{\xi}_0\,,\quad \beta \to \beta - \xi_0\,,\quad \delta\pi \to \delta\pi + \xi_0\dot{\pi}\,,\quad \zeta \to \zeta + \xi_0 H\,, \tag{18}$$

where H is the Hubble parameter, ξ_0 is the gauge function, and, as before, an overdot stands for the time derivative.

Substituting metric (16) into the action $S_{\rm H} + S_{\rm BH}$ (see Eqns (5) and (11)) and expanding it to the second order in perturbations, we obtain $S^{(2)} = S^{(2)}_{\rm tensor} + S^{(2)}_{\rm scalar}$, where

$$S^{(2)}_{\rm tensor} = \frac{1}{8}\int dt\,d^3x\,a^3\bigg[\mathcal{G}_\mathcal{T}\big(\dot{h}_{ij}^T\big)^2 - \frac{\mathcal{F}_\mathcal{T}}{a^2}\big(\nabla h_{ij}^T\big)^2\bigg]\,, \tag{19}$$

$$\mathcal{G}_\mathcal{T} = 2G_4 - 4G_{4X}X + G_{5\pi}X - 2HG_{5X}X\dot{\pi} + 2F_4X^2 + 6HF_5X^2\dot{\pi}\,, \tag{20}$$

$$\mathcal{F}_\mathcal{T} = 2G_4 - 2G_{5X}X\ddot{\pi} - G_{5\pi}X\,. \tag{21}$$

[6] In the general case, when the gravity gets modified (this is the case for the Horndeski theory and its generalizations), the NEC (1) is replaced with the Null Convergence Condition (NCC) [62]: $R_{\mu\nu}n^\mu n^\nu > 0$, where $R_{\mu\nu}$ is the Ricci tensor.

As for the scalar sector and the corresponding quadratic action $S^{(2)}_{\text{scalar}}$, its explicit form depends on the residual gauge fixing choice in Eqn (18). One of the standard approaches is to remove the perturbations of the scalar field $\delta\pi = 0$ (the unitary gauge). Then, the quadratic action for the scalar sector reads[7]

$$S^{(2)}_{\text{scalar}} = \int \mathrm{d}t\, \mathrm{d}^3x\, a^3 \Bigg[-3\mathcal{G}_{\mathcal{T}}\dot{\zeta}^2 + \mathcal{F}_{\mathcal{T}}\,\frac{(\nabla\zeta)^2}{a^2} - 2(\mathcal{G}_{\mathcal{T}} + \mathcal{D}\dot{\pi})\alpha\,\frac{\Delta\zeta}{a^2} + 2\mathcal{G}_{\mathcal{T}}\dot{\zeta}\,\frac{\Delta\beta}{a^2} + 6\Theta\alpha\dot{\zeta} - 2\Theta\alpha\,\frac{\Delta\beta}{a^2} + \Sigma\alpha^2 \Bigg], \tag{22}$$

where $(\nabla\zeta)^2 = \delta^{ij}\partial_i\zeta\partial_j\zeta$, $\Delta = \delta^{ij}\partial_i\partial_j$, while $\mathcal{G}_{\mathcal{T}}$, $\mathcal{F}_{\mathcal{T}}$ are given in (20), (21) and

$$\mathcal{D} = -2F_4X\dot{\pi} - 6HF_5X^2\,, \tag{23}$$

$$\begin{aligned}\Theta = {} & -K_XX\dot{\pi} + 2G_4H - 8HG_{4X}X - 8HG_{4XX}X^2 \\ & + G_{4\pi}\dot{\pi} + 2G_{4\pi X}X\dot{\pi} - 5H^2G_{5X}X\dot{\pi} - 2H^2G_{5XX}X^2\dot{\pi} \\ & + 3HG_{5\pi}X + 2HG_{5\pi X}X^2 + 10HF_4X^2 + 4HF_{4X}X^3 \\ & + 21H^2F_5X^2\dot{\pi}\,,\end{aligned} \tag{24}$$

$$\begin{aligned}\Sigma = {} & F_XX + 2F_{XX}X^2 + 12HK_XX\dot{\pi} + 6HK_{XX}X^2\dot{\pi} - K_\pi X \\ & - K_{\pi X}X^2 - 6H^2G_4 + 42H^2G_{4X}X + 96H^2G_{4XX}X^2 \\ & + 24H^2G_{4XXX}X^3 - 6HG_{4\pi}\dot{\pi} - 30HG_{4\pi X}X\dot{\pi} \\ & - 12HG_{4\pi XX}X^2\dot{\pi} + 30H^3G_{5X}X\dot{\pi} + 26H^3G_{5XX}X^2\dot{\pi} \\ & + 4H^3G_{5XXX}X^3\dot{\pi} - 18H^2G_{5\pi}X - 27H^2G_{5\pi X}X^2 \\ & - 6H^2G_{5\pi XX}X^3 - 90H^2F_4X^2 - 78H^2F_{4X}X^3 \\ & - 12H^2F_{4XX}X^4 - 168H^3F_5X^2\dot{\pi} - 102H^3F_{5X}X^3\dot{\pi} \\ & - 12H^3F_{5XX}X^4\dot{\pi}\,.\end{aligned} \tag{25}$$

Actions (19) and (22) were derived for the general Horndeski theory in [66] and later generalized to the beyond Horndeski class, e.g., in [67].

Action (22) does not involve time derivatives of either α or β and, hence, the variation in the action WRT α and β provides two constraint equations:

$$\frac{\Delta\beta}{a^2} = \frac{1}{\Theta}\left(3\Theta\dot{\zeta} - (\mathcal{G}_{\mathcal{T}} + \mathcal{D}\dot{\pi})\,\frac{\Delta\zeta}{a^2} + \Sigma\alpha\right), \tag{26a}$$

$$\alpha = \frac{\mathcal{G}_{\mathcal{T}}\dot{\zeta}}{\Theta}\,. \tag{26b}$$

Making use of these constraints to express α and $\Delta\beta$, we rewrite action (22) in terms of one dynamical DOF, which is the curvature perturbation ζ:

$$S^{(2)}_{\zeta} = \int \mathrm{d}t\, \mathrm{d}^3x\, a^3 \left[\mathcal{G}_{\mathcal{S}}\dot{\zeta}^2 - \mathcal{F}_{\mathcal{S}}\,\frac{(\nabla\zeta)^2}{a^2}\right], \tag{27}$$

where the following notations are introduced:

$$\mathcal{G}_{\mathcal{S}} = \frac{\Sigma\mathcal{G}_{\mathcal{T}}^2}{\Theta^2} + 3\mathcal{G}_{\mathcal{T}}\,, \tag{28}$$

$$\mathcal{F}_{\mathcal{S}} = \frac{1}{a}\frac{\mathrm{d}\xi}{\mathrm{d}t} - \mathcal{F}_{\mathcal{T}}\,, \tag{29}$$

$$\xi = \frac{a(\mathcal{G}_{\mathcal{T}} + \mathcal{D}\dot{\pi})\,\mathcal{G}_{\mathcal{T}}}{\Theta}\,. \tag{30}$$

[7] We note that, here and in what follows, we make use of the background equations of motion, whose explicit form may be found, e.g., in [67].

The quadratic actions (19) and (27) describe one scalar (ζ) and two tensor (h^T_{ij}) DOFs. The propagation speed squared of the scalar and tensor perturbations reads, respectively,

$$c_{\mathcal{T}}^2 = \frac{\mathcal{F}_{\mathcal{T}}}{\mathcal{G}_{\mathcal{T}}}\,, \qquad c_{\mathcal{S}}^2 = \frac{\mathcal{F}_{\mathcal{S}}}{\mathcal{G}_{\mathcal{S}}}\,. \tag{31}$$

Let us briefly comment on the main types of instabilities, which can possibly arise at the linearized level of perturbations (see [16] for details). In the case of a homogeneous and isotropic background, the coefficients $\mathcal{G}_{\mathcal{T}}$, $\mathcal{F}_{\mathcal{T}}$, $\mathcal{G}_{\mathcal{S}}$, and $\mathcal{F}_{\mathcal{S}}$ are the functions of time. The most dangerous instabilities are those arising in the high energy regime, i.e., when the characteristic scales of temporal and spatial variations of ζ and h^T_{ij} are considerably smaller than that of the homogeneous background. In the high energy approximation, the coefficients $\mathcal{G}_{\mathcal{S},\mathcal{T}}$ and $\mathcal{F}_{\mathcal{S},\mathcal{T}}$ can be treated as time independent at relevant time intervals. Then, the following situations are possible (the notations $\mathcal{G}_{\mathcal{S},\mathcal{T}}$, $\mathcal{F}_{\mathcal{S},\mathcal{T}}$ refer to pairs of coefficients $\mathcal{G}_{\mathcal{S}}$, $\mathcal{F}_{\mathcal{S}}$ or $\mathcal{G}_{\mathcal{T}}$, $\mathcal{F}_{\mathcal{T}}$):

(1) Gradient instabilities (exponential growth of perturbations):

$$\mathcal{G}_{\mathcal{S},\mathcal{T}} > 0\,, \quad \mathcal{F}_{\mathcal{S},\mathcal{T}} < 0\,, \quad \text{or} \quad \mathcal{G}_{\mathcal{S},\mathcal{T}} < 0\,, \quad \mathcal{F}_{\mathcal{S},\mathcal{T}} > 0\,. \tag{32}$$

(2) Ghosts (catastrophic instability of the vacuum state; see Ref. [16] for discussion):

$$\mathcal{G}_{\mathcal{S},\mathcal{T}} < 0\,, \quad \mathcal{F}_{\mathcal{S},\mathcal{T}} < 0\,. \tag{33}$$

(3) Stable solution:

$$\mathcal{G}_{\mathcal{S},\mathcal{T}} > 0\,, \quad \mathcal{F}_{\mathcal{S},\mathcal{T}} > 0\,. \tag{34}$$

Let us note that we do not analyze the tachyonic instabilities, i.e., the modes with negative mass or imaginary frequencies at low momenta, since this case is likely to require performing the full-fledged stability analysis.

Another potential issue is related to superluminal propagation. Here and in what follows, we choose the tactics of constraining superluminal modes from appearance. It should be noted, however, that the superluminality and stability problems are not directly related to each other: superluminality may occur in a vicinity of a stable part of the solution (see, e.g., [15, 68]). So, the superluminality issue requires separate analysis and, in order to ensure that the speeds of both tensor and scalar modes (31) are luminal at most, the stability conditions in Eqn (34) have to be modified as follows:

$$\mathcal{G}_{\mathcal{T}} \geqslant \mathcal{F}_{\mathcal{T}} > 0\,, \quad \mathcal{G}_{\mathcal{S}} \geqslant \mathcal{F}_{\mathcal{S}} > 0\,. \tag{35}$$

Hence, satisfying the set of constraints (35) allows guaranteeing that the cosmological solution in (beyond) Horndeski theories is stable and does not allow the propagation of superluminal modes. We address the issue of superluminalities in further detail in Section 4.2.

As a result, within different Horndeski subclasses, a great number of nonsingular cosmological settings have been suggested with either bouncing dynamics, e.g., in [69–78], or the Genesis epoch [14, 15, 79–93], which comply with the above stability conditions at least throughout and around the NEC-violating phase.

Although (beyond) Horndeski theories are suitable for nonsingular cosmologies with no obvious pathologies during the NEC-violating stage, ensuring that the solution is free from instabilities during the entire evolution turns out to be a nontrivial task. The problem with global or 'complete' stability of nonsingular cosmological scenarios was first addressed in the $\mathcal{L}_3$ subclass of Horndeski theory (7) and formulated in the form of the no-go theorem in [94], and later generalized for the case of general Horndeski theory (5) in [95] (see also [96]). In Section 3, we revisit the no-go theorem for completely stable nonsingular FLRW solutions in Horndeski theory and then discuss existing ways to circumvent it.

3. Stability of nonsingular cosmologies: no-go theorem and ways to circumvent it

3.1 No-go theorem in Horndeski theory

Let us start by setting the terms and specifying the notion of a completely stable solution in the FLRW background. Namely, in what follows, 'complete stability' implies that pathological perturbations do not develop around the background solution during the entire period of evolution in time. According to Eqn (34), the absence of ghosts and gradient instabilities as well as superluminal propagation over a homogeneous background solution suggests the following restrictions on the coefficients in the quadratic actions (19) and (27):

$$\mathcal{G}_{\mathcal{T}} \geqslant \mathcal{F}_{\mathcal{T}} > \epsilon > 0\,, \quad \mathcal{G}_{\mathcal{S}} \geqslant \mathcal{F}_{\mathcal{S}} > \epsilon > 0\,. \tag{36}$$

Hereafter, ϵ denotes a positive constant, whose actual value is irrelevant for our reasoning, so it may vary in different formulas below. The reason behind introducing this constant in Eqn (36) is to avoid the situations where $\mathcal{G}_{\mathcal{S},\mathcal{T}}, \mathcal{F}_{\mathcal{S},\mathcal{T}} \to 0$ in the asymptotic past and/or future, which we address separately in Section 3.2. In what follows, we focus on nonsingular cosmological solutions only with $a > \epsilon > 0$, where ϵ ensures that the scale factor is bounded from below and, hence, guarantees the geodesic completeness (we comment on the case of a quickly vanishing scale factor in the asymptotic past or future in Section 3.2)

The no-go theorem states that there are no completely stable nonsingular cosmological solutions with $a > \epsilon > 0$ in Horndeski theory, since the instability inevitably shows up in the scalar sector of perturbations provided one considers the entire evolution of the solution.

We prove the theorem by contradiction and start with the assumption that a nonsingular cosmology exists in Horndeski theory which complies with the stability constraints (36) at all times. Let us now analyze the constraint for the gradient coefficient $\mathcal{F}_{\mathcal{S}}$ in the quadratic action for scalar modes (see Eqns (27) and (29), (30)):

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = a(\mathcal{F}_{\mathcal{S}} + \mathcal{F}_{\mathcal{T}}) > \epsilon > 0\,, \tag{37}$$

with

$$\xi = \frac{a\mathcal{G}_{\mathcal{T}}^2}{\Theta}\,. \tag{38}$$

Recall that $\mathcal{D} = 0$ for Horndeski theory (see Eqn (23)). According to stability criteria (36) and definition (37), $\mathrm{d}\xi/\mathrm{d}t$ is strictly positive at all times; hence, ξ is a monotonic function of time. Let us now show that this monotonic function ξ inevitably crosses zero at some moment of time, which is in direct contradiction to the definition of ξ in Eqn (38), since $\mathcal{G}_{\mathcal{T}} > \epsilon > 0$, $a > \epsilon > 0$, and Θ is a smooth function of time, finite at all points (see Eqn (24) for the definition).

We start by integrating (37) over time from t_{i} to some $t_{\mathrm{f}} > t_{\mathrm{i}}$ and get:

$$\int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} a(\mathcal{F}_{\mathcal{S}} + \mathcal{F}_{\mathcal{T}})\,\mathrm{d}t = \xi(t_{\mathrm{f}}) - \xi(t_{\mathrm{i}})\,. \tag{39}$$

It is natural to assume that the integral on the left-hand side is divergent at time infinities, i.e., as $t_{\mathrm{i}} \to -\infty$ and $t_{\mathrm{f}} \to +\infty$ (the contrary would require both $\mathcal{F}_{\mathcal{T}}$ and $\mathcal{F}_{\mathcal{S}}$ to approach zero sufficiently fast in the asymptotic past and/or future, which is in contradiction to (36)). Now, suppose that $\xi(t_{\mathrm{i}}) < 0$; then,

$$\int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} a(\mathcal{F}_{\mathcal{S}} + \mathcal{F}_{\mathcal{T}})\,\mathrm{d}t - \left|\xi(t_{\mathrm{i}})\right| = \xi(t_{\mathrm{f}})\,, \tag{40}$$

and since the integral is divergent in the asymptotic future, there exists a sufficiently large t_{f}, so that $\xi(t_{\mathrm{f}}) > 0$. Hence, based on the continuity of the monotonic function, $\xi(t)$ crosses zero at some moment between t_{i} and t_{f}. However, as we pointed out above, according to the definition of ξ (38) and constraints (36), this is impossible for nonsingular cosmologies with $a > \epsilon > 0$. The only option for ξ to cross zero is when $\Theta \to \infty$, which corresponds to a singularity in the classical solution, since Θ is a combination of Lagrangian functions and background values (see Eqn (24)). Therefore, we deduce that ξ has to be positive at all times. But, upon writing

$$\xi(t_{\mathrm{i}}) = \xi(t_{\mathrm{f}}) - \int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} a(\mathcal{F}_{\mathcal{S}} + \mathcal{F}_{\mathcal{T}})\,\mathrm{d}t\,, \tag{41}$$

we find that the right-hand side becomes negative for $t_{\mathrm{i}} \to -\infty$; hence, $\xi(t_{\mathrm{i}}) < 0$. This contradiction proves the theorem.

The no-go theorem was proved to hold even with additional matter present along with the Horndeski scalar [95, 97, 98], as well as in the case of the open Universe [99].

Let us now discuss the existing loopholes and possible ways to evade the no-go theorem. In practice, all strategies of getting around the no-go theorem suggest the ways to resolve the contradiction between the fact that, by definition, ξ (38) cannot cross zero but, at the same time, crossing zero is necessary to avoid gradient instabilities (37).

3.2 Evading no-go theorem: nonzero ξ and potential strong coupling

One of the ways to resolve the contradiction is to protect ξ from crossing zero. This loophole is related to relaxing the assumption that the central integral in (39) is divergent as $t_{\mathrm{i}} \to -\infty$ and/or $t_{\mathrm{f}} \to +\infty$, which naturally followed from the requirement $\mathcal{F}_{\mathcal{T}}$, $\mathcal{F}_{\mathcal{S}} > \epsilon > 0$ in Eqn (36). As was already noted in [95], one may allow sufficiently rapid decay $\mathcal{G}_{\mathcal{S},\mathcal{T}}, \mathcal{F}_{\mathcal{S},\mathcal{T}} \to 0$ in the asymptotic past and/or future, so that the integral in (39) is convergent as $t_{\mathrm{i}} \to -\infty$ and/or $t_{\mathrm{f}} \to +\infty$ and, hence, the function ξ does not necessarily cross zero. This loophole was utilized in [77, 93, 95] in constructing the specific Genesis scenarios and the bouncing Universe in Horndeski theory without the apparent gradient instability.

It should be noted that this approach is tricky since at least naively $\mathcal{G}_{\mathcal{S},\mathcal{T}}, \mathcal{F}_{\mathcal{S},\mathcal{T}} \to 0$ signals that a strong coupling regime takes place in the asymptotic past and/or future. However, it was explicitly shown in [100–103] that there is a region in the parameter space of the Horndeski Lagrangian where the problem of strong coupling does not occur and the completely stable nonsingular solution can be constructed in Horndeski theory.

Another way to make the integral in (39) convergent is to abandon the requirement for the scale factor to be strictly nonvanishing and allow asymptotically $a \to 0$. This is exactly the strategy that was suggested in [94] within the modified Genesis scenario. It should be noted that the case of the convergent integral suffers from geodesic incompleteness [63], which is not necessarily an obstacle but requires additional study [104].

3.3 Evading no-go theorem: divergent ξ- and γ-crossing

An alternative approach to evading the no-go theorem is to make ξ in (38) cross zero by adjusting the behavior of the coefficient Θ. There is an option of allowing Θ and $\mathcal{G}_{\mathcal{T}}$ to vanish simultaneously, i.e., $\Theta(t_*) = 0$ and $\mathcal{G}_{\mathcal{T}}(t_*) = 0$, so that ξ could cross zero at the moment t_*. However, this option implies fine-tuning and faces the problem of strong coupling, as $\mathcal{G}_{\mathcal{T}}$ vanishes in a finite moment of time t_*.

Let us note that the option of $\Theta = 0$ at any moment of time might seem generally unacceptable without fine-tuning due to the fact that Θ is involved in denominators of both $\mathcal{G}_{\mathcal{S}}$ and $\mathcal{F}_{\mathcal{S}}$ (28), (29). What makes things look even worse is that Θ is present in the denominators of both constraints (26), which questions the validity of the derived quadratic action for the scalar mode (27). These problems related to Θ crossing zero were addressed in [105] where the moment of $\Theta = 0$ was referred to as γ-crossing (the naming is motivated by [68, 77, 106], where the issue was initially addressed in different notations, i.e., $\Theta \equiv \gamma$). It was explicitly shown that all perturbation variables, including the nondynamical α and β in Eqns (26), are in fact regular even at γ-crossing and no other divergences occur; hence, $\Theta = 0$ is totally acceptable. Let us stress that γ-crossing itself does not help to evade the no-go theorem, but this phenomenon turns out to be crucial for constructing cosmological solutions with bounce and Genesis, whose asymptotic forms are simple as $t \to \pm\infty$, i.e., where the theory reduces to GR.

A peculiar approach for evading the no-go theorem inspired by γ-crossing was suggested in [107], where Θ was identically vanishing, i.e., $\Theta \equiv 0$ at all times. This case implies imposing a specific constraint on the Lagrangian functions and a different strategy of removing nondynamical variables α and β in the quadratic equation (22). It was shown that in the resulting Horndeski theory there is no dynamical scalar DOF; hence, the theory propagates only 2 DOFs like the GR. However, allowing anisotropy in the cosmological setting revives the scalar DOF in this type of constrained Horndeski subclass and immediately invokes the instability [108].

3.4 Evading no-go theorem: cuscuton

Yet other special subclasses of Horndeski theory that allow one to evade the no-go theorem are given by the cuscuton [109, 110] and the extended cuscuton [111, 112] theories. The underlying idea is quite similar to that with $\Theta \equiv 0$ and amounts to constraining the Lagrangian functions in a special manner so that the scalar DOF becomes nondynamical. Originally, this option was realized within a subclass of k-essence, where an additional relation is imposed on the Lagrangian functions (see Eqns (5)): $F_X X + 2F_{XX}X^2 = 0$. Indeed, this constraint follows from the requirement

$$\Sigma = -\frac{3\Theta^2}{\mathcal{G}_{\mathcal{T}}}\,, \tag{42}$$

which gives $\Sigma = -6H^2 G_4$ based on the definitions of Θ and $\mathcal{G}_{\mathcal{T}}$ in Eqns (20) and (24) for the k-essence case. Upon comparing this result with the definition of Σ (25), one comes to the constraint for the Lagrangian function in question. Most importantly, this specific choice in Eqn (42) results in $\mathcal{G}_{\mathcal{S}} = 0$ (see Eqn (28)) and, hence, the scalar mode is indeed nondynamical [110]. This type of constraint was further generalized to the case of the general Horndeski and even beyond Horndeski theories [112]. Being a singular case of Horndeski theory, just like the option with $\Theta \equiv 0$ above, the cuscuton theory and its additional degeneracy condition (42) are more robust since, for example, the anisotropy in the cosmological background does not affect the number of dynamical DOFs. Moreover, the Hamiltonian analysis for this type of theory shows that above any cosmological background there is no propagating scalar mode [112, 113]. This result relies on unitary slicing (i.e., the scalar field $\pi(t)$ is time dependent only), which can always be chosen for the everywhere time-like gradient of a scalar field. However, in a general configuration of the scalar field, adopting unitary slicing is not possible and the propagating mode reappears and might be pathological. This situation resembles the defining feature of the U-DHOST theories [114]: these theories seem to be degenerate when written in the unitary slicing, but there is an apparent extra DOF as soon as one considers their fully covariant version with no specific slicing or gauge choice.

As a result, several specific examples of the bouncing solutions were suggested within the cuscuton theory (see, e.g., [115–118]).

3.5 Evading no-go theorem: going beyond Horndeski theory

Finally, to circumvent the no-go theorem, it is possible to extend the Horndeski theory and consider more general scalar-tensor theories. In particular, one may consider the more general geometry with torsion in Horndeski–Cartan theory [119, 120, 123] or teleparallel Horndeski gravity [121, 122]. The conventional option, however, is to consider beyond Horndeski theory (11) or DHOST theories (13). Indeed, already in beyond Horndeski theory, the definition of ξ is modified as compared to the general Horndeski case, since the coefficient $\mathcal{D} \neq 0$ in (30) due to the nontrivial functions $F_4(\pi, X)$ and $F_5(\pi, X)$. While the coefficient $\mathcal{G}_{\mathcal{T}}$ is still responsible for stability in the tensor sector and has to be always positive, the combination $(\mathcal{G}_{\mathcal{T}} + \mathcal{D}\dot{\pi})$ can take any values, including zero and negative ones. Hence, thanks to the unconstrained $\mathcal{D}$, the variable ξ can monotonically grow and cross zero at some moment of time. In this case, the no-go theorem no longer holds.

This opportunity to go beyond Horndeski was explored both within the EFT framework [63, 64, 124–127] and in the covariant language [67, 105, 128–134], where explicit examples of stable nonsingular cosmologies with the bouncing or Genesis stages were put forward.

3.6 No-go theorem and disformal transformations

Evading the no-go theorem by extending the theory to the beyond Horndeski class (11) might seem somewhat contradictory in view of the disformal relation between Horndeski and beyond Horndeski theories mentioned above and discussed in detail, e.g., in [47, 53, 59]. Indeed, the disformal transformation (10) is a field redefinition, and, while it is invertible, it cannot change the number of DOFs [135–137] and cannot affect the stability of the solution.

Therefore, the natural question is how it is possible that beyond Horndeski theory allows completely stable nonsingular solutions while general Horndeski theory does not. The answer is that the disformal transformation, which turns the beyond Horndeski Lagrangian allowing a stable solution into the general Horndeski form, becomes singular at some moment of time. This result was first obtained within the EFT approach in [63] and later confirmed in the covariant formalism [55, 130]. In a nutshell, the finding is based on the form of transformation rules for X-derivatives of the Lagrangian functions G_4 and G_5 under disformal transformation (10) with $\Omega(\pi, X) = 1$:

$$\bar{G}_{4\bar{X}} \equiv \frac{\partial \bar{G}_4}{\partial \bar{X}} = \left(\hat{G}_4 (1 + \Gamma X) - \frac{1}{2}\, \hat{G}_4 (\Gamma + X \Gamma_X) \right) \frac{\sqrt{1 + \Gamma X}}{1 - \Gamma_X X^2} \,,$$
$$\bar{G}_{5\bar{X}} \equiv \frac{\partial \bar{G}_5}{\partial \bar{X}} = G_{5X}\, \frac{(1 + \Gamma X)^{5/2}}{1 - \Gamma_X X^2} \,, \tag{43}$$

where the functions $\bar{G}_{4\bar{X}}$ and $\bar{G}_{5\bar{X}}$ belong to Horndeski theory, while $\hat{G}_4$ [8] and G_{5X} are their counterparts from the beyond Horndeski Lagrangian. Both transformation rules involve the same denominator, which can be rewritten as follows (see [55] for details):

$$\frac{1}{1 - \Gamma_X X^2} = \frac{\mathcal{G}_{\mathcal{T}}}{\mathcal{G}_{\mathcal{T}} + \mathcal{D}\dot{\pi}} \,, \tag{44}$$

where $\mathcal{G}_{\mathcal{T}}$ and $\mathcal{D}$ are similar to those involved in the definition of ξ in Eqn (30). As discussed in Section 3.5, the combination $(\mathcal{G}_{\mathcal{T}} + \mathcal{D}\dot{\pi})$ is supposed to cross zero at some moment, so that ξ could cross zero unimpededly and satisfy stability constraint (37) at any time. We see that the denominator of the transformation rules (43) goes through zero at the same moment as ξ does. This means that both Lagrangian functions $\bar{G}_{4\bar{X}}$ and $\bar{G}_{5\bar{X}}$ diverge at that moment of time. This proves that, once there is a completely stable nonsingular cosmological model in beyond Horndeski theory, the theory cannot be disformally transformed into the Horndeski class due to singularity in the corresponding transformation rules. So, in fact there is no contradiction between the no-go theorem and the existence of completely stable cosmologies in seemingly disformally related theories.

As mentioned in Section 1.2, the invertible conformal-disformal transformation of the general form (10) connects the Horndeski, beyond Horndeski, and DHOST Ia theories. Lately, there have been certain developments in this area and new classes of scalar-tensor theories were discovered with the help of even more general transformations of a metric. In particular, generalized disformal Horndeski (GDH) theories that were put forward are connected to Horndeski via an invertible generalized disformal transformation [138–141] of the following form:

$$g_{\mu\nu} \to \bar{F}_0 g_{\mu\nu} + \bar{F}_1 \pi_\mu \pi_\nu + \bar{F}_2 (\pi_\mu X_\nu + \pi_\nu X_\mu) + \bar{F}_3 X_\mu X_\nu \,, \tag{45}$$

with functions $\bar{F}_i = \bar{F}_i(\pi, X, Y, Z)$ $(i = 0, 1, 2, 3)$, where $Y = \pi_\mu X_\mu$ and $Z = X_\mu X_\mu$, and which are constrained by the invertibility condition, imposed on transformation (45) (see, e.g., [140]). The Lagrangian of the GDH theories explicitly contains third derivatives of the scalar field, but the theory still propagates $2 + 1$ DOFs, since it can be converted into Horndeski via an invertible field redefinition. The GDH theory naturally contains the DHOST Ia subclass as a special case, but the theory is more general and has non-DHOST subclasses as well. Notably, the GDH theory does not involve other subclasses of DHOST theory, which are not disformally related to Horndeski theory; however, those extra subclasses do not have a viable cosmological application, so we do not discuss them in this review.

It turns out that the disformal transformations can be generalized even further, so that still more general scalar-tensor theories can be constructed. There is a possibility of including into the disformal transformation arbitrarily high derivatives of the scalar field [142] or even metric derivatives [143].

[8] Here, we preserve the notations of [55], where the hat over $\hat{G}_4$ emphasizes that this contribution originates from the disformal transformation of $\bar{G}_4$, but not $\bar{G}_5$.

4. Stability of nonsingular cosmological models and matter coupling

This whole machinery related to the disformal transformations and the degenerate theories works perfectly for pure scalar-tensor theories. However, different problems often arise in theories beyond the Horndeski subclass when the coupling to other matter fields is introduced, while the Horndeski theory itself is safe in this regard.

In this section, we discuss two types of problems related to matter coupling. The first one has to do with the fact that the introduction of an additional set of fields can spoil the degeneracy conditions for the scalar sector, which protects the system from the Ostrogradsky ghost. The second problem is the appearance of a superluminally propagating mode in the scalar sector, which is the case when matter with a speed of sound close to the speed of light interacts even minimally with the gravitational sector within the scalar-tensor theory.

4.1 Matter coupling and potentially ruined degeneracy

The issue of spoiled degeneracy upon matter coupling can be investigated in two different ways. First, one may study the spectrum of the joint theory, which consists of the general scalar-tensor part and additional matter, and find out if the Ostrogradsky mode is present. The alternative way, which underlies the more common approach, is to analyze the theory, which is disformally related to the original one. In particular, since all the phenomenologically promising scalar-tensor theories are related to Horndeski via the invertible (general) disformal transformations, one can make use of this feature and disformally reduce any generalization of Horndeski theory with additional matter into the Horndeski subclass, where the additional matter is coupled to the disformally related metric, i.e., the coupling is no longer minimal in terms of the original metric.

A detailed study showed that the matter which is coupled to the standard disformal metric (10) does not spoil the degeneracy of the Horndeski theory [144, 145]. In other words, both beyond Horndeski and DHOST Ia theories allow healthy matter coupling. This result holds if the matter does not feature higher derivatives itself (i.e., is not the Horndeski type) and if the coupling terms do not involve derivatives. So, for example, couplings to a conventional scalar field and to classical fermions are allowed. Counterexamples are the lower subclasses of Horndeski theory (7) (and higher), which exemplify the matter that involves higher derivatives, and the nongauge invariant vector field, which is coupled through the derivatives — in both cases the resulting kinetic matrix is not degenerate anymore. In the more general theories like GDH, the situation gets more complicated: unless one adopts unitary slicing, the only subclass that preserves its degeneracy upon coupling to scalars and fermions is the DHOST Ia theory itself [146, 147].

Let us note that the reasoning above is formulated with no specific choice of a background and implies that the invertible disformal transformations do exist at all moments, which is not the case for completely stable nonsingular cosmological models. Indeed, in Section 3.6, it was shown that the beyond Horndeski theory, which allows a completely stable nonsingular solution, cannot be disformally reduced into the Horndeski subclass due to the existence of singular point(s) in the corresponding disformal transformation rules. Therefore, in this case, the analysis based on disformal transformations cannot be used to check the degeneracy of beyond Horndeski theory with additional matter in a finite number of points. This indicates that the theory's degeneracy in these singular points has to be studied separately (e.g., by analyzing the spectrum of the linearized theory).

Moreover, the discussion above is relevant for scalar-tensor theories with $2+1$ DOFs coupled to the additional matter. However, the situation is somewhat more subtle within 'double degenerate' cases of such scalar-tensor theories, where the 'first' degeneracy in the kinetic matrix removes the would-be Ostrogradsky ghost, while the 'second' one makes the scalar mode nondynamical, as in the cuscuton theory (see Section 3.4). This kind of special subclasses of scalar-tensor theories is more susceptible to the appearance of a new propagating mode after matter coupling, since the additional degeneracy condition (like (42) in the case of the cuscuton) is ruined even by minimal coupling. This was explicitly shown for the case of 'veiled' gravity theory [144], but other cases like the cuscuton may also be problematic in this sense.

4.2 Matter coupling and potentially induced superluminality

Let us now revisit the issue of superluminal propagation, which was briefly mentioned while formulating the set of stability constraints in Section 2 (see Eqn (35)). Even though, as discussed above, the beyond Horndeski and DHOST theories remain degenerate when coupled to additional matter and are, hence, protected from the Ostrogradsky ghost, the properties of the coupled matter itself could be altered: in particular, the corresponding sound speed may exceed that of light.

The potential appearance of superluminalities was shown to be a characteristic feature of scalar-tensor theories like the Horndeski family (see, e.g., [24, 148–156]). Generally, the modes propagating at superluminal speed over an arbitrary background solution are considered undesirable, since their existence indicates that the theory cannot descend, being a low energy effective field theory, from any UV-complete, Lorentz-covariant theory [148]. However, one seldom claims to define the Lagrangian in the entire phase space: it is usually adequate to keep only those terms in the Lagrangian which are sufficient for developing the solution and analyzing its stability (i.e., the terms that do not vanish in the given solution and its close neighborhood). In that case, the minimal requirement is the absence of superluminality for perturbations about the cosmological solution of interest and in its vicinity. In particular, this way of treating superluminalities was discussed in both the Genesis and bouncing scenarios (see, e.g., [15, 68, 83, 157, 158])

Another tricky point is related to the way coupling of the additional matter to Horndeski theories and their generalizations affects the propagation speed of perturbations over a cosmological background. For example, it was explicitly shown for the Genesis model in Horndeski theory [83] that, upon adding whatever tiny amount of external matter (ideal fluid) with a normal equation of state ($0 < w < 1$), the superluminal mode appears in some otherwise healthy region of a phase space [158]. The relation between the emergent superluminality and the coupling of additional matter is based on the kinetic mixing or braiding of the Horndeski scalar with metric [36], which implies that the sound speed of the Horndeski scalar changes in the presence of other matter components.

Things were shown to be different once one goes beyond Horndeski: e.g., in [157], it was proved that a specifically designed beyond Horndeski Lagrangian, which on its own allows a stable and subluminal bouncing solution, remains free of superluminalities upon adding extra matter in the form of a perfect fluid with the equation of state parameter $w \leqslant 1/3$ (or even somewhat larger). However, upon analyzing the sound speeds of scalar modes over a cosmological background in the general setting of beyond Horndeski + perfect fluid, it was found that, for w equal or close to 1, one of the scalar propagation speeds inevitably becomes superluminal (this does not necessarily happen for w substantially smaller than 1) [157]. This effect has to do with the fact, already noticed in [57, 59, 61, 159], that, due to the specific structure of the beyond Horndeski Lagrangian, there is kinetic mixing between matter and scalar field perturbations, and hence the sound speeds of both scalar modes get modified (the superluminal one is predominantly the sound wave in matter). This finding of emergent superluminality has been supported by a similar result when, instead of a perfect fluid, a conventional, minimally coupled scalar field (whose propagation is luminal, $c_m = 1$) is added to a cosmological setup in beyond Horndeski theory [160] and also generalized for the DHOST Ia case [161].

In the following sections, we explicitly demonstrate how the propagation speeds of the scalar perturbations are interrelated in the joint system of DHOST Ia theory and a minimally coupled additional scalar field of the most general type.

4.3 DHOST Ia with an extra scalar: emergent superluminality

First, we briefly discuss the quadratic action for perturbations within the pure DHOST Ia subclass (13)–(15) and, hence, generalize the results for the linearized beyond Horndeski case given in Section 2. We then introduce coupling of additional matter to the DHOST Ia setup and discuss its impact on stability. In this section we closely follow the reasoning of [161].

Our starting point is the (perturbed) spatially flat FLRW background (16) with a rolling DHOST field $\pi = \pi(t)$. Similarly to Section 2, we consider the unitary gauge $\delta\pi = 0$, so that the dynamical perturbations in the DHOST sector are still given by the tensor modes h_{ij}^T and the scalar mode $\tilde{\zeta}$ (which differs from the original ζ in Section 2 and is explicitly defined below). In pure DHOST Ia theory, the unconstrained second order action reads [9]

$$S_{\pi}^{(2)} = \int \mathrm{d}t\, \mathrm{d}^3x\, a^3 \left[\left(\frac{\mathcal{G}_{\mathcal{T}}}{8} \left(\dot{h}_{ik}^T \right)^2 - \frac{\mathcal{F}_{\mathcal{T}}}{8a^2} (\partial_i h_{kl}^T)^2 \right) + \left(\mathcal{G}_{\mathcal{S}} \dot{\tilde{\zeta}}^2 - \frac{1}{a^2} \mathcal{F}_{\mathcal{S}} (\partial_i \tilde{\zeta})^2 \right) \right], \tag{46}$$

where

$$\mathcal{G}_{\mathcal{T}} = -2F_2 + 2A_1 X, \tag{47a}$$

$$\mathcal{F}_{\mathcal{T}} = -2F_2, \tag{47b}$$

$$\mathcal{G}_{\mathcal{S}} = \frac{\tilde{\Sigma} \mathcal{G}_{\mathcal{T}}^2}{\tilde{\Theta}^2} + 3\mathcal{G}_{\mathcal{T}}, \tag{47c}$$

$$\mathcal{F}_{\mathcal{S}} = \frac{1}{a} \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{a\, \mathcal{G}_{\mathcal{T}} (\mathcal{G}_{\mathcal{T}} + \mathcal{D}\dot{\pi} + \mathcal{F}_{\mathcal{T}} \Gamma)}{\tilde{\Theta}} \right] - \mathcal{F}_{\mathcal{T}} \tag{47d}$$

with

$$\mathcal{D} = -2A_1 \dot{\pi} + 4F_{2X} \dot{\pi}, \tag{48a}$$

$$\Gamma = \frac{X}{2\mathcal{G}_{\mathcal{T}}} (2A_1 - 4F_{2X} - A_3 X), \tag{48b}$$

$$\tilde{\Theta} = \Theta - \mathcal{G}_{\mathcal{T}} \dot{\Gamma}, \tag{48c}$$

$$\tilde{\Sigma} = \Sigma + 3\mathcal{G}_{\mathcal{T}} \dot{\Gamma}^2 + 6\tilde{\Theta}\dot{\Gamma} - \frac{3}{a^3} \frac{\mathrm{d}}{\mathrm{d}t} \left[a^3 (\tilde{\Theta} + \mathcal{G}_{\mathcal{T}} \dot{\Gamma}) \Gamma \right], \tag{48d}$$

where the expressions for Σ and Θ are quite lengthy and are given in the Appendix for completeness. In these notations, the scalar perturbation is $\tilde{\zeta} = \zeta - \alpha\Gamma$. The result for the beyond Horndeski case can be restored upon the following choice of Lagrangian functions [49]:

$$F_2 = -G_4, \quad A_1 = -A_2 = -(2G_{4X} - XF_4), \quad A_3 = -A_4 = -2F_4, \quad A_5 = 0. \tag{49}$$

Note that Γ identically vanishes in this case and we get back to the original curvature perturbation $\tilde{\zeta} \equiv \zeta$. We note in passing that hereafter we do not use the background equations of motion when deriving the action for perturbations.

As before, the stability of the background requires $\mathcal{G}_{\mathcal{T}}$, $\mathcal{F}_{\mathcal{T}}$, $\mathcal{G}_{\mathcal{S}}$, $\mathcal{F}_{\mathcal{S}} > \epsilon > 0$ in (46), while the sound speeds squared are restricted to be luminal at most:

$$c_{\mathcal{S},0}^2 = \frac{\mathcal{F}_{\mathcal{S}}}{\mathcal{G}_{\mathcal{S}}} \leqslant 1, \qquad c_{\mathcal{T}}^2 = \frac{\mathcal{F}_{\mathcal{T}}}{\mathcal{G}_{\mathcal{T}}} \leqslant 1. \tag{50}$$

Now, let us turn to the additional matter component and consider another scalar field χ in the form of k-essence [34]:

$$S_\chi = \int \mathrm{d}^4x \sqrt{-g}\, P(\chi, Y), \quad Y = g^{\mu\nu} \chi_{,\mu} \chi_{,\nu}. \tag{51}$$

The Lagrangian in Eqn (51) describes a minimally coupled scalar field χ of the most general type (assuming the absence of second derivatives in the Lagrangian). In this context, k-essence successfully mimics an irrotational, barotropic fluid. The field equation for the homogeneous scalar $\chi = \chi(t)$ in the flat FLRW setting reads

$$\ddot{\chi} + 3\frac{P_Y}{Q} H\dot{\chi} - \frac{P_\chi - 2YP_{\chi Y}}{2Q} = 0, \tag{52}$$

where $Q = P_Y + 2YP_{YY}$. In a spatially homogeneous background (possibly rolling, $Y = \dot{\chi}^2 \neq 0$), the stability conditions for the scalar field χ have a standard form,

$$P_Y > 0, \qquad Q > 0, \tag{53}$$

while the propagation speed of perturbations equals

$$c_m^2 = \frac{P_Y}{Q}. \tag{54}$$

Our main result on the emergent superluminality in what follows applies most straightforwardly to the conventional scalar field with

$$P(\chi, Y) = \frac{1}{2} Y - V(\chi) \tag{55}$$

and $c_m^2 = 1$, but for now we retain full generality and do not make any assumptions about the form of the function $P(\chi, Y)$.

We now combine the two theories and consider DHOST Ia along with the extra scalar theory (51) over the background where neither $\dot{\pi}$ nor $\dot{\chi}$ vanish (in particular, $Y = \dot{\chi}^2 \neq 0$). The expressions for $\mathcal{G}_{\mathcal{T}}$ and $\mathcal{F}_{\mathcal{T}}$ in Eqns (47a), (47b) do not get modified, so the tensor perturbations remain (sub)luminal according to our requirement in (50). On the contrary, the situation in the scalar sector changes dramatically. The nonvanishing background $\dot{\chi}$ induces mixing between the scalars $\tilde{\zeta}$ and $\delta\chi$ [61], so the unconstrained quadratic action in the joint scalar sector reads

$$S_{\pi+\chi}^{(2)\,\mathrm{scalar}} = \int \mathrm{d}t\, \mathrm{d}^3x\, a^3 \left[G_{AB} \dot{v}^A \dot{v}^B - \frac{1}{a^2} F_{AB} \partial_i v^A \partial_i v^B + \ldots \right], \tag{56}$$

where $A, B = 1, 2$, $v^1 = \tilde{\zeta}$, $v^2 = \delta\chi$, and the ellipsis stands for terms with fewer than two derivatives. The matrices G_{AB} and F_{AB} have the following forms:

$$G_{AB} = \begin{pmatrix} \mathcal{G}_{\mathcal{S}} + \dfrac{\mathcal{G}_{\mathcal{T}}^2}{\tilde{\Theta}^2} YQ & \dot{\chi}Qg \\ \dot{\chi}Qg & Q \end{pmatrix}, \quad F_{AB} = \begin{pmatrix} \mathcal{F}_{\mathcal{S}} & \dot{\chi}P_Y f \\ \dot{\chi}P_Y f & P_Y \end{pmatrix}, \tag{57}$$

where

$$g = -\frac{\mathcal{G}_{\mathcal{T}}}{\tilde{\Theta}} \left(1 - 3\frac{P_Y}{Q} \Gamma \right), \tag{58a}$$

$$f = -\frac{(\mathcal{G}_{\mathcal{T}} + \mathcal{D}\dot{\pi} + \mathcal{F}_{\mathcal{T}} \Gamma)}{\tilde{\Theta}}. \tag{58b}$$

Note that these expressions are valid for any Y. The two sound speeds squared $c_{\mathcal{S},\pm}^2$ are given by eigenvalues of the matrix $G^{-1}F$, i.e., they satisfy

$$\det(F_{AB} - c_{\mathcal{S}}^2 G_{AB}) = 0. \tag{59}$$

Before proceeding with the DHOST Ia case, let us address the results for the (beyond) Horndeski theories [160].

4.3.1 Beyond Horndeski case: superluminality due to a conventional scalar field. For the beyond Horndeski case, both matrices G_{AB} and F_{AB} simplify since $\Gamma \equiv 0$ (see Eqns (48b), (49), and (58)). Hence, Eqn (59) gives the

[9] Here, we adopt the notations for coefficients similar to those in the beyond Horndeski case (see Eqns (19) and (27)).

following propagation speeds of $\tilde{\zeta} \equiv \zeta$ and $\delta\chi$ (recall that $c_m^2 = P_Y/Q$):

$$c_{\mathcal{S}\pm}^2 = \frac{1}{2}(c_m^2 + \mathcal{A}) \pm \frac{1}{2}\sqrt{(c_m^2 - \mathcal{A})^2 + \mathcal{B}}\,, \tag{60}$$

where

$$\mathcal{A} = \frac{\mathcal{F_S}}{\mathcal{G_S}} - \frac{YP_Y}{\mathcal{G_S}}\frac{\mathcal{G_T}(\mathcal{G_T}+2\mathcal{D}\dot{\pi})}{\Theta^2}\,, \quad \mathcal{B} = 4c_m^2\frac{YP_Y}{\mathcal{G_S}}\frac{(\mathcal{D}\dot{\pi})^2}{\Theta^2}\,.$$

In a stable and rolling background, both matrices G_{AB} and F_{AB} must be positive definite ($G_{11}, G_{22} > 0$, $\det G > 0$ and $F_{11}, F_{22} > 0$, $\det F > 0$):

$$\mathcal{G_S} > 0\,, \quad \mathcal{F_S} > 0\,, \quad P_Y > 0\,, \quad Q > 0\,,$$
$$\mathcal{F_S} - YP_Y\frac{(\mathcal{G_T} + \mathcal{D}\dot{\pi})^2}{\Theta^2} > 0\,, \tag{61}$$

and, hence, with $Y > 0$, the coefficient $\mathcal{B}$ is positive. This immediately gives

$$c_{\mathcal{S}+}^2 > c_m^2 \quad \text{for} \quad Y \neq 0\,. \tag{62}$$

So, if the propagation of the scalar perturbation $\delta\chi$ is luminal, $c_m = 1$, then according to (62) it becomes superluminal in the 'beyond Horndeski + scalar field' system. The interpretation of $c_{\mathcal{S}+}^2$ as the speed of perturbations in the additional matter can be supported by restoration of the Horndeski limit.

Indeed, in the unextended Horndeski case (5), the coefficient $\mathcal{D}$ vanishes (see Eqn (23)), so that $g = f$ in matrices G_{AB} and F_{AB} in (57) and the matrix $G^{-1}F$ becomes triangular. As a result, the speed of $\delta\chi$ recovers its standard value c_m^2, while the propagation speed of the curvature perturbation ζ is modified. Indeed, with $\mathcal{D} = 0$, the propagation speeds in (60) get simplified as follows:

$$c_{\mathcal{S}-}^2\Big|_{\mathcal{D}=0} = \frac{\mathcal{F_S}}{\mathcal{G_S}} - \frac{YP_Y}{\mathcal{G_S}}\frac{\mathcal{G_T}^2}{\Theta^2}\,, \quad c_{\mathcal{S}+}^2\Big|_{\mathcal{D}=0} = c_m^2\,, \tag{63}$$

where we explicitly see that $c_{\mathcal{S}+}^2$ belongs to $\delta\chi$.

In particular, result (62) indicates that, in the case of a conventional scalar field (55), where $c_m^2 = 1$ for any Y, the scalar sector of beyond Horndeski + $P(\chi, Y)$ system features a superluminal mode even with a tiny amount of kinetic energy associated with a rolling scalar $\chi(t)$. Therefore, we conclude that the multi-scalar system with the minimal coupling of a conventional scalar field to beyond Horndeski theory is exposed to the emergent superluminality.

4.3.2 DHOST Ia case: superluminality and an exceptional subclass. We now get back to the full-fledged DHOST Ia case and assume that the background is stable, and DHOST perturbations in it are not superluminal. The stability conditions for the scalar sector in the combined system of DHOST Ia and k-essence generalize those for beyond Horndeski theory + k-essence in (61) and read explicitly

$$\mathcal{G_S} > 0\,, \quad \mathcal{F_S} > 0\,, \quad P_Y > 0\,, \quad Q > 0\,,$$
$$\mathcal{F_S} - YP_Y\frac{(\mathcal{G_T} + \mathcal{D}\dot{\pi} + \mathcal{F_T}\Gamma)^2}{\tilde{\Theta}^2} > 0\,, \tag{64}$$
$$1 + \frac{6\mathcal{G_T}^2}{\tilde{\Theta}^2}\frac{YP_Y}{\mathcal{G_S}}\left(1 - \frac{3}{2}\frac{P_Y}{Q}\Gamma\right)\Gamma > 0\,,$$

where the second and third lines are characteristic of the combined theory, while the first line coincides with the stability conditions for separated DHOST Ia theory and k-essence. Note that the beyond Horndeski case (61) is straightforwardly restored upon taking $\Gamma = 0$. At this point, we neither impose any further constraints on the background nor assume any relations other than (15) between the Lagrangian functions in (13) yet.

Let us consider the general situation with $f \neq g$ and take Y to be small in Eqns (57), (58). Then, we distinguish two situations:

(i) If $c_m^2 \neq c_{\mathcal{S},0}^2$ (i.e., $P_Y/Q \neq \mathcal{F_S}/\mathcal{G_S}$), then one of the sound speeds, following from Eqn (59), is $c_{\mathcal{S},-} = c_{\mathcal{S},0} + \mathcal{O}(Y)$, while the other is given by

$$c_{\mathcal{S},+}^2 = c_m^2\left(1 + \frac{Y(f-g)^2}{\mathcal{G_S}(c_m^2 - c_{\mathcal{S},0}^2)}\right) + \mathcal{O}(Y^2)\,. \tag{65}$$

This means, among other things, that, in the theory of the conventional scalar field (55) with $c_m = 1$ and subluminal DHOST Ia with $c_{\mathcal{S},0} < 1$, the mode which is predominantly $\delta\chi$ becomes superluminal at small but nonzero background values of Y.

(ii) For $c_m^2 = c_{\mathcal{S},0}^2$, the sound speeds are given by

$$c_{\mathcal{S},\pm}^2 = c_m^2\left[1 \pm \left(\frac{YP_Y(f-g)^2}{\mathcal{G_S}}\right)^{1/2}\right] + \mathcal{O}(Y)\,, \tag{66}$$

which again shows that, in the case of a conventional scalar field with $c_m = 1$, one of the modes becomes superluminal at small Y.

Let us finally turn to the special case $f = g$. By making use of Eqn (57), it is found that in this case the matrix $G^{-1}F$ is triangular *for any value of* Y, so that one of the sound speeds remains unmodified, $c_{\mathcal{S},+}^2 = c_m^2 = P_Y/Q$, while the other is not necessarily superluminal ($c_{\mathcal{S},-} = c_{\mathcal{S},0} + \mathcal{O}(Y)$ for small Y). For the luminal extra scalar with $c_m = 1$, the condition $f = g$ provides a constraint $\mathcal{D}\dot{\pi} = -(3\mathcal{G_T} + \mathcal{F_T})\Gamma$ on the DHOST Lagrangian, or, explicitly,

$$A_3 = \frac{2(A_1 - 2F_{2X})(A_1X - 2F_2)}{X(3A_1X - 4F_2)}\,. \tag{67}$$

This is an exceptional subclass of DHOST Ia theories in which adding an extra luminal scalar field does not necessarily lead to superluminality. Note that this subclass includes the theory with $A_1 = 2F_{2X}$ and $A_3 = 0$, which is Horndeski, and does *not* include beyond Horndeski (GLPV) theories with $XA_3 = 2A_1 - 4F_{2X}$ (this relation is inconsistent with (67) for $\mathcal{G_T} \neq 0$; see Eqn (47a)). This observation is in agreement with the result for beyond Horndeski in Section 4.3.1.

Thus, the emergent superluminality over the cosmological background generally takes place in the DHOST Ia class upon adding to the system even a tiny amount of kinetic energy $Y > 0$ related to the additional scalar field χ (55) with $c_m^2 = 1$. There is, however, an exceptional subclass of DHOST Ia theories given by (67), where the superluminality of perturbations in the scalar sector is not induced.

5. Conclusions

In this mini-review, we have briefly discussed updates to the cosmological scenarios without the initial singularity in Horndeski theories and their generalizations. The review is focused on ensuring linear stability at the perturbation level WRT ghosts and gradient instabilities, which has been shown to pose significant challenges to the construction of nonsingular cosmological models that are stable throughout the entire period of evolution. We have revisited the no-go theorem, which rules out the existence of completely stable nonsingular solutions within Horndeski theory, and discussed in detail the existing approaches to evade it, highlighting the specific subclasses of non-extended Horndeski theory where the no-go can be circumvented. Furthermore, we addressed the role of disformal transformations, which interconnect various viable subclasses of Horndeski and generalized theories like DHOST Ia, and discussed the effect of additional matter coupling on the degeneracy of the joint field theory. We also drew special attention to the potential appearance of superluminal modes in the DHOST Ia + extra matter system: it was shown that a superluminal scalar mode generally arises in the cosmological background in the DHOST Ia class upon adding to the system even a tiny amount of kinetic energy sourced by the additional scalar field with the sound speed equal to that of light. This effect of the emergent superluminality, however, does not take place for a specific DHOST Ia subclass, which was identified explicitly and includes Horndeski theory as a special case but does not include the beyond Horndeski (GLPV) subclass.

Scalar-tensor theories like Horndeski theories and their generalizations constitute a particularly rich framework of modified gravity theories, which are suitable for creating and analyzing various cosmological scenarios, describing both the early and late-time Universe. As a result of studying specific cosmological models like those without the initial singularity, one might be hopeful not only that fundamental aspects behind the theories they are built on will be discovered, but also that further insights will be gained into the theoretical aspects of gravity and physics of the early Universe.

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6. Appendix

Here, we provide the explicit expressions for Θ and Σ involved in the quadratic action for DHOST Ia theories (see Eqns (48c), (48d)):

$$\begin{aligned}\Theta &= (-A_{3X} + A_5)\ddot{\pi}\dot{\pi}X^2 - \dot{\pi}\big(F_{2\pi} + \ddot{\pi}(-3A_1 + 6F_{2X})\big) \\ &- 2F_2H + X(3A_1 + 2F_{2X})H \\ &+ X^2\left(-2A_{1X}H + \frac{3}{2}(4A_{1X} + A_3)H\right) \\ &- \dot{\pi}X\left(\ddot{\pi}\left(-2A_{1X} + \frac{3}{2}A_3 - A_4 + 4F_{2XX}\right) + 2F_{2\pi X} + K_X\right),\end{aligned} \tag{68}$$

$$\begin{aligned}\Sigma &= 6F_2H^2 - 2\ddot{\pi}(A_{5\pi}X + A_{5XX}\ddot{\pi})\dot{\pi}^8 \\ &- \dot{\pi}^7(2A_{5X}\dddot{\pi} + 3A_{3\pi X}H + 6A_{5X}\ddot{\pi}H) \\ &+ 6\dot{\pi}[F_{2\pi}H - 2\ddot{\pi}(A_1 - 2F_{2X})H] \\ &+ \dot{\pi}^6\big[-2(A_{3\pi X} + A_{4\pi X} + 4A_{5\pi})\ddot{\pi} \\ &- (2A_{3XX} + 2A_{4XX} + 13A_5X)\ddot{\pi}^2 - 3A_{3X}\dot{H} \\ &- 3(4A_{1XX} + 3A_{3X})H^2\big] + \dot{\pi}^2\big[-3(A_3 + A_4)\ddot{\pi}^2 \\ &- 12\dot{H}(-A_1 + 2F_{2X}) + F_X - 42F_{2X}H^2 - K_\pi\big] \\ &+ \dot{\pi}^4\big[-6(A_{3\pi} + A_{4\pi})\ddot{\pi} - (9A_{3X} + 9A_{4X} + 12A_5)\ddot{\pi}^2 \\ &- 3\dot{H}(-2A_{1X} + 3A_3 + 4F_{2XX}) + 2F_{XX} - 36A_{1X}H^2 \\ &- 27A_3H^2 - 24F_{2XX}H^2 - K_{\pi X}\big] - \dot{\pi}^3\,\ddot{\pi}\big[6(A_3 + A_4)\,\dddot{\pi} \\ &+ 3\ddot{\pi}(2A_{1X} + 3A_3 + 6A_4 - 4F_{2XX})H \\ &+ 6H(-2A_{1\pi} - F_{2\pi X} - 2K_X)\big] \\ &- \dot{\pi}^5\big[2(A_{3X} + A_{4X} + 4A_5) + 3(A_{3X} + 2A_{4X} + 8A_5)\ddot{\pi}H \\ &+ 3H(-2A_{1\pi X} + 3A_{3\pi} - 2K_{XX})\big],\end{aligned} \tag{69}$$

where the functions A_4 and A_5 are given by (15b) and (15c), respectively.

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