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METHODOLOGICAL NOTES

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Noether formalism for constructing conserved quantities in teleparallel equivalents of general relativity

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Contents

Abstract. This paper has a methodological character, where we present a comprehensive formalism for constructing conserved quantities in the Teleparallel Equivalent of General Relativity (TEGR) and Symmetric Teleparallel Equivalent of General Relativity (STEGR). It was developed in a series of our earlier studies. and here, we combine them into a complete form. By employing the Noether method within a tensor formalism, conserved currents, superpotentials, and charges are constructed. These are shown to be covariant under coordinate transformations and local Lorentz rotations in TEGR, while, in STEGR, they are covariant under coordinate transformations. The teleparallel (flat) connections in both theories are defined using the 'turning off gravity' principle. Uniting such defined flat connections with the tetrad in TEGR and metric in STEGR, a new notion — 'gauge' — fruitful in applications, is introduced. The choice of various initial tetrads in TEGR or initial coordinates in STEGR leads to different gauges, giving different conserved quantities. Finally, we discuss an appropriate choice of gauges from a possible set of them.

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1. Introduction

Teleparallel theories of gravity have been actively developed in recent years. A main feature of these theories is the use of the connection with zero Riemann curvature. These theories include the Teleparallel Equivalent of General Relativity (TEGR), the Symmetric Teleparallel Equivalent of General Relativity (STEGR), and modifications of these theories [1–5]. Such modifications have the advantage that their field equations are of the second-order, which imparts similarities with gauge field theories and potentially links gravity to other theories of fundamental interactions in nature.

In TEGR and its modifications, a flat metric compatible connection is used. In STEGR and its modifications, a flat connection with zero torsion is used. TEGR and STEGR are fully equivalent to General Relativity (GR) at the level of field equations; thus, solutions of the field equations in TEGR and STEGR are exactly the same as those in GR. The teleparallel connections in TEGR and STEGR are not dynamical quantities and cannot be determined by field equations. This fact does not influence the dynamics of gravitational interacting objects but introduces ambiguities to the values of the main quantities which define gravitational energy–momentum.

Despite the fact that the focus is currently on how accurately the modified teleparallel theories can describe the observed phenomena, not all the issues have been resolved in TEGR and STEGR themselves. One such issue is the definition of gravitational energy–momentum and other conserved quantities. There are various approaches for constructing conserved quantities in both TEGR and STEGR. They have already been tested to construct them, for example, for the Schwarzschild solution and cosmological models [6–11].

In most of the aforementioned work on TEGR, observers measuring conserved quantities (like masses, angular momentum) or testing the Einstein equivalence principle are associated with the time-like tetrad vector that destructs a Lorentz covariance (or invariance) of quantities, of course. Many of the earlier approaches are based on the reconstruction of field equations in the form of conservation laws [3]. However, this way, for example, in TEGR, leads to problems when constructing charges which are not simultaneously coordinate covariant and invariant under local Lorentz rotations of the tetrad. In the framework of the formalism of differential forms, this problem has been resolved in [9, 12]. However, these results have not been further developed and have been tested on a limited number of models. Construction of conserved quantities in STEGR has not been so active; on this topic, see for STEGR and its modifications review [13] and numerous references therein.

Nevertheless, as far as is known, the definition of conserved quantities is one of the most important tasks in the development of the foundations of any theory. In recent years, we have been developing methods to construct conserved quantities in TEGR and STEGR [14–22]. Our approaches are completely based on the classical Noether theorem. Diffeomorphism invariance of TEGR and STEGR actions is taken as a basis for its applications. Both of the theories are considered classical field theories where the tetrad components in TEGR and metric components in STEGR are interpreted as dynamical variables. We take into account that diffeomorphisms act on all geometrical objects, including nondynamical teleparallel (flat) connections, not only on tetrads and metrics. Observers are associated with displacement vectors of diffeomorphisms, and conserved quantities are interpreted depending on the choice of these vectors. Both in TEGR and STEGR, a generalized principle of 'turning off gravity' was elaborated, and a new notion of 'gauges' was introduced. All of these help us to study the problem of constructing conserved quantities systematically.

Our methods, principles, notions, and assumptions were introduced step by step in developing our approach; however, they were not formulated as a unique approach. Thus, the purpose of this article is methodological. Here, we unite all items of our approach in constructing conserved quantities in TEGR and STEGR presented in [14–22] into a separate formalism. It could be used more effectively for TEGR and STEGR in applications. Moreover, it has substantial potential for a possible development itself in the framework of TEGR and STEGR modifications.

The rest of the article is organized as follows. In Section 2, the foundational elements of teleparallel equivalences of GR, TEGR, and STEGR are introduced. Underlying principles and such notions as the torsion scalar, nonmetricity tensor, and their role in the respective theories are highlighted.

Section 3 delves into the Noether formalism, deriving general conservation laws and conserved quantities in an arbitrary covariant field theory. This forms the basis for constructing conserved currents, superpotentials, and charges.

Section 4 applies the general formalism to TEGR and STEGR, presenting detailed constructions of conserved quantities and exploring their covariance properties under coordinate transformations and local Lorentz rotations.

Section 5 focuses on defining teleparallel connections using the 'turning off gravity' principle, discussing the dependence of conserved quantities on the choice of initial tetrads in TEGR or coordinates in STEGR. As a necessary element of our formalism, a new notion named 'gauges' in TEGR and STEGR is introduced.

Finally, in Section 6 of the article, we place the conclusion, summarizing the results and highlighting their significance for teleparallel gravity.

All definitions in TEGR correspond to [3]. All definitions in STEGR correspond to [1].

2. Main elements of teleparallel equivalent of general relativity and symmetric teleparallel equivalent of general relativity

2.1 Teleparallel equivalent of general relativity

The most popular gravitational Lagrangian in TEGR is presented in the form given, for example, in book [3]:

$$\overset{\bullet}{\mathcal{L}} = \frac{h}{2\kappa}\overset{\bullet}{T} \equiv \frac{h}{2\kappa}\left(\frac{1}{4}\overset{\bullet}{T}{}^{\rho}{}_{\mu\nu}\overset{\bullet}{T}{}_{\rho}{}^{\mu\nu} + \frac{1}{2}\overset{\bullet}{T}{}^{\rho}{}_{\mu\nu}\overset{\bullet}{T}{}^{\nu\mu}{}_{\rho} - \overset{\bullet}{T}{}^{\rho}{}_{\mu\rho}\overset{\bullet}{T}{}^{\nu\mu}{}_{\nu}\right). \tag{1}$$

We set $G = c = 1$ units, for which the Einstein constant $\kappa = 8\pi$. To present one of the main quantities in TEGR, namely, the torsion tensor $\overset{\bullet}{T}{}^{a}{}_{\mu\nu}$, one has to introduce the following ideas and notations.

First, there are tetrads with components $h^{a}{}_{\nu}$, where Latin indices numerate tetrad vectors, and Greek indices are related to spacetime coordinates. They are connected to metric $g_{\mu\nu}$ by the standard relation

$$g_{\mu\nu} = \eta_{ab}\, h^{a}{}_{\mu} h^{b}{}_{\nu}\,, \tag{2}$$

where η_{ab} is the Minkowskian metric in the tetrad space, $h = \det h^{a}{}_{\nu}$. In addition, we use the signature $(-,+,+,+)$ both for $g_{\mu\nu}$ and for η_{ab}. The transformation of the tetrad indices into spacetime ones and vice versa is performed by contraction with tetrad vectors, for example, $\overset{\bullet}{T}{}^{\rho}{}_{\mu\nu} = \overset{\bullet}{T}{}^{a}{}_{\mu\nu}\, h_{a}{}^{\rho}$, etc.

Second, another important quantity in TEGR is the Weitzenbök (teleparallel) connection $\overset{\bullet}{\Gamma}{}^{\alpha}{}_{\kappa\lambda}$ [3]. Quantities constructed with the use of $\overset{\bullet}{\Gamma}{}^{\alpha}{}_{\kappa\lambda}$ are denoted by '•' above a symbol, for example, a related covariant derivative $\overset{\bullet}{\nabla}_{\mu}$. The connection $\overset{\bullet}{\Gamma}{}^{\alpha}{}_{\kappa\lambda}$, as stated in the Introduction, is flat, meaning that the corresponding curvature is equal to zero:

$$\overset{\bullet}{R}{}^{\alpha}{}_{\beta\mu\nu} \equiv \partial_{\mu}\overset{\bullet}{\Gamma}{}^{\alpha}{}_{\beta\nu} - \partial_{\nu}\overset{\bullet}{\Gamma}{}^{\alpha}{}_{\beta\mu} + \overset{\bullet}{\Gamma}{}^{\alpha}{}_{\kappa\mu}\overset{\bullet}{\Gamma}{}^{\kappa}{}_{\beta\nu} - \overset{\bullet}{\Gamma}{}^{\alpha}{}_{\kappa\nu}\overset{\bullet}{\Gamma}{}^{\kappa}{}_{\beta\mu} = 0\,. \tag{3}$$

Also note that $\overset{\bullet}{\Gamma}{}^{\alpha}{}_{\kappa\lambda}$ is compatible with the physical metric. This means that corresponding nonmetricity $\overset{\bullet}{Q}_{\mu\alpha\beta}$ is zero:

$$\overset{\bullet}{Q}_{\mu\alpha\beta} \equiv \overset{\bullet}{\nabla}_{\mu} g_{\alpha\beta} = 0\,. \tag{4}$$

Third, special attention has to be paid to the inertial spin connection (ISC) $\overset{\bullet}{A}{}^{a}{}_{c\nu}$, defined as

$$\overset{\bullet}{A}{}^{a}{}_{b\mu} = -h_{b}{}^{\nu}\,\overset{\bullet}{\nabla}_{\mu}\, h^{a}{}_{\nu}\,. \tag{5}$$

Then, the definition of $\overset{\bullet}{T}{}^a{}_{\mu\nu}$ can be presented evidently:

$$\overset{\bullet}{T}{}^a{}_{\mu\nu} = \partial_\mu h^a{}_\nu - \partial_\nu h^a{}_\mu + \overset{\bullet}{A}{}^a{}_{c\mu} h^c{}_\nu - \overset{\bullet}{A}{}^a{}_{c\nu} h^c{}_\mu \,. \tag{6}$$

Namely, the presence of $\overset{\bullet}{A}{}^a{}_{c\nu}$ makes the torsion (6) covariant with respect to local Lorentz rotations of the tetrad.

It is instructive to introduce some quantities constructed with the use of the Levi-Civita connection (Christoffel symbols) $\overset{\circ}{\Gamma}{}^\alpha{}_{\kappa\lambda}$. Following [3], we denote such quantities by '∘' above a symbol. Thus, $\overset{\circ}{A}{}^a{}_{b\rho}$ presents the Levi-Civita spin connection (L-CSC) defined as

$$\overset{\circ}{A}{}^a{}_{b\mu} = -h_b{}^\nu \overset{\circ}{\nabla}_\mu h^a{}_\nu \,, \tag{7}$$

where the covariant derivative $\overset{\circ}{\nabla}_\mu$ is related to $\overset{\circ}{\Gamma}{}^\alpha{}_{\kappa\lambda}$.

Let us list the main tensors in TEGR. An important quantity in TEGR is the contortion tensor $\overset{\bullet}{K}{}^a{}_{b\rho}$, which is the difference between (5) and (7); thus,

$$\overset{\bullet}{K}{}^a{}_{b\rho} = \overset{\bullet}{A}{}^a{}_{b\rho} - \overset{\circ}{A}{}^a{}_{b\rho} \,. \tag{8}$$

It is expressed through the torsion tensor as

$$\overset{\bullet}{K}{}^\rho{}_{\mu\nu} = \frac{1}{2}\left(\overset{\bullet}{T}{}_\mu{}^\rho{}_\nu + \overset{\bullet}{T}{}_\nu{}^\rho{}_\mu - \overset{\bullet}{T}{}^\rho{}_{\mu\nu}\right). \tag{9}$$

The torsion scalar in (1) is rewritten as

$$\overset{\bullet}{T} = \frac{1}{2}\,\overset{\bullet}{S}{}_a{}^{\rho\sigma}\,\overset{\bullet}{T}{}^a{}_{\rho\sigma} \,, \tag{10}$$

making the use of the tensor

$$\overset{\bullet}{S}{}_a{}^{\rho\sigma} = \overset{\bullet}{K}{}^{\rho\sigma}{}_a + h_a{}^\sigma \overset{\bullet}{K}{}^{\theta\rho}{}_\theta - h_a{}^\rho \overset{\bullet}{K}{}^{\theta\sigma}{}_\theta \,, \tag{11}$$

which is called the teleparallel superpotential and is antisymmetric in the upper indexes.

All of the aforementioned tensors $\overset{\bullet}{T}{}^a{}_{\mu\nu}$, $\overset{\bullet}{K}{}^{\rho\sigma}{}_a$, and $\overset{\bullet}{S}{}_a{}^{\rho\sigma}$ are covariant with respect to both coordinate transformations and local Lorentz rotations. Local Lorentz covariance of these quantities is achieved by simultaneous transformation of both the tetrad and the ISC:

$$h'^a{}_\mu = \Lambda^a{}_b(x)\, h^b{}_\mu \,, \tag{12}$$

$$\overset{\bullet}{A}{}'^a{}_{b\mu} = \Lambda^a{}_c(x)\,\overset{\bullet}{A}{}^c{}_{d\mu}\,\Lambda_b{}^d(x) + \Lambda^a{}_c(x)\,\partial_\mu \Lambda_b{}^c(x) \,, \tag{13}$$

where $\Lambda^a{}_c(x)$ is the matrix of a local Lorentz rotation, and $\Lambda_a{}^c(x)$ is an inverse matrix of the latter. Transformation (13) tells us that the ISC is not a tensor and can be equalized to zero by an appropriate local Lorentz transformation. Then, by an appropriate Lorentz rotation, it can be represented in the form

$$\overset{\bullet}{A}{}^a{}_{c\nu} = \tilde{\Lambda}^a{}_b\,\partial_\nu\,(\tilde{\Lambda}^{-1})^b{}_c \,. \tag{14}$$

Now, let us vary the action with Lagrangian (1) with respect to tetrad components:

$$E_a{}^\rho = -\frac{\delta\overset{\bullet}{\mathcal{L}}}{\delta h^a{}_\rho} = -\left[\frac{\partial\overset{\bullet}{\mathcal{L}}}{\partial h^a{}_\rho} - \partial_\sigma\left(\frac{\partial\overset{\bullet}{\mathcal{L}}}{\partial h^a{}_{\rho,\sigma}}\right)\right]. \tag{15}$$

The above information is enough to rewrite Lagrangian (1) in the form [3]

$$\overset{\bullet}{\mathcal{L}} = \overset{\circ}{\mathcal{L}} - \frac{1}{\kappa}\,\partial_\mu\left(h\,\overset{\bullet}{T}{}^{\nu\mu}{}_\nu\right), \tag{16}$$

where the first term is the Hilbert Lagrangian

$$\overset{\circ}{\mathcal{L}} = -\frac{h}{2\kappa}\,\overset{\circ}{R} \tag{17}$$

with the Riemannian curvature scalar $\overset{\circ}{R}$ expressed through the tetrad components by (2) (see [23]). Then, because the TEGR Lagrangian (1) contains the ISC in the divergence, only $E_a{}^\rho$ in (15) does not contain the ISC totally:

$$E_a{}^\rho = -\frac{\delta\overset{\bullet}{\mathcal{L}}}{\delta h^a{}_\rho} = -\frac{\delta\overset{\circ}{\mathcal{L}}}{\delta h^a{}_\rho} \,. \tag{18}$$

Let us add to Lagrangian (1) a matter Lagrangian $\overset{m}{\mathcal{L}}$, where matter fields ϕ are coupled minimally to metric (2): $\overset{\bullet}{\mathcal{L}} + \overset{m}{\mathcal{L}}$. Varying the action with such a Lagrangian with respect to tetrad components, one obtains gravitational field equations

$$E_a{}^\rho = \theta_a{}^\rho \,, \tag{19}$$

where $\theta_a{}^\rho = 2\delta\overset{m}{\mathcal{L}}/\delta h^a{}_\rho$ is the matter energy–momentum tensor. Due to (18), we are convinced that the TEGR field equations (19) and the GR field equations are equivalent.

It is important to discuss the place of the ISC in the above scheme. Varying the action with Lagrangian (1), and with (16) as well, with respect to $\overset{\bullet}{A}{}^a{}_{b\rho}$, one obtains $0 = 0$. This means that the ISC cannot be determined in the framework of TEGR itself. Then, if necessary, it has to be defined by additional requirements, for example, by a construction of acceptable conserved quantities for a concrete solution, as will be shown below.

2.2 Symmetric teleparallel equivalent of general relativity

Usually, a Lagrangian in STEGR is considered in the form given, for example, in [1]:

$$\mathcal{L} = \frac{\sqrt{-g}}{2\kappa}\, g^{\mu\nu}\left(L^\alpha{}_{\beta\mu} L^\beta{}_{\nu\alpha} - L^\alpha{}_{\beta\alpha} L^\beta{}_{\mu\nu}\right). \tag{20}$$

Here, the disformation tensor $L^\alpha{}_{\mu\nu}$ is introduced by

$$L^\alpha{}_{\mu\nu} = \frac{1}{2}\,Q^\alpha{}_{\mu\nu} - \frac{1}{2}\,Q_\mu{}^\alpha{}_\nu - \frac{1}{2}\,Q_\nu{}^\alpha{}_\mu \,, \tag{21}$$

where the nonmetricity tensor $Q_{\alpha\mu\nu}$ is defined, as usual:

$$Q_{\alpha\mu\nu} \equiv \nabla_\alpha g_{\mu\nu} \,. \tag{22}$$

The covariant derivative ∇_α is defined making use of the teleparallel connection $\Gamma^\alpha{}_{\mu\nu}$. It is symmetric in lower indexes, and we introduce the abbreviation STC (symmetric teleparallel connection). The STC is torsionless because $T^\alpha{}_{\mu\nu} \equiv \Gamma^\alpha{}_{\mu\nu} - \Gamma^\alpha{}_{\nu\mu} = 0$ by definition. Of course, STC is flat, which means that the related curvature tensor is zero:

$$R^\alpha{}_{\beta\mu\nu}(\Gamma) = \partial_\mu \Gamma^\alpha{}_{\nu\beta} - \partial_\nu \Gamma^\alpha{}_{\mu\beta} + \Gamma^\alpha{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\beta} - \Gamma^\alpha{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\beta} = 0 \,. \tag{23}$$

One can easily check that (21) can be rewritten in the form

$$L^\beta{}_{\mu\nu} \equiv \Gamma^\beta{}_{\mu\nu} - \overset{\circ}{\Gamma}{}^\beta{}_{\mu\nu} \,, \tag{24}$$

where $\overset{\circ}{\Gamma}{}^\beta{}_{\mu\nu}$ is the Levi-Civita connection. Next, with the use of (21)–(24), Lagrangian (20) acquires the form

$$\mathcal{L} = \overset{\circ}{\mathcal{L}} + \frac{\sqrt{-g}g^{\mu\nu}}{2\kappa} R_{\mu\nu} + \mathcal{L}' \,. \tag{25}$$

Just like (17), the first term is the Hilbert Lagrangian; however, depending on the metric components (not tetrad components),

$$\overset{\circ}{\mathcal{L}} = -\frac{\sqrt{-g}}{2\kappa}\overset{\circ}{R}\,. \tag{26}$$

The second term in (25) is equal to zero due to (23). However, if we preserve it, we need to vary it. On the one hand, variation (25) together with the second term with respect to the metric means variation (20) exactly. At the same time, variation (20) with respect to $\Gamma^{\beta}{}_{\mu\nu}$ gives $\Gamma^{\beta}{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\beta}{}_{\mu\nu}$. This means that the flat STC $\Gamma^{\beta}{}_{\mu\nu}$ has to be equal to the Levi-Civita connection that is not flat in general. Since this is not permissible, the second term in (25) has to be cancelled. The third term $\mathcal{L}'$ is the total divergence:

$$\mathcal{L}' = \frac{\sqrt{-g}}{2\kappa}\overset{\circ}{\nabla}_{\alpha}(\hat{Q}^{\alpha} - Q^{\alpha}) = \partial_{\alpha}\left[\frac{1}{2\kappa}\sqrt{-g}\,(\hat{Q}^{\alpha} - Q^{\alpha})\right] = \partial_{\alpha}\mathcal{D}^{\alpha}, \tag{27}$$

where $Q_{\alpha} = g^{\mu\nu}Q_{\alpha\mu\nu}$, $\hat{Q}_{\alpha} = g^{\mu\nu}Q_{\mu\alpha\nu}$. As a result, instead of (20) or (25) in the STEGR, we choose the Lagrangian

$$\mathcal{L} = -\frac{\sqrt{-g}}{2\kappa}\overset{\circ}{R} + \partial_{\alpha}\mathcal{D}^{\alpha}\,, \tag{28}$$

where

$$\mathcal{D}^{\alpha} \equiv -\frac{\sqrt{-g}}{2\kappa}\,(Q^{\alpha} - \hat{Q}^{\alpha})\,. \tag{29}$$

Because the STEGR Lagrangian contains the STC in the divergence only varying the action with Lagrangian (28) with respect to the metric components, one obtains the GR field equations exactly. Keeping in mind the possible presence of matter variables, one has

$$\overset{\circ}{G}_{\mu\nu} = \kappa\theta_{\mu\nu}\,, \tag{30}$$

where $\overset{\circ}{G}_{\mu\nu}$ is the usual Einstein tensor, and $\theta_{\mu\nu}$ is the usual symmetric (metric) matter energy–momentum tensor. Moreover, it is another form of (19), and, again, using (30), we stress the equivalence between STEGR and GR.

Finally, it is important to remark on the following. Varying the action with Lagrangian (28) with respect to $\Gamma^{\alpha}{}_{\mu\nu}$, one gets $0 = 0$. That is, the STC cannot be determined in the framework of STEGR itself. This means that the STC, just like the ISC in TEGR, is an external structure and can be defined by additional requirements only, like a construction of conserved quantities with acceptable values.

3. Noether conserved quantities in arbitrary field theory

To derive conservation laws which follow from the diffeomorphism invariance of an arbitrary covariant field theory of fields ψ^A, we turn, first of all, to Mitskevich's classical book [24] (see also [25, 26]). Let us consider the action

$$S = \int d^4x\,\mathcal{L}(\psi^A; \psi^A{}_{,\alpha}; \psi^A{}_{,\alpha\beta})\,, \tag{31}$$

where the symbol ψ^A means an arbitrary tensor density or set of such densities with A being a collective index. Setting the invariance of the action (31) under diffeomorphisms with the displacement vectors ξ^{α}, we assume variations in ψ^A in the form of the Lie derivatives:

$$\delta\psi^A = \mathscr{L}_{\xi}\psi^A = -\xi^{\alpha}\partial_{\alpha}\psi^A + \psi^A|^{\alpha}_{\beta}\,\partial_{\alpha}\xi^{\beta}\,. \tag{32}$$

We borrow the notation $\psi^A|^{\alpha}_{\beta}$ from [24], its concrete presentation defined by the transformation properties of ψ^A. For example, for a tensor $\psi^A = \psi^{\sigma}{}_{\rho}$, we have $\psi^{\sigma}{}_{\rho}|^{\alpha}_{\beta} = \delta^{\sigma}_{\beta}\psi^{\alpha}{}_{\rho} - \delta^{\alpha}_{\rho}\psi^{\sigma}{}_{\beta}$. From the diffeomorphism invariance of (31), it follows that Lagrangian $\mathcal{L}$ is a scalar density with the mathematical weight $+1$. It defines the main Noether identity:

$$\mathscr{L}_{\xi}\mathcal{L} = -\partial_{\alpha}(\xi^{\alpha}\mathcal{L})\,. \tag{33}$$

Keeping in mind (32), the main Noether identity is rewritten in the form

$$\frac{\delta\mathcal{L}}{\delta\psi^B}\mathscr{L}_{\xi}\psi^B + \partial_{\alpha}\left[\frac{\delta\mathcal{L}}{\delta\psi^B{}_{,\alpha}}\mathscr{L}_{\xi}\psi^B + \frac{\partial\mathcal{L}}{\partial\psi^B{}_{,\beta\alpha}}(\mathscr{L}_{\xi}\psi^B)_{,\beta} + \xi^{\alpha}\mathcal{L}\right] \equiv 0, \tag{34}$$

where the standard notations are used:

$$\frac{\delta\mathcal{L}}{\delta\psi^B} = \frac{\partial\mathcal{L}}{\partial\psi^B} - \partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial\psi^B{}_{,\mu}}\right) + \partial_{\mu\nu}\left(\frac{\partial\mathcal{L}}{\partial\psi^B{}_{,\mu\nu}}\right),$$

$$\frac{\delta\mathcal{L}}{\delta\psi^B{}_{,\alpha}} = \frac{\partial\mathcal{L}}{\partial\psi^B{}_{,\alpha}} - \partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial\psi^B{}_{,\alpha\mu}}\right),$$

where we denote the double partial derivatives of $\partial^2/(\partial x^{\mu}\partial x^{\nu})$ as $\partial_{\mu\nu}$ or $_{,\mu\nu}$. We use partial derivatives of variables since, in metric theories, a set of field variables ψ^A includes the metric tensor $g_{\mu\nu}$ anyway, whose covariant derivatives identically vanish. Indeed, it is not permissible to differentiate with respect to $\overset{\circ}{\nabla}g_{\mu\nu} \equiv 0$; thus, the formalism fails.

Substituting (32) into (34), regrouping terms, applying the Leibniz rule, and transforming divergences, we obtain

$$-\left[\frac{\delta\mathcal{L}}{\delta\psi^B}\psi^B{}_{,\alpha} + \partial_{\beta}\left(\frac{\delta\mathcal{L}}{\delta\psi^B}\psi^B|^{\beta}_{\alpha}\right)\right]\xi^{\alpha}$$
$$+\,\partial_{\alpha}[\mathcal{U}_{\sigma}{}^{\alpha}\xi^{\sigma} + \mathcal{M}_{\sigma}{}^{\alpha\tau}\partial_{\tau}\xi^{\sigma} + \mathcal{N}_{\sigma}{}^{\alpha\tau\beta}\partial_{\beta\tau}\xi^{\sigma}] \equiv 0\,. \tag{35}$$

Here, the coefficients

$$\mathcal{U}_{\sigma}{}^{\alpha} \equiv \mathcal{L}\delta^{\alpha}_{\sigma} + \frac{\delta\mathcal{L}}{\delta\psi^B}\psi^B|^{\alpha}_{\sigma} - \frac{\delta\mathcal{L}}{\delta\psi^B{}_{,\alpha}}\partial_{\sigma}\psi^B - \frac{\partial\mathcal{L}}{\partial\psi^B{}_{,\beta\alpha}}\partial_{\beta\sigma}\psi^B, \tag{36}$$

$$\mathcal{M}_{\sigma}{}^{\alpha\tau} \equiv \frac{\delta\mathcal{L}}{\delta\psi^B{}_{,\alpha}}\psi^B|^{\tau}_{\sigma} - \frac{\partial\mathcal{L}}{\partial\psi^B{}_{,\tau\alpha}}\partial_{\sigma}\psi^B + \frac{\partial\mathcal{L}}{\partial\psi^B{}_{,\beta\alpha}}\partial_{\beta}\left(\psi^B|^{\tau}_{\sigma}\right), \tag{37}$$

$$\mathcal{N}_{\sigma}{}^{\alpha\tau\beta} \equiv \frac{1}{2}\left[\frac{\partial\mathcal{L}}{\partial\psi^B{}_{,\beta\alpha}}\psi^B|^{\tau}_{\sigma} + \frac{\partial\mathcal{L}}{\partial\psi^B{}_{,\tau\alpha}}\psi^B|^{\beta}_{\sigma}\right] \tag{38}$$

are fully determined by Lagrangian (31) and its derivatives. To obtain (38), the symmetry property $\mathcal{N}_{\sigma}{}^{\alpha\tau\beta} = \mathcal{N}_{\sigma}{}^{\alpha\beta\tau}$ was used, which follows directly from (35) due to the commutativity of second partial derivatives.

Applying a partial derivative to each term in the square brackets of identity (35) and remembering that the vector field

ξ^σ and all its partial derivatives are independent and arbitrary at every point of the spacetime manifold, one concludes that all coefficients of ξ^σ and its derivatives must independently vanish identically. This leads to the system of identities

$$\partial_\alpha \mathcal{U}_\sigma{}^\alpha \equiv \frac{\delta\mathcal{L}}{\delta\psi^B}\,\psi^B{}_{,\alpha} + \partial_\beta\left(\frac{\delta\mathcal{L}}{\delta\psi^B}\,\psi^B\big|^\beta_\alpha\right), \tag{39}$$

$$\mathcal{U}_\sigma{}^\alpha + \partial_\lambda \mathcal{M}_\sigma{}^{\lambda\alpha} \equiv 0\,, \tag{40}$$

$$\mathcal{M}_\sigma{}^{(\alpha\beta)} + \partial_\lambda \mathcal{N}_\sigma{}^{\lambda(\alpha\beta)} \equiv 0\,, \tag{41}$$

$$\mathcal{N}_\sigma^{(\alpha\beta\gamma)} \equiv 0\,. \tag{42}$$

A system like (39)–(42) was first derived by Klein [27] and is usually called *Klein identities*. Differentiating (40) and using (41) and (42) yields $\partial_\alpha \mathcal{U}_\sigma{}^\alpha \equiv 0$. This implies that the right-hand side of (39) must identically vanish as well:

$$\frac{\delta\mathcal{L}}{\delta\psi^B}\,\psi^B{}_{,\alpha} + \partial_\beta\left(\frac{\delta\mathcal{L}}{\delta\psi^B}\,\psi^B\big|^\beta_\alpha\right) \equiv 0\,. \tag{43}$$

This is essentially a statement of Noether's second theorem [26] and a generalization of the Bianchi identity. Considering the historical development of the theory, we refer to the system (39)–(43) as *Klein-Noether identities*.

Identity (43) enables us to use independently the identity

$$\partial_\alpha\left[\mathcal{U}_\sigma{}^\alpha\xi^\sigma + \mathcal{M}_\sigma{}^{\alpha\tau}\partial_\tau\xi^\sigma + \mathcal{N}_\sigma{}^{\alpha\tau\beta}\,\partial_{\beta\tau}\xi^\sigma\right] \equiv 0 \tag{44}$$

instead of (35). Here, the vector density under the divergence is usually classified as a current:

$$\mathcal{I}^\alpha(\xi) \equiv -\left[\mathcal{U}_\sigma{}^\alpha\xi^\sigma + \mathcal{M}_\sigma{}^{\alpha\tau}\,\partial_\tau\xi^\sigma + \mathcal{N}_\sigma{}^{\alpha\tau\beta}\,\partial_{\beta\tau}\xi^\sigma\right]. \tag{45}$$

The negative sign is chosen to match the conventional negative sign in front of the gravitational (metric) action (see, for example, (17) and (26)). As a result, identity (44) is rewritten in a compact form:

$$\partial_\alpha \mathcal{I}^\alpha(\xi) \equiv 0\,. \tag{46}$$

Since (46) is an identity, the current must be expressed through a tensorial quantity (superpotential), $\mathcal{I}^\alpha(\xi) \equiv \partial_\beta \mathcal{I}^{\alpha\beta}(\xi)$, whose double divergence must identically vanish: $\partial_{\alpha\beta}\mathcal{I}^{\alpha\beta}(\xi) \equiv 0$. Let us show this. Keeping in mind the symmetry in the last two indexes of (38) and identity (42), we obtain

$$\mathcal{N}_\sigma{}^{\alpha\tau\beta} + \mathcal{N}_\sigma{}^{\tau\beta\alpha} + \mathcal{N}_\sigma{}^{\beta\alpha\tau} \equiv 0\,. \tag{47}$$

Substituting (40) into (45) and using (41) and (47), we derive

$$\mathcal{I}^\alpha(\xi) \equiv \partial_\beta\left(\mathcal{M}_\sigma{}^{\beta\alpha}\xi^\sigma + 2\mathcal{N}_\sigma{}^{\beta\alpha\lambda}\,\partial_\lambda\xi^\sigma\right). \tag{48}$$

As follows from (46), the divergence of the right-hand side of (48) is expected to vanish. To show this, we add a term identically equal to zero,

$$\frac{4}{3}\,\partial_{\beta\lambda}\left(\hat{N}_\sigma{}^{[\lambda\beta]\alpha}\xi^\sigma\right) \equiv 0\,,$$

to the right-hand side of (48). Then, using (41) and (47), we find

$$\mathcal{I}^\alpha(\xi) \equiv \partial_\beta\left(-\mathcal{M}_\sigma{}^{[\alpha\beta]}\xi^\sigma + \frac{2}{3}\,\partial_\lambda\mathcal{N}_\sigma{}^{[\alpha\beta]\lambda}\xi^\sigma - \frac{4}{3}\mathcal{N}_\sigma{}^{[\alpha\beta]\lambda}\,\partial_\lambda\xi^\sigma\right). \tag{49}$$

The expression in square brackets in (49) is explicitly antisymmetric in α and β, and therefore its double divergence indeed vanishes.

As a result, the current expression (49) can be written in the initially assumed form:

$$\mathcal{I}^\alpha(\xi) \equiv \partial_\beta \mathcal{I}^{\alpha\beta}(\xi)\,, \tag{50}$$

where

$$\mathcal{I}^{\alpha\beta}(\xi) \equiv -\left(\mathcal{M}_\sigma{}^{[\alpha\beta]}\xi^\sigma - \frac{2}{3}\partial_\lambda\mathcal{N}_\sigma{}^{[\alpha\beta]\lambda}\xi^\sigma + \frac{4}{3}\mathcal{N}_\sigma{}^{[\alpha\beta]\lambda}\,\partial_\lambda\xi^\sigma\right) \tag{51}$$

is called a (Noether) *superpotential*. Of course, identity (50) can be considered to be equivalent to the conservation law (46) for the current.

One can show that both the above constructed current and superpotential are tensorial quantities (for details, see [26]). Thus, $\mathcal{I}^\alpha(\xi)$ is a vector density with weight $+1$, whereas $\mathcal{I}^{\alpha\beta}(\xi)$ is an antisymmetric tensor density with weight $+1$. For such quantities, $\partial_\mu \equiv \overset{\circ}{\nabla}_\mu$. Through this, conservation laws (46) and (50) can be rewritten in an evidently covariant form:

$$\overset{\circ}{\nabla}_\alpha \mathcal{I}^\alpha(\xi) \equiv 0\,, \tag{52}$$

$$\mathcal{I}^\alpha(\xi) \equiv \overset{\circ}{\nabla}_\beta \mathcal{I}^{\alpha\beta}(\xi)\,. \tag{53}$$

Recall that, here, a covariant derivative $\overset{\circ}{\nabla}_\alpha$ is constructed with the use of the Levi-Civita connection.

Finally, let us define integral conserved quantities. Integrating through a 4-volume identity (46), one obtains a conserved (in time with related boundary conditions) quantity $\mathcal{P}(\xi)$ placed in a 3-dimensional section $\Sigma := x^0 = t = \text{const}$:

$$\mathcal{P}(\xi) = \int_\Sigma \mathrm{d}^3x\,\mathcal{I}^0(\xi)\,; \tag{54}$$

here and below, the '0'-component is related to a time coordinate, whereas Latin indexes from the middle of the alphabet, for example, 'i,' are related to space coordinates. The quantity (54), making use of (50), is reduced to a surface integral that is called the Noether charge:

$$\mathcal{P}(\xi) = \oint_{\partial\Sigma} \mathrm{d}s_i\,\mathcal{I}^{0i}(\xi)\,, \tag{55}$$

where $\partial\Sigma$ is a boundary of Σ, and $\mathrm{d}s_i$ is an element of integration on $\partial\Sigma$. Using the construction, the quantity $\mathcal{P}(\xi)$ both in (54) and in (55) is a scalar: it is invariant with respect to coordinate transformations.

It is important to outline the role of divergence in the Lagrangian in constructing conserved quantities. According to Chapter 7 in book [26], for every total divergence $\partial_\alpha\mathcal{D}^\alpha = \overset{\circ}{\nabla}_\alpha\mathcal{D}^\alpha$ of a vector density $\mathcal{D}^\alpha$ (not considering its inner structure), in the Lagrangian, it is possible to construct the conservation law in the form (50) or (53):

$$\mathcal{I}^\alpha_{\text{div}}(\xi) = \partial_\beta\mathcal{I}^{\alpha\beta}_{\text{div}}(\xi) = \overset{\circ}{\nabla}_\beta\mathcal{I}^{\alpha\beta}_{\text{div}}(\xi)\,. \tag{56}$$

The current $\mathcal{I}^\alpha_{\text{div}}$ and the superpotential $\mathcal{I}^{\alpha\beta}_{\text{div}}$ for the divergence of $\mathcal{D}^\alpha$ are defined as

$$\mathcal{I}^\alpha_{\text{div}} = \overset{\circ}{\nabla}_\beta(-\mathcal{M}_{(\text{div})\sigma}{}^{[\alpha\beta]}\xi^\sigma) = -\mathcal{U}_{(\text{div})\sigma}{}^\alpha\xi^\sigma - \mathcal{M}_{(\text{div})\sigma}{}^{[\alpha\beta]}\overset{\circ}{\nabla}_\beta\xi^\sigma\,, \tag{57}$$

$$\mathcal{I}^{\alpha\beta}_{\text{div}} = -\mathcal{M}_{(\text{div})\sigma}{}^{[\alpha\beta]}\xi^\sigma \tag{58}$$

with

$$\mathcal{M}_{(\mathrm{div})\sigma}{}^{[\alpha\beta]} = 2\delta_\sigma^{[\alpha}\mathcal{D}^{\beta]}\,, \tag{59}$$

$$\mathcal{U}_{(\mathrm{div})\sigma}{}^{\alpha} = 2\overset{\circ}{\nabla}_\beta(\delta_\sigma^{[\alpha}\mathcal{D}^{\beta]}) \equiv 2\partial_\beta(\delta_\sigma^{[\alpha}\mathcal{D}^{\beta]})\,. \tag{60}$$

The related Noether charge is defined by analogy with (54) and (55):

$$\mathcal{P}_{\mathrm{div}}(\xi) = \int_\Sigma \mathrm{d}^3x\,\mathcal{I}^0_{\mathrm{div}}(\xi) = \oint_{\partial\Sigma} \mathrm{d}s_i\mathcal{I}^{0i}_{\mathrm{div}}(\xi)\,. \tag{61}$$

Finally, let us remark that, despite the divergence in the Lagrangian not contributing to the field equations, it explicitly adds terms to the Noether current, superpotential, and charge.

4. Covariant conserved quantities in teleparallel equivalent of general relativity and symmetric teleparallel equivalent of general relativity

4.1 Teleparallel equivalent of general relativity

The TEGR Lagrangian (1) actually belongs to the class of Lagrangians (31). Therefore, the procedure for constructing currents and superpotentials, as described in the previous section, can be applied to (1). However, (1) does not contain second derivatives that make the procedure simpler.

In the TEGR case, the collective field ψ^A is represented as a set of covariant tetrad vectors and the inertial spin connection: $\psi^A = \{h^a{}_\rho; \overset{\bullet}{A}{}^a{}_{b\mu}\}$. Despite the fact that the inertial spin connection $\overset{\bullet}{A}{}^a{}_{b\mu}$ is not a dynamical field, its variation must be taken into account, because diffeomorphisms act on all geometric objects under consideration. Thus, because the ISC is a vector in a lower spacetime index, the variations (32) are rewritten as

$$\delta h^a{}_\rho = \mathscr{L}_\xi h^a{}_\rho = -\xi^\alpha h^a{}_{\rho,\alpha} - \xi^\alpha{}_{,\rho}\, h^a{}_\alpha\,, \tag{62}$$

$$\delta\overset{\bullet}{A}{}^a{}_{b\mu} = \mathscr{L}_\xi \overset{\bullet}{A}{}^a{}_{b\mu} = -\xi^\alpha \overset{\bullet}{A}{}^a{}_{b\mu,\alpha} - \xi^\alpha{}_{,\mu}\,\overset{\bullet}{A}{}^a{}_{b\alpha}\,. \tag{63}$$

Then, the coefficients (36)–(38) constructed with the covariant TEGR Lagrangian (1) acquire the form

$$\overset{\bullet}{\mathcal{U}}_\sigma{}^\alpha \equiv \overset{\bullet}{\mathcal{L}}\delta^\alpha_\sigma + \frac{\delta\overset{\bullet}{\mathcal{L}}}{\delta h^a{}_\rho}\, h^a{}_\rho\big|^\alpha_\sigma + \frac{\partial\overset{\bullet}{\mathcal{L}}}{\partial\overset{\bullet}{A}{}^a{}_{b\mu}}\,\overset{\bullet}{A}{}^a{}_{b\mu}\big|^\alpha_\sigma - \frac{\partial\overset{\bullet}{\mathcal{L}}}{\partial h^a{}_{\rho,\alpha}}\, h^a{}_{\rho,\sigma}\,, \tag{64}$$

$$\overset{\bullet}{\mathcal{M}}_\sigma{}^{\alpha\tau} \equiv \frac{\partial\overset{\bullet}{\mathcal{L}}}{\partial\, h^a{}_{\rho,\alpha}}\, h^a{}_\rho\big|^\tau_\sigma\,, \tag{65}$$

$$\overset{\bullet}{\mathcal{N}}_\sigma{}^{\alpha\tau\beta} \equiv 0\,. \tag{66}$$

The third term in (64), making the use of the structure of (1) and (6), is transformed as

$$\frac{\partial\overset{\bullet}{\mathcal{L}}}{\partial\overset{\bullet}{A}{}^a{}_{b\mu}}\,\overset{\bullet}{A}{}^a{}_{b\mu}\big|^\alpha_\sigma = -\frac{\partial\overset{\bullet}{\mathcal{L}}}{\partial h^a{}_{\rho,\alpha}}\,\overset{\bullet}{A}{}^a{}_{b\sigma}\, h^b{}_\rho\,. \tag{67}$$

Then, current (45) acquires the form

$$\overset{\bullet}{\mathcal{J}}{}^\alpha(\xi) \equiv \left[h\overset{\bullet}{\theta}_\sigma{}^\alpha + \frac{\delta\overset{\bullet}{\mathcal{L}}}{\delta h^a{}_\alpha} h^a{}_\sigma\right]\xi^\sigma + \frac{\partial\overset{\bullet}{\mathcal{L}}}{\partial h^a{}_{\rho,\alpha}}\, h^a{}_\sigma\partial_\rho\xi^\sigma\,, \tag{68}$$

where the term $\overset{\bullet}{\theta}_\sigma{}^\alpha$ is interpreted as the energy–momentum tensor of the gravitational field in the covariant TEGR:

$$\overset{\bullet}{\theta}_\sigma{}^\alpha \equiv \frac{1}{h}\left[\frac{\partial\overset{\bullet}{\mathcal{L}}}{\partial h^a{}_{\rho,\alpha}}\left(\partial_\sigma h^a{}_\rho + \overset{\bullet}{A}{}^a{}_{b\sigma}\, h^b{}_\rho\right) - \overset{\bullet}{\mathcal{L}}\delta^\alpha_\sigma\right]. \tag{69}$$

Current (68) and energy–momentum (69) are expressed through partial derivatives $\partial_\sigma h^a{}_\rho$ and $\partial_\rho\xi^\sigma$. However, they are easily rewritten in the explicitly covariant form, replacing partial derivatives in (68) and (69) simultaneously by $\overset{\bullet}{\nabla}_\sigma h^a{}_\rho$ and $\overset{\circ}{\nabla}_\rho\xi^\sigma$, where a covariant derivative $\overset{\circ}{\nabla}_\alpha$ is constructed with the metric (2). Due to definitions (7), (8) and

$$\frac{\partial\overset{\bullet}{\mathcal{L}}}{\partial h^a{}_{\rho,\sigma}} \equiv -\frac{h}{\kappa}\,\overset{\bullet}{S}_a{}^{\rho\sigma}\,, \tag{70}$$

current (68) and the energy–momentum tensor (69) are represented as

$$\overset{\bullet}{\mathcal{J}}{}^\alpha(\xi) \equiv \left[h\overset{\bullet}{\theta}_\sigma{}^\alpha + \frac{\delta\overset{\bullet}{\mathcal{L}}}{\delta h^a{}_\alpha}\, h^a{}_\sigma\right]\xi^\sigma + \frac{h}{\kappa}\,\overset{\bullet}{S}_\sigma{}^{\alpha\rho}\overset{\circ}{\nabla}_\rho\xi^\sigma\,, \tag{71}$$

$$\overset{\bullet}{\theta}_\sigma{}^\alpha \equiv \frac{1}{\kappa}\,\overset{\bullet}{S}_a{}^{\alpha\rho}\overset{\bullet}{K}{}^a{}_{\sigma\rho} - \frac{1}{h}\,\overset{\bullet}{\mathcal{L}}\delta^\alpha_\sigma\,. \tag{72}$$

It can easily be seen that they are explicitly covariant with respect to coordinate transformations and Lorentz rotations.

Current (71) is identically conserved by (46) and (52):

$$\partial_\alpha\overset{\bullet}{\mathcal{J}}{}^\alpha(\xi) \equiv \overset{\circ}{\nabla}_\alpha\overset{\bullet}{\mathcal{J}}{}^\alpha(\xi) \equiv 0\,. \tag{73}$$

Similarly, by conservation laws (50) and (53), the TEGR Noether current is expressed by the divergence of the TEGR Noether superpotential:

$$\overset{\bullet}{\mathcal{J}}{}^\alpha(\xi) \equiv \partial_\alpha\overset{\bullet}{\mathcal{J}}{}^{\alpha\beta}(\xi) \equiv \overset{\circ}{\nabla}_\alpha\overset{\bullet}{\mathcal{J}}{}^{\alpha\beta}(\xi)\,, \tag{74}$$

where the superpotential (see (51)) is presented with the use of (65) and (70):

$$\overset{\bullet}{\mathcal{J}}{}^{\alpha\beta}(\xi) = -\overset{\bullet}{\mathcal{M}}_\sigma{}^{\alpha\beta}\xi^\sigma = \frac{\partial\overset{\bullet}{\mathcal{L}}}{\partial h^a{}_{\beta,\alpha}}\, h^a{}_\sigma\xi^\sigma = \frac{h}{\kappa}\,\overset{\bullet}{S}_a{}^{\alpha\beta}\, h^a{}_\sigma\xi^\sigma\,. \tag{75}$$

Relations (73) and (74) are still identities at this stage; they lack physical significance, because the field equations have not yet been used. After applying the field equations (19), current (71) transforms into

$$\overset{\bullet}{\mathcal{J}}{}^\alpha(\xi) = h\left[\overset{\bullet}{\theta}_\sigma{}^\alpha + \theta_\sigma{}^\alpha\right]\xi^\sigma + \frac{h}{\kappa}\,\overset{\bullet}{S}_\sigma{}^{\alpha\rho}\,\overset{\circ}{\nabla}_\rho\xi^\sigma\,, \tag{76}$$

where $\theta_\sigma{}^\alpha$ is the matter energy–momentum tensor introduced in (19). Then, identities (73) and (74) become physically meaningful differential conservation laws:

$$\partial_\alpha\overset{\bullet}{\mathcal{J}}{}^\alpha(\xi) = \overset{\circ}{\nabla}_\alpha\overset{\bullet}{\mathcal{J}}{}^\alpha(\xi) = 0\,, \tag{77}$$

$$\overset{\bullet}{\mathcal{J}}{}^\alpha(\xi) = \partial_\beta\overset{\bullet}{\mathcal{J}}{}^{\alpha\beta}(\xi) = \overset{\circ}{\nabla}_\beta\overset{\bullet}{\mathcal{J}}{}^{\alpha\beta}(\xi)\,. \tag{78}$$

All the local conserved quantities are covariant under coordinate transformations and invariant under local Lorentz rotations. Consequently, integral conserved quantities that can be constructed by the aforementioned standard rules exhibit the same properties:

$$\mathcal{P}(\xi) = \int_{\Sigma} \mathrm{d}^3 x\, \overset{\bullet}{\mathcal{J}}{}^0(\xi) = \oint_{\partial\Sigma} \mathrm{d}s_i\, \overset{\bullet}{\mathcal{J}}{}^{0i}(\xi)\,. \tag{79}$$

Finally, it can be concluded that the problem of constructing fully covariant conservation laws, noted during the discussion in the Introduction, has been *solved through the construction* (77)–(79). It is worth noting that this result was achieved not only through the use of the inertial spin connection but also by retaining in expressions the displacement vector ξ after applying the Noether theorem.

4.2 Symmetric teleparallel equivalent of general relativity

All items in the STEGR Lagrangian (28) are scalar densities; therefore, it is logical to consider each of them separately when the Noether theorem is applied. Concerning the Hilbert Lagrangian (26), the collective field ψ^A is represented simply as $\psi^A \in \{g_{\mu\nu}\}$. As a result, the current and superpotential are classical, and the reader can be sent to books [24] and [26]. The related Noether current is derived as

$$\mathcal{J}^{\alpha}_{GR}(\xi) \equiv -(\mathcal{U}_{\sigma}{}^{\alpha}\xi^{\sigma} + \mathcal{M}_{\sigma}{}^{\alpha\tau}\partial_{\tau}\xi^{\sigma} + \mathcal{N}_{\sigma}{}^{\alpha\tau\beta}\partial_{\tau\beta}\xi^{\sigma})\,, \tag{80}$$

with the coefficients

$$\mathcal{N}_{\sigma}{}^{\alpha\tau\beta} = \frac{\sqrt{-g}}{4\kappa}\,(2g^{\tau\beta}\delta^{\alpha}_{\sigma} - g^{\alpha\beta}\delta^{\tau}_{\sigma} - g^{\alpha\tau}\delta^{\beta}_{\sigma})\,, \tag{81}$$

$$\mathcal{M}_{\sigma}{}^{\alpha\tau} = \frac{\sqrt{-g}}{2\kappa}(2\overset{\circ}{\Gamma}{}^{\alpha}{}_{\sigma\omega}g^{\tau\omega} - \overset{\circ}{\Gamma}{}^{\omega}{}_{\sigma\omega}\, g^{\alpha\tau} - \overset{\circ}{\Gamma}{}^{\tau}{}_{\omega\epsilon}g^{\omega\epsilon}\delta^{\alpha}_{\sigma})\,, \tag{82}$$

$$\begin{aligned}\mathcal{U}_{\sigma}{}^{\alpha} = {} & \frac{\sqrt{-g}}{2\kappa}(g^{\alpha\lambda}g^{\omega\epsilon} - g^{\alpha\epsilon}g^{\omega\lambda})(g_{\lambda\sigma,\,\omega\epsilon} + \overset{\circ}{\Gamma}{}^{\nu}{}_{\omega\epsilon}\overset{\circ}{\Gamma}{}_{\nu|\lambda\sigma} \\ & - \overset{\circ}{\Gamma}{}^{\nu}{}_{\omega\epsilon}\overset{\circ}{\Gamma}{}_{\lambda|\nu\sigma}) = -\frac{\sqrt{-g}}{\kappa}\Bigg(G^{\alpha}_{\sigma} + \frac{1}{2}\,\delta^{\alpha}_{\sigma}R + \frac{1}{2}g^{\alpha\omega}\overset{\circ}{\Gamma}{}^{\rho}{}_{\rho(\omega,\,\sigma)} \\ & - \frac{1}{2}\,g^{\omega\epsilon}\overset{\circ}{\Gamma}{}^{\alpha}{}_{\omega\epsilon,\,\sigma}\Bigg)\,,\end{aligned} \tag{83}$$

where the notation $\overset{\circ}{\Gamma}{}_{\alpha|\beta\gamma} = g_{\alpha\rho}\,\overset{\circ}{\Gamma}{}^{\rho}{}_{\beta\gamma}$ is used.

It is important to represent the current (and the related superpotential) in an explicitly covariant form. The evident identities

$$\partial_{\beta}\xi^{\sigma} \equiv \overset{\circ}{\nabla}_{\beta}\xi^{\sigma} - \overset{\circ}{\Gamma}{}^{\sigma}{}_{\lambda\beta}\xi^{\lambda}\,, \tag{84}$$

$$\begin{aligned}\partial_{\tau}\partial_{\beta}\xi^{\sigma} \equiv {} & \overset{\circ}{\nabla}_{\tau}\overset{\circ}{\nabla}_{\beta}\xi^{\sigma} + \big[\delta^{\sigma}_{\lambda}\overset{\circ}{\Gamma}{}^{\rho}{}_{\beta\tau} - \delta^{\rho}_{\beta}\,\overset{\circ}{\Gamma}{}^{\sigma}{}_{\lambda\beta} - \delta^{\rho}_{\beta}\overset{\circ}{\Gamma}{}^{\sigma}{}_{\lambda\tau}\big]\overset{\circ}{\nabla}_{\rho}\xi^{\lambda} \\ & - \big[\overset{\circ}{\Gamma}{}^{\sigma}{}_{\lambda\beta,\,\tau} - \overset{\circ}{\Gamma}{}^{\sigma}{}_{\rho\beta}\,\overset{\circ}{\Gamma}{}^{\rho}{}_{\lambda\tau}\big]\xi^{\lambda}\end{aligned} \tag{85}$$

are used. Substituting (84) and (85) into (80), one just rewrites the GR current in a covariant form:

$$\mathcal{J}^{\alpha}_{GR}(\xi) \equiv -(\mathcal{U}{*}_{\sigma}{}^{\alpha}\xi^{\sigma} + \mathcal{M}{*}_{\sigma}{}^{\alpha\tau}\overset{\circ}{\nabla}_{\tau}\xi^{\sigma} + \mathcal{N}{*}_{\sigma}{}^{\alpha\tau\beta}\,\overset{\circ}{\nabla}_{(\tau}\overset{\circ}{\nabla}_{\beta)}\xi^{\sigma})\,, \tag{86}$$

with covariantized coefficients

$$\mathcal{N}{*}_{\sigma}{}^{\alpha\tau\beta} = \mathcal{N}_{\sigma}{}^{\alpha\tau\beta}\,, \tag{87}$$

$$\mathcal{M}{*}_{\sigma}{}^{\alpha\tau} = \mathcal{M}_{\sigma}{}^{\alpha\tau} - 2\mathcal{N}_{\lambda}{}^{\alpha\tau\beta}\overset{\circ}{\Gamma}{}^{\lambda}{}_{\sigma\beta} + \mathcal{N}_{\sigma}{}^{\alpha\lambda\beta}\overset{\circ}{\Gamma}{}^{\tau}{}_{\beta\lambda} = 0\,, \tag{88}$$

$$\begin{aligned}\mathcal{U}{*}_{\sigma}{}^{\alpha} = {} & \mathcal{U}_{\sigma}{}^{\alpha} - \mathcal{M}_{\lambda}{}^{\alpha\tau}\overset{\circ}{\Gamma}{}^{\lambda}{}_{\sigma\tau} - \mathcal{N}_{\lambda}{}^{\alpha\tau\beta}\partial_{\tau}\overset{\circ}{\Gamma}{}^{\lambda}{}_{\sigma\beta} \\ & + \mathcal{N}_{\kappa}{}^{\alpha\tau\beta}\overset{\circ}{\Gamma}{}^{\kappa}{}_{\lambda\beta}\overset{\circ}{\Gamma}{}^{\lambda}{}_{\sigma\tau} = -\frac{\sqrt{-g}}{4\kappa}g^{\alpha\omega}\overset{\circ}{R}_{\omega\sigma}\,.\end{aligned} \tag{89}$$

Thus, the current (86) acquires an evidently covariant form:

$$\mathcal{J}^{\alpha}_{GR}(\xi) \equiv \frac{\sqrt{-g}}{4\kappa}\left(\overset{\circ}{R}{}^{\alpha}{}_{\sigma}\xi^{\sigma} + 2g^{\alpha\beta}\overset{\circ}{\nabla}_{(\beta}\overset{\circ}{\nabla}_{\sigma)}\xi^{\sigma} - 2\overset{\circ}{\nabla}_{\beta}\overset{\circ}{\nabla}{}^{\beta}\xi^{\alpha}\right). \tag{90}$$

To obtain a covariantized superpotential related to the Hilbert Lagrangian (26), one substitutes (83), (82), and (81) into definition (49) and gets

$$\mathcal{J}^{\alpha\beta}_{GR} \equiv \mathcal{K}^{\alpha\beta} = \frac{\sqrt{-g}}{\kappa}\overset{\circ}{\nabla}{}^{[\alpha}\xi^{\beta]}\,, \tag{91}$$

which is Komar's well-known superpotential [24, 26].

Let us turn to the divergence in the STEGR Lagrangian (28). The formalism given in (57)–(60), of course, can be applied to the divergence $\mathcal{L}'$ given in (27) with (29). Thus, one calculates the related Noether current (57) and superpotential (58) for $\mathcal{L}'$ in STEGR:

$$\mathcal{J}^{\alpha}_{\rm div} \equiv \frac{\sqrt{-g}}{\kappa}\,\delta^{[\alpha}_{\sigma}\overset{\circ}{\nabla}_{\beta}\big[(Q^{\beta]} - \hat{Q}^{\beta]})\xi^{\sigma}\big]\,, \tag{92}$$

$$\mathcal{J}^{\alpha\beta}_{\rm div} \equiv \frac{\sqrt{-g}}{\kappa}\,\delta^{[\alpha}_{\sigma}(Q^{\beta]} - \hat{Q}^{\beta]})\xi^{\sigma}\,. \tag{93}$$

Finally, the total currents and superpotentials of Lagrangian (28) in STEGR are

$$\mathcal{J}^{\alpha} \equiv \mathcal{J}^{\alpha}_{GR} + \mathcal{J}^{\alpha}_{\rm div}\,, \tag{94}$$

$$\mathcal{J}^{\alpha\beta} \equiv \mathcal{J}^{\alpha\beta}_{GR} + \mathcal{J}^{\alpha\beta}_{\rm div}\,. \tag{95}$$

Up to now, we have considered identities only. Thus, $\mathcal{J}^{\alpha}$ and $\mathcal{J}^{\alpha\beta}$ in (94) and (95) satisfy conservation laws like (46), (52) and (50), (53). They are identities as well. To obtain physically meaningful conservation laws, one has to use the field equations. Thus, after using (30), one rewrites (90) as

$$\begin{aligned}\mathcal{J}^{\alpha}_{GR}(\xi) \equiv \frac{\sqrt{-g}}{4\kappa}\Bigg[& \kappa\left(\theta^{\alpha}{}_{\sigma} - \frac{1}{2}\delta^{\alpha}_{\sigma}\theta^{\beta}{}_{\beta}\right)\xi^{\sigma} \\ & + 2g^{\alpha\beta}\overset{\circ}{\nabla}_{(\beta}\overset{\circ}{\nabla}_{\sigma)}\xi^{\sigma} - 2\overset{\circ}{\nabla}_{\alpha}\overset{\circ}{\nabla}{}^{\alpha}\xi^{\sigma}\Bigg]\,.\end{aligned} \tag{96}$$

After that, the currents and superpotentials (94) and (95) satisfy the physically meaning conservation laws:

$$\partial_{\alpha}\mathcal{J}^{\alpha}(\xi) = \overset{\circ}{\nabla}_{\alpha}\mathcal{J}^{\alpha}(\xi) = 0\,, \tag{97}$$

$$\mathcal{J}^{\alpha}(\xi) = \partial_{\beta}\mathcal{J}^{\alpha\beta}(\xi) = \overset{\circ}{\nabla}_{\beta}\mathcal{J}^{\alpha\beta}(\xi) = 0\,. \tag{98}$$

All the local conserved quantities are covariant under coordinate transformations, and conservation laws for them make it possible to construct integral conserved quantities in a noncontradictory way:

$$\mathcal{P}(\xi) = \int_{\Sigma} \mathrm{d}^3 x\, \mathcal{J}^0(\xi) = \oint_{\partial\Sigma} \mathrm{d}s_i\, \mathcal{J}^{0i}(\xi)\,. \tag{99}$$

This result was achieved by making use of the Noether theorem and preserving in expressions the displacement vector ξ after its application.

4.3 Interpretation of conserved quantities

To interpret conserved quantities in TEGR and STEGR defined above, it is instructive to outline Minkowski space with matter propagating through it in the simplest case. Let us consider a flat spacetime in coordinates where the metric has the form

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\,. \tag{100}$$

Coordinates are numbered as $(t, x, y, z) = (x^0, x^i) = (x^\alpha)$, where $i = 1, 2, 3$. To organize this spacetime as a reference frame, one has to add observers to it. Let observers be static with proper vectors

$$\xi^\alpha = (-1, 0, 0, 0)\,. \tag{101}$$

One assumes that the matter in the Minkowski space has differentially conserved symmetric energy–momentum tensor $\Theta^\alpha{}_\beta := \partial_\alpha \Theta^\alpha{}_\beta = 0$. As a result, the related current $J^\alpha(\xi) = \Theta^\alpha{}_\beta \xi^\beta$ is differentially conserved, as well, $\partial_\alpha J^\alpha(\xi) = 0$. Its components present the energy density $J^0 = \Theta^0{}_0 \xi^0$ and the momentum density $J^i = \Theta^i{}_0 \xi^0$ measured by the aforementioned observers.

It is evident that the current $\overset{\bullet}{\mathcal{J}}{}^\alpha(\xi)$ defined in (76) in TEGR and the current $\mathcal{J}^\alpha(\xi)$ defined in (94) with (96) in STEGR generalize the simplest current definition $J^\alpha(\xi)$. Consequently, their components have an analogous interpretation for observers. The generalization is related to the observer proper vectors ξ given in (101). They can be arbitrary timelike vectors with which a reference frame can be associated. A proper vector of a freely falling observer can be chosen as an extreme vector ξ. Then, owing to the Einstein equivalence principle, all the components of the currents have to be equal to zero: $\overset{\bullet}{\mathcal{J}}{}^\alpha(\xi) = \mathcal{J}^\alpha(\xi) = 0$.

The Noether charges $\mathcal{P}(\xi)$ defined in TEGR (79) and in STEGR (99) also depend on the vector field ξ. Because, as a rule, they are defined on surfaces abounding (possibly at infinity) isolated systems, ξ is chosen as a vector reflecting symmetries of a spacetime (for example, the Killing vector) at the boundary. When the vector ξ is timelike, the corresponding conserved charge, constrained to the surface $\partial\Sigma$, represents the energy of this restricted (or infinite) volume. In this case, it can be classified as energy measured by a set of observers on $\partial\Sigma$ with their own vectors ξ.

5. 'Turning off' gravity principle and 'gauges'

In Sections 1 and 2, it was remarked that teleparallel connections both in TEGR and in STEGR are external structures and cannot be determined inside the theory itself. One has to chose them using additional principles introduced for each concrete solution [1, 28]. Moreover, such a choice is not unique. Our principles generalize those in earlier studies by other authors and are based on simple, natural requirements as follows. All the currents, superpotentials, and charges constructed above must vanish for systems where the gravitational field is absent.

5.1 Teleparallel equivalent of general relativity

In TEGR, Noether's current (71) or (76), $\overset{\bullet}{\mathcal{J}}{}^\alpha(\xi)$, with energy–momentum (72), $\overset{\bullet}{\theta}_\sigma{}^\alpha$, and superpotential (75), $\overset{\bullet}{\mathcal{J}}{}^{\alpha\beta}(\xi)$ are proportional to contortion components $\overset{\bullet}{K}{}^a{}_{c\mu}$ (or, alternatively $\overset{\bullet}{T}{}^\alpha{}_{\mu\nu}$ or $\overset{\bullet}{S}_a{}^{\mu\nu}$), and these quantities must vanish in the absence of gravity. Following this requirement, we turn to formula (8) and formulate the list of rules for determining ISC $\overset{\bullet}{A}{}^a{}_{c\mu}$:

(1) for the GR solution under consideration, we choose a convenient tetrad and define L-CSC $\overset{\circ}{A}{}^a{}_{c\mu}$ by (7);

(2) then, we construct a related curvature tensor of the L-CSC determined above:

$$\overset{\circ}{R}{}^i{}_{j\mu\nu} = \partial_\mu \overset{\circ}{A}{}^i{}_{j\nu} - \partial_\nu \overset{\circ}{A}{}^i{}_{j\mu} + \overset{\circ}{A}{}^i{}_{k\mu} \overset{\circ}{A}{}^k{}_{j\nu} - \overset{\circ}{A}{}^i{}_{k\nu} \overset{\circ}{A}{}^k{}_{j\mu}\,;$$

(3) to 'switch off' gravity, we solve the absent gravity equation $\overset{\circ}{R}{}^a{}_{b\gamma\delta} = 0$ for the parameters in the chosen GR solution;

(4) then, for the parameters satisfying $\overset{\circ}{R}{}^a{}_{b\gamma\delta} = 0$, we take $\overset{\circ}{A}{}^a{}_{c\mu} = \overset{\bullet}{A}{}^a{}_{c\mu}$.

Recall that both $\overset{\bullet}{\mathcal{J}}{}^\alpha(\xi)$, and $\overset{\bullet}{\mathcal{J}}{}^{\alpha\beta}(\xi)$ are explicitly spacetime covariant and Lorentz invariant. Therefore, if the chosen tetrad and the determined (as above) ISC are transformed by local Lorentz rotations as in (12) and (13) simultaneously and/or arbitrary coordinate transformations, then conserved quantities are left covariant (invariant). All of these can be reformulated as follows. The initial pair of the tetrad and ISC defines a set of the pairs connected with the initial one by Lorentz rotations and/or arbitrary coordinate transformations for which conserved quantities are covariant (invariant). We call such a set of pairs 'gauge.'

However, it turns out that, after applying the 'turning off' gravity principle in TEGR, the result of determining the ISC is ambiguous. It depends on the tetrad that we choose from the start. As a result, we obtain different pairs of the tetrad and ISC, that is, different 'gauges,' where pairs are not connected by local Lorentz rotations and/or arbitrary coordinate transformations. For different 'gauges,' $\overset{\bullet}{\mathcal{J}}{}^\alpha(\xi)$ and $\overset{\bullet}{\mathcal{J}}{}^{\alpha\beta}(\xi)$ are different as well, which gives us different values of conserved quantities. Therefore, one of the main purposes in constructing conserved quantities is to find the appropriate gauges in TEGR in which we would have physically meaningful results for the concrete solutions.

Finally, traditionally, the Wietzenböck gauge idea is used [3]. In this case, the word 'gauge' is used in a different sense: when one says 'Wietzenböck gauge,' only one pair — the tetrad and zero ISC — is meant; in our definition, the 'gauge' means the whole equivalence class of pairs of tetrads and ISCs connected as defined above.

5.2 Symmetric teleparallel equivalent of general relativity

In STEGR, to define the undetermined STC, we use the 'turning off' gravity principle, adapted from TEGR. It is based on the assumption that the total current (94), $\mathcal{J}^\alpha(\xi)$, and superpotential (95), $\mathcal{J}^{\alpha\beta}(\xi)$, have to vanish in the absence of gravity. It is reasonable to propose that all of the terms in (94) and (95) vanish separately. Then, $Q_{\alpha\mu\nu}$ or $L^\alpha{}_{\mu\nu}$, in addition to $\overset{\circ}{R}{}^\alpha{}_{\beta\mu\nu}$ in GR, have to vanish in the absence of gravity. As a result, we formulate the next steps for finding the STC in STEGR:

(1) for a known GR solution, we construct the related Riemannian curvature tensor of the Levi-Civita connection:

$$\overset{\circ}{R}{}^\alpha{}_{\beta\mu\nu} = \partial_\mu \overset{\circ}{\Gamma}{}^\alpha{}_{\beta\nu} - \partial_\nu \overset{\circ}{\Gamma}{}^\alpha{}_{\beta\mu} + \overset{\circ}{\Gamma}{}^\alpha{}_{\kappa\mu} \overset{\circ}{\Gamma}{}^\kappa{}_{\beta\nu} - \overset{\circ}{\Gamma}{}^\alpha{}_{\kappa\nu} \overset{\circ}{\Gamma}{}^\kappa{}_{\beta\mu}\,;$$

(2) to 'switch off' gravity, we solve the absent gravity equation $\overset{\circ}{R}{}^\alpha{}_{\beta\mu\nu} = 0$ for the parameters in the chosen GR solution;

(3) then, for the found parameters satisfying $\overset{\circ}{R}{}^\alpha{}_{\beta\mu\nu} = 0$, $\Gamma^\alpha{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^\alpha{}_{\mu\nu}$ is taken for the chosen solution.

The torsion of the found connection should be zero automatically, because we take it from the Levi-Civita connection for some parameter values, and the Levi-Civita connection is always symmetric. The curvature of the found connection should be zero too, because we found it from the equation $\overset{\circ}{R}{}^\alpha{}_{\beta\gamma\delta} = 0$.

To have the same terminology in STEGR as in TEGR, we formally define a pair of coordinates (x^α) and connection $\Gamma^\alpha{}_{\mu\nu}$, together with the set of pairs which are connected to it by the transformations for coordinates $x^\alpha = x^\alpha(\bar{x})$ and STC,

$$\Gamma^\alpha{}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \bar{x}^{\bar{\alpha}}} \frac{\partial \bar{x}^{\bar{\mu}}}{\partial x^\mu} \frac{\partial \bar{x}^{\bar{\nu}}}{\partial x^\nu} \bar{\Gamma}^{\bar{\alpha}}{}_{\bar{\mu}\bar{\nu}} + \frac{\partial x^\alpha}{\partial \bar{x}^{\bar{\lambda}}} \frac{\partial}{\partial x^\mu}\left(\frac{\partial \bar{x}^{\bar{\lambda}}}{\partial x^\nu}\right),$$

as a 'gauge.' When we say 'gauge' in our definition, we mean the set of all possible coordinates and the values of STCs in them, such that the above relation $\Gamma^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu}(\bar{\Gamma})$ is satisfied under all possible coordinate only transformations. It is the equivalence class. It differs only from the case of a zero STC, which is called 'coincident gauge' [29, 30]. It is only one case of coordinates.

Just as in TEGR, the result of determining the STC in STEGR is ambiguous: it depends on the coordinates that we choose from the start. This leads to different 'gauges' for which one obtains different values for conserved quantities. Therefore, again, one of the main purposes in constructing conserved quantities in STEGR is to find appropriate 'gauges' in which we would have physically meaningful results for the concrete solutions.

5.3 Appropriate gauges

A requirement when constructing *covariant conserved quantities* in TEGR and STEGR for this or that solution (model) is that the possible choice of ISCs and STCs be restricted. Such a situation is not new in metric GR, where the classical energy–momentum pseudotensors and superpotentials are not covariant (see Chapter 1 in book [26]). One of the many ways to improve the situation is to use the bi-metric representation of GR, for example, as suggested in [31]. Introducing an auxiliary background metric $\bar{g}_{\mu\nu}$ and making use of the Noether theorem, one can construct covariant energy–momentum tensors and superpotentials. The Einstein equations are preserved, but the currents and superpotentials greatly depend on $\bar{g}_{\mu\nu}$. Although, in the general formalism, $\bar{g}_{\mu\nu}$ is arbitrary, an appropriate choice of $\bar{g}_{\mu\nu}$ essentially depends on the solution under consideration. For example, it can be a flat background if one considers an isolated system with an asymptotically flat metric. However, even if a concrete system is chosen for the analysis, ambiguity in the choice of $\bar{g}_{\mu\nu}$ can remain.

Returning to TEGR and STEGR, we stress that, on the one hand, our formalism for constructing conserved quantities in TEGR and STEGR is a general one, because it presents a general method. On the other hand, it is highly solution-dependent and is thus not generally applicable, because gauges have to be determined for each concrete solution separately. The role of the ISC and STC in TEGR and STEGR is analogous to $\bar{g}_{\mu\nu}$ in the metric GR. Thus, there is no unique energy–momentum tensor, unlike in electrodynamics. Such a situation is not surprising in GR according to the Einstein equivalence principle.

In concluding this section, we establish a rigorous framework for conserved quantities in TEGR and STEGR. The method of 'turning off gravity' is applied to define the teleparallel connections in both TEGR and STEGR, where the choice of the initial tetrad in TEGR or the initial coordinates in STEGR determines the resulting gauge that has to be found in the end as an appropriate one for a concrete solution.

6. Conclusion

In this paper, we have presented a unified, fully covariant formalism for constructing conserved quantities in TEGR and STEGR in a tensor form. On the whole, our methods differ from generally accepted ones. It has been demonstrated that the Noether theorem approach holds great potential for development. It already has important achievements in applications. We have at least covered all the application results of earlier authors and added new ones. Thus, we have described all the known astrophysical and cosmological examples, with special attention paid to the Schwarzschild solution (see for all of these [14–20]). We were the first to state the equivalence principle for the gravitational wave [21] and the first to calculate the angular momentum for the Kerr solution [22] in teleparallel equivalents of GR.

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