

METHODOLOGICAL NOTES

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Mathematical paradoxes of Dirac equation representations

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Contents

Abstract. This paper examines the Foldy–Wouthuysen and Feynman–Gell-Mann representations of the Dirac equation. The analysis is conducted for electrons and positrons interacting with electromagnetic fields. Versions of quantum electrodynamics are considered both within the scope of perturbation theory and in the nonperturbative case with strong electromagnetic fields. Mathematical artifacts that contradict the physical premises of the theory are identified in the studied representations of the Dirac equation. These mathematical paradoxes are resolved if the theory only employs amplitude states (real and virtual) with positive energies.

Keywords: Dirac equation representations, quantum electrodynamics, fermion vacuum, positive and negative energy states, mathematical paradoxes of the theory

1. Introduction

The Dirac equation with a bispinor wave function is used in standard quantum electrodynamics (QED). The Dirac equation for an electron with mass m and electric charge $e < 0$, interacting with an electromagnetic field $A^{\mu}(\mathbf{x}, t)$, can be written in the form

$$p^0 \psi_{\mathrm{D}}(\mathbf{x}, t) = H_{\mathrm{D}}(\mathbf{x}, t)\psi_{\mathrm{D}}(\mathbf{x}, t) = \big(\boldsymbol{\alpha}(\mathbf{p} - e\mathbf{A}(\mathbf{x}, t)) + \beta m + eA^0(\mathbf{x}, t)\big)\psi_{\mathrm{D}}(\mathbf{x}, t)\,. \qquad (1)$$

Here and below, the unit system $\hbar = c = 1$ is used; $H_{\mathrm{D}}(\mathbf{x}, t)$ is the Dirac Hamiltonian; $p^{\mu} = \mathrm{i}(\partial/\partial x_{\mu})$, $\mu = 0, 1, 2, 3$; $A^{\mu}(\mathbf{x}, t)$ are the electromagnetic potentials; and α^i, β are the four-dimensional Dirac matrices. In the standard representation, the matrices α^i, β, Σ^i, γ^5, and γ^i have the form

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \beta = \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$
$$\Sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \gamma^0 \alpha^i\,. \qquad (2)$$

The bispinor $\psi_{\mathrm{D}}(\mathbf{x}, t)$ can be written as

$$\psi_{\mathrm{D}}(\mathbf{x}, t) = \begin{pmatrix} \varphi(\mathbf{x}, t) \\ \chi(\mathbf{x}, t) \end{pmatrix}. \qquad (3)$$

In a free case (without interaction), the Dirac equation has the following normalized solutions with positive and negative energies ε:

$$(\psi_{\mathrm{D}})_0^{(+)}(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}}\left(1 + \frac{\mathbf{p}^2}{(|E| + m)^2}\right)^{-1/2} \begin{pmatrix} U_{\mathrm{S}} \\ \dfrac{\boldsymbol{\sigma}\mathbf{p}}{|E| + m}\, U_{\mathrm{S}} \end{pmatrix} \times \exp\big(-\mathrm{i}\,|E|t + \mathrm{i}\,\mathbf{px}\big)\,, \quad \varepsilon = |E| > 0\,,$$
$$(\psi_{\mathrm{D}})_0^{(-)}(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}}\left(1 + \frac{\mathbf{p}^2}{(|E| + m)^2}\right)^{-1/2} \begin{pmatrix} \dfrac{\boldsymbol{\sigma}\mathbf{p}}{|E| + m}\, U_{\mathrm{S}} \\ U_{\mathrm{S}} \end{pmatrix} \times \exp\big(+\mathrm{i}\,|E|t - \mathrm{i}\,\mathbf{px}\big)\,, \quad \varepsilon = -|E| < 0\,. \qquad (4)$$

Here, U_{S} are the normalized Pauli spinors (for $S_z = 1/2$, $U_{\mathrm{S}} = \binom{1}{0}$, and for $S_z = -1/2$, $U_{\mathrm{S}} = \binom{0}{1}$).

The Dirac equation also has solutions with positive and negative energies for stationary states in the presence of static electromagnetic fields.

On the one hand, the set of solutions with positive and negative energies provides mathematical completeness. On the other hand, the solutions with negative energies are not directly the solutions of the Dirac equation for antiparticles. The author of this equation, P.A.M. Dirac, understood this well (see, e.g., [1]). Two interpretations of the solutions with negative energy have become widely known.

(1) The physical vacuum of the Dirac equation is described using the concept of fully occupied states with negative energies (the Dirac sea). Holes in the Dirac sea are interpreted as antiparticles [1].

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(2) In the Stuekelberg and Feynman positron theory [2–4], positrons are electrons with negative energies that move in opposite directions in spacetime.

In standard QED, the fermion vacuum is nonvoid; virtual birth and annihilation of particles and antiparticles are theoretically allowed in it.

There are versions of QED with a void fermion vacuum. These versions are based on using certain representations of the Dirac equation. They include the representation of Foldy and Wouthuysen (FW) [5], the representation of Feynman and Gell-Mann (FG) [6], and the representation with the Klein–Gordon (KG)-type fermion equations [7, 8].

For these representations, in the framework of perturbation theory, the formalisms $(\mathrm{QED})_{\mathrm{FW}}$ [9, 10], $(\mathrm{QED})_{\mathrm{FG}}$ [11], and $(\mathrm{QED})_{\mathrm{KG}}$ [12, 13] were developed, and some physical effects have been calculated. The final physical results fully agree with the respective results of standard QED with the Dirac equation.

Closed equations for fermions in the FW and FG representations were also formulated for nonperturbative QED in strong electromagnetic fields [14, 15].

Using the aforementioned representations of the Dirac equation, it is sufficient to consider solutions with positive fermion energies when calculating physical effects. This applies to both real and intermediate virtual fermion states. In these cases, two separate equations are needed for fermions and antifermions. These equations differ by the sign of electric charge.

The use of these representation of the Dirac equation to calculate QED effects revealed some contradictions between the physical premises of the theory and its mathematical results. All of these contradictions arise from the use of states with negative energies of fermions in calculations.

This paper analyzes the cause of these mathematical paradoxes.

It is organized as follows. Section 2 analyzes using $(\mathrm{QED})_{\mathrm{FW}}$ in the applicability domain of perturbation theory. Section 3 presents an analysis for the representations of nonperturbative QED with strong electromagnetic fields. We consider the standard QED and the representations of Foldy–Wouthuysen and Feynman–Gell-Mann. We discuss the conclusions from our analysis in Section 4.

2. Perturbation theory in Foldy–Wouthuysen representation

Two conditions should hold in the Foldy–Wouthuysen representation (see, e.g., [16]):

(1) The Hamiltonian or the energy operators are diagonal with respect to the upper and lower spinors of the wave function $\psi_{\mathrm{FW}}(\mathbf{x})$, i.e., these operators do not mix the upper and lower components of $\psi_{\mathrm{FW}}(\mathbf{x})$.

(2) The condition of wave function reduction holds under the Foldy–Wouthuysen transformation. When the Dirac Hamiltonian does not depend on time (the case of external static fields), the condition of reduction can be written as [1]

$$\psi_{\mathrm{D}}^{(+)}(\mathbf{x},t)=\exp(-\mathrm{i}\varepsilon t)A_{(+)}\begin{pmatrix}\varphi(\mathbf{x})\\ \chi(\mathbf{x})\end{pmatrix}\to\psi_{\mathrm{FW}}^{(+)}(\mathbf{x},t)=U_{\mathrm{FW}}^{(+)}(\mathbf{x})\,\psi_{\mathrm{D}}^{(+)}(\mathbf{x},t)=\exp(-\mathrm{i}\varepsilon t)\begin{pmatrix}\varphi(\mathbf{x})\\ 0\end{pmatrix},\quad(5)$$

[1] Wave functions are normalized by unit probability in a box with volume V below. For brevity, multipliers $1/\sqrt{V}$ are absent from our expressions.

where $\varepsilon>0$, and

$$\psi_{\mathrm{D}}^{(-)}(\mathbf{x},t)=\exp(-\mathrm{i}\varepsilon t)A_{(-)}\begin{pmatrix}\varphi(\mathbf{x})\\ \chi(\mathbf{x})\end{pmatrix}\to\psi_{\mathrm{FW}}^{(-)}(\mathbf{x},t)=U_{\mathrm{FW}}^{(-)}(\mathbf{x})\,\psi_{\mathrm{D}}^{(-)}(\mathbf{x},t)=\exp(-\mathrm{i}\varepsilon t)\begin{pmatrix}0\\ \chi(\mathbf{x})\end{pmatrix},\quad(6)$$

where $\varepsilon<0$.

In (5) and (6), $A_{(+)}$, and $A_{(-)}$ are the normalizing operators, and $U_{\mathrm{FW}}^{(+)}$ and $U_{\mathrm{FW}}^{(-)}$ are the FW transformation operators. The operators $A_{(+)}$, $A_{(-)}$, $U_{\mathrm{FW}}^{(+)}$, $U_{\mathrm{FW}}^{(-)}$ are not necessarily the same for positive and negative energies.

In the FW representation, the Dirac equation for an electron interacting with an electromagnetic field $A^{\mu}(\mathbf{x},t)$ can be obtained as a series in powers of the electromagnetic coupling constant by applying a series of unitary transformations to equation (1) (see [9]):

$$U_{\mathrm{FW}}=(1+e\delta_1+e^2\delta_2+e^3\delta_3+\ldots)U_{\mathrm{FW}}^0\,.\quad(7)$$

Here, $U_{\mathrm{FW}}^{+}=U_{\mathrm{FW}}^{-1}$.

As a result, we obtain the equation

$$\varepsilon\psi_{\mathrm{FW}}=H_{\mathrm{FW}}\psi_{\mathrm{FW}}=\big(\beta E_{\mathrm{p}}+eK_1^{\mathrm{FW}}(+m,A^{\mu})+e^2K_2^{\mathrm{FW}}(+m,A^{\mu},A^{\nu})+e^3K_3^{\mathrm{FW}}(+m,A^{\mu},A^{\nu},A^{\gamma})+\ldots\big)\psi_{\mathrm{FW}}\,.\quad(8)$$

Here, $E_{\mathrm{p}}=\sqrt{m^2+\mathbf{p}^2}$. The notation $+m$ in K_n^{FW} indicates that the positive sign before βm is taken in equation (1). Equation (8) does not contain terms with a negative sign before the mass m. This follows from the structure of the expressions for K_1^{FW}, $K_2^{\mathrm{FW}}\ldots$ (see (39)–(41), (20), (21) in [9]).

In equation (8),

$$\psi_{\mathrm{FW}}=U_{\mathrm{FW}}\psi_{\mathrm{D}}\,.\quad(9)$$

In the free case,

$$\varepsilon(\psi_{\mathrm{FW}})_0=\beta E_{\mathrm{p}}(\psi_{\mathrm{FW}})_0\,,\quad(10)$$

where, for the positive energy $\varepsilon=|E|>0$,

$$(\psi_{\mathrm{FW}})_0^{(+)}(\mathbf{x},t)=U_{\mathrm{FW}}^0(\psi_{\mathrm{D}})_0^{(+)}=\frac{1}{(2\pi)^{3/2}}\begin{pmatrix}U_{\mathrm{S}}\\ 0\end{pmatrix}\exp(-\mathrm{i}|E|t+\mathrm{i}\mathbf{px})\,,\quad(11)$$

and for the negative energy $\varepsilon=-|E|<0$,

$$(\psi_{\mathrm{FW}})_0^{(-)}(\mathbf{x},t)=U_{\mathrm{FW}}^0(\psi_{\mathrm{D}})_0^{(-)}=\frac{1}{(2\pi)^{3/2}}\begin{pmatrix}0\\ U_{\mathrm{S}}\end{pmatrix}\exp(+\mathrm{i}|E|t-\mathrm{i}\,\mathbf{px})\,.\quad(12)$$

In the FW representation, equation (8) has a noncovariant form, and the Hamiltonian H_{FW} is nonlocal. Using standard methods of second quantization in quantum field theory is difficult in this case. However, the S-matrix approach and the Feynman method of the propagator function [2–4, 17] can be used instead. In this method, QED processes are described by integral equations.

Equation (8) can be written for the four-dimensional x, y in the form

$$\psi_{\mathrm{FW}}(x)=(\psi_{\mathrm{FW}})_0^{(\pm)}(x)+\int\mathrm{d}^4y\,S_{\mathrm{FW}}(x-y)K^{\mathrm{FW}}(y)\psi_{\mathrm{FW}}(y),\quad(13)$$

where $K^{\mathrm{FW}}(y)=\sum_{n=1}^{\infty}e^nK_n^{\mathrm{FW}}(y)$ is the interaction Hamiltonian in equation (8) and $S_{\mathrm{FW}}(x-y)$ is the Feynman propagator in the Foldy–Wouthuysen representation,

$$S_{\rm FW}(x-y)=\frac{1}{(2\pi)^4}\int {\rm d}^4p\,\exp\left(-\,{\rm i}p(x-y)\right)\frac{p^0+\beta E}{p^2-m^2+{\rm i}\varepsilon}. \quad (14)$$

The elements of the S-matrix can be written as

$$S_{fi}=\delta_{fi}-{\rm i}\varepsilon_f\int {\rm d}^4y\left[(\bar{\psi}_{\rm FW})_0^{(\pm)}(y)\right]_f K^{\rm FW}(y)\left[\psi_{\rm FW}(y)\right]_i . \quad (15)$$

Here, the bar over the function $\psi_{\rm FW}$ denotes the Hermitian conjugate, $\varepsilon_f=+1$ for the solution $[(\bar{\psi}_{\rm FW})_0^{(+)}(y)]_f$ and $\varepsilon_f=-1$ for the solution $[(\bar{\psi}_{\rm FW})_0^{(-)}(y)]_f$.

For standard QED, equations (13) and (15) were derived in the works of R. Feynman [3, 4] and also, for example, in monograph [17].

Additionally, we mention several important points.

(1) The Hamiltonians $H_{\rm FW}$ in (8) and $K^{\rm FW}(y)$ in (13) are diagonal with respect to the mixing of the upper and lower components of the bispinor $\psi_{\rm FW}$. Each of the equations (8) and (13) includes two independent equations with the spinor wave functions $\sim U_{\rm S}$. One equation contains the states with positive energies, and the other one contains the states with negative energies. The elements of the S-matrix in (15) can be calculated by using only the states with positive energies. In this case, the states with negative energies are not used in calculations of physical processes in the QED: they are only needed for mathematical completeness in the expansion of operators and wave functions.

(2) In standard QED with the Dirac equation, positrons are electrons with negative energies moving in opposite directions in spacetime. In the Foldy–Wouthuysen representation, this situation changes. If in (15) we use $[(\bar{\psi}_{\rm FW})_0^{(+)}]_f$ on the left-hand side and $[(\psi_{\rm FW})_0^{(-)}]_i$ on the right-hand side, then, due to the structure of the bispinors (11), (12), the respective elements of the S-matrix will be zero in all orders of perturbation theory. Taking (11) and (12) into account, we can write

$$\left\langle \frac{1}{(2\pi)^{3/2}}\exp\left({\rm i}|E_f|t-{\rm i}\mathbf{p}_f\mathbf{x}\right)\left(\bar{U}_{\rm S}\quad 0\right)|M_{\rm FW}|\begin{pmatrix}0\\ U_{\rm S}\end{pmatrix}\frac{1}{(2\pi)^{3/2}} \times\exp\left({\rm i}|E_i|t-{\rm i}\mathbf{p}_i\,\mathbf{x}\right)\right\rangle=0\,.$$

Here, by definition, $M_{\rm FW}$ is a diagonal operator.

A similar result is obtained when in (15) we use $[(\bar{\psi}_{\rm FW})_0^{(-)}]_f$ on the left-hand side and $[(\psi_{\rm FW})_0^{(+)}]_i$ on the right-hand side.

Thus, positrons in the FW representation cannot be described by electron states with negative energy. Positrons in this representation should be described by positive energy states of a special equation for positrons.

We have therefore obtained the first paradox. *By performing a unitary transformation of the Dirac equation in the FW representation, we lost the interaction between the states with positive and negative energies.*

3. Nonperturbative quantum electrodynamics and representations of Dirac equation

Consider the case of electrostatic fields of hydrogen-like ions.

3.1 Standard quantum electrodynamics in fields of hydrogen-like ions with large charge number Z

Figure 1 shows the lower energy levels of a hydrogen-like ion as a function of the charge number Z for the standard QED with a fluctuating fermion vacuum. This figure is adapted from monograph [18].

Consider the level $1{\rm s}_{1/2}$. In the Coulomb field of a point nucleus charge $+Z|e|$, the level $1{\rm s}_{1/2}$ disappears on reaching $Z=137$. If finite dimensions of atomic nuclei are considered [19–22], the energy of the $1{\rm s}_{1/2}$ state becomes negative for $Z>146$. For $Z_{\rm cr}\approx 171$, the level $1{\rm s}_{1/2}$ becomes immersed in a negative-energy continuum. Similarly, the energy of the level $2{\rm p}_{1/2}$ becomes negative for $Z>168$; for $Z_{\rm cr}\approx 184$, level $2{\rm p}_{1/2}$ becomes immersed in the negative-energy continuum. According to theoretical predictions of standard QED, when a level becomes immersed in the negative-energy continuum, the neutral vacuum decays, emitting two electron–positron pairs [20, 21].

3.2 Representations of Feynman–Gell-Mann and Foldy–Wouthuysen

We write the Dirac equation in an external electromagnetic field in a covariant form:

$$\left(\gamma^0(p^0-eA^0)-\boldsymbol{\gamma}(\mathbf{p}-e\mathbf{A})-m\right)\psi_{\rm D}(\mathbf{x},t)=0\,. \quad (16)$$

We multiply the left-hand side of (16) by the operator with the changed sign of the fermion mass,

$$\left(\gamma^0(p^0-eA^0)-\boldsymbol{\gamma}(\mathbf{p}-e\mathbf{A})+m\right)\left(\gamma^0(p^0-eA^0)-\boldsymbol{\gamma}(\mathbf{p}-e\mathbf{A})-m\right)\psi(\mathbf{x},t)=0\,. \quad (17)$$

The result is a second order equation whose solutions are degenerate with respect to the sign before the mass m:

$$\left[(p^0-eA^0)^2-(\mathbf{p}-e\mathbf{A})^2-m^2+e\boldsymbol{\Sigma}\,\mathbf{H}-{\rm i}e\,\boldsymbol{\alpha}\mathbf{E}\right]\psi(\mathbf{x},t)=0. \quad (18)$$

Here, $\mathbf{H}={\rm rot}\,\mathbf{A}$, and $\mathbf{E}=-(\partial\mathbf{A}/\partial t)-\boldsymbol{\nabla}A^0$ are the magnetic and electric fields, respectively.

We limit ourselves to the case of static electromagnetic fields when $p^0\psi=\varepsilon\psi$.

To transition to the FG representation, one needs to use the Dirac matrices in the chiral representation in equation (18). This is achieved by the unitary transformation

$$S=S^{-1}=\frac{1}{\sqrt{2}}\begin{pmatrix}I & I\\ I & -I\end{pmatrix}, \quad (19)$$

$$\psi_{\rm FG}(\mathbf{x},t)=S\psi(\mathbf{x},t)=\begin{pmatrix}\varphi_{\rm FG}(\mathbf{x})\\ \chi_{\rm FG}(\mathbf{x})\end{pmatrix}\exp\left(-{\rm i}\varepsilon t\right), \quad (20)$$

$$\alpha^i_{\rm c}=S\alpha^iS^{-1}=\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix},\quad \Sigma_{\rm c}=S\Sigma^iS^{-1}=\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}. \quad (21)$$

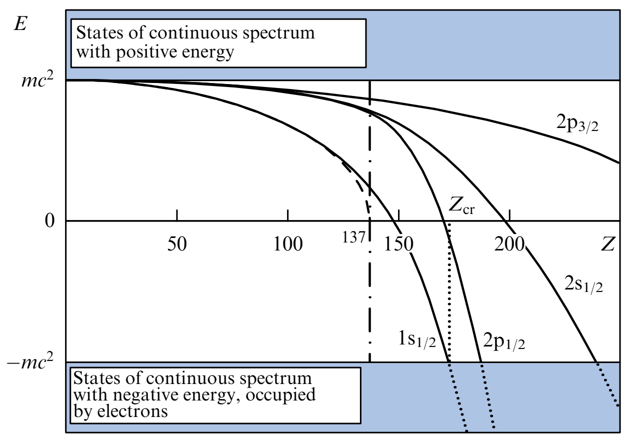

Figure 1. Lower energy levels of hydrogen-like ion as function of nuclear charge number Z.

In the FG representation, there is no mixing of the upper and lower components of the bispinor $\psi_{\rm FG}(\mathbf{x}, t)$ in equation (18). In the case of stationary states, equation (18) is reduced to two separate equations for spinors $\varphi_{\rm FG}(\mathbf{x})$ and $\chi_{\rm FG}(\mathbf{x})$:

$$\left[(\varepsilon - eA^0)^2 - (\mathbf{p} - e\mathbf{A})^2 - m^2 + e\boldsymbol{\sigma}\mathbf{H} - \mathrm{i}e\boldsymbol{\sigma}\mathbf{E}\right]\varphi_{\rm FG}(\mathbf{x}) = 0, \quad (22)$$

$$\left[(\varepsilon - eA^0)^2 - (\mathbf{p} - e\mathbf{A})^2 - m^2 + e\boldsymbol{\sigma}\mathbf{H} + \mathrm{i}e\boldsymbol{\sigma}\mathbf{E}\right]\chi_{\rm FG}(\mathbf{x}) = 0. \quad (23)$$

Analogous equations were considered earlier by Feynman and Gell-Mann [6].

It is noteworthy that equations (22) and (23) are connected with equations in the Foldy–Wouthuysen representation [14, 15]. Equation (22) in the FW representation is obtained for positive energies $\varepsilon = |E| > 0$. In this case, $\varphi_{\rm FG}(\mathbf{x}) = A_{(+)}\varphi_{\rm c}(\mathbf{x})$, where $\varphi_{\rm c}(\mathbf{x})$ is the upper spinor in the Foldy–Wouthuysen representation,

$$\psi_{\rm FW}^{(+)}(\mathbf{x}) = \begin{pmatrix} \varphi_{\rm c}(\mathbf{x}) \\ 0 \end{pmatrix}.$$

Equation (23) in the FW representation is obtained for negative energies $\varepsilon = -|E| < 0$. In this case, $\chi_{\rm FG}(\mathbf{x}) = A_{(-)}\chi_{\rm c}(\mathbf{x})$, where $\chi_{\rm c}(\mathbf{x})$ is the lower spinor in the FW representation,

$$\psi_{\rm FW}^{(-)}(\mathbf{x}) = \begin{pmatrix} 0 \\ \chi_{\rm c}(\mathbf{x}) \end{pmatrix}.$$

We write, in agreement with [14], certain equations for electrons and positrons in the FG and FW representations.

(1) Equation for electrons with positive energies $(\varepsilon = |E| > 0, e = -|e| < 0)$,

$$\left(\left(|E| + |e|A^0\right)^2 - \mathbf{p}^2 - m^2 - \mathrm{i}|e|\boldsymbol{\sigma}\nabla A^0\right)\varphi_{\rm FG}^{\rm e} = 0. \quad (24)$$

(2) Equation for electrons with negative energies $(\varepsilon = -|E| < 0, e = -|e| < 0)$,

$$\left(\left(|E| - |e|A^0\right)^2 - \mathbf{p}^2 - m^2 + \mathrm{i}|e|\boldsymbol{\sigma}\nabla A^0\right)\chi_{\rm FG}^{\rm e} = 0. \quad (25)$$

(3) Equation for positrons with positive energies $(\varepsilon = |E| > 0, e = |e| > 0)$,

$$\left(\left(|E| - |e|A^0\right)^2 - \mathbf{p}^2 - m^2 + \mathrm{i}|e|\boldsymbol{\sigma}\nabla A^0\right)\varphi_{\rm FG}^{\rm p} = 0. \quad (26)$$

In equation (24), the spinor $\varphi_{\rm FG}^{\rm e}(\mathbf{x})$ in the Feynman–Gell-Mann representation is proportional to the upper spinor in the Foldy–Wouthuysen representation:

$$\varphi_{\rm FG}^{\rm e}(\mathbf{x}) = A_{(+)}^{\rm e}\varphi_{\rm c}^{\rm e}(\mathbf{x}); \quad \left(\psi_{\rm FW}^{(+)}(\mathbf{x})\right)^{\rm e} = \begin{pmatrix} \varphi_{\rm c}^{\rm e}(\mathbf{x}) \\ 0 \end{pmatrix}. \quad (27)$$

In equation (25), the spinor $\chi_{\rm FG}^{\rm e}(\mathbf{x})$ in the Feynman–Gell-Mann representation is proportional to the lower spinor $\chi_{\rm c}^{\rm e}(\mathbf{x})$ in the Foldy–Wouthuysen representation:

$$\chi_{\rm FG}^{\rm e}(\mathbf{x}) = A_{(-)}^{\rm e}\chi_{\rm c}^{\rm e}(\mathbf{x}); \quad \left(\psi_{\rm FW}^{(-)}(\mathbf{x})\right)^{\rm e} = \begin{pmatrix} 0 \\ \chi_{\rm c}^{\rm e}(\mathbf{x}) \end{pmatrix}. \quad (28)$$

In equation (26), the spinor $\varphi_{\rm FG}^{\rm p}(\mathbf{x})$ in the Feynman–Gell-Mann representation is proportional to the upper spinor $\varphi_{\rm c}^{\rm p}(\mathbf{x})$ in the Foldy–Wouthuysen representation:

$$\varphi_{\rm FG}^{\rm p}(\mathbf{x}) = A_{(+)}^{\rm p}\varphi_{\rm c}^{\rm p}(\mathbf{x}); \quad \left(\psi_{\rm FW}^{(+)}(\mathbf{x})\right)^{\rm p} = \begin{pmatrix} \varphi_{\rm c}^{\rm p}(\mathbf{x}) \\ 0 \end{pmatrix}. \quad (29)$$

Here,

$$A_{(+)}^{\rm e} = \left(1 + \frac{m^2}{\left(|E| + \boldsymbol{\sigma}\mathbf{p} + |e|A^0\right)^2}\right)^{-1/2},$$

$$A_{(-)}^{\rm e} = \left(1 + \frac{m^2}{\left(|E| + \boldsymbol{\sigma}\mathbf{p} - |e|A^0\right)^2}\right)^{-1/2},$$

$$A_{(+)}^{\rm p} = \left(1 + \frac{m^2}{\left(|E| + \boldsymbol{\sigma}\mathbf{p} - |e|A^0\right)^2}\right)^{-1/2}.$$

As a result, we see that equation (26) for positrons with positive energies $\varepsilon > 0$ coincides with equation (25) for electrons with negative energies $\varepsilon < 0$. This conclusion is confirmed by separating the variables. The systems of equations for the radial functions, derived from equations (25) and (26), are fully consistent with each other.

In our case, the positrons are in a repulsive Coulomb field of ionized nuclei. For them, an upper continuum exists with a continuous energy spectrum $\varepsilon > m$. However, stationary bound states with $\varepsilon < m$ are absent in this case.

The equality of equations (25) and (26) assumes that electrons with negative energies and positrons with positive energies have an equivalent (up to the sign of energy) continuous energy spectrum with the same stationary wave functions. However, the spectrum of equation (25) contains the negative energy level $1\mathrm{s}_{1/2}$ in the interval $Z_\Sigma = 147 - 170$ and the negative energy level $2\mathrm{p}_{1/2}$ in the interval $Z_\Sigma = 169 - 183$ (see Fig. 1).

Simple physical considerations prohibit such bound states for equation (26). Therefore, the existence of bound states with negative energies is a mathematical artifact.

Since equations (24)–(26) are obtained through unitary transformations of the Dirac equation, the conclusion that there is no physical (and not mathematical) contribution of bound states with negative energies in effects calculated in QED is also valid for the original Dirac equation.

Note that equation (26) with a changed sign by $A^0(\mathbf{x})$ (the motion of a positron in an attractive Coulomb field) coincides with equation (24) for electrons. As expected, discrete and continuous energy spectra of electrons and positrons moving in an attractive Coulomb field coincide with each other.

3.3 Dirac equation with nonrelativistic Hamiltonian in Foldy–Wouthuysen representation

The conclusions in Section 3.2 are confirmed by analyzing the nonrelativistic Hamiltonian in the FW representation, which was already obtained in initial work on the Foldy–Wouthuysen transformation (see [5]).

Consider nonrelativistic motion of electrons and positrons in an external electrostatic field $eA^0(\mathbf{x})$. According to [5], the nonrelativistic Hamiltonian $H_{\rm FW}$ takes the form

$$H_{\rm FW} = \beta\left(m + \frac{\mathbf{p}^2}{2m}\right) + eA^0 - \frac{\mathrm{i}e}{8m^2}\,\boldsymbol{\sigma}\,\mathrm{rot}\,\mathbf{E} - \frac{e}{4m^2}\boldsymbol{\sigma}\,(\mathbf{E}\times\mathbf{p}) - \frac{e}{8m^2}\,\mathrm{div}\,\mathbf{E}. \quad (30)$$

Here, $\mathbf{E} = -\nabla A^0$ is the electric field.

The Dirac equation with the Hamiltonian $H_{\rm FW}$ is written in the form

$$\left[(\varepsilon - eA^0) - \beta\left(m + \frac{\mathbf{p}^2}{2m}\right) + \frac{\mathrm{i}e}{8m^2}\,\boldsymbol{\sigma}\,\mathrm{rot}\,\mathbf{E} + \frac{e}{4m^2}\boldsymbol{\sigma}\,(\mathbf{E}\times\mathbf{p}) + \frac{e}{8m^2}\,\mathrm{div}\,\mathbf{E}\right]\psi_{\rm FW}(\mathbf{x}, t) = 0. \quad (31)$$

Here, for $\varepsilon > 0$,

$$\psi_{\rm FW}^{(+)}(\mathbf{x}, t) = \begin{pmatrix} \varphi(\mathbf{x}) \\ 0 \end{pmatrix} \exp(-\mathrm{i}\varepsilon t)\,,$$

and, in this case,

$$\beta\psi_{\rm FW}^{(+)}(\mathbf{x}, t) = \psi_{\rm FW}^{(+)}(\mathbf{x}, t);$$

for $\varepsilon < 0$,

$$\psi_{\rm FW}^{(-)}(\mathbf{x}, t) = \begin{pmatrix} 0 \\ \chi(\mathbf{x}) \end{pmatrix} \exp(-\mathrm{i}\varepsilon t)\,,$$

and, in this case, $\beta\psi_{\rm FW}^{(-)}(\mathbf{x}, t) = -\psi_{\rm FW}^{(-)}(\mathbf{x}, t)$.

Just as in Section 3.2, consider Eqn (31) for electrons and positrons.

(1) The equation for electrons with positive energies ($\varepsilon = |E| > 0, e = -|e| < 0$):

$$\left[\left(|E|+|e|A^0\right)-\left(m^2+\frac{\mathbf{p}^2}{2m}\right)-\frac{\mathrm{i}|e|}{8m^2}\,\boldsymbol{\sigma}\,\mathrm{rot}\,\mathbf{E} -\frac{|e|}{4m^2}\boldsymbol{\sigma}(\mathbf{E}\times\mathbf{p})-\frac{|e|}{8m^2}\,\mathrm{div}\,\mathbf{E}\right]\varphi^{\rm e}(\mathbf{x})=0. \quad (32)$$

(2) The equation for electrons with negative energies ($\varepsilon = -|E| < 0, e = -|e| < 0$):

$$\left[\left(|E|-|e|A^0\right)-\left(m^2+\frac{\mathbf{p}^2}{2m}\right)+\frac{\mathrm{i}|e|}{8m^2}\,\boldsymbol{\sigma}\,\mathrm{rot}\,\mathbf{E} +\frac{|e|}{4m^2}\boldsymbol{\sigma}(\mathbf{E}\times\mathbf{p})+\frac{|e|}{8m^2}\,\mathrm{div}\,\mathbf{E}\right]\chi^{\rm e}(\mathbf{x})=0. \quad (33)$$

(3) The equation for positrons with positive energies ($\varepsilon = |E| > 0, e = |e| > 0$):

$$\left[\left(|E|-|e|A^0\right)-\left(m^2+\frac{\mathbf{p}^2}{2m}\right)+\frac{\mathrm{i}|e|}{8m^2}\,\boldsymbol{\sigma}\,\mathrm{rot}\,\mathbf{E} +\frac{|e|}{4m^2}\boldsymbol{\sigma}(\mathbf{E}\times\mathbf{p})+\frac{|e|}{8m^2}\,\mathrm{div}\,\mathbf{E}\right]\varphi^{\rm p}(\mathbf{x})=0. \quad (34)$$

It can be seen that equations (33) and (34) coincide, i.e., just as in the preceding sections, the equation for positrons with $\varepsilon > 0$ coincides with the equation for electrons with $\varepsilon < 0$ in the Coulomb field of atomic nuclei.

However, in the nonrelativistic case with $|E| - m \ll 1$, the coincidence of equations (33) and (34) does not lead to a conflict with physical reality. Indeed, in this case, equation (33) does not produce discrete levels with negative energies (see Fig. 1).

Such levels arise mathematically in the domain of strong electrostatic fields with $Z > 146$. This contradicts clear physical arguments about the absence of discrete levels with negative energies.

4. Conclusions

Certain paradoxes of the Dirac equation have been explained in the FW representation earlier.

(1) In the Dirac equation without interaction, the velocity of fermions is $\mathbf{v}_{\rm D} = c\boldsymbol{\alpha}$ [1]. In the FW representation, the velocity of fermions takes the classical form $\mathbf{v}_{\rm FW} = c\mathbf{p}/E$ [5].

(2) In the FW representation, the 'jitter' (Zitterbewegung) of the fermion coordinates is absent [5]. See also [23, 24] on the problem of Zitterbewegung.

(3) As many authors have mentioned, a positive aspect of the FW representation is that the correspondence principle between quantum-mechanical operators and analogous classical quantities is explicitly observed in nonrelativistic quantum mechanics. As can be seen, for example from pp. 1, 2, such a correspondence is often absent in the Dirac representation.

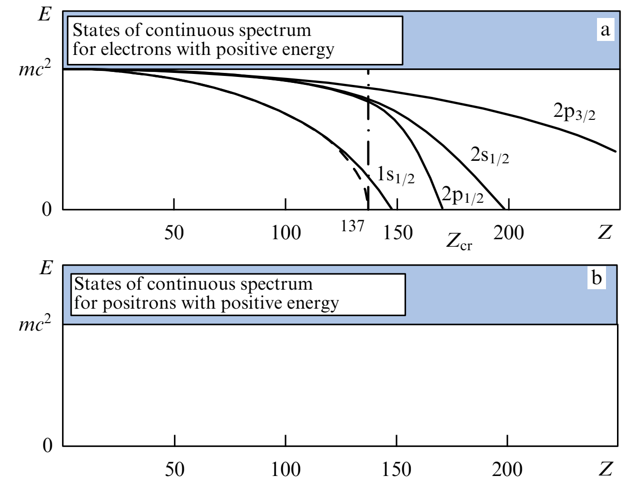

Figure 2. Energy spectrum of (a) equation for electrons (24) and (b) equation for positrons (25).

These facts are related to the absence of virtual interaction between fermions with positive and negative energies in the Foldy–Wouthuysen representation.

The physical community has become accustomed to the paradoxes of the Dirac equation, because they do not affect the results of calculations of physical effects in QED.

Our analysis revealed two new paradoxes, whose solution influences the physical effects of QED.

Paradox No. 1. After applying a unitary transformation to the Dirac equation in an external electromagnetic field and transition to the FW representation, the elements of the *S*-matrix lose the interaction between the states with positive and negative energy. To restore the interaction, an additional equation for positrons with positive energies needs to be introduced into the theory.

Paradox No. 2. Contrary to a clear physical picture, mathematical calculations of the energy spectrum for hydrogen-like ions with $Z_\Sigma = 147-183$ produce levels with negative energies.[2]

Both the paradoxes are related to negative-energy solutions of the Dirac equation.

All paradoxes disappear when only states with positive energies are used to calculate the physical effects of QED. The states with negative energy should only be taken into account to ensure completeness in the expansions of wave functions and operators. In this case, in addition to the equation for electrons with positive energies, an equation for positrons with positive energies is introduced.

This approach applies for $\mathrm{QED_{FW}}$, $\mathrm{QED_{KG}}$, and also for the standard QED with the Dirac equation (see [25]). Within the domain of applicability of perturbation theory, the final physical results coincide with those of standard QED.

In nonperturbative QED, one should consider the spectrum shown in Fig. 2 instead of the spectrum of hydrogen-like ions with charges $Z|e|$ (see Fig. 1).

[2] Discrete levels with negative energies appear in calculations that consider the finite size of nuclei [19–22]. In calculations with the Coulomb field of point nucleus charge $+Z|e|$, such states are absent.

In nonperturbative QED, the calculated spectrum shown in Fig. 2a (without discrete and continuous spectra with negative energies) can be confirmed experimentally. Reference [26] suggests performing a series of experiments on heavy ion colliders to confirm this.

We note that Prof. Dirac, dissatisfied with the presence of states with negative energies in his equation, already at the end of his life turned to searching for a relativistic wave equation with only positive energy solutions [27, 28]. However, he did not succeed in carrying his ideas to a logical end. In this work, we continue the path to solving physical problems of quantum electrodynamics caused by the presence of fermion states with negative energies.

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